Scaling limit of the cluster size distribution for the random current measure on the complete graph[*]

Dmitrii Krachun[†]   Christoforos Panagiotis[‡]   Romain Panis[§]

Abstract

We study the percolation configuration arising from the random current representation of the near-critical Ising model on the complete graph. We compute the scaling limit of the cluster size distribution for an arbitrary set of sources in the single and the double current measures. As a byproduct, we compute the tangling probabilities recently introduced by Gunaratnam, Panagiotis, Panis, and Severo in [GPPS22]. This provides a new perspective on the switching lemma for the φ^4 model introduced in the same paper: in the Gaussian limit we recover Wick's law, while in the Ising limit we recover the corresponding tool for the Ising model.

1 Introduction

For $n \geq 1$, consider the complete graph $K_n = (V_n, \mathscr{P}_2(V_n))$ on n vertices, where $\mathscr{P}_2(V_n)$ is the set of pairs of elements of V_n. Let $\lambda \in \mathbb{R}$, and define the inverse temperature

$$d_n(\lambda) := \frac{1}{n}\left(1 - \frac{\lambda}{\sqrt{n}}\right).$$

For any $\sigma \in \{\pm 1\}^{V_n}$, define the Hamiltonian of the Ising model on the complete graph K_n by

$$\mathscr{H}_{n,\lambda}(\sigma) := - \sum_{\{i,j\} \in \mathscr{P}_2(V_n)} d_n(\lambda)\sigma_i\sigma_j.$$

[†]Princeton University, United States of America. E-mail: dk9781@princeton.edu
[‡]Bath University, England. E-mail: cp2324@bath.ac.uk
[§]Institut Camille Jordan, France. E-mail: panis@math.univ-lyon1.fr

The associated Ising measure is defined via the expectation values

$$\langle F(\sigma) \rangle_{n,\lambda} := \frac{1}{\mathfrak{Z}_n(\lambda)} \sum_{\sigma \in \{\pm 1\}^{V_n}} F(\sigma) \exp(-\mathscr{H}_{n,\lambda}(\sigma))$$

for $F : \{\pm 1\}^{V_n} \to \mathbb{R}^+$, where $\mathfrak{Z}_n(\lambda)$ is the partition function of the model. This model, which is also known as the *near-critical Curie–Weiss* model, has been introduced independently by many people, including [Tem54, Hus54], and since then, it has been extensively studied. See [FV17, Chapter 2] and references therein.

We are interested in a particular expansion of the partition function of the model called the *random current expansion* which provides an interesting geometric representation of the correlation functions of the model. This expansion was initially introduced in [GHS70] to prove concavity of the magnetisation with a positive external field, but the probabilistic picture attached to it was developed in [Aiz82]. We refer to [DC17] for more information about this expansion.

A *current* on K_n is a function $\mathbf{n} : \mathscr{P}_2(V_n) \to \mathbb{N} := \{0, 1, \ldots\}$. For $\{i, j\} \in \mathscr{P}_2(V_n)$ and $\mathbf{n} \in \Omega_n$, we will write $\mathbf{n}_{i,j} = \mathbf{n}(i, j)$ for the value of $\mathbf{n}$ on the edge $\{i, j\}$. We denote by Ω_n the set of currents on K_n. The set of *sources* of $\mathbf{n}$, denoted by $\partial \mathbf{n}$, is defined as

$$\partial \mathbf{n} := \left\{ x \in V_n \ \Big| \ \sum_{y \in V_n \setminus \{x\}} \mathbf{n}_{x,y} \text{ is odd} \right\}.$$

It is an easy exercise to check that $|\partial \mathbf{n}|$ is always even. For a current $\mathbf{n}$ on K_n, introduce the weight

$$w(\mathbf{n}) = w_{n,\lambda}(\mathbf{n}) := \prod_{\{i,j\} \in \mathscr{P}_2(V_n)} \frac{d_n(\lambda)^{\mathbf{n}_{i,j}}}{\mathbf{n}_{i,j}!}.$$

For S an even subset of V_n (i.e. of even cardinality), let $Z_n^S(\lambda) := \sum_{\partial \mathbf{n} = S} w(\mathbf{n})$. When $S = \emptyset$, we let $Z_n^{\emptyset}(\lambda) = Z_n(\lambda)$. It is classical [GHS70, Aiz82] that

$$\mathfrak{Z}_n(\lambda) = 2^n Z_n(\lambda), \tag{1.1}$$

and furthermore, if $\sigma_S := \prod_{j \in S} \sigma_j$,

$$\langle \sigma_S \rangle_{n,\lambda} = \frac{Z_n^S(\lambda)}{Z_n^{\emptyset}(\lambda)}.$$

To each current $\mathbf{n}$, we can associate a percolation configuration by looking at the trace of the edges that carry a non-zero weight, i.e. $(\mathbb{1}[\mathbf{n}_{i,j} > 0])_{\{i,j\} \in \mathscr{P}_2(V_n)}$. As is standard in percolation theory, our interest lies in the connectivity properties of this percolation configuration. Let us import some standard terminology. If $x, y \in V_n$, we say that x is *connected* to y in $\mathbf{n}$ and write $x \xleftrightarrow{\mathbf{n}} y$, if there exists a sequence of points $x_0 = x, x_1, \ldots, x_m = y$ such that $\mathbf{n}_{x_i, x_{i+1}} > 0$ for $0 \leq i \leq m - 1$. For $x \in V_n$, we let $\mathscr{C}_x(\mathbf{n})$ be the set of vertices of K_n which are connected to x in $\mathbf{n}$. Note that for any $x \in V_n$, $\mathscr{C}_x(\mathbf{n})$ contains an even number of sources.

We naturally define a probability measure $\mathbb{P}_{n,\lambda}^S$ on the set of currents with source set S by letting

$$\mathbb{P}_{n,\lambda}^S[\mathbf{n}] := \frac{1}{Z_n^S(\lambda)} w(\mathbf{n}) \mathbb{1}[\partial \mathbf{n} = S].$$

The main interest in the current expansion of the Ising model arises from the existence of a combinatorial tool called the *switching lemma*, which was initially introduced in

[GHS70] and used in many subsequent works to capture fine properties of the Ising model [Aiz82, AG83, AF86, ABF87, DCT16, ADCTW19, Rao20, ADC21, Pan23], see [DC17] for more information. This tool relates ratios and differences of correlation functions to a system of two independent currents. This motivates us to also consider, for $S_1, S_2 \subset V_n$, the measure

$$\mathbb{P}_{n,\lambda}^{S_1,S_2} := \mathbb{P}_{n,\lambda}^{S_1} \otimes \mathbb{P}_{n,\lambda}^{S_2}. \tag{1.2}$$

Applying the switching lemma [DC17, Lemma 4.3], we find that for any event $\mathcal{A}$ which depends on $\mathbf{n}_1 + \mathbf{n}_2$,

$$\mathbb{P}_{n,\lambda}^{S_1,S_2}[\mathcal{A}] = \frac{\langle \sigma_{S_1 \Delta S_2} \rangle_{n,\lambda}}{\langle \sigma_{S_1} \rangle_{n,\lambda} \langle \sigma_{S_2} \rangle_{n,\lambda}} \mathbb{P}_{n,\lambda}^{S_1 \Delta S_2, \emptyset}[\mathcal{A}, \mathscr{F}_{S_2}], \tag{1.3}$$

where $\mathscr{F}_{S_2}$ is the event that there exists a sub-current $\mathbf{m} \leq \mathbf{n}_1 + \mathbf{n}_2$ satisfying $\partial \mathbf{m} = S_2$. Note that if $S_2 = \{i, j\}$, $\mathscr{F}_{S_2}$ coincides with the event that $i \xleftrightarrow{\mathbf{n}} j$. This observation and Lemma 2.1 (which provides asymptotics as $n \to \infty$ for the ratio of correlation functions appearing in the right-hand side of (1.3)) reduce the analysis to the case $S_2 = \emptyset$.

Let S be an even subset of V_n. Let $\mathscr{P}_{\text{even}}(S)$ be the set of even partitions of S, i.e. the set made of partitions of S consisting only of even subsets of S. If $\mathbf{n}$ is a current on K_n with sources S, we denote by $\mathfrak{P}(\mathbf{n})$ the partition of S in the associated percolation configuration. Note that for a current $\mathbf{n}$ one always has that $\mathfrak{P}(\mathbf{n}) \in \mathscr{P}_{\text{even}}(\partial \mathbf{n})$.

In [GPPS22, Proposition 4.3], the authors proved[1] that the clusters of the sources have size of order $\sqrt{n}$. Our main result strengthens this observation. For simplicity we focus on the clusters of the sources. Beyond these clusters, there could be several other clusters of size of order $\sqrt{n}$. Their study reduces to the case below as explained in Remark 3.5.

Theorem 1.1. *Let $\lambda \in \mathbb{R}$ and $k \geq 1$. Set $S = \{1, \ldots, 2k\}$. Let $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) be distributed according to $\mathbb{P}_{n,\lambda}^S$ (resp. $\mathbb{P}_{n,\lambda}^\emptyset$). There exist two random variables $X_{k,\lambda}^{\mathsf{s}}, X_{k,\lambda}^{\mathsf{d}}$ on $(\mathbb{R}_{\geq 0})^{2k} \times \mathscr{P}_{\text{even}}(S)$ such that*

$$\left(\frac{|\mathscr{C}_1(\mathbf{n}_1)|}{\sqrt{n}}, \ldots, \frac{|\mathscr{C}_{2k}(\mathbf{n}_1)|}{\sqrt{n}}, \mathfrak{P}(\mathbf{n}_1) \right) \xrightarrow[n\to\infty]{} X_{k,\lambda}^{\mathsf{s}},$$

$$\left(\frac{|\mathscr{C}_1(\mathbf{n}_1 + \mathbf{n}_2)|}{\sqrt{n}}, \ldots, \frac{|\mathscr{C}_{2k}(\mathbf{n}_1 + \mathbf{n}_2)|}{\sqrt{n}}, \mathfrak{P}(\mathbf{n}_1 + \mathbf{n}_2) \right) \xrightarrow[n\to\infty]{} X_{k,\lambda}^{\mathsf{d}},$$

where both convergences are in distribution. Moreover, each of the first $2k$ coordinates of $X_{k,\lambda}^{\mathsf{s}}$ (resp. $X_{k,\lambda}^{\mathsf{d}}$) is strictly positive almost surely.

Remark 1.2. *In Section 3, we will obtain exact formulas for the density of these random variables (see Proposition 3.2).*

To prove Theorem 1.1, we will first analyse the structure of the clusters of the sources in the limit $n \to \infty$, see Section 2. As it turns out, the clusters have a simple combinatorial description in terms of a family of mutligraphs we call *backbone multigraphs* which in particular carry the information of how the sources are connected together, see Definition 3.4. Conditionally on this backbone multigraph, the scaling limit of the cluster size distribution can be computed exactly, see (3.6) and (3.7).

In comparison with our model, for the well-studied (near-)critical[2] Erdős–Rényi model, the size of the largest cluster is of order $n^{2/3}$. The asymptotic joint distribution of components sizes after normalising by $n^{-2/3}$ was obtained by Aldous [Ald97], which lead

---

[1]This observation was already made in [Aiz82].

[2]The critical threshold for the Erdős–Rényi model on K_n is $\frac{1}{n}$. The near-critical regime is given by a parameter of the form $\frac{1}{n}\left(1 - \frac{\lambda}{n^{1/3}}\right)$ with $\lambda \in \mathbb{R}$.

to the continuum limit of Erdős–Rényi components [ABBG12], as well as to results in the spirit of Theorem 1.1 for several other models, see e.g. [BGJ96, AP00, NP10]. See also [Hof16] and references therein.

As a corollary of Theorem 1.1 and Proposition 2.6, where we obtain the scaling limit of the total number of vertices of positive degree, we get the following result. Recall that a random variable X on $\mathbb{R}^d$ is said to have Gaussian tails if there exists $c > 0$ such that for all $t \geq 1$,

$$\mathbb{P}[|X| \geq t] \leq \frac{1}{c} e^{-ct^2},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Corollary 1.3 (Scaling limit of the cluster size distribution). *Let $\lambda \in \mathbb{R}$ and $k \geq 1$. Set $S = \{1, \ldots, 2k\}$. Let $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) be distributed according to $\mathbb{P}^S_{n,\lambda}$ (resp. $\mathbb{P}^{\emptyset}_{n,\lambda}$). Let $Y^{\mathsf{s}}_{k,\lambda}$ (resp. $Y^{\mathsf{d}}_{k,\lambda}$) be the projection on the first $2k$ coordinates of $X^{\mathsf{s}}_{k,\lambda}$ (resp. $X^{\mathsf{d}}_{k,\lambda}$). Then, $Y^{\mathsf{s}}_{k,\lambda}, Y^{\mathsf{d}}_{k,\lambda}$ have Gaussian tails and,*

$$\left(\frac{|\mathscr{C}_1(\mathbf{n}_1)|}{\sqrt{n}}, \ldots, \frac{|\mathscr{C}_{2k}(\mathbf{n}_1)|}{\sqrt{n}} \right) \underset{n \to \infty}{\longrightarrow} Y^{\mathsf{s}}_{k,\lambda}, \tag{1.4}$$

$$\left(\frac{|\mathscr{C}_1(\mathbf{n}_1 + \mathbf{n}_2)|}{\sqrt{n}}, \ldots, \frac{|\mathscr{C}_{2k}(\mathbf{n}_1 + \mathbf{n}_2)|}{\sqrt{n}} \right) \underset{n \to \infty}{\longrightarrow} Y^{\mathsf{d}}_{k,\lambda}, \tag{1.5}$$

where both convergences are in distribution.

One striking property of the near-critical Ising model introduced above is that its scaling limit is (under appropriate scaling) the φ^4 measure, see [SG73] or Appendix A. This observation plays a central role in [GPPS22] where a switching lemma for the discrete φ^4 model (see Section 4 for a definition) is obtained by approximating the latter model with a well-chosen sequence of Ising models of the above type. As it turns out, the limiting objects are related to the distribution of $\mathfrak{P}(\mathbf{n})$. In the latter paper, even partitions of S are referred to as *tanglings*, see Section 4.

In the proof of Theorem 1.1, we will also obtain that the projections of $X^{\mathsf{s}}_{k,\lambda}$ and $X^{\mathsf{d}}_{k,\lambda}$ on $\mathscr{P}_{\mathrm{even}}(S)$ give positive probability to any even partition. This yields the following result which in particular improves on [GPPS22, Lemma 4.7], where the full convergence of the tangling probabilities is missing.

Corollary 1.4 (Convergence of the tanglings probabilities). *Let $\lambda \in \mathbb{R}$ and $k \geq 1$. Set $S = \{1, \ldots, 2k\}$. Fix $\mathsf{P} \in \mathscr{P}_{\mathrm{even}}(S)$. Let $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) be distributed according to $\mathbb{P}^S_{n,\lambda}$ (resp. $\mathbb{P}^{\emptyset}_{n,\lambda}$). Let $P^{\mathsf{s}}_{k,\lambda}$ (resp. $P^{\mathsf{d}}_{k,\lambda}$) be the projection on $\mathscr{P}_{\mathrm{even}}(S)$ of $X^{\mathsf{s}}_{k,\lambda}$ (resp. $X^{\mathsf{d}}_{k,\lambda}$). Then,*

$$\mathfrak{P}(\mathbf{n}_1) \underset{n \to \infty}{\longrightarrow} P^{\mathsf{s}}_{k,\lambda},$$

$$\mathfrak{P}(\mathbf{n}_1 + \mathbf{n}_2) \underset{n \to \infty}{\longrightarrow} P^{\mathsf{d}}_{k,\lambda},$$

where both convergences are in distribution. Moreover, the support of $P^{\mathsf{s}}_{k,\lambda}$ and $P^{\mathsf{d}}_{k,\lambda}$ is equal to $\mathscr{P}_{\mathrm{even}}(S)$.

As mentioned above, this last corollary has interesting consequences regarding the study of the discrete φ^4 model. In particular, it will be used to provide a new perspective on the switching lemma developed in [GPPS22], by showing that this combinatorial tool interpolates between the corresponding tool for Ising (as λ tends to $-\infty$) and Wick's law (as λ tends to $+\infty$). See Section 4 for more details.

Notation 1.5. *We will use Landau's formalism: if $(u_n)_{n \geq 0}, (v_n)_{n \geq 0} \in (\mathbb{R}^*_+)^{\mathbb{N}}$, write $u_n = O(v_n)$ (resp. $u_n = o(v_n)$) if there exists $C > 0$ such that for all $n \geq 1$, $u_n \leq C v_n$ (resp. $\lim_{n \to \infty} u_n / v_n = 0$).*

2 The random current model on the complete graph

We begin by importing a result from [GPPS22]. For $g > 0$ and $a \in \mathbb{R}$, let $\rho_{g,a}$ be the probability measure on $\mathbb{R}$ defined by

$$\mathrm{d}\rho_{g,a}(t) = \frac{1}{z_{g,a}} e^{-gt^4 - at^2} \mathrm{d}t,$$

where $z_{g,a} = \int_{\mathbb{R}} e^{-gt^4 - at^2} \mathrm{d}t$. If $g > 0$ and $\lambda \in \mathbb{R}$, let $\tilde{g} = (12g)^{1/4}$ and choose $a = a(g, \lambda)$ such that $\lambda = 2a\tilde{g}^{-2}$. We now fix $g > 0$ for the rest of the paper. Let

$$c_n := \tilde{g}^{-1} n^{-3/4}, \tag{2.1}$$

so that $(c_n n)^2 = \tilde{g}^{-2}\sqrt{n}$. The following result is a consequence of [SG73] which states that the scaling limit of the Ising model under consideration is the φ^4 single-site measure. We reprove this result for sake of completeness.

Lemma 2.1 ([GPPS22, Lemma 3.24]). *Let $g > 0$ and $\lambda \in \mathbb{R}$. For any $p \in \mathbb{N}$,*

$$(c_n n)^p \left\langle \prod_{j=1}^{p} \sigma_j \right\rangle_{n,\lambda} \xrightarrow[n \to \infty]{} \langle \varphi^p \rangle_0 \,,$$

where $\langle \cdot \rangle_0 = \langle \cdot \rangle_{0,g,a(\lambda)}$ is the average with respect to the probability measure $\rho_{g,a(\lambda)}$.

Proof. For $p = 1$, there is nothing to show because both $\langle \sigma_1 \rangle_{n,\lambda}$ and $\langle \varphi \rangle_0$ are equal to 0, so let us assume that $p \geq 2$.

Applying Proposition A.1 for $\lambda = 2a\tilde{g}^{-2}$ and doing the change of variables $s = \tilde{g}^2 u$ we get that for every $k \geq 0$

$$c_n^k \left\langle \left(\sum_{i=1}^{n} \sigma_i \right)^k \right\rangle_{n,\lambda} \xrightarrow[n \to \infty]{} \langle \varphi^k \rangle_0.$$

Let $n \geq p$. Expanding $\left(\sum_{i=1}^{n} \sigma_i \right)^p$ and taking cases according to whether some index repeats or not we obtain

$$\left\langle \left(\sum_{i=1}^{n} \sigma_i \right)^p \right\rangle_{n,\lambda} = \sum_{\substack{(i_1,\ldots,i_p) \in V_n^p \\ \forall k \neq \ell, \ i_k \neq i_\ell}} \left\langle \prod_{j=1}^{p} \sigma_{i_j} \right\rangle_{n,\lambda} + \sum_{\substack{(i_1,\ldots,i_p) \in V_n^p \\ \exists k < \ell, \ i_k = i_\ell}} \left\langle \prod_{j=1}^{p} \sigma_{i_j} \right\rangle_{n,\lambda}.$$

Note that

$$\sum_{\substack{(i_1,\ldots,i_p) \in V_n^p \\ \forall k \neq \ell, \ i_k \neq i_\ell}} \left\langle \prod_{j=1}^{p} \sigma_{i_j} \right\rangle_{n,\lambda} = p! \binom{n}{p} \left\langle \prod_{j=1}^{p} \sigma_j \right\rangle_{n,\lambda} = (1 + o(1)) n^p \left\langle \prod_{j=1}^{p} \sigma_j \right\rangle_{n,\lambda}$$

by the symmetries of the complete graph. Furthermore,

$$0 \leq \sum_{\substack{(i_1,\ldots,i_p) \in V_n^p \\ \exists k < \ell, \ i_k = i_\ell}} \left\langle \prod_{j=1}^{p} \sigma_{i_j} \right\rangle_{n,\lambda} \leq \binom{p}{2} n \left\langle \left(\sum_{i=1}^{n} \sigma_i \right)^{p-2} \right\rangle_{n,\lambda} \xrightarrow[n \to \infty]{} 0$$

by choosing 2 indices k, ℓ for which $i_k = i_\ell$, and the element of V_n that they are equal to. Here we have also used that each summand is non-negative by Griffiths' inequality. $\square$

We now turn to some useful estimates on the geometry of the clusters under the measures $\mathbb{P}_{n,\lambda}^S$ and $\mathbb{P}_{n,\lambda}^{S,\emptyset}$. Below, we let $k \geq 1$ and $S = \{1, \ldots, 2k\}$. The constant C may depend on λ. We let $\mathbb{P}$ denote either $\mathbb{P}_{n,\lambda}^S$ or $\mathbb{P}_{n,\lambda}^{S,\emptyset}$, and we write $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$.

Lemma 2.2 (Multiplicity of the edges). *There exists $C > 0$ such that for all n large enough:*

(i) $\mathbb{E}\left[\sum_{1\leq i<j\leq n} \mathbb{1}[\mathbf{n}_{i,j}\geq 1]\right] \leq C\sqrt{n},$

(ii) $\mathbb{E}\left[\sum_{1\leq i<j\leq n} \mathbb{1}[\mathbf{n}_{i,j}\geq 2]\right] \leq C,$

(iii) $\mathbb{E}\left[\sum_{1\leq i<j\leq n} \mathbb{1}[\mathbf{n}_{i,j}\geq 3]\right] \leq \frac{C}{n^{3/2}}.$

Remark 2.3. The above result implies that with high probability: there are at most $O(\sqrt{n})$ open edges, there is a bounded number of edges of weight 2, and there are no edges of weight at least 3.

Proof. We first show the result for $\mathbb{P} = \mathbb{P}_{n,\lambda}^S$. We begin by proving (i). Consider $1 \leq i < j \leq n$. Let $\mathbf{n}$ be a current such that $\mathbf{n}_{i,j} \geq 1$, and define the current $\mathbf{m}$ that satisfies $\mathbf{m}_{i,j} = \mathbf{n}_{i,j} - 1$ and otherwise coincides with $\mathbf{n}$. Then, $\mathbf{m}$ has sources $S\Delta\{i,j\}$ and $w(\mathbf{n}) \leq d_n(\lambda)w(\mathbf{m})$. Thus,

$$\mathbb{P}_{n,\lambda}^S\left[\mathbf{n}_{i,j}\geq 1\right] \leq d_n(\lambda)\frac{Z_n^{S\Delta\{i,j\}}(\lambda)}{Z_n^S(\lambda)} = (1+o(1))\,d_n(\lambda)\,(c_n n)^{|S|-|S\Delta\{i,j\}|}\frac{\left\langle\varphi^{|S\Delta\{i,j\}|}\right\rangle_0}{\left\langle\varphi^{|S|}\right\rangle_0}.$$

Summing over all $i < j$, and splitting the sum according to the three possibilities $|S| - |S\Delta\{i,j\}| \in \{-2, 0, 2\}$, we see by Lemma 2.1 that

$$\mathbb{E}_{n,\lambda}^S\left[\sum_{1\leq i<j\leq n} \mathbb{1}[\mathbf{n}_{i,j}\geq 1]\right]$$
$$\leq O\left(n^2 d_n(\lambda)\frac{1}{\sqrt{n}}\right) + O(nd_n(\lambda)) + O(d_n(\lambda)\sqrt{n}) = O(\sqrt{n}). \quad (2.2)$$

We now turn to (ii). As above, let $\mathbf{n}$ be a current such that $\mathbf{n}_{i,j} \geq 2$, and define the current $\mathbf{m}$ that satisfies $\mathbf{m}_{i,j} = \mathbf{n}_{i,j} - 2$ and otherwise coincides with $\mathbf{n}$. Then, $\mathbf{m}$ has sources S and $w(\mathbf{n}) \leq d_n(\lambda)^2 w(\mathbf{m})$. Thus $\mathbb{P}_{n,\lambda}^S\left[\mathbf{n}_{i,j}\geq 2\right] \leq d_n(\lambda)^2$, which implies that

$$\mathbb{E}_{n,\lambda}^S\left[\sum_{1\leq i<j\leq n} \mathbb{1}[\mathbf{n}_{i,j}\geq 2]\right] = O(1).$$

Finally, we prove (iii). We have

$$\mathbb{P}_{n,\lambda}^S\left[\mathbf{n}_{i,j}\geq 3\right] \leq d_n(\lambda)^3\frac{Z_n^{S\Delta\{i,j\}}(\lambda)}{Z_n^S(\lambda)} = (1+o(1))d_n(\lambda)^3(c_n n)^{|S|-|S\Delta\{i,j\}|}\frac{\left\langle\varphi^{|S\Delta\{i,j\}|}\right\rangle_0}{\left\langle\varphi^{|S|}\right\rangle_0}.$$

Hence, proceeding as above,

$$\mathbb{E}_{n,\lambda}^S\left[\sum_{1\leq i<j\leq n} \mathbb{1}[\mathbf{n}_{i,j}\geq 3]\right] \leq O(n^{-3/2}).$$

The proof for $\mathbb{E} = \mathbb{P}_{n,\lambda}^{S,\emptyset}$ is very similar. For (i), it follows from the proof above together with the observation that

$$\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_{i,j}\geq 1] \leq \mathbb{P}_{n,\lambda}^S[\mathbf{n}_{i,j}\geq 1] + \mathbb{P}_{n,\lambda}^{\emptyset}[\mathbf{n}_{i,j}\geq 1].$$

For (ii), we observe that,

$$\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_{i,j}\geq 2] \leq \mathbb{P}_{n,\lambda}^S[\mathbf{n}_{i,j}\geq 2] + \mathbb{P}_{n,\lambda}^{\emptyset}[\mathbf{n}_{i,j}\geq 2] + \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1(i,j)\geq 1,\,\mathbf{n}_2(i,j)\geq 1].$$

We handle the sum over $1 \leq i < j \leq n$ of the first two probabilities using the same method as above. For the last one, notice that

$$
\begin{aligned}
\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1(i,j) \geq 1,\, \mathbf{n}_2(i,j) \geq 1] &\leq d_n(\lambda)^2 \frac{Z_n^{S\Delta\{i,j\},\{i,j\}}(\lambda)}{Z_n^{S,\emptyset}(\lambda)} \\
&= (1 + o(1))d_n(\lambda)^2 (c_n n)^{|S|-|S\Delta\{i,j\}|-2} \frac{\langle \varphi^{|S\Delta\{i,j\}|}\rangle_0 \langle \varphi^2\rangle_0}{\langle \varphi^{|S|}\rangle_0}.
\end{aligned}
$$

Finally, we proceed similarly for (iii) by noticing that

$$
\begin{aligned}
\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_{i,j} \geq 3] \leq \mathbb{P}_{n,\lambda}^{S}[\mathbf{n}_{i,j} \geq 3] + \mathbb{P}_{n,\lambda}^{\emptyset}[\mathbf{n}_{i,j} \geq 3] &+ \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1(i,j) \geq 2,\, \mathbf{n}_2(i,j) \geq 1] \\
&+ \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1(i,j) \geq 1,\, \mathbf{n}_2(i,j) \geq 2].
\end{aligned}
$$

$\square$

For $x \in V_n$ and a current $\mathbf{n}$, we let $\Delta_{\mathbf{n}}(x) := \sum_{y \neq x} \mathbf{n}_{x,y}$ be the degree of x in the multigraph associated with $\mathbf{n}$.

Lemma 2.4 (The degree is either $1, 2$ or 4)**.** *There exists $C > 0$ such that for all n large enough:*

(i) $\mathbb{E}\left[\sum_{1 \leq i < j \leq n} \mathbb{1}[\mathbf{n}_{i,j} \geq 2]\mathbb{1}[\Delta_{\mathbf{n}}(i) \geq 3 \text{ or } \Delta_{\mathbf{n}}(j) \geq 3]\right] \leq \frac{C}{n^{1/2}}$,

(ii) *for all* $i \in S$, $\mathbb{P}[\Delta_{\mathbf{n}}(i) \geq 2] \leq \frac{C}{n^{1/2}}$,

(iii) $\mathbb{E}\left[\sum_{x \in V_n} \mathbb{1}[\Delta_{\mathbf{n}}(x) \geq 4]\right] \leq C$,

(iv) $\mathbb{E}\left[\sum_{x \in V_n} \mathbb{1}[\Delta_{\mathbf{n}}(x) \geq 5]\right] \leq \frac{C}{n^{1/2}}$.

Remark 2.5. The first property above proves that double edges (i.e. of multiplicity at least two) are isolated with high probability (w.h.p). The second one shows that the sources have degree one w.h.p., the third one states the tightness of the total number of vertices of degree 4, and finally, the fourth inequality above rules out w.h.p the existence of vertices of degree at least 5.

Proof. We prove the result for $\mathbb{P} = \mathbb{P}_{n,\lambda}^{S}$. The proof for $\mathbb{P}_{n,\lambda}^{S,\emptyset}$ uses similar arguments as in the proof of Lemma 2.2.

For (i) we already proved in Lemma 2.2 that there exists $C_1 > 0$ such that

$$
\mathbb{E}_{n,\lambda}^{S}\left[\sum_{1 \leq i < j \leq n} \mathbb{1}[\mathbf{n}_{i,j} \geq 3]\right] \leq \frac{C_1}{n^{3/2}}.
$$

Hence, it remains to handle $\mathbb{E}_{n,\lambda}^{S}\left[\sum_{1 \leq i < j < k \leq n} \mathbb{1}[\mathbf{n}_{i,j} = 2,\, \mathbf{n}_{j,k} \geq 1]\right]$. We have

$$
\mathbb{P}_{n,\lambda}^{S}[\mathbf{n}_{i,j} = 2,\, \mathbf{n}_{j,k} \geq 1] \leq d_n(\lambda)^3 \frac{Z_n^{S\Delta\{j,k\}}(\lambda)}{Z_n^{S}(\lambda)},
$$

hence, taking into account whether j, k belong to S or not (as in (2.2)),

$$
\mathbb{E}_{n,\lambda}^{S}\left[\sum_{1 \leq i < j < k \leq n} \mathbb{1}[\mathbf{n}_{i,j} = 2,\, \mathbf{n}_{j,k} \geq 1]\right] \leq O(n^{-1/2}).
$$

For (ii), we have already showed that

$$
\mathbb{P}_{n,\lambda}^{S}[\mathbf{n}_{i,j} \geq 2] \leq d_n(\lambda)^2,
$$

hence, for a fixed $i \in S$,

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{j \neq i} \mathbb{1}[\mathbf{n}_{i,j} \geq 2] \right] = O(1/n).$$

It remains to estimate $\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq j < k \leq n} \mathbb{1}[\mathbf{n}_{i,j} \geq 1, \, \mathbf{n}_{i,k} \geq 1] \right]$. Note that

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i,j} \geq 1, \, \mathbf{n}_{i,k} \geq 1] \leq d_n(\lambda)^2 \frac{Z_n^{S\Delta\{j,k\}}(\lambda)}{Z_n^S(\lambda)},$$

which yields,

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{j < k} \mathbb{1}[\mathbf{n}_{i,j} \geq 1, \, \mathbf{n}_{i,k} \geq 1] \right] \leq O(n^{-1/2}).$$

For (iii), we only need to estimate $\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j < k < \ell < m \leq n} \mathbb{1}[\mathbf{n}_{i,x} \geq 1, \; \forall x \in \{j,k,\ell,m\}] \right]$. Indeed, if i has an outgoing double edge, then with high probability it has degree 2 by (i). To estimate the latter expectation, note that

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i,x} \geq 1, \, \forall x \in \{j,k,\ell,m\}] \leq d_n(\lambda)^4 \frac{Z_n^{S\Delta\{j,k,\ell,m\}}(\lambda)}{Z_n^S(\lambda)},$$

hence, letting $r = |\{j,k,\ell,m\} \cap S^c|$ we see that the main contribution comes from when $r = 4$, and we get using Lemma 2.1 again

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j < k < \ell < m \leq n} \mathbb{1}[\mathbf{n}_{i,x} \geq 1, \, \forall x \in \{j,k,\ell,m\}] \right] \leq (1 + o(1)) \tilde{g}^4 \frac{\langle \varphi^{|S|+4} \rangle_0}{\langle \varphi^{|S|} \rangle_0}. \quad (2.3)$$

Similarly, for (iv), we have

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i,x} \geq 1, \, \forall x \in \{j,k,\ell,m,y\}] \leq d_n(\lambda)^5 \frac{Z_n^{S\Delta\{i,j,k,\ell,m,y\}}(\lambda)}{Z_n^S(\lambda)}$$

hence

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j < k < \ell < m < y \leq n} \mathbb{1}[\mathbf{n}_{i,x} \geq 1, \, \forall x \in \{j,k,\ell,m,y\}] \right] = O \left(d_n(\lambda)^5 \sum_{r=0}^{6} n^r (c_n n)^{6-2r} \right)$$
$$= O(n^{-1/2}).$$

$\square$

The two above results have the following interesting consequence for the number of vertices with non-zero degree. For a current $\mathbf{n}$, let $\mathcal{N}(\mathbf{n})$ be the number of vertices $i \in V_n$ such that $\Delta_{\mathbf{n}}(i) > 0$. Recall that for $g > 0$, $\tilde{g} = (12g)^{1/4}$.

Proposition 2.6. *Let $\lambda \in \mathbb{R}$, $g > 0$, and $k \geq 0$. Set $S = \{1, \dots, 2k\}$ if $k \geq 1$ and $S = \emptyset$ otherwise. Let $\mathbf{n}$ be distributed according to $\mathbb{P}_{n,\lambda}^S$. Then,*

$$\frac{\mathcal{N}(\mathbf{n})}{(c_n n)^2} \xrightarrow[n \to \infty]{} \frac{\tilde{g}^4}{2} Z_k^2,$$

where the convergence holds in distribution, and Z_k is the random variable with density[3]

$$\frac{\varphi^{2k}}{\langle \varphi^{2k} \rangle_0} \mathrm{d}\rho_{g,a(\lambda)}.$$

---

[3]Recall that $a = a(\lambda)$ was defined in the beginning of Section 2.

Proof. By Lemma 2.4, "most vertices" of positive degree have degree 2. As a result, with high probability we have that $\mathscr{E}(\mathbf{n}) := \sum_{1 \leq i < j \leq n} \mathbb{1}[\mathbf{n}_{i,j} \geq 1] = (1 + o(1))\mathscr{N}(\mathbf{n})$. Hence, it suffices to prove the convergence (in distribution) of $\mathscr{E}(\mathbf{n})/(c_n n)^2$.

We will first show convergence of all moments. Let $r > 0$. Recall that by Lemma 2.2, most open edges are simple (i.e. carry weight 1) with high probability. As a result, in order to estimate $\mathbb{E}_{n,\lambda}^S\left[\frac{\mathscr{E}^r}{(c_n n)^{2r}}\right]$, it is sufficient to consider the probabilities $\mathbb{P}_{n,\lambda}^S\left[\mathbf{n}_{i_s,j_s} = 1, \forall s \in \{1, 2 \ldots, r\}\right]$ for $i_1, j_1, i_2, j_2, \ldots, i_r, j_r \in V_n$ not necessarily distinct. Letting ℓ be the number of distinct edges $\{i_s, j_s\}$ and $m = |S\Delta\{i_1, j_1, \ldots, i_r, j_r\}|$ we have

$$
\mathbb{P}_{n,\lambda}^S\left[\mathbf{n}_{i_s,j_s} = 1, \forall s \in \{1, 2 \ldots, r\}\right] = d_n(\lambda)^\ell \frac{Z_n^{S\Delta\{i_1,j_1,\ldots,i_r,j_r\}}(\lambda)[\mathbf{n}_{i_s,j_s} = 0, \forall s \in \{1, 2 \ldots, r\}]}{Z_n^S(\lambda)}
$$

$$
= (1 + o(1)) \, d_n(\lambda)^\ell \frac{Z_n^{S\Delta\{i_1,j_1,\ldots,i_r,j_r\}}(\lambda)}{Z_n^S(\lambda)}
$$

$$
= (1 + o(1)) \, d_n(\lambda)^\ell \, (c_n n)^{2k-m} \frac{\langle \varphi^m \rangle_0}{\langle \varphi^{2k} \rangle_0},
$$

where we used that,

$$
\frac{Z_n^{S\Delta\{i_1,j_1,\ldots,i_r,j_r\}}(\lambda)[\mathbf{n}_{i_s,j_s} = 0, \forall s \in \{1, 2 \ldots, r\}]}{Z_n^{S\Delta\{i_1,j_1,\ldots,i_r,j_r\}}(\lambda)} = \mathbb{P}_{n,\lambda}^S\left[\mathbf{n}_{i_s,j_s} = 0, \forall s \in \{1, 2 \ldots, r\}\right]
$$

$$
= 1 - o(1).
$$

Summing these probabilities we see that the main contribution comes from the case where $i_1, j_1, i_2, j_2, \ldots, i_r, j_r \in V_n \setminus S$ are all distinct. Thus,

$$
\mathbb{E}_{n,\lambda}^S\left[\frac{\mathscr{E}^r}{(c_n n)^{2r}}\right] = (1 + o(1))\frac{n^{2r}}{2^r} d_n(\lambda)^r (c_n n)^{-4r} \frac{\langle \varphi^{2k+2r} \rangle_0}{\langle \varphi^{2k} \rangle_0}
$$

$$
= (1 + o(1))\frac{\tilde{g}^{4r}}{2^r} \frac{\langle \varphi^{2k+2r} \rangle_0}{\langle \varphi^{2k} \rangle_0}
$$

$$
= (1 + o(1))\mathbb{E}\left[\frac{\tilde{g}^{4r}}{2^r} Z_k^{2r}\right],
$$

where Z_k is a random variable with density $\frac{\varphi^{2k}}{\langle \varphi^{2k} \rangle_0}\mathrm{d}\rho_{g,a}$.

The convergence of all moments implies that the sequence $\left(\mathscr{N}(\mathbf{n})/(c_n n)^2\right)_{n \geq 1}$ is tight, and any limiting random variable has r-th moment given by $\frac{\tilde{g}^{4r}}{2^r} \frac{\langle \varphi^{2k+2r} \rangle_0}{\langle \varphi^{2k} \rangle_0}$. Since $\frac{\tilde{g}^4}{2} Z_k^2$ has finite exponential moments, it follows that there exists a unique distribution with r-th moment given by $\frac{\tilde{g}^{4r}}{2^r} \frac{\langle \varphi^{2k+2r} \rangle_0}{\langle \varphi^{2k} \rangle_0}$. Thus $\left(\mathscr{N}(\mathbf{n})/(c_n n)^2\right)_{n \geq 1}$ converges to $\frac{\tilde{g}^4}{2} Z_k^2$, as desired. □

3 Scaling limit of the cluster size distribution

In this section, we will use the results of Section 2 and Appendix A to prove Theorem 1.1.

We begin with a definition.

Definition 3.1. *Let $n \geq 1$. Let S be an even subset of V_n and $\mathsf{P} = \{P_1, \ldots, P_r\} \in \mathscr{P}_{\mathrm{even}}(S)$. Let $\mathbf{a}, \mathbf{b} \in (\mathbb{R}_{\geq 0})^r$ with $a_i \leq b_i$ for each $1 \leq i \leq r$. Define the subset $\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P})$ of Ω_n as*

$$
\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}) := \left\{\mathbf{n} \in \Omega_n, \mathfrak{P}(\mathbf{n}) = \mathsf{P}, \ a_i\sqrt{n} \leq |\mathscr{C}_{P_i}(\mathbf{n})| \leq b_i\sqrt{n} \, \forall \, 1 \leq i \leq r\right\},
$$

where for each $1 \leq i \leq r$, $\mathscr{C}_{P_i}(\mathbf{n})$ is the cluster of $\mathbf{n}$ containing the elements of P_i.

As we will see, it is sufficient to study the above events to obtain Theorem 1.1. In fact, the latter will be a consequence of the following result.

Proposition 3.2. *Let* $\lambda \in \mathbb{R}$ *and* $k \geq 1$. *Set* $S := \{1, \ldots, 2k\}$ *and consider* $\mathsf{P} = \{P_1, \ldots, P_r\} \in \mathscr{P}_{\mathrm{even}}(S)$. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{\geq 0}^r$ *with* $a_i \leq b_i$ *for each* $1 \leq i \leq r$. *Then, the following limits exist*

$$\lim_{n \to \infty} \mathbb{P}_{n,\lambda}^S[\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P})], \qquad \lim_{n \to \infty} \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P})].$$

Moreover, both limits are continuous as functions of $\mathbf{a}$ *and* $\mathbf{b}$.

We now show how to obtain our main result using Proposition 3.2.

Proof of Theorem 1.1. We give the proof for the case of $\mathbb{P}_{n,\lambda}^S$, with the proof for the double current measure being similar. The main observation we need to make is that with high probability the clusters of the sources always remain of order at most $(c_n n)^2 \asymp \sqrt{n}$. This fact follows immediately from the results of Section 2. Indeed, using Lemma 2.2 (i) and Markov's inequality, we get the existence of $C > 0$ such that, for any $A > 0$,

$$\limsup_{n \to \infty} \mathbb{P}_{n,\lambda}^S \left[\cup_{1 \leq i \leq 2k} \{ |\mathscr{C}_i(\mathbf{n})| \geq A\sqrt{n} \} \right] \leq \frac{2kC}{A}.$$

In particular, this proves that the sequence of random variables

$$\left(\frac{|\mathscr{C}_1(\mathbf{n})|}{\sqrt{n}}, \ldots, \frac{|\mathscr{C}_{2k}(\mathbf{n})|}{\sqrt{n}}, \mathfrak{P}(\mathbf{n}) \right)_{n \geq 1}$$

is tight on $(\mathbb{R}_{\geq 0})^{2k} \times \mathscr{P}_{\mathrm{even}}(S)$ (equipped with the standard product σ-algebra). Proposition 3.2 then fully characterises the limit which yields the desired convergence. The fact that the first $2k$ coordinates of the limiting random variables $X_{k,\lambda}^{\mathsf{s}}$ and $X_{k,\lambda}^{\mathsf{d}}$ are strictly positive almost surely is a consequence of the second part of Proposition 3.2, since the event $\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P})$ is empty if one of the b_i's is 0. $\qquad\square$

It remains to prove Proposition 3.2. Below, we fix $g > 0$, $\lambda \in \mathbb{R}$ and $k \geq 1$. We recall that $S = \{1, \ldots, 2k\}$. We work on the complete graph K_n with inverse temperature $d_n(\lambda)$ and study the measures $\mathbb{P}_{n,\lambda}^S$ and $\mathbb{P}_{n,\lambda}^{S,\emptyset}$.

Let $\mathsf{P} = \{P_1, \ldots, P_r\} \in \mathscr{P}_{\mathrm{even}}(S)$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{\geq 0}^r$ with $a_i \leq b_i$ for each $1 \leq i \leq r$. We begin by analysing the typical form of a current $\mathbf{n}$ which belongs to $\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P})$. The results of Section 2 motivate the introduction of the following event.

Definition 3.3 (Good event). *For* $A > 0$, *let* $\mathscr{G}_A$ *be the event that the current* $\mathbf{n}$ *satisfies the following properties: for every* $1 \leq i \leq 2k$ *the cluster* $\mathscr{C}_i(\mathbf{n})$ *has only simple edges (i.e. edges* e *with* $\mathbf{n}_e = 1$) *and* $\Delta_{\mathbf{n}}(i) = 1$, *there are no vertices of degree 5 or larger, and the number of vertices of degree 4 is at most* A.

Recall that the elements of S^c always have even degree. This means that if $\mathbf{n} \in \mathscr{G}_A$, vertices in S^c have degree 0, 2 or 4 in $\mathbf{n}$. Lemmas 2.2 and 2.4 imply that there exists $C > 0$ such that for all $n \geq 1$,

$$\mathbb{P}_{n,\lambda}^S[\mathscr{G}_A] \geq 1 - \frac{C}{A} - \frac{C}{\sqrt{n}}, \qquad \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathscr{G}_A] \geq 1 - \frac{C}{A} - \frac{C}{\sqrt{n}}. \tag{3.1}$$

As a result, we can restrict our attention to currents $\mathbf{n}$ satisfying $\mathbf{n} \in \mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}) \cap \mathscr{G}_A$. Let $\mathbf{n}$ be such a current. Denote by $B(\mathbf{n})$ the set of vertices of degree 4 in $\cup_{i=1}^r \mathscr{C}_{P_i}(\mathbf{n})$. Then, each cluster $\mathscr{C}_{P_i}(\mathbf{n})$ consists of a union of self-avoiding walks whose internal (meaning not the endpoints) vertices have degree 2 in $\mathbf{n}$, together with a collection of cycles of

length at least 3 which contain exactly one element of $B(\mathbf{n}) \cap \mathscr{C}_{P_i}(\mathbf{n})$. Note that these self-avoiding walks begin and end at either an element of $B(\mathbf{n})$ or at a source. See Figure 1.

This observation motivates the following definition.

Definition 3.4 (Backbone multigraph). *Let $\mathbf{n} \in \mathscr{G}_A$ satisfy $\mathfrak{P}(\mathbf{n}) = \mathsf{P}$. Let $G = (V, E)$ be a multigraph (i.e. we allow multiple edges with the same endpoints and loops attached to a vertex) such that $V = \{1, 2, \ldots, v\}$ for some $2k \leq v \leq n$, all vertices in $\{1, 2, \ldots, 2k\}$ have degree 1, and all vertices in $\{2k+1, \ldots, v\}$ have degree 4, where we use the standard convention that loops contribute twice to the degree of a vertex. We say that G is a backbone multigraph of $\mathbf{n}$ if there exists an automorphism π of K_n which fixes each element of S such that: $\pi(V)$ consists of the union of the source set S with the set of vertices of degree 4 lying in one of the clusters $\mathscr{C}_{P_i}(\mathbf{n})$; the set $\pi(E)$ is obtained by contracting each self-avoiding walk or cycle in the above description of $\cup_{i=1}^{r} \mathscr{C}_{P_i}(\mathbf{n})$.*

We call G_i the connected component of G which contains P_i, and we write $\mathscr{B}(G)$ for the event that G is a backbone multigraph of $\mathbf{n}$. Moreover, we define $\Gamma(k, A, \mathsf{P})$ as the set of isomorphism classes of backbone multigraphs for the currents $\mathbf{n} \in \mathscr{G}_A$ such that $\mathfrak{P}(\mathbf{n}) = \mathsf{P}$, where we identify two backbone multigraphs G and G' if there is an isomorphism from G to G' which fixes each element of S.
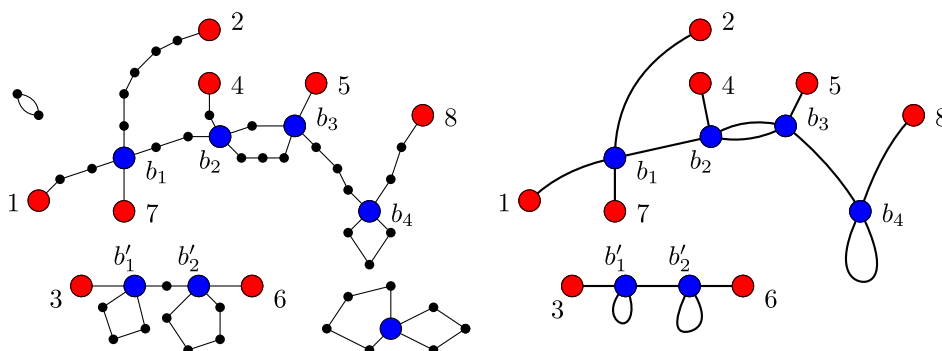

Figure 1: Left: the multigraph associated with a typical sample $\mathbf{n}$ of $\mathbb{P}_{n,\lambda}^{S}$ with $S = \{1, \ldots, 8\}$. Right: the backbone multigraph associated to $\mathbf{n}$. The sources are represented in red, the set $B(\mathbf{n})$ of vertices of degree 4 is represented in blue, and the vertices of degree 2 in black. The edges of multiplicity 2 do not appear in the backbone graph since they do not contribute to the cluster of the sources with high probability.

We are now equipped to prove Proposition 3.2. We will use the above observations to show that, under the measures of interest, the probabilities of events of the type $\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}) \cap \mathscr{G}_A \cap \mathscr{B}(G)$ for a fixed *admissible* (i.e. compatible with the event $\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}) \cap \mathscr{G}_A$) backbone multigraph G converge as $n \to \infty$. Proposition 3.2 will then follow from summing the above quantities among the (finite) set of admissible backbone multigraphs.

Proof of Proposition 3.2. We start with the simpler case of a single random current. As we will see, the case of double random currents can be handled similarly, with only a few modifications. We work on the complete graph K_n and consider the random current measure $\mathbb{P}_{n,\lambda}^{S}$ with inverse temperature $d_n(\lambda)$. Recall that $S := \{1, \ldots, 2k\}$ and that we fixed $\mathsf{P} = \{P_1, \ldots, P_r\} \in \mathscr{P}_{\mathrm{even}}(S)$ and $\mathbf{a} \leq \mathbf{b} \in (\mathbb{R}_{\geq 0})^r$.

With the above definitions in hands we have,

$$\mathbb{P}_{n,\lambda}^{S}[\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}), \mathscr{G}_A] = \sum_{[G] \in \Gamma(k, A, \mathsf{P})} \mathbb{P}_{n,\lambda}^{S}[\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}), \mathscr{G}_A, \mathscr{B}(G)],$$

where G is an arbitrary representative of the corresponding isomorphism class.

It remains to estimate the probability $\mathbb{P}^S_{n,\lambda}[\mathcal{I}_n(\mathbf{a},\mathbf{b},\mathsf{P}),\,\mathcal{G}_A,\,\mathcal{B}(G)]$. To this end, let us write m_i for the number of vertices of degree 2 in $\mathcal{C}_{P_i}(\mathbf{n})$, and m for the number of vertices of degree 2 in $\cup_{i=1}^r \mathcal{C}_{P_i}(\mathbf{n})$. Note that $m = \sum_{i=1}^r m_i$. We also let ℓ_i be the number of edges of G_i, so that $\ell := \sum_{i=1}^r \ell_i$ is the number of edges in G. Define, $\mathcal{H}_n(\mathbf{a},\mathbf{b},A,G,(m_i)_{1\le i\le r})$ to be the set of currents $\mathbf{n}$ which satisfy: $\mathbf{n} \in \mathcal{I}_n(\mathbf{a},\mathbf{b},\mathsf{P}) \cap \mathcal{G}_A \cap \mathcal{B}(G)$, and $\mathcal{C}_{P_i}(\mathbf{n})$ contains m_i vertices of degree 2.

Let $\mathcal{C}_S(\mathbf{n}) = \cup_{i=1}^{2k} \mathcal{C}_i(\mathbf{n})$. A current $\mathbf{n} \in \mathcal{H}_n(\mathbf{a},\mathbf{b},A,G,(m_i)_{1\le i\le r})$ can be written as:

$$\mathbf{n} = \mathbf{m}_1(\mathbf{n}) + \mathbf{m}_2(\mathbf{n}),$$

where $\mathbf{m}_1 = \mathbf{m}_1(\mathbf{n})$ has source set S and is supported on $\mathscr{P}_2(\mathcal{C}_S(\mathbf{n}))$, and $\mathbf{m}_2 = \mathbf{m}_2(\mathbf{n})$ is sourceless and is supported on $\mathscr{P}_2(\mathcal{C}_S(\mathbf{n})^c)$. Moreover, one has that $w(\mathbf{n}) = w(\mathbf{m}_1)w(\mathbf{m}_2)$ with $w(\mathbf{m}_1) = d_n(\lambda)^{m+\ell}$. Indeed, by definition $\mathbf{m}_1(\mathbf{n})$ only contains edges of multiplicity one. Hence, its weight is exactly $d_n(\lambda)^{|\{e,\,\mathbf{m}_1(e)>0\}|}$, and $|\{e,\,\mathbf{m}_1(e)>0\}|$ is nothing but the number of open edges in $\mathcal{C}_S(\mathbf{n})$. By the handshake lemma applied to G we get that $2\ell = 2k + 4(v-2k)$. Now, applying the handshake lemma to $\mathcal{C}_S(\mathbf{n})$ we get that its number of edges is equal to $k + m + 2(v-2k) = m + \ell$.[4] Hence, for each possible value $\mathbf{m}$ that $\mathbf{m}_1(\mathbf{n})$ may take,

$$\sum_{\mathbf{n}\in\mathcal{H}_n(\mathbf{a},\mathbf{b},A,G,(m_i)_{1\le i\le r}),\,\mathbf{m}_1(\mathbf{n})=\mathbf{m}} w(\mathbf{n}) = d_n(\lambda)^{m+\ell} \times \sum_{\mathbf{m}_2\in\Omega_{n-v-m},\,\partial\mathbf{m}_2=\emptyset} w_{n,\lambda}(\mathbf{m}_2). \quad (3.2)$$

Let $x = m/\sqrt{n}$. Notice that,

$$d_n(\lambda) = \frac{1}{n}\left(1 - \frac{\lambda}{\sqrt{n}}\right) = \frac{1}{n-v-m}\left(1 - (1+o(1))\frac{\lambda+x}{\sqrt{n-v-m}}\right),$$

where the $o(1)$ term goes to 0 uniformly over x in a bounded interval. Thus, one obtains that

$$\sum_{\mathbf{m}_2\in\Omega_{n-v-m},\,\partial\mathbf{m}_2=\emptyset} w_{n,\lambda}(\mathbf{m}_2) = Z^{\emptyset}_{n-v-m}\big[(\lambda+x)(1+o(1))\big]. \quad (3.3)$$

Since the right-hand side of (3.2) does not depend on the value of $\mathbf{m}_1(\mathbf{n})$, it remains to count the number of values $\mathbf{m}_1(\mathbf{n})$ may take for $\mathbf{n} \in \mathcal{H}_n(\mathbf{a},\mathbf{b},A,G)$. We claim that the number of possibilities for $\mathbf{m}_1(\mathbf{n})$ is equal to

$$\frac{1+o(1)}{2^L|\mathfrak{A}_G|}\sum_{m_1=a_1\sqrt{n}}^{b_1\sqrt{n}}\cdots\sum_{m_r=a_r\sqrt{n}}^{b_r\sqrt{n}}\binom{n-2k}{v-2k}(v-2k)!\binom{n-v}{m}$$

$$\times\binom{m}{m_1,\ldots,m_r}\prod_{i=1}^r m_i!\frac{m_i^{\ell_i-1}}{(\ell_i-1)!} =$$

$$\frac{1+o(1)}{2^L|\mathfrak{A}_G|}\sum_{m_1=a_1\sqrt{n}}^{b_1\sqrt{n}}\cdots\sum_{m_r=a_r\sqrt{n}}^{b_r\sqrt{n}}\binom{n-2k}{v-2k}(v-2k)!\binom{n-v}{m}m!\prod_{i=1}^r\frac{m_i^{\ell_i-1}}{(\ell_i-1)!}, \quad (3.4)$$

where L is the number of loops of G, and $\mathfrak{A}_G$ is the set of automorphisms of G which fix each element of S. To see this, note that we have $\binom{n-2k}{v-2k}$ ways to choose the vertices of degree 4 in $\mathcal{C}_S(\mathbf{n})$, and $\frac{(v-2k)!}{|\mathfrak{A}_G|}$ ways to associate each of these vertices to a vertex of G in a unique way, see Figure 2.

---

[4]An alternative way to see this is to observe that every self-avoiding walk and every cycle that corresponds to an edge or a loop of G contains one more edge than vertices of degree 2.
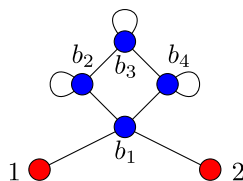
Figure 2: A backbone multigraph G with two automorphisms in $\mathfrak{A}_G$.

Moreover, there are $\binom{n-v}{m}$ ways to choose the m vertices of degree 2. Next, we split these m vertices into ordered sets of size $m_1, m_2, \ldots, m_r$, corresponding to the vertices of degree 2 in each $\mathscr{C}_{P_i}(\mathbf{n})$, for which there are $\binom{m}{m_1, \ldots, m_r}$ possibilities. Then, we split each of these m_i vertices into ℓ_i sets corresponding to the ℓ_i edges of G_i, and order them within the sets. These ordered sets may be empty except for those corresponding to the cycles of $\mathscr{C}_S(\mathbf{n})$ that are attached to a vertex of degree 4. Thus, the number of possibilities to split them into ℓ_i ordered sets is sandwiched between[5] $m_i!\binom{m_i+\ell_i-1}{\ell_i-1} - Lm_i!\binom{m_i+\ell_i-2}{\ell_i-2} = (1+o(1))\, m_i!\frac{m_i^{\ell_i-1}}{(\ell_i-1)!}$ and $m_i!\binom{m_i+\ell_i-1}{\ell_i-1} = (1+o(1))\, m_i!\frac{m_i^{\ell_i-1}}{(\ell_i-1)!}$. Finally, the term 2^{-L} accounts for the cyclic ordering of the vertices in the cycles corresponding to the loops of G.

Gathering (3.2), (3.3), and (3.4) yields that $\mathbb{P}_{n,\lambda}^S[\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}),\ \mathscr{G}_A,\ \mathscr{B}(G)]$ is equal to

$$
\frac{1+o(1)}{Z_n^S(\lambda)2^L|\mathfrak{A}_G|} \sum_{m_1=a_1\sqrt{n}}^{b_1\sqrt{n}} \cdots \sum_{m_r=a_r\sqrt{n}}^{b_r\sqrt{n}} \binom{n-2k}{v-2k}(v-2k)!
$$
$$
\times \binom{n-v}{m} m! \prod_{i=1}^{r} \frac{m_i^{\ell_i-1}}{(\ell_i-1)!}\, d_n(\lambda)^{m+\ell} Z_{n-v-m}^{\emptyset}\big[(\lambda+x)(1+o(1))\big]. \quad (3.5)
$$

We now proceed with the asymptotic analysis of each of these terms. Note that for $x_i = m_i/\sqrt{n}$ we have

$$
\binom{n-2k}{v-2k}(v-2k)! = (1+o(1))n^{v-2k}, \quad d_n(\lambda)^{m+\ell} = (1+o(1))\, n^{-m-\ell}e^{-\lambda x},
$$
$$
m_i^{\ell_i-1} = x_i^{\ell_i-1}n^{(\ell_i-1)/2}.
$$

Then, by the asymptotic estimate of the partition function obtained in Proposition A.2,

$$
Z_{n-v-m}^{\emptyset}\big[(\lambda+x)(1+o(1))\big] = (1+o(1))\frac{n^{1/4}}{\sqrt{2\pi e}}\int_{\mathbb{R}} e^{-\frac{1}{12}s^4-\frac{\lambda+x}{2}s^2}\,\mathrm{d}s,
$$

where the $o(1)$ term on the right-hand side is uniform in x thanks to the monotonicity of $Z_k^{\emptyset}(u)$ in u. Moreover, we have

$$
\log\left\{\frac{(n-v)!}{(n-v-m)!n^m}\right\} = \sum_{i=v}^{v+m-1}\log\left(\frac{n-i}{n}\right) = \sum_{i=v}^{v+m-1}\left(-\frac{i}{n}+O\left(\frac{i^2}{n^2}\right)\right) = -\frac{m^2}{2n}+o(1),
$$

hence,

$$
\binom{n-v}{m}m! = (1+o(1))\, e^{-x^2/2}n^m.
$$

---

[5]Here the term $m_i!\binom{m_i+\ell_i-2}{\ell_i-2}$ upper bounds the number of possibilities in which a particular set corresponding to one of the loops is empty, and we recall that L is the number of loops.

Multiplying all these together, and recalling that $2\ell = 4(v - 2k) + 2k$, we see that the exponent of n is equal to

$$v - 2k + (\ell - r)/2 - \ell + 1/4 = v - 2k - \ell/2 - (r/2 - 1/4) = -k/2 - (r/2 - 1/4).$$

Thus, using Riemann approximation, we obtain that (3.5) is equal to

$$(1 + o(1)) \frac{n^{-k/2+1/4}}{\sqrt{2\pi e} 2^L |\mathfrak{A}_G|} \frac{1}{n^{r/2}} \sum_{m_1 = a_1 \sqrt{n}}^{b_1 \sqrt{n}} \cdots \sum_{m_r = a_r \sqrt{n}}^{b_r \sqrt{n}} e^{-x^2/2 - \lambda x} \prod_{i=1}^{r} \frac{x_i^{\ell_i - 1}}{(\ell_i - 1)!} \int_{\mathbb{R}} e^{-\frac{1}{12} s^4 - \frac{\lambda + x}{2} s^2} \mathrm{d}s$$

$$= (1 + o(1)) \frac{n^{-k/2+1/4}}{\sqrt{2\pi e} 2^L |\mathfrak{A}_G|} \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} \int_{\mathbb{R}} \prod_{i=1}^{r} \frac{x_i^{\ell_i - 1}}{(\ell_i - 1)!} e^{-\frac{1}{2} x^2 - \lambda x - \frac{1}{12} s^4 - \frac{\lambda + x}{2} s^2} \prod_{i=1}^{r} \mathrm{d}x_i \mathrm{d}s.$$

By Proposition A.2, $Z_n^S(\lambda) = (1 + o(1)) K_S(\lambda) n^{-k/2+1/4}$ for some constant $K_S(\lambda) > 0$, hence $\mathbb{P}_{n,\lambda}^S[\mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}), \mathscr{G}_A, \mathscr{B}(G)]$ is asymptotically equal to

$$\frac{1 + o(1)}{K_S(\lambda) \sqrt{2\pi e} 2^L \mathfrak{A}_G} \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} \int_{\mathbb{R}} \prod_{i=1}^{r} \frac{x_i^{\ell_i - 1}}{(\ell_i - 1)!} e^{-\frac{1}{2} x^2 - \lambda x - \frac{1}{12} s^4 - \frac{\lambda + x}{2} s^2} \prod_{i=1}^{r} \mathrm{d}x_i \mathrm{d}s. \quad (3.6)$$

This completes the proof in the case of a single random current, and now we proceed with the proof in the case of the measure $\mathbb{P}_{n,\lambda}^{S,\emptyset}$. Introduce the event $\mathscr{F}_n$ that $\Delta_{\mathbf{n}}(i) = 0$ for every $i \in S$, and recall that Lemma 2.4 (ii) implies that $\mathbb{P}_{n,\lambda}^{\emptyset}[\mathscr{F}_n] = 1 - O(1/\sqrt{n})$. Thus, it suffices to estimate the probability $\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}), \mathbf{n}_1 + \mathbf{n}_2 \in \mathscr{G}_A, \mathbf{n}_2 \in \mathscr{F}_n, \mathscr{B}(G)]$. Using the same notations as above, we see that the weight of $\mathbf{m}_1(\mathbf{n}_1 + \mathbf{n}_2)$ is again equal to $d_n(\lambda)^{m+\ell}$, and the main differences in the estimates come from the number of possibilities for $\mathbf{m}_1(\mathbf{n}_1 + \mathbf{n}_2)$.

Given a realisation of the clusters $\mathscr{C}_{P_i}(\mathbf{n}_1 + \mathbf{n}_2)$ there are several ways to assign each open edge of these clusters to either $\mathbf{n}_1$ or $\mathbf{n}_2$. Observe that if $u \in \mathscr{C}_{P_i}(\mathbf{n}_1 + \mathbf{n}_2)$ is a vertex of degree 2 with neighbours z and w, then either $\mathbf{n}_1(u, z) = \mathbf{n}_1(u, w) = 1$ or $\mathbf{n}_2(u, z) = \mathbf{n}_2(u, w) = 1$. As a consequence, $\mathscr{C}_{P_i}(\mathbf{n}_1 + \mathbf{n}_2)$ consists of self-avoiding walks and cycles which are open either in $\mathbf{n}_1$ or in $\mathbf{n}_2$. This implies that the number of ways to assign each open edge of $\mathscr{C}_S(\mathbf{n}_1 + \mathbf{n}_2)$ to either $\mathbf{n}_1$ or $\mathbf{n}_2$ is equal to number of ways to colour the edges G red and blue in such a way that the edges incident to S are all coloured red, and the graph consisting of red edges has even degree for every $u \in V \setminus S$, where we recall that loops contribute twice. We denote this number C_G. We can now conclude that the number of possibilities for $\mathbf{m}_1(\mathbf{n})$ is equal to

$$\frac{(1 + o(1)) C_G}{2^L |\mathfrak{A}_G|} \sum_{m_1 = a_1 \sqrt{n}}^{b_1 \sqrt{n}} \cdots \sum_{m_r = a_r \sqrt{n}}^{b_r \sqrt{n}} \binom{n - 2k}{v - 2k} (v - 2k)! \binom{n - v}{m} m! \prod_{i=1}^{r} \frac{m_i^{\ell_i - 1}}{(\ell_i - 1)!}.$$

Using once again Proposition A.2,

$$Z_{n-v-m}^{\emptyset,\emptyset}[(\lambda + x)(1 + o(1)] = (1 + o(1)) \frac{n^{1/2}}{2\pi e} \left(\int_{\mathbb{R}} e^{-\frac{1}{12} s^4 - \frac{\lambda + x}{2} s^2} \mathrm{d}s \right)^2.$$

Arguing as above, we find that $\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathscr{I}_n(\mathbf{a}, \mathbf{b}, \mathsf{P}), \mathbf{n}_1 + \mathbf{n}_2 \in \mathscr{G}_A, \mathbf{n}_2 \in \mathscr{F}_n, \mathscr{B}(G)]$ is equal to

$$\frac{(1 + o(1)) C_G 2^{-L}}{K_S'(\lambda) \sqrt{2\pi e} |\mathfrak{A}_G|} \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} \left(\int_{\mathbb{R}} e^{-\frac{1}{12} s^4 - \frac{\lambda + x}{2} s^2} \mathrm{d}s \right)^2 \prod_{i=1}^{r} \frac{x_i^{\ell_i - 1}}{(\ell_i - 1)!} e^{-\frac{1}{2} x^2 - \lambda x} \prod_{i=1}^{r} \mathrm{d}x_i \quad (3.7)$$

for some constant $K_S'(\lambda) > 0$. $\qquad\square$

We now turn to the proofs of Corollaries 1.3 and 1.4.

Proof of Corollary 1.3. The convergence in distribution is a direct consequence of Theorem 1.1. The existence of Gaussian tails for the limiting measures follows from Proposition 2.6. Indeed, looking at the case of a single current (the double current case is similar), one has

$$\left| \left(\frac{|\mathscr{C}_1(\mathbf{n}_1)|}{\sqrt{n}}, \ldots, \frac{|\mathscr{C}_{2k}(\mathbf{n}_1)|}{\sqrt{n}} \right) \right|_\infty \leq \frac{\mathscr{N}(\mathbf{n}_1)}{\sqrt{n}}, \tag{3.8}$$

where $\mathscr{N}(\mathbf{n}_1)$ was defined in Proposition 2.6 and where $|\cdot|_\infty$ is the ℓ^∞ norm on $\mathbb{R}^d$. Now, by Proposition 2.6, the sequence of random variables $(\frac{\mathscr{N}(\mathbf{n}_1)}{\sqrt{n}})_{n \geq 1}$ converges in law to a multiple of Z_k^2, where Z_k has quartic tails, and hence has Gaussian tails. $\qquad \square$

Proof of Corollary 1.4. Again, Theorem 1.1 gives convergence in law of $(\mathfrak{P}(\mathbf{n}))_{n \geq 1}$. The second part of the statement follows from the explicit formula for the limit obtained in the proof of Proposition 3.2. $\qquad \square$

Remark 3.5 (Study of the clusters of points which are not sources)**.** The above analysis does not exclude the possibility of existence of clusters of size of order $\sqrt{n}$ which contain no sources. It is possible to study these clusters via the following observation. Notice that removing an edge of weight 1 creates at most two new sources. More precisely, if $i, j \in V_n$, we can asymptotically relate the measure $\mathbb{P}_{n,\lambda}^S[\cdot \mid \mathbf{n}_{i,j} = 1]$ to the measure $\mathbb{P}_{n,\lambda}^{S\Delta\{i,j\}}$. This allows to study the scaling limit of the cluster size distribution of i (or j) under the measure $\mathbb{P}_{n,\lambda}^S[\cdot \mid \mathbf{n}_{i,j} = 1]$.

4 Consequences for the φ^4 model

The goal of this section is to give a new perspective on the switching lemma for the φ^4 model obtained in [GPPS22], and recalled below. We begin with some useful definitions.

Let $\Lambda = (V, E)$ be a finite graph. Let $g > 0$ and $a \in \mathbb{R}$. The ferromagnetic φ^4 model on Λ with parameters (g, a) at inverse temperature $\beta > 0$ is defined by the finite volume Gibbs equilibrium state: for $F : \mathbb{R}^V \to \mathbb{R}$,

$$\langle F(\varphi) \rangle_{\Lambda, g, a, \beta} = \frac{1}{Z_{\Lambda, g, a, \beta}} \int F(\varphi) \exp\left(-\beta H_\Lambda(\varphi)\right) \prod_{x \in V} \mathrm{d}\rho_{g,a}(\varphi_x),$$

where $Z_{\Lambda, g, a, \beta}$ is the partition function and

$$H_\Lambda(\varphi) := - \sum_{\{x,y\} \in E} \varphi_x \varphi_y.$$

Choosing $a > 0$, it is clear that the limit of $\rho_{g,a}$ as g goes to 0 is the Gaussian law ρ_a given by

$$\mathrm{d}\rho_a(t) = \frac{1}{z_a} e^{-at^2} \mathrm{d}t.$$

We will denote by $\langle \cdot \rangle_{0,a}$ the average with respect to ρ_a. Hence, if we additionally require that $a > 4d\beta$, the corresponding φ^4 measure on Λ converges (as g goes to 0) to the rescaled massive discrete Gaussian free field (GFF) of mass $m = a - 4d\beta$, defined via the expectation values

$$\langle F \rangle_{\Lambda, a, \beta}^{\mathsf{GFF}} := \frac{1}{Z_{\Lambda, a, \beta}^{\mathsf{GFF}}} \int_{\varphi \in \mathbb{R}^V} F(\varphi) \exp\left(-\beta H_\Lambda(\varphi)\right) \prod_{x \in V} \mathrm{d}\rho_a(\varphi_x), \tag{4.1}$$

where $Z^{\mathsf{GFF}}_{\Lambda,a,\beta}$ is the partition function.

Moreover, choosing $a = -2g$ and taking the limit as $g \to \infty$,

$$\frac{\delta_{-1} + \delta_1}{2} = \lim_{g \to \infty} \frac{1}{z_{g,-2g}} e^{-g(\varphi^2 - 1)^2 + g} \mathrm{d}\varphi,$$

the corresponding φ^4 measure on Λ converges to the Ising model on Λ defined via the expectation values

$$\langle F \rangle^{\mathsf{Ising}}_{\Lambda,\beta} := \frac{1}{Z^{\mathsf{Ising}}_{\Lambda,\beta}} \sum_{\sigma \in \{\pm 1\}^V} F(\sigma) \exp\left(-\beta H_\Lambda(\sigma)\right). \tag{4.2}$$

The above observations suggest that the φ^4 model interpolates between the discrete GFF and the Ising model. Interestingly, Theorem 1.1 allows to extend this observation at the level of the switching lemma for φ^4: under the appropriate limits mentioned above, one recovers Wick's law (on the GFF side), or the switching lemma (on the Ising model side). More precisely, the proof will reduce to showing that the random variable $P^{\mathsf{d}}_{k,\lambda}$ converges in distribution to either: the uniform measure over pairings in the GFF limit, or the Dirac measure $\delta_{\{1,\dots,2k\}}$ in the Ising limit.

This gives an interpretation[6] of the switching lemma for φ^4 as a type of Wick's law for the non-Gaussian φ^4 field.

4.1 Switching lemma for φ^4 and full description of the tangling probabilities

We recall the switching lemma for φ^4 in its simplest case and import some terminology from [GPPS22].

Let $\Lambda = (V, E)$ be a finite graph. Write $\Omega_\Lambda := \mathbb{N}^E$ for the space of currents on Λ. Given $\mathbf{n} \in \Omega_\Lambda$, let $\Delta_{\mathbf{n}}(x) := \sum_{y \in V} \mathbf{n}_{x,y}$ be the $\mathbf{n}$-degree of x. We define

$$\mathscr{M}(\Lambda) := \left\{ A \in \mathbb{N}^V, \sum_{x \in V} A_x \text{ is even} \right\}$$

to be the set of admissible moments on Λ, and write $\partial A := \{x \in V : A_x \text{ is odd}\}$. We will write $A = \emptyset$ if $A_x = 0$ for all $x \in \Lambda$. Given $A, B \in \mathscr{M}(\Lambda)$, define $A + B \in \mathscr{M}(\Lambda)$ by

$$(A + B)_x := A_x + B_x$$

for all $x \in V$.

Fix $A, B \in \mathscr{M}(\Lambda)$. Let $\mathbf{n}_1, \mathbf{n}_2 \in \Omega_\Lambda$ satisfy $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$. For each $x \in V$, we define the block $\mathscr{B}_x(\mathbf{n}_1, A)$ as follows: for each $y \in V$, it contains $\mathbf{n}_{x,y}$ points labelled $(xy(k))_{1 \le k \le \mathbf{n}_1(x,y)}$, and A_x points labelled $(ya(k))_{1 \le k \le A_x}$. We also define $\mathscr{B}_x(\mathbf{n}_2, B)$ similarly. To define a notion of tangled currents for $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, we define $\mathscr{B}_x(\mathbf{n}, A, B)$ in the following way: for every $y \in \Lambda$, it contains $\mathbf{n}_{x,y}$ points labelled $(xy(k))_{1 \le k \le \mathbf{n}_{x,y}}$, A_x points labelled $(xa(k))_{1 \le k \le A_x}$, and B_x points labelled $(xb(k))_{1 \le k \le B_x}$. There is a natural injection of $\mathscr{B}_x(\mathbf{n}_1, A)$ and $\mathscr{B}_x(\mathbf{n}_2, B)$ in $\mathscr{B}_x(\mathbf{n}, A, B)$. We write $\mathscr{T}_{\mathbf{n},A,B}(x)$ for the set of even partitions of the set $\mathscr{B}_x(\mathbf{n}, A, B)$ whose restriction to $\mathscr{B}_x(\mathbf{n}_1, A)$ and $\mathscr{B}_x(\mathbf{n}_2, B)$ is also an even partition, i.e. for every $\mathsf{P} = \{P_1, P_2, \dots, P_r\} \in \mathscr{T}_{\mathbf{n},A,B}(x)$, both $|P_i \cap \mathscr{B}_x(\mathbf{n}_1, A)|$ and $|P_i \cap \mathscr{B}_x(\mathbf{n}_2, B)|$ are even numbers. The elements of $\mathscr{T}_{\mathbf{n},A,B}(x)$ are called *admissible* partitions. We also define

$$\mathscr{T}_{\mathbf{n},A,B} = \bigotimes_{x \in V} \mathscr{T}_{\mathbf{n},A,B}(x)$$

---

[6]This property can already be seen at the level of the formulas obtained for the tangling probabilities — see (3.6) and (3.7), which are expressed as a sum over backbone multigraphs, similarly to Gaussian correlations being expressed as a sum over pairings.
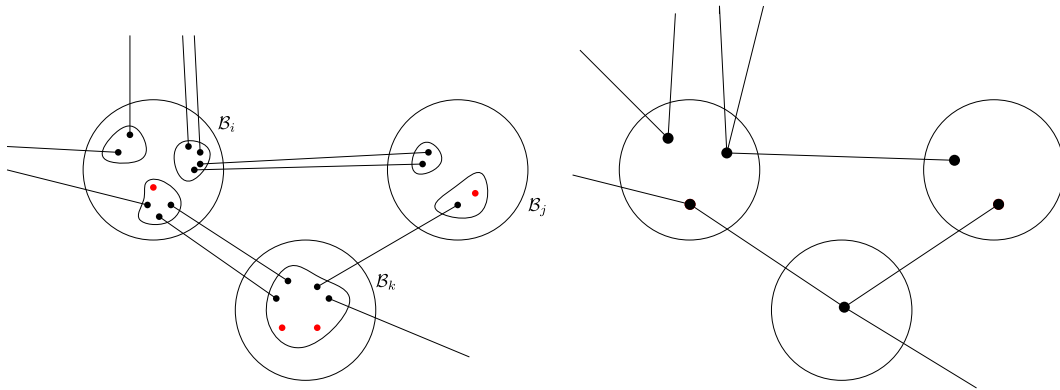
Figure 3: An example of a tangled current $(\mathbf{n}, t)$ with a moment function A satisfying $A_i = 1, A_j = 1$, and $A_k = 2$. On the left picture, the circles represent the blocks $\mathscr{B}(\mathbf{n}, A)$; the small ellipses inside represent the elements of the partitions t. Notice that i, j are sources of $\mathbf{n}$, and the red vertices correspond to the vertices associated with A. On the right picture, the multigraph $\mathscr{H}(\mathbf{n}, t, A)$ is obtained by identifying vertices in the same ellipses.

to be the set of admissible *tanglings*. For any admissible A and B and any current $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$ with $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$, we let $\mathscr{H}(\mathbf{n}, A, B)$ be the graph with vertex set

$$\bigcup_{z \in V} \mathscr{B}_x(\mathbf{n}, A, B),$$

and edge set

$$\bigcup_{\{x,y\} \in E} \{\{xy(k), yx(k)\},\ 1 \le k \le \mathbf{n}_{x,y}\}.$$

Given any tangling $t \in \mathscr{T}_{\mathbf{n}, A, B}$, the graph $\mathscr{H}(\mathbf{n}, A, B)$ naturally projects to a multigraph $\mathscr{H}(\mathbf{n}, t, A, B)$ where for each block $\mathscr{B}_x(\mathbf{n}, A, B)$, vertices in the same partition class given by t_x are identified as one. See Figure 3.

We define $\Omega_\Lambda^{\mathscr{T}}$ to be the set of tangled currents on Λ. Fix $\beta > 0$. Before recalling the switching lemma for φ^4, we need a definition.

Definition 4.1. Let $B \in \mathscr{M}(\Lambda)$. Define $\mathscr{F}_B^\Lambda$ to be the subset of $\Omega_\Lambda^{\mathscr{T}}$ consisting of all tangled currents $(\mathbf{n}, t)$ for which every connected component intersects $\{xb(k) : x \in \Lambda, 1 \le k \le B_x\}$ an even number of times. In words, $\mathscr{F}_B^\Lambda$ consists of all tangled currents that pair the elements of B in its associated multigraph. The set $\mathscr{F}_B^\Lambda$ is measurable with respect to the natural σ-algebra on $\Omega_\Lambda^{\mathscr{T}}$.

We let

$$w_{A,\beta}(\mathbf{n}) := \prod_{\{x,y\} \in E} \frac{\beta^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!} \left\langle \varphi^{\Delta_{\mathbf{n}}(x) + A_x} \right\rangle_{0,g,a}. \tag{4.3}$$

Theorem 4.2 (Switching lemma for φ^4, [GPPS22, Theorem 3.11])**.** *Let $A, B \in \mathscr{M}(\Lambda)$. For every $\mathbf{n}_1, \mathbf{n}_2 \in \Omega_\Lambda$ satisfying $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$ and for every $z \in \Lambda$, there are probability measures $\rho_{z,A,B}^{\mathbf{n}_1,\mathbf{n}_2} = \rho_{z,A,B,g,a}^{\mathbf{n}_1,\mathbf{n}_2}$ on $\mathscr{T}_{\mathbf{n},A,B}(z)$ such that the following holds. For any measurable and bounded $F : \Omega_\Lambda^{\mathscr{T}} \to \mathbb{R}$,*

$$\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A,\beta}(\mathbf{n}_1) w_{B,\beta}(\mathbf{n}_2) \rho_{A,B}^{\mathbf{n}_1,\mathbf{n}_2} \big[F(\mathbf{n}_1 + \mathbf{n}_2, t) \big]$$

$$= \sum_{\substack{\partial \mathbf{n}_1 = \partial(A+B) \\ \partial \mathbf{n}_2 = \emptyset}} w_{A+B,\beta}(\mathbf{n}_1) w_{\emptyset,\beta}(\mathbf{n}_2) \rho_{A+B,\emptyset}^{\mathbf{n}_1,\mathbf{n}_2} \big[F(\mathbf{n}_1 + \mathbf{n}_2, t) \mathbb{1}_{\mathscr{F}_B^\Lambda} \big],$$

where

$$\rho_{A,B}^{\mathbf{n}_1,\mathbf{n}_2} := \bigotimes_{z \in \Lambda} \rho_{z,A,B}^{\mathbf{n}_1,\mathbf{n}_2}, \text{ and } \rho_{A+B,\emptyset}^{\mathbf{n}_1,\mathbf{n}_2} := \bigotimes_{z \in \Lambda} \rho_{z,A+B,\emptyset}^{\mathbf{n}_1,\mathbf{n}_2},$$

are measures on $\mathscr{T}_{\mathbf{n},A,B}$ and $\mathscr{T}_{\mathbf{n},A+B,\emptyset}$ respectively, and $\mathscr{F}_B^\Lambda$ is the event defined above.

In [GPPS22], the probability measures $\rho_{z,A,B}^{\mathbf{n}_1,\mathbf{n}_2}$ were obtained by taking subsequential limits of certain measures defined below. Our main result implies full convergence of these measures. We will need the following definitions.

If S_1, S_2 are two even and disjoint subsets of V_n, we let $\tilde{\rho}_{n,\lambda}^{S_1,S_2}$ be the measure $\mathbb{P}_{n,\lambda}^{S_1,S_2}$ quotiented by the relation:

$$\mathbf{n} \sim \mathbf{m} \iff \mathfrak{P}(\mathbf{n}) = \mathfrak{P}(\mathbf{m}).$$

In words, $\tilde{\rho}_{n,\lambda}^{S_1,S_2}$ is the law of $\mathfrak{P}(\mathbf{n})$ for $\mathbf{n}$ sampled according to $\mathbb{P}_{n,\lambda}^{S_1,S_2}$. Note that the measure $\tilde{\rho}_{n,\lambda}^{S_1,S_2}$ is supported on the set $\mathscr{P}_{\text{even}}(S_1, S_2)$ which consists of even partitions $\mathsf{P} = \{P_1, \ldots, P_r\}$ of $S_1 \sqcup S_2$ which are compatible[7] with both S_1 and S_2 in the sense that: for all $1 \le i \le r$, $|P_i \cap S_1|$ and $|P_i \cap S_2|$ are even.

For $x \in V$, the measure $\rho_{x,A,B}^{\mathbf{n}_1,\mathbf{n}_2}$ is defined in [GPPS22] as a subsequential limit of the sequence of measures $(\tilde{\rho}_{n,\lambda}^{S_1,S_2})_{n \ge 1}$ where $|S_1| = A_x + \Delta_{\mathbf{n}_1}(x)$ and $|S_2| = B_x + \Delta_{\mathbf{n}_2}(x)$ (and S_1, S_2 are disjoint).

As mentioned in (1.2), the switching lemma (for the Ising model), gives a connection between $\tilde{\rho}_{n,\lambda}^{S_1,S_2}$ and $\tilde{\rho}_{n,\lambda}^{S_1 \sqcup S_2,\emptyset}$. Indeed, notice that the event $\mathscr{F}_{S_2}$ defined below (1.2) satisfies $\{\mathbf{n}, \mathfrak{P}(\mathbf{n}) = \mathsf{P}\} \subset \mathscr{F}_{S_2}$ for any $\mathsf{P} \in \mathscr{P}_{\text{even}}(S_1, S_2)$. As a result, for any such P,

$$\tilde{\rho}_{n,\lambda}^{S_1,S_2}[\mathsf{P}] = \frac{\langle \sigma_{S_1 \sqcup S_2} \rangle_{n,\lambda}}{\langle \sigma_{S_1} \rangle_{n,\lambda} \langle \sigma_{S_2} \rangle_{n,\lambda}} \tilde{\rho}_{n,\lambda}^{S_1 \sqcup S_2,\emptyset}[\mathsf{P}]. \tag{4.4}$$

Proposition 4.3. *Let $\lambda \in \mathbb{R}$. For any S_1, S_2 even and disjoint subsets of V_n. The sequence of measures $(\tilde{\rho}_{n,\lambda}^{S_1,S_2})_{n \ge 1}$ converges in the sense of distribution.*

Proof. Since the sequence of measures $(\tilde{\rho}_{n,\lambda}^{S_1,S_2})_{n \ge 1}$ lives on the finite space $\mathscr{P}_{\text{even}}(S_1, S_2)$ we just need to prove the convergence for $\mathsf{P} \in \mathscr{P}_{\text{even}}(S_1, S_2)$ of $(\tilde{\rho}_{n,\lambda}^{S_1,S_2}[\mathsf{P}])_{n \ge 1}$. Using Corollary 1.4, Lemma 2.1, and (4.4), we find that

$$\lim_{n \to \infty} \tilde{\rho}_{n,\lambda}^{S_1,S_2}[\mathsf{P}] = \frac{\langle \varphi^{|S_1|+|S_2|} \rangle_0}{\langle \varphi^{|S_1|} \rangle_0 \langle \varphi^{|S_2|} \rangle_0} \mathbb{P}[P_{k,\lambda}^{\mathsf{d}} = \mathsf{P}],$$

where $\langle \cdot \rangle_0 = \langle \cdot \rangle_{0,g,a(\lambda)}$ was defined in Lemma 2.1, and $k = (|S_1| + |S_2|)/2$. $\qquad \square$

As mentioned in [GPPS22], we can also give a probabilistic interpretation of Theorem 4.2. We begin by defining the natural probability measures on the set of tangled currents.

Definition 4.4. *Let $A, B \in \mathcal{M}(\Lambda)$. We define a measure $\mathbf{P}_{\Lambda,g,a\beta}^{A,B}$ on the set of tangled currents on Λ with source set $A + B$ as follows: for $(\mathbf{n}, t) \in \Omega_{\Lambda,\mathscr{T}}$,*

$$\mathbf{P}_{\Lambda,g,a,\beta}^{A,B}[(\mathbf{n}, t)] = \mathbb{1}_{\partial \mathbf{n} = \partial(A+B)} \frac{\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A,\beta}(\mathbf{n}_1) w_{B,\beta}(\mathbf{n}_2) \mathbb{1}_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}} \rho_{A,B,g,a}^{\mathbf{n}_1,\mathbf{n}_2}[(\mathbf{n}, t)]}{\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A,\beta}(\mathbf{n}_1) w_{B,\beta}(\mathbf{n}_2)} \tag{4.5}$$

We write $\mathbf{E}_{\Lambda,\beta}^A$ for the expectation with respect to the measure $\mathbf{P}_{\Lambda,\beta}^A$.

---

[7] Note also that $\mathscr{P}_{\text{even}}(S_1, \emptyset) = \mathscr{P}_{\text{even}}(S_1)$.

Corollary 4.5 (Probabilistic version of the switching lemma). *Let $A, B \in \mathcal{M}(\Lambda)$. For any measurable and bounded $F : \Omega_{\Lambda, \mathcal{T}} \to \mathbb{R}$,*

$$\frac{\langle \varphi_A \rangle_{\Lambda, g, a, \beta} \langle \varphi_B \rangle_{\Lambda, g, a, \beta}}{\langle \varphi_{A+B} \rangle_{\Lambda, g, a, \beta}} \mathbf{E}_{\Lambda, \beta}^{A, B}[F(\mathbf{n}, t)] = \mathbf{E}_{\Lambda, g, a, \beta}^{A+B, \emptyset}[F(\mathbf{n}, t) \mathbb{1}_{\mathcal{F}_B^\Lambda}],$$

where for $S \in \mathcal{M}(\Lambda)$, $\langle \varphi_S \rangle_{\Lambda, g, a, \beta} := \langle \prod_{x \in \Lambda} \varphi_x^{S_x} \rangle_{\Lambda, g, a, \beta}$.

4.2 An alternative proof of Wick's law

Let $U_{x,A,B}^{\mathbf{n}_1, \mathbf{n}_2}$ be the uniform measure on pairings in $\mathcal{T}_{\mathbf{n}, A, B}(x)$.

Proposition 4.6. *Let $a > 0$ and $x \in V$. As g tends to 0, $\rho_{x, A, B, g, a}^{\mathbf{n}_1, \mathbf{n}_2}$ converges in distribution to $U_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$.*

Proof. Fix S_1, S_2 even and disjoint subsets of V_n of size $A_x + \Delta_{\mathbf{n}_1}(x)$ and $B_x + \Delta_{\mathbf{n}_2}(x)$, respectively. Let $\mu_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ be any limiting measure as g tends to 0. Due to the invariance of $\mathbb{P}_{n, \lambda}^{S_1, S_2}$ with respect to permuting the elements of S_1 and the elements of S_2, $\mu_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ also has this property, and the desired result will follow once we prove that $\mu_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ is supported on (admissible) pairings.

To this end, note that if there are no vertices of degree 4 (or more) under $\mathbb{P}_{n, \lambda}^{S_1, S_2}$, then $\mathfrak{P}(\mathbf{n})$ consists of pairs, see the discussion above Definition 3.4. Let us estimate the probability of having a vertex of degree at least 4. By (4.4), it is sufficient to study this question for the measure $\mathbb{P}_{n, \lambda}^{S_1 \sqcup S_2, \emptyset}$. Let $S := S_1 \sqcup S_2$.

By Lemma 2.4 (ii), all sources have degree 1 with high probability, and it remains to consider the degree of the vertices in $V_n \setminus S$. Let $i \in V_n \setminus S$. If $\Delta_{\mathbf{n}}(i) \geq 4$ then one of the following occurs: (i) $\Delta_{\mathbf{n}_1}(i) \geq 4$ or $\Delta_{\mathbf{n}_2}(i) \geq 4$, (ii) $\Delta_{\mathbf{n}_1}(i) \geq 2$ and $\Delta_{\mathbf{n}_2}(i) \geq 2$. The contribution coming from case (i) is handled by (2.3). The contribution coming from (ii) is handled by noticing that for $j, k \notin S$,

$$\begin{aligned}
\mathbb{P}_{n, \lambda}^{S, \emptyset}[\mathbf{n}_1(i, j) \geq 1, \, \mathbf{n}_2(i, k) \geq 1] &\leq (1 + o(1)) d_n(\lambda)^2 (c_n n)^{-4} \frac{\langle \varphi^{|S|+2} \rangle_0 \langle \varphi^2 \rangle_0}{\langle \varphi^{|S|} \rangle_0} \\
&= (1 + o(1)) \tilde{g}^4 n^{-3} \frac{\langle \varphi^{|S|+2} \rangle_0 \langle \varphi^2 \rangle_0}{\langle \varphi^{|S|} \rangle_0},
\end{aligned}$$

which implies that

$$\mathbb{E}_{n, \lambda}^{S, \emptyset}\left[\sum_{i \notin S} \mathbb{1}\left[\Delta_{\mathbf{n}_1}(i) \geq 2, \, \Delta_{\mathbf{n}_2}(i) \geq 2\right]\right] \leq (1 + o(1)) \tilde{g}^4 \frac{\langle \varphi^{|S|+2} \rangle_0 \langle \varphi^2 \rangle_0}{\langle \varphi^{|S|} \rangle_0}. \tag{4.6}$$

Hence, one can find $C > 0$ (which does not depend on g) such that,

$$\limsup_{n \to \infty} \mathbb{E}_{n, \lambda}^{S, \emptyset}\left[|\{i \in V_n, \, \Delta_{\mathbf{n}}(i) \geq 4\}|\right] \leq C \tilde{g}^4 \left(\frac{\langle \varphi^{|S|+2} \rangle_{0, g, a} \langle \varphi^2 \rangle_{0, g, a}}{\langle \varphi^{|S|} \rangle_{0, g, a}} + \frac{\langle \varphi^{|S|+4} \rangle_{0, g, a}}{\langle \varphi^{|S|} \rangle_{0, g, a}} + \langle \varphi^4 \rangle_{0, g, a}\right).$$

If $p \geq 0$, as g tends to 0, $\langle \varphi^{2p} \rangle_{0, g, a}$ tends to $\langle \varphi^{2p} \rangle_{0, a}$. As a consequence, using Markov's inequality,

$$\lim_{g \to 0} \limsup_{n \to \infty} \mathbb{P}_{n, \lambda}^{S, \emptyset}[\exists \, i \in V_n, \, \Delta_{\mathbf{n}}(i) \geq 4] = 0.$$

The desired result follows. $\qquad\square$

Combining Theorem 4.2 with the above result yields a new proof of Wick's law for the GFF.

Corollary 4.7 (Wick's law). *Let $\Lambda = (V, E)$ be a finite graph. Let $a > 0$ and $\beta \geq 0$ be such that $a > 4d\beta$. Consider the Gaussian free field at parameter a and inverse temperature β on Λ. Then for any $p \geq 1$, for any $(i_k)_{1 \leq k \leq 2p} \in V^{2k}$, one has,*

$$\mathbb{E}_{\Lambda,a,\beta}^{\mathsf{GFF}}\left[\prod_{k=1}^{2p} \varphi_{i_k}\right] = \sum_{\pi} \prod_{k=1}^{p} \mathbb{E}_{\Lambda,a,\beta}^{\mathsf{GFF}}\left[\varphi_{i_{\pi(2k-1)}}\varphi_{i_{\pi(2k)}}\right],$$

where the sum is over pairings π of $\{1, \ldots, 2p\}$.

Proof. For $x \in \Lambda$, set $A_x := |\{k, \ i_k = x\}|$. This defines an $A \in \mathcal{M}(\Lambda)$. Fix $u \in \Lambda$ such that $A_u > 0$. If $v \in \Lambda$ satisfies $A_v > 0$, define $C = C_{u,v} := \mathbb{1}_u + \mathbb{1}_v$ and $B = B_{u,v} := A - C_{u,v}$.

By Corollary 4.5 applied to $F \equiv 1$: for all $1 \leq j \leq A_v$ (except if $u = v$ in which case[8] we require $j \neq 1$),

$$\frac{\langle\varphi_B\rangle_{\Lambda,a,\beta}^{\mathsf{GFF}}\langle\varphi_C\rangle_{\Lambda,a,\beta}^{\mathsf{GFF}}}{\langle\varphi_A\rangle_{\Lambda,a,\beta}^{\mathsf{GFF}}} = \mathbf{P}_{\Lambda,a,\beta}^{A,\emptyset}[\mathscr{F}_C^\Lambda] = \mathbf{P}_{\Lambda,a,\beta}^{A,\emptyset}[ua(1) \overset{\mathscr{H}(\mathbf{n},t,A,\emptyset)}{\longleftrightarrow} va(j)], \tag{4.7}$$

where $\mathbf{P}_{\Lambda,a,\beta}^{A,\emptyset} := \lim_{g\to 0} \mathbf{P}_{\Lambda,g,a,\beta}^{A,\emptyset}$. In particular, by Proposition 4.6 this measure is defined similarly as in (4.5) replacing the measures $\rho_{x,A,\emptyset,g,a}^{\mathbf{n_1}+\mathbf{n_2}}$ by $U_{x,A,\emptyset}^{\mathbf{n_1},\mathbf{n_2}}$. Now, since $U_{x,A,\emptyset}^{\mathbf{n_1},\mathbf{n_2}}$ is supported on pairings, almost surely under $\mathscr{F}_B^\Lambda$, all vertices in $\mathscr{H}(\mathbf{n}, t, A, \emptyset)$ have degree 2 except for the sources that have degree 1. This implies that $ua(k)$ is connected to only one element of $\mathcal{S}_u := \{va(t), 1 \leq t \leq A_v, v \in \Lambda \setminus \{u\}\} \cup \{ua(t), 2 \leq t \leq A_u\}$ in $\mathscr{H}(\mathbf{n}, t, A, \emptyset)$. Thus,

$$\sum_{s\in\mathcal{S}_u} \mathbf{P}_{\Lambda,a,\beta}^{A,\emptyset}[ua(1) \overset{\mathscr{H}(\mathbf{n},t,A,\emptyset)}{\longleftrightarrow} s] = 1. \tag{4.8}$$

As a result, summing (4.7) over $va(j) \in \mathcal{S}_u$ and using (4.8) yields

$$\langle\varphi_A\rangle_{\Lambda,a,\beta}^{\mathsf{GFF}} = \sum_{v\in\Lambda} \sum_{j=1+\delta_u(v)}^{A_v} \langle\varphi_{B_{u,v}}\rangle_{\Lambda,a,\beta}^{\mathsf{GFF}}\langle\varphi_{C_{u,v}}\rangle_{\Lambda,a,\beta}^{\mathsf{GFF}},$$

which implies the result by induction. $\square$

4.3 Recovering the switching lemma for the Ising model

Below, we call ECT (for "Everybody Connected Together") the event that $\mathfrak{P}(\mathbf{n}) = \{\partial\mathbf{n}\}$.

Proposition 4.8. *Let $a = -2g$. Then, as g tends to ∞, $\rho_{x,A,B}^{\mathbf{n_1},\mathbf{n_2}}$ converges in distribution to the Dirac measure δ_{ECT}.*

Proof. First, observe that if $p \geq 0$, as g tends to ∞, $\langle\varphi^{2p}\rangle_{0,g,-2g}$ tends to 1. Fix S_1, S_2 even and disjoint subsets of V_n of size $A_x + \Delta_{\mathbf{n_1}}(x)$ and $B_x + \Delta_{\mathbf{n_2}}(x)$, respectively. Using (4.4) and the observation above, we see that for any $\mathsf{P} \in \mathscr{P}_{\mathsf{even}}(S_1, S_2)$,

$$\lim_{g\to\infty} \rho_{x,A,B,g,-2g}^{\mathbf{n_1},\mathbf{n_2}}[\mathsf{P}] = \lim_{g\to\infty}\lim_{n\to\infty} \mathbb{P}_{n,\lambda(g)}^{S,\emptyset}[\mathfrak{P}(\mathbf{n_1}+\mathbf{n_2}) = \mathsf{P}],$$

where $S = S_1 \sqcup S_2$ and $\lambda(g)$ is such that $a(\lambda(g)) = -2g$. We can always assume that $S = \{1, \ldots, |S|\}$. Clearly, if $\mathsf{P} \neq \{S\}$, there exists $1 \leq i < j \leq |S|$ such that i is not connected to j in $\mathbf{n_1} + \mathbf{n_2}$. As a result, for such a P,

$$\mathbb{P}_{n,\lambda(g)}^{S,\emptyset}[\mathfrak{P}(\mathbf{n_1}+\mathbf{n_2}) = \mathsf{P}] \leq \sum_{1\leq i<j\leq|S|} \left(1 - \mathbb{P}_{n,\lambda(g)}^{S,\emptyset}[i \overset{\mathbf{n_1}+\mathbf{n_2}}{\longleftrightarrow} j]\right).$$

---

[8]For this case to occur, one must have $A_u \geq 2$.

Using the switching lemma for the Ising model (1.3) and Lemma 2.1, we get that

$$\mathbb{P}_{n,\lambda(g)}^{S,\emptyset}[i \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} j] = (1+o(1))\frac{\langle\varphi^{|S|-2}\rangle_{0,g,-2g}\langle\varphi^2\rangle_{0,g,-2g}}{\langle\varphi^{|S|}\rangle_{0,g,-2g}}.$$

This readily implies that for $\mathsf{P} \neq \{S\}$,

$$\lim_{g\to\infty}\lim_{n\to\infty}\mathbb{P}_{n,\lambda(g)}^{S,\emptyset}[\mathfrak{P}(\mathbf{n}_1+\mathbf{n}_2) = \mathsf{P}] = 0$$

and the result follows. $\qquad\square$

The above result immediately implies that one can recover the switching lemma for the Ising model when taking the Ising limit in the identity of Theorem 4.2. Indeed, the measures $\rho_{x,A,B}^{\mathbf{n}_1,\mathbf{n}_2}$ converge to δ_{ECT} which has as a consequence that the vertex set of the multigraph $\mathscr{H}(\mathbf{n},t,A,B)$ coincides with V, hence we can identify $\mathscr{H}(\mathbf{n},t,A,B)$ with the current $\mathbf{n}$.

A Explicit Griffiths-Simon approximation for ϕ^4

In this appendix, we obtain asymptotic estimates for the partition functions of the Ising and random current models on the complete graph. Along the way, we reprove the main result of [SG73].

Proposition A.1. *Let $\lambda \in \mathbb{R}$. Let $(\sigma_1,\ldots,\sigma_n)$ be distributed as spins of the Ising model on the complete graph K_n with inverse temperature $d_n(\lambda)$. Then, one has*

$$\frac{\sigma_1 + \cdots + \sigma_n}{n^{3/4}} \underset{n\to\infty}{\longrightarrow} \frac{1}{z(\lambda)}e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2}\,\mathrm{d}s,$$

where the convergence is in distribution and $z(\lambda) := \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2}\mathrm{d}s$ is a renormalising factor. Moreover, for the partition function of the Ising model on K_n we have

$$\mathscr{Z}_n(\lambda) = (1+o(1))\frac{2^n n^{1/4}}{\sqrt{2\pi e}}\int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2}\mathrm{d}s.$$

Proof. We assume that n is even, the other case is similar. We denote by $\mathbf{P}_{n,\lambda}$ the measure of the Ising model on K_n with inverse temperature $d_n(\lambda)$. Note that

$$2\sum_{1\le i<j\le n}\sigma_i\sigma_j = \left(\sum_{i=1}^{n}\sigma_i\right)^2 - n,$$

so that, for any $m \in [0,n/2]$ we have,

$$\mathbf{P}_{n,\lambda}\left[\sum_{i=1}^{n}\sigma_i = 2m\right] = \frac{1}{\mathscr{Z}_n(\lambda)}\binom{n}{n/2+m}\exp\left(d_n(\lambda)\frac{4m^2-n}{2}\right) = \frac{\exp(-d_n(\lambda)\frac{n}{2})}{\mathscr{Z}_n(\lambda)}f_n(m),$$

where $f_n(m) := \binom{n}{n/2+m}\exp\left(2d_n(\lambda)m^2\right)$. We have

$$\log\frac{f_n(m)}{f_n(m-1)} = \log\frac{n/2-m+1}{n/2+m} + (1-\lambda/\sqrt{n})\frac{4m-2}{n}$$

$$= \log\frac{1-2m/n}{1+2m/n} + (1-\lambda/\sqrt{n})\frac{4m}{n} + O\left(\frac{1}{n}\right).$$

Expanding in powers of m/n and using that $\log(1-2t)-\log(1+2t) = -4t-16t^3/3+O(t^5)$, we observe that the first order term cancels and we get

$$\log\frac{f_n(m)}{f_n(m-1)} = -\frac{16m^3}{3n^3} - \lambda\frac{4m}{n\sqrt{n}} + O\left(\frac{m^5}{n^5} + \frac{1}{n}\right).$$

By telescoping, we obtain that

$$\log \frac{f_n(m)}{f_n(0)} = -\frac{4m^4}{3n^3} - \lambda \frac{2m^2}{n\sqrt{n}} + O\left(\frac{m^6}{n^5} + \frac{m}{n}\right).$$

By symmetry, the above holds also for negative m up to an error term $O\left(\frac{m^6}{n^5} + \frac{|m|}{n}\right)$. Letting $g(x) = \exp\left(-\frac{4}{3}x^4 - 2\lambda x^2\right)$, for any $a, b \in \mathbb{R}$ with $a < b$ we have

$$\mathbf{P}_{n,\lambda}\left[an^{3/4} \leq \sum_{i=1}^n \sigma_i \leq bn^{3/4}\right] = (1 + o(1)) \frac{\exp(-d_n(\lambda)\frac{n}{2}) f_n(0)}{\mathfrak{Z}_n(\lambda)} \sum_{\frac{a}{2}n^{3/4} \leq m \leq \frac{b}{2}n^{3/4}} g\left(\frac{m}{n^{3/4}}\right)$$

$$= (1 + o(1)) \frac{e^{-1/2} f_n(0)}{\mathfrak{Z}_n(\lambda)} n^{3/4} \int_{\frac{a}{2}}^{\frac{b}{2}} g(\lambda) \, d\lambda$$

$$= (1 + o(1)) \frac{e^{-1/2} f_n(0)}{2\mathfrak{Z}_n(\lambda)} n^{3/4} \int_a^b e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} \, ds.$$

It remains to estimate $\mathfrak{Z}_n(\lambda)$, and for that we need to handle the case where m is much larger than $n^{3/4}$. Since all terms in the Taylor expansion of $\log(1 - 2t) - \log(1 + 2t)$ are non-positive it follows that

$$\log \frac{f_n(m)}{f_n(m-1)} \leq -\frac{16m^3}{3n^3} - \lambda \frac{4m}{n^{3/2}} + O\left(\frac{1}{n}\right).$$

Thus, there exists $C > 0$ such that

$$f_n(m) \leq C f_n(0) g\left(\frac{m}{n^{3/4}}\right), \tag{A.1}$$

from which we can conclude by arguing as above that

$$\mathfrak{Z}_n(\lambda) = (1 + o(1)) \frac{e^{-1/2} f_n(0)}{2} n^{3/4} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} \, ds.$$

Sending n to infinity we get the desired convergence in distribution.

Furthermore, (A.1) and the fact that $\mathfrak{Z}_n(\lambda) \geq \exp(-d_n(\lambda)\frac{n}{2}) f_n(0)$ imply the Gaussian upper bound

$$\mathbf{P}_{n,\lambda}\left[\sum_{i=1}^n = 2m\right] \leq C \exp\left(d_n(\lambda)\frac{n}{2}\right) g\left(\frac{m}{n^{3/4}}\right) = C \exp\left(\frac{1}{2}\left(1 - \frac{\lambda}{n}\right)\right) g\left(\frac{m}{n^{3/4}}\right).$$

The latter inequality together with the convergence in distribution imply the convergence of all moments.

Finally, we have $f_n(0) = (1 + o(1)) \, 2^n / \sqrt{\pi n/2}$ by Stirling's approximation, hence

$$\mathfrak{Z}_n(\lambda) = (1 + o(1)) \frac{2^n n^{1/4}}{\sqrt{2\pi e}} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} \, ds,$$

as desired. $\square$

Recall that for $g > 0$ and $\lambda \in \mathbb{R}$, the measure $\langle \cdot \rangle_0$ is the average with respect to $\rho_{g,a(\lambda)}$ with $a(\lambda) = \frac{\lambda \tilde{g}^2}{2}$.

Proposition A.2 (Estimate of the partition function of the current). *Let $\lambda \in \mathbb{R}$, $g > 0$, and $k \geq 1$. Set $S = \{1, \ldots, 2k\}$. Then,*

$$Z_n^S(\lambda) = (1 + o(1))\tilde{g}^{2k} \frac{\langle \varphi^{2k} \rangle_0}{\sqrt{2\pi e}} \, n^{1/4 - k/2} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} \, ds.$$

Moreover, for every $n \geq 1$, $Z_n^S(\lambda)$ is decreasing as a function of $\lambda \in (-\infty, \sqrt{n}]$.

Proof. Recall that by Lemma 2.1

$$\frac{Z_n^S(\lambda)}{Z_n(\lambda)} = \langle \sigma_S \rangle_{n,\lambda} = (1 + o(1))\, (c_n n)^{-2k} \left\langle \varphi^{2k} \right\rangle_0,$$

and as mentioned in (1.1), $Z_n(\lambda) = 2^{-n}\mathscr{Z}_n(\lambda)$. Thus, by Proposition A.1, and since $(c_n n)^2 = \tilde{g}^{-2}\sqrt{n}$,

$$Z_n^S(\lambda) = (1 + o(1))\, (c_n n)^{-2k} \frac{\left\langle \varphi^{2k} \right\rangle_0}{\sqrt{2\pi e}}\, n^{1/4} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2}\, \mathrm{d}s$$

$$= (1 + o(1))\tilde{g}^{2k} \frac{\left\langle \varphi^{2k} \right\rangle_0}{\sqrt{2\pi e}}\, n^{1/4 - k/2} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2}\, \mathrm{d}s.$$

The second part of the statement follows from the monotonicity of $d_n(\lambda)$ as a function of λ. □

References

[ABBG12] L. Addario-Berry, N. Broutin, and C. Goldschmidt. The continuum limit of critical random graphs. *Probability Theory and Related Fields*, **152**(3-4):367–406, 2012. MR2892951

[ABF87] M. Aizenman, D.J. Barsky, and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *Journal of Statistical Physics*, **47**:343–374, 1987. MR0894398

[ADC21] M. Aizenman and H. Duminil-Copin. Marginal triviality of the scaling limits of critical $4d$ Ising and φ_4^4 models. *Annals of Mathematics*, **194**(1):163–235, 2021. MR4276286

[ADCTW19] H. Aizenman, H. Duminil-Copin, V. Tassion, and S. Warzel. Emergent planarity in two-dimensional Ising models with finite-range interactions. *Inventiones Mathematicae*, **216**(3):661–743, 2019. MR3955708

[AF86] M. Aizenman and R. Fernández. On the critical behavior of the magnetization in high-dimensional Ising models. *Journal of Statistical Physics*, **44**(3-4):393–454, 1986. MR0857063

[AG83] M. Aizenman and R. Graham. On the renormalized coupling constant and the susceptibility in φ_4^4 field theory and the Ising model in four dimensions. *Nuclear Physics B*, **225**(2):261–288, 1983. MR0720344

[Aiz82] M. Aizenman. Geometric analysis of φ^4 fields and Ising models. parts I and II. *Communications in Mathematical Physics*, **86**(1):1–48, 1982. MR0678000

[Ald97] D. Aldous. Brownian excursions, critical random graphs and the multiplicative coalescent. *The Annals of Probability*, **25**:812–854, 1997. MR1434128

[AP00] D.J. Aldous and B. Pittel. On a random graph with immigrating vertices: Emergence of the giant component. *Random Structures & Algorithms*, **17**(2):79–102, 2000. MR1774745

[BGJ96] B. Bollobás, G. Grimmett, and S. Janson. The random-cluster model on the complete graph. *Probability Theory and Related Fields*, **104**:283–317, 1996. MR1376340

[DC17] H. Duminil-Copin. Lectures on the Ising and Potts models on the hypercubic lattice. In *PIMS-CRM Summer School in Probability*, pages 35–161. Springer, 2017. MR4043224

[DCT16] H. Duminil-Copin and V. Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Communications in Mathematical Physics*, **343**(2):725–745, 2016. MR3477351

[FV17] S. Friedli and Y. Velenik. *Statistical mechanics of lattice systems: a concrete mathematical introduction*. Cambridge University Press, 2017. MR3752129

[GHS70]   R.B. Griffiths, C.A. Hurst, and S. Sherman. Concavity of magnetization of an Ising ferromagnet in a positive external field. *Journal of Mathematical Physics*, **11**(3):790–795, 1970. MR0266507

[GPPS22] T. S. Gunaratnam, C. Panagiotis, R. Panis, and F. Severo. Random tangled currents for φ^4: translation invariant gibbs measures and continuity of the phase transition. *arXiv:2211.00319*, 2022.

[Hof16]   R. Van Der Hofstad. *Random graphs and complex networks*, volume **43**. Cambridge University Press, 2016. MR3617364

[Hus54]   K. Husimi. Statistical mechanics of condensation. In *Proceedings of the International Conference of Theoretical Physics*, 1954.

[NP10]   A. Nachmias and Y. Peres. Critical percolation on random regular graphs. *Random Structures & Algorithms*, **36**(2):111–148, 2010. MR2583058

[Pan23]   R. Panis. Triviality of the scaling limits of critical Ising and φ^4 models with effective dimension at least four. *arXiv:2309.05797*, 2023.

[Rao20]   A. Raoufi. Translation-invariant Gibbs states for the Ising model: general setting. *The Annals of Probability*, **48**(2):760–777, 2020. MR4089494

[SG73]   B. Simon and R.B. Griffiths. The φ_2^4 field theory as a classical Ising model. *Communications in Mathematical Physics*, **33**:145–164, 1973. MR0428998

[Tem54]   H.N.V. Temperley. The Mayer theory of condensation tested against a simple model of the imperfect gas. *Proceedings of the Physical Society. Section A*, **67**(3):233, 1954.