Mean square displacement of Brownian paths perturbed by bounded pair potentials*

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Abstract

We study Brownian paths perturbed by semibounded pair potentials and prove upper bounds on the mean square displacement. As a technical tool we derive infinite dimensional versions of key inequalities that were first used in [15] in order to study the effective mass of the Fröhlich polaron.

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1 Introduction

Perturbations of d-dimensional Brownian motion by a density that changes the behaviour of paths appear in many applications. A simple class of these measures is obtained by subjecting the Brownian paths to an external potential $V : \mathbb{R}^d \to \mathbb{R}$, resulting in probability measures of the type

$$\mathbb{P}_{V,T}(\mathrm{d}x) = \frac{1}{Z_T} \exp\Big(- \int_0^T V(x_s)\,\mathrm{d}s \Big) \mathbb{P}_{[0,T]}(\mathrm{d}x),$$

where $\mathbb{P}_{[0,T]}$ is the Brownian path measure on $C\big([0,T]; \mathbb{R}^d\big)$, and Z_T is the normalization. Under suitable conditions on the function V the Feynman-Kac formula provides a link to semigroups of Schrödinger operators, and to the theory of Itô-diffusions, and allows for an essentially complete understanding of these measures and their $T \to \infty$ limits. We refer to [11] for details.

In this paper, we are concerned with the more difficult case where the perturbation appears through a pair potential W, meaning that we study probability measures of the form

$$\hat{\mathbb{P}}_{\alpha,T}(\mathrm{d}x) = \frac{1}{\hat{Z}_T} \mathrm{e}^{\alpha \int_0^T \int_0^T W(\|x_t - x_s\|, |t-s|)\mathrm{d}t\mathrm{d}s} \mathbb{P}_{[0,T]}(\mathrm{d}x).$$

The potential acts on the Brownian increments, so that paths with increments which lead to large values of W are favoured. W usually decays to zero in the second argument fairly quickly, meaning that this influence is localized. We will be concerned with cases where $W(x,t)$ is maximal near $x = 0$ for all t, which intuitively leads to a self-attractive force on Brownian paths: one would then expect that the mean square displacement $m_T(\alpha) = \hat{\mathbb{P}}_{\alpha,T}\left(\|x_T\|^2\right)$ grows less quickly for large α than it does for small ones.

Arguably the most important, and best studied case is where $W(x,t) = \frac{1}{|x|}\mathrm{e}^{-|t|}$ and the dimension is at least 3, which corresponds to the Fröhlich polaron, a basic model for matter interacting with a quantized field. We refer to [7] and the references therein for more background on these connections. A conjecture of Landau and Pekar [10] states that in this case, $\lim_{T\to\infty}\frac{m_T(\alpha)}{T} = C_{\mathrm{LP}}\alpha^{-4}(1+O_{\alpha\to\infty}(\alpha^{-1}))$, with an explicit constant C_{LP}. Proving this conjecture has been on the agenda of mathematical physicists at least since the seminal paper of Spohn [16]. However, significant progress was only made recently. First, using a point process representation of $\hat{\mathbb{P}}_{\alpha,T}$ introduced in [12], an upper bound of the order $\limsup_{T\to\infty}\frac{m_T(\alpha)}{T} \leq C\alpha^{-2/5}$ was shown in [3], and then, using Gaussian domination, an almost optimal upper bound of the form $\limsup_{T\to\infty}\frac{m_T(\alpha)}{T} \leq C\log(\alpha)^6/\alpha^4$ was shown in [15]. In parallel, Brooks and Seiringer [6] showed the sharp lower bound $\liminf_{T\to\infty}\frac{m_T(\alpha)}{T} \geq C_{\mathrm{LP}}\alpha^{-4}(1 + O_{\alpha\to\infty}(\alpha^{-c}))$ (for some absolute constant $c > 0$) by employing functional analytic methods on the quantum model of the polaron. It is, however, worthwhile to note that their approach requires results obtained via the probabilistic route in [13]. In conjunction with the recent preprint [5] the Landau-Pekar conjecture is now fully settled.

The present paper builds on the techniques introduced in [15] and has two main objectives: the first is concerned with the central tool introduced in [15], which is a way to leverage the Gaussian correlation inequality in order to obtain concentration inequalities for a rather large class of probability measures. In [15], these inequalities were introduced for finite dimensional measures, and their application to path measures was achieved via approximating the latter and by checking that the bounds obtained did not depend on the discretization parameter. Since these concentration inequalities may become a useful tool in various applications, it seems worth-while to develop a version that directly works in infinite dimensional spaces, and is thus directly applicable to path spaces. This is done in Theorems 2.6 and 2.8. The second objective is to use these inequalities in order to get upper bounds on $m_T(\alpha)$ for potentials W that are quite different from the Polaron case, in particular ones which are bounded from above and have compact support. This is the content of Theorem 2.11. The main conditions we require are that W must be symmetric and quasi-concave. Thus in many cases, fortunately including the Polaron, we now have powerful tools to estimate the effect of pair potentials on Brownian paths. However in several other cases the situation is just as unclear as it was before. In particular we currently have no tools to meaningfully estimate the effect of *repulsive* pair interactions.

The paper is organized as follows: in Section 2, we introduce all relevant quantities and give precise statements of our results. In Section 3, we prove them. In an appendix, we collect and prove some facts about infinite dimensional measures that are most certainly known, but of which we were unable to find a satisfactory account in the literature.

2 Definitions and results

Unless specified otherwise Ω is assumed to be a real, separable Banach space with Borel-σ-field $\mathcal{B}$. A measurable function $f : \Omega \to \mathbb{R}$ is said to be symmetric whenever $f(x) = f(-x)$ for all $x \in \Omega$, and quasi-concave whenever $\{x : f(x) \geq r\}$ is a convex set

for all $r \in \mathbb{R}$.

Definition 2.1 (QC)**.** *A function $f : \Omega \to \mathbb{R}$ is defined to be (QC) whenever f is symmetric and quasi-concave. Furthermore, a function is also said to be (QC) whenever f is a finite product of non-negative symmetric and quasi-concave functions or the uniformly bounded limit of such.*

We say that a function $f : \Omega \to \mathbb{R}$ is (QC) on (a symmetric and convex set) A whenever $\mathbb{1}_A f$ is (QC).

The reader will have no problem to verify that the following functions are (QC):

- any symmetric function $m : \mathbb{R} \to \mathbb{R}$ which is decreasing on $\mathbb{R}_{\geq 0}$;
- $h \circ g$, where $g : \Omega \to \mathbb{R}$ is convex and symmetric and $h : \mathbb{R} \to \mathbb{R}$ is decreasing;
- $\mathbb{1}_K$, where $K \subseteq \Omega$ is a convex, symmetric set;
- $f \mathbb{1}_K$, where $K \subseteq \Omega$ is a convex, symmetric set and $f \geq 0$ is (QC).

Example 2.2. Consider the special case $\Omega = C\left([0,1]; \mathbb{R}^d\right)$ and let $f \in C(\mathbb{R}^d; \mathbb{R})$ be bounded from above, symmetric and quasi-concave. The property (QC) is inherited by

$$x \mapsto \mathrm{e}^{\int_0^1 f(x_t)\mathrm{d}t} = \lim_{n \to \infty} \prod_{j=1}^{n} \mathrm{e}^{\frac{1}{n}f(x_{j/n})}.$$

This can be seen as follows:

$$x \mapsto \tfrac{1}{n}f(x_{j/n})$$

is (QC) per assumption. It follows that

$$x \mapsto \mathrm{e}^{\frac{1}{n}f(x_{j/n})}$$

is (QC), as well. But then the entire product is uniformly bounded as the integral is bounded, and the claim is shown. The same reasoning lets us conclude that double integrals over increments are (QC), e.g.

$$x \mapsto \mathrm{e}^{\int_0^1 \int_0^1 f(x_t - x_s)\mathrm{d}s\mathrm{d}t}.$$

Definition 2.3. *Let μ, ν be two probability measures on $(\Omega, \mathcal{B})$. Write*

$$\nu \preceq \mu$$

whenever $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is (QC).

The significance of Definition 2.3 stems from the following result.

Proposition 2.4. *Let μ, ν be two probability measures on $(\Omega, \mathcal{B})$. Furthermore, let μ be a centred Gaussian measure and let $\nu \preceq \mu$.*

1. *For a (QC) function $f : \Omega \to \mathbb{R}$ with $f \geq 0$ or $\|f\|_\infty < \infty$,*

$$\mu(f) \leq \nu(f). \tag{2.1}$$

2. *For $g : \Omega \to \mathbb{R}_{\geq 0}$ symmetric and convex,*

$$\mu(g) \geq \nu(g). \tag{2.2}$$

Proof. A functional version of the Gaussian correlation inequality (GCI, [14]) states that for any centered Gaussian measure μ and any (QC) functions $f, g \geq 0$, we have $\mu(fg) \geq \mu(f)\mu(g)$; see [2, Proposition 5.1]. This directly gives (2.1) for non-negative f:

$$\mu(f)\mu\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \leq \mu\left(f\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right). \tag{2.3}$$

For f bounded and (QC) the claim follows by applying (2.1) to the non-negative (QC) function $f + \|f\|_\infty$.

For (2.2) note that, for any $R > 0$, $-\min\{R, g\}$ is (QC) as $\min(R, \cdot)$ is non-decreasing. Clearly this function is bounded and so inequality (2.1) can be applied to obtain

$$-\mu\left(\min\{R, g\}\right) \leq -\nu\left(\min\{R, g\}\right).$$

An application of monotone convergence finishes the proof. $\qquad\square$

Define for a probability measure μ and for each Borel set $A \in \mathcal{B}$

$$\mu^{\times 2}(A) := \mu(A/2).$$

The next Lemma is essential in the proofs of Theorem 2.6 and Theorem 2.8. It provides us with a tool that makes it possible to work with a centred Gaussian measure μ on a finite-dimensional vector space as if it would be supported on a convex, symmetric set K which has most of its mass. The important part is that the "bad part" of μ, which is not supported on K, is still dominated by $\mu^{\times 2}$; this makes it possible to obtain easy error estimates. We note that the proof given in [15] is not applicable whenever the support of μ is infinite-dimensional due to the fact that $\mu^{\times 2}$ and μ are singular; otherwise, the result of Theorem 2.6 would be immediate.

Lemma 2.5 ([15, Lemma 3.1]). *Let μ be a centred Gaussian measure on a finite-dimensional real vector-space X and let $K \subseteq X$ be a symmetric convex set with $\mu(K) > 1 - \delta$ for some $\delta \leq 0.1$. Then, there exists a decomposition*

$$\mu = (1 - \delta')\nu + \delta'\overline{\nu}$$

of μ into a mixture of probability measures such that for some $c_1 \geq 100$:

1. *$\delta' \leq \delta$,*
2. *supp(ν) $\subseteq c_1 K$,*
3. *$\nu \preceq \mu$,*
4. *$\overline{\nu} \preceq \mu^{\times 2}$.*

For a non-negative function f with $\mu(f) < \infty$ we write

$$\mu^{(f)}(\mathrm{d}x) := \frac{f(x)}{\mu(f)}\mu(\mathrm{d}x).$$

We will use the notation $\mu \propto \nu$ for two positive measures if ν is a finite measure and $\mu = \frac{\nu}{\nu(\Omega)}$. In this notation,

$$\mu^{(f)}(\mathrm{d}x) \propto f(x)\mu(\mathrm{d}x).$$

It is possible to approximate any centred, non-degenerate Gaussian measure μ by centred Gaussian measures with finite-dimensional support using Theorem 3.1. In what follows $(\mu_n)_{n\in\mathbb{N}}$ references this approximation sequence. We can now state the first result:

Theorem 2.6. *Let μ be a non-degenerate, centred Gaussian measure and let $A \in \mathcal{B}$ be a closed, symmetric convex set with $\mu(A^c) < 0.1$ and $\mu(\partial A) = 0$.*

1. *There exists a constant $c_1 < \tilde{c} < c_1 + 1$ s.t. for any (QC) function $f \in C_b(\Omega; \mathbb{R}_{\geq 0})$ and $\delta \leq \mu(A^c)$,*
$$\mu(f) \geq (1 - \delta)\mu(f; \tilde{c}A) + \delta\mu^{\times 2}(f). \tag{2.4}$$

2. *Take two (QC) functions $g \in C_b(\Omega; \mathbb{R}_{\geq 0})$ and $h \in C_b(\Omega; \mathbb{R}_{>0})$. If, for all $n \in \mathbb{N}$, $\mu_n^{(h)} \preceq \mu_n$ holds and $\mu_n^{(h)}$ is still a centred Gaussian measure and g/h is (QC) on $c_1 A$, then for any (QC) function $f \in C_b(\Omega; \mathbb{R})$*

$$\mu^{(g)}(f) \geq (1-\delta)\mu^{(h)}(f) + \delta\mu^{\times 2}(f). \tag{2.5}$$

Our next result is an extension of Theorem 2.6 to the T-fold product of a centred Gaussian measure, $\tilde{\mu} = \mu^{\otimes T}$. The strategy to prove Theorem 2.8 will be to split each component of $\mu^{\otimes T}$ using Lemma 2.5. We then have to deal with 2^T product measures indexed by $\gamma \in \{0,1\}^T$, where on each component the support differs.

Fix a closed, convex symmetric set $A \subseteq \mathcal{B}$. For $\gamma \in \{0,1\}^T$ define

$$A^\gamma := B_{\gamma(0)} \times ... \times B_{\gamma(T-1)},$$

where $B_0 = \Omega$, $B_1 = A$. Also,

$$S(\gamma) := \{i < T - 1 : \gamma(i) = 1 = \gamma(i+1)\}$$

and, for $j = 0, 1$,

$$M_j(\gamma) := \{i : \gamma(i) = j\}.$$

In the following, a vector $\mathbf{h}$ is defined such that finite dimensional approximations of μ remain centred Gaussian after reweighting. Then, if a given perturbation g of $\tilde{\mu}$ is sufficiently confining, it is possible to apply (GCI) to swap the density from g to $\mathbf{h}$.

Definition 2.7. *Define*

$$\mathbf{h} := (h_1, h_2)$$

for some non-negative symmetric quadratic forms $h_1 : \Omega \to \mathbb{R}$, $h_2 : \Omega^2 \to \mathbb{R}$.

*We say $\mathbf{h}$ **dominates** g on A whenever*

$$\mathrm{e}^{g(x_0,...,x_{T-1}) + \sum_{j=0}^{T-1} \mathbb{1}_{\gamma(j)=1} h_1(x_j) + \sum_{i \in S(\gamma)} h_2(x_i, x_{i+1})}$$

is (QC) on A^γ for each $\gamma \in \{0,1\}^T$.

We conclude with a convenient notation for dominating measures. For any measure μ on Ω and $\mathbf{x} := (x_0, ..., x_{T-1})$ define

$$\mu^{\langle \mathbf{h}(\gamma) \rangle}(\mathrm{d}\mathbf{x}) \propto \mathrm{e}^{\sum_{j=0}^{T-1} \mathbb{1}_{\gamma(j)=1} h_1(x_j) + \sum_{i \in S(\gamma)} h_2(x_i, x_{i+1})} \mu_\gamma(\mathrm{d}\mathbf{x})$$

with

$$\mu_\gamma := \bigotimes_{j=0}^{T-1} \mu_{\gamma(j)}$$

where $\mu_0 := \mu^{\times 2}$ and $\mu_1 := \mu$. Also,

$$\mu^{-\langle \mathbf{h}(\gamma) \rangle} := \mu^{\langle -\mathbf{h}(\gamma) \rangle}.$$

Similarly, for any function g which is bounded from above,

$$\mu^{\langle g \rangle}(\mathrm{d}x) \propto \mathrm{e}^{g(x)} \mu(\mathrm{d}x).$$

In this notation it follows that if $\mathbf{h}$ dominates g on A, then $\tilde{\mu}^{\langle g \rangle} \preceq \tilde{\mu}^{-\langle \mathbf{h}(\gamma) \rangle}$ on A^γ. This observation leads to the second Theorem.

Theorem 2.8. *Let μ be a centred, non-degenerate Gaussian measure. Let $A \in \mathcal{B}$ be a closed, symmetric convex set with $\mu(A^c) < 0.1$ and $\mu(\partial A) = 0$. Fix some $T \in \mathbb{N}$, let $f \in C_b(\Omega^T; \mathbb{R})$ be (QC), and let $g \in C(\Omega^T; \mathbb{R})$ be bounded above with e^g (QC). If $\mathbf{h}$ dominates g on $c_1 A$, then there exist weights [1] $\{w(\gamma) : \gamma \in \{0,1\}^T\}$ satisfying*

$$\sum_{\gamma \in \{0,1\}^T} w(\gamma)|M_1(\gamma)| \geq T(1 - \delta)$$

and

$$(\mu^{\otimes T})^{\langle g \rangle}(f) \geq \sum_{\gamma \in \{0,1\}^T} w(\gamma)\mu^{-\langle \mathbf{h}(\gamma) \rangle}(f). \tag{2.6}$$

As an application of the developed theory we use Theorem 2.8 to derive an upper bound on the mean square displacement of Brownian paths perturbed by a class of pair potentials W.

Assumption 2.9. *$W : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is bounded from above, jointly continuous and*

$$x \mapsto W(x,t)$$

is (QC) for each fixed $t > 0$. Also, there exists $\varepsilon > 0$, $\delta > 0$ and a constant $C_{2\varepsilon} > 0$ s.t.

$$x \mapsto W(x,t) + C_{2\varepsilon}x^2$$

is decreasing on $[0, 2\varepsilon]$ for each $t \in [0, 2\delta]$.

Let $T > 0$ and denote by $\mathbb{P}_{[0,T]}$ the law of d-dimensional Brownian Motion (BM) on $C([0,T]; \mathbb{R}^d)$. We call the measure

$$\hat{\mathbb{P}}_{\alpha,T}(\mathrm{d}x) \propto e^{\alpha \int_0^T \int_0^T W(\|x_t - x_s\|, |t-s|)\mathrm{d}t\mathrm{d}s} \mathbb{P}_{[0,T]}(\mathrm{d}x) \tag{2.7}$$

perturbed measure with coupling strength $\alpha > 0$ and pair potential W.

Remark 2.10. We require the boundedness of W due to the fact that our arguments require weak convergence to go from finite-dimensional approximations to the infinite-dimensional case. Moreover, by definition of (QC), the density in (2.7) is required to be uniformly bounded, which would not be the case for W unbounded. That being said, whenever the partition function of the model is finite one can define a sequence of truncated W^n with the property that $W^n \uparrow W$. When W^n is unimodal with a single singularity at 0 (as in the case of the Fröhlich polaron), one can hope to take the approximating W^n to be monotone, and then apply Theorem 2.6 and Theorem 2.8 to the densities induced by W^n. After that, one can use dominated convergence in (2.5) and (2.6) to remove the cut-off after passing to infinite dimensions. The final bounds one obtains from Theorem 2.11 can be, however, far from optimal. Indeed, in [15, Proposition 2.5] a specific α-dependent truncation was used, which enables genuinely different asymptotic behavior.

Theorem 2.11. *Let W be s.t. Assumption 2.9 is satisfied. For $T > 0$ and $\alpha > C(W)$, where $C(W)$ is a constant depending on the pair interaction, it holds that*

$$\hat{\mathbb{P}}_{\alpha,[0,T]}\left(\|x_{0,T}\|^2\right) \leq O\left(\frac{T \log(\alpha)^3}{\alpha} + \left(\frac{\alpha}{\log(\alpha)^3}\right)^{-1/2}\right).$$

At least for quadratic potentials, this result is optimal up to the $\log(\alpha)$ factors.

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[1] We define weights as $w(\gamma) \geq 0$ for all $\gamma \in \{0,1\}^T$ and $\sum_{\gamma \in \{0,1\}^T} w(\gamma) = 1$.

Proposition 2.12. *Let* $W(x,t) = -x^2 g(t)$ *for* $g \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *with* $C_g = \int_0^\infty t^2 g(t) \mathrm{d}t < \infty$. *Then*

$$\hat{\mathbb{P}}_{\alpha,[0,T]}\left(\|x_{0,T}\|^2/d\right) \geq \frac{T}{1 + 2\alpha C_g}.$$

Proof. By a continuous analog of [12, Eq. (3.3)], we have the identity

$$\hat{\mathbb{P}}_{\alpha,[0,T]}\left(\frac{\|x_{0,T}\|^2}{dT}\right)$$
$$= \sup_{f \in C^\infty([0,T])} \left\{ \frac{2(f(T) - f(0))}{\sqrt{T}} - \int_0^T f'(t)^2 \mathrm{d}t - \int_0^T \int_0^T \alpha |f(t) - f(s)|^2 g(|t-s|) \mathrm{d}t \mathrm{d}s \right\}.$$

Specializing to linear $f(t) = Lt$, the right-hand side is at least

$$\sup_{L \geq 0} \left\{ 2L\sqrt{T} - L^2 T - L^2 \alpha \int_0^T \int_0^T (t-s)^2 g(|t-s|) \mathrm{d}t \mathrm{d}s \right\} = \frac{T}{T + \alpha \int_0^T \int_0^T (t-s)^2 g(|t-s|) \mathrm{d}t \mathrm{d}s}$$

which is achieved by

$$L = \frac{\sqrt{T}}{T + \alpha \int_0^T \int_0^T (t-s)^2 g(|t-s|) \mathrm{d}t \mathrm{d}s}.$$

Finally since g is non-negative,

$$\int_0^T \int_0^T (t-s)^2 g(|t-s|) \mathrm{d}t \mathrm{d}s \leq \int_0^T \int_{-\infty}^\infty (t-s)^2 g(|t-s|) \mathrm{d}t \mathrm{d}s = 2T \int_0^\infty u^2 g(u) \mathrm{d}u = 2TC_g.$$

Combining completes the proof. $\qquad\square$

3 Proofs of Theorem 2.6 and Theorem 2.8

3.1 Proof of Theorem 2.6

Recall that Ω is a separable Banach space. It is known that any centred, non-degenerate Gaussian measure μ on Ω is the weak limit of centred Gaussian measures with finite-dimensional support. A proof can be found in the Appendix.

Theorem 3.1. *Let* μ *be a centred, non-degenerate Gaussian measure. Then, there exists a sequence of Gaussian measures* $(\mu_n)_{n \in \mathbb{N}}$ *with* $\operatorname{supp}(\mu_n) = \Omega_n$ *finite-dimensional and*

$$\mu_n \to \mu$$

weakly.

For the proof of the next Lemma we also refer to the Appendix.

Lemma 3.2. *Let* μ *be a Borel probability measure. Moreover, let* A *be a symmetric, closed convex set s.t.* $\mu(A) > 0$ *and* $\mu(\partial A) = 0$. *Then, for arbitrary* $C > 1$ *and* $\delta > 0$ *there exists* $C < c < C + \delta$ *such that* $\mu(\partial cA) = 0$.

Before going into the proofs, let us briefly discuss the strategy to show Theorem 2.6, more specifically equation (2.5). The proof for Theorem 2.8 and equation (2.4) similar in spirit, but require different amounts of additional technicalities. Our starting point is, as just stated, that every centred Gaussian measure μ can be approximated by a sequence of Gaussians $(\mu_n)_{n \geq 1}$, which are supported on a finite-dimensional subspace. On these subspaces we may use Lemma 2.5 to split the measure μ_n into a *good part* ν_n and a *bad part* $\overline{\nu}_n$. The good part is the one with most mass, and on which we can also change densities from g to h. The bad part can be dominated by $\mu^{\times 2}$, thanks to the result in Lemma 2.5. The remaining challenge is to show that this approximate mixture decomposition survives the limit $n \to \infty$.

Proof of Theorem 2.6. We start with showing (2.4). Use Theorem 3.1 to find a sequence $(\mu_n)_{n\in\mathbb{N}}$ such that $\mu_n \to \mu$. Then, $\mu_n(A) \to \mu(A)$ so eventually $\mu_n(A) > 0.9$. Therefore, for n large enough it is possible to apply Lemma 2.5 to get a decomposition

$$\mu_n = (1-\delta_n)\nu_n + \delta_n\overline{\nu}_n \tag{3.1}$$

with $\delta_n \leq \mu_n(A^c)$. It follows that $\delta_n \to \delta$ for some $0 < \delta \leq \mu(A^c)$ along some subsequence as $\limsup_{n\to\infty} \mu_n(A^c) \leq \mu(A^c)$. We abuse notation and take this subsequence as n. By construction $\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n}$ is (QC) and $\mathrm{supp}(\nu_n) \subseteq \tilde{c}A$, where $\tilde{c}$ is chosen with Lemma 3.2 s.t. $\mu(\tilde{c}A) = 0$. Clearly

$$\nu_n(\mathrm{d}x) = \mathbb{1}_{\tilde{c}A}(x)\nu_n(\mathrm{d}x)$$

which yields

$$\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n} = \frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n}\mathbb{1}_{\tilde{c}A}.$$

Now, $\mu_n(\tilde{c}A) \to \mu(\tilde{c}A)$ implies that

$$\mu_n^{(\mathbb{1}_{\tilde{c}A})} \to \mu^{(\mathbb{1}_{\tilde{c}A})}.$$

This is seen by an application of the Portmanteau theorem and

$$\mu_n^{(\mathbb{1}_{\tilde{c}A})}(B) = \frac{\mu_n(\tilde{c}A \cap B)}{\mu_n(\tilde{c}A)}.$$

We deduce that for any $f \in C_b(\Omega; \mathbb{R})$

$$\mu_n(f; \tilde{c}A) \to \mu(f; \tilde{c}A). \tag{3.2}$$

We are now ready to show (2.4). Taking a (QC) function $f \in C_b(\Omega; \mathbb{R}_{\geq 0})$ and using the decomposition of μ_n, it is easily seen that

$$\begin{aligned} \mu_n(f) &= (1-\delta_n)\nu_n(f) + \delta_n\overline{\nu_n}(f) \\ &= (1-\delta_n)\nu_n(f; \tilde{c}A) + \delta_n\overline{\nu_n}(f) \\ &\geq (1-\delta_n)\mu_n(f; \tilde{c}A) + \delta_n\mu_n^{\times 2}(f). \end{aligned}$$

By (3.2) all expressions on the right hand side converge. The convergence of the left hand side is immediate.

To show (2.5), reweight both sides of equation (3.1) by g. Thanks to (GCI)

$$\frac{\nu_n(g)}{\mu_n(g)} \geq 1.$$

That is, the weight for the "good part" of μ_n only improves. In other words,

$$\mu_n^{(g)} = (1-\delta_n)\frac{\nu_n(g)}{\mu_n(g)}\nu_n^{(g)} + \delta_n\frac{\overline{\nu_n}(g)}{\mu_n(g)}\overline{\nu_n}^{(g)}$$

yields

$$\mu_n^{(g)} = (1-\delta_n')\nu_n^{(g)} + \delta_n'\overline{\nu_n}^{(g)}$$

with $\delta_n' \leq \delta_n \to \delta$.

Take a (QC) function $f \in C_b(\Omega; \mathbb{R})$. Notice that $\mu_n^{(h)}(f) \geq \mu_n(f) \geq \mu_n^{\times 2}(f)$ and

$$\frac{\mathrm{d}\nu_n^{(g)}}{\mathrm{d}\mu_n^{(h)}} = \frac{\mathrm{d}\nu_n^{(g)}}{\mathrm{d}\nu_n^{(h)}}\frac{\mathrm{d}\nu_n^{(h)}}{\mathrm{d}\mu_n^{(h)}} \propto (g/h)\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n}.$$

Per assumption g/h is (QC) on the support of ν_n and we may calculate

$$
\begin{aligned}
\mu_n^{(g)}(f) &= (1 - \delta_n')\nu_n^{(g)}(f) + \delta_n'\overline{\nu_n}^{(g)}(f) \\
&\geq (1 - \delta_n')\mu_n^{(h)}(f) + \delta_n'\mu_n^{\times 2}(f) \\
&\geq (1 - \delta_n)\mu_n^{(h)}(f) + \delta_n\mu_n^{\times 2}(f) \\
&\to (1 - \delta)\mu^{(h)}(f) + \delta\mu^{\times 2}(f)
\end{aligned}
$$

The fact that $\mu_n^{\times 2} \to \mu^{\times 2}$ and $\mu_n^{(g)} \to \mu^{(g)}$ follows directly from $\mu_n \to \mu$. $\qquad\square$

3.2 Proof of Theorem 2.8

Lemma 3.3. *Let $(\nu_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be two sequences of probability measures on Ω such that for each convex, symmetric set K*

$$
\nu_n(K) \geq \mu_n(K)
$$

and $\mu_n \to \mu$ weakly. Then, the sequence $(\nu_n)_{n\in\mathbb{N}}$ is tight.

Proof. Since $\mu_n \to \mu$, the sequence $(\mu_n)_{n\in\mathbb{N}}$ is tight. Take $\delta > 0$ and let $K \subseteq \Omega$ be compact s.t. $\sup_{n\in\mathbb{N}} \mu_n(K^c) \leq \delta$. In a first step we symmetrise K by defining

$$
K_1 := K \cup -K.
$$

This operation clearly preserves compactness. In a second step, take

$$
K_2 := \overline{\mathrm{conv}(K_1)}.
$$

It is known that this set is again compact as Ω is a Banach space ([1], Theorem 5.35). The assumption yields for all $n \in \mathbb{N}$

$$
\nu_n(K_2^c) \leq \mu_n(K_2^c) \leq \mu_n(K^c)
$$

and so

$$
\sup_{n\in\mathbb{N}} \nu_n(K_2^c) \leq \sup_{n\in\mathbb{N}} \mu_n(K_2^c) \leq \delta,
$$

showing the claim. $\qquad\square$

Proof of Theorem 2.8. Use Theorem 3.1 to get a sequence of centred Gaussian measures $(\mu_n)_{n\in\mathbb{N}}$ with finite-dimensional support such that $\mu_n \to \mu$. By the same argument as in Theorem 2.6, eventually

$$
\mu_n = (1 - \delta_n)\nu_n + \delta_n\overline{\nu}_n
$$

is possible with $\mathrm{supp}(\nu_n) \subseteq c_1 A$ and $\delta_n \leq \mu_n(A^c)$.

Recall that the measure of interest is $\mu^{\otimes T}$. Doing the split of measures just mentioned in each coordinate leads to weighted measures of type

$$
\Psi_{n,\gamma} = \bigotimes_{i=0}^{T-1} \psi_{n,\gamma(i)}
$$

where $\psi_{n,0} := \overline{\nu}_n$ and $\psi_{n,1} := \nu_n$. Indeed, it holds that

$$
\mu_n^{\otimes T} = \sum_{\gamma\in\{0,1\}^T} \delta_n^{|M_0(\gamma)|}(1 - \delta_n)^{|M_1(\gamma)|}\Psi_{n,\gamma}. \tag{3.3}
$$

Reweight both sides of (3.3) by e^g and note that the new weights for a specific γ are

$$w_n(\gamma) := \delta_n^{|M_0(\gamma)|}(1 - \delta_n)^{|M_1(\gamma)|}\frac{\Psi_{n,\gamma}(\mathrm{e}^g)}{\mu_n^{\otimes T}(\mathrm{e}^g)}.$$

It then follows that $\sum\limits_{\gamma:\gamma(i)=1} w_n(\gamma) = (1 - \delta_n)\frac{\mu_n\otimes...\otimes\nu_n\otimes...(\mathrm{e}^g)}{\mu_n^{\otimes T}(\mathrm{e}^g)}$, where the quotient is ≥ 1 by (GCI). This yields

$$\sum_{\gamma\in\{0,1\}^T} w_n(\gamma)|M_1(\gamma)| = \sum_{i=0}^{T-1}\sum_{\gamma:\gamma(i)=1} w_n(\gamma) \geq T(1 - \delta_n).$$

Use Lemma 3.3 to deduce that $(\nu_n)_{n\in\mathbb{N}}$ is tight. Then, there exists a weak limit measure along some subsequence for which $\delta_n \to \delta \leq \mu(A^c)$ (which we will also denote by n). Along this sequence $(\overline{\nu}_n)_{n\in\mathbb{N}}$ also converges weakly to some limit measure since

$$\mu_n = (1 - \delta_n)\nu_n + \delta_n\overline{\nu}_n.$$

As the underlying space is assumed to be separable, product measures of weakly convergent measures also converge weakly to the product of the respective limits [4]. Therefore, for each γ, there exists Ψ_γ s.t.

$$\Psi_{n,\gamma} \to \Psi_\gamma$$

from which it follows that along this subsequence all weights $w_n(\gamma)$ converge by construction and are bounded from below as claimed.

Now,

$$\frac{\mathrm{d}\Psi_{n,\gamma}^{\langle g\rangle}}{\mathrm{d}\mu_n^{-\langle\mathbf{h}(\gamma)\rangle}} = \frac{\mathrm{d}\Psi_{n,\gamma}^{\langle g\rangle}}{\mathrm{d}\Psi_{n,\gamma}}\frac{\mathrm{d}\Psi_{n,\gamma}}{\mathrm{d}\mu_{n,\gamma}}\frac{\mathrm{d}\mu_{n,\gamma}}{\mathrm{d}\mu_n^{-\langle\mathbf{h}(\gamma)\rangle}}$$

$$= \left(\frac{\mathrm{d}\Psi_{n,\gamma}^{\langle g\rangle}}{\mathrm{d}\Psi_{n,\gamma}}\frac{\mathrm{d}\mu_{n,\gamma}}{\mathrm{d}\mu_n^{-\langle\mathbf{h}(\gamma)\rangle}}\right)\frac{\mathrm{d}\Psi_{n,\gamma}}{\mathrm{d}\mu_{n,\gamma}}.$$

The first term is proportional to

$$\exp\left(g(x_0,...,x_{T-1}) + \sum_{j=0}^{T-1}\mathbb{1}_{\gamma(j)=1}h_1(x_j) + \sum_{i\in S(\gamma)}h_2(x_i, x_{i+1})\right)$$

which is (QC) on $(c_1 A)^\gamma = \mathrm{supp}\,(\Psi_{n,\gamma})$ by assumption. The second RN-derivative is (QC) due to the fact that each coordinate is. Remembering that $\mu_n^{-\langle\mathbf{h}(\gamma)\rangle}$ is centred Gaussian, an application of (GCI) yields

$$(\mu_n^{\otimes T})^{\langle g\rangle})(f) = \sum_{\gamma\in\{0,1\}^T} w_n(\gamma)\Psi_{n,\gamma}^{\langle g\rangle}(f)$$

$$\geq \sum_{\gamma\in\{0,1\}^T} w_n(\gamma)\mu_n^{-\langle\mathbf{h}(\gamma)\rangle}(f)$$

By taking the subsequence found before to infinity, the left hand side converges as weak convergence survives under taking products and both f and e^g are continuous and bounded. For the right hand side, the same argument holds by recalling that along the chosen subsequence all $w_n(\gamma)$ converge, as well. □

We want to note that the assumptions in the previous Theorem can be modified. It is only required that $\mathbf{h}$ consists of quadratic, symmetric non-negative forms on the finite

dimensional subspaces supp $(\Psi_{n,\gamma})$. This is because we only use (GCI) for the finite dimensional measures.

Additionally, we may trade the condition $\mu(\partial A) = 0$ for

$$\limsup_{n\to\infty} \mu_n(A^c) < 0.1.$$

Indeed, if the above condition is satisfied, one may still apply Lemma 2.5 to the approximating sequence eventually; in comparison to Theorem 2.6 the convergence of $\mathbb{1}_{\tilde{c}A}$ is not required.

4 Application on path space

The goal of this section is to prove Theorem 2.11. The proof is a direct application of Theorem 2.8 after specifying all the relevant data going into the Theorem. The following Corollary is easily deduced from Theorem 2.8 by a standard approximation argument.

Corollary 4.1. *Let the situation in Theorem 2.8 be given. In the special case that* $\Omega = C\left([0,1]; \mathbb{R}^d\right)$ *is the path space for some* $d \in \mathbb{N}$ *and* $f : C\left([0,1]; \mathbb{R}^d\right) \to \mathbb{R}^d$ *is linear,*

$$\left(\mu^{\otimes T}\right)^{(g)} \left(\|f(x)\|^2\right) \leq \sum_{\gamma \in \{0,1\}^T} w(\gamma) \mu^{-\langle \mathbf{h}(\gamma) \rangle} \left(\|f(x)\|^2\right).$$

Let $c > 0$ be arbitrary. We will choose a specific c shortly. In this section take $\Omega = C\left([0,c]; \mathbb{R}^3\right)$. Whenever convenient identify $C\left([0,c]; \mathbb{R}^3\right)^2 \simeq C\left([0,2c]; \mathbb{R}^3\right)$. Write

$$K_R := \left\{ (x_t)_{t \leq c} \in C\left([0,c]; \mathbb{R}^3\right) : \sup_{s \in [0,c]} \|x_{0,s}\| \leq R \right\}.$$

It is easily seen that $\mathbb{P}_{[0,c]}\left(\partial K_R\right) = 0$ for any $R > 0$. In the following we fix W such that Assumption 2.9 is satisfied. We choose the corresponding $\varepsilon > 0$ and $\delta > 0$ accordingly.

Take $c_b > 0$ s.t. for all $\alpha \geq 2$

$$\mathbb{P}_{[0,1]} \left(\sup_{s \leq 1} \|x_s\| \geq c_b \sqrt{\log(\alpha)} \right) \leq \alpha^{-10}.$$

It is possible to change the diffusion radius and still obtain the same bound as in the last display if we restrict the path length. Diffusive scaling of BM

$$\frac{1}{\sqrt{c}} x_{ct} \sim x_t \iff x_{ct} \sim \sqrt{c} x_t$$

implies that

$$\mathbb{P}_{[0,c(\alpha)]} \left(\sup_{s \leq c(\alpha)} \|x_s\| \geq \varepsilon/2 \right) \leq \alpha^{-10} \tag{4.1}$$

with

$$c = c(\alpha) := \frac{\varepsilon^2}{4 c_b^2 \log(\alpha)}.$$

Fix $\alpha \geq 2$ and let $T > 0$ be a time s.t. $\frac{1}{c}T \in \mathbb{N}$. One can scale the BM in the underlying measure so that this condition is not restrictive. Split $[0,T]$ in $\frac{1}{c}T$ smaller intervals of size c. By (4.1) we know that on each of these small intervals BM is essentially concentrated on K_ε; i.e.,

$$\mathbb{P}_{[0,c]} \left(K_{\varepsilon/2}^c \right) \leq \alpha^{-10}.$$

Define the vector $\mathbf{h}^\beta$ with $\beta = (\beta_0, \beta_1)$ by

$$h_0^{\beta_0} : (x_t)_{t \in [0,c]} \mapsto -\beta_0 \int_0^c \int_0^c \|x_t - x_s\|^2 \mathrm{d}t \mathrm{d}s,$$

$$h_1^{\beta_1} : (x_t)_{t \in [0,2c]} \mapsto -\beta_1 \int_0^c \int_c^{2c} \|x_t - x_s\|^2 \mathrm{d}t \mathrm{d}s.$$

It should be noted that both functions are symmetric, quadratic non-negative forms on path space.

Our next goal is to show that αW is dominated by $\mathbf{h}^\beta$ on K_ε for certain parameters β, considered on the product space $\Omega^{\otimes \frac{1}{c}T}$. To do this, let $\lfloor x \rfloor$ be the biggest number $l < x$ with $l = cn$ for some $n \in \mathbb{N}$. Also, for $l \in \mathbb{N}$, $\gamma(lc) = \gamma(l)$.

Proposition 4.2. *Let $T\frac{1}{c} \in \mathbb{N}$ and $\gamma \in \{0,1\}^{T\frac{1}{c}}$. Define*

$$F(s,t) := \begin{cases} \beta_0 \text{ if } \lfloor s \rfloor = \lfloor t \rfloor \text{ and } \gamma(\lfloor s \rfloor) = 1, \\ \beta_1 \text{ if } \lfloor s \rfloor \in S(\gamma) \text{ and } t \in \big[\lfloor s \rfloor + c, \lfloor s \rfloor + 2c\big], \\ 0 \text{ else.} \end{cases}$$

For $0 \le \beta_0, \beta_1 \le \alpha C_{2\varepsilon}$ and $c = c(\alpha) < \delta$, the map

$$x \mapsto \exp\left(\int_0^T \int_0^T \alpha W(\|x_t - x_s\|, |t - s|) + F(s,t)\|x_t - x_s\|^2 \mathrm{d}t \mathrm{d}s\right)$$

is (QC) on K_ε^γ. It follows that $\mathbf{h}^\beta$ dominates

$$g(x) := \int_0^T \int_0^T \alpha W(\|x_t - x_s\|, |t - s|) \mathrm{d}t \mathrm{d}s$$

on K_ε.

Proof. The proof is given in Example 2.2 by noting that the function F was chosen in such a way that

$$x \mapsto \alpha W(\|x_t - x_s\|, |t - s|) + F(s,t)\|x_t - x_s\|^2$$

is symmetric and quasi-concave on K_ε^γ. This is because only nearest neighbor intervals are considered (whenever fluctuations are both bounded). That is, for all $s, t \in [0,T]$ with $F(s,t) \ne 0$ we have $|t - s| \le 2\delta$ (for sufficiently big α) and $\|x_t - x_s\| \le 2\varepsilon$, which implies per assumption that the map in the above display is decreasing. The assumption that W is jointly continuous justifies the approximation by Riemann sums. □

In the following we write c used in the definition of $\mathbf{h}^\beta$ in the subscript of $\mathbb{P}$. It is very helpful to think of c as the basic block size in which the perturbed measure is eventually split. It is therefore required that the entire time interval T is a multiple of the block size c. Define for $s \in \mathbb{N}$ and $c > 0$

$$\bar{\mathbb{P}}^\beta_{c,[0,cs]}(\mathrm{d}x) \propto e^{-\beta \sum_{j=0}^{s-1} \int_{cj}^{c(j+1)} \int_{cj}^{c(j+1)} \|x_t - x_s\|^2 \mathrm{d}t \mathrm{d}s - \beta \sum_{j=0}^{s-2} \int_{cj}^{c(j+1)} \int_{c(j+1)}^{c(j+2)} \|x_t - x_s\|^2 \mathrm{d}t \mathrm{d}s} \mathbb{P}_{[0,cs]}(\mathrm{d}x)$$

$$(4.2)$$

$$\propto e^{-\beta \sum_{j=0}^{s-1} \int_{cj}^{c(j+1)} \int_{cj}^{c(j+1)} \|x_t - x_s\|^2 \mathrm{d}t \mathrm{d}s - \beta \sum_{j=0}^{s-2} \int_{cj}^{c(j+1)} \int_{c(j+1)}^{c(j+2)} \|x_t + x_{c(j+1)} - x_s\|^2 \mathrm{d}t \mathrm{d}s} \mathbb{P}_{[0,c]}^{\otimes s}(\mathrm{d}x).$$

The following result is deduced in the same manner as in [15, Section 6].

Lemma 4.3 ([15, Lemma 6.5])**.** *Let* $s \in \mathbb{N}$ *and* $\beta \geq 2$*. Then,*

$$\bar{\mathbb{P}}^{\beta}_{1,[0,s]} \left(\|x_{0,s}\|^2 \right) \leq O\left(\frac{s}{\beta} + \beta^{-1/2} \right).$$

By an application of Brownian rescaling one obtains

$$\bar{\mathbb{P}}^{\beta}_{c,[0,cs]} \left(\|x_{0,cs}\|^2 \right) = c\bar{\mathbb{P}}^{\beta c^3}_{1,[0,s]} \left(\|x_{0,s}\|^2 \right). \tag{4.3}$$

Take $\gamma \in \{0,1\}^T$. We define **good blocks** to be maximal subintervals $[i,j] \subseteq [0,T]$ for which $\gamma(i) = \gamma(i+1) = ... = \gamma(j) = 1$, i.e. also $\gamma(i-1) = 0 = \gamma(j+1)$ (with convention $\gamma(-1) = \gamma(T+1) = 0$). Let $B(\gamma)$ be the set of good blocks associated to γ, where each block is identified with its length. A **bad block** is an interval with $\gamma(k) = 0$.

The next calculation will apply Theorem 2.8 to $\hat{\mathbb{P}}_{\alpha,[0,T]}$. By joining the endpoints of the path from previous intervals to the next one it is possible to write $\hat{\mathbb{P}}_{\alpha,[0,T]}$ as a reweighted product measure. As the notation is quite cumbersome, we omit the details. For a simple example we refer to equation (4.2).

Proof for Theorem 2.11. W.l.o.g. let $\frac{1}{c}T \in \mathbb{N}$. Choose $\beta = \beta_0 = \beta_1 = \alpha C_{2\varepsilon}$. For fixed γ there are $|M_0(\gamma)| + 1$ good blocks at most. By independent increments of Brownian Motion, the structure of $\mathbb{P}^{-\langle \mathbf{h}^{\beta}(\gamma) \rangle}_{c,[0,T]}$ and the fact that the law of $\mathbb{P}^{-\langle \mathbf{h}^{\beta}(\gamma) \rangle}_{c,[0,T]}$ on a good block of length s is given by $\bar{\mathbb{P}}^{\beta}_{c,[0,cs]}$, we split the integral along its blocks to obtain

$$\mathbb{P}^{-\langle \mathbf{h}^{\beta}(\gamma) \rangle}_{c,[0,T]} \left(\|x_{0,T}\|^2 \right) \leq \sum_{l \in B(\gamma)} \bar{\mathbb{P}}^{\beta}_{c[0,cl]} \left(\|x_{0,cl}\|^2 \right) + |M_0(\gamma)| \mathbb{P}^{\times 2}_{[0,c]} \left(\|x_{0,c}\|^2 \right).$$

Use Proposition 4.2 to apply Corollary 4.1 with

- $\mu = \mathbb{P}_{[0,c]}$,
- $\mathbf{h} = \mathbf{h}^{\beta}$ with $\beta = (\alpha C_{2\varepsilon}, \alpha C_{2\varepsilon})$,
- $g(x) = \alpha \int_0^T \int_0^T W(\|x_t - x_s\|, |t-s|) \mathrm{d}t\mathrm{d}s$,
- $A = K_{\varepsilon}$,
- $\tilde{T} = \frac{1}{c}T$

and estimate with $\delta \leq \alpha^{-10}$ and (4.3)

$$\hat{\mathbb{P}}_{[0,T]} \left(\|x_{0,T}\|^2 \right) \leq \sum_{\gamma \in \{0,1\}^{c^{-1}T}} w(\gamma) \mathbb{P}^{-\langle \mathbf{h}^{\beta}(\gamma) \rangle}_{c,[0,T]} \left(\|x_{0,T}\|^2 \right)$$

$$\leq \sum_{\gamma \in \{0,1\}^{c^{-1}T}} w(\gamma) \Big[\sum_{l \in B(\gamma)} \bar{\mathbb{P}}^{\beta}_{c,[0,cl]} \left(\|x_{0,cl}\|^2 \right) + |M_0(\gamma)| \mathbb{P}^{\times 2}_{[0,c]} \left(\|x_{0,c}\|^2 \right) \Big]$$

$$\leq \sum_{\gamma \in \{0,1\}^{c^{-1}T}} w(\gamma) \left[O\left(\frac{T}{c^3\beta} + (|M_0(\gamma)| + 1) (\beta c^3)^{-1/2} \right) + |M_0(\gamma)| \mathbb{P}^{\times 2}_{[0,1]} \left(\|x_{0,1}\|^2 \right) \right]$$

$$\leq O\left(\frac{T}{\beta c^3} + (\alpha^{-10}T + 1) (\beta c^3)^{-1/2} \right) + \sum_{\gamma \in \{0,1\}^{c^{-1}T}} w(\gamma) |M_0(\gamma)| \mathbb{P}^{\times 2}_{[0,1]} \left(\|x_{0,1}\|^2 \right)$$

$$\leq O\left(\frac{T}{c^3\beta} + (\alpha^{-10}T + 1) (\beta c^3)^{-1/2} \right) + \alpha^{-10} \frac{T}{c} \mathbb{P}^{\times 2}_{[0,1]} \left(\|x_{0,1}\|^2 \right)$$

$$= O\left(\frac{T \log(\alpha)^3}{\alpha} + \left(\frac{\alpha}{\log(\alpha)^3} \right)^{-1/2} \right). \qquad \square$$

A Appendix: Gaussian approximation

We will require the notion of an abstract Wiener Space. For us, an abstract Wiener Space is a triplet which consists of a separable Banach space Ω, a non-degenerate centred Gaussian μ on Ω and the corresponding Cameron Martin space H. Recall that H is dense in Ω, uniquely determined by μ and that $H \hookrightarrow \Omega$ is continuous [17, Theorem 3.3.2].

The continuity of the embedding implies via the Riesz Representation Theorem that for each $\xi \in \Omega^*$ there exists a corresponding $h_\xi \in H$ s.t. $\xi(h) = \langle h_\xi, h \rangle_H$ for each $h \in H$. Moreover, one can show that the map

$$\xi \mapsto h_\xi$$

is linear, continuous and one-to-one [17, Lemma 3.3.1 **(i)**]. Also, for each $f \in \Omega^*$ it holds that ([17, Theorem 3.3.2])

$$\int e^{if(x)} \mu(\mathrm{d}x) = e^{-\frac{1}{2}\|h_f\|_H^2}. \tag{A.1}$$

Finally, it can be concluded that there exists a sequence $\{\xi_n : n \in \mathbb{N}\} \subseteq \Omega^*$ s.t. the set $\{h_{\xi_n} : n \in \mathbb{N}\}$ is an ONB for H [17, Lemma 3.3.1 **(iii)**]. Hence for each $h \in H$,

$$h = \sum_{i=1}^{\infty} \langle h_{\xi_i}, h \rangle_H h_{\xi_i} = \sum_{i=1}^{\infty} \xi_i(h) h_{\xi_i}.$$

Define for each $x \in \Omega$ the projection

$$P_n x := \sum_{i=1}^{n} \xi_i(x) h_{\xi_i}.$$

It is easily seen that P_n extends the projection of H onto span $\{h_{\xi_1}, ..., h_{\xi_n}\}$ to Ω.

We define

$$\mu_n := \mu \circ (P_n)^{-1}.$$

By definition it is clear that supp $(\mu_n) \subseteq \Omega$ is finite-dimensional. Moreover, it is also immediate that μ_n is centred Gaussian. To find its covariance take $f \in \Omega^*$ and calculate

$$\int e^{if(x)} \mu_n(\mathrm{d}x) = \int e^{i \sum_{j=1}^{n} \xi_j(x) f(h_{\xi_j})} \mu(\mathrm{d}x)$$

$$= \exp\left(-\frac{1}{2}\left\langle \sum_{j=1}^{n} h_{\xi_j} f(h_{\xi_j}), \sum_{j=1}^{n} h_{\xi_j} f(h_{\xi_j}) \right\rangle_H\right)$$

$$= \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \langle h_f, h_{\xi_j} \rangle_H^2\right).$$

It can be concluded that for $f \in \Omega^*$

$$\int e^{if(x)} \mu_n(\mathrm{d}x) \to \int e^{if(x)} \mu(\mathrm{d}x) \tag{A.2}$$

via (A.1). Define the random variables

$$X_j : \Omega \to \Omega, \quad x \mapsto \xi_j(x) h_{\xi_j}.$$

Notice that X_i is independent of X_j whenever $i \neq j$ and that X_i is symmetrically distributed because ξ_i is. Then, using the Itô-Nisio Theorem ([9, Theorem 4.1]), we see that (A.2) implies μ-a.e.

$$\sum_{j=1}^{n} X_j \to I,$$

where I is the identity on Ω. We collect this result in the following Lemma.

Lemma A.1. *Let Ω be a separable Banach space endowed with Borel-σ-field $\mathcal{B}$ and let μ be a non-degenerate, centred Gaussian measure. Then, there exists a sequence of projections $(P_n)_{n \in \mathbb{N}}$ such that $\Omega_n := P_n(\Omega)$ is finite-dimensional and μ-a.e.*

$$\lim_{n \to \infty} P_n x = x.$$

Proposition A.2. $\mu_n \to \mu$ *weakly.*

Proof. Let $A \in \mathcal{B}$ be an open set and let $(x_n)_{n \in \mathbb{N}} \subseteq \Omega$ be a sequence s.t. $x_n \to x$. We claim that

$$\mathbb{1}_A(x_n) \geq \mathbb{1}_A(x)$$

for all $n \geq n_0 \in \mathbb{N}$ which implies that

$$\liminf_{n \to \infty} \mathbb{1}_A(x_n) \geq \mathbb{1}_A(x).$$

Indeed, if $x \in A$, there exists some ball with radius $\varepsilon > 0$ centred at x that is also contained in A. x_n converges to x, so eventually $x_n \in B(x, \varepsilon) \subseteq A$ for all $n \geq n_0$. The case $x \notin A$ is trivial. An application of Fatou's Lemma in conjunction with Lemma A.1 (use $x_n := P_n x$) yields

$$\liminf_{n \to \infty} \mu_n(A) = \liminf_{n \to \infty} \int \mathbb{1}_A(x) \mu_n(\mathrm{d}x) \geq \int \liminf_{n \to \infty} \mathbb{1}_A(P_n x) \mu(\mathrm{d}x)$$
$$\geq \int \mathbb{1}_A(x) \mu(\mathrm{d}x) = \mu(A). \tag{A.3}$$

Using the Portmanteau Theorem shows the claim. $\square$

B Appendix: measure of boundary of symmetric, convex sets

We use $\sqcup$ to denote disjoint unions, and always assume μ to be a Borel probability measure on a real Banach space.

Proposition B.1. *Let μ be a Borel probability measure. Moreover, let A be a closed, symmetric and convex set s.t. $\mu(A) > 0$ and $\mu(\partial A) = 0$. Then, there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subseteq A$.*

Proof. Since $A = \partial A \sqcup A^\circ$, the assumption gives that $\mu(A^\circ) > 0$. Then, $A^\circ \neq \emptyset$ and so there exists $x \in A^\circ$ with $B(x, \varepsilon) \subseteq A$. By symmetry of A, $B(-x, \varepsilon) \subseteq A$. Using the convexity we know that $\mathrm{conv}\big(B(x, \varepsilon), B(-x, \varepsilon)\big) \subseteq A$ and so $B(0, \varepsilon) \subseteq A$. $\square$

Proposition B.2. *If A is convex, $x \in A^\circ$, $y \in \overline{A}$, then*

$$[x, y) \subseteq A^\circ.$$

Proof. We follow the proof given in [8]. Take $x \in A^\circ$ and $y \in \overline{A}$. For $\alpha \in (0,1)^2$ write $z = \alpha x + (1-\alpha)y$. We wish to show that $z \in A^\circ$. To this end, take $(y_n)_{n \in \mathbb{N}} \subseteq A$ with $y_n \to y$. Let $\varepsilon > 0$ so that $B(x, \varepsilon) \subseteq A$. Note that the sequence $(x_n)_{n \in \mathbb{N}}$ defined by

$$x_k := \frac{1}{\alpha}\big(z - (1-\alpha)y_k\big)$$

---

[2]The case $\alpha = 1$ is trivial.

converges to x. That is, there exists $n_0 \in \mathbb{N}$ s.t.,for all $n \geq n_0$, $x_n \in B(x, \varepsilon)$ and so

$$x_k \in B(x, \varepsilon) \iff z - (1 - \alpha)y_k \in \alpha B(x, \varepsilon) \iff z \in \alpha B(x, \varepsilon) + (1 - \alpha)y_k.$$

By convexity $\alpha B(x, \varepsilon) + (1 - \alpha)y_k \subseteq A$, which shows the claim. $\square$

Proposition B.3. *Let A be an open, symmetric and convex set. Moreover, assume that $B(0, \varepsilon) \subseteq A$ for some $\varepsilon > 0$. Then, for any $t > 1$,*

$$\overline{A} \subseteq tA.$$

Proof. Write $\overline{A} = \partial A \sqcup A$. $A \subseteq tA$ is obvious by $0 \in A$ and the convexity. To show the inclusion for ∂A define for $p > 0$ and $x \in B$ the scaled convex cone

$$M(p, x) := \operatorname{conv}\big(B(0, p\varepsilon), px\big)$$

so that $M(1, x) \backslash x \subseteq A$ whenever $x \in \partial A$ (use Proposition B.2). Notice that $M(1, x) \subseteq M(t, x) \backslash tx$ for $t > 1^3$. But now

$$M(1, x) \subseteq M(t, x) \backslash tx \subseteq tA$$

for $x \in \partial A$. In particular x is now an inner point of tA, which shows the claim. $\square$

Proof of Lemma 3.2. Because A is symmetric and convex, for $t > 1$ it holds that $tA \supseteq A$ and so the map

$$t \mapsto \mu(tA^\circ)$$

is monotone increasing. It is well known that a monotone increasing function is continuous, up to a countable set of discontinuities. Then, we can find $c \in (C, C + \delta)$ which is a point of continuity for the map just introduced. Write $cA = \partial cA \sqcup (cA)^\circ$. Use Proposition B.1 with A to deduce that A° contains $B(0, \varepsilon')$ for some $\varepsilon' > 0$. This ball is then also contained in $(cA)^\circ \supseteq A^\circ$. Obtain with Proposition B.3 and the fact that cA is convex that for any $\varepsilon > 0$

$$(c + \varepsilon)A^\circ = (1 + \tfrac{\varepsilon}{c})(cA)^\circ \supseteq \overline{(cA)^\circ} = cA$$

and so

$$\partial cA \subseteq (c + \varepsilon)A^\circ \backslash cA^\circ.$$

We conclude that for $\varepsilon > 0$ arbitrary

$$\mu(\partial cA) \leq \mu((c + \varepsilon)A^\circ) - \mu(cA^\circ).$$

Since c is assumed to be a point of continuity the claim follows. $\square$

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