DOI: 10.1214/ECP.v12-1293

Elect. Comm. in Probab. **12** (2007), 248–256

ELECTRONIC COMMUNICATIONS in PROBABILITY

ON A THEOREM IN MULTI-PARAMETER POTENTIAL THEORY

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Submitted March 23, 2007, *accepted in final form* July 17, 2007

AMS 2000 Subject classification: 60G60, 60G51, 60G17
Keywords: Additive Lévy processes, Hausdorff dimension, multiple points.

Abstract
Let X be an N-parameter additive Lévy process in $\mathbb{R}^d$ with Lévy exponent $(\Psi_1, \cdots, \Psi_N)$ and let λ_d denote Lebesgue measure in $\mathbb{R}^d$. We show that

$$E\{\lambda_d(X(\mathbb{R}_+^N))\} > 0 \Longleftrightarrow \int_{\mathbb{R}^d} \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi < \infty.$$

This was previously proved by Khoshnevisan, Xiao and Zhong [1] under a sector condition.

1 Introduction and Proof

Let $X^1_{t_1},\ X^2_{t_2}, \cdots, X^N_{t_N}$ be N independent Lévy processes in $\mathbb{R}^d$ with their respective Lévy exponents $\Psi_j,\ j = 1, 2, \cdots, N$. The random field

$$X_t = X^1_{t_1} + X^2_{t_2} + \cdots + X^N_{t_N}, \qquad t = (t_1, t_2, \cdots, t_N) \in \mathbb{R}_+^N$$

is called the additive Lévy process. Let λ_d denote Lebesgue measure in $\mathbb{R}^d$.

Theorem 1.1 *Let X be an additive Lévy process in $\mathbb{R}^d$ with Lévy exponent $(\Psi_1, \cdots, \Psi_N)$. Then*

$$E\{\lambda_d(X(\mathbb{R}_+^N))\} > 0 \Longleftrightarrow \int_{\mathbb{R}^d} \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi < \infty. \tag{1.1}$$

Recently, Khoshnevisan, Xiao and Zhong [1] proved that if

$$\mathrm{Re}\left(\prod_{j=1}^{N} \frac{1}{1+\Psi_j(\xi)}\right) \geq \theta \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) \tag{1.2}$$

for some constant $\theta > 0$ then Theorem 1.1 holds. In fact the proof of Theorem 1.1 does not need any condition.

Proof of Theorem 1.1: Define

$$\mathcal{E}_\Psi(\mu) = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi$$

where μ is a probability measure on a compact set $F \subset \mathbb{R}^d$ and $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\mu(dx)$. Let $F = \{0\} \subset \mathbb{R}^d$ and δ_0 be the point mass at $0 \in \mathbb{R}^d$. We first quote a key lemma of [1]:

Lemma 5.5 *Suppose X is an additive Lévy process in $\mathbb{R}^d$ that satisfies Condition (1.3), and that $\int_{\mathbb{R}^d} \prod_{j=1}^N |1+\Psi_j(\xi)|^{-1} d\xi < +\infty$, where $\Psi = (\Psi_1, \cdots, \Psi_N)$ denotes the Lévy exponent of X. Then, for all compact sets $F \subset \mathbb{R}^d$, and for all $r > 0$,*

$$E\{\lambda_d(X([0,r]^N \oplus F)\} \le \theta^{-2}(4e^{2r})^N \cdot \mathcal{C}_\Psi(F),$$

where $\theta > 0$ is the constant in Condition (1.3).

By reviewing the whole process of the proof of Theorem 1.1 of [1] given by Khoshnevisan, Xiao and Zhong, our Theorem 1.1 certainly follows if we instead prove the following statement:

Let X be any additive Lévy process in $\mathbb{R}^d$. If $\int_{\mathbb{R}^d} \prod_{j=1}^N |1+\Psi_j(\xi)|^{-1} d\xi < +\infty$, then

$$E\{\lambda_d(X([0,r]^N))\} \le \frac{c_{N,d,r}}{\mathcal{E}_\Psi(\delta_0)} \tag{1.3}$$

for some constant $c_{N,d,r} \in (0,\infty)$ depending on $N,\ d,\ r$ only.

Clearly, all we have to do is to complete Eq. (5.11) of [1] without bothering ourselves with Condition (1.3) of [1]. Since δ_0 is the only probability measure on $F = \{0\}$, letting $\eta \to 0,\ k \to \infty$, and $\varepsilon \to 0$ and using the integrability condition $\int_{\mathbb{R}^d} \prod_{j=1}^N |1+\Psi_j(\xi)|^{-1} d\xi < +\infty$ yield

$$\mathcal{E}_\Psi(\delta_0) \ge c_1 \left| \int_{\mathbb{R}^d} \mathrm{Re}\left(\prod_{i=1}^N \frac{1}{1+\Psi_i(\xi)}\right) d\xi \right|^2 E\{\lambda_d(X([0,r]^N))\} \tag{1.4}$$

where $c_1 \in (0,\infty)$ is a constant depending on $N,\ d,\ r$ only.
Consider the 2^{N-1} similar additive Lévy processes (including X_t itself) $X_t^\pm = X_{t_1}^1 \pm X_{t_2}^2 \pm \cdots \pm X_{t_N}^N$. Here, $\pm$ is merely a symbol for each possible arrangement of the minus signs; e.g., $X^1 - X^2 + X^3$, $X^1 - X^2 - X^3$, $X^1 + X^2 + X^3$ and so on. Let $\Psi^\pm$ be the Lévy exponent for $X_t^\pm$. Since $-X^j$ has Lévy exponent $\overline{\Psi_j}$, $\mathcal{E}_{\Psi^\pm}(\mu) = \mathcal{E}_\Psi(\mu)$ for all $X_t^\pm$ and

$$\sum \mathrm{Re}\left(\int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j - s\cdot\Psi^\pm(\xi)} ds\right) = 2^{N-1} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) > 0$$

where the first summation $\sum$ is taken over the collection of all the $X_t^\pm$. On the other hand,

$$Q_\mu(\xi) = \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N |t_j - s_j| \Psi_j(\mathrm{sgn}(t_j - s_j)\xi)} \mu(ds)\mu(dt)$$

remains unchanged for all $X_t^{\pm}$ as long as μ is an $N-$fold product measure on $\mathbb{R}_+^N$. Proposition 10.3 of [1] and Theorem 2.1 of [1] together state that for any additive Lévy process X,

$$k_1\left(\int_{\mathbb{R}^d} Q_{\lambda^r}(\xi)d\xi\right)^{-1} \le E\{\lambda_d(X([0,r]^N))\} \le k_2\left(\int_{\mathbb{R}^d} Q_{\lambda^r}(\xi)d\xi\right)^{-1},$$

where λ^r is the restriction of the Lebesgue measure λ_N in $\mathbb{R}^N$ to $[0,r]^N$ and $k_1,\ k_2 \in (0,\infty)$ are two constants depending only on $r,\ N,\ d,\ \pi$. Note that λ^r is an $N-$fold product measure on $\mathbb{R}_+^N$. Thus, there exists a constant $c_2 \in (0,\infty)$ depending only on N and r such that

$$E\{\lambda_d(X([0,r]^N))\} \le c_2 E\{\lambda_d(X^{\pm}([0,r]^N))\}$$

for all $X_t^{\pm}$. Since $|1+z| = |1+\bar{z}|$ where z is a complex number, $\int_{\mathbb{R}^d}\prod_{j=1}^N |1+\Psi_j^{\pm}(\xi)|^{-1}d\xi < +\infty$ as well. Therefore, by (1.4),

$$\begin{aligned}
&2^{N-1}\sqrt{c_2}\sqrt{\frac{\mathcal{E}_\Psi(\delta_0)}{E\{\lambda_d(X([0,r]^N))\}}}\\
&\ge \sum \sqrt{\frac{\mathcal{E}_{\Psi^{\pm}}(\delta_0)}{E\{\lambda_d(X^{\pm}([0,r]^N))\}}}\\
&\ge \sqrt{c_1}\sum\left|\int_{\mathbb{R}^d}\mathrm{Re}\left(\int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j - s\cdot\Psi^{\pm}(\xi)}ds\right)d\xi\right|\\
&\ge \sqrt{c_1}\left|\sum\int_{\mathbb{R}^d}\mathrm{Re}\left(\int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j - s\cdot\Psi^{\pm}(\xi)}ds\right)d\xi\right|\\
&= 2^{N-1}\sqrt{c_1}\int_{\mathbb{R}^d}\prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right)d\xi\\
&= 2^{N-1}\sqrt{c_1}(2\pi)^d\mathcal{E}_\Psi(\delta_0).
\end{aligned}$$

(1.3) follows, so does the theorem. □

2 Applications

2.1 The Range of An Additive Lévy Process

As the first application, we use Theorem 1.1 to compute $\dim_H X(\mathbb{R}_+^N)$. Here, $\dim_H$ denotes the Hausdorff dimension. To begin, we introduce the standard d-parameter additive α-stable Lévy process in $\mathbb{R}^d$ for $\alpha \in (0,1)$:

$$S_t^\alpha = S_{t_1}^1 + S_{t_2}^2 + \cdots + S_{t_d}^d,$$

that is, the S^j are independent standard α-stable Lévy processes in $\mathbb{R}^d$ with the common Lévy exponent $|\xi|^\alpha$.

Theorem 2.1 *Let X be any N-parameter additive Lévy process in $\mathbb{R}^d$ with Lévy exponent $(\Psi_1,\cdots,\Psi_N)$. Then*

$$\dim_H X(\mathbb{R}_+^N) = \sup\left\{\beta\in(0,d) : \int_{\mathbb{R}^d}|\xi|^{\beta-d}\prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right)d\xi < \infty\right\}\ a.s. \tag{2.1}$$

Proof Let $\mathcal{C}_\beta$ denote the Riesz capacity. By Theorem 7.2 of [1], for all $\beta \in (0,d)$ and $S^{1-\beta/d}$ independent of X,

$$E\mathcal{C}_\beta(X(\mathbb{R}_+^N)) > 0 \Longleftrightarrow E\{\lambda_d(S^{1-\beta/d}(\mathbb{R}_+^d) + X(\mathbb{R}_+^N))\} > 0. \tag{2.2}$$

Note that $S^{1-\beta/d} + X$ is a $(d+N,\ d)-$additive Lévy process. Thus, by Theorem 1.1 and the fact that $\beta < d$ and $\mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) \in (0,1]$, we have for all $\beta \in (0,d)$,

$$E\mathcal{C}_\beta(X(\mathbb{R}_+^N)) > 0 \Longleftrightarrow \int_{\mathbb{R}^d} |\xi|^{\beta-d} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi < \infty. \tag{2.3}$$

Thanks to the Frostman theorem, it remains to show that $\mathcal{C}_\beta(X(\mathbb{R}_+^N)) > 0$ is a trivial event. Let $\mathcal{E}_\beta$ denote the Riesz energy. By Plancherel's theorem, given any $\beta \in (0,d)$, there is a constant $c_{d,\beta} \in (0,\infty)$ such that

$$\mathcal{E}_\beta(\nu) = c_{d,\beta} \int_{\mathbb{R}^d} |\hat{\nu}(\xi)|^2 |\xi|^{\beta-d} d\xi \tag{2.4}$$

holds for all probability measures ν in $\mathbb{R}^d$; see Mattila [3; Lemma 12.12]. Consider the 1-killing occupation measure

$$O(A) = \int_{\mathbb{R}_+^N} 1(X_t \in A) e^{-\sum_{j=1}^N t_j} dt, \quad A \subset \mathbb{R}^d.$$

Clearly, O is a probability measure supported on $X(\mathbb{R}_+^N)$. It is easy to verify that

$$E|\widehat{O}(\xi)|^2 = \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right).$$

It follows from (2.4) that

$$E\mathcal{E}_\beta(O) = c_{d,\beta} \int_{\mathbb{R}^d} |\xi|^{\beta-d} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi < \infty$$

when $E\mathcal{C}_\beta(X(\mathbb{R}_+^N)) > 0$. Therefore, $\mathcal{E}_\beta(O) < \infty$ a.s. Hence, $\mathcal{C}_\beta(X(\mathbb{R}_+^N)) > 0$ a.s. □

2.2 The Set of k-Multiple Points

First, we mention a q-potential density criterion: Let X be an additive Lévy process and assume that X has an a.e. positive q-potential density on $\mathbb{R}^d$ for some $q \geq 0$. Then for all Borel sets $F \subset \mathbb{R}^d$,

$$P\left\{F \bigcap X((0,\infty)^N) \neq \emptyset\right\} > 0 \Longleftrightarrow E\left\{\lambda_d(F - X((0,\infty)^N))\right\} > 0. \tag{2.5}$$

The argument is elementary but crucially hinges on the property: $X_{b+t} - X_b,\ t \in \mathbb{R}_+^N$ (independent of X_b) can be replaced by X for all $b \in \mathbb{R}_+^N$; moreover, the second condition "a.e. positive on $\mathbb{R}^d$" is absolutely necessary for the direction $\Longleftarrow$ in (2.5); see for example Proposition 6.2 of [1].

Let $X^1, \cdots, X^k$ be k independent Lévy processes in $\mathbb{R}^d$. Define

$$Z_t = (X^2_{t_2} - X^1_{t_1}, \cdots, X^k_{t_k} - X^{k-1}_{t_{k-1}}), \qquad t = (t_1, t_2, \cdots, t_k) \in \mathbb{R}^k_+.$$

Z is a k-parameter additive Lévy process taking values in $\mathbb{R}^{d(k-1)}$.

Theorem 2.2 *Let $(X^1;\ \Psi_1),\ \cdots,\ (X^k;\ \Psi_k)$ be k independent Lévy processes in $\mathbb{R}^d$ for $k \geq 2$. Assume that Z has an a.e. positive q-potential density for some $q \geq 0$. [A special case is that if for each $j = 1, \cdots, k$, X^j has a one-potential density $u^1_j > 0$, λ_d-a.e., then Z has an a.e. positive 1-potential density on $\mathbb{R}^{d(k-1)}$.] Then*

$$P(\bigcap_{j=1}^k X^j((0,\infty)) \neq \emptyset) > 0 \Longleftrightarrow$$

$$\int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})}\right) d\xi_1 \cdots d\xi_{k-1} < \infty \tag{2.6}$$

with $\xi_0 = \xi_k = 0$.

Proof For any $\mathbb{R}^d$-valued random variable X and $\xi_1,\ \xi_2 \in \mathbb{R}^d$, $e^{i[(\xi_1,\xi_2)\cdot(X,-X)]} = e^{i(\xi_1-\xi_2)\cdot X}$. In particular, the Lévy process $(X^j, -X^j)$ has Lévy exponent $\Psi_j(\xi_1 - \xi_2)$. It follows that the corresponding integral in (1.1) for Z equals

$$\int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})}\right) d\xi_1 \cdots d\xi_{k-1}$$

with $\xi_0 = \xi_k = 0$. Clearly,

$$P(\bigcap_{j=1}^k X^j((0,\infty)) \neq \emptyset) > 0 \Longleftrightarrow P(0 \in Z((0,\infty)^k)) > 0.$$

Since Z has an a.e. positive q-potential density, by (2.5)

$$P(0 \in Z((0,\infty)^k)) > 0 \Longleftrightarrow E\{\lambda_{d(k-1)}(Z((0,\infty)^k))\} > 0.$$

(2.6) now follows from Theorem 1.1. $\square$

For each $\beta \in (0, d)$ and $S^{1-\beta/d}$ independent of $X^1, \cdots, X^k$, define

$$Z^{S,\beta}_t = (X^1_{t_1} - S^{1-\beta/d}_{t_0}, X^2_{t_2} - X^1_{t_1}, \cdots, X^k_{t_k} - X^{k-1}_{t_{k-1}}),$$

$$t = (t_0, t_1, t_2, \cdots, t_k) \in \mathbb{R}^{d+k}_+, \quad t_0 \in \mathbb{R}^d_+.$$

$Z^{S,\beta}$ is a $k + d$ parameter additive Lévy process taking values in $\mathbb{R}^{dk}$.

Theorem 2.3 *Let $(X^1;\ \Psi_1),\ \cdots,\ (X^k;\ \Psi_k)$ be k independent Lévy processes in $\mathbb{R}^d$ for $k \geq 2$. Assume that for each $\beta \in (0, d)$, $Z^{S,\beta}$ has an a.e. positive q-potential density on $\mathbb{R}^{dk}$ for some $q \geq 0$. (q might depend on β.) [A special case is that if for each $j = 1, \cdots, k$, X^j has a* *one-potential density* $u_j^1 > 0$, λ_d*-a.e., then* $Z^{S,\beta}$ *has an a.e. positive* 1*-potential density on* $\mathbb{R}^{dk}$ *for all* $\beta \in (0,d)$*.] If* $P(\bigcap_{j=1}^k X^j((0,\infty)) \neq \emptyset) > 0$*, then almost surely* $\dim_H \bigcap_{j=1}^k X^j((0,\infty))$ *is a constant on* $\{\bigcap_{j=1}^k X^j((0,\infty)) \neq \emptyset\}$ *and*

$$\dim_H \bigcap_{j=1}^k X^j((0,\infty)) = \sup\{\beta \in (0,d) :$$

$$\int_{\mathbb{R}^{dk}} (1 + |\sum_{j=1}^k \xi_j|)^{\beta-d} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi_j)}\right) d\xi_1 d\xi_2 \cdots d\xi_k < \infty\}. \tag{2.7}$$

Proof According to the argument, Eq. (4.96)-(4.102), in *Proof of Theorem 3.2.* of Khoshnevisan, Shieh, and Xiao [2], it suffices to show that for all $\beta \in (0,d)$ and $S^{1-\beta/d}$ independent of $X^1, \cdots, X^k$,

$$P\left[\bigcap_{j=1}^k X^j((0,\infty)) \bigcap S^{1-\beta/d}((0,\infty)^d) \neq \emptyset\right] > 0 \Longleftrightarrow$$

$$\int_{\mathbb{R}^{dk}} (1 + |\sum_{j=1}^k \xi_j|)^{\beta-d} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi_j)}\right) d\xi_1 d\xi_2 \cdots d\xi_k < \infty. \tag{2.8}$$

Similarly, the corresponding integral in (1.1) for $Z^{S,\beta}$ equals

$$\int_{\mathbb{R}^{dk}} \frac{1}{(1+|\xi_0|^{1-\beta/d})^d} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi_j - \xi_{j-1})}\right) d\xi_0 d\xi_1 \cdots d\xi_{k-1}$$

with $\xi_k = 0$. Since $Z^{S,\beta}$ has an a.e. positive q-potential density, by (2.5) and Theorem 1.1

$$P\left[\bigcap_{j=1}^k X^j((0,\infty)) \bigcap S^{1-\beta/d}((0,\infty)^d) \neq \emptyset\right] > 0 \Longleftrightarrow$$

$$P(0 \in Z^{S,\beta}((0,\infty)^{k+d})) > 0 \Longleftrightarrow E\{\lambda_{dk}(Z^{S,\beta}((0,\infty)^{k+d}))\} > 0 \Longleftrightarrow$$

$$\int_{\mathbb{R}^{dk}} \frac{1}{(1+|\xi_0|^{1-\beta/d})^d} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi_j - \xi_{j-1})}\right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty$$

with $\xi_k = 0$. Note that

$$\int_{\mathbb{R}^{dk}} \frac{1}{(1+|\xi_0|^{1-\beta/d})^d} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi_j - \xi_{j-1})}\right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty$$

$$\Longleftrightarrow \int_{\mathbb{R}^{dk}} (1+|\xi_0|)^{\beta-d} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi_j - \xi_{j-1})}\right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty.$$

Finally, use the cyclic transformation: $\xi_j - \xi_{j-1} = \xi'_j$, $j = 1, \cdots, k-1$, $\xi_{k-1} = \xi'_k$ to obtain

$$\int_{\mathbb{R}^{dk}} (1 + |\xi_0|)^{\beta-d} \prod_{j=1}^{k} \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})}\right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty$$

$$\Longleftrightarrow \int_{\mathbb{R}^{dk}} (1 + |\sum_{j=1}^{k} \xi'_j|)^{\beta-d} \prod_{j=1}^{k} \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi'_j)}\right) d\xi'_1 d\xi'_2 \cdots d\xi'_k < \infty. \qquad \square$$

Let X be a Lévy process in $\mathbb{R}^d$. Fix any path $X_t(\omega)$. A point $x^\omega \in \mathbb{R}^d$ is said to be a k-multiple point of $X(\omega)$ if there exist k distinct times $t_1, t_2, \cdots, t_k$ such that $X_{t_1}(\omega) = X_{t_2}(\omega) = \cdots = X_{t_k}(\omega) = x^\omega$. Denote by E_k^ω the set of k-multiple points of $X(\omega)$. It is well known that E_k can be identified with $\bigcap_{j=1}^{k} X^j((0,\infty))$ where the X^j are i.i.d. copies of X. Thus, Theorem 2.2 and Theorem 2.3 imply the next theorem.

Theorem 2.4 *Let $(X,\ \Psi)$ be any Lévy process in $\mathbb{R}^d$. Assume that X has a one-potential density $u^1 > 0$, λ_d-a.e. Let E_k be the k-multiple-point set of X. Then*

$$P(E_k \neq \emptyset) > 0 \Longleftrightarrow \int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^{k} \mathrm{Re}\left(\frac{1}{1 + \Psi(\xi_j - \xi_{j-1})}\right) d\xi_1 \cdots d\xi_{k-1} < \infty \qquad (2.9)$$

with $\xi_0 = \xi_k = 0$. If $P(E_k \neq \emptyset) > 0$, then almost surely $\dim_H E_k$ is a constant on $\{E_k \neq \emptyset\}$ and

$$\dim_H E_k = \sup\{\beta \in (0, d) :$$

$$\int_{\mathbb{R}^{dk}} (1 + |\sum_{j=1}^{k} \xi_j|)^{\beta-d} \prod_{j=1}^{k} \mathrm{Re}\left(\frac{1}{1 + \Psi(\xi_j)}\right) d\xi_1 d\xi_2 \cdots d\xi_k < \infty\}. \qquad (2.10)$$

2.3 Intersection of Two Independent Subordinators

Let $X_t,\ t \geq 0$ be a process with $X_0 = 0$, taking values in $\mathbb{R}_+$. First, we ask this question: What is a condition on X such that for all sets $F \subset (0,\infty)$,

$$P(F \bigcap X((0,\infty)) \neq \emptyset) > 0 \Longleftrightarrow E\{\lambda_1(F - X((0,\infty)))\} > 0 \ \ ?$$

For subordinators, still the existence and positivity of a q-potential density ($q \geq 0$) is the only known useful condition to this question.

Let σ be a subordinator. Take an independent copy σ^- of $-\sigma$. We then define a process $\tilde{\sigma}$ on $\mathbb{R}$ by $\tilde{\sigma}_s = \sigma_s$ for $s \geq 0$ and $\tilde{\sigma}_s = \sigma^-_{-s}$ for $s < 0$. Note that $\tilde{\sigma}$ is a process of the property: $\tilde{\sigma}_{t+b} - \tilde{\sigma}_b,\ t \geq 0$ (independent of $\tilde{\sigma}_b$) can be replaced by σ for all $b \in \mathbb{R}$.

Let $X_t,\ t \geq 0$ be any process in $\mathbb{R}^d$. Then the q-potential density is nothing but the density of the expected q-occupation measure with respected to the Lebesgue measure. (When $q = 0$, assume that the expected 0-occupation measure is finite on the balls.) Since the reference measure is Lebesgue, one can easily deduce that if u is a q-potential density of X, then $u(-x)$ is a q-potential density of $-X$. Consequently, if we define $\widetilde{X}_s = X_s$ for $s \geq 0$ and $\widetilde{X}_s = X^-_{-s}$ for $s < 0$ where X^- is an independent copy of $-X$, then $u(x) + u(-x)$ is a q-potential density of $\widetilde{X}$. Conversely, if $\widetilde{X}$ has a q-potential density, then it has to be the form $u(x) + u(-x)$, where u

is a q-potential density of X. If σ is a subordinator, after a little thought we can conclude that $\tilde{\sigma}$ has an a.e. positive q-potential density on $\mathbb{R}$ if and only if σ has an a.e. positive q-potential density on $\mathbb{R}_+$.

Lemma 2.5 *If a subordinator σ has an a.e. positive q-potential density for some $q \geq 0$ on $\mathbb{R}_+$, then for all Borel sets $F \subset (0, \infty)$,*

$$P(F \bigcap \sigma((0,\infty)) \neq \emptyset) > 0 \Longleftrightarrow E\{\lambda_1(F - \sigma((0,\infty)))\} > 0. \tag{2.11}$$

Proof Assume that $E\{\lambda_1(F - \sigma((0,\infty)))\} > 0$. From the above discussion, $\tilde{\sigma}$ has an a.e. positive q-potential density. Moreover, $\tilde{\sigma}$ is a process of the property: $\tilde{\sigma}_{t+b} - \tilde{\sigma}_b$, $t \geq 0$ (independent of $\tilde{\sigma}_b$) can be replaced by σ for all $b \in \mathbb{R}$. It follows from the standard q-potential density argument that $P(F \bigcap \tilde{\sigma}(\mathbb{R} \backslash \{0\}) \neq \emptyset) > 0$. But $F \subset (0,\infty)$ and $\tilde{\sigma}((-\infty, 0]) \subset (-\infty, 0]$. Thus, $P(F \bigcap \sigma((0,\infty)) \neq \emptyset) > 0$. The direction $\Longrightarrow$ in (2.11) is elementary since σ has a q-potential density. $\square$

Theorem 2.6 *Let σ^1 and σ^2 be two independent subordinators having the Lévy exponents Ψ_1 and Ψ_2, respectively. Assume that σ^1 has an a.e. positive q-potential density for some $q \geq 0$ on $\mathbb{R}_+$. Then*

$$P[\sigma^1((0,\infty)) \bigcap \sigma^2((0,\infty)) \neq \emptyset] > 0 \Longleftrightarrow$$

$$\int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{1}{\Psi_1(x)}\right) \mathrm{Re}\left(\frac{1}{1+\Psi_2(x)}\right) dx < \infty. \tag{2.12}$$

Note that our result does not require any continuity condition on the q-potential density.

Proof By Lemma 2.5 and Theorem 1.1,

$$P[\sigma^1((0,\infty)) \bigcap \sigma^2((0,\infty)) \neq \emptyset] > 0 \Longleftrightarrow$$

$$\int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{1}{1+\Psi_1(x)}\right) \mathrm{Re}\left(\frac{1}{1+\Psi_2(x)}\right) dx < \infty.$$

Since σ^1 is transient, $\int_{|x| \leq 1} \mathrm{Re}\left(\frac{1}{\Psi_1(x)}\right) dx < \infty$. The proof is therefore completed. $\square$

2.4 A Fourier Integral Problem

This part of content can be found in Section 6 of [1]. It is an independent Fourier integral problem. Neither computing the Hausdorff dimension nor proving the existence of 1-potential density needs the discussion below. [But this Fourier integral problem might be of novelty to those who want to replace the Lévy exponent by the 1-potential density.] Let X be an additive Lévy process. Here is the question. Suppose that $K : \mathbb{R}^d \to [0, \infty]$ is a symmetric function with $K(x) < \infty$ for $x \neq 0$ that satisfies $K \in L^1$ and $\widehat{K}(\xi) = k_1 \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right)$. Under what conditions, can

$$\int \int K(x-y)\mu(dx)\mu(dy) = k_2 \int |\hat{\mu}(\xi)|^2 \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi \tag{2.13}$$

hold for all probability measures μ in $\mathbb{R}^d$? Here, $k_1,\ k_2 \in (0,\infty)$ are two constants. Consider the function K in the following example. Define $\widetilde{X}^j_{t_j} = -Y^j_{-t_j}$ for $t_j < 0$ and $\widetilde{X}^j_{t_j} = X^j_{t_j}$ for $t_j \geq 0$, where Y^j is an independent copy of X^j and the Y^j are independent of each other and of X as well. Then $\widetilde{X}_t = \widetilde{X}^1_{t_1} + \widetilde{X}^2_{t_2} + \cdots + \widetilde{X}^N_{t_N}$, $t \in \mathbb{R}^N$ is a random field on $\mathbb{R}^N$. Assume that $\widetilde{X}$ has a 1-potential density K. So, $K \in L^1$ and a direct check verifies that K is symmetric. By the definition of K, $\widehat{K}(\xi) = \int_{\mathbb{R}^N} e^{-\sum_{j=1}^N |t_j|} E e^{i\xi\cdot\widetilde{X}_t} dt$. Evaluating this integral quadrant by quadrant and using the identity $\sum \prod_{j=1}^N \frac{1}{1+z_j^{\pm}} = 2^N \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+z_j}\right)$ for $\mathrm{Re}(z_j) \geq 0$ (where $\sum$ is taken over the 2^N permutations of conjugate) yield $\widehat{K}(\xi) = k_1 \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) > 0$. If $\widehat{K} \in L^1$ (even though this case is less interesting), on one hand by Fubini,

$$\int |\hat{\mu}(\xi)|^2 \widehat{K}(\xi) d\xi = \int\int\int e^{-i\xi\cdot(x-y)} \widehat{K}(\xi) d\xi \mu(dx)\mu(dy)$$

and on the other hand by inversion (assuming the inversion holds everywhere by modification on a null set),

$$\int\int K(x-y)\mu(dx)\mu(dy) = (2\pi)^{-d} \int\int\int e^{-i\xi\cdot(x-y)} \widehat{K}(\xi) d\xi \mu(dx)\mu(dy).$$

Thus, (2.13) holds automatically in this case. If K is continuous at 0 and $K(0) < \infty$, then $\widehat{K} \in L^1$. This is a standard fact. Since $K \in L^1$ and $\widehat{K} > 0$, a bottom line condition needed to prove (2.13) is that K is continuous at 0 on $[0,\infty]$. This paper makes no attempt to solve the general case $K(0) = \infty$.

Remark Lemma 6.1 of [1] is not valid. The assumption that $\mathrm{Re}\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right) > 0$ cannot justify either equation in (6.4) of [1]. Fortunately, Lemma 6.1 played no role in [1], because Theorem 7.2 of [1] is an immediate consequence of the well-known identity (2.4) of the present paper and Theorem 1.5 of [1]. Nevertheless [1] indeed showed that the 1-potential density of an isotropic stable additive process is comparable to the Riesz kernel at 0, and therefore the 1-potential density is continuous at 0 on $[0,\infty]$.

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