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ON THE ZERO-ONE LAW AND THE LAW OF LARGE NUMBERS FOR RANDOM WALK IN MIXING RANDOM ENVIRONMENT

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Abstract
We prove a weak version of the law of large numbers for multi-dimensional finite range random walks in certain mixing elliptic random environments. This already improves previously existing results, where a law of large numbers was known only under strong enough transience. We also prove that for such walks the zero-one law implies a law of large numbers.

1. Introduction

Random walk in random environment is one of the basic models of the field of disordered systems of particles. In this model, an environment is a collection of transition probabilities $\omega = (\omega_x)_{x\in\mathbb{Z}^d} \in \mathcal{P}^{\mathbb{Z}^d}$, where $\mathcal{P} = \{(p_z)_{z\in\mathbb{Z}^d} : p_z \geq 0, \sum_z p_z = 1\}$. We will denote the coordinates of ω_x by $\omega_x = (\pi_{xy})_{y\in\mathbb{Z}^d}$. Let us denote by $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ the space of all such transition probabilities. The space Ω is equipped with the canonical product σ-field $\mathfrak{S}$, and with the natural shift $\pi_{xy}(T_z\omega) = \pi_{x+z,y+z}(\omega)$, for $z \in \mathbb{Z}^d$. On the space of environments $(\Omega, \mathfrak{S})$, we are given a certain T-invariant probability measure $\mathbb{P}$, with $(\Omega, \mathfrak{S}, (T_z)_{z\in\mathbb{Z}^d}, \mathbb{P})$ ergodic. We will say that the environment is i.i.d. when $\mathbb{P}$ is a product measure on $\mathcal{P}^{\mathbb{Z}^d}$. Throughout this work, we assume the following ellipticity condition on $\mathbb{P}$:

Hypothesis (E) *There exists a deterministic function $p_0 : \mathbb{Z}^d \to [0,1]$ and two deterministic constants $M > 0$ (the range of the increments), and $c > 0$, such that $p_0(z) = 0$ for $|\,z\,| > M$, $p_0(e) > 0$ for $|\,e\,| = 1$, and for all $z \in \mathbb{Z}^d$*

$$\mathbb{P}(p_0(z) \leq \pi_{0,z} \leq cp_0(z)) = 1.$$

Here, and in the rest of this paper, $|\cdot|$ denotes the l^1-norm on $\mathbb{Z}^d$, so that $M = 1$ means the walk is nearest-neighbor. The above ellipticity hypothesis basically provides a uniform lower bound on the random transitions. Moreover, if a transition is not allowed for some environment, then it is not allowed for any other environment.

Let us now describe the process. First, the environment ω is chosen from the distribution $\mathbb{P}$. Once this is done, it remains fixed for all times. The random walk in environment ω is then the canonical Markov chain $(X_n)_{n\geq 0}$ with state space $\mathbb{Z}^d$ and transition probability

$$\begin{aligned} P_0^\omega(X_0 = 0) &= 1, \\ P_0^\omega(X_{n+1} = y \,|\, X_n = x) &= \pi_{xy}(\omega). \end{aligned}$$

The process P_0^ω is called the *quenched law*. The *annealed law* is then

$$P_0 = \int P_0^\omega \, \mathbb{P}(d\omega).$$

One of the most fundamental questions one can ask is:

Question 1 (Directional 0-1 law) *Is it true that*

$$\forall \ell \in \mathbb{R}^d - \{0\} : P_0(\lim_{n\to\infty} X_n \cdot \ell = \infty) \in \{0,1\}? \tag{1}$$

For $d = 2$, Question 1 was first asked by Kalikow in [5]. Recently, Merkl and Zerner [13] answered it positively for two dimensional nearest neighbor walks ($M = 1$) in an i.i.d. random environment. It is noteworthy that the ellipticity hypothesis in [13] is weaker than our Hypothesis (E). However, they also provide a counter-example indicating that in order to extend the result to more general environments one needs to assume stronger conditions on the environment. Hypothesis (E) is one such possibility.
We have learnt, through a private communication with O. Zeitouni, of counter-examples for any $d \geq 3$ with Hypothesis (E) being satisfied. In these examples, $\mathbb{P}$ is ergodic but not mixing. Therefore, one has to make some assumptions on the mixing properties of $\mathbb{P}$. In Section 2 below, we will introduce our mixing Hypothesis (M). This is the Dobrushin-Shlosman strong mixing condition IIIc in [3].
In this paper, we will not answer the above important question. Instead, we will address its relation to another fundamental question:

Question 2 (The law of large numbers) *Is there a deterministic vector v such that*

$$P_0(\lim_{n\to\infty} n^{-1} X_n = v) = 1?$$

Proving the law of large numbers for random walks in a random environment has been the subject of several works. When $d = 1$, the law of large numbers is known to hold; see [8] for a product environment, and [1] for the ergodic case. When $d \geq 2$, a law of large numbers for a general ergodic environment is out of question due to the counter-examples to Question 1. The law of large numbers has been proven to hold under some strong directional transience assumptions; see [9, 10] for the product case, and [2, 6] for environments satisfying Hypothesis (M) below, as well as other weaker mixing assumptions. Recently, Zerner [12] proved that in a product environment the directional 0-1 law implies the law of large numbers. This result closed the question of the law of large numbers for random walks in two-dimensional product environments. It has also reduced proving the law of large numbers, e.g. when $d \geq 3$, to answering Question 1. In fact, Zerner's result uses a weak version of the law of large numbers (see Theorem 1 below) for i.i.d. environments, which was also proved by him [12] and is interesting by itself, in the absence of a proof of the directional 0-1 law (1). Roughly, if we define for $\ell \in \mathbb{R}^d - \{0\}$ the events

$$A_\ell = \{\lim_{n\to\infty} X_n \cdot \ell = \infty\} \quad \text{and}$$

$$B_\ell \quad = \quad \{ \varliminf_{n\to\infty} X_n \cdot \ell = -\infty, \varlimsup_{n\to\infty} X_n \cdot \ell = \infty \},$$

then Zerner shows that, conditioned on A_ℓ, one has a law of large numbers in direction ℓ, for all ℓ. Therefore, to prove that the 0-1 law implies a law of large numbers Zerner also shows that when $P_0(B_\ell) = 1$ for all ℓ, the limit of $n^{-1}X_n$ exists. Zerner's arguments, however, use regeneration times and are best suited for an i.i.d. environment. We will have a different approach.

In Section 3, we will recall some large deviations results from [7] and use them to prove Lemma 1 of Section 4, which gives a lower bound on the probability of escaping with a non-zero velocity, when starting at a fresh point; i.e. when $X_n \cdot \ell$ reaches a new maximum. This lower bound turns out to be uniform in the history of the walk, and positive when $P_0(\lim_{n\to\infty} n^{-1} X_n \cdot \ell = 0) < 1$. Thus, the walker will have infinitely many chances to escape with the non-zero velocity. Using this in Section 5, we prove the following law of large numbers.

Theorem 1 *Assume that $\mathbb{P}$ satisfies Hypotheses* (E) *and* (M). *Then there exist two deterministic vectors $v_-, v_+ \in \mathbb{R}^d$, such that*

(i) If $\ell \in \mathbb{R}^d$ is such that $\ell \cdot v_+ > 0$ or $\ell \cdot v_- < 0$, then

$$\lim_{n\to\infty} n^{-1} X_n = v_+ 1\!\!1_{A_\ell} + v_- 1\!\!1_{A_{-\ell}}, \; P_0\text{-a.s.}$$

(ii) If $\ell \in \mathbb{R}^d$ is such that $\ell \cdot v_+ = \ell \cdot v_- = 0$, then

$$\lim_{n\to\infty} n^{-1} X_n \cdot \ell = 0, \; P_0\text{-a.s.}$$

In Section 5, we also prove our main result, which comes as a consequence of Theorem 1:

Theorem 2 *Assume that $\mathbb{P}$ satisfies Hypotheses* (E) *and* (M). *Assume also that the directional 0-1 law* (1) *holds. Then there exists a deterministic $v \in \mathbb{R}^d$, such that*

$$P_0(\lim_{n\to\infty} n^{-1} X_n = v) = 1.$$

Remark 1 *The importance of Theorem* 2 *stems from the fact that it reduces the problem of proving a law of large numbers, for environments satisfying Hypotheses* (E) *and* (M), *to that of proving* (1).

An interesting corollary follows from Theorem 1:

Corollary 1 *Assume that $\mathbb{P}$ satisfies Hypotheses* (E) *and* (M), *and that there exists an $\ell \in \mathbb{R}^d - \{0\}$ such that*

$$P_0(\varliminf_{n\to\infty} n^{-1} X_n \cdot \ell > 0) = 1. \tag{2}$$

Then there exists a deterministic vector v such that

$$P_0(\lim_{n\to\infty} n^{-1} X_n = v) = 1.$$

Remark 2 *Apart from the fact that the mixing conditions in [6] and [2] are a bit weaker than our mixing condition* (M), *this corollary essentially improves the results therein by relaxing the so-called "Kalikow condition" into just a directional ballistic transience condition. Observe also that if a law of large numbers is satisfied, then either the velocity is* 0 *or* (2) *holds for some ℓ. The above corollary states that the converse is also true.*

2. Mixing assumption

For a set $V \subset \mathbb{Z}^d$, let us denote by Ω_V the set of possible configurations $\omega_V = (\omega_x)_{x\in V}$, and by $\mathfrak{S}_V$ the σ-field generated by the environments $(\omega_x)_{x\in V}$. For a probability measure $\mathbb{P}$ we will denote by $\mathbb{P}_V$ the projection of $\mathbb{P}$ onto $(\Omega_V, \mathfrak{S}_V)$. For $\omega \in \Omega$, denote by $\mathbb{P}_V^\omega$ the regular conditional probability, knowing $\mathfrak{S}_{\mathbb{Z}^d - V}$, on $(\Omega_V, \mathfrak{S}_V)$. Furthermore, for $\Lambda \subset V$, $\mathbb{P}_{V,\Lambda}^\omega$ will denote the projection of $\mathbb{P}_V^\omega$ onto $(\Omega_\Lambda, \mathfrak{S}_\Lambda)$. Also, we will use the notations $V^c = \mathbb{Z}^d - V$, $\partial_r V = \{x \in \mathbb{Z}^d - V : \operatorname{dist}(x, V) \le r\}$, with $r \ge 0$. Note that sometimes we will write $\mathbb{P}_V^{\omega_{V^c}}$ (resp. $\mathbb{P}_{V,\Lambda}^{\omega_{V^c}}$) to emphasize the dependence on ω_{V^c} in $\mathbb{P}_V^\omega$ (resp. $\mathbb{P}_{V,\Lambda}^\omega$).
Consider a reference product measure α on $(\Omega, \mathfrak{S})$ and a family of functions $U = (U_A)_{A\subset\mathbb{Z}^d}$, called an interaction, such that $U_A \equiv 0$ if $|A| > r$ (finite range), $U_A(\omega)$ only depends on ω_A, $\beta = \sup_{A,\omega} |U_A(\omega)| < \infty$ (bounded interaction), and $U_{\theta^x A}(\theta^x \omega) = U_A(\omega)$ (shift invariant). One then can define the specification

$$\frac{d\mathbb{P}_V^{\omega_{V^c}}}{d\alpha_V}(\omega_V) = \frac{e^{-H_V(\omega_V|\omega_{V^c})}}{Z_V(\omega_{V^c})},$$

where

$$Z_V(\omega_{V^c}) = \mathbb{E}^\alpha\left(e^{-H_V(\omega_V|\omega_{V^c})}\right)$$

is the partition function, and

$$H_V(\omega_V|\omega_{V^c}) = \sum_{A: A\cap V \neq \phi} U_A(\omega)$$

is the conditional Hamiltonian. The parameter $\beta > 0$ is called the inverse temperature. One can ask whether or not this system of conditional probabilities arises from a probability measure, and if such a measure is unique. In [3] the authors introduce a sufficient condition for this to happen. The Dobrushin–Shlosman strong decay property holds if there exist $G, g > 0$ such that for all $\Lambda \subset V \subset \mathbb{Z}^d$ finite, $x \in \partial_r V$, and $\omega, \bar{\omega} \in \Omega$, with $\omega_y = \bar{\omega}_y$ when $y \neq x$, we have

$$d_{\mathrm{var}}(\mathbb{P}_{V,\Lambda}^\omega, \mathbb{P}_{V,\Lambda}^{\bar{\omega}}) \le G e^{-g \operatorname{dist}(x,\Lambda)}, \tag{3}$$

where $d_{\mathrm{var}}(\cdot\,,\cdot)$ is the variational distance $d_{\mathrm{var}}(\mu,\nu) = \sup_{E\in\mathfrak{S}}(\mu(E) - \nu(E))$. If the above condition holds, then there exists a unique $\mathbb{P}$ that is consistent with the specification $(\mathbb{P}_V^{\omega_{V^c}})$; see Comment 2.3 in [3]. If the interaction is translation-invariant, and the specification satisfies (3), then the unique field $\mathbb{P}$ is also shift-invariant (see [4], Section 5.2). One should note that (3) is satisfied for several classes of Gibbs fields. Namely, in the high-temperature region (that is when β is small enough; class $\mathcal{A}$ in [3]), in the case of a large magnetic field (class $\mathcal{B}$ in [3]), and in the case of one-dimensional and almost one-dimensional interactions (class $\mathcal{E}$ in [3]); see Theorem 2.2 in [3] for the proof, and for the precise definitions of the above classes. It is worthwhile to note that from the definitions of the above classes, it follows that adding any 0-range interaction to an interaction in one of these classes, results in a new interaction belonging to the same class; see the definitions on pages 378-379 of [3]. This will be our second condition on the environment $\mathbb{P}$.

Hypothesis (M) *The probability measure $\mathbb{P}$ is the unique Gibbs field corresponding to a finite range interaction such that any perturbation of it by a 0-range interaction satisfies* (3).

3. Some large deviations results

In this section, we will recall some results from [7, 11] that will be used in the rest of the paper. First, we give some definitions. For $n \geq 1$, let

$$\mathrm{W}_n = \{(z_{-n+1}, \cdots, z_0) \in (\mathbb{Z}^d)^n : |z_i| \leq M\},$$

and let $\mathrm{W}_0 = \{\phi\}$, with ϕ representing an empty history. Similarly, let

$$\mathrm{W}_\infty^{\mathrm{tr}} = \left\{(z_i)_{i \leq 0} : |z_i| \leq M, \left|\sum_{j=i}^{0} z_j\right| \underset{j \to -\infty}{\to} \infty\right\} \text{ and } \mathrm{W}^{\mathrm{tr}} = \bigcup_{n \geq 0} \mathrm{W}_n \cup \mathrm{W}_\infty^{\mathrm{tr}}.$$

For $\mathrm{w} \in \mathrm{W}_n$ and $-n \leq i \leq 0$, define

$$x_i(\mathrm{w}) = -\sum_{j=i+1}^{0} z_j,$$

as the walk with increments $(z_i)_{i=-n+1}^{0}$, shifted to end at 0. For $n < \infty$ and $\mathrm{w} \in \mathrm{W}_n$, let Q_w be the annealed random walk on $\mathbb{Z}^d$, conditioned on the first n steps being given by w. More precisely, for $(x_m)_{m \geq 0} \in (\mathbb{Z}^d)^{\mathbb{N}}$, $x_0 = 0$, we have $Q_\mathrm{w}(X_0 = 0) = 1$, and $Q_\mathrm{w}(X_{m+1} = x_{m+1} \,|\, X_m = x_m, \cdots, X_1 = x_1)$ is given by

$$\frac{\mathbb{E}(\pi_{x_{-n}x_{-n+1}} \cdots \pi_{x_{-1}x_0} \pi_{x_0 x_1} \cdots \pi_{x_m, x_{m+1}})}{\mathbb{E}(\pi_{x_{-n}x_{-n+1}} \cdots \pi_{x_{-1}x_0} \pi_{x_0 x_1} \cdots \pi_{x_{m-1} x_m})}.$$

In [7] we have shown that if $\mathbb{P}$ satisfies hypotheses (E) and (M), then Q_w can still be well defined, even for $\mathrm{w} \in \mathrm{W}_\infty^{\mathrm{tr}}$; see Lemma 4.1 in [7]. Moreover, there exists a constant C such that, for any $\mathrm{w}_1, \mathrm{w}_2 \in \mathrm{W}^{\mathrm{tr}}$, one has

$$\left| \log \left. \frac{dQ_{\mathrm{w}_1}}{dQ_{\mathrm{w}_2}} \right|_{\mathcal{F}_n} (x_0, \cdots, x_n) \right| \leq C \left(\sum_{i=0}^{n} e^{-g\, \mathrm{dist}(x_i, S(\mathrm{w}_1))} + \sum_{i=0}^{n} e^{-g\, \mathrm{dist}(x_i, S(\mathrm{w}_2))} \right), \tag{4}$$

where $\mathcal{F}_n$ is the σ-field generated by $X_1, \cdots, X_n$, and

$$S(\mathrm{w}) = \left\{x : \sum_{i \leq 0} \mathbb{1}_{\{x\}}(x_i(\mathrm{w})) > 0\right\} = \{x_i(\mathrm{w}) : i \leq 0\}$$

is the range of the walk; see Lemma 5.3 and its proof in [7].
Using this estimate, we have then shown in [7] that the annealed process satisfies a large deviations principle with a rate function H that is zero either at a single point or on a line segment containing the origin; see Theorem 5.1 and Remark 2.4 in [7].
Furthermore, for each extreme point v of the zero-set of H, there exists a unique measure μ on the space $(\mathbb{Z}^d)^{\mathbb{Z}}$ that is ergodic with respect to the natural shift on $(\mathbb{Z}^d)^{\mathbb{Z}}$, with $E^\mu(Z_1) = v$, and

$$\mu(Z_1 = z_1 \,|\, (Z_i)_{i \leq 0} = \mathrm{w}) = Q_\mathrm{w}(X_1 = z_1); \tag{5}$$

i.e. μ is also invariant with respect to the Markov process on W, defined by Q_w. See Remark 2.4 in [7].

4. Two lemmas

For $\ell \in \mathbb{R}^d - \{0\}$, define

$$\mathrm{W}_\ell^- = \bigcup_{n\geq 0} \left\{ \mathrm{w} \in \mathrm{W}_n : \sup_{-n\leq i\leq 0} x_i(\mathrm{w})\cdot \ell \leq 0 \right\}.$$

The following lemma is the heart of the proof of Theorems 1 and 2.

Lemma 1 *Assume $\mathbb{P}$ satisfies Hypotheses* (E) *and* (M)*, and let v be a non-zero extreme point of the zero set of the rate function H. Then we have for each $\ell \in \mathbb{R}^d$ such that $\ell \cdot v > 0$*

$$\delta_\ell = \inf_{\mathrm{w}\in \mathrm{W}_\ell^-} Q_\mathrm{w}(\lim_{n\to\infty} n^{-1} X_n = v) > 0.$$

Proof. Due to (4), one has for $\mathrm{w}_1 \in \mathrm{W}_\ell^-$ and $\mathrm{w}_2 \in \mathrm{W}_\infty^{\mathrm{tr}}$

$$\begin{aligned} \inf_{n\geq 1} \frac{dQ_{\mathrm{w}_1}}{dQ_{\mathrm{w}_2}}\bigg|_{\mathcal{F}_n} (x_0, \cdots, x_n) &\geq \exp\left(-C \sum_{i\geq 0} \left(e^{-g|\ell|^{-1} x_i\cdot \ell} + e^{-g\,\mathrm{dist}(x_i, S(\mathrm{w}_2))} \right)\right) \\ &= F_\ell(\mathrm{w}_2, (x_i)_{i\geq 0}), \end{aligned}$$

and, therefore, if one defines $A = \{\lim_{n\to\infty} n^{-1} X_n = v\}$, then

$$Q_{\mathrm{w}_1}(A) \geq \int_A F_\ell(\mathrm{w}_2, \cdot)\, dQ_{\mathrm{w}_2}(\cdot). \tag{6}$$

Note that we can consider F_ℓ as a function on $(\mathbb{Z}^d)^{\mathbb{Z}}$. Now let μ be as in (5). Then by the ergodic theorem one has

$$\mu(\lim_{n\to\infty} n^{-1} X_n = v, \lim_{n\to\infty} n^{-1} X_{-n} = -v) = 1.$$

Therefore, since $\ell \cdot v > 0$, the sum in the definition of F_ℓ is a converging geometric series, μ-a.s. and $\mu(F_\ell > 0) = 1$. Integrating (6) against $\mu(d\mathrm{w}_2)$, then taking the infimum over $\mathrm{w}_1 \in \mathrm{W}_\ell^-$, one has

$$\delta_\ell \geq \int F_\ell \, d\mu > 0,$$

and the proof is complete. □

The next lemma is a consequence of Lemma 1 and will be useful in the proof of Theorem 1.

Lemma 2 *Assume that $\mathbb{P}$ satisfies Hypotheses* (E) *and* (M)*, and let v_-, v_+ be the two, possibly equal, extremes of the zero-set of the rate function H. Then for all $\ell \in \mathbb{R}^d$ such that $|\,\ell \cdot v_+\,| + |\,\ell \cdot v_-\,| \neq 0$, one has*

$$P_0(B_\ell) = 0 \text{ and } P_0(A_\ell \cup A_{-\ell}) = 1.$$

Proof. First, notice that if Hypothesis (E) holds, then every time $|X_n \cdot \ell| \leq L$, the quenched walker has a fresh chance of at least $(\min_{|e|=1} p_0(e))^{2L} > 0$ to dip below level $-L$, in direction ℓ. Using the conditional version of Borel-Cantelli's lemma, one then can show that

$$P_0(X_n \cdot \ell < -L \text{ finitely often}, \; -L \leq X_n \cdot \ell \leq L \text{ infinitely often}) = 0,$$

for all $L \geq 1$ and $\ell \in \mathbb{R}^d - \{0\}$. For a more detailed argument, see the proof of (1.4) in [10]. But this shows us that

$$P_0(|\varliminf_{n\to\infty} X_n \cdot \ell| \leq L) = 0.$$

Taking L to infinity shows that

$$P_0(|\varliminf_{n\to\infty} X_n \cdot \ell| < \infty) = 0.$$

By a similar argument, one also has

$$P_0(|\varlimsup_{n\to\infty} X_n \cdot \ell| < \infty) = 0. \tag{7}$$

Combining the two, we get that Hypothesis (E) implies that, for all $\ell \in \mathbb{R}^d - \{0\}$, we have $P_0(A_\ell \cup A_{-\ell} \cup B_\ell) = 1$ and to prove the lemma one only needs to show that $P_0(A_\ell \cup A_{-\ell}) = 1$. Now fix ℓ as in the statement of the lemma. Notice that the claim of the lemma is the same whether one considers ℓ or $-\ell$. Since $|\,v_+ \cdot \ell\,| + |\,v_- \cdot \ell\,| \neq 0$, one can assume, without loss of generality, that $\ell \cdot v_+ > 0$. But then, by Lemma 1,

$$P_0(A_\ell \,|\, X_1, \cdots, X_n) \geq P_0(\lim_{n\to\infty} n^{-1} X_n = v_+ \,|\, X_1, \cdots, X_n) \geq \delta_\ell > 0$$

whenever

$$\sup_{0\leq i\leq n} X_i \cdot \ell \leq X_n \cdot \ell.$$

Observe that this will happen infinitely often on $G_\ell = \{\varlimsup_{n\to\infty} X_n \cdot \ell = \infty\}$. Now since $P_0(A_\ell \,|\, X_1, \cdots, X_n)$ is a bounded martingale, with respect to $\{\mathcal{F}_n\}_{n\geq 0}$, it converges to $\mathbb{I}_{A_\ell}$, P_0-a.s. Thus $P_0(\mathbb{I}_{A_\ell}\mathbb{I}_{G_\ell} \geq \delta_\ell \mathbb{I}_{G_\ell}) = 1$ and

$$\mathbb{I}_{A_\ell}\mathbb{I}_{G_\ell} = \mathbb{I}_{G_\ell},\ P_0\text{-a.s.}$$

On the other hand, (7) shows that

$$\mathbb{I}_{A_{-\ell}} = \mathbb{I}_{G_\ell^c} = 1 - \mathbb{I}_{G_\ell},\ P_0\text{-a.s.}$$

Since $P_0(A_\ell \cap A_{-\ell}) = 0$, a simple computation yields

$$\mathbb{I}_{A_\ell} + \mathbb{I}_{A_{-\ell}} = \mathbb{I}_{A_\ell}\mathbb{I}_{G_\ell} + \mathbb{I}_{A_\ell}\mathbb{I}_{A_{-\ell}} + 1 - \mathbb{I}_{G_\ell} = 1,\ P_0\text{-a.s.},$$

and we are done. $\square$

5. Proofs of Theorems 1 and 2

If the zero-set of the rate function H is a singleton, then we have a law of large numbers and we are done. Therefore, let us assume it is a line segment and let v_-, v_+ be its extreme points, $v_0 = 0$, and $C_\epsilon = \{\lim_{n\to\infty} n^{-1} X_n = v_\epsilon\}$, where $\epsilon \in \{-, +, 0\}$.
Clearly, any limit point v of $n^{-1}X_n$ is a zero of H. This already proves point (ii) of Theorem 1, since $\ell \cdot v = 0$ for all v with $H(v) = 0$.
Next, fix ℓ such that $\ell \cdot v_+ > 0$. Then by Lemma 1,

$$P_0(C_+ \,|\, X_1, \cdots, X_n) \geq \delta_\ell > 0,$$

whenever

$$\sup_{0\le i\le n} X_i\cdot\ell \le X_n\cdot\ell.$$

This will happen infinitely often on A_ℓ. But $P_0(C_+\,|\,X_1,\cdots,X_n)$, being a bounded martingale with respect to $\{\mathcal{F}_n\}_{n\ge0}$, converges to $\mathbb{1}_{C_+}$, P_0-a.s. Thus, $\mathbb{1}_{C_+}\mathbb{1}_{A_\ell}\ge\delta_\ell\mathbb{1}_{A_\ell}$, P_0-a.s., and

$$P_0(C_+\cap A_\ell)=P_0(A_\ell). \tag{8}$$

Now, if $v_-\ne 0$, then $\ell\cdot v_-<0$ and by the same reasoning as for v_+, one has

$$P_0(C_-\cap A_{-\ell})=P_0(A_{-\ell}). \tag{9}$$

On the other hand, if $v_-=0$, then (9) is trivial. Indeed, let v be a limit point of $n^{-1}X_n$ on $A_{-\ell}$. Then $\ell\cdot v\le 0$ because of the restriction $A_{-\ell}$ imposes. But $v=tv_+$ with $t\in[0,1]$, since $v_-=0$. Thus $\ell\cdot v_+>0$ implies $t=0$ and $v=0$. Since $C_-=C_0$ when $v_-=0$, we have (9). Finally, by Lemma 2, we have $P_0(A_\ell\cup A_{-\ell})=1$. Adding up (8) and (9) proves point (i) of Theorem 1. □

To prove Theorem 2, one again needs only to look at the case where the zero-set of H is a line segment. Without loss of generality, one can assume that $v_+\ne0$. Lemma 1 tells us then that $P_0(A_{v_+})\ge P_0(C_+)>0$. If (1) holds, then one has $P_0(A_{v_+})=1$ and Theorem 1 concludes the proof, with $v=v_+$. □

Remark 3 *The above argument also shows that if* (1) *holds, then there cannot be more than one non-zero extreme point of the zero-set of H. In other words, if this set is a line segment, then it cannot extend on both sides of* 0*; i.e. $|\,v_-\,|\cdot|\,v_+\,|=0$, and there is no ambiguity in choosing v.*

Remark 4 *It is noteworthy that unlike [12], we do not need, in the proof of Theorem* 2*, to discuss the case when B_ℓ always happens, since due to the large deviations results (Section* 3*) it follows that in that case the zero set of H reduces to $\{0\}$ and the velocity of escape exists and is* 0.

Remark 5 *For a function $h:\mathbb{Z}^d\times\Omega\to\mathbb{R}$, we say it is harmonic if*

$$\sum_y \pi_{xy}(\omega)h(y,\omega)=h(x,\omega),\mathbb{P}\text{-a.s.},$$

and covariant if $h(x,T_y\omega)=h(x+y,\omega)$, for all $x,y\in\mathbb{Z}^d$, and $\mathbb{P}$-a.e. ω. Question 1 *then is itself a consequence of a more general question:*

Question 3 (Harmonic 0-1 law) *Are constants the only bounded harmonic covariant functions?*

This is because, by the martingale convergence theorem, we know that $h(X_n,\omega)=P^\omega_{X_n}(A_\ell)$ converges to $\mathbb{1}_{A_\ell}$, P_0-a.s. This reduces proving the law of large numbers, for environments satisfying Hypotheses (E) *and* (M)*, to just answering Question* 3.

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