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# A COUNTEREXAMPLE FOR THE OPTIMALITY OF KENDALL-CRANSTON COUPLING

KAZUMASA KUWADA
*Department of Mathematics, Faculty of Science, Ochanomizu University, Tokyo 112-8610, Japan*
email: `kuwada@math.ocha.ac.jp`

KARL-THEODOR STURM
*Institute for Applied Mathematics, University of Bonn, Wegelerstrasse 6, 53115 Bonn, Germany*
email: `sturm@uni-bonn.de`

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*Abstract*
We construct a Riemannian manifold where the Kendall-Cranston coupling of two Brownian particles does not maximize the coupling probability.

## 1 Introduction

Given two stochastic processes $X_t$ and $Y_t$ on a state space $M$, a coupling $Z_t = (Z_t^{(1)}, Z_t^{(2)})$ is a process on $M \times M$ so that $Z^{(1)}$ or $Z^{(2)}$ has the same distribution as $X$ or $Y$ respectively. Of particular interest in many applications is the distribution of the coupling time $T(Z) := \inf\{t > 0 \,;\, Z_s^{(1)} = Z_s^{(2)} \text{ for all } s > t\}$. The goal is to make the coupling probability $\mathbb{P}[T(Z) \le t]$ as large as possible by taking a suitable coupling. When $X$ and $Y$ are Brownian motions on a Riemannian manifold, Kendall [3] and Cranston [1] constructed a coupling by using the Riemannian geometry of the underlying space. Roughly speaking, under their coupling, infinitesimal motion $\Delta Y_t \in T_{Y_t}M$ at time $t$ is given as a sort of reflection of $\Delta X_t$ via the minimal geodesic joining $X_t$ and $Y_t$. Their coupling has the advantage of controlling the coupling probability by using geometric quantities such as the Ricci curvature. As a result, Kendall-Cranston coupling produces various estimates for heat kernels, harmonic maps, eigenvalues etc. under natural geometric assumptions.
On the other hand, there is the question of optimality. We say that a coupling $Z$ of $X$ and $Y$ is *optimal* at time $t$ if

$$\mathbb{P}[T(Z) \le t] \ge \mathbb{P}[T(\tilde{Z}) \le t]$$

holds for any other coupling $\tilde{Z}$. Though Kendall-Cranston coupling has a good feature as mentioned, in general there is no reason why it should be optimal.

The Kendall-Cranston coupling is optimal if the underlying space has a good symmetry. For example, in the case $M = \mathbb{R}^d$, the Kendall-Cranston coupling $(Z^{(1)}, Z^{(2)})$ is nothing but the mirror coupling. It means that $Z_t^{(2)} = \Psi(Z_t^{(1)})$ up to the time they meet, where $\Psi$ is a reflection with respect to a hyperplane in $\mathbb{R}^d$ so that $\Psi(X_0) = Y_0$. It is well known that the mirror coupling is optimal. Indeed, it is the only coupling which is optimal and Markovian [2]. More generally, the same result holds if there is a sort of reflection structure like a map $\Psi$ on $\mathbb{R}^d$ (see [4]).
In this paper, we show that the Kendall-Cranston coupling is *not* optimal in general.

**Theorem 1.1** *For each $t > 0$, there is a complete Riemannian manifold $M$ where the Kendall-Cranston coupling of two Brownian motions $X.$ and $Y.$ with specified starting points is not optimal.*

The proof of Theorem 1.1 is reduced to the case $t = 1$ by taking a scaling of Riemannian metric. We construct a manifold $M$ in the next section and prove Theorem 1.1 in section 3.

*Notation:* Given a Riemannian manifold $N$ we denote by $B_r^N(x)$ or simply $B_r(x)$ the open ball in $N$ of radius $r$ centered at $x$.
Given a Brownian motion $(X_t)_{t\geq 0}$ on $N$ we denote by $\tau_A = \inf\{t > 0 \,:\, X_t \in A\}$ the hitting time of a set $A \subset N$. We remark that, throughout this article, $\tau_A$ always stands for the hitting time for the process $(X_t)_{t\geq 0}$ even when we consider a coupled motion $(X_t, Y_t)_{t\geq 0}$.

# 2 Construction of the manifold

We take three parameter $R > 0, \zeta > 0$ and $\delta > 0$ such that $\zeta < R/4$ and $\delta < \zeta/3$. Let $C = \mathbb{R} \times S^1$ be a cylinder with a flat metric such that the length of a circle $S^1$ equals $\zeta$. For simplicity of notation, we write $z = (r, \theta)$ for $z \in C$ where $r \in \mathbb{R}$ and $\theta \in (-\zeta/2, \zeta/2]$ such that the Riemannian metric is written as $dr^2 + d\theta^2$. If appropriate, any $\theta \in \mathbb{R}$ will be regarded mod $\zeta$ and considered as element of $(-\zeta/2, \zeta/2]$. We put

$$M_1 := ([-R, \infty) \times S^1) \setminus B_\delta^C((0, \zeta/2)) \subset C$$

and write $\partial_{1,0} := \partial B_\delta^C((0, \zeta/2))$ as well as $\partial_{1,2} := \{-R\} \times S^1$ (see Fig.1). Let $C'$ be a copy of $C$. Then we put analogously

$$M_2 := ((-\infty, R] \times S^1) \setminus B_\delta^{C'}((0, 0)) \subset C'$$

and write $\partial_{2,0} := \partial B_\delta^{C'}((0, 0))$ as well as $\partial_{2,1} := \{R\} \times S^1$. Let $M_0 = S^1 \times [-1, 1]$ be another cylinder. We write $z \in M_0$ by $z = (\varphi, r)$ where $\varphi \in (0, 2\pi]$ and $r \in [-1, 1]$. Now we define a $C^\infty$-manifold $M$ (see Fig.2) by $M = M_0 \sqcup M_1 \sqcup M_2/\sim$, where the identification "$\sim$" means

$$\begin{aligned} \partial_{1,2} \ni (-R,\, \theta) &\sim (R,\, \zeta/2 - \theta) \in \partial_{2,1} && \text{for } \theta \in (-\zeta/2,\, \zeta/2]\ , \\ \partial_{1,0} \ni (\delta\cos\varphi,\, \zeta/2 - \delta\sin\varphi) &\sim (\varphi,\, -1) \in M_0 && \text{for } \varphi \in (0,\, 2\pi]\ , \\ \partial_{2,0} \ni (\delta\cos\varphi,\, \delta\sin\varphi) &\sim (\varphi,\, 1) \in M_0 && \text{for } \varphi \in (0,\, 2\pi]\ . \end{aligned}$$

We endow $M$ with a $C^\infty$-metric $g$ such that $(M, g)$ becomes a complete Riemannian manifold and:

(i) $g|_{M_1}$ coincides with the metric on $M_1$ inherited from $C$,

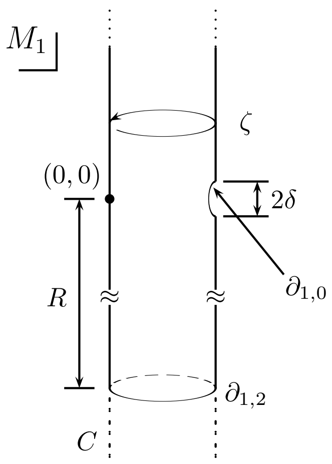


Fig.1

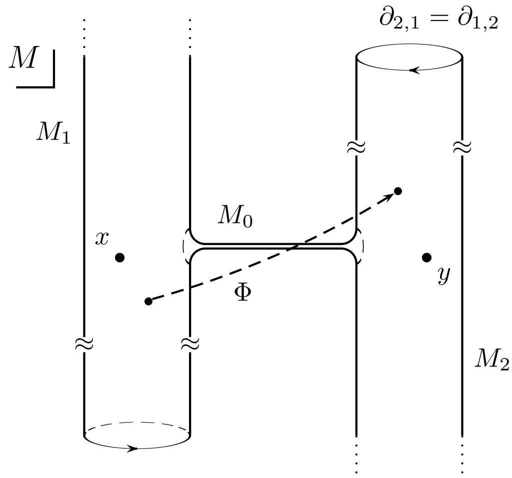


Fig.2

(ii) $g|_{M_2}$ coincides with the metric on $M_2$ inherited from $C'$,

(iii) $g|_{M_0}$ is invariant under maps $(\theta, r) \mapsto (\theta, -r)$ and $(\theta, r) \mapsto (\theta + \varphi, r)$ on $M_0$,

(iv) $d((-1,0),(1,0)) = \zeta$ for $z_1 = (-1,0), z_2 = (1,0) \in M_0$

where $d$ is the distance function on $M$.

# 3 Comparison of coupling probabilities

Let $M$ be the manifold constructed above (with suitably chosen parameters $R$, $\zeta$ and $\delta$) and fix two points $x = (0, \zeta/6) \in M_1$ and $y = (0, \zeta/3) \in M_2$.
In this paper, the construction of Kendall-Cranston coupling is due to von Renesse [5]. We will try to explain his idea briefly. His approach is based on the approximation by coupled geodesic random walks $\{\hat{\Xi}^k\}_{k\in\mathbb{N}}$ starting in $(x, y)$ whose sample paths are piecewise geodesic. Given their positions after $(n-1)$-th step, one determines its next direction $\xi_n$ according to the uniform distribution on a small sphere in the tangent space and the other does it as the reflection of $\xi_n$ along a minimal geodesic joining their present positons. We obtain a Kendall-Cranston coupling $(X_t, Y_t)$ by taking the (subsequential) limit in distribution of them. We will construct another Brownian motion $(\hat{Y}_t)_{t\geq 0}$ on $M$ starting in $y$, again defined on the same probability space as we construct $(X_t, Y_t)$ such that

$$\mathbb{P}\left(X \text{ and } Y \text{ meet before time } 1\right) \;<\; \mathbb{P}\left(X \text{ and } \hat{Y} \text{ meet before time } 1\right).$$

In other words, if $\mathbb{Q}$ denotes the distribution of $(X, Y)$ and $\hat{\mathbb{Q}}$ denotes the distribution of $(X, \hat{Y})$ then

**Proposition 3.1** $\mathbb{Q}[T \leq 1] < \hat{\mathbb{Q}}[T \leq 1]$ .

Our construction of the process $\hat{Y}$ will be as follows. We define a map $\Phi : M_1 \to M_2$ by $\Phi((r,\theta)) = (-r, \zeta/2 - \theta)$ and then put

(i) $\hat{Y}_t = \Phi(X_t)$ for $t \in [0, \tau_{\partial_{1,0}} \wedge T)$;

(ii) $X$ and $\hat{Y}$ move independently for $t \in [\tau_{\partial_{1,0}}, T)$ in case $\tau_{\partial_{1,0}} < T$;

(iii) $\hat{Y}_t = X_t$ for $t \in [T, \infty)$.

Note that $\tau_{\partial_{1,2}} = T$ holds when $\tau_{\partial_{1,2}} \le \tau_{\partial_{1,0}}$ under $\hat{\mathbb{Q}}$.
Set $H = S^1 \times \{0\} \subset M_0 \subset M$. For $z_1, z_2 \in M$ and $A \subset M$, minimal length of paths joining $z_1$ and $z_2$ which intersect $A$ is denoted by $d(z_1, z_2\,; A)$. We define a constant $L_0$ by

$$L_0 := \inf \left\{ L \in (\delta, R]\,;\ \begin{array}{l} d(z_1, z_2\,; H) \ge d(z_1, z_2\,; \partial_{1,2}) \\ \text{for some } z_1 = (L, \theta) \in M_1, z_2 = (L, \zeta/2 - \theta) \in M_2 \end{array} \right\}.$$

**Lemma 3.2** $R - \zeta < L_0 < R$.

**Proof.** First we show $L_0 < R$. Let $z_1 = (R, 0) \in M_1$ and $z_2 = (R, \zeta/2) \in M_2$. Obviously there is a path of length $2R$ joining $z_1$ and $z_2$ across $\partial_{1,2}$. Thus we have $d(z_1, z_2\,; \partial_{1,2}) \le 2R$. By symmetry of $M$,

$$d(z_1, z_2\,; H) = 2d(z_1, H) = 2\left(d(z_1, \partial_{1,0}) + \frac{\zeta}{2}\right) = 2\left(\sqrt{R^2 + \zeta^2/4} - \delta\right) + \zeta > 2R,$$

where the second equality follows from the third and fourth properties of $g$ and the last inequality follows from the choice of $\delta$. These estimates imply $L_0 < R$.
Next, let $z_1' = (R - \zeta, \theta) \in M_1$ and $z_2' = (R - \zeta, \zeta/2 - \theta) \in M_2$. In the same way as observed above, we have

$$d(z_1', z_2'\,; H) = 2\left(\sqrt{(R-\zeta)^2 + \theta^2} - \delta\right) + \zeta \le 2R - 2\delta.$$

Note that the length of a path joining $z_1'$ and $z_2'$ which intersects both of $\partial_{1,2}$ and $H$ is obviously greater than $d(z_1', z_2'\,; H)$. Thus, in estimating $d(z_1', z_2'\,; \partial_{1,2})$, it is sufficient to consider all paths joining $z_1'$ and $z_2'$ across $\partial_{1,2}$ which do not intersect $H$. Such a path must intersect both $\{\delta\} \times S^1 \subset M_1$ and $\{-\delta\} \times S^1 \subset M_1$ (see Fig.3). Thus we have

$$\begin{aligned} d(z_1', z_2'\,; \partial_{1,2}) &\ge d(z_1', \{\delta\} \times S^1) + d(\{-\delta\} \times S^1, \partial_{1,2}) + d(\partial_{2,1}, z_2') \\ &\ge (R - \zeta - \delta) + (R - \delta) + \zeta \\ &= 2R - 2\delta. \end{aligned}$$

Hence, the conclusion follows. □

Set $M_1' := M_1 \cap [-L_0, L_0] \times S^1 \subset C$ and $M_2' := M_2 \cap [-L_0, L_0] \times S^1 \subset C'$. We define a submanifold $M' \subset M$ with boundary by $M' = M_0 \sqcup M_1' \sqcup M_2' / \sim$ (see Fig.4). Let $\Psi : M' \to M'$ be the reflection with respect to $H$. For instance, for $z = (r, \theta) \in M_1'$, $\Psi(z) = (r, \zeta/2 - \theta) \in M_2'$. Note that $\Psi$ is an isometry, $\Psi \circ \Psi = \mathrm{id}$ and $\{z \in M'\,;\ \Psi(z) = z\} = H$.
Let $X'$ be the given Brownian motion starting in $x$ and now stopped at $\partial M'$, i.e. $X_t' = X_{t \wedge \tau_{\partial M'}}$. Define a stopped Brownian motion starting in $y$ by $Y_t' = \Psi(X_t')$ for $t < \tau_H$ and by $Y_t = X_t$ for $t \ge \tau_H$ (that is, the two Brownian particles coalesce after $\tau_H$). Then we can prove the following lemma.

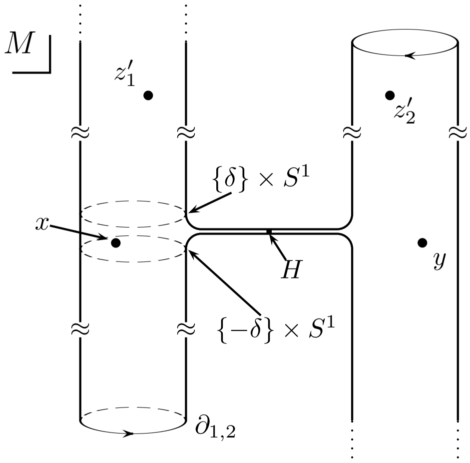


Fig.3

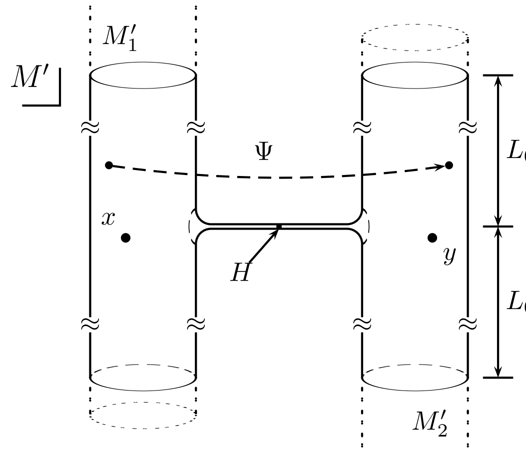


Fig.4

**Lemma 3.3** *The law of* $(X_{t\wedge\tau_{\partial M'}}, Y_{t\wedge\tau_{\partial M'}})_{t\geq 0}$ *coincides with that of* $(X'_t, Y'_t)_{t\geq 0}$.

**Proof.** Note that the minimal geodesic in $M$ joining $z$ and $\Psi(z)$ must intersect $H$ for every $z \in M'$ by virtue of the choice of $L_0$. Thus, by the symmetry of $M'$ with respect to $H$, coupled geodesic random walks $\hat{\Xi}^k$ are in $E$ defined by

$$E := \left\{ (z_\cdot^{(1)}, z_\cdot^{(2)}) \in C([0,\infty) \to M \times M) \; ; \; z_t^{(2)} = \Psi(z_t^{(1)}) \text{ before } z_\cdot^{(1)} \text{ exits from } M' \right\}$$

(cf. Theorem 5.1 in [4]). Since $E$ is closed in $C([0,\infty) \to M \times M)$, $(X_\cdot, Y_\cdot) \in E$ holds $\mathbb{P}$-almost surely by taking a (subsequential) limit in distribution of $\{\hat{\Xi}^k\}_{k\in\mathbb{N}}$. Thus the conclusion follows. $\square$

We now begin to show Proposition 3.1. First we give a lower estimate of $\hat{\mathbb{Q}}\left[T \leq 1\right]$. Let

$$\gamma(a) := \left\{ (x_1, x_2) \in \mathbb{R}^2 \; ; \; x_2 = a \right\}, \; a \in \mathbb{R},$$
$$A(\delta) := \bigcup_{n\in\mathbb{Z}} B_\delta^{\mathbb{R}^2} \left( \left( \zeta \left( n + \frac{1}{3} \right), 0 \right) \right).$$

The remark after the definition of $\hat{\mathbb{Q}}$ implies

$$\hat{\mathbb{Q}}\left[T \leq 1\right] \geq \hat{\mathbb{Q}}\left[T \leq 1, \, \tau_{\partial_{1,2}} < \tau_{\partial_{1,0}}\right] = \hat{\mathbb{Q}}\left[\tau_{\partial_{1,2}} \leq 1 \wedge \tau_{\partial_{1,0}}\right].$$

By lifting $X_t$ to $\mathbb{R}^2$, the universal cover of $C$,

$$\begin{aligned} \hat{\mathbb{Q}}\left[\tau_{\partial_{1,2}} \leq 1 \wedge \tau_{\partial_{1,0}}\right] &= \mathbb{P}^{\mathbb{R}^2}\left[\tau_{\gamma(R)} \leq 1 \wedge \tau_{A(\delta)}\right] \\ &\geq \mathbb{P}^{\mathbb{R}^2}\left[\tau_{\gamma(R)} \leq 1, \, \tau_{A(\delta)} > 1\right] \\ &\geq \mathbb{P}^{\mathbb{R}}\left[\tau_R \leq 1\right] - \mathbb{P}^{\mathbb{R}^2}\left[\tau_{A(\delta)} \leq 1\right]. \end{aligned} \tag{3.1}$$

Here $\mathbb{P}^{\mathbb{R}^2}$ and $\mathbb{P}^{\mathbb{R}}$ denote the usual Wiener measure for Brownian motion (starting at the origin) on $\mathbb{R}^2$ or $\mathbb{R}$, resp. For simplicity, we write $\tau_R$ instead of $\tau_{\{R\}}$.

Next we give an upper estimate of $\mathbb{Q}[T \le 1]$. Let $E := \left\{\tau_{\partial_{1,0}} < 1 \wedge \tau_{\partial M'}\right\}$. Then

$$\mathbb{Q}[E] = \mathbb{P}[E] \le \mathbb{P}^{\mathbb{R}^2}\left[\tau_{A(\delta)} < 1\right].$$

Note that, on $\{T \le 1\} \cap E^c$, $X$ must hit $\partial M'$ before $T$. It means

$$\mathbb{Q}[\{T \le 1\} \cap E^c] = \mathbb{Q}[\{\tau_{\partial M'} < T \le 1\} \cap E^c].$$

By Lemma 3.3, $Y_{\tau_{\partial M'}} = \Psi(X_{\tau_{\partial M'}})$ on $E^c$ under $\mathbb{Q}$. In order to collide two Brownian motions starting at $X_{\tau_{\partial M'}}$ and $\Psi(X_{\tau_{\partial M'}})$, either of them must escape from the flat cylinder of length $2(L_0 - \delta)$ where its starting point has distance $L_0 - \delta$ from the boundary. This observation together with the strong Markov property yields

$$\begin{aligned}
\mathbb{Q}[\{\tau_{\partial M'} < T \le 1\} \cap E^c] &= \mathbb{Q}\left[\mathbb{Q}_{(X_{\tau_{\partial M'}}, \Psi(X_{\tau_{\partial M'}}))}[T \le 1 - s]\,|_{s=\tau_{\partial M'}}\,; \tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right] \\
&\le 2\mathbb{Q}\left[\mathbb{P}^{\mathbb{R}}\left[\tau_{-(L_0-\delta)} \wedge \tau_{L_0-\delta} < 1 - s\right]|_{s=\tau_{\partial M'}}\,; \tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right] \\
&\le 4\mathbb{Q}\left[\mathbb{P}^{\mathbb{R}}\left[\tau_{L_0-\delta} < 1 - s\right]|_{s=\tau_{\partial M'}}\,; \tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right].
\end{aligned}$$

By Lemma 3.2 and the definition of $\zeta$ and $\delta$, we have $L_0 - \delta \ge R - \zeta - \delta > 2R/3$. Thus

$$\begin{aligned}
\mathbb{Q}\left[\mathbb{P}^{\mathbb{R}}\left[\tau_{L_0-\delta} < 1 - s\right]|_{s=\tau_{\partial M'}}\,; \tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right] &\le 2\exp\left(-\frac{(L_0-\delta)^2}{2}\right)\mathbb{P}\left[\tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right] \\
&\le 2\exp\left(-\frac{2R^2}{9}\right)\mathbb{P}\left[\tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right].
\end{aligned}$$

By lifting $X_t$ to $\mathbb{R}^2$, we have

$$\mathbb{P}\left[\tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right] \le \mathbb{P}^{\mathbb{R}^2}\left[\tau_{\gamma(L_0)} \wedge \tau_{\gamma(-L_0)} < 1 \wedge \tau_{A(\delta)}\right] \le 2\mathbb{P}^{\mathbb{R}}\left[\tau_{L_0} < 1\right] \le 2\mathbb{P}^{\mathbb{R}}\left[\tau_{R-\zeta} < 1\right].$$

Here the last inequality follows from Lemma 3.2. Consequently, we obtain

$$\mathbb{Q}[T \le 1] \le \mathbb{P}^{\mathbb{R}^2}\left[\tau_{A(\delta)} < 1\right] + 16\exp\left(-\frac{2R^2}{9}\right)\mathbb{P}^{\mathbb{R}}\left[\tau_{R-\zeta} < 1\right]. \tag{3.2}$$

Now take $R > 3\sqrt{2\log 2}$. After that we choose $\zeta$ so small that $\mathbb{P}^{\mathbb{R}}[\tau_{R-\zeta} < 1] \approx \mathbb{P}^{\mathbb{R}}[\tau_R < 1]$. Finally we choose $\delta$ so small that $\mathbb{P}^{\mathbb{R}^2}\left[\tau_{A(\delta)} < 1\right] \approx 0$. Then Proposition 3.1 follows from (3.1) and (3.2).

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