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Harmonizable fractional stable motion: Asymptotically normal estimators for both parameters

Antoine Ayache

Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France,
e-mail: antoine.ayache@univ-lille.fr

Abstract: There are two classical very different extensions of the well-known Gaussian fractional Brownian motion to non-Gaussian frameworks of heavy-tailed stable distributions: the harmonizable fractional stable motion (HFSM) and the linear fractional stable motion (LFSM). As far as we know, while several articles in the literature, some of which appeared a long time ago, have proposed statistical estimators for parameters of LFSM, no estimator has yet been proposed in the framework of HFSM. Among other things, what makes statistical estimation of parameters of HFSM to be a difficult problem is that, in contrast to LFSM, HFSM is not ergodic. The main goal of our work is to propose a new strategy for dealing with this problem and constructing strongly consistent and asymptotically normal statistical estimators for both parameters of HFSM. The keystone of our new strategy consists in the construction of new transforms of HFSM which allow to obtain, at any dyadic level and also at any two consecutive dyadic levels, sequences of *independent* stable random variables. This new strategy might allow to make statistical inference for more general non-ergodic harmonizable stable processes and fields than HFSM. Moreover, it could turn out to be useful in study of other issues related to them.

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1. Introduction and statement of the main results

A real-valued harmonizable fractional stable motion[1] (HFSM), denoted by $\{X(t)\}_{t\in\mathbb{R}}$, is a paradigmatic example of a continuous symmetric stable self-similar stochastic process with stationary increments. It was introduced, about 35 years ago, by Cambanis and Maejima in [10]. Basically, it depends on two parameters: the Hurst parameter H belonging to the open interval $(0,1)$, and the stability parameter α belonging to the interval $(0,2]$. Among other things, the parameter H governs roughness of sample paths of $\{X(t)\}_{t\in\mathbb{R}}$ and its self-

[1] Notice that the HFSM is sometimes called harmonizable fractional stable process.

similarity property:

$$\left\{a^{-H}X(at)\right\}_{t\in\mathbb{R}} \overset{d}{=} \{X(t)\}_{t\in\mathbb{R}}, \quad \text{for any fixed } a \in (0,+\infty),$$

where $\overset{d}{=}$ means that the equality holds in the sense of the finite-dimensional distributions. While the parameter α determines, for each $t \neq 0$ (notice that $X(0) \overset{a.s.}{=} 0$), heaviness of the tail of the distribution of the random variable $X(t)$, whose characteristic function $\Phi_{X(t)}$ satisfies $\Phi_{X(t)}(\lambda) = \exp\big(-\sigma(X(t))^\alpha |\lambda|^\alpha\big)$, for all $\lambda \in \mathbb{R}$, where $\sigma(X(t)) > 0$ is the scale parameter of $X(t)$. Indeed, except in the very particular Gaussian case $\alpha = 2$ in which the probability $\mathbb{P}\big(|X(t)| \geq z\big)$ vanishes exponentially fast when $z \to +\infty$, for any other value of α, one knows from Relation (1.2.10) in the well-known book [21], that one has, for some finite constants $0 < c'(t) < c''(t)$,

$$c'(t)z^{-\alpha} \leq \mathbb{P}\big(|X(t)| \geq z\big) \leq c''(t)z^{-\alpha}, \quad \text{for all } z \in [1,+\infty),$$

which in particular implies that $\mathbb{E}\big(|X(t)|^\gamma\big) < +\infty$ only when $\gamma \in (-1,\alpha)$.

The HFSM $\{X(t)\}_{t\in\mathbb{R}}$ is defined, for all $t \in \mathbb{R}$, through the stable stochastic integral in the frequency domain:

$$X(t) := \mathrm{Re}\Big(\int_{\mathbb{R}} \frac{e^{it\xi}-1}{|\xi|^{H+1/\alpha}}\, d\widetilde{M}_\alpha(\xi)\Big), \tag{1.1}$$

where $\widetilde{M}_\alpha$ is a complex-valued rotationally invariant α-stable random measure with Lebesgue control measure. A detailed presentation of such a random measure and the corresponding stable stochastic integral and related topics can for instance be found in Chapter 6 of the book [21]. The following remark, which provides two very important properties of this stochastic integral, will play a fundamental role in our work.

Remark 1.1. (*i*) The stable stochastic integral $\int_{\mathbb{R}}(\,\cdot\,)\,d\widetilde{M}_\alpha$ is a linear map on the Lebesgue space $L^\alpha(\mathbb{R})$ such that, for any deterministic function $g \in L^\alpha(\mathbb{R})$, the real part $\mathrm{Re}\big\{\int_{\mathbb{R}} g(\xi)\,d\widetilde{M}_\alpha(\xi)\big\}$ is a real-valued Symmetric α-Stable (SαS) random variable with a scale parameter satisfying

$$\sigma\Big(\mathrm{Re}\Big\{\int_{\mathbb{R}} g(\xi)\,d\widetilde{M}_\alpha(\xi)\Big\}\Big)^\alpha = \int_{\mathbb{R}} \big|g(\xi)\big|^\alpha\, d\xi. \tag{1.2}$$

The equality (1.2) is reminiscent of the classical isometry property of stochastic Wiener integrals; it implies that $\mathrm{Re}\big\{\int_{\mathbb{R}} g_n(\xi)\,d\widetilde{M}_\alpha(\xi)\big\}$ converges to $\mathrm{Re}\big\{\int_{\mathbb{R}} g(\xi)\,d\widetilde{M}_\alpha(\xi)\big\}$ in probability, when a sequence $(g_n)_n$ converges to g in $L^\alpha(\mathbb{R})$.

(*ii*) Let $m \in \mathbb{N}$ be arbitrary and let $f_1,\ldots,f_m$ be arbitrary functions of $L^\alpha(\mathbb{R})$ whose supports are disjoint up to Lebesgue negligible sets, then the SαS random variables $\mathrm{Re}\big\{\int_{\mathbb{R}} f_1(\xi)\,d\widetilde{M}_\alpha(\xi)\big\},\ldots,\mathrm{Re}\big\{\int_{\mathbb{R}} f_m(\xi)\,d\widetilde{M}_\alpha(\xi)\big\}$ are independent.

In the very particular Gaussian case where the stability parameter $\alpha = 2$, the HFSM represented by (1.1) reduces to the very classical Gaussian fractional Brownian motion (FBM) with Hurst parameter H, denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$. One refers to the two books [21, 13] for detailed presentations of FBM and many other related topics. It is well-known that, up to a deterministic multiplicative constant, the Gaussian process $\{B_H(t)\}_{t\in\mathbb{R}}$ can also be represented as a moving average stochastic Wiener integral in the time domain, whose integrand is no longer the complex-valued function $\xi \mapsto \frac{e^{it\xi}-1}{|\xi|^{H+1/2}}$ but the real-valued function $s \mapsto \Big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\Big)$. One recalls, in passing, the usual convention that, for all $(x,\beta)\in\mathbb{R}^2$, one has $(x)_+^\beta := x^\beta$ if $x>0$ and $(x)_+^\beta := 0$ else.

When the stability parameter $\alpha \neq 2$, the HFSM in (1.1) can no longer be represented as a moving average stable stochastic integral in the time domain. Actually, it is very different from the real-valued linear fractional stable motion (LFSM) $\{L(t)\}_{t\in\mathbb{R}}$ defined, for each $t\in\mathbb{R}$, by

$$L(t) := \int_{\mathbb{R}} \Big((t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\Big)\, dM_\alpha(s), \tag{1.3}$$

where M_α is a real-valued α-stable random measure. The large differences between the two processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{L(t)\}_{t\in\mathbb{R}}$ can be explained by several reasons. Two important ones of them are: (i) in contrast to the process $\{L(t)\}_{t\in\mathbb{R}}$ the process $\{X(t)\}_{t\in\mathbb{R}}$ is not ergodic (the latter fact results from Theorem 4 in [9]), (ii) behavior of sample paths of $\{L(t)\}_{t\in\mathbb{R}}$ and $\{X(t)\}_{t\in\mathbb{R}}$ is far from being the same. Indeed, sample paths of $\{L(t)\}_{t\in\mathbb{R}}$ are multifractal functions (see [6]), which become discontinuous when $H \le 1/\alpha$ and even unbounded on any interval when $H < 1/\alpha$ (see for example [21, 13]). While those of $\{X(t)\}_{t\in\mathbb{R}}$ are, on each compact interval, Hölder continuous of any order strictly less than H, for every value of $H\in(0,1)$ (see [14, 15, 21, 13]); namely, for each fixed $\delta>0$ and $T>0$, one has almost surely

$$\sup_{-T\le t'<t''\le T} \left\{ \frac{\big|X(t')-X(t'')\big|}{|t'-t''|^{H-\delta}} \right\} < +\infty\,. \tag{1.4}$$

Moreover, sample paths of $\{X(t)\}_{t\in\mathbb{R}}$ are monofractal functions; the latter fact results from their Hölderianity property combined with Corollary 4.4 in [5]. Also, for later purposes, one mentions that as regards their behavior at infinity, one can derive from Corollary 4.2 in [1], that, for all fixed $\delta>0$, one has, almost surely,

$$\sup_{|t|\ge 1} \left\{ \frac{\big|X(t)\big|}{|t|^{H+\delta}} \right\} < +\infty\,. \tag{1.5}$$

Let us now present the main motivations behind our present work and its main contributions. Statistical estimators for the parameters H and α of the LFSM $\{L(t)\}_{t\in\mathbb{R}}$ and related moving average stable processes have been proposed in several articles in the literature (see for instance [24, 20, 3, 2, 11, 18, 17, 16]), some of which appeared a long time ago. However, as far as we know, in the framework of the HFSM $\{X(t)\}_{t\in\mathbb{R}}$ and related harmonizable stable processes and fields no statistical estimator for any one of these two parameters has yet been proposed in the literature. Also, according to what is mentioned in Remark 1.2 (D) of the very recent article [7], their statistical estimation in such a framework is far from being an obvious problem due to the fact that HFSM and related harmonizable stable processes and fields are not ergodic. The main idea behind our strategy for dealing with the latter problem is to construct new transforms of HFSM which allow to obtain, at any dyadic level $j \in \mathbb{N}$, a sequence $\{Y_{j,k}\}_{k\in\mathbb{N}}$ of *independent* real-valued SαS random variables whose scale parameters $\sigma(Y_{j,k})$, $k \in \mathbb{N}$, are closely connected to the unknown parameters H and α of HFSM through simple formulas which are rather easy to handle. Also, we emphasize that the two sequences of random variables $\{Y_{j,2p-1}\}_{p\in\mathbb{N}}$ and $\{Y_{j+1,2p-1}\}_{p\in\mathbb{N}}$ are *independent*. Roughly speaking, these new transforms $Y_{j,k}$, $(j,k) \in \mathbb{N}^2$, of HFSM are at the same time inspired by discrete wavelet transforms $\mathcal{W}_{j,k}$, $(j,k) \in \mathbb{N}^2$, of HFSM and significantly different from them. Indeed, while $\mathcal{W}_{j,k}$ is defined (see e.g. [12, 19]), sometimes up to normalizing factor, as the pathwise Lebesgue integral

$$\mathcal{W}_{j,k} := \int_{\mathbb{R}} \psi_{j,k}(t) X(t)\, dt,$$

where

$$\psi_{j,k}(t) := \psi(2^j t - k), \quad \text{for all } t \in \mathbb{R}, \tag{1.6}$$

with ψ being a "nice" real-valued function; we define $Y_{j,k}$ as the pathwise Lebesgue integral

$$Y_{j,k} := \frac{2}{2\pi} \int_{\mathbb{R}} \operatorname{Re}\Big(\widehat{\psi}_{j,k}(t)\Big) X(t)\, dt, \tag{1.7}$$

where $\widehat{\psi}_{j,k}$ is the Fourier transform of $\psi_{j,k}$.

Remark 1.2. Throughout our work, we use the very common convention that, for any function $f \in L^1(\mathbb{R})$, the Fourier transform $\widehat{f}$, also denoted by $\mathcal{F}(f)$, is defined as

$$\mathcal{F}(f)(t) = \widehat{f}(t) := \int_{\mathbb{R}} e^{-it\xi} f(\xi)\, d\xi, \quad \text{for all } t \in \mathbb{R}. \tag{1.8}$$

While, the inverse Fourier transform of f, denoted by $\mathcal{F}^{-1}(f)$, is defined as

$$\mathcal{F}^{-1}(f)(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} f(t)\, dt, \quad \text{for all } \xi \in \mathbb{R}. \tag{1.9}$$

It is well-known (see e.g. Chapter 1 of the book [23]) that the two maps $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to $L^2(\mathbb{R})$, and satisfy

$$\mathcal{F}^{-1}\big(\mathcal{F}(g)\big) = \mathcal{F}\big(\mathcal{F}^{-1}(g)\big) = g, \quad \text{for every } g \in L^2(\mathbb{R}). \tag{1.10}$$

We always assume that the function ψ in (1.6) and (1.7) satisfies the two general assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ that we are now going to give.

($\mathcal{A}_1$) ψ is an even (i.e. $\psi(-\xi) = \psi(\xi)$ for all $\xi \in \mathbb{R}$), real-valued, continuous function on $\mathbb{R}$, with a compact non-empty support, denoted by I, such that

$$I := \operatorname{supp} \psi := \overline{\{\xi \in \mathbb{R},\, \psi(\xi) \neq 0\}} \subseteq \left[-4^{-1}, 4^{-1}\right]. \tag{1.11}$$

Observe that this assumption ($\mathcal{A}_1$) implies that $\widehat{\psi}$ (the Fourier transform of ψ) is a real-valued, even, continuous function on $\mathbb{R}$. Thus, it follows from (1.6), (1.7) and elementary properties of Fourier transform that

$$Y_{j,k} = \frac{2^{1-j}}{2\pi} \int_{\mathbb{R}} \cos(2^{-j} k\, t) \widehat{\psi}(2^{-j} t) X(t)\, dt\,, \quad \text{for all } (j,k) \in \mathbb{N}^2. \tag{1.12}$$

In view of (1.5) and continuity on $\mathbb{R}$ of sample paths of the HFSM $\{X(t)\}_{t\in\mathbb{R}}$, in order to guarantee the existence and finiteness of the Lebesgue pathwise integral in (1.12), we impose the following assumption ($\mathcal{A}_2$) to the Fourier transform of ψ:

($\mathcal{A}_2$) There exists a constant c such that

$$\left|\widehat{\psi}(t)\right| \leq c\big(1 + |t|\big)^{-2}, \quad \text{for every } t \in \mathbb{R}. \tag{1.13}$$

Remark 1.3. Observe that there are many functions satisfying the two general assumptions ($\mathcal{A}_1$) and ($\mathcal{A}_2$), as for instance the piecewise linear triangle function:

$$\psi(\xi) := \big(\mathbb{1}_{[-1,1]} * \mathbb{1}_{[-1,1]}\big)(8\xi) = \big(2 - |8\xi|\big)\mathbb{1}_{[-4^{-1},4^{-1}]}(\xi), \quad \text{for all } \xi \in \mathbb{R},$$

where "$*$" denotes the usual convolution product. Also, observe that there is no need to impose to ψ to have any vanishing moment, while such a condition on moment(s) of ψ plays a crucial role in the case of the discrete wavelet transform $\mathcal{W}_{j,k}$.

Before stating the first main theorem of our work, one needs to make the following remark.

Remark 1.4. For all $\alpha \in (0,2]$, one denotes by $W_{(\alpha)}$ an arbitrary real-valued SαS random variable with scale parameter equals to 1; thus, its characteristic function $\Phi_{W_{(\alpha)}}$ satisfies, for all $\lambda \in \mathbb{R}$, $\Phi_{W_{(\alpha)}}(\lambda) = \exp(-|\lambda|^\alpha)$ (see e.g. [21]). Notice that in the very special Gaussian case $\alpha = 2$, the random variable $W_{(2)}$ has a centered Gaussian distribution with standard deviation equals to $2^{1/2}$ (and not 1). One knows from Theorem 3 in [22] and from the classical equality $\mathbb{E}\big(|W_{(2)}|^\rho\big) = 2^\rho\, \Gamma\big((1+\rho)/2\big)/\Gamma(1/2)$ that, for every $\alpha \in (0,2]$,

$$\mathcal{M}(\rho, \alpha^{-1}) := \mathbb{E}\big(|W_{(\alpha)}|^\rho\big) = \frac{2^\rho\, \Gamma\big((1+\rho)/2\big)\Gamma\big(1 - \rho\alpha^{-1}\big)}{\Gamma(1/2)\Gamma\big(1 - \rho/2\big)}\,, \quad \text{for all } \rho \in (-1, \alpha), \tag{1.14}$$

where $\alpha^{-1} = 1/\alpha$ and Γ is the usual "Gamma" function. Moreover, denoting by $\log_2$ the binary logarithm (that is $\log_2(x) := \log(x)/\log(2)$, for all $x \in (0, +\infty)$, where log is the Napierian logarithm) and by ∂_ρ the partial derivative operator with respect to the variable ρ, one has, for all $\alpha \in (0,2]$, that

$$\big(\log(2)\big)\mathbb{E}\Big(\log_2|W_{(\alpha)}|\Big) = (\partial_\rho \mathcal{M})(0,\alpha^{-1}) \tag{1.15}$$
$$\text{and} \quad \big(\log(2)\big)^2\,\mathbb{E}\Big(\big(\log_2|W_{(\alpha)}|\big)^2\Big) = (\partial_\rho^2 \mathcal{M})(0,\alpha^{-1});$$

since

$$\rho \mapsto |G'_{W_{(\alpha)}}(\rho)| = |W_{(\alpha)}|^\rho \big|\log|W_{(\alpha)}|\big|$$

and

$$\rho \mapsto |G''_{W_{(\alpha)}}(\rho)| = |W_{(\alpha)}|^\rho \big(\log|W_{(\alpha)}|\big)^2,$$

the absolute values of the derivative functions of orders 1 and 2 of the function $\rho \mapsto G_{W_{(\alpha)}}(\rho) := |W_{(\alpha)}|^\rho$, are, on a deterministic neighborhood of 0, bounded by an integrable random variable not depending on ρ; more precisely, let $\rho_0 \in (0, 4^{-1}\alpha)$ be arbitrary and fixed, and let B_{ρ_0} be the positive integrable random variable defined, almost surely, as

$$B_{\rho_0} := \Big(|W_{(\alpha)}|^{-\rho_0} + |W_{(\alpha)}|^{\rho_0}\Big)\Big(\big|\log|W_{(\alpha)}|\big| + \big(\log|W_{(\alpha)}|\big)^2\Big),$$

using the inequality $|x|^\rho \le |x|^{-\rho_0} + |x|^{\rho_0}$, for all $x \in \mathbb{R}\setminus\{0\}$ and $\rho \in [-\rho_0, \rho_0]$, one has almost surely,

$$\sup_{\rho\in[-\rho_0,\rho_0]} \Big\{|G'_{W_{(\alpha)}}(\rho)| + |G''_{W_{(\alpha)}}(\rho)|\Big\} \le B_{\rho_0}.$$

One can derive from (1.15), (1.14) and standard calculations, that $\alpha^{-1} \mapsto \mathrm{Var}\big(\log_2|W_{(\alpha)}|\big)$ is an explicit continuous polynomial positive function of degree 2 in the variable $\alpha^{-1} \in [2^{-1}, +\infty)$, whose coefficients can be expressed in terms of the "Gamma" function. The positive continuous function G is defined as

$$G(\alpha^{-1}) := \Big(2\,\mathrm{Var}\big(\log_2|W_{(\alpha)}|\big)\Big)^{-1/2}, \quad \text{for all } \alpha^{-1} \in [2^{-1}, +\infty). \tag{1.16}$$

Theorem 1.5. *For every $n \in \mathbb{N}$, one sets*

$$\widehat{\alpha}^{-1}_{n,\log_2} := \frac{1}{n}\left(\sum_{p=1}^{n}\Big(\log_2|Y_{1,2p-1}| - \log_2|Y_{2,4p-1}|\Big)\right), \tag{1.17}$$

where $Y_{1,2p-1}$ and $Y_{2,4p-1}$ are defined through (1.12). Then, the following two results hold.

(i) $\widehat{\alpha}^{-1}_{n,\log_2}$ is a strongly consistent (almost surely convergent) estimator of the inverse α^{-1} of the stability parameter α of HFSM.

(ii) For all $n \in \mathbb{N}$, the random variable $D_{2,n,\log_2}$ is defined as

$$D_{2,n,\log_2} := G\Big(\max\{\widehat{\alpha}_{n,\log_2}^{-1}, 2^{-1}\}\Big) n^{1/2}\big(\widehat{\alpha}_{n,\log_2}^{-1} - \alpha^{-1}\big), \tag{1.18}$$

where the positive continuous function G is as in (1.16). When n goes to $+\infty$, the random variable $D_{2,n,\log_2}$ converges in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.

In order to state the second and the third main theorems of our work, one needs the following definition.

Definition 1.6. For every $(j,m) \in \mathbb{N}^2$, the statistics $V_{j,\log_2}^m$ and $V_{j,\gamma}^m$, γ being a fixed positive real number, are defined as

$$V_{j,\log_2}^m := \sum_{p=1}^{m} \log_2 \left|Y_{j,2p-1}\right| \tag{1.19}$$

and

$$V_{j,\gamma}^m := \sum_{p=1}^{m} \left|Y_{j,2p-1}\right|^{\gamma}, \tag{1.20}$$

where $Y_{j,2p-1}$ is defined through (1.12).

Theorem 1.7. *For every $n \in \mathbb{N}$, one sets*

$$\widehat{H}_{n,\log_2} := \frac{1}{n}\Big(V_{2,\log_2}^n - V_{1,\log_2}^n\Big), \tag{1.21}$$

where $V_{1,\log_2}^n$ and $V_{2,\log_2}^n$ are defined through (1.19). Then, the following two results hold.

(i) $\widehat{H}_{n,\log_2}$ is a strongly consistent (almost surely convergent) estimator of the Hurst parameter H of HFSM.

(ii) For all $n \in \mathbb{N}$, the random variable $D_{1,n,\log_2}$ is defined as

$$D_{1,n,\log_2} := G\Big(\max\{\widehat{\alpha}_{n,\log_2}^{-1}, 2^{-1}\}\Big) n^{1/2}\big(\widehat{H}_{n,\log_2} - H\big), \tag{1.22}$$

where G and $\widehat{\alpha}_{n,\log_2}^{-1}$ are as in (1.16) and (1.17). When n goes to $+\infty$, the random variable $D_{1,n,\log_2}$ converges in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.

Theorem 1.8. *One assumes that $\alpha \in [\underline{\alpha}, 2]$, where the lower bound $\underline{\alpha} \in (0,2]$ is known. Also, one assumes that $\gamma \in (0, 4^{-1}\underline{\alpha})$ is arbitrary and fixed. Let $(m_j)_{j\in\mathbb{N}}$ be an arbitrary non-decreasing sequence (that is $m_j \le m_{j+1}$, for all $j \in \mathbb{N}$) of integers larger than 2 which satisfy the condition*

$$m_j \ge j\,, \quad \text{for all } j \in \mathbb{N}. \tag{1.23}$$

For all $j \in \mathbb{N}$, one sets

$$\widehat{H}_{j,\gamma} := \gamma^{-1} \log_2 \left(\frac{V_{2,\gamma}^{m_j}}{V_{1,\gamma}^{m_j}}\right) \tag{1.24}$$

and

$$\widehat{\alpha}_{j,\gamma}^{-1} := \gamma^{-1}\left(1-\log_2\left(\frac{V_{2,\gamma}^{m_{j+1}}}{V_{1,\gamma}^{m_j}}\right)\right), \tag{1.25}$$

where $V_{1,\gamma}^{m_j}$, $V_{2,\gamma}^{m_j}$ *and* $V_{2,\gamma}^{m_{j+1}}$ *are defined through (1.20). Then, the following two results hold.*

(i) $\widehat{H}_{j,\gamma}$ *is a strongly consistent (almost surely convergent) estimator of the Hurst parameter* H *of HFSM.*
(ii) Under the condition

$$\lim_{j\to+\infty}\log_2\left(\frac{m_{j+1}}{m_j}\right)=1, \tag{1.26}$$

$\widehat{\alpha}_{j,\gamma}^{-1}$ *is a strongly consistent (almost surely convergent) estimator of the inverse* α^{-1} *of the stability parameter* α *of HFSM.*

Before stating the fourth and the last main theorem of our work, one needs to make the following remark.

Remark 1.9. Let $\underline{\alpha}\in(0,2]$ be as in Theorem 1.8 and let γ be arbitrary and such that

$$0<\gamma<\frac{\underline{\alpha}}{2\underline{\alpha}+2}\,, \tag{1.27}$$

which implies that

$$2\gamma(H+1/\alpha)<1, \quad \text{for all } (H,\alpha)\in[0,1]\times[\underline{\alpha},2]. \tag{1.28}$$

For every $(H,\alpha^{-1})\in[0,1]\times[2^{-1},\underline{\alpha}^{-1}]$, one sets

$$F_\gamma(H,\alpha^{-1}) := \frac{\mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\big(1-2\gamma(H+\alpha^{-1})\big)^{1/2}}{\big(\mathrm{Var}\big(|W_{(\alpha)}|^\gamma\big)\big)^{1/2}\big(1-\gamma(H+\alpha^{-1})\big)}\,, \tag{1.29}$$

where $W_{(\alpha)}$ denotes an arbitrary real-valued SαS random variable with scale parameter equals to 1. One can derive from (1.29), (1.28) and (1.14) that the positive function F_γ can be expressed in an explicit way in terms of the "Gamma" function, which allows to show that F_γ is continuous on the compact rectangle $[0,1]\times[2^{-1},\underline{\alpha}^{-1}]$.

The following theorem provides Central Limit Theorems for the two estimators introduced in Theorem 1.8.

Theorem 1.10. *Let* $(m_{1,j})_{j\in\mathbb{N}}$ *and* $(m_{2,j})_{j\in\mathbb{N}}$ *be two arbitrary non-decreasing sequences of integers larger than* 2 *which satisfy the condition (1.23). Also, one assumes that* $(m_{2,j})_{j\in\mathbb{N}}$ *satisfies the following strengthened version of the condition (1.26):*

$$\lim_{j\to+\infty}(m_{2,j})^{1/2}\left|\log_2\left(\frac{m_{2,j+1}}{m_{2,j}}\right)-1\right|=0\,. \tag{1.30}$$

Let $\underline{\alpha}$ be as in Theorem 1.8 and let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and such that (1.27) holds. For each $j \in \mathbb{N}$, one denotes by $\widehat{H}_{1,j,\gamma}$ the strongly consistent estimator of the Hurst parameter H of HFSM defined through (1.24) with $m_j = m_{1,j}$, and one denotes by $\widehat{\alpha}_{2,j,\gamma}^{-1}$ the strongly consistent estimator of the inverse α^{-1} of the stability parameter α of HFSM defined through (1.25) with $m_j = m_{2,j}$ and $m_{j+1} = m_{2,j+1}$. For all $j \in \mathbb{N}$, the two random variables $D_{1,j,\gamma}$ and $D_{2,j,\gamma}$ are defined as

$$D_{1,j,\gamma} := 2^{-1/2}\big(\log(2)\big)\gamma F_\gamma\big(\tau_1(\widehat{H}_{1,j,\gamma}), \tau_2(\widehat{\alpha}_{2,j,\gamma}^{-1})\big)(m_{1,j})^{1/2}\Big(\widehat{H}_{1,j,\gamma} - H\Big) \quad (1.31)$$

and

$$D_{2,j,\gamma} := (2/3)^{1/2}\big(\log(2)\big)\gamma F_\gamma\big(\tau_1(\widehat{H}_{1,j,\gamma}), \tau_2(\widehat{\alpha}_{2,j,\gamma}^{-1})\big)(m_{2,j})^{1/2}\Big(\widehat{\alpha}_{2,j,\gamma}^{-1} - \alpha^{-1}\Big), \quad (1.32)$$

where F_γ is the positive continuous function on $[0,1] \times [2^{-1}, \underline{\alpha}^{-1}]$ defined in Remark 1.9, and τ_1 and τ_2 are the two continuous "truncation" functions defined, for all $x \in \mathbb{R}$, as

$$\tau_1(x) := \max\Big\{0, \min\big\{x, 1\big\}\Big\} \quad \textit{and} \quad \tau_2(x) := \max\Big\{2^{-1}, \min\big\{x, \underline{\alpha}^{-1}\big\}\Big\}. \quad (1.33)$$

When j goes to $+\infty$, the two random variables $D_{1,j,\gamma}$ and $D_{2,j,\gamma}$ converge in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.

Remark 1.11. (i) We believe that the new strategy introduced in our present work would open the door to statistical estimation of parameters of harmonizable stable processes and fields extending the HFSM, as for instance the harmonizable fractional stable field studied in e.g. [5], or the harmonizable fractional stable sheet studied in e.g. [4].

(ii) In our present work, the four estimators $\widehat{\alpha}_{n,\log_2}^{-1}$, $\widehat{H}_{n,\log_2}$, $\widehat{H}_{j,\gamma}$ and $\widehat{\alpha}_{j,\gamma}^{-1}$, are obtained from the observation of a sample path of the HFSM X in continuous time, we believe that it would be possible to extend our estimation procedures and the associated Central Limit Theorems to frameworks where only a discretized sample path of X is observed.

We intend to study these two issues (i) and (ii) in future works.

The remaining of our present work is organized in the following way. In Section 2, basically we show that the real-valued SαS random variables $Y_{j,k}$, $(j,k) \in \mathbb{N}^2$, defined in (1.12), can be represented in terms of the stable stochastic integral $\int_{\mathbb{R}} (\cdot)\, d\widetilde{M}_\alpha$ (see Lemma 2.1); two important consequences, for any fixed $j \in \mathbb{N}$, of this representation are: the independence property of the random variables $Y_{j,k}$, $k \in \mathbb{N}$, and the independence property of the two sequences of random variables $\{Y_{j,2p-1}\}_{p\in\mathbb{N}}$ and $\{Y_{j+1,2p-1}\}_{p\in\mathbb{N}}$. Section 3 is devoted to the proofs of Theorems 1.5 and 1.7, Section 4 to that of Theorem 1.8, and Section 5 to that of Theorem 1.10.

2. The keystone

The main goal of the present section is to prove the following very crucial lemma and to derive important consequences of it.

Lemma 2.1. *For all $(j,k) \in \mathbb{N}^2$, $Y_{j,k}$ (defined in (1.12)) is a real-valued SαS random variable which can almost surely be expressed as*

$$Y_{j,k} = \mathrm{Re}\Big(\int_{\mathbb{R}} \frac{\widetilde{\psi}_{j,k}(\xi)}{|\xi|^{H+1/\alpha}}\, d\widetilde{M}_\alpha(\xi)\Big), \tag{2.1}$$

where

$$\widetilde{\psi}_{j,k}(\xi) := \psi(2^j\xi + k) + \psi(2^j\xi - k), \quad \text{for every } \xi \in \mathbb{R}. \tag{2.2}$$

Remark 2.2. One knows from (2.2) and the assumption $(\mathcal{A}_1)$ on ψ (see Section 1) that, for all $(j,k) \in \mathbb{N}^2$, $\widetilde{\psi}_{j,k}$ is a real-valued even continuous function on $\mathbb{R}$ with compact support, denoted by $I_{j,k}$, such that (see (1.11))

$$\begin{aligned} I_{j,k} &:= \operatorname{supp} \widetilde{\psi}_{j,k} \\ &\subseteq \Big[-2^{-j-2} - k2^{-j}, 2^{-j-2} - k2^{-j}\Big] \cup \Big[-2^{-j-2} + k2^{-j}, 2^{-j-2} + k2^{-j}\Big]. \end{aligned} \tag{2.3}$$

Then, in view of the fact that $k \geq 1$, it turns out that the function $\xi \mapsto |\xi|^{-H-1/\alpha}\, \widetilde{\psi}_{j,k}(\xi)$ belongs to the Lebesgue space $L^\alpha(\mathbb{R})$, which guarantees that the SαS stochastic integral in (2.1) is well-defined.

The proof of Lemma 2.1 will be given by the end of the section. For the time being, we focus on the following very fundamental lemma which provides 3 important consequences of Lemma 2.1.

Lemma 2.3. *For each fixed $j \in \mathbb{N}$, the following three results hold:*

- *(i) $\{Y_{j,k}\}_{k\in\mathbb{N}}$ is a sequence of independent random variables.*
- *(ii) The two sequences of random variables $\{Y_{j,2p-1}\}_{p\in\mathbb{N}}$ and $\{Y_{j+1,2p-1}\}_{p\in\mathbb{N}}$ are independent. One points out that the latter fact implies, for all $(m',m'') \in \mathbb{N}^2$, that the two random variables $V^{m'}_{j,\log_2}$ and $V^{m''}_{j+1,\log_2}$ (see (1.19)) are independent, and also that the two random variables $V^{m'}_{j,\gamma}$ and $V^{m''}_{j+1,\gamma}$ (see (1.20)) are independent; actually the sums in (1.19) and (1.20) have been restricted to odd integers for having this independence property with respect to j.*
- *(iii) The two sequences of random variables $\{Y_{j,k}\}_{k\in\mathbb{N}}$ and $\{2^{(j-1)H}Y_{1,k}\}_{k\in\mathbb{N}}$ have the same distribution.One points out that the latter fact entails, for every $m \in \mathbb{N}$, that the two random variables $V^m_{j,\log_2}$ and $V^m_{1,\log_2} + m(j-1)H$ have the same distribution, and also that the two random variables $V^m_{j,\gamma}$ and $2^{(j-1)\gamma H}V^m_{1,\gamma}$ have the same distribution.*

Proof. Part (i) of the lemma follows from (2.3), which clearly implies that the compact supports of the functions $\widetilde{\psi}_{j,k}$, $k \in \mathbb{N}$, are disjoint. Indeed, in view of (2.1) and of Remark 1.1 (*ii*), the latter fact entails that the SαS random variables $Y_{j,k}$, $k \in \mathbb{N}$, are independent.

For proving Part (*ii*) of the lemma, one has to show that for each two finite non-empty sets $\mathcal{P}_0 \subset \mathbb{N}$ and $\mathcal{P}_1 \subset \mathbb{N}$ and for any two finite collections of real numbers $(\lambda_{0,p})_{p\in\mathcal{P}_0}$ and $(\lambda_{1,p})_{p\in\mathcal{P}_1}$, the two real-valued S$\alpha$S random variables

$$\sum_{p\in\mathcal{P}_0} \lambda_{0,p} Y_{j,2p-1} = \mathrm{Re}\left(\int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \Big(\sum_{p\in\mathcal{P}_0} \lambda_{0,p} \widetilde{\psi}_{j,2p-1}(\xi) \Big) \, d\widetilde{M}_\alpha(\xi) \right) \tag{2.4}$$

and

$$\sum_{p\in\mathcal{P}_1} \lambda_{1,p} Y_{j+1,2p-1} = \mathrm{Re}\left(\int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \Big(\sum_{p\in\mathcal{P}_1} \lambda_{1,p} \widetilde{\psi}_{j+1,2p-1}(\xi) \Big) \, d\widetilde{M}_\alpha(\xi) \right) \tag{2.5}$$

are independent. Thus, in view of Remark 1.1 (*ii*), it is enough to prove that the compact supports of the functions $\sum_{p\in\mathcal{P}_0} \lambda_{0,p}\widetilde{\psi}_{j,2p-1}$ and $\sum_{p\in\mathcal{P}_1} \lambda_{1,p}\widetilde{\psi}_{j+1,2p-1}$ are disjoint. One knows from Remark 2.2 that

$$\begin{aligned} &\mathrm{supp} \sum_{p\in\mathcal{P}_0} \lambda_{0,p}\widetilde{\psi}_{j,2p-1} \subseteq \bigcup_{p\in\mathcal{P}_0} I_{j,2p-1} \\ &\text{and} \quad \mathrm{supp} \sum_{p\in\mathcal{P}_1} \lambda_{1,p}\widetilde{\psi}_{j+1,2p-1} \subseteq \bigcup_{p\in\mathcal{P}_1} I_{j+1,2p-1} \,. \end{aligned}$$

Then, for proving that these two supports are disjoint, it is sufficient to show that

$$I_{j,2p_0-1} \cap I_{j+1,2p_1-1} = \emptyset \,, \quad \text{for all } (p_0,p_1) \in \mathcal{P}_0 \times \mathcal{P}_1. \tag{2.6}$$

One knows from (2.3) that

$$\begin{aligned} &I_{j,2p_0-1} \cap I_{j+1,2p_1-1} \\ &= \Big\{ x \in \mathbb{R},\ \big| |x| - (2p_0-1)2^{-j} \big| \le 2^{-j-2} \\ &\qquad\qquad \text{and } \big| |x| - (2p_1-1)2^{-j-1} \big| \le 2^{-j-3} \Big\} . \end{aligned} \tag{2.7}$$

Suppose ad absurdum that, for some $(\overline{p}_0, \overline{p}_1) \in \mathcal{P}_0 \times \mathcal{P}_1$, there exists some $\overline{x} \in I_{j,2\overline{p}_0-1} \cap I_{j+1,2\overline{p}_1-1}$. Then one can derive from (2.7) and the triangle inequality that

$$\begin{aligned} &\big| (2\overline{p}_0-1)2^{-j} - (2\overline{p}_1-1)2^{-j-1} \big| \\ &\le \big| (2\overline{p}_0-1)2^{-j} - |\overline{x}| \big| + \big| |\overline{x}| - (2\overline{p}_1-1)2^{-j-1} \big| \le 3 \cdot 2^{-j-3} \,, \end{aligned}$$

which implies that

$$\big| 2(2\overline{p}_0-1) - (2\overline{p}_1-1) \big| \le 3/4 < 1 \,.$$

This cannot happen since $2(2\overline{p}_0-1)$ is an even integer while $2\overline{p}_1-1$ is an odd integer.

For proving Part (*iii*) of the lemma, one has to show that for each finite non-empty set $\mathcal{P} \subset \mathbb{N}$ and for any finite collection of real numbers $(\lambda_k)_{k\in\mathcal{P}}$, the two real-valued S$\alpha$S random variables $\sum_{k\in\mathcal{P}} \lambda_k Y_{j,k}$ and $2^{(j-1)H}\sum_{k\in\mathcal{P}} \lambda_k Y_{1,k}$ have the same distribution, which, in view of Remark 3.1 in Section 3, amounts to prove that their scale parameters are equal. One can derive from (2.1), (2.2) and Remark 1.1 (*i*) that these scale parameters satisfy

$$\sigma\Big(\sum_{k\in\mathcal{P}} \lambda_k Y_{j,k}\Big)^{\alpha} = \int_{\mathbb{R}} |\xi|^{-\alpha H-1}\Big|\sum_{k\in\mathcal{P}} \lambda_k \big(\psi(2^j\xi+k)+\psi(2^j\xi-k)\big)\Big|^{\alpha}\,d\xi$$

and

$$\begin{aligned}&\sigma\Big(2^{(j-1)H}\sum_{k\in\mathcal{P}} \lambda_k Y_{1,k}\Big)^{\alpha}\\ &= 2^{(j-1)\alpha H}\int_{\mathbb{R}} |\eta|^{-\alpha H-1}\Big|\sum_{k\in\mathcal{P}} \lambda_k \big(\psi(2\eta+k)+\psi(2\eta-k)\big)\Big|^{\alpha}\,d\eta.\end{aligned}$$

Thus, it results from the change of variable $\eta = 2^{j-1}\xi$ that

$$\sigma\Big(\sum_{k\in\mathcal{P}} \lambda_k Y_{j,k}\Big) = \sigma\Big(2^{(j-1)H}\sum_{k\in\mathcal{P}} \lambda_k Y_{j,k}\Big). \qquad \square$$

From now on, our aim is to prove Lemma 2.1, to this end we need the following definition and some preliminary results.

Definition 2.4. For all $(j,k)\in\mathbb{N}^2$, let $\widetilde{\psi}_{j,k}$ be the same function as in (2.2) and let $\widehat{\widetilde{\psi}}_{j,k}$ be its Fourier transform. For every $n\in\mathbb{N}$, the random variable $Y_{j,k}^n$ is defined as the finite sum:

$$\begin{aligned}&Y_{j,k}^n\\ &:= \frac{1}{2\pi}\sum_{|m|\le 4^n} X(d_{n,m})\int_{d_{n,m}}^{d_{n,m+1}} \widehat{\widetilde{\psi}}_{j,k}(t)\,dt \qquad (2.8)\\ &= \mathrm{Re}\left(\int_{\mathbb{R}} |\xi|^{-H-1/\alpha}\left(\frac{1}{2\pi}\sum_{|m|\le 4^n}\int_{d_{n,m}}^{d_{n,m+1}} \big(e^{id_{n,m}\xi}-1\big)\widehat{\widetilde{\psi}}_{j,k}(t)\,dt\right) d\widetilde{M}_{\alpha}(\xi)\right),\end{aligned}$$

where, for each $(n,m)\in\mathbb{N}\times\mathbb{Z}$, the dyadic number $d_{n,m} := 2^{-n}\,m$.

Remark 2.5. Notice that the last equality in (2.8) results from (1.1) which allows to express $Y_{j,k}^n$ as

$$\begin{aligned}Y_{j,k}^n &= \sum_{|m|\le 4^n}\Big(\frac{1}{2\pi}\int_{d_{n,m}}^{d_{n,m+1}} \widehat{\widetilde{\psi}}_{j,k}(t)\,dt\Big)\mathrm{Re}\Big(\int_{\mathbb{R}} \frac{e^{id_{n,m}\xi}-1}{|\xi|^{H+1/\alpha}}\,d\widetilde{M}_{\alpha}(\xi)\Big)\\ &= \mathrm{Re}\left(\sum_{|m|\le 4^n}\Big(\frac{1}{2\pi}\int_{d_{n,m}}^{d_{n,m+1}} \widehat{\widetilde{\psi}}_{j,k}(t)\,dt\Big)\int_{\mathbb{R}} \frac{e^{id_{n,m}\xi}-1}{|\xi|^{H+1/\alpha}}\,d\widetilde{M}_{\alpha}(\xi)\right), \qquad (2.9)\end{aligned}$$

where the last equality results from the fact that the deterministic Lebesgue integrals $\int_{d_{n,m}}^{d_{n,m+1}} \widehat{\widetilde{\psi}}_{j,k}(t)\,dt$ are real numbers. Then, using the fact that the sum $\sum_{|m|\le 4^n} \cdots$ in (2.9) consists in only a finite number of terms, and the linearity property of the stable stochastic integral $\int_{\mathbb{R}} (\,\cdot\,)\, d\widetilde{M}_\alpha$ and of the deterministic Lebesgue integrals $\int_{d_{n,m}}^{d_{n,m+1}} (\,\cdot\,)\,dt$, one obtains the last equality in (2.8).

Remark 2.6. Let $(j,k) \in \mathbb{N}^2$ be arbitrary and fixed. One can derive from (2.2) and standard calculations that the Fourier transform of $\widetilde{\psi}_{j,k}$ is given by

$$\widehat{\widetilde{\psi}}_{j,k}(t) = 2^{1-j}\cos(2^{-j}kt)\widehat{\psi}(2^{-j}t) \quad \text{for all } t \in \mathbb{R}. \tag{2.10}$$

Thus, (1.12) reduces to

$$Y_{j,k} = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\widetilde{\psi}}_{j,k}(t) X(t)\,dt\,. \tag{2.11}$$

Moreover, it follows from (2.10) and (1.13), that, for some finite deterministic constant $c(j) > 0$, only depending on j, one has

$$\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big| \le c(j)\big(1 + |t|\big)^{-2}, \quad \text{for every } t \in \mathbb{R}, \tag{2.12}$$

which implies that $\widehat{\widetilde{\psi}}_{j,k}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Lemma 2.7. *Let $(j,k) \in \mathbb{N}^2$ be arbitrary and fixed. The continuous function $\widetilde{\psi}_{j,k}$ can be expressed as*

$$\widetilde{\psi}_{j,k}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \big(e^{i\xi t} - 1\big)\widehat{\widetilde{\psi}}_{j,k}(t)\,dt, \quad \textit{for all } \xi \in \mathbb{R}. \tag{2.13}$$

Proof. First, observe that the fact that $\widetilde{\psi}_{j,k}$ is a continuous function on $\mathbb{R}$ results from (2.2) and the assumption $(\mathcal{A}_1)$ in Section 1. Then, since $\widehat{\widetilde{\psi}}_{j,k}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, one knows from Remark 1.2, and more particularly from (1.10) in it (see also Corollary 1.21 in Chapter 1 of the book [23]), that the function $\widetilde{\psi}_{j,k}$ can be expressed as

$$\widetilde{\psi}_{j,k}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{\widetilde{\psi}}_{j,k}(t)\,dt, \quad \text{for all } \xi \in \mathbb{R}. \tag{2.14}$$

Thus combining (2.14) with (2.3) and the fact that the integer $k \ge 1$, one obtains that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\widetilde{\psi}}_{j,k}(t)\,dt = \widetilde{\psi}_{j,k}(0) = 0. \tag{2.15}$$

Then (2.14) and (2.15) imply that (2.13) holds. □

Lemma 2.8. *Let $(j,k) \in \mathbb{N}^2$ be arbitrary and fixed, one denotes by $\check{Y}_{j,k}$ the right-hand side of (2.1), namely $\check{Y}_{j,k}$ is the $S\alpha S$ random variable defined as*

$$\check{Y}_{j,k} := \mathrm{Re}\Big(\int_{\mathbb{R}} \frac{\widetilde{\psi}_{j,k}(\xi)}{|\xi|^{H+1/\alpha}}\, d\widetilde{M}_\alpha(\xi)\Big). \tag{2.16}$$

Moreover, using the same notation as in (2.8), for each $n \in \mathbb{N}$, the $S\alpha S$ random variable $R^n_{j,k}$ is defined as

$$R^n_{j,k} := Y^n_{j,k} - \check{Y}_{j,k}. \tag{2.17}$$

Then, for any arbitrary fixed real number τ satisfying

$$0 < \tau < \min\{H, 1-H\}, \tag{2.18}$$

there is a finite constant $c > 0$, which does not depend on n and k, such that the scale parameter $\sigma(R^n_{j,k})$ satisfies

$$\sigma(R^n_{j,k}) \le c\, 2^{-n\tau}. \tag{2.19}$$

Remark 2.9. Let $\Phi_{R^n_{j,k}}$ be the characteristic function of the SαS random variable $R^n_{j,k}$ defined in (2.17). Using (2.19) and the equality

$$\Phi_{R^n_{j,k}}(\lambda) = \exp\big(-\sigma(R^n_{j,k})^\alpha |\lambda|^\alpha\big), \quad \text{for all } \lambda \in \mathbb{R},$$

one obtains, for every $\lambda \in \mathbb{R}$, that $\lim_{n\to+\infty} \Phi_{R^n_{j,k}}(\lambda) = 1$. Therefore, when n goes to $+\infty$, the random variable $R^n_{j,k}$ converges in distribution to 0, which also means that it converges in probability to 0. Thus, it results from (2.17) that $Y^n_{j,k}$ converges in probability to $\check{Y}_{j,k}$ when n goes to $+\infty$.

Proof of Lemma 2.8. Let $(j,k) \in \mathbb{N}^2$ an $n \in \mathbb{N}$ be arbitrary and fixed. Putting together (2.17), the second equality in (2.8) and (2.16), it follows that the SαS random variable $R^n_{j,k}$ can be expressed as

$$R^n_{j,k} = \mathrm{Re}\Bigg(\int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \Bigg(\frac{1}{2\pi} \sum_{|m|\le 4^n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{id_{n,m}\xi} - 1\big) \widehat{\widetilde{\psi}}_{j,k}(t)\, dt \\ - \widetilde{\psi}_{j,k}(\xi)\Bigg)\, d\widetilde{M}_\alpha(\xi)\Bigg).$$

Therefore, (2.13) implies that

$$R^n_{j,k} \\ = \mathrm{Re}\Bigg(\int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \Bigg(\frac{1}{2\pi} \sum_{|m|\le 4^n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{id_{n,m}\xi} - 1\big) \widehat{\widetilde{\psi}}_{j,k}(t)\, dt$$

$$- \frac{1}{2\pi} \int_{\mathbb{R}} \big(e^{i\xi t} - 1\big) \widehat{\widehat{\psi}}_{j,k}(t)\, dt \Big)\, d\widetilde{M}_\alpha(\xi) \Bigg)$$

$$= \frac{1}{2\pi} \mathrm{Re} \Bigg(\int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \Big(\sum_{|m| \le 4^n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{id_{n,m}\xi} - e^{it\xi}\big) \widehat{\widehat{\psi}}_{j,k}(t)\, dt - \int_{\{t \notin [-2^n, 2^n + 2^{-n})\}} \big(e^{it\xi} - 1\big) \widehat{\widehat{\psi}}_{j,k}(t)\, dt \Big)\, d\widetilde{M}_\alpha(\xi) \Bigg).$$

Thus, one can derive from Remark 1.1 (i) that

$$\begin{aligned} &\sigma(R_{j,k}^n)^\alpha \\ &= \frac{1}{(2\pi)^\alpha} \int_{\mathbb{R}} |\xi|^{-\alpha H - 1} \Big| \sum_{|m| \le 4^n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{id_{n,m}\xi} - e^{it\xi}\big) \widehat{\widehat{\psi}}_{j,k}(t)\, dt \\ &\qquad - \int_{\{t \notin [-2^n, 2^n + 2^{-n})\}} \big(e^{it\xi} - 1\big) \widehat{\widehat{\psi}}_{j,k}(t)\, dt \Big|^\alpha d\xi. \end{aligned} \tag{2.20}$$

Then, combining (2.20) with the triangle inequality and the inequality $|a+b|^\alpha \le 2^\alpha\big(|a|^\alpha + |b|^\alpha\big)$, for all complex numbers a and b, one gets that

$$\sigma(R_{j,k}^n)^\alpha \le 2^\alpha \big(\mathcal{U}_{j,k}^n + \mathcal{V}_{j,k}^n\big), \tag{2.21}$$

where

$$\mathcal{U}_{j,k}^n := \int_{\mathbb{R}} |\xi|^{-\alpha H - 1} \Big(\sum_{|m| \le 4^n} \int_{d_{n,m}}^{d_{n,m+1}} \big|e^{i(d_{n,m} - t)\xi} - 1\big| \big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\, dt \Big)^\alpha d\xi \tag{2.22}$$

and

$$\mathcal{V}_{j,k}^n := \int_{\mathbb{R}} |\xi|^{-\alpha H - 1} \Big(\int_{\{|t| \ge 2^n\}} \big|e^{it\xi} - 1\big| \big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\, dt \Big)^\alpha d\xi. \tag{2.23}$$

Let $B_0 := [-1, 1]$ and $B_1 := \mathbb{R} \setminus B_0$. For deriving appropriate upper bounds for the integrals $\mathcal{U}_{j,k}^n$ and $\mathcal{V}_{j,k}^n$, we need to split them as

$$\mathcal{U}_{j,k}^n = \mathcal{U}_{j,k}^{n,0} + \mathcal{U}_{j,k}^{n,1} \quad \text{and} \quad \mathcal{V}_{j,k}^n = \mathcal{V}_{j,k}^{n,0} + \mathcal{V}_{j,k}^{n,1}, \tag{2.24}$$

where, for $l \in \{0, 1\}$,

$$\mathcal{U}_{j,k}^{n,l} := \int_{B_l} |\xi|^{-\alpha H - 1} \Big(\sum_{|m| \le 4^n} \int_{d_{n,m}}^{d_{n,m+1}} \big|e^{i(d_{n,m} - t)\xi} - 1\big| \big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\, dt \Big)^\alpha d\xi \tag{2.25}$$

and

$$\mathcal{V}_{j,k}^{n,l} := \int_{B_l} |\xi|^{-\alpha H - 1} \Big(\int_{\{|t| \ge 2^n\}} \big|e^{it\xi} - 1\big| \big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\, dt \Big)^\alpha d\xi. \tag{2.26}$$

Let us now bound $\mathcal{U}_{j,k}^{n,0}$ and $\mathcal{V}_{j,k}^{n,1}$. To this end, we will make use of the classical inequality

$$\left|e^{i\theta}-1\right| \le \min\left\{|\theta|,2\right\}, \quad \text{for all } \theta\in\mathbb{R}. \tag{2.27}$$

Combining (2.25), with $l=0$, and (2.27), one gets that

$$\begin{aligned}\mathcal{U}_{j,k}^{n,0} &\le \int_{B_0}|\xi|^{\alpha(1-H)-1}\bigg(\sum_{|m|\le 4^n}\int_{d_{n,m}}^{d_{n,m+1}}\left|t-d_{n,m}\right|\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\bigg)^{\alpha}d\xi\\ &=\bigg(\int_{B_0}|\xi|^{\alpha(1-H)-1}\,d\xi\bigg)\bigg(\sum_{|m|\le 4^n}\int_{d_{n,m}}^{d_{n,m+1}}\left|t-d_{n,m}\right|\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\bigg)^{\alpha}.\end{aligned}$$

Then, the inequality

$$|t-d_{n,m}|\le 2^{-n}, \quad \text{for all } (n,m)\in\mathbb{N}\times\mathbb{Z} \text{ and } t\in[d_{n,m},d_{n,m+1}], \tag{2.28}$$

and the inequality (2.12) imply that

$$\mathcal{U}_{j,k}^{n,0}\le\bigg(\int_{B_0}|\xi|^{\alpha(1-H)-1}\,d\xi\bigg)\bigg(\int_{\mathbb{R}}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\bigg)^{\alpha}2^{-n\alpha}\le c_1\,2^{-n\alpha}, \tag{2.29}$$

where c_1 is a finite positive constant which does not depend on n and k. On another hand, combining (2.26), with $l=1$, and (2.27), one obtains that

$$\begin{aligned}\mathcal{V}_{j,k}^{n,1} &\le 2^{\alpha}\int_{B_1}|\xi|^{-\alpha H-1}\bigg(\int_{\{|t|\ge 2^n\}}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\bigg)^{\alpha}d\xi\\ &=2^{\alpha}\bigg(\int_{B_1}|\xi|^{-\alpha H-1}\,d\xi\bigg)\bigg(\int_{\{|t|\ge 2^n\}}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\bigg)^{\alpha}.\end{aligned}$$

Then, the inequality (2.12) and standard calculations entail that

$$\mathcal{V}_{j,k}^{n,1}\le c_2\,2^{-n\alpha}, \tag{2.30}$$

where c_2 is a finite positive constant which does not depend on n and k.

In all the sequel, the fixed positive real number $\tau\in(0,1)$ is as in (2.18). Let us now bound $\mathcal{U}_{j,k}^{n,1}$. Using (2.27) and (2.28), one has, for all $\xi\in B_1$ and for every $(n,m)\in\mathbb{N}\times\mathbb{Z}$,

$$\begin{aligned}&\int_{d_{n,m}}^{d_{n,m+1}}\left|e^{i(d_{n,m}-t)\xi}-1\right|\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\\ &=\int_{d_{n,m}}^{d_{n,m+1}}\left|e^{i(d_{n,m}-t)\xi}-1\right|^{1-\tau}\left|e^{i(d_{n,m}-t)\xi}-1\right|^{\tau}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\\ &\le 2^{1-\tau}|\xi|^{\tau}\int_{d_{n,m}}^{d_{n,m+1}}|t-d_{n,m}|^{\tau}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\\ &\le\bigg(2^{1-\tau}|\xi|^{\tau}\int_{d_{n,m}}^{d_{n,m+1}}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\bigg)2^{-n\tau}.\end{aligned}$$

Then, one can derive from (2.25), with $l = 1$, that

$$\mathcal{U}_{j,k}^{n,1} \le \left(2^{\alpha(1-\tau)} \int_{B_1} |\xi|^{-\alpha(H-\tau)-1}\, d\xi\right)\left(\int_{\mathbb{R}} \big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\, dt\right)^{\alpha} 2^{-n\alpha\tau} \le c_3\, 2^{-n\alpha\tau}, \tag{2.31}$$

where c_3 is a finite positive constant which does not depend on n and k, the latter fact follows from (2.12) and the inequality $\tau < H$ (see (2.18)).

Let us now bound $\mathcal{V}_{j,k}^{n,0}$. Using (2.27) and (2.12), one has, for all $\xi \in B_0$ and for every $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\{|t|\ge 2^n\}} \big|e^{it\xi}-1\big|\big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\, dt = \int_{\{|t|\ge 2^n\}} \big|e^{it\xi}-1\big|^{\tau}\big|e^{it\xi}-1\big|^{1-\tau}\big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\, dt \\
&\le 2^{\tau}\,|\xi|^{1-\tau}\int_{\{|t|\ge 2^n\}} |t|^{1-\tau}\big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\, dt \le c_4\, 2^{\tau}\,|\xi|^{1-\tau}\int_{\{|t|\ge 2^n\}} \big(1+|t|\big)^{-1-\tau}\, dt \\
&\le c_5\,|\xi|^{1-\tau}\,2^{-n\tau},
\end{aligned}$$

where c_4 and c_5 are two positive finite constants not depending on n, k and ξ. Then, one can derive from (2.26), with $l = 0$, that

$$\mathcal{V}_{j,k}^{n,0} \le c_6\, 2^{-n\alpha\tau}, \tag{2.32}$$

where c_6 is the positive finite constant, not depending on n and k, defined as $c_6 := c_5^{\alpha}\int_{B_0} |\xi|^{\alpha(1-\tau-H)-1}\, d\xi$. Notice that the finiteness of c_6 results from the inequality $1-\tau > H$ which is a consequence of (2.18).

Finally, putting together (2.21), (2.24), (2.29), (2.30), (2.31) and (2.32), it follows that (2.19) is satisfied. □

We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1. In view of Lemma 2.8 and Remark 2.9, it turns out that for proving Lemma 2.1, it is enough to show that, when n goes to $+\infty$, the SαS random variable $Y_{j,k}^{n}$ (see (2.8)) converges almost surely to the random variable $Y_{j,k}$ (see (2.11)).

One recalls that, for each $(n,m) \in \mathbb{N}\times\mathbb{Z}$, the dyadic number $d_{n,m} := 2^{-n} m$. Let N be an arbitrary fixed positive integer. One sets

$$Y_{j,k}^{N} := \frac{1}{2\pi}\int_{-N}^{N} \widehat{\widehat{\psi}}_{j,k}(t) X(t)\, dt \tag{2.33}$$

and, for every integer $n \ge N$,

$$Y_{j,k}^{N,n} := \frac{1}{2\pi}\sum_{m=-N2^n}^{N2^n-1} X(d_{n,m})\int_{d_{n,m}}^{d_{n,m+1}} \widehat{\widehat{\psi}}_{j,k}(t)\, dt. \tag{2.34}$$

Notice that, it easily results from (2.33) that

$$Y_{j,k}^{N} = \frac{1}{2\pi}\sum_{m=-N2^n}^{N2^n-1}\int_{d_{n,m}}^{d_{n,m+1}} X(t)\widehat{\widehat{\psi}}_{j,k}(t)\, dt\,.$$

Then, using (2.34), (1.4) with $T = N$, and the inclusion $[d_{n,m}, d_{n,m+1}] \subset [-N, N]$ when $-N2^n \le m \le N2^n - 1$, one gets, on an event of probability 1 depending only on N and denoted by Ω_N^*, that

$$\begin{aligned} \left|Y_{j,k}^N - Y_{j,k}^{N,n}\right| &\le \sum_{m=-N2^n}^{N2^n-1} \int_{d_{n,m}}^{d_{n,m+1}} \left|X(t) - X(d_{n,m})\right| \left|\widehat{\widetilde{\psi}}_{j,k}(t)\right| dt \\ &\le C_{N,\delta}^* \sum_{m=-N2^n}^{N2^n-1} \int_{d_{n,m}}^{d_{n,m+1}} \left|t - d_{n,m}\right|^{H-\delta} \left|\widehat{\widetilde{\psi}}_{j,k}(t)\right| dt \\ &\le \left(C_{N,\delta}^* \int_{\mathbb{R}} \left|\widehat{\widetilde{\psi}}_{j,k}(t)\right| dt \right) 2^{-n(H-\delta)}, \end{aligned} \tag{2.35}$$

where δ is an arbitrarily small fixed positive real number, and $C_{N,\delta}^*$ is a positive finite random variable, only depending on N and δ. Also, notice that (2.12) entails that the integral $\int_{\mathbb{R}} \left|\widehat{\widetilde{\psi}}_{j,k}(t)\right| dt$ in (2.35) is finite. Next, let Ω^* be the event of probability 1 defined as the countable intersection $\Omega^* := \bigcap_{N \in \mathbb{N}} \Omega_N^*$. One can derive from (2.35) that

$$\lim_{n \to +\infty} \left|Y_{j,k}^N(\omega) - Y_{j,k}^{N,n}(\omega)\right| = 0, \quad \text{for all } N \in \mathbb{N} \text{ and } \omega \in \Omega^*. \tag{2.36}$$

On another hand, it follows from (1.5) that there are Ω^{**} an event of probability 1 and C_δ^{**} a positive finite random variable only depending on δ, such that

$$\left|X(t, \omega)\right| \le C_\delta^{**}(\omega) |t|^{H+\delta}, \quad \text{for all } t \notin (-1, 1) \text{ and } \omega \in \Omega^{**}. \tag{2.37}$$

Combining the first equality in (2.8) and (2.34) one obtains, for every $\omega \in \Omega^{**}$ and positive integers N and $n \ge N$,

$$\begin{aligned} &\left|Y_{j,k}^n(\omega) - Y_{j,k}^{N,n}(\omega)\right| \\ &\le \sum_{m=N2^n}^{+\infty} \left|X(d_{n,m}, \omega)\right| \int_{d_{n,m}}^{d_{n,m+1}} \left|\widehat{\widetilde{\psi}}_{j,k}(t)\right| dt \\ &\qquad + \sum_{m=-\infty}^{-N2^n-1} \left|X(d_{n,m}, \omega)\right| \int_{d_{n,m}}^{d_{n,m+1}} \left|\widehat{\widetilde{\psi}}_{j,k}(t)\right| dt. \end{aligned}$$

Therefore, (2.37) and (2.12) entail, for all $\omega \in \Omega^{**}$ and positive integers N and $n \ge N$, that

$$\begin{aligned} &\left|Y_{j,k}^n(\omega) - Y_{j,k}^{N,n}(\omega)\right| \\ &\le C_\delta^{**}(\omega) \left(\sum_{m=N2^n}^{+\infty} \left|d_{n,m}\right|^{H+\delta} \int_{d_{n,m}}^{d_{n,m+1}} \left|\widehat{\widetilde{\psi}}_{j,k}(t)\right| dt \right. \\ &\qquad \left. + \sum_{m=-\infty}^{-N2^n-1} \left|d_{n,m}\right|^{H+\delta} \int_{d_{n,m}}^{d_{n,m+1}} \left|\widehat{\widetilde{\psi}}_{j,k}(t)\right| dt \right) \end{aligned}$$

$$\leq C_\delta^{**}(\omega)\left(\sum_{m=N2^n}^{+\infty}\int_{d_{n,m}}^{d_{n,m+1}}|t|^{H+\delta}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\right.$$
$$\left.+\sum_{m=-\infty}^{-N2^n-1}\int_{d_{n,m}}^{d_{n,m+1}}\big(1+|t|\big)^{H+\delta}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|dt\right)$$
$$\leq 2c(j)C_\delta^{**}(\omega)\int_N^{+\infty}\big(1+t\big)^{H+\delta-2}\,dt$$
$$\leq 2(1-H-\delta)^{-1}\,c(j)C_\delta^{**}(\omega)N^{-(1-H-\delta)}, \tag{2.38}$$

where $c(j)$ is the same finite constant as in (2.12). On another hand, putting together (2.11), (2.33), (2.37) and (2.12), it follows that, for all $\omega\in\Omega^{**}$ and positive integers N, one has

$$\begin{aligned}\left|Y_{j,k}(\omega)-Y_{j,k}^N(\omega)\right| &\leq \int_{\{|t|\geq N\}}\left|\widehat{\widehat{\psi}}_{j,k}(t)\right|\left|X(t,\omega)\right|dt\\ &\leq 2c(j)C_\delta^{**}(\omega)\int_N^{+\infty}\big(1+t\big)^{H+\delta-2}\,dt\\ &\leq 2(1-H-\delta)^{-1}\,c(j)C_\delta^{**}(\omega)N^{-(1-H-\delta)}.\end{aligned} \tag{2.39}$$

Finally, observe that, for all positive integers N and $n\geq N$, and for each $\omega\in\Omega^*\cap\Omega^{**}$ (the event $\Omega^*\cap\Omega^{**}$ is clearly of probability 1), using the triangle inequality, (2.38) and (2.39), one gets that

$$\begin{aligned}&\left|Y_{j,k}(\omega)-Y_{j,k}^n(\omega)\right|\\ &\leq \left|Y_{j,k}(\omega)-Y_{j,k}^N(\omega)\right|+\left|Y_{j,k}^N(\omega)-Y_{j,k}^{N,n}(\omega)\right|+\left|Y_{j,k}^{N,n}(\omega)-Y_{j,k}^n(\omega)\right|\\ &\leq 4(1-H-\delta)^{-1}\,c(j)C_\delta^{**}(\omega)N^{-(1-H-\delta)}+\left|Y_{j,k}^N(\omega)-Y_{j,k}^{N,n}(\omega)\right|.\end{aligned}$$

Thus, one can derive from (2.36) that, for all positive integers N and for each $\omega\in\Omega^*\cap\Omega^{**}$,

$$\limsup_{n\to+\infty}\left|Y_{j,k}(\omega)-Y_{j,k}^n(\omega)\right|\leq 4(1-H-\delta)^{-1}\,c(j)C_\delta^{**}(\omega)N^{-(1-H-\delta)}. \tag{2.40}$$

Finally, when N goes to $+\infty$, (2.40) implies that

$$\limsup_{n\to+\infty}\left|Y_{j,k}(\omega)-Y_{j,k}^n(\omega)\right|=0,\quad \text{for all } \omega\in\Omega^*\cap\Omega^{**},$$

which shows that $Y_{j,k}^n$ converges almost surely to $Y_{j,k}$, when n tends to $+\infty$. □

3. Proofs of Theorems 1.5 and 1.7

For proving Theorems 1.5 and 1.7, one needs the following three preliminary results.

Remark 3.1. For any arbitrary $\alpha \in (0,2]$, let W be an arbitrary real-valued SαS random variable with scale parameter equals to 1. Then, for each real-valued SαS random variable Z with scale parameter $\sigma(Z)$, one has $Z \stackrel{d}{=} \sigma(Z) W$ (equality in distribution). This equality is a straightforward consequence of the fact that Φ_Z and Φ_W, the characteristic functions of Z and W, satisfy (see e.g. [21]), for all $\lambda \in \mathbb{R}$, $\Phi_Z(\lambda) = \exp\big(-\sigma(Z)^\alpha |\lambda|^\alpha\big)$ and $\Phi_W(\lambda) = \exp\big(-|\lambda|^\alpha\big)$.

Lemma 3.2. *For all $(j,k) \in \mathbb{N}^2$, the scale parameter of the real-valued SαS random variable $Y_{j,k}$ (see (1.12) and (2.1)) satisfies*

$$\sigma(Y_{j,k}) = 2^{jH+1/\alpha} \left(\int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^\alpha}{(\eta+k)^{\alpha H+1}}\, d\eta \right)^{1/\alpha}, \tag{3.1}$$

which clearly implies (see (1.11)) that

$$\begin{aligned} &\|\psi\|_{L^\alpha(\mathbb{R})} 2^{jH+1/\alpha} \big(k+4^{-1}\big)^{-(H+1/\alpha)} \\ &\le \sigma(Y_{j,k}) \le \|\psi\|_{L^\alpha(\mathbb{R})} 2^{jH+1/\alpha} \big(k-4^{-1}\big)^{-(H+1/\alpha)}, \end{aligned} \tag{3.2}$$

where

$$\big\|\psi\big\|_{L^\alpha(\mathbb{R})} := \left(\int_{\mathbb{R}} \big|\psi(\eta)\big|^\alpha\, d\eta \right)^{1/\alpha} = \left(\int_{-4^{-1}}^{4^{-1}} \big|\psi(\eta)\big|^\alpha\, d\eta \right)^{1/\alpha}.$$

Proof. One knows from (2.1), (2.2), Remark 1.1 (i) and (1.11) that, for all $(j,k) \in \mathbb{N}^2$, one has

$$\sigma(Y_{j,k})^\alpha = \int_{2^{-j}(-k-4^{-1})}^{2^{-j}(-k+4^{-1})} \frac{\big|\psi(2^j\xi+k)\big|^\alpha}{|\xi|^{\alpha H+1}}\, d\xi + \int_{2^{-j}(k-4^{-1})}^{2^{-j}(k+4^{-1})} \frac{\big|\psi(2^j\xi-k)\big|^\alpha}{|\xi|^{\alpha H+1}}\, d\xi.$$

Then, using the fact that ψ is an even function (see the assumption $(\mathcal{A}_1)$ in Section 1) and the change of variable $\eta = 2^j\xi - k$ i.e. $\xi = 2^{-j}(\eta+k)$, one gets that

$$\sigma(Y_{j,k})^\alpha = 2\int_{2^{-j}(k-4^{-1})}^{2^{-j}(k+4^{-1})} \frac{\big|\psi(2^j\xi-k)\big|^\alpha}{(\xi)^{\alpha H+1}}\, d\xi = 2^{j\alpha H+1} \int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^\alpha}{(\eta+k)^{\alpha H+1}}\, d\eta.$$

□

Lemma 3.3. *For any fixed positive real number γ, there exists a finite constant $c = c(\gamma)$, such that, for all $p \in \mathbb{N}$, one has*

$$\left| \left(\int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^\alpha}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\, d\eta \right)^{\gamma/\alpha} - \|\psi\|_{L^\alpha(\mathbb{R})}^\gamma \right| \le c\, p^{-1}. \tag{3.3}$$

Proof. For each $p \in \mathbb{N}$, one sets

$$r_p := \int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^\alpha}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\, d\eta - \int_{-4^{-1}}^{4^{-1}} \big|\psi(\eta)\big|^\alpha\, d\eta\,. \tag{3.4}$$

Using the fact that there exists a positive finite constant c_1 such that one has

$$\left|(1+x)^{-\alpha H-1}-1\right|\le c_1|x|\,,\quad \text{for every } x\in[-5/8,5/8],$$

one gets, for all $p\in\mathbb{N}$ and $\eta\in[-1/4,1/4]$, that

$$\left|\big(1+(2p)^{-1}(\eta-1)\big)^{-\alpha H-1}-1\right|\le c_1(2p)^{-1}|\eta-1|\le c_1p^{-1},$$

and consequently (see (3.4)) that

$$|r_p|\le c_2\,p^{-1}\,,\quad \text{for all } p\in\mathbb{N}, \tag{3.5}$$

where the finite constant $c_2:=c_1\int_{-4^{-1}}^{4^{-1}}\big|\psi(\eta)\big|^{\alpha}\,d\eta = c_1\|\psi\|^{\alpha}_{L^{\alpha}(\mathbb{R})}$. Next, one notices that, since $\|\psi\|_{L^{\alpha}(\mathbb{R})}>0$, there are two positive finite constants y_0 and c_3 such that

$$\left|\big(\|\psi\|^{\alpha}_{L^{\alpha}(\mathbb{R})}+y\big)^{\gamma/\alpha}-\|\psi\|^{\gamma}_{L^{\alpha}(\mathbb{R})}\right|\le c_3|y|\,,\quad \text{for every } y\in[-y_0,y_0]. \tag{3.6}$$

Moreover, one knows from (3.5) that there exists $p_0\in\mathbb{N}$ such that, for all $p\ge p_0$, one has $|r_p|\le y_0$. Thus, one can derive from (3.4), (3.6) with $y=r_p$, and (3.5) that, for all $p\ge p_0$,

$$\begin{aligned}&\left|\left(\int_{-4^{-1}}^{4^{-1}}\frac{\big|\psi(\eta)\big|^{\alpha}}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\,d\eta\right)^{\gamma/\alpha}-\|\psi\|^{\gamma}_{L^{\alpha}(\mathbb{R})}\right|\\&=\left|\big(\|\psi\|^{\alpha}_{L^{\alpha}(\mathbb{R})}+r_p\big)^{\gamma/\alpha}-\|\psi\|^{\gamma}_{L^{\alpha}(\mathbb{R})}\right|\\&\le c_3|r_p|\le c_3c_2\,p^{-1},\end{aligned}$$

which shows that

$$\sup_{p\in\mathbb{N}}\left\{p\times\left|\left(\int_{-4^{-1}}^{4^{-1}}\frac{\big|\psi(\eta)\big|^{\alpha}}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\,d\eta\right)^{\gamma/\alpha}-\|\psi\|^{\gamma}_{L^{\alpha}(\mathbb{R})}\right|\right\}<+\infty,$$

which means that (3.3) is satisfied. □

Proof of Theorem 1.5. One knows from Lemma 3.2 that the scale parameters of the SαS random variables $Y_{j,k}$ are strictly positive. For each $p\in\mathbb{N}$, the two SαS random variables with scale parameters equal to 1, $W_{1,2p-1}$ and $W_{2,4p-1}$, are defined (see Remark 3.1) as

$$W_{1,2p-1}=\frac{Y_{1,2p-1}}{\sigma(Y_{1,2p-1})}\quad\text{and}\quad W_{2,4p-1}=\frac{Y_{2,4p-1}}{\sigma(Y_{2,4p-1})}=\frac{Y_{2,4p-1}}{2^{H}\sigma(Y_{1,4p-1})}. \tag{3.7}$$

Observe that the equality $\sigma(Y_{2,4p-1})=2^{H}\sigma(Y_{1,4p-1})$ results from (3.1). Next, it follows from (1.17) and (3.7) that, for all $n\in\mathbb{N}$, one has

$$\widehat{\alpha}^{-1}_{n,\log_2}-\alpha^{-1}$$

$$= \frac{1}{n}\sum_{p=1}^{n}\Big(\log_2|Y_{1,2p-1}| - \log_2|Y_{2,4p-1}| - \log_2\big(2^{1/\alpha}\big)\Big) \tag{3.8}$$
$$= \frac{1}{n}\sum_{p=1}^{n}\left(\log_2|W_{1,2p-1}| - \log_2|W_{2,4p-1}| + \log_2\left(\frac{\sigma(Y_{1,2p-1})}{2^{H+1/\alpha}\,\sigma(Y_{1,4p-1})}\right)\right).$$

Observe that one knows from the very fundamental Lemma 2.3 and (3.7) that

$$\Big(\log_2|W_{1,2p-1}| - \log_2|W_{2,4p-1}|\Big)_{p\in\mathbb{N}}$$

is a sequence of independent, identically distributed, centered and square integrable random variables. Thus, one can derive from the very classical strong law of large numbers that

$$\frac{1}{n}\sum_{p=1}^{n}\Big(\log_2|W_{1,2p-1}| - \log_2|W_{2,4p-1}|\Big) \xrightarrow[n\to+\infty]{a.s.} 0. \tag{3.9}$$

Moreover, one can derive from the very classical Central Limit Theorem that

$$\left(\frac{1}{2\mathrm{Var}\big(\log_2|W_{(\alpha)}|\big)n}\right)^{1/2}\sum_{p=1}^{n}\Big(\log_2|W_{1,2p-1}| - \log_2|W_{2,4p-1}|\Big) \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1), \tag{3.10}$$

where $W_{(\alpha)}$ is as in Remark 1.4, and where $\xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1)$ denotes the convergence in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution. Thus, in view of (3.8), (3.9) and (3.10), it turns out that for proving the theorem it is enough to show that

$$\lim_{n\to+\infty}\frac{1}{n^{1/2}}\sum_{p=1}^{n}\left|\log_2\left(\frac{\sigma(Y_{1,2p-1})}{2^{H+1/\alpha}\,\sigma(Y_{1,4p-1})}\right)\right| = 0. \tag{3.11}$$

Indeed, it easily follows from (3.8), (3.9) and (3.11) that

$$\widehat{\alpha}_{n,\log_2}^{-1} - \alpha^{-1} \xrightarrow[n\to+\infty]{a.s.} 0, \tag{3.12}$$

which shows that Part (i) of the theorem is satisfied. Moreover, (3.8), (3.10) and (3.11) clearly entail that

$$\left(\frac{n}{2\mathrm{Var}\big(\log_2|W_{(\alpha)}|\big)}\right)^{1/2}\big(\widehat{\alpha}_{n,\log_2}^{-1} - \alpha^{-1}\big) \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1). \tag{3.13}$$

Therefore, combining (3.13) with the fact that

$$G\Big(\max\{\widehat{\alpha}_{n,\log_2}^{-1}, 2^{-1}\}\Big)\Big(2\mathrm{Var}\big(\log_2|W_{(\alpha)}|\big)\Big)^{1/2} \xrightarrow[n\to+\infty]{a.s.} 1, \tag{3.14}$$

one gets Part (*ii*) of the theorem. Notice that (3.14) results from (3.12) and Remark 1.4.

From now on, one focuses on the proof of (3.11). Observe that (3.1) implies, for all $p \in \mathbb{N}$, that

$$
\begin{aligned}
&\frac{\sigma(Y_{1,2p-1})}{2^{H+1/\alpha}\,\sigma(Y_{1,4p-1})} - 1 \\
&= 2^{-(H+1/\alpha)} \left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(4p+\eta-1\right)^{\alpha H+1}}\, d\eta \right)^{-1/\alpha} \\
&\quad \times \left(\left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(2p+\eta-1\right)^{\alpha H+1}}\, d\eta \right)^{1/\alpha} \right. \\
&\qquad \left. - 2^{H+1/\alpha} \left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(4p+\eta-1\right)^{\alpha H+1}}\, d\eta \right)^{1/\alpha} \right) \\
&= (2p)^{H+1/\alpha} \left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(1+(4p)^{-1}(\eta-1)\right)^{\alpha H+1}}\, d\eta \right)^{-1/\alpha} \\
&\quad \times \left((2p)^{-(H+1/\alpha)} \left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(1+(2p)^{-1}(\eta-1)\right)^{\alpha H+1}}\, d\eta \right)^{1/\alpha} \right. \\
&\qquad \left. - (2p)^{-(H+1/\alpha)} \left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(1+(4p)^{-1}(\eta-1)\right)^{\alpha H+1}}\, d\eta \right)^{1/\alpha} \right) \\
&= \left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(1+(4p)^{-1}(\eta-1)\right)^{\alpha H+1}}\, d\eta \right)^{-1/\alpha} \\
&\quad \times \left(\left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(1+(2p)^{-1}(\eta-1)\right)^{\alpha H+1}}\, d\eta \right)^{1/\alpha} \right. \\
&\qquad \left. - \left(\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(1+(4p)^{-1}(\eta-1)\right)^{\alpha H+1}}\, d\eta \right)^{1/\alpha} \right).
\end{aligned}
\tag{3.15}
$$

On another hand, for all $p \in \mathbb{N}$, one has

$$
\begin{aligned}
\int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(1+(4p)^{-1}(\eta-1)\right)^{\alpha H+1}}\, d\eta &\geq \int_{-4^{-1}}^{4^{-1}} \frac{\left|\psi(\eta)\right|^{\alpha}}{\left(1+4^{-1}|\eta-1|\right)^{\alpha H+1}}\, d\eta \\
&\geq \left(\frac{16}{21}\right)^{\alpha H+1} \|\psi\|_{L^{\alpha}(\mathbb{R})}^{\alpha}.
\end{aligned}
$$

Thus, one can derives from (3.15) and the triangle inequality that there exists a positive finite constant c_1 such that, for all $p \in \mathbb{N}$,

$$\begin{aligned} &\left| \frac{\sigma(Y_{1,2p-1})}{2^{H+1/\alpha}\,\sigma(Y_{1,4p-1})} - 1 \right| \qquad (3.16) \\ &\le c_1 \left| \left(\int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^{\alpha}}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\, d\eta \right)^{1/\alpha} - \|\psi\|_{L^{\alpha}(\mathbb{R})} \right| \\ &\qquad + c_1 \left| \left(\int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^{\alpha}}{\big(1+(4p)^{-1}(\eta-1)\big)^{\alpha H+1}}\, d\eta \right)^{1/\alpha} - \|\psi\|_{L^{\alpha}(\mathbb{R})} \right|. \end{aligned}$$

Then, combining (3.16) and (3.3) (with $\gamma = 1$), one gets, for some positive finite constant c_2 that

$$\left| \frac{\sigma(Y_{1,2p-1})}{2^{H+1/\alpha}\,\sigma(Y_{1,4p-1})} - 1 \right| \le c_2\, p^{-1}, \quad \text{for all } p \in \mathbb{N},$$

which in turn implies that there exists a positive finite constant c_3 such that

$$\left| \log_2 \left(\frac{\sigma(Y_{1,2p-1})}{2^{H+1/\alpha}\,\sigma(Y_{1,4p-1})} \right) \right| \le c_3\, p^{-1}, \quad \text{for every } p \in \mathbb{N}.$$

Therefore, one obtains, for all $n \in \mathbb{N}$, that

$$\begin{aligned} \sum_{p=1}^{n} \left| \log_2 \left(\frac{\sigma(Y_{1,2p-1})}{2^{H+1/\alpha}\,\sigma(Y_{1,4p-1})} \right) \right| &\le c_3 \sum_{p=1}^{n} p^{-1} \qquad (3.17) \\ &\le c_3 \left(1 + \int_1^n x^{-1}\, dx \right) \le c_3 \big(1 + \log(n) \big). \end{aligned}$$

Then, one can derive from (3.17) that (3.11) holds. □

Proof of Theorem 1.7. One knows from Lemma 3.2 that the scale parameters of the SαS random variables $Y_{j,k}$ are strictly positive. For each $p \in \mathbb{N}$, the two SαS random variables with scale parameters equal to 1, $W_{1,2p-1}$ and $W_{2,2p-1}$ are defined (see Remark 3.1) as

$$W_{1,2p-1} = \frac{Y_{1,2p-1}}{\sigma(Y_{1,2p-1})} \quad \text{and} \quad W_{2,2p-1} = \frac{Y_{2,2p-1}}{\sigma(Y_{2,2p-1})} = \frac{Y_{2,2p-1}}{2^H \sigma(Y_{1,2p-1})}. \qquad (3.18)$$

Observe that the equality $\sigma(Y_{2,2p-1}) = 2^H \sigma(Y_{1,2p-1})$ follows from (3.1). Next, using (1.21), (1.19) and (3.18) one has, for all $n \in \mathbb{N}$, that

$$\begin{aligned} &\widehat{H}_{n,\log_2} - H \\ &= \frac{1}{n} \sum_{p=1}^{n} \Big(\log_2 |Y_{2,2p-1}| - \log_2 |Y_{1,2p-1}| - \log_2 \big(2^H\big) \Big) \\ &= \frac{1}{n} \sum_{p=1}^{n} \Big(\log_2 |W_{2,2p-1}| - \log_2 |W_{1,2p-1}| + \log_2 \big(\sigma(Y_{2,2p-1})\big) \Big) \end{aligned}$$

$$-\log_2\big(2^H\sigma(Y_{1,2p-1})\big)\Big)$$

$$= \frac{1}{n}\sum_{p=1}^{n}\Big(\log_2|W_{2,2p-1}| - \log_2|W_{1,2p-1}|\Big). \tag{3.19}$$

Next, observe that one knows from the very fundamental Lemma 2.3 and (3.18) that

$$\Big(\log_2|W_{2,2p-1}| - \log_2|W_{1,2p-1}|\Big)_{p\in\mathbb{N}}$$

is a sequence of independent, identically distributed, centered and square integrable random variables. Thus using (3.19), the very classical Strong Law of Large Numbers, and the very classical Central Limit Theorem, one obtains Part (i) of the theorem, and that

$$\left(\frac{n}{2\mathrm{Var}\big(\log_2|W_{(\alpha)}|\big)}\right)^{1/2}\big(\widehat{H}_{n,\log_2} - H\big) \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1), \tag{3.20}$$

where $W_{(\alpha)}$ is as in Remark 1.4. Finally combining (3.20) with (3.14), one gets Part (ii) of the theorem. □

4. Proof of Theorem 1.8

The two main ingredients of the proof of Theorem 1.8 are the following two lemmas.

Lemma 4.1. *Under the sole condition (1.23) on the sequence $(m_j)_{j\in\mathbb{N}}$, one has, for all fixed $\gamma \in (0, 4^{-1}\underline{\alpha})$,*

$$\lim_{j\to+\infty}\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} = \lim_{j\to+\infty}\frac{V_{2,\gamma}^{m_j}}{\mathbb{E}(V_{2,\gamma}^{m_j})} = 1, \tag{4.1}$$

where the convergences hold almost surely.

Lemma 4.2. *Let $\underline{\alpha} \in (0, 2]$ be as in Theorem 1.8 and let γ be arbitrary and such that*

$$0 < \gamma < \frac{\underline{\alpha}}{\underline{\alpha}+1}\,, \tag{4.2}$$

which clearly implies that

$$\gamma(H + 1/\alpha) < 1, \quad \text{for all } (H,\alpha) \in [0,1]\times[\underline{\alpha}, 2]. \tag{4.3}$$

When the stability parameter α of the HFSM belongs to $[\underline{\alpha}, 2]$, there exists a finite constant c such that, for all $(j, m, n) \in \mathbb{N}^3$, one has

$$\big|\Delta_j^{m,n}(\gamma, H, \alpha)\big| \le c\Big(m^{\gamma(H+1/\alpha)-1} + n^{\gamma(H+1/\alpha)-1}\Big), \tag{4.4}$$

where

$$\Delta_j^{m,n}(\gamma, H, \alpha) := \log_2\left(\frac{\mathbb{E}\big(V_{j+1,\gamma}^{n}\big)}{\mathbb{E}\big(V_{j,\gamma}^{m}\big)}\right) - \gamma H - \big(1-\gamma(H+1/\alpha)\big)\log_2\left(\frac{n}{m}\right). \tag{4.5}$$

For proving Lemmas 4.1 and 4.2 one needs two preliminary results. The following remark is a straightforward consequence of Remark 3.1 and of Lemma 3.2.

Remark 4.3. Let $\gamma \in (0,\alpha)$ be arbitrary. For all $(j,p) \in \mathbb{N}^2$, one has

$$\begin{aligned} &\mathbb{E}\big(|Y_{j,2p-1}|^\gamma\big) \\ &= \mathbb{E}\big(|W_{(\alpha)}|^\gamma\big) 2^{\gamma(jH+1/\alpha)} \bigg(\int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^\alpha}{(\eta+2p-1)^{\alpha H+1}}\, d\eta \bigg)^{\gamma/\alpha} \\ &= \mathbb{E}\big(|W_{(\alpha)}|^\gamma\big) 2^{(j-1)\gamma H} p^{-\gamma(H+1/\alpha)} \bigg(\int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^\alpha}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\, d\eta \bigg)^{\gamma/\alpha}, \end{aligned} \tag{4.6}$$

where $W_{(\alpha)}$ denotes an arbitrary real-valued SαS random variable with scale parameter equals to 1.

Lemma 4.4. *Let γ be such that (4.2) holds. When the stability parameter α of the HFSM belongs to $[\underline{\alpha}, 2]$, there exists a finite constant c such that one has, for all $(j,m) \in \mathbb{N}^2$,*

$$\Big| 2^{-(j-1)\gamma H} m^{\gamma(H+1/\alpha)-1}\, \mathbb{E}(V_{j,\gamma}^m) - A(\gamma,H,\alpha) \Big| \le c\, m^{\gamma(H+1/\alpha)-1}, \tag{4.7}$$

where the finite positive constant

$$A(\gamma,H,\alpha) := \mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\big(1-\gamma(H+1/\alpha)\big)^{-1} \|\psi\|_{L^\alpha(\mathbb{R})}^\gamma. \tag{4.8}$$

As usual, $W_{(\alpha)}$ denotes an arbitrary real-valued SαS random variable with scale parameter equals to 1.

Proof. It follows from (1.20), (4.6) and the triangle inequality that, for all $(j,m) \in \mathbb{N}^2$, one has

$$\begin{aligned} &2^{-(j-1)\gamma H} m^{\gamma(H+1/\alpha)-1} \Big| \mathbb{E}(V_{j,\gamma}^m) - \mathbb{E}\big(|W_{(\alpha)}|^\gamma\big) \|\psi\|_{L^\alpha(\mathbb{R})}^\gamma 2^{(j-1)\gamma H} \sum_{p=1}^m p^{-\gamma(H+1/\alpha)} \Big| \\ &\le \mathbb{E}\big(|W_{(\alpha)}|^\gamma\big) m^{\gamma(H+1/\alpha)-1} \\ &\qquad \times \sum_{p=1}^m p^{-\gamma(H+1/\alpha)} \bigg| \bigg(\int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^\alpha}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\, d\eta \bigg)^{\gamma/\alpha} - \|\psi\|_{L^\alpha(\mathbb{R})}^\gamma \bigg|. \end{aligned} \tag{4.9}$$

Moreover, (3.3) entails, for every $m \in \mathbb{N}$, that

$$\begin{aligned} &\sum_{p=1}^m p^{-\gamma(H+1/\alpha)} \bigg| \bigg(\int_{-4^{-1}}^{4^{-1}} \frac{\big|\psi(\eta)\big|^\alpha}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\, d\eta \bigg)^{\gamma/\alpha} - \|\psi\|_{L^\alpha(\mathbb{R})}^\gamma \bigg| \\ &\le c_0 \sum_{p=1}^m p^{-\gamma(H+1/\alpha)-1} < c_0 \sum_{p=1}^{+\infty} p^{-\gamma(H+1/\alpha)-1} < +\infty, \end{aligned} \tag{4.10}$$

where c_0 denotes the finite constant c in (3.3). Then, one can derive from (4.9) and (4.10) that there is a finite constant c_1 such that, for all $(j,m)\in\mathbb{N}^2$,

$$2^{-(j-1)\gamma H} m^{\gamma(H+1/\alpha)-1}\left|\mathbb{E}(V_{j,\gamma}^m)-\mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\|\psi\|_{L^\alpha(\mathbb{R})}^\gamma 2^{(j-1)\gamma H}\sum_{p=1}^m p^{-\gamma(H+1/\alpha)}\right| \\ \le c_1\, m^{\gamma(H+1/\alpha)-1}\,. \tag{4.11}$$

Next, observe that, for every $m\in\mathbb{N}$, one has that

$$\left|m^{\gamma(H+1/\alpha)-1}\sum_{p=1}^m p^{-\gamma(H+1/\alpha)}-\int_{1/m}^{(m+1)/m} x^{-\gamma(H+1/\alpha)}\,dx\right| \\ \le \sum_{p=1}^m\int_{p/m}^{(p+1)/m}\left|\big(p/m\big)^{-\gamma(H+1/\alpha)}-x^{-\gamma(H+1/\alpha)}\right|dx\,.$$

Moreover, the mean value theorem allows to show, for all $p\in\{1,\dots,m\}$ and $x\in\big[p/m,(p+1)/m\big]$, that

$$\left|\big(p/m\big)^{-\gamma(H+1/\alpha)}-x^{-\gamma(H+1/\alpha)}\right|\le m^{-1}\big(p/m\big)^{-\gamma(H+1/\alpha)-1}\,.$$

Then, using the previous two inequalities, one gets, for every $m\in\mathbb{N}$, that

$$\left|m^{\gamma(H+1/\alpha)-1}\sum_{p=1}^m p^{-\gamma(H+1/\alpha)}-\int_{1/m}^{(m+1)/m} x^{-\gamma(H+1/\alpha)}\,dx\right| \\ \le m^{-2}\sum_{p=1}^m\big(p/m\big)^{-\gamma(H+1/\alpha)-1}\le c_2\, m^{\gamma(H+1/\alpha)-1}\,, \tag{4.12}$$

where the finite constant $c_2 := \sum_{p=1}^{+\infty} p^{-\gamma(H+1/\alpha)-1}$. Also, notice that, in view of (4.3), one has, for all $m\in\mathbb{N}$, that

$$\Big|\int_{1/m}^{(m+1)/m} x^{-\gamma(H+1/\alpha)}\,dx-\big(1-\gamma(H+1/\alpha)\big)^{-1}\Big| \\ =\big(1-\gamma(H+1/\alpha)\big)^{-1}\left|\big(1+m^{-1}\big)^{1-\gamma(H+1/\alpha)}-1-m^{\gamma(H+1/\alpha)-1}\right| \\ \le 2\big(1-\gamma(H+1/\alpha)\big)^{-1} m^{\gamma(H+1/\alpha)-1}\,. \tag{4.13}$$

Next, observe that, one can derive from the triangle inequality that, for all $(j,m)\in\mathbb{N}^2$,

$$\left|2^{-(j-1)\gamma H} m^{\gamma(H+1/\alpha)-1}\,\mathbb{E}(V_{j,\gamma}^m)-\mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\big(1-\gamma(H+1/\alpha)\big)^{-1}\|\psi\|_{L^\alpha(\mathbb{R})}^\gamma\right| \\ \le 2^{-(j-1)\gamma H} m^{\gamma(H+1/\alpha)-1}\Big|\mathbb{E}(V_{j,\gamma}^m)$$

$$
\begin{aligned}
&\qquad - \mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\|\psi\|_{L^\alpha(\mathbb{R})}^\gamma 2^{(j-1)\gamma H}\sum_{p=1}^m p^{-\gamma(H+1/\alpha)}\Bigg| \\
&+ \mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\|\psi\|_{L^\alpha(\mathbb{R})}^\gamma \Bigg| m^{\gamma(H+1/\alpha)-1}\sum_{p=1}^m p^{-\gamma(H+1/\alpha)} \\
&\qquad\qquad - \int_{1/m}^{(m+1)/m} x^{-\gamma(H+1/\alpha)}\,dx\Bigg| \\
&+ \mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\|\psi\|_{L^\alpha(\mathbb{R})}^\gamma \Bigg|\int_{1/m}^{(m+1)/m} x^{-\gamma(H+1/\alpha)}\,dx - \big(1-\gamma(H+1/\alpha)\big)^{-1}\Bigg|.
\end{aligned}
\tag{4.14}
$$

Finally, putting together (4.11), (4.12), (4.13) and (4.14), one gets (4.7). □

Proof of Lemma 4.2. Observe that, one knows from Part (*iii*) of Lemma 2.3 that, for all $(j,m,n)\in\mathbb{N}^3$, one has

$$
\log_2\left(\frac{\mathbb{E}\big(V_{j+1,\gamma}^n\big)}{\mathbb{E}\big(V_{j,\gamma}^m\big)}\right) = \log_2\left(\frac{2^{j\gamma H}\mathbb{E}\big(V_{1,\gamma}^n\big)}{2^{(j-1)\gamma H}\mathbb{E}\big(V_{1,\gamma}^m\big)}\right) = \log_2\left(\frac{\mathbb{E}\big(V_{1,\gamma}^n\big)}{\mathbb{E}\big(V_{1,\gamma}^m\big)}\right) + \gamma H.
$$

Thus, letting $A^{-1}(\gamma,H,\alpha)$ be the inverse of the positive constant $A(\gamma,H,\alpha)$ (see (4.8)), and using (4.5) and standard calculations, one obtains, for every $(j,m,n)\in\mathbb{N}^3$, that

$$
\Delta_j^{m,n}(\gamma,H,\alpha) = \log_2\left(\frac{A^{-1}(\gamma,H,\alpha)n^{\gamma(H+1/\alpha)-1}\,\mathbb{E}\big(V_{1,\gamma}^n\big)}{A^{-1}(\gamma,H,\alpha)m^{\gamma(H+1/\alpha)-1}\,\mathbb{E}\big(V_{1,\gamma}^m\big)}\right),
$$

which implies that

$$
\begin{aligned}
\big|\Delta_j^{m,n}(\gamma,H,\alpha)\big| \le {}& \left|\log_2\left(A^{-1}(\gamma,H,\alpha)n^{\gamma(H+1/\alpha)-1}\,\mathbb{E}\big(V_{1,\gamma}^n\big)\right)\right| \\
&+ \left|\log_2\left(A^{-1}(\gamma,H,\alpha)m^{\gamma(H+1/\alpha)-1}\,\mathbb{E}\big(V_{1,\gamma}^m\big)\right)\right|.
\end{aligned}
\tag{4.15}
$$

Next observe that one knows from (4.7) and (4.3) that

$$
\lim_{m\to+\infty}\left|A^{-1}(\gamma,H,\alpha)m^{\gamma(H+1/\alpha)-1}\,\mathbb{E}\big(V_{1,\gamma}^m\big)-1\right| = 0. \tag{4.16}
$$

Moreover, it can easily be seen that one has, for some finite constant c_1,

$$
\big|\log_2(1+x)\big| \le c_1|x|, \quad \text{for every } x\in[-2^{-1},+\infty). \tag{4.17}
$$

Thus, one can derive from (4.16), (4.17) and (4.7) that there exists $m_0\in\mathbb{N}$ and a finite constant $c_2>0$, such that for all $m\ge m_0$,

$$
\left|\log_2\left(A^{-1}(\gamma,H,\alpha)m^{\gamma(H+1/\alpha)-1}\,\mathbb{E}\big(V_{1,\gamma}^m\big)\right)\right|
$$

$$\leq c_1 \left| A^{-1}(\gamma, H, \alpha) m^{\gamma(H+1/\alpha)-1}\, \mathbb{E}\big(V_{1,\gamma}^m\big) - 1 \right|$$
$$\leq c_2 m^{\gamma(H+1/\alpha)-1},$$

which entails that

$$\sup_{m\in\mathbb{N}} m^{1-\gamma(H+1/\alpha)} \left|\log_2 \Big(A^{-1}(\gamma, H, \alpha) m^{\gamma(H+1/\alpha)-1}\, \mathbb{E}\big(V_{1,\gamma}^m\big)\Big)\right| < +\infty. \tag{4.18}$$

Finally, it follows from (4.15) and (4.18) that (4.4) is satisfied. □

Proof of Lemma 4.1. One will only show that

$$\lim_{j\to+\infty} \frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} = 1 \quad \text{(almost surely)}, \tag{4.19}$$

since the proof of the fact that

$$\lim_{j\to+\infty} \frac{V_{2,\gamma}^{m_j}}{\mathbb{E}(V_{2,\gamma}^{m_j})} = 1 \quad \text{(almost surely)},$$

can be done in the same way. First notice that using Markov inequality, for each $j \in \mathbb{N}$, one has

$$\mathbb{P}\left(\left| \frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} - 1 \right| \geq m_j^{-\rho} \right) = \mathbb{P}\left(\left| V_{1,\gamma}^{m_j} - \mathbb{E}(V_{1,\gamma}^{m_j}) \right| \geq m_j^{-\rho}\, \mathbb{E}(V_{1,\gamma}^{m_j}) \right)$$
$$\leq m_j^{4\rho} \times \frac{\mathbb{E}\Big(\big| V_{1,\gamma}^{m_j} - \mathbb{E}(V_{1,\gamma}^{m_j}) \big|^4 \Big)}{\big(\mathbb{E}(V_{1,\gamma}^{m_j})\big)^4}, \tag{4.20}$$

where ρ is a fixed positive constant small enough, which will be chosen more precisely later.

Let us now provide an appropriate upper bound for the expectation

$$\mathbb{E}\Big(\big| V_{1,\gamma}^{m_j} - \mathbb{E}(V_{1,\gamma}^{m_j}) \big|^4 \Big),$$

which is finite because of the assumption $\gamma \in (0, 4^{-1}\underline{\alpha})$. One can derive from (1.20) and the fact that, for any fixed $j \in \mathbb{N}$, the centered random variables $|Y_{j,2p-1}|^\gamma - \mathbb{E}(|Y_{j,2p-1}|^\gamma)$, $p \in \mathbb{N}$, are independent (see Part (i) of the very fundamental Lemma 2.3) that

$$\mathbb{E}\Big(\big| V_{1,\gamma}^{m_j} - \mathbb{E}(V_{1,\gamma}^{m_j}) \big|^4 \Big) = \sum_{p_1,\ldots,p_4=1}^{m_j} \mathbb{E}\left(\prod_{l=1}^{4} \Big(|Y_{1,2p_l-1}|^\gamma - \mathbb{E}(|Y_{1,2p_l-1}|^\gamma) \Big) \right)$$
$$\leq \left(\sum_{p=1}^{m_j} \mathbb{E}\Big(\big| |Y_{1,2p-1}|^\gamma - \mathbb{E}(|Y_{1,2p-1}|^\gamma) \big|^4 \Big) \right) \tag{4.21}$$
$$+3 \left(\sum_{p=1}^{m_j} \mathbb{E}\Big(\big| |Y_{1,2p-1}|^\gamma - \mathbb{E}(|Y_{1,2p-1}|^\gamma) \big|^2 \Big) \right)^2.$$

Moreover, one knows from Remark 3.1 that, for each $(j,p)\in\mathbb{N}^2$ and for all $q\in\{1,2\}$, one has

$$\mathbb{E}\Big(\big||Y_{j,2p-1}|^\gamma-\mathbb{E}(|Y_{j,2p-1}|^\gamma)\big|^{2q}\Big)=c_q\,\sigma(Y_{j,2p-1})^{2q\gamma}\,, \tag{4.22}$$

where the positive finite constant $c_q:=\mathbb{E}\Big(\big||W|^\gamma-\mathbb{E}(|W|^\gamma)\big|^{2q}\Big)$ does not depend on (j,p). Then, it follows from (4.21), (4.22) and the second inequality in (3.2) that

$$\begin{aligned}
&\mathbb{E}\Big(\big|V_{1,\gamma}^{m_j}-\mathbb{E}(V_{1,\gamma}^{m_j})\big|^4\Big)\\
&\le c_2\left(\sum_{p=1}^{m_j}\sigma(Y_{1,2p-1})^{4\gamma}\right)+3c_1^2\left(\sum_{p=1}^{m_j}\sigma(Y_{1,2p-1})^{2\gamma}\right)^2\\
&\le c_3\left(\sum_{p=1}^{m_j}\sigma(Y_{1,2p-1})^{2\gamma}\right)^2\le c_4\left(\sum_{p=1}^{m_j}\big(2p-5/4\big)^{-2\gamma(H+1/\alpha)}\right)^2\\
&\le c_4\left((4/3)^{2\gamma(H+1/\alpha)}+\int_1^{m_j}\big(2x-5/4\big)^{-2\gamma(H+1/\alpha)}\,dx\right)^2\\
&\le c_5\left(1+\mathbb{1}_{\{1\}}\big(2\gamma(H+1/\alpha)\big)\log\big(m_j\big)+m_j^{1-2\gamma(H+1/\alpha)}\right)^2,
\end{aligned}\tag{4.23}$$

where c_3, c_4 and c_5 are three positive finite constants not depending on j. On another hand, observe that, in view of the fact that $\gamma\in(0,4^{-1}\underline{\alpha})\subseteq(0,4^{-1}\alpha)$, one knows from (4.7) and (1.23) that

$$\lim_{j\to+\infty}m_j^{4\gamma(H+1/\alpha)-4}\left(\mathbb{E}(V_{1,\gamma}^{m_j})\right)^4=A^4(\gamma,H,\alpha)>0,$$

which implies that there exists a positive finite constant c_6 such that, for all $j\ge 1$,

$$\left(\mathbb{E}(V_{1,\gamma}^{m_j})\right)^{-4}\le c_6\,m_j^{4\gamma(H+1/\alpha)-4}. \tag{4.24}$$

Next, combining (4.23) with (4.24), one gets, for some finite constant c_7 and for all $j\in\mathbb{N}$, that

$$\begin{aligned}
&m_j^{4\rho}\times\frac{\mathbb{E}\Big(\big|V_{1,\gamma}^{m_j}-\mathbb{E}(V_{1,\gamma}^{m_j})\big|^4\Big)}{\left(\mathbb{E}(V_{1,\gamma}^{m_j})\right)^4}\\
&\le c_7\,m_j^{-4(1-\rho-\gamma(H+1/\alpha))}\left(1+\mathbb{1}_{\{1\}}\big(2\gamma(H+1/\alpha)\big)\log\big(m_j\big)+m_j^{1-2\gamma(H+1/\alpha)}\right)^2\\
&\le c_7\Bigg(m_j^{-2(1-\rho-\gamma(H+1/\alpha))}
\end{aligned}$$

$$+ \mathbb{1}_{\{1\}}\big(2\gamma(H+1/\alpha)\big) m_j^{-2(1-\rho-\gamma(H+1/\alpha))} \log\big(m_j\big) + m_j^{-(1-2\rho)}\Bigg)^2$$

$$\le 3c_7\Bigg(m_j^{-4(1-\rho-\gamma(H+1/\alpha))} + \mathbb{1}_{\{1\}}\big(2\gamma(H+1/\alpha)\big) m_j^{-4(1-\rho-\gamma(H+1/\alpha))} \log^2\big(m_j\big) + m_j^{-2(1-2\rho)}\Bigg). \tag{4.25}$$

Next, notice that, since $4\big(1-\gamma(H+1/\alpha)\big) > 4\big(1-4^{-1}\alpha(H+1/\alpha)\big) > 1$, the positive constant ρ can be chosen small enough so that one has

$$4\big(1-\rho-\gamma(H+1/\alpha)\big) > 1 \quad \text{and} \quad 2(1-2\rho) > 1\,. \tag{4.26}$$

Then, it follows from (4.25), (4.26), (1.23) and (4.20) that

$$\sum_{j=1}^{+\infty} \mathbb{P}\Bigg(\Big| \frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} - 1 \Big| \ge m_j^{-\rho} \Bigg) < +\infty.$$

Therefore, (4.19) results from Borel-Cantelli Lemma. □

We are now in position to complete the proof of Theorem 1.8.

End of the proof of Theorem 1.8. In view of the two equalities

$$\frac{V_{2,\gamma}^{m_j}}{V_{1,\gamma}^{m_j}} = \frac{V_{2,\gamma}^{m_j}}{\mathbb{E}(V_{2,\gamma}^{m_j})} \cdot \frac{\mathbb{E}(V_{2,\gamma}^{m_j})}{\mathbb{E}(V_{1,\gamma}^{m_j})} \cdot \frac{\mathbb{E}(V_{1,\gamma}^{m_j})}{V_{1,\gamma}^{m_j}}$$

and

$$\frac{V_{2,\gamma}^{m_{j+1}}}{V_{1,\gamma}^{m_j}} = \frac{V_{2,\gamma}^{m_{j+1}}}{\mathbb{E}(V_{2,\gamma}^{m_{j+1}})} \cdot \frac{\mathbb{E}(V_{2,\gamma}^{m_{j+1}})}{\mathbb{E}(V_{1,\gamma}^{m_j})} \cdot \frac{\mathbb{E}(V_{1,\gamma}^{m_j})}{V_{1,\gamma}^{m_j}}\,,$$

it results from (1.24), (1.25), (4.5) and standard calculations that, for all $j \in \mathbb{N}$, one has

$$\gamma \widehat{H}_{j,\gamma} - \gamma H = \log_2\left(\frac{V_{2,\gamma}^{m_j}}{\mathbb{E}(V_{2,\gamma}^{m_j})}\right) - \log_2\left(\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})}\right) + \Delta_1^{m_j,m_j}(\gamma,H,\alpha) \tag{4.27}$$

and

$$\begin{aligned} &\gamma\, \widehat{\alpha}_{j,\gamma}^{-1} - \gamma\alpha^{-1} \\ &= \log_2\left(\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})}\right) - \log_2\left(\frac{V_{2,\gamma}^{m_{j+1}}}{\mathbb{E}(V_{2,\gamma}^{m_{j+1}})}\right) \\ &\quad -\Delta_1^{m_j,m_{j+1}}(\gamma,H,\alpha) + \big(1-\gamma(H+1/\alpha)\big)\left(1 - \log_2\left(\frac{m_{j+1}}{m_j}\right)\right). \end{aligned} \tag{4.28}$$

Moreover, Lemma 4.2, the inequality $\gamma(H+1/\alpha)-1<-1/4$ and (1.23) entail that

$$\lim_{j\to+\infty} \Delta_1^{m_j,m_j}(\gamma,H,\alpha)=0 \tag{4.29}$$

and

$$\lim_{j\to+\infty} \Delta_1^{m_j,m_{j+1}}(\gamma,H,\alpha)=0\,. \tag{4.30}$$

Then, one can derive from (4.27), Lemma 4.1 and (4.29) that

$$\widehat{H}_{j,\gamma}-H \xrightarrow[j\to+\infty]{a.s.} 0.$$

Moreover, it follows from (4.28), Lemma 4.1, (4.30) and (1.26) that

$$\widehat{\alpha}_{j,\gamma}^{-1}-\alpha^{-1} \xrightarrow[j\to+\infty]{a.s.} 0. \qquad \square$$

5. Proof of Theorem 1.10

For proving Theorem 1.10 one needs several preliminary results.

Proposition 5.1. *Let γ be arbitrary and such that (1.27) holds. For every $(j,m)\in\mathbb{N}^2$, let $V_{j,\gamma}^m$ be as in Definition 1.6. The random variable $R_{j,\gamma}^m$ is defined as*

$$R_{j,\gamma}^m := \frac{V_{j,\gamma}^m-\mathbb{E}(V_{j,\gamma}^m)}{\big(\mathrm{Var}(V_{j,\gamma}^m)\big)^{\frac12}} = \left(\frac{\mathbb{E}(V_{j,\gamma}^m)}{\big(\mathrm{Var}(V_{j,\gamma}^m)\big)^{\frac12}}\right)\left(\frac{V_{j,\gamma}^m}{\mathbb{E}(V_{j,\gamma}^m)}-1\right). \tag{5.1}$$

Let $(m_j)_{j\in\mathbb{N}}$ be an arbitrary non-decreasing sequence of integers larger than 2 which satisfies the condition (1.23). When j goes to $+\infty$, the random variables $R_{1,\gamma}^{m_j}$ and $R_{2,\gamma}^{m_j}$ converge in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.

Proof. First notice that it follows from Lemma 2.3 (iii) and (5.1) that, for every $j\in\mathbb{N}$, $R_{1,\gamma}^{m_j} \overset{d}{=} R_{2,\gamma}^{m_j}$. Thus, we give the proof only in the case of $R_{1,\gamma}^{m_j}$. In view of (5.1), (1.20) and of the fact that the random variables $\big|Y_{1,2p-1}\big|^\gamma$, $p\in\mathbb{N}$, are independent, one knows from the Lyapunov Central Limit Theorem (see for instance Theorem 7.3 on page 44 in [8]) that it is enough to show that, for some fixed $\delta>0$ small enough so that $\gamma(2+\delta)<\underline{\alpha}$, one has

$$\lim_{j\to+\infty} \big(\mathrm{Var}(V_{1,\gamma}^{m_j})\big)^{-\frac{2+\delta}{2}} \sum_{p=1}^{m_j} \mathbb{E}\left(\Big|\big|Y_{1,2p-1}\big|^\gamma-\mathbb{E}\big(\big|Y_{1,2p-1}\big|^\gamma\big)\Big|^{2+\delta}\right)=0\,. \tag{5.2}$$

The independence property of the random variables $\big|Y_{1,2p-1}\big|^\gamma$, $p\in\{1,\dots,m_j\}$, implies that

$$\mathrm{Var}(V_{1,\gamma}^{m_j}) = \sum_{p=1}^{m_j} \mathrm{Var}\big(\big|Y_{1,2p-1}\big|^\gamma\big). \tag{5.3}$$

Then, one can derive from (5.3) and Remark 3.1 that

$$\mathrm{Var}(V_{1,\gamma}^{m_j}) = \mathrm{Var}\big(\big|W\big|^{\gamma}\big)\sum_{p=1}^{m_j}\sigma(Y_{1,2p-1})^{2\gamma}. \tag{5.4}$$

Also, Remark 3.1 entails that

$$\begin{aligned}&\sum_{p=1}^{m_j}\mathbb{E}\bigg(\Big|\big|Y_{1,2p-1}\big|^{\gamma}-\mathbb{E}\big(\big|Y_{1,2p-1}\big|^{\gamma}\big)\Big|^{2+\delta}\bigg)\\ &=\mathbb{E}\bigg(\Big|\big|W\big|^{\gamma}-\mathbb{E}\big(\big|W\big|^{\gamma}\big)\Big|^{2+\delta}\bigg)\sum_{p=1}^{m_j}\sigma(Y_{1,2p-1})^{\gamma(2+\delta)}.\end{aligned} \tag{5.5}$$

Next notice that it follows from (5.4), the first inequality in (3.2), (1.28) and standard calculations, that one has for some constant $c_1 > 0$ and, for every $j \in \mathbb{N}$,

$$\big(\mathrm{Var}(V_{1,\gamma}^{m_j})\big)^{-\frac{2+\delta}{2}} \le c_1 m_j^{\frac{(2+\delta)}{2}(2\gamma(H+1/\alpha)-1)}. \tag{5.6}$$

Also notice that it results from (5.5), the second inequality in (3.2) and standard calculations, that one has for some constant $c_2 > 0$ and, for every $j \in \mathbb{N}$,

$$\sum_{p=1}^{m_j}\mathbb{E}\bigg(\Big|\big|Y_{1,2p-1}\big|^{\gamma}-\mathbb{E}\big(\big|Y_{1,2p-1}\big|^{\gamma}\big)\Big|^{2+\delta}\bigg) \le c_2\Big(\log(m_j)+m_j^{(1-\gamma(H+1/\alpha)(2+\delta))}\Big). \tag{5.7}$$

Finally, combining (5.6) and (5.7) one obtains, for some constant $c_3 > 0$ and for all $j \in \mathbb{N}$, that

$$\begin{aligned}&\big(\mathrm{Var}(V_{1,\gamma}^{m_j})\big)^{-\frac{2+\delta}{2}}\sum_{p=1}^{m_j}\mathbb{E}\bigg(\Big|\big|Y_{1,2p-1}\big|^{\gamma}-\mathbb{E}\big(\big|Y_{1,2p-1}\big|^{\gamma}\big)\Big|^{2+\delta}\bigg)\\ &\le c_3\Big(m_j^{\frac{(2+\delta)}{2}(2\gamma(H+1/\alpha)-1)}\log(m_j)+m_j^{-\frac{\delta}{2}}\Big).\end{aligned}$$

Thus, one can derive from (1.28) and (1.23) that (5.2) is satisfied. □

The following remark is a straightforward consequence of Remark 3.1 and of Lemma 3.2.

Remark 5.2. Let $\gamma \in (0, 2^{-1}\alpha)$ be arbitrary and let $W_{(\alpha)}$ be an arbitrary real-valued SαS random variable with scale parameter equals to 1. For all $(j,p) \in \mathbb{N}^2$, one has

$$\begin{aligned}&\mathrm{Var}\big(|Y_{j,2p-1}|^{\gamma}\big)\\ &=\mathrm{Var}\big(|W_{(\alpha)}|^{\gamma}\big)2^{2\gamma(jH+1/\alpha)}\bigg(\int_{-4^{-1}}^{4^{-1}}\frac{\big|\psi(\eta)\big|^{\alpha}}{(\eta+2p-1)^{\alpha H+1}}\,d\eta\bigg)^{2\gamma/\alpha} \qquad (5.8)\\ &=\mathrm{Var}\big(|W_{(\alpha)}|^{\gamma}\big)2^{(j-1)2\gamma H}p^{-2\gamma(H+1/\alpha)}\\ &\qquad\times\bigg(\int_{-4^{-1}}^{4^{-1}}\frac{\big|\psi(\eta)\big|^{\alpha}}{\big(1+(2p)^{-1}(\eta-1)\big)^{\alpha H+1}}\,d\eta\bigg)^{2\gamma/\alpha}.\end{aligned}$$

Lemma 5.3. *Let γ be arbitrary and such that (1.27) holds, and let $W_{(\alpha)}$ be an arbitrary real-valued $S\alpha S$ random variable with scale parameter equals to* 1*. There is a finite constant c such that, for all $(j,m)\in\mathbb{N}^2$, one has*

$$\Big| 2^{-(j-1)2\gamma H} m^{2\gamma(H+1/\alpha)-1}\,\mathrm{Var}(V_{j,\gamma}^m) - \mathrm{Var}\big(|W_{(\alpha)}|^\gamma\big)\big(1-2\gamma(H+1/\alpha)\big)^{-1}\|\psi\|_{L^\alpha(\mathbb{R})}^{2\gamma}\Big| \le c\,m^{2\gamma(H+1/\alpha)-1}. \tag{5.9}$$

Proof. For proving (5.9), one uses the equality

$$\mathrm{Var}(V_{j,\gamma}^m) = \sum_{p=1}^{m} \mathrm{Var}\big(\big|Y_{j,2p-1}\big|^\gamma\big),$$

the second equality in (5.8) and arguments similar to those which allowed to obtain (4.7). □

Proposition 5.4. *Let γ be arbitrary and such that (1.27) holds. For all $(j,m)\in\mathbb{N}^2$, the random variable $\widetilde{R}_{j,\gamma}^m$ is defined as*

$$\widetilde{R}_{j,\gamma}^m := F_\gamma(H,\alpha^{-1})\,m^{1/2}\left(\frac{V_{j,\gamma}^m}{\mathbb{E}(V_{j,\gamma}^m)}-1\right), \tag{5.10}$$

where $H\in(0,1)$ and $\alpha\in[\underline{\alpha},2]$ are the unknown Hurst parameter and stability parameter of the HFSM $\{X(t)\}_{t\in\mathbb{R}}$, and F_γ is the positive continuous function introduced in Remark 1.9. Let $(m_j)_{j\in\mathbb{N}}$ be an arbitrary non-decreasing sequence of integers larger than 2 *which satisfies the condition (1.23). When j goes to $+\infty$, the random variables $\widetilde{R}_{1,\gamma}^{m_j}$ and $\widetilde{R}_{2,\gamma}^{m_j}$ converge in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.*

Proof. First notice that it follows from Lemma 2.3 (*iii*) and (5.10) that, for every $j\in\mathbb{N}$, $\widetilde{R}_{1,\gamma}^{m_j} \overset{d}{=} \widetilde{R}_{2,\gamma}^{m_j}$. Thus, we give the proof only in the case of $\widetilde{R}_{1,\gamma}^{m_j}$. In view of Proposition 5.1, it is enough to show that

$$\lim_{j\to+\infty}\mathbb{E}\Big(\big|R_{1,\gamma}^{m_j}-\widetilde{R}_{1,\gamma}^{m_j}\big|^2\Big)=0\,. \tag{5.11}$$

It follows from (5.1), (5.10) and (1.29) that, for all $j\in\mathbb{N}$,

$$R_{1,\gamma}^{m_j}-\widetilde{R}_{1,\gamma}^{m_j} = \nu_j\,m_j^{1/2}\left(\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{j,\gamma}^{m_j})}-1\right)$$

and consequently that

$$\mathbb{E}\Big(\big|R_{1,\gamma}^{m_j}-\widetilde{R}_{1,\gamma}^{m_j}\big|^2\Big) = \nu_j^2\times\frac{m_j\,\mathrm{Var}(V_{1,\gamma}^{m_j})}{\big(\mathbb{E}(V_{1,\gamma}^{m_j})\big)^2}\,, \tag{5.12}$$

where

$$\nu_j := \left(\frac{\mathbb{E}(V_{1,\gamma}^{m_j})}{\big(m_j \operatorname{Var}(V_{1,\gamma}^{m_j})\big)^{\frac{1}{2}}} - \frac{\mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\big(1-2\gamma(H+1/\alpha)\big)^{1/2}}{\big(\operatorname{Var}\big(|W_{(\alpha)}|^\gamma\big)\big)^{1/2}\big(1-\gamma(H+1/\alpha)\big)} \right). \tag{5.13}$$

Observe that Lemma 4.4, (1.28) and (1.23) imply that

$$\lim_{j\to+\infty} m_j^{\gamma(H+1/\alpha)-1}\,\mathbb{E}(V_{1,\gamma}^{m_j}) = \frac{\mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\|\psi\|_{L^\alpha(\mathbb{R})}^\gamma}{\big(1-\gamma(H+1/\alpha)\big)}.$$

Also, observe that Lemma 5.3, (1.28) and (1.23) entail that

$$\lim_{j\to+\infty} \big(m_j^{2\gamma(H+1/\alpha)-1}\operatorname{Var}(V_{1,\gamma}^{m_j})\big)^{\frac{1}{2}} = \frac{\big(\operatorname{Var}\big(|W_{(\alpha)}|^\gamma\big)\big)^{1/2}\|\psi\|_{L^\alpha(\mathbb{R})}^\gamma}{\big(1-2\gamma(H+1/\alpha)\big)^{1/2}}.$$

Thus, using the equality

$$\frac{\mathbb{E}(V_{1,\gamma}^{m_j})}{\big(m_j \operatorname{Var}(V_{1,\gamma}^{m_j})\big)^{\frac{1}{2}}} = \frac{m_j^{\gamma(H+1/\alpha)-1}\,\mathbb{E}(V_{1,\gamma}^{m_j})}{\big(m_j^{2\gamma(H+1/\alpha)-1}\operatorname{Var}(V_{1,\gamma}^{m_j})\big)^{\frac{1}{2}}}, \quad \text{for all } j \in \mathbb{N},$$

and (5.13) one obtains that

$$\lim_{j\to+\infty} \nu_j = 0 \tag{5.14}$$

and consequently that

$$\lim_{j\to+\infty} \frac{m_j \operatorname{Var}(V_{1,\gamma}^{m_j})}{\big(\mathbb{E}(V_{1,\gamma}^{m_j})\big)^2} = \frac{\operatorname{Var}\big(|W_{(\alpha)}|^\gamma\big)\big(1-\gamma(H+1/\alpha)\big)^2}{\big(\mathbb{E}\big(|W_{(\alpha)}|^\gamma\big)\big)^2\big(1-2\gamma(H+1/\alpha)\big)}. \tag{5.15}$$

Finally, putting together (5.12), (5.14) and (5.15), one gets (5.11). □

Corollary 5.5. *Let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and such that (1.27) holds. For all $(j,m) \in \mathbb{N}^2$, the random variable $\Lambda_{j,\gamma}^m$ is defined as*

$$\Lambda_{j,\gamma}^m := \log(2) F_\gamma(H, \alpha^{-1}) m^{1/2} \log_2\left(\frac{V_{j,\gamma}^m}{\mathbb{E}(V_{j,\gamma}^m)} \right), \tag{5.16}$$

where $H \in (0,1)$ and $\alpha \in [\underline{\alpha}, 2]$ are the unknown Hurst parameter and stability parameter of the HFSM $\{X(t)\}_{t\in\mathbb{R}}$, and F_γ is the positive continuous function introduced in Remark 1.9. Let $(m_j)_{j\in\mathbb{N}}$ be an arbitrary non-decreasing sequence of integers larger than 2 which satisfies the condition (1.23). When j goes to $+\infty$, the random variables $\Lambda_{1,\gamma}^{m_j}$ and $\Lambda_{2,\gamma}^{m_j}$ converge in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.

Proof. First notice that it follows from Lemma 2.3 (*iii*) and (5.16) that, for every $j \in \mathbb{N}$, $\Lambda_{1,\gamma}^{m_j} \overset{d}{=} \Lambda_{2,\gamma}^{m_j}$. Thus, we give the proof only in the case of $\Lambda_{1,\gamma}^{m_j}$. In view of Proposition 5.4, it is enough to show that

$$\Lambda_{1,\gamma}^{m_j} - \widetilde{R}_{1,\gamma}^{m_j} \xrightarrow[j\to+\infty]{\mathbb{P}} 0, \tag{5.17}$$

where $\xrightarrow[j\to+\infty]{\mathbb{P}}$ denotes the convergence in probability. Let ε be an arbitrary fixed positive real number. One clearly has, for all $j \in \mathbb{N}$, that

$$\begin{aligned}&\mathbb{P}\Big(\big|\Lambda_{1,\gamma}^{m_j} - \widetilde{R}_{1,\gamma}^{m_j}\big| \geq \varepsilon\Big)\\ &\leq \mathbb{P}\bigg(\Big\{\big|\Lambda_{1,\gamma}^{m_j} - \widetilde{R}_{1,\gamma}^{m_j}\big| \geq \varepsilon\Big\} \cap \Big\{\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} \geq 2^{-1}\Big\}\bigg) + \mathbb{P}\bigg(\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} < 2^{-1}\bigg).\end{aligned}$$

Moreover, one knows from Lemma 4.1 that

$$\lim_{j\to+\infty} \mathbb{P}\bigg(\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} < 2^{-1}\bigg) = 0\,.$$

Thus, for proving (5.17) it is enough to show that

$$\lim_{j\to+\infty} \mathbb{P}\bigg(\Big\{\big|\Lambda_{1,\gamma}^{m_j} - \widetilde{R}_{1,\gamma}^{m_j}\big| \geq \varepsilon\Big\} \cap \Big\{\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} \geq 2^{-1}\Big\}\bigg) = 0\,. \tag{5.18}$$

It can easily be shown that one has, for some deterministic constant $c_1 > 0$, that

$$\big|\log(2)\log_2(1+x) - x\big| \leq c_1 x^2\,, \quad \text{for all } x \in [-2^{-1}, +\infty). \tag{5.19}$$

Then, in view of (5.10) and (5.16), setting $c_2 := c_1 F_\gamma(H, \alpha^{-1}) > 0$, one can derive from (5.19) with $x = \frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} - 1$ that

$$\begin{aligned}&\mathbb{P}\bigg(\Big\{\big|\Lambda_{1,\gamma}^{m_j} - \widetilde{R}_{1,\gamma}^{m_j}\big| \geq \varepsilon\Big\} \cap \Big\{\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} \geq 2^{-1}\Big\}\bigg)\\ &\leq \mathbb{P}\bigg(\Big\{c_2 (m_j)^{1/2}\Big(\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} - 1\Big)^2 \geq \varepsilon\Big\} \cap \Big\{\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} \geq 2^{-1}\Big\}\bigg)\\ &\leq \mathbb{P}\bigg((m_j)^{1/2}\Big(\frac{V_{1,\gamma}^{m_j}}{\mathbb{E}(V_{1,\gamma}^{m_j})} - 1\Big)^2 \geq c_2^{-1}\varepsilon\bigg)\\ &\leq c_2\,\varepsilon^{-1}\,(m_j)^{1/2}\,\frac{\operatorname{Var}(V_{1,\gamma}^{m_j})}{\big(\mathbb{E}(V_{1,\gamma}^{m_j})\big)^2}\,,\end{aligned} \tag{5.20}$$

where the last inequality follows from Markov inequality. Finally, combining (5.20) and (5.15), one obtains (5.18). □

We are now in position to complete the proof of Theorem 1.10.

End of the proof of Theorem 1.10. One can derive from Theorem 1.8 and from the continuity property of the functions F_γ, τ_1 and τ_2 (see Remark 1.9 and (1.33)) that

$$\lim_{j\to+\infty} F_\gamma\big(\tau_1(\widehat{H}_{1,j,\gamma}), \tau_2(\widehat{\alpha}_{2,j,\gamma}^{-1})\big)\big(F_\gamma(H,\alpha^{-1})\big)^{-1} = 1\,, \quad \text{(almost surely)}.$$

Thus, for proving the theorem it is enough to show that

$$2^{-1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(m_{1,j})^{1/2}\Big(\widehat{H}_{1,j,\gamma}-H\Big)\xrightarrow[j\to+\infty]{d}\mathcal{N}(0,1) \tag{5.21}$$

and

$$(2/3)^{1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(m_{2,j})^{1/2}\Big(\widehat{\alpha}_{2,j,\gamma}^{-1}-\alpha^{-1}\Big)\xrightarrow[j\to+\infty]{d}\mathcal{N}(0,1), \tag{5.22}$$

where $(m_{1,j})_{j\in\mathbb{N}}$, $(m_{2,j})_{j\in\mathbb{N}}$, $\widehat{H}_{1,j,\gamma}$ and $\widehat{\alpha}_{2,j,\gamma}^{-1}$ are as in the statement of Theorem 1.10. One knows from (4.27) and (5.16) that, for all $j\in\mathbb{N}$,

$$\begin{aligned}&2^{-1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(m_{1,j})^{1/2}\Big(\widehat{H}_{1,j,\gamma}-H\Big) \\ &= 2^{-1/2}\left(\Lambda_{2,\gamma}^{m_{1,j}}-\Lambda_{1,\gamma}^{m_{1,j}}+\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(m_{1,j})^{1/2}\Delta_1^{m_{1,j},m_{1,j}}(\gamma,H,\alpha)\right).\end{aligned} \tag{5.23}$$

Moreover, it follows from (4.4), (1.28) and (1.23) that the deterministic quantity

$$2^{-1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(m_{1,j})^{1/2}\Delta_1^{m_{1,j},m_{1,j}}(\gamma,H,\alpha)\xrightarrow[j\to+\infty]{}0. \tag{5.24}$$

On another hand, Corollary 5.5 and the fact that, for all $j\in\mathbb{N}$, the two random variables $\Lambda_{1,\gamma}^{m_{1,j}}$ and $\Lambda_{2,\gamma}^{m_{1,j}}$ are independent (see (5.16) and the fundamental Lemma 2.3 (ii)) imply that

$$2^{-1/2}\left(\Lambda_{2,\gamma}^{m_{1,j}}-\Lambda_{1,\gamma}^{m_{1,j}}\right)\xrightarrow[j\to+\infty]{d}\mathcal{N}(0,1). \tag{5.25}$$

Thus, combining (5.24) and (5.25), one obtains (5.21).

Let us now prove that (5.22) holds. It follows from (4.28) and (5.16) that, for all $j\in\mathbb{N}$,

$$\begin{aligned}&(2/3)^{1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(m_{2,j})^{1/2}\Big(\widehat{\alpha}_{2,j,\gamma}^{-1}-\alpha^{-1}\Big) \\ &= (2/3)^{1/2}\Bigg(\Lambda_{1,\gamma}^{m_{2,j}}-\left(\frac{m_{2,j}}{m_{2,j+1}}\right)^{1/2}\Lambda_{2,\gamma}^{m_{2,j+1}} \\ &\qquad -\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(m_{2,j})^{1/2}\Delta_1^{m_{2,j},m_{2,j+1}}(\gamma,H,\alpha) \\ &\quad +\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)\big(1-\gamma(H+1/\alpha)\big)(m_{2,j})^{1/2}\left(1-\log_2\left(\frac{m_{2,j+1}}{m_{2,j}}\right)\right)\Bigg).\end{aligned} \tag{5.26}$$

Moreover, (4.4), (1.28), (1.23) and the inequality $m_{2,j}\le m_{2,j+1}$ imply that the deterministic quantity

$$\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(m_{2,j})^{1/2}\Delta_1^{m_{2,j},m_{2,j+1}}(\gamma,H,\alpha)\xrightarrow[j\to+\infty]{}0, \tag{5.27}$$

and (1.30) entails that the deterministic quantity

$$\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)\big(1-\gamma(H+1/\alpha)\big) \tag{5.28}$$

$$\times\,(m_{2,j})^{1/2}\left(1-\log_2\left(\frac{m_{2,j+1}}{m_{2,j}}\right)\right)\Bigg)\xrightarrow[j\to+\infty]{}0.$$

On another hand, it results from Corollary 5.5, (1.30) and from the fact that, for all $j\in\mathbb{N}$, the two random variables $\Lambda_{1,\gamma}^{m_{2,j}}$ and $\Lambda_{2,\gamma}^{m_{2,j+1}}$ are independent (see (5.16) and the fundamental Lemma 2.3 (ii)), that

$$(2/3)^{1/2}\left(\Lambda_{1,\gamma}^{m_{2,j}}-\left(\frac{m_{2,j}}{m_{2,j+1}}\right)^{1/2}\Lambda_{2,\gamma}^{m_{2,j+1}}\right)\xrightarrow[j\to+\infty]{d}\mathcal{N}(0,1). \tag{5.29}$$

Finally, putting together (5.26), (5.27), (5.28) and (5.29), one gets (5.22). □

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