

## Convergence of simulated annealing using kinetic Langevin dynamics

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### Abstract

We study the simulated annealing algorithm based on the kinetic Langevin dynamics, in order to find the global minimum of a non-convex potential function. For both the continuous time formulation and a discrete time analogue, we obtain the convergence rate results under technical conditions on the potential function, together with an appropriate choice of the cooling schedule and the time discretization parameters.

**Keywords:** simulated annealing; kinetic Langevin dynamics; hypocoercivity; discretization; convergence rate.

**MSC2020 subject classifications:** 60J25; 60J60; 46N30.

Submitted to EJP on November 20, 2023, final version accepted on May 2, 2024.

## 1 Introduction

Simulated annealing has always been an important method to find the global minimum of a given function  $U : \mathbb{R}^d \rightarrow \mathbb{R}$ , especially when  $U$  is non-convex. Classical studies on the simulated annealing have been mainly focused on an algorithm based on the overdamped Langevin dynamic:

$$dX_t = -\nabla U(X_t)dt + \sqrt{2\varepsilon_t}dB_t, \tag{1.1}$$

where  $(B_t)_{t \geq 0}$  is a standard  $d$ -dimensional Brownian motion and  $(\varepsilon_t)_{t \geq 0}$  is a time-dependent temperature parameter that turns to 0 as  $t \rightarrow \infty$ . Notice that, with constant temperature  $\varepsilon_t \equiv \varepsilon$  and under mild conditions on  $U$ , the process  $X$  in (1.1) is the standard overdamped Langevin dynamic and has the invariant measure  $\mu_{0,\varepsilon}^*(dx) \propto \exp(-U(x)/\varepsilon) dx$ . With small  $\varepsilon > 0$ , samples from  $\mu_{0,\varepsilon}^*$  would approximately concentrate around the global minimum of function  $U$ , which is the intuition of the simulated annealing algorithm.

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Since the introduction of the simulated annealing algorithm by Kirkpatrick, Gellatt and Vecchi [23], many works have been devoted to the convergence analysis of (1.1); see e.g., Geman and Hwang [16], Chiang, Hwang and Sheu [9], Royer [37], Holley, Kusuoka and Stroock [20], Miclo [29], Zitt [41], Fournier and Tardif [13], Tang and Zhou [39], etc. It has been shown that, the cooling schedule  $\varepsilon_t$  should be at least of the order  $\frac{E}{\log t}$  as  $t \rightarrow \infty$  for some constant  $E > 0$ , in order to ensure the convergence of  $X_t$  to the global minimum of  $U$  as  $t \rightarrow \infty$ . Intuitively, this cooling schedule allows the diffusion process to have enough time to escape from the local minima and at the same time to explore the whole space in order to find the global minimum of  $U$ ; finally, the annealing process will “freeze” at the global minimum of  $U$  as  $\varepsilon_t \rightarrow 0$ . We would like to mention in particular the recent paper by Tang and Zhou [39], where the authors derived a convergence rate result of (1.1), where a fine estimation of the log-Sobolev inequality in Menz and Schlichting [27] for invariant measure  $\mu_{\varepsilon,0}^*$  with low-temperature (small  $\varepsilon > 0$ ) has been crucially used. Moreover, they have also analyzed a corresponding discrete time scheme of (1.1) and obtained a convergence rate result. Notice that in practice it is the discrete time scheme which is implemented to find the optimizer of  $U$ , for some sufficient large constant  $E > 0$ .

Motivated by the above works, we will study in this paper the simulated annealing based on the kinetic (underdamped) Langevin dynamics, that is, the process  $(X, Y) = (X_t, Y_t)_{t \geq 0}$  defined by

$$X_t = X_0 + \int_0^t Y_s ds, \quad Y_t = Y_0 - \int_0^t \nabla_x U(X_s) ds - \int_0^t \theta Y_s ds + \int_0^t \sqrt{2\varepsilon_s} dB_s, \quad (1.2)$$

where  $\theta > 0$  is a fixed constant and  $(\varepsilon_t)_{t \geq 0}$  is a cooling schedule satisfying  $\varepsilon_t \rightarrow 0$  as  $t \rightarrow \infty$ . Moreover, we will also study a discrete time scheme of (1.2). More precisely, consider a sequence  $(\Delta t_k)_{k \geq 0}$  of time steps and define the discrete time grid  $0 = T_0 < T_1 < \dots$  by

$$T_k := \sum_{j=0}^{k-1} \Delta t_j.$$

The discrete time scheme will be defined on the grid  $(T_k)_{k \geq 0}$ . For ease of presentation and the convergence analysis later, we will write this scheme as a continuous time process  $(\bar{X}, \bar{Y}) = (\bar{X}_t, \bar{Y}_t)_{t \geq 0}$  by using the time freezing function  $\eta(t) := \sum_{k=0}^{\infty} T_k \mathbf{1}_{\{t \in [T_k, T_{k+1})\}}$ . Then, the discrete time scheme process  $(\bar{X}, \bar{Y})$  is defined by

$$\bar{X}_t = X_0 + \int_0^t \bar{Y}_s ds, \quad \bar{Y}_t = Y_0 + \int_0^t (-\nabla_x U(\bar{X}_{\eta(s)}) - \theta \bar{Y}_s) ds + \int_0^t \sqrt{2\varepsilon_{\eta(s)}} dB_s. \quad (1.3)$$

Notice that the above scheme is the second-order scheme, rather than the Euler scheme of (1.2). Compare to the Euler scheme, the second-order scheme (1.3) is preferred for its convenience on both analytical and computational aspects. While (1.3) can be explicitly re-written on the time grid  $(T_k)_{k \geq 0}$  and hence is implementable (see (2.3) and (2.4) for details). This second-order scheme has also been introduced and studied for standard kinetic Langevin dynamics, i.e., for (1.2) with constant temperature  $\varepsilon_t \equiv \varepsilon_0$ ; see, e.g., Cheng, Chatterji, Bartlett and Jordan [8], Zou, Xu and Gu [42], Gao, Gürbüzbalaban and Zhu [14] and Ma, Chatterji, Cheng, Flammarión, Bartlett and Jordan [25].

For the kinetic simulated annealing process  $(X, Y)$  in (1.2), a convergence result without convergence rate has already been established by Journé and Monmarché [22]. In the present paper, we aim at obtaining some convergence rate results for both the simulated annealing process  $(X, Y)$  in (1.2) and the discrete time scheme  $(\bar{X}, \bar{Y})$

in (1.3). To the best of our knowledge, we are the first to study the convergence of the kinetic simulated annealing algorithm in the discrete time framework. Let us also mention that, by cooling the parameter  $\theta$  instead of  $\varepsilon$  in (1.2), Monmarché [30] studied an alternative kinetic simulated annealing process and derived a convergence rate for it (see Remark 2.11 in the following for a detailed comparison of his work and ours).

The remainder of the paper is organized as follows. We first introduce some notations. In Section 2, we state the assumptions and our main results, and provide the main idea of the proofs, together with some discussions on the related literature. The proof of the convergence rate of (1.2) is given in Section 3, and the convergence rate of (1.3) is given in Section 4.

**Notations.** (i) Denote by  $C^\infty(\mathbb{R}^d)$ , or simply  $C^\infty$ , the collection of all smooth (i.e., infinitely differentiable) functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . For  $f \in C^\infty$ , let  $\nabla f$ ,  $\nabla_x^2 f$ , and  $\Delta f$  denote the gradient, Hessian, and Laplacian of  $f$ , respectively. For a smooth vector field  $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\nabla \cdot v$  denotes the divergence of  $v$ . For vectors  $a, b \in \mathbb{R}^d$ ,  $\langle a, b \rangle$  is their inner product and  $|a| = \sqrt{\langle a, a \rangle}$  is the Euclidean norm of  $a$ . For two matrices  $M, N \in \mathcal{M}_{d \times d}(\mathbb{R})$  their Frobenius inner product is defined as  $\langle M, N \rangle_F = \text{tr}(M^T N) = \sum_{i,j=1}^d M_{ij} N_{ij}$ , and  $\|M\|_F = \sqrt{\langle M, M \rangle_F}$  is the Frobenius norm of  $M$ . A function  $\varepsilon \mapsto w(\varepsilon)$  is said to be sub-exponential if  $\varepsilon \log w(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

(ii) We denote by  $\mathcal{P}(\mathbb{R}^d)$  (resp.  $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ ) the collection of all probability measures on  $\mathbb{R}^d$  (resp.  $\mathbb{R}^d \times \mathbb{R}^d$ ). For functions  $f$  and  $g$  defined on  $\mathbb{R}_+$ , the symbol  $f = \mathcal{O}(g)$  means that  $f/g$  is bounded when some problem parameter tends to 0 or  $\infty$ .

## 2 Main results and literature

We will state our main convergence rate results and then discuss the main idea of proof as well as some related literature.

### 2.1 Main results

We first provide some conditions on the (potential) function  $U : \mathbb{R}^d \rightarrow \mathbb{R}$ . Without loss of generality, we assume that  $\min_{x \in \mathbb{R}^d} U(x) = 0$  throughout the paper.

**Assumption 2.1.** (i) *The function  $U \in C^\infty(\mathbb{R}^d)$  and all its derivatives have at most polynomial growth. The gradient  $\nabla U$  is  $L$ -Lipschitz for constant  $L > 0$ . Moreover,  $U$  is  $(r, m)$ -dissipative in the sense that for some positive constants  $r > 0$  and  $m > 0$ ,*

$$\nabla U(x) \cdot x \geq r|x|^2 - m, \quad \text{for all } x \in \mathbb{R}^d.$$

(ii) *The function  $U$  has multiple but a finite number of local minimizers, and  $\nabla^2 U$  is non-degenerate at the local minimizers.*

**Remark 2.2.** (i) In the literature of the standard kinetic (underdamped) Langevin dynamics (i.e., (1.2) with constant temperature  $\varepsilon_t \equiv \varepsilon_0$ ), the dissipative condition on  $U$  is a standard Lyapunov condition to ensure the ergodicity of the process; see e.g., Eberle, Guillin and Zimmer [12] and Mattingly, Stuart and Higham [26]. The Lipschitz condition on  $\nabla U$  is also usually imposed to obtain quantitative exponential convergence rate of the law of standard kinetic Langevin dynamics to the invariant measure; see, e.g., [12, 14, 25]. In particular, this condition ensures that the process  $(X, Y)$  in (1.2) is well defined.

(ii) The Lipschitz condition on  $\nabla U$ , together with the dissipative condition, implies that  $U$  to have quadratic growth at infinity. More precisely, there exists a constant  $K > 0$

such that

$$\frac{r}{3}|x|^2 - K \leq U(x) \leq L|x|^2 + K, \text{ for all } x \in \mathbb{R}^d, \tag{2.1}$$

see e.g. Raginsky, Rakhlin and Telgarsky [36, Lemma 2].

Let  $m_U$  and  $M_U$  denote respectively the set of local minima and the set of global minima of  $U$ , so that  $M_U \subseteq m_U$  in particular. We then define the constant  $E_* > 0$ , the so-called critical depth of  $U$ , by

$$E_* := \max_{\substack{x \in m_U \\ y \in M_U}} \inf \left\{ \max_{s \in [0,1]} U(\gamma(s)) - U(x) : \gamma \in \mathcal{C}([0,1], \mathbb{R}^d), \gamma(0) = x, \gamma(1) = y \right\}.$$

**Assumption 2.3.** (i) *The initial distribution of  $(X_0, Y_0)$ , denoted as  $\mu_0 = \mathcal{L}(X_0, Y_0)$ , has a  $C^\infty$  density function  $p_0$ . Moreover, the initial Fisher information  $\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla p_0(x,y)|^2}{p_0(x,y)} dx dy$  is finite and  $\mathbb{E}[|X_0|^m + |Y_0|^m] < \infty$  for each  $m \geq 1$ .*

(ii) *The function  $t \mapsto \varepsilon_t$  is positive, non-increasing and differentiable. Moreover, for some time  $t_0$  and a constant  $E > E_*$ , one has  $\varepsilon_t = \frac{E}{\log t}$  for all  $t > t_0$ .*

**Remark 2.4.** (i) For the overdamped Langevin dynamic (1.1), the above logarithmic cooling schedule has been widely used to obtain the convergence of the simulated annealing; see, e.g., [16, 9, 37, 20, 29, 39]. In Holley, Kusuoka and Stroock [20], the authors proved the failure of the simulated annealing algorithm with faster cooling schedule. For the kinetic simulated annealing process (1.2), such logarithmic cooling schedule has also been used in Journeé and Monmarché [22] to deduce a convergence result, where it is also proved that the convergence may fail for the cooling schedule faster than logarithmic. In Monmarché [30], for an alternative kinetic simulated annealing process with cooling schedule on parameter  $\theta$ , a similar cooling schedule is also assumed. And our technical conditions on  $U$  in Assumption 2.1 are also generally motivated by those in [30].

(ii) Nevertheless, the logarithmic cooling schedule seems seldom used in practice because of its low decreasing speed. In this aspect, some efforts are also devoted to design better temperature schedule. In Menz, Schlichting, Tang and Wu [28], the authors used parallel tempering technique to improve the performance of simulated annealing algorithm. In Gao, Xu, Zhou [15], the authors considered the temperature  $\varepsilon_t$  as a control variable and reformulate the problem of designing the cooling schedule as a solvable stochastic control problem after suitable regularization.

Let us now provide a first convergence rate result on  $(X, Y)$  defined by (1.2). Recall that the condition  $\min_{x \in \mathbb{R}^d} U(x) = 0$  is assumed throughout the paper.

**Theorem 2.5.** *Let Assumptions 2.1 and 2.3 hold true. Then, for any constants  $\delta > 0$  and  $\alpha > 0$ , there exists some constant  $C > 0$  such that*

$$\mathbb{P}(U(X_t) > \delta) \leq Ct^{-\min(\frac{\delta}{E}, \frac{1}{2}(1 - \frac{E_*}{E})) + \alpha}, \text{ for all } t > 0.$$

**Remark 2.6.** The above convergence rate in Theorem 2.5 is the same to that in Miclo [29] and Tang and Zhou [39] for the simulated annealing using overdamped Langevin dynamic, and also to that in Chak, Kantas and Pavliotis [7] for the simulated annealing using generalized Langevin process. While higher order Langevin dynamics are often used in MCMC as accelerated versions compare to the overdamped Langevin dynamic (see for instance [25, 14, 32]), this is not the case in the simulated annealing problem. Indeed, it has been observed that the convergence behavior of the annealed process will be mainly determined by the potential function  $U$ , but not the process used; see for instance the discussion in [30, Remarks after Theorem 1].

**Remark 2.7.** (i) Notice that, by a classical time change argument (with  $t \mapsto \tau(t) := t/\theta$ ), one can reduce the same problem to the case with  $\theta = 1$  in (1.2). For this reason, we will directly assume w.l.o.g. that  $\theta = 1$  in the proofs.

(ii) However, one cannot reduce the cooling mechanism on parameter  $\varepsilon$  as in (1.2) to the setting with cooling mechanism on  $\theta$  as in Monmarché [30]. Therefore, the above convergence rate result in Theorem 2.5 is new, although it is formally similar to Monmarché [30, Theorem 1]. Let us also refer to Remark 2.11 for more detailed comparison. In fact, our cooling mechanism is the same to that in Journal and Monmarché [22], where the authors derive a convergence result without convergence rate, but under considerably weaker conditions on  $U$  than those in Assumption 2.1.

We now present the convergence rate result for the discrete time scheme  $(\bar{X}, \bar{Y})$  as defined in (1.3).

**Theorem 2.8.** *Let Assumptions 2.1 and 2.3 hold true. Assume in addition that  $\nabla_x^2 U$  is  $L'$ -Lipschitz, i.e.  $\|\nabla_x^2 U(x) - \nabla_x^2 U(y)\|_F \leq L'|x - y|$  for all  $x, y \in \mathbb{R}^d$ , for some constant  $L' > 0$ . And that the time step size parameters  $(\Delta t_k)_{k \geq 0}$  satisfies*

$$\lim_{k \rightarrow \infty} T_k = \infty \quad \text{and} \quad \limsup_{k \rightarrow \infty} \Delta t_k \sqrt{T_k} < \infty. \tag{2.2}$$

Then for all constants  $\delta > 0$  and  $\alpha > 0$ , there exists a constant  $C > 0$ , such that

$$\mathbb{P}(U(\bar{X}_{T_k}) > \delta) \leq CT_k^{-\min(\frac{\delta}{E}, \frac{1}{2}(1 - \frac{E_*}{E})) + \alpha}, \quad \text{for all } k \geq 1.$$

**Remark 2.9.** (i) The additional Lipschitz condition on  $\nabla_x^2 U$  will be essentially used to control the discretization error in the discrete time scheme (1.3).

(ii) We need that  $\Delta t_k \rightarrow 0$  as  $T_k \rightarrow \infty$  to control the (cumulative) discretization error in the scheme (1.3). At the same time,  $\Delta t_k$  should not decrease too fast, so that  $T_k \rightarrow \infty$  as  $k \rightarrow \infty$  and thus  $(X_{T_k})_{k \geq 0}$  can reach the global minima of  $U$ . This explains the condition (2.2).

(iii) Let  $C_1 > 0$ ,  $C_2 > 0$  be two constants, let us define  $(\Delta t_k)_{k \geq 0}$  by  $\Delta t_0 := C_1$  and  $\Delta t_k := C_2 T_k^{-1/2}$  for  $k \geq 1$ . Then, it is straightforward to check that

$$T_{k+1}^2 = (T_k + \Delta t_k)^2 = T_k^2 + C_2^2/T_k + 2C_2 T_k^{1/2} \geq T_k^2 + 2C_2 C_1^{1/2},$$

as  $T_k^{1/2} \geq C_1^{1/2}$ . Therefore,  $T_k \rightarrow \infty$  as  $k \rightarrow \infty$ , and thus condition (2.2) holds.

(iv) In Tang and Zhou [39], for the discrete simulated annealing based on the overdamped Langevin dynamic (1.1), the authors required the step size  $\Delta t_k$  satisfying

$$\lim_{k \rightarrow \infty} T_k = \infty \quad \text{and} \quad \limsup_{k \rightarrow \infty} \Delta t_k T_k < \infty,$$

which is a little stronger than our condition (2.2). Of course, (2.2) is only a sufficient condition to ensure the convergence rate result.

Notice that (1.3) is a linear SDE on each time interval  $[T_k, T_{k+1}]$ , so we can solve it explicitly. By Ma, Chatterji, Cheng, Flammarion, Bartlett and Jordan [25, Appendix B], given the value  $(\bar{X}_{T_k}, \bar{Y}_{T_k})$ , we have for all  $t \in [T_k, T_{k+1}]$ ,

$$\begin{cases} \bar{X}_t = \bar{X}_{T_k} + (1 - e^{-(t-T_k)}) \bar{Y}_{T_k} - (t - T_k - (1 - e^{-(t-T_k)})) \nabla_x U(\bar{X}_{T_k}) + D_x(t), \\ \bar{Y}_t = e^{-(t-T_k)} \bar{Y}_{T_k} - (1 - e^{-(t-T_k)}) \nabla_x U(\bar{X}_{T_k}) + D_y(t), \end{cases} \tag{2.3}$$

with

$$D_x(t) = \int_{T_k}^t D_y(s) ds, \quad D_y(t) = \sqrt{2\varepsilon_{T_k}} \int_{T_k}^t e^{-(t-s)} dB_s,$$

where  $B_t$  is the Brownian motion in (1.3). Therefore, we can implement  $(\bar{X}, \bar{Y})$  on the discrete time grid  $(T_k)_{k \geq 0}$  in an exact way. More precisely, by abbreviating  $(\bar{X}_{T_k}, \bar{Y}_{T_k}, \varepsilon_{T_k})$  to  $(\bar{X}_k, \bar{Y}_k, \varepsilon_k)$ , we have

$$\begin{cases} \bar{X}_{k+1} = \bar{X}_k + (1 - e^{-\Delta t_k}) \bar{Y}_k - (\Delta t_k - (1 - e^{-\Delta t_k})) \nabla_x U(\bar{X}_k) + D_x(k), \\ \bar{Y}_{k+1} = e^{-\Delta t_k} \bar{Y}_k - (1 - e^{-\Delta t_k}) \nabla_x U(\bar{X}_k) + D_y(k), \end{cases} \quad (2.4)$$

where  $(D_x(k), D_y(k))$  is Gaussian vector in  $\mathbb{R}^d \times \mathbb{R}^d$  independent of  $(\bar{X}_k, \bar{Y}_k)$  with mean zero and covariance matrix

$$\Sigma_k = \begin{pmatrix} \Sigma_{11}(k)I_d & \Sigma_{12}(k)I_d \\ \Sigma_{12}(k)I_d & \Sigma_{22}(k)I_d \end{pmatrix}, \quad (2.5)$$

with  $\Sigma_{11}(k) := \varepsilon_k (2\Delta t_k - 3 + 4e^{-\Delta t_k} - e^{-2\Delta t_k})$ ,  $\Sigma_{12}(k) := \varepsilon_k (1 - 2e^{-\Delta t_k} + e^{-2\Delta t_k})$  and  $\Sigma_{22}(k) := \varepsilon_k (1 - e^{-2\Delta t_k})$ .

## 2.2 Main idea of proofs and related literature

Recall that for two probability measures  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$  (or in  $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ ) such that  $\mu \ll \nu$ , the relative entropy  $\text{KL}(\mu|\nu)$  and the Fisher information  $I(\mu|\nu)$  are defined by

$$\text{KL}(\mu|\nu) := \int \log \left( \frac{d\mu}{d\nu} \right) d\mu, \quad I(\mu|\nu) := \int \left| \nabla \log \frac{d\mu}{d\nu} \right|^2 d\mu.$$

For the simulated annealing process (1.1) using overdamped Langevin dynamic, let us denote  $\mu_{o,t} := \mathcal{L}(X_t)$  and  $\mu_{o,\varepsilon}^*(dx) \propto \exp(-U(x)/\varepsilon)dx$  the invariant measure of standard overdamped Langevin dynamic with constant temperature  $\varepsilon$ . To deduce their convergence rate result, Tang and Zhou [39] analyze the evolution of  $\text{KL}(\mu_{o,t}|\mu_{\varepsilon_t,o}^*)$ . A key step consists in obtaining

$$\frac{d}{dt} \text{KL}(\mu_{o,t}|\nu) \Big|_{\nu=\mu_{\varepsilon_t,o}^*} = -I(\mu_{o,t}|\mu_{\varepsilon_t,o}^*) \leq -\rho(\varepsilon_t) \text{KL}(\mu_{o,t}|\mu_{\varepsilon_t,o}^*) \quad (2.6)$$

for a good constant  $\rho(\varepsilon_t)$  (depending on  $\varepsilon_t$ ). The equality in (2.6) follows the so-called de Bruijn's identity, which is the entropy dissipation equation for the standard overdamped Langevin dynamic; see for instance Chapter 5.2. in the monograph Bakry, Ledoux and Gentil [2] or Otto and Villani [33] for a more general setting. The inequality in (2.6) follows from the log-Sobolev inequality (LSI). This step is also the classical way to deduce the exponential ergodicity of the standard overdamped Langevin dynamic; see for instance Arnold, Markowich, Toscani and Unterreiter [1] and Pavliotis [34].

For the kinetic simulated annealing process (1.2) (with  $\theta = 1$ ), let us denote  $\mu_t := \mathcal{L}(X_t, Y_t)$  and by

$$\mu_\varepsilon^*(dx dy) \propto \exp \left( -\frac{1}{\varepsilon} \left( U(x) + \frac{|y|^2}{2} \right) \right) dx dy$$

the invariant measure of standard kinetic Langevin dynamics with constant temperature  $\varepsilon$  (i.e. (1.2) with  $\varepsilon_t \equiv \varepsilon$ ). It is, however, no longer helpful to use the relative entropy  $\text{KL}(\mu_t|\mu_{\varepsilon_t}^*)$  for the convergence analysis. Indeed, because the Brownian motion only

appears in the  $y$ -direction in (1.2), the time derivative of the relative entropy only gives a part of the Fisher information:

$$\frac{d}{dt} \text{KL}(\mu_t | \nu) \Big|_{\nu=\mu_\varepsilon^*} = -\varepsilon \int \left| \nabla_y \log \frac{d\mu_t}{d\mu_\varepsilon^*} \right|^2 d\mu_t \neq -\varepsilon I(\mu_t | \mu_\varepsilon^*).$$

Thus, we cannot use the LSI to proceed as in [39] for the overdamped simulated annealing.

This is also the main reason why we cannot use the relative entropy to deduce the exponential ergodicity of the standard kinetic Langevin dynamics, for which a successful alternative method is the so-called hypocoercivity method, initiated in Desvillettes and Villani [10], Hérau [18, 19] (see also Villani [40] for a detailed presentation). This method has also been used recently to derive convergence rates for various kinetic-type equations to their stationary distributions; see for instance, Baudoin, Gordina and Herzog [3] and Camrud, Herzog, Stoltz, and Gordina [6] for the convergence of kinetic Langevin dynamics with singular potentials, Guillin, Liu, Wu and Zhang [17] and Bayraktar, Feng and Li [4] for the convergence of kinetic Langevin dynamics with mean-field interaction. Here we will adopt the hypocoercivity method in the entropic sense, as in Monmarché [30], Monmarché [31] and Camrud, Durmus, Monmarché and Stoltz [5] to study our kinetic simulated annealing processes (1.2) and (1.3). More precisely, for two probability measures  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ , we consider the distorted relative entropy  $H_\gamma(\mu | \nu)$  defined by

$$H_\gamma(\mu | \nu) := \int \left( \left| \nabla_x \log \frac{d\mu}{d\nu} + \nabla_y \log \frac{d\mu}{d\nu} \right|^2 + \gamma \log \frac{d\mu}{d\nu} \right) d\mu, \tag{2.7}$$

where  $\gamma > 0$  is a constant. To deduce the convergence rate result of our kinetic simulated annealing process (1.2) by using the hypocoercivity method, we need to choose  $\gamma$  to be a function of  $\varepsilon_t$ , and then analyze the evolution of  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$ :

$$\frac{d}{dt} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) = \partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) + \partial_{\varepsilon,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*), \tag{2.8}$$

where

$$\partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) := \frac{d}{dt} H_\gamma(\mu_t | \nu) \Big|_{\nu=\mu_{\varepsilon_t}^*, \gamma=\gamma(\varepsilon_t)} \tag{2.9}$$

and

$$\partial_{\varepsilon,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) := \frac{d}{dt} H_{\gamma(\varepsilon_t)}(\mu | \mu_{\varepsilon_t}^*) \Big|_{\mu=\mu_t}. \tag{2.10}$$

The term (2.9) is from the (instantaneous) evolution of  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$  along the kinetic diffusion (1.2) for fixed temperature  $\varepsilon_t$  and the term (2.10) arises from the influence of (instantaneous) invariant measure  $\mu_{\varepsilon_t}^*$  and  $\gamma(\varepsilon_t)$  on  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$ .

For the term (2.9), with a carefully chosen  $\varepsilon_t \mapsto \gamma(\varepsilon_t)$  and by adapting a computation strategy in Ma, Chatterji, Cheng, Flammarion, Bartlett and Jordan [25] for standard kinetic Langevin dynamics, we obtain

$$\partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) \leq -c(\varepsilon_t) \rho_{\varepsilon_t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*), \tag{2.11}$$

where  $c(\varepsilon_t) = \gamma^{-1}(\varepsilon_t)$  and  $\rho_{\varepsilon_t} > 0$  is the constant in the log-Sobolev inequality (LSI) satisfied by  $\mu_{\varepsilon_t}^*$ .

**Remark 2.10.** As in [30] and [39], it is crucial to use a fine estimation on the constant  $\rho_{\varepsilon_t}$  in the LSI for  $\mu_{\varepsilon_t}^*$  (under Assumption 2.1), in particular for the case with small  $\varepsilon_t$ . We refer to Menz and Schlichting [27], especially, Section 2.3.3 therein, for this issue.

The treatment of the term (2.10) is classical (see e.g. Holley, Kusuoka and Stroock [20], Miclo [29], Tang and Zhou [39] in the case of the overdamped simulated annealing, or Monmarché [30] in the case of an alternative kinetic simulated annealing). First, we will apply the Lyapunov function technique as in Talay [38] to obtain a uniform bound on the moment of  $(X_t, Y_t)$  in (1.2) for all  $t \geq 0$ . Using this uniform moment bound, together with the explicit expression of  $\mu_{\varepsilon_t}^*$  for all  $t \geq 0$ , we can directly compute that, for some sub-exponential  $\omega(\varepsilon)$ ,

$$\partial_{\varepsilon,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) \leq \omega(\varepsilon_t) |\varepsilon_t'| (1 + H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)), \quad \text{for all } t > 0. \quad (2.12)$$

where  $\varepsilon_t'$  is the derivative of  $t \mapsto \varepsilon_t$ . The term  $\omega(\varepsilon_t) |\varepsilon_t'|$  at the right-hand-side (r.h.s.) of (2.12) is non-negative. In order to make  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$  decrease along the time and to get an explicit decay rate, we need to make sure that the term  $c(\varepsilon_t) \rho_{\varepsilon_t}$  in (2.11) is greater than the term  $\omega(\varepsilon_t) |\varepsilon_t'|$  in (2.12). This requires a careful choice of  $\varepsilon \mapsto \gamma(\varepsilon)$  as well as some conditions on the cooling schedule  $t \mapsto \varepsilon_t$  as in Assumption 2.3.

**Remark 2.11.** (i) The above sketch of proof, as well as the use of the hypocoercivity method with the distorted entropy (2.7), is similar to that in Monmarché [30] for a different kinetic annealing process. More precisely, [30] studies the process

$$X_t = X_0 + \int_0^t Y_s ds, \quad Y_t = Y_0 - \int_0^t \nabla U(X_s) ds - \int_0^t \frac{1}{\varepsilon_s} Y_s ds + \int_0^t \sqrt{2} dB_s, \quad (2.13)$$

with  $\varepsilon_s \rightarrow 0$  as  $s \rightarrow \infty$ . As our cooling schedule is on the diffusion parameter, we need to design another function  $\varepsilon_t \mapsto \gamma(\varepsilon_t)$  to compute the distorted entropy  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$ . More importantly, [30] reformulated the problem into a Bakry-Emery framework to establish a contraction inequality similar to (2.11), while we apply a different computation strategy adapted from [25]. In particular, as shown in [25], this computation strategy can be adapted to the discrete time setting for convergence analysis of numerical schemes, a subject that is not addressed in [30].

(ii) From a numerical point of view, our kinetic annealing process (1.2) seems to be more convenient for discrete time simulation than that in (2.13). Indeed, as  $\varepsilon_s \rightarrow 0$ , the Lipschitz constant of the coefficient functions in SDE (2.13) will explode, so the corresponding time discretization error in the numerical analysis can be much harder to control.

For the convergence analysis of the discrete time process  $(\bar{X}, \bar{Y})$  in (1.3), the main idea will be quite similar. Denote  $\bar{\mu}_t := \mathcal{L}(\bar{X}_t, \bar{Y}_t)$ . For ease of presentation, let us abbreviate  $(\varepsilon_{T_k}, \bar{\mu}_{T_k}, \mu_{\varepsilon_{T_k}}^*)$  to  $(\varepsilon_k, \bar{\mu}_k, \mu_k^*)$ . Then, with the same distorted entropy function defined by (2.8), we need to compute the difference, for each  $k \geq 0$ ,

$$H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1} | \mu_{k+1}^*) - H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*) = H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1} | \mu_k^*) - H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*) \quad (2.14)$$

$$+ H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1} | \mu_{k+1}^*) - H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1} | \mu_k^*). \quad (2.15)$$

For the term (2.14), we can adapt the computation in [25] for the standard kinetic Langevin process. Concretely, we can interpolate  $H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*)$  and  $H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1} | \mu_k^*)$  by the function  $t \mapsto H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*)$  with  $t \in [T_k, T_{k+1}]$ . Notice that  $(\bar{\mu}_t)_{t \in [T_k, T_{k+1}]}$  satisfies a Fokker-Planck equation with a (partially) frozen coefficient function. We can then apply the same computation strategy as in the continuous time setting to obtain a (asymptotic) contraction estimation.

Similarly, the term (2.15) can be handled by interpolating  $H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1} | \mu_k^*)$  and  $H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1} | \mu_{k+1}^*)$  with  $\varepsilon \mapsto H_{\gamma(\varepsilon)}(\bar{\mu}_{k+1} | \mu_\varepsilon^*)$  for  $\varepsilon \in [\varepsilon_{k+1}, \varepsilon_k]$ , together with a uniform moment estimation on  $(\bar{X}, \bar{Y})$ . Finally, with a good choice of the time step size



parameters  $(\Delta t_k)_{k \geq 0}$ , we can combine the estimations of (2.14) and (2.15) to obtain a convergence rate result for the process  $(\bar{X}, \bar{Y})$  on the discrete time grid  $(T_k)_{k \geq 0}$ .

In particular, the above analysis on  $H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*)$  extends the hypocoercivity method for discrete time Langevin dynamics in [25] to the setting with time-dependent coefficient. We also notice that most existing papers on the numerical aspects of kinetic type equations by hypocoercivity method mainly focused on the discretization of corresponding Fokker-Planck equations, as in Dujardin, Hérau and Lafitte [11] and Porretta and Zuazua [35].

### 3 Convergence of the continuous-time kinetic simulated annealing

The main goal of this section is to prove Theorem 2.5. The idea is similar to that in Monmarché [30], which is mainly based on hypocoercivity computation with time-varying coefficients as discussed in Section 2.2. We nevertheless adapt the computation framework in Ma, Chatterji, Cheng, Flammarion, Bartlett and Jordan [25], which seems more convenient for the computation in the discrete-time setting in our Section 4.

For simplicity of presentation, we will assume that  $\theta = 1$ . Denote  $z = (x, y)$  as a single variable and  $Z = (X, Y)$  as the process satisfying (1.2). For each  $t \geq 0$ , denote by  $\mu_t = \mathcal{L}(Z_t)$  the marginal distribution of  $Z$ . Similar to [30, Proposition 4], under the smoothness and growth conditions of  $U$  in Assumption 2.1,  $\mu_t$  has a strictly positive smooth density function on  $\mathbb{R}^{2d}$ , denoted by  $p_t$ . Recall also that, for each  $\varepsilon > 0$ , the invariant probability measure of (1.2) with constant temperature  $\varepsilon_t \equiv \varepsilon$  is denoted by  $\mu_\varepsilon^*$ , i.e., with some renormalized constant  $C_\varepsilon > 0$ ,

$$\mu_\varepsilon^*(dz) = p_\varepsilon^*(z)dz, \quad \text{with } p_\varepsilon^*(x, y) := C_\varepsilon \exp\left(-\left(U(x) + |y|^2/2\right)/\varepsilon\right).$$

#### 3.1 Preliminary analysis of continuous kinetic annealing process

Let us first recall a fine result on the log-Sobolev inequality from [30, Proposition 2], which is mainly based on Menz and Schlichting [27, Corollary 2.18].

**Proposition 3.1.** *Let  $U$  satisfy Assumption 2.1. Then, for all  $\varepsilon > 0$ ,*

$$\text{KL}(\nu | \mu_\varepsilon^*) \leq \max\left\{\frac{\varepsilon}{2}, \frac{1}{2\rho_\varepsilon}\right\} I(\nu | \mu_\varepsilon^*), \quad \text{for all } \nu \ll \mu_\varepsilon^*,$$

where  $\rho_\varepsilon = \chi(\varepsilon) \exp\left(-\frac{E_*}{\varepsilon}\right)$  and  $\chi(\varepsilon)$  is sub-exponential in the sense that  $\varepsilon \log \chi(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Next, we give a moment estimation on the solution  $(X_t, Y_t)_{t \geq 0}$  to (1.2).

**Proposition 3.2.** *Let Assumptions 2.1 and 2.3 hold. Then,*

$$C_0 := \sup_{t \geq 0} \mathbb{E} [U(X_t) + |Y_t|^2] < \infty.$$

*Proof.* The proof is very similar to [30, Lemma 7], we consider the Lyapunov function

$$R(x, y) := U(x) + \phi_1(x, y) + \beta \phi_2(x, y),$$

where  $\phi_1(x, y) := \frac{|y|^2}{2}$ ,  $\phi_2(x, y) := x \cdot y$ , and  $\beta := \frac{1}{2} \frac{1}{1+\frac{1}{2r}} \frac{1}{1+\frac{1}{\sqrt{r/3}}} < \frac{1}{2}$  with the constant  $r > 0$  as given in Assumption 2.1.

For any smooth function  $\phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ , let  $\mathcal{L}_\varepsilon \phi(x, y) := y \cdot \nabla_x \phi(x, y) - (\nabla_x U + y) \cdot \nabla_y \phi(x, y) + \varepsilon \Delta_y \phi(x, y)$ , which corresponds to the generator of the diffusion process (1.2) at temperature  $\varepsilon > 0$ . Then

$$\mathcal{L}_\varepsilon(U(x) + \phi_1(x, y)) = -|y|^2 + \varepsilon d.$$

Further, by Assumption 2.1,

$$\mathcal{L}_\varepsilon \phi_2(x, y) = |y|^2 - x \cdot y - x \cdot \nabla_x U(x) \leq \left(1 + \frac{1}{2c_1}\right) |y|^2 - \frac{r}{2} |x|^2 + m.$$

By (2.1) as well as the fact that  $\beta < \frac{1}{2}$ , it follows that

$$\begin{aligned} \mathcal{L}_\varepsilon R(x, y) &\leq -\left(1 - \beta \left(1 + \frac{1}{2r}\right)\right) |y|^2 - \beta \frac{r}{2} |x|^2 + \beta m + \varepsilon d \\ &\leq -c_3 \beta (U(x) + |y|^2) + c_2, \end{aligned}$$

for some positive constants  $c_2, c_3$  independent of  $\varepsilon$ . Using again  $\beta < \frac{1}{2}$  and that  $U$  is of quadratic growth (see (2.1)), we obtain that

$$R(x, y) \leq U(x) + \frac{|y|^2}{2} + \frac{1}{2} x \cdot y \leq c_4 (1 + U(x) + |y|^2),$$

for some constant  $c_4 > 0$ . Thus,

$$\mathcal{L}_\varepsilon R(x, y) \leq -c_3 \beta \left(\frac{R(x, y)}{c_4} - 1\right) + c_2 \leq -c_5 R(x, y) + c_6,$$

for some positive constants  $c_5$  and  $c_6$  independent of  $\varepsilon > 0$ . Notice that  $R(x, y)$  does not depend on  $\varepsilon$ , so

$$\frac{d}{dt} \mathbb{E}[R(X_t, Y_t)] = \mathbb{E}[\mathcal{L}_{\varepsilon_t} R(X_t, Y_t)] \leq -c_5 \mathbb{E}[R(X_t, Y_t)] + c_6.$$

Now  $(\mathbb{E}[R(X_t, Y_t)])_{t \geq 0}$  is uniformly bounded by Grönwall Lemma.

To conclude, by [30, Lemma 7] there is some constant  $C > 0$  such that

$$U(x) + |y|^2 \leq C(R(x, y) + 1), \text{ for all } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Therefore,

$$C_0 := \sup_{t \geq 0} \mathbb{E}[U(X_t) + |Y_t|^2] \leq C + C \sup_{t \geq 0} \mathbb{E}[R(X_t, Y_t)] < \infty.$$

□

### 3.2 Estimates of distorted entropy in continuous-time setting

We now compute the time derivative of  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$ , which consists of computing  $\partial_{\mu, t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$  and  $\partial_{\varepsilon, t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$ , defined in (2.9) and (2.10) respectively. For simplicity, for a function  $f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ , we abbreviate  $\int_{\mathbb{R}^d \times \mathbb{R}^d} f(z) dz$  to  $\int f dz$ . Recalling that  $p_t$  (resp.  $p_{\varepsilon_t}^*$ ) represents the density function of  $\mu_t$  (resp.  $\mu_{\varepsilon_t}^*$ ), we also write

$$\mathbb{E}_{\mu_t}[f] := \int f p_t dz, \quad \mathbb{E}_{\mu_{\varepsilon_t}^*}[f] := \int f p_{\varepsilon_t}^* dz.$$

Computing  $\partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*)$  can be easily addressed by directly using methods in [25]. Notice that the marginal distribution  $p_t$  satisfies the kinetic Fokker-Planck equation

$$\partial_t p_t = -y \cdot \nabla_x p_t + \nabla_x U \cdot \nabla_y p_t + \nabla_y \cdot (y p_t) + \varepsilon_t \Delta_y p_t. \tag{3.1}$$

Denote  $\nabla_z = (\nabla_x, \nabla_y)^T$ . Because  $p_{\varepsilon_t}^* \propto \exp\left(-\frac{1}{\varepsilon_t} \left(U(x) + \frac{|y|^2}{2}\right)\right)$ , we can rewrite (3.1) as

$$\partial_t p_t + \nabla_z \cdot (p_t v_t) = 0, \tag{3.2}$$

where  $v_t$  is the vector flow that transports  $\mu_t$  towards  $\mu_{\varepsilon_t}^*$ :

$$v_t = \varepsilon_t \begin{pmatrix} 0 & I_d \\ -I_d & -I_d \end{pmatrix} \begin{pmatrix} \nabla_x \log \frac{p_t}{p_{\varepsilon_t}^*} \\ \nabla_y \log \frac{p_t}{p_{\varepsilon_t}^*} \end{pmatrix}. \tag{3.3}$$

The same as the technique applied in [25], let  $h_t := \sqrt{\frac{p_t}{p_{\varepsilon_t}^*}}$  and  $S := \begin{pmatrix} I_d & I_d \\ I_d & I_d \end{pmatrix}$ . Then the distorted relative entropy  $H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*)$  can be written as

$$H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*) = 4 \mathbb{E}_{\mu_t} [\langle \nabla_z \log h_t, S \nabla_z \log h_t \rangle] + 2\gamma(\varepsilon_t) \mathbb{E}_{\mu_t} [\log h_t]. \tag{3.4}$$

Denote as  $\nabla_z^* = (\nabla_x^*, \nabla_y^*) := \left(-\nabla_x \cdot + \frac{1}{\varepsilon_t} \nabla_x U \cdot, -\nabla_y \cdot + \frac{y}{\varepsilon_t} \cdot\right)$  the adjoint operator of  $\nabla_z$  w.r.t.  $\mu_{\varepsilon_t}^*$ , in the sense that for any  $f, g \in L^2(\mu_{\varepsilon_t}^*)$ ,  $\mathbb{E}_{\mu_{\varepsilon_t}^*} [f \cdot \nabla_z g] = \mathbb{E}_{\mu_{\varepsilon_t}^*} [g \nabla_z^* f]$ . Using (3.2) and integration by parts, we obtain

$$\begin{aligned} \partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*) &= \int \frac{\delta H_{\gamma}(\mu_t|\nu)}{\delta \mu_t} \Big|_{\nu=\mu_{\varepsilon_t}^*, \gamma=\gamma(\varepsilon_t)} \partial_t p_t \, dz \\ &= \int \nabla_z \left( \frac{\delta H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*)}{\delta \mu_t} \right) \cdot v_t \, p_t \, dz, \end{aligned}$$

where  $\frac{\delta H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*)}{\delta \mu_t}$  is the first order variational derivative of  $H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*)$  at  $\mu_t$ . After a similar calculation as in [25], we obtain the following result.

**Proposition 3.3.** *One has*

$$\begin{aligned} &\partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*) \\ &= -8\varepsilon_t \mathbb{E}_{\mu_t} [\langle \nabla_z \nabla_y \log h_t, S \nabla_z \nabla_y \log h_t \rangle_F] - 4 \mathbb{E}_{\mu_t} [\langle \nabla_z \log h_t, M_t \nabla_z \log h_t \rangle], \end{aligned}$$

with  $M_t = \begin{pmatrix} 2I_d & 2I_d - \nabla_x^2 U \\ 2I_d - \nabla_x^2 U & (2 + \gamma(\varepsilon_t)\varepsilon_t) I_d - 2\nabla_x^2 U \end{pmatrix} \in \mathcal{M}_{2d \times 2d}(\mathbb{R})$ .

A sketch of proof could be found in Appendix A.1.

Notice that  $S = \begin{pmatrix} I_d & I_d \\ I_d & I_d \end{pmatrix} \succeq 0$ , thus we obtain

$$\begin{aligned} \partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t|\mu_{\varepsilon_t}^*) &\leq -4 \mathbb{E}_{\mu_t} [\langle \nabla_z \log h_t, M_t \nabla_z \log h_t \rangle] \\ &= -\mathbb{E}_{\mu_t} \left[ \left\langle \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*}, M_t \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*} \right\rangle \right]. \end{aligned}$$

The r.h.s. of the above inequality resembles the Fisher information  $I(\mu_t|\mu_{\varepsilon_t}^*) = \mathbb{E}_{\mu_t} \left[ \left\langle \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*}, \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*} \right\rangle \right]$ . In order to use the log-Sobolev inequality satisfied by  $\mu_{\varepsilon_t}^*$

(Proposition 3.1) and connect it to the distorted relative entropy (3.4), we will prove in Lemma 3.4 that, for sufficiently large  $t \geq 0$ ,

$$M_t \succeq c(\varepsilon_t)\rho_{\varepsilon_t} \left( S + \frac{\gamma(\varepsilon_t)}{2\rho_{\varepsilon_t}} I_{2d} \right), \tag{3.5}$$

where  $c(\varepsilon_t) := \frac{1}{\gamma(\varepsilon_t)}$ . It follows that

$$\begin{aligned} & \partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) \\ & \leq -c(\varepsilon_t)\rho_{\varepsilon_t} \mathbb{E}_{\mu_t} \left[ \left\langle \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*}, S \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*} \right\rangle + \frac{\gamma(\varepsilon_t)}{2\rho_{\varepsilon_t}} \left\langle \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*}, \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*} \right\rangle \right] \\ & \leq -c(\varepsilon_t)\rho_{\varepsilon_t} \mathbb{E}_{\mu_t} \left[ \left\langle \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*}, S \nabla_z \log \frac{p_t}{p_{\varepsilon_t}^*} \right\rangle + \gamma(\varepsilon_t) \text{KL}(\mu_t | \mu_{\varepsilon_t}^*) \right] \\ & = -c(\varepsilon_t)\rho_{\varepsilon_t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*). \end{aligned}$$

**Lemma 3.4.** *Let Assumptions 2.1 and 2.3 hold true. Choose  $\gamma(\varepsilon_t) := \frac{4}{\varepsilon_t}(1 + L^2)$  and  $c(\varepsilon_t) := \frac{1}{\gamma(\varepsilon_t)}$ . Then, for sufficiently large  $t$ ,  $M_t$  satisfies (3.5).*

*Proof.* The proof is very similar to that of [30, Lemma 8], so we only present the main idea here.

By the choice of  $\gamma(\varepsilon_t)$  and  $c(\varepsilon_t)$ , we have that  $(\varepsilon_t, c(\varepsilon_t), \rho_{\varepsilon_t}, \gamma(\varepsilon_t)) \rightarrow (0, 0, 0, \infty)$  as  $t \rightarrow \infty$ . Thus, it is sufficient to prove that for sufficiently large  $t \geq 0$ ,

$$M_t - \frac{1}{2}c(\varepsilon_t)\gamma(\varepsilon_t)I_{2d} = M_t - \frac{1}{2}I_{2d} \succ 0.$$

The characteristic equation of  $M_t - \frac{1}{2}I_{2d}$  is:

$$\det \left( M_t - \frac{1}{2}I_{2d} - \lambda I_{2d} \right) = \det \begin{pmatrix} \left( \frac{3}{2} - \lambda \right) I_d & 2I_d - \nabla_x^2 U \\ 2I_d - \nabla_x^2 U & \left( \frac{11}{2} - \lambda + 4L^2 \right) I_d - 2\nabla_x^2 U \end{pmatrix},$$

and by direct computation

$$\begin{aligned} & \det \left( M_t - \frac{1}{2}I_{2d} - \lambda I_{2d} \right) \\ & = \det \left( \left( \frac{3}{2} - \lambda \right) \left( \frac{11}{2} - \lambda + 4L^2 \right) I_d - 2 \left( \frac{3}{2} - \lambda \right) \nabla_x^2 U - (2I_d - \nabla_x^2 U)^2 \right). \end{aligned}$$

By diagonalizing  $\nabla_x^2 U$ , with  $(\nabla_x^2 U)_i$  denoting the  $i$ -th eigenvalue of  $\nabla_x^2 U$ , we obtain  $d$  quadratic equations on  $\lambda$  in the form

$$\lambda^2 - (7 + 4L^2 - 2(\nabla_x^2 U)_i) \lambda + \left( \frac{17}{4} + 6L^2 - (\nabla_x^2 U)_i^2 + (\nabla_x^2 U)_i \right) = 0,$$

whose roots are the eigenvalues of  $M_t - \frac{1}{2}I_{2d}$ . Using the fact that  $|(\nabla_x^2 U)_i| \leq L$  (as  $\nabla U$  is  $L$ -Lipschitz), we can check that all roots of the above equations are positive, so all eigenvalues of  $M_t - \frac{1}{2}I_{2d}$  are positive. This concludes the proof.  $\square$

**Remark 3.5.** The previous lemma only ensures  $M_t \succeq c(\varepsilon_t)\rho_{\varepsilon_t} \left( S + \frac{\gamma(\varepsilon_t)}{2\rho_{\varepsilon_t}} I_{2d} \right)$  for sufficiently large  $t$ . However, this is sufficient to obtain the convergence of our kinetic annealing process (1.2), as long as we can ensure the distorted relative entropy  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$  is finite in any finite time interval (see Lemma 3.8 below). We also want to point out that the choice of  $\gamma(\varepsilon_t)$  and  $c(\varepsilon_t)$  is not unique to ensure that  $M_t$  satisfies (3.5).

We summarize the above results in the following proposition:

**Proposition 3.6.** *Let Assumptions 2.1 and 2.3 hold. Then, for sufficiently large  $t \geq 0$ , we have*

$$\partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) \leq -c(\varepsilon_t) \rho_{\varepsilon_t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*),$$

where  $c(\varepsilon_t) := \frac{\varepsilon_t}{4(1+L^2)}$  and  $\rho_{\varepsilon_t}$  is given in Proposition 3.1.

Now for  $\partial_{\varepsilon,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_t^*)$  defined in (2.10), denote  $\varepsilon'_t$  the derivative of  $t \mapsto \varepsilon_t$ , Proposition 3.2 together with [30, Lemma 15] give

**Lemma 3.7.** *Let Assumptions 2.1 and 2.3 hold true. Then, there exists some sub-exponential function  $\varepsilon \mapsto \omega(\varepsilon)$ , such that*

$$\partial_{\varepsilon,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) \leq |\varepsilon'_t| \omega(\varepsilon_t) (H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) + 1 + C_0),$$

where  $C_0$  is the upper bound of  $\mathbb{E}[U(X_t) + |Y_t|^2]$  defined in Proposition 3.2.

Finally, we claim  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$  is uniformly bounded on any finite time horizon  $[0, t]$ , the proof could be found in Appendix A.2.

**Lemma 3.8.** *Let Assumptions 2.1 and 2.3 hold true. Then, for all  $t \geq 0$ , we have*

$$\sup_{0 \leq s \leq t} H_{\gamma(\varepsilon_s)}(\mu_s | \mu_{\varepsilon_s}^*) < \infty.$$

### 3.3 Proof of Theorem 2.5

For the estimation of  $\mathbb{P}(U(\tilde{X}_t) > \delta)$ , we have the following classical result (see e.g. [30, Lemma 3], or [39, Lemma 3]).

**Lemma 3.9.** *Let  $U$  satisfy Assumption 2.1. Then, for all constants  $\delta$  and  $\alpha > 0$ , there exists some constant  $C > 0$  (which depends only on  $\delta, \alpha, U, d$ ) such that*

$$\mathbb{P}(U(\tilde{X}_t) > \delta) \leq C e^{-\frac{\delta - \alpha}{\varepsilon_t}}.$$

In particular, let  $\varepsilon_t$  satisfy Assumption 2.3. Then for sufficiently large  $t > 0$ , we have

$$\mathbb{P}(U(\tilde{X}_t) > \delta) \leq C t^{-\frac{\delta - \alpha}{E}}.$$

**Remark 3.10.** The proof of Lemma 3.9 relies on the Laplace approximation method which may fail when both the temperature parameter  $\varepsilon_t$  goes to 0 and the dimension  $d$  goes to infinity. In Tang and Zhou [39, Remark 1], the authors got a concrete dependence of the constant  $C$  in above Lemma on  $d$ . Their estimates shows the simulated annealing algorithm is exponentially slow as the dimension increasing. This result is accordance with Jain and Kar [21] that it is NP-hard for finding the global minimum of a general non-convex function.

The estimates on  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)$  can be obtained by using the same arguments as in [30, Lemma 19], together with the results in Section 3.2. We conclude them as the following proposition.

**Proposition 3.11.** *Let Assumptions 2.1 and 2.3 hold and choose  $\gamma(\varepsilon_t) = \frac{4}{\varepsilon_t} (1 + L^2)$ . Then, for any  $\alpha > 0$ , there exists  $C > 0$ , such that for sufficiently large  $t$ ,*

$$H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) \leq C t^{-(1 - \frac{E}{E^*} - \alpha)}.$$

*Proof of Theorem 2.5.* For any  $t \geq 0$ , let  $\tilde{Z}_t = (\tilde{X}_t, \tilde{Y}_t)$  be a random variable (on the same initial probability space) following distribution  $\mu_{\varepsilon_t}^*$ . Then, for any  $\delta > 0$ ,

$$\begin{aligned} \mathbb{P}(U(X_t) > \delta) &= \mathbb{P}(U(X_t) > \delta, U(\tilde{X}_t) > \delta) + \mathbb{P}(U(X_t) > \delta, U(\tilde{X}_t) \leq \delta) \\ &\leq \mathbb{P}(U(\tilde{X}_t) > \delta) + \text{TV}(\mu_t | \mu_{\varepsilon_t}^*) \leq \mathbb{P}(U(\tilde{X}_t) > \delta) + \sqrt{2\text{KL}(\mu_t | \mu_{\varepsilon_t}^*)} \\ &\leq \mathbb{P}(U(\tilde{X}_t) > \delta) + \sqrt{2H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)}, \end{aligned} \tag{3.6}$$

where we have used the Csiszár-Kullback-Pinsker inequality, the definition of the distorted relative entropy (2.7), and the fact that  $\gamma(\varepsilon_t) = \frac{4}{\varepsilon_t}(1 + L^2) \geq 1$  for sufficiently large  $t \geq 0$ . We then conclude the proof by Lemma 3.9 and Proposition 3.11.  $\square$

#### 4 Convergence of the discrete-time kinetic simulated annealing

Following the sketch of proof in Section 2.2, we provide in this section the convergence analysis of the discrete time kinetic simulated annealing process  $(\bar{X}, \bar{Y})$  in (1.3) and prove Theorem 2.8.

Without loss of generality, we assume that the Lipschitz constant of  $\nabla U(x)$  satisfies  $L \geq 1$ . We also stay in the context with  $\theta = 1$  for the proof. Denote  $z = (x, y)$  and  $\bar{Z} := (\bar{X}, \bar{Y})$ . By the explicit solution of  $(\bar{X}_t, \bar{Y}_t)$  in (2.3), we can see that the marginal distribution  $\bar{\mu}_t := \mathcal{L}(\bar{Z}_t)$  is denoted by  $\bar{p}_t$  has a strictly positive and smooth density function  $\bar{p}_t(z)$ . Also recall that  $\mu_\varepsilon^*(dz) = p_\varepsilon^*(z)dz$  with  $p_\varepsilon^*(x, y) \propto \exp\left(-\frac{1}{\varepsilon}(U(x) + \frac{|y|^2}{2})\right) dz$ . For the ease of notation, we abbreviate

$$(\bar{X}_{T_k}, \bar{Y}_{T_k}, \bar{Z}_{T_k}, \varepsilon_{T_k}, \bar{\mu}_{T_k}, \mu_{\varepsilon_{T_k}}^*, \bar{p}_{T_k}, p_{\varepsilon_{T_k}}^*) \text{ to } (\bar{X}_k, \bar{Y}_k, \bar{Z}_k, \varepsilon_k, \bar{\mu}_k, \mu_k^*, \bar{p}_k, p_k^*).$$

##### 4.1 Preliminary analysis of discrete kinetic annealing process

We first provide a uniform boundedness result on the moment of  $(\bar{X}, \bar{Y})$ . Although the coupling method (as in [25, Lemma 11]) is widely used for getting the moment bound of  $(\bar{X}, \bar{Y})$ , it could not be used in our problem. Since by using such method, the log-Sobolev constant  $\rho_\varepsilon$  will appear in the denominator. By Proposition 3.1,  $\rho_\varepsilon \rightarrow 0$  as temperature  $\varepsilon \rightarrow 0$ . Thus the moment bound get in this way is too crude to use in our simulated annealing problem. Instead, a more involved analysis using Lyapunov function in discrete setting as in [14, Lemma EC.5] will be carried out. To this end, we introduce two positive constants

$$\eta^* := \frac{\min\left\{\frac{\tilde{\beta}r}{2L}, 1 - \frac{\tilde{\beta}}{r}\right\} \min\left\{1 - \frac{\tilde{\beta}^2}{r/3}, \frac{1}{2}\right\}}{\max\left\{\frac{9L^2}{r} + \frac{4}{3}, L + 6\right\} \max\left\{1 + \frac{3}{4r}, \frac{3}{2}\right\}}, \quad \tilde{\beta} := \frac{1}{2(1 + \frac{1}{r})(1 + \frac{1}{\sqrt{r/3}})},$$

where  $r > 0$  is the constant given in Assumption 2.1. One can check that  $\tilde{\beta} \leq \frac{1}{2}$ ,  $\tilde{\beta} < r$  and  $\frac{\tilde{\beta}^2}{r/3} < 1$ .

**Proposition 4.1.** *Let Assumptions 2.1 and 2.3 hold. Assume that  $\Delta t_k \leq \min\left\{\frac{\eta^*}{2}, \frac{1}{L}\right\}$  for sufficiently large  $k \geq 0$ . Then*

$$\bar{C}_0 := \sup_{k \geq 0} \mathbb{E}[U(\bar{X}_k) + |\bar{Y}_k|^2] < \infty. \tag{4.1}$$

*Proof.* Similar to the proof of Proposition 3.2, we introduce the Lyapunov function

$\tilde{R}(x, y) = U(x) + \frac{|y|^2}{2} + \tilde{\beta}x \cdot y$  for all  $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ . Then, to prove (4.1), it is equivalent to prove that  $\left(\mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right]\right)_{k \geq 0}$  is uniformly bounded.

Denote  $\theta_k := 1 - e^{-\Delta t_k} \leq \Delta t_k$ , so that the discrete time scheme (2.4) can be rewritten as

$$\begin{cases} \bar{X}_{k+1} = \bar{X}_k + \theta_k \bar{Y}_k - (\Delta t_k - \theta_k) \nabla_x U(\bar{X}_k) + D_x(k), \\ \bar{Y}_{k+1} = e^{-\Delta t_k} \bar{Y}_k - \theta_k \nabla_x U(\bar{X}_k) + D_y(k), \end{cases}$$

where  $(D_x(k), D_y(k))$  is a Gaussian vector in  $\mathbb{R}^d \times \mathbb{R}^d$  with mean zero and covariance matrix  $\Sigma_k$  given by (2.5). Then,

$$\begin{aligned} & \mathbb{E} [U(\bar{X}_{k+1})] \\ &= \mathbb{E} [U(\bar{X}_k)] + \mathbb{E} \left[ \int_0^1 \nabla_x U(\bar{X}_k + t(\bar{X}_{k+1} - \bar{X}_k)) \cdot (\bar{X}_{k+1} - \bar{X}_k) dt \right] \\ &= \mathbb{E} [U(\bar{X}_k)] + \mathbb{E} \left[ \int_0^1 \nabla_x U(\bar{X}_k) \cdot (\bar{X}_{k+1} - \bar{X}_k) dt \right] \\ & \quad + \mathbb{E} \left[ \int_0^1 (\nabla_x U(\bar{X}_k + t(\bar{X}_{k+1} - \bar{X}_k)) - \nabla_x U(\bar{X}_k)) \cdot (\bar{X}_{k+1} - \bar{X}_k) dt \right] \end{aligned}$$

Then by the Lipschitz property of  $\nabla_x U$ , we derive

$$\begin{aligned} \mathbb{E} [U(\bar{X}_{k+1})] &\leq \mathbb{E} [U(\bar{X}_k)] + \frac{L}{2} \mathbb{E} [|\bar{X}_{k+1} - \bar{X}_k|^2] + \mathbb{E} [\nabla_x U(\bar{X}_k) \cdot (\bar{X}_{k+1} - \bar{X}_k)] \\ &= \mathbb{E} [U(\bar{X}_k)] + \frac{L}{2} \theta_k^2 \mathbb{E} [|\bar{Y}_k|^2] \\ & \quad + \left( \frac{L(\Delta t_k - \theta_k)^2}{2} - (\Delta t_k - \theta_k) \right) \mathbb{E} [|\nabla_x U(\bar{X}_k)|^2] \\ & \quad + (\theta_k - L\theta_k(\Delta t_k - \theta_k)) \mathbb{E} [\bar{Y}_k \cdot \nabla_x U(\bar{X}_k)] + \frac{L}{2} \Sigma_{11}(k)d. \end{aligned}$$

Further, we compute directly that

$$\begin{aligned} \mathbb{E} [|\bar{Y}_{k+1}|^2] &= \mathbb{E} \left[ |e^{-\Delta t_k} \bar{Y}_k - \theta_k \nabla_x U(\bar{X}_k) + D_y(k)|^2 \right] \\ &= e^{-2\Delta t_k} \mathbb{E} [|\bar{Y}_k|^2] + \theta_k^2 \mathbb{E} [|\nabla_x U(\bar{X}_k)|^2] \\ & \quad - 2\theta_k e^{-\Delta t_k} \mathbb{E} [\bar{Y}_k \cdot \nabla_x U(\bar{X}_k)] + \Sigma_{22}(k)d \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} [\bar{X}_{k+1} \cdot \bar{Y}_{k+1}] &= \mathbb{E} \left[ (\bar{X}_k + \theta_k \bar{Y}_k - (\Delta t_k - \theta_k) \nabla_x U(\bar{X}_k) \right. \\ & \quad \left. + D_x(k)) \cdot (e^{-\Delta t_k} \bar{Y}_k - \theta_k \nabla_x U(\bar{X}_k) + D_y(k)) \right] \\ &= e^{-\Delta t_k} \mathbb{E} [\bar{X}_k \cdot \bar{Y}_k] - \theta_k \mathbb{E} [\bar{X}_k \cdot \nabla_x U(\bar{X}_k)] + \theta_k e^{-\Delta t_k} \mathbb{E} [|\bar{Y}_k|^2] \\ & \quad - (\theta_k^2 + (\Delta t_k - \theta_k)e^{-\Delta t_k}) \mathbb{E} [\bar{Y}_k \cdot \nabla_x U(\bar{X}_k)] \\ & \quad + \theta_k(\Delta t_k - \theta_k) \mathbb{E} [|\nabla_x U(\bar{X}_k)|^2] + \Sigma_{12}(k)d. \end{aligned}$$

Therefore, by the definition of  $\tilde{R}$ , it follows that

$$\begin{aligned} & \frac{1}{\Delta t_k} \left( \mathbb{E} \left[ \tilde{R}(\bar{X}_{k+1}, \bar{Y}_{k+1}) \right] - \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] \right) \\ \leq & \frac{1}{\Delta t_k} \left( \left( \frac{L}{2} \theta_k^2 + \frac{1}{2} (e^{-2\Delta t_k} - 1) + \tilde{\beta} \theta_k e^{-\Delta t_k} \right) \mathbb{E} \left[ |\bar{Y}_k|^2 \right] + \tilde{\beta} (e^{-\Delta t_k} - 1) \mathbb{E} \left[ \bar{X}_k \cdot \bar{Y}_k \right] \right. \\ & + \left( \frac{L(\Delta t_k - \theta_k)^2}{2} - (\Delta t_k - \theta_k) + \frac{1}{2} \theta_k^2 + \tilde{\beta} \theta_k (\Delta t_k - \theta_k) \right) \mathbb{E} \left[ |\nabla_x U(\bar{X}_k)|^2 \right] \\ & + \left( \theta_k - L\theta_k(\Delta t_k - \theta_k) - \theta_k e^{-\Delta t_k} - \tilde{\beta} (\theta_k^2 + (\Delta t_k - \theta_k) e^{-\Delta t_k}) \right) \mathbb{E} \left[ \bar{Y}_k \cdot \nabla_x U(\bar{X}_k) \right] \\ & \left. - \tilde{\beta} \theta_k \mathbb{E} \left[ \bar{X}_k \cdot \nabla_x U(\bar{X}_k) \right] + \frac{L}{2} \Sigma_{11}(k)d + \frac{1}{2} \Sigma_{22}(k)d + \tilde{\beta} \Sigma_{12}(k)d \right). \end{aligned}$$

Using the fact that  $\tilde{\beta} \leq \frac{1}{2}$ ,  $\theta_k \leq \Delta t_k \leq \frac{1}{L} \leq 1$ , and the expressions of  $\Sigma_{ij}(k)$  in (2.5), we have

$$\frac{L}{2} \theta_k^2 + \frac{1}{2} (e^{-2\Delta t_k} - 1) + \tilde{\beta} \theta_k e^{-\Delta t_k} \leq -\frac{1}{2} \Delta t_k + \left( \frac{L}{2} + 2 - \frac{3}{2} \tilde{\beta} \right) \Delta t_k^2,$$

$$\frac{L(\Delta t_k - \theta_k)^2}{2} - (\Delta t_k - \theta_k) + \frac{1}{2} \theta_k^2 + \tilde{\beta} \theta_k (\Delta t_k - \theta_k) \leq \frac{1}{6} \Delta t_k^3 + \frac{L}{8} \Delta t_k^4,$$

$$\theta_k - L\theta_k(\Delta t_k - \theta_k) - \theta_k e^{-\Delta t_k} - \tilde{\beta} (\theta_k^2 + (\Delta t_k - \theta_k) e^{-\Delta t_k}) \leq \left( 1 - \frac{3}{2} \tilde{\beta} \right) \Delta t_k^2 + \left( 2\tilde{\beta} - \frac{L}{2} \right) \Delta t_k^3,$$

and

$$\frac{L}{2} \Sigma_{11}(k)d + \frac{1}{2} \Sigma_{22}(k)d + \tilde{\beta} \Sigma_{12}(k)d \leq \frac{5}{2} \varepsilon_k \Delta t_k d.$$

Thus, by  $(r, m)$ -dissipativity of  $U$  and the  $L$ -Lipschitz of  $\nabla_x U$ , one has

$$\begin{aligned} & \frac{1}{\Delta t_k} \left( \mathbb{E} \left[ \tilde{R}(\bar{X}_{k+1}, \bar{Y}_{k+1}) \right] - \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] \right) \\ \leq & \frac{1}{\Delta t_k} \left\{ \left( -\frac{1}{2} \Delta t_k + \left( \frac{L}{2} + 2 - \frac{3}{2} \tilde{\beta} \right) \Delta t_k^2 \right) \mathbb{E} \left[ |\bar{Y}_k|^2 \right] + \left( \frac{1}{6} \Delta t_k^3 + \frac{L}{8} \Delta t_k^4 \right) \mathbb{E} \left[ |\nabla_x U(\bar{X}_k)|^2 \right] \right. \\ & \left( \left( 1 - \frac{3}{2} \tilde{\beta} \right) \Delta t_k^2 + \left( 2\tilde{\beta} - \frac{L}{2} \right) \Delta t_k^3 \right) \frac{\mathbb{E} \left[ |\bar{Y}_k|^2 \right] + \mathbb{E} \left[ |\nabla_x U(\bar{X}_k)|^2 \right]}{2} \\ & \left. + \tilde{\beta} \left( -\Delta t_k + \frac{\Delta t_k^2}{2} \right) \mathbb{E} \left[ \bar{X}_k \cdot \bar{Y}_k \right] - \tilde{\beta} \left( \Delta t_k - \frac{1}{2} \Delta t_k^2 \right) \left( r \mathbb{E} \left[ |\bar{X}_k|^2 \right] - m \right) + \frac{5}{2} \varepsilon_k \Delta t_k d \right\} \end{aligned}$$

and hence there exists some constant  $b$  such that

$$\begin{aligned} & \frac{1}{\Delta t_k} \left( \mathbb{E} \left[ \tilde{R}(\bar{X}_{k+1}, \bar{Y}_{k+1}) \right] - \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] \right) \\ \leq & -\tilde{\beta} r \mathbb{E} \left[ |\bar{X}_k|^2 \right] - \frac{1}{2} \mathbb{E} \left[ |\bar{Y}_k|^2 \right] - \tilde{\beta} \left( 1 - \frac{1}{2} \Delta t_k \right) \mathbb{E} \left[ \bar{X}_k \cdot \bar{Y}_k \right] \\ & + \left( \left( L^2 + \frac{1}{2} \tilde{\beta} r \right) \Delta t_k + L^2 (2\tilde{\beta} + 1) \Delta t_k^2 \right) \mathbb{E} \left[ |\bar{X}_k|^2 \right] \\ & + \left( \frac{L+5}{2} \Delta t_k + \tilde{\beta} \Delta t_k^2 \right) \mathbb{E} \left[ |\bar{Y}_k|^2 \right] + b \\ \leq & -\tilde{\beta} r \mathbb{E} \left[ |\bar{X}_k|^2 \right] - \frac{1}{2} \mathbb{E} \left[ |\bar{Y}_k|^2 \right] + \tilde{\beta} \left( c \mathbb{E} \left[ |\bar{X}_k|^2 \right] + \frac{1}{4c} \mathbb{E} \left[ |\bar{Y}_k|^2 \right] \right) \\ & + \Delta t_k \left( \frac{L+6}{2} \mathbb{E} \left[ |\bar{Y}_k|^2 \right] + \left( 3L^2 + \frac{r}{4} \right) \mathbb{E} \left[ |\bar{X}_k|^2 \right] \right) + b \end{aligned}$$



for arbitrary  $c$ . Now choose  $c = \frac{r}{2}$ . Notice that  $\tilde{\beta} < r$  by its definition, then

$$\begin{aligned} & \frac{1}{\Delta t_k} \left( \mathbb{E} \left[ \tilde{R}(\bar{X}_{k+1}, \bar{Y}_{k+1}) \right] - \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] \right) \\ & \leq -\frac{1}{2} \tilde{\beta} r \mathbb{E} \left[ |\bar{X}_k|^2 \right] - \frac{1}{2} \left( 1 - \frac{\tilde{\beta}}{r} \right) \mathbb{E} \left[ |\bar{Y}_k|^2 \right] \\ & \quad + \Delta t_k \left( \frac{L+6}{2} \mathbb{E} \left[ |\bar{Y}_k|^2 \right] + \left( 3L^2 + \frac{r}{4} \right) \mathbb{E} \left[ |\bar{X}_k|^2 \right] \right) + b. \end{aligned}$$

Using (2.1), we derive

$$\begin{aligned} & \frac{1}{\Delta t_k} \left( \mathbb{E} \left[ \tilde{R}(\bar{X}_{k+1}, \bar{Y}_{k+1}) \right] - \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] \right) \\ & \leq -\frac{\tilde{\beta} r}{2L} \mathbb{E} \left[ U(\bar{X}_k) \right] - \frac{1}{2} \left( 1 - \frac{\tilde{\beta}}{r} \right) \mathbb{E} \left[ |\bar{Y}_k|^2 \right] \\ & \quad + \Delta t_k \left( \left( \frac{9L^2}{r} + \frac{3}{4} \right) \mathbb{E} \left[ U(\bar{X}_k) \right] + \frac{L+6}{2} \mathbb{E} \left[ |\bar{Y}_k|^2 \right] \right) + b_3 \\ & \leq -b_1 \left( \mathbb{E} \left[ U(\bar{X}_k) \right] + \frac{1}{2} \mathbb{E} \left[ |\bar{Y}_k|^2 \right] \right) + b_2 \Delta t_k \left( \mathbb{E} \left[ U(\bar{X}_k) \right] + \frac{1}{2} \mathbb{E} \left[ |\bar{Y}_k|^2 \right] \right) + b_3, \end{aligned}$$

where  $b_1 := \min \left\{ \frac{\tilde{\beta} r}{2L}, 1 - \frac{\tilde{\beta}}{r} \right\}$ ,  $b_2 := \max \left\{ \frac{9L^2}{r} + \frac{4}{3}, L + 6 \right\}$ , and  $b_3 := b + \frac{\tilde{\beta} r K}{2L} + \left( \frac{9L^2}{r} + \frac{4}{3} \right) K$ .

Set  $b_4 := \max \left\{ 1 + \frac{3}{4r}, \frac{3}{2} \right\}$  and  $b_5 := \min \left\{ \frac{1}{2}, 1 - \frac{\tilde{\beta}^3}{r/3} \right\} > 0$ . Then, we have

$$b_5 \left( U(x) + \frac{1}{2} |y|^2 \right) - \frac{3\tilde{\beta}^2 K}{r} \leq \tilde{R}(x, y) \leq b_4 \left( U(x) + \frac{1}{2} |y|^2 \right) + \frac{3K}{4r}.$$

We thus obtain

$$\begin{aligned} & \frac{1}{\Delta t_k} \left( \mathbb{E} \left[ \tilde{R}(\bar{X}_{k+1}, \bar{Y}_{k+1}) \right] - \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] \right) \\ & \leq -\frac{b_1}{b_4} \left( \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] - \frac{3K}{4r} \right) + \Delta t_k \frac{b_2}{b_5} \left( \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] + \frac{3\tilde{\beta}^2 K}{r} \right) + b_3. \end{aligned}$$

Rearrange the above, we derive, for some constant  $C > 0$ , that

$$\mathbb{E} \left[ \tilde{R}(\bar{X}_{k+1}, \bar{Y}_{k+1}) \right] \leq \left( 1 - \frac{b_1}{b_4} \Delta t_k + \frac{b_2}{b_5} \Delta t_k^2 \right) \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] + C \Delta t_k.$$

When  $\Delta t_k \leq \frac{\eta^*}{2}$ , we have  $1 - \frac{b_1}{b_4} \Delta t_k + \frac{b_2}{b_5} \Delta t_k^2 \leq 1 - \frac{b_1}{2b_4} \Delta t_k$ , which implies that  $\left( \mathbb{E} \left[ \tilde{R}(\bar{X}_k, \bar{Y}_k) \right] \right)_{k \geq 0}$  is uniformly bounded.  $\square$

**Remark 4.2.** Under the conditions in Theorem 2.8, we have

$$\lim_{k \rightarrow \infty} T_k = \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \Delta t_k = 0,$$

so the conditions of Proposition 4.1 hold.

By Proposition 4.1, we can easily obtain that  $(\mathbb{E} [|\bar{Y}_t|^2])_{t \geq 0}$  is also uniformly bounded. Moreover, we have the following fine estimation on the increment of  $\bar{X}$ :

**Proposition 4.3.** *There exists a constant  $C > 0$ , such that*

$$\sup_{t \in [T_k, T_{k+1}]} \mathbb{E} \left[ |\bar{X}_t - \bar{X}_k|^2 \right] \leq C \Delta t_k^2, \quad \text{for all } k \geq 0.$$

*Proof.* For  $t \in [T_k, T_{k+1}]$ , notice that  $\bar{X}_t = \bar{X}_k + \int_{T_k}^t \bar{Y}_s ds$ . It then follows by Cauchy-Schwarz inequality that

$$\mathbb{E} \left[ |\bar{X}_t - \bar{X}_k|^2 \right] = \mathbb{E} \left[ \left| \int_{T_k}^t \bar{Y}_s ds \right|^2 \right] \leq (t - T_k) \int_{T_k}^t \mathbb{E} \left[ |\bar{Y}_s|^2 \right] ds \leq C \Delta t_k^2$$

for some constant  $C$  independent of  $k \geq 0$ . □

#### 4.2 One-step entropy evolution with fixed temperature

In this subsection, we aim at adapt the method in [25] to estimate the difference term (2.14), which is the one-step evolution of  $H_{\gamma(\varepsilon)}(\bar{\mu}_t | \mu_k^*)$  with fixed  $\varepsilon$ . The computation framework is similar to what we have done in Section 3 for continuous-time setting. Though there are some differences, the computation details in this subsection is similar to [25, Section 4]. We leave most of the proof in the Appendix B for completeness.

We first provide the Fokker-Planck equation satisfied by  $\bar{\mu}_t$  on each interval  $[T_k, T_{k+1}]$ . Recall that  $\bar{p}_t$  denotes the density function of  $\bar{\mu}_t$ . For  $t \in [T_k, T_{k+1}]$  and  $z_k \in \mathbb{R}^d \times \mathbb{R}^d$ , denote by

$$z \mapsto \bar{p}_{t|T_k}(z|z_k) \text{ the density function of } \mathcal{L}(\bar{Z}_t | \bar{Z}_{T_k} = z_k),$$

that is, the conditional distribution of  $\bar{Z}_t$  knowing  $\bar{Z}_{T_k} = z_k$ . Further, similar to (3.3), define

$$v_{t,k}(z) := \begin{pmatrix} y \\ -\nabla_x U(x) - y - \varepsilon_k \nabla_y \log \bar{p}_t \end{pmatrix} \text{ and } \hat{v}_{t,k}(z) := \begin{pmatrix} y \\ -\nabla_x U(x_k) - y - \varepsilon_k \nabla_y \log \bar{p}_t \end{pmatrix}.$$

**Lemma 4.4.** For all  $k \geq 0$ ,  $\bar{p}_t$  satisfies

$$\partial_t \bar{p}_t = -\nabla_z \cdot (\bar{p}_t v_{t,k}) - \int \left[ \nabla_z \cdot (\bar{p}_{t|T_k}(\cdot|z_k) (\hat{v}_{t,k} - v_{t,k})) \right] d\bar{\mu}_k(z_k), \quad t \in [T_k, T_{k+1}].$$

*Proof.* Conditioned on the  $\sigma$ -field  $\bar{\mathcal{F}}_{T_k} := \sigma(\bar{Z}_t : 0 \leq t \leq T_k)$ ,  $\bar{Z}$  follows a linear SDE with deterministic parameters on  $[T_k, T_{k+1}]$  (see (1.3)). Thus, the conditional density function  $z \mapsto p(z | z_k)$  satisfies the Fokker-Planck equation

$$\begin{aligned} \partial_t \bar{p}_{t|T_k}(z|z_k) &= -\nabla_z \cdot (\bar{p}_{t|T_k}(z|z_k) \hat{v}_{t,k}(z)) \\ &= -\nabla_z \cdot (\bar{p}_{t|T_k}(z|z_k) v_{t,k}(z)) - \nabla_z \cdot (\bar{p}_{t|T_k}(z|z_k) (\hat{v}_{t,k}(z) - v_{t,k}(z))). \end{aligned}$$

Notice that  $v_{t,k}$  is independent of  $z_k$ . We can then complete the proof by integrating both sides of the above equality with respect to  $z_k$  under the measure  $\bar{\mu}_k(z_k)$ . □

For  $t \in [T_k, T_{k+1}]$ , let  $\tilde{h}_t = \sqrt{\frac{\bar{p}_t}{p_k}}$ , we write the distorted relative entropy

$$H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*) = 4 \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \tilde{h}_t, S \nabla_z \log \tilde{h}_t \right\rangle \right] + 2\gamma(\varepsilon_k) \mathbb{E}_{\bar{\mu}_t} \left[ \log \tilde{h}_t \right].$$

Denote  $(\tilde{\nabla}_z)^* = (\tilde{\nabla}_x^*, \tilde{\nabla}_y^*) := \left( -\nabla_x \cdot + \frac{1}{\varepsilon_k} \nabla_x U \cdot, -\nabla_y \cdot + \frac{y}{\varepsilon_k} \cdot \right)$  as the adjoint operator of  $\nabla_z$  w.r.t.  $\mu_k^*$ , in the sense that for any  $f, g \in L^2(\mu_k^*)$ ,  $\mathbb{E}_{\mu_k^*} [f \cdot \nabla_z g] = \mathbb{E}_{\mu_k^*} [g \tilde{\nabla}_z^* f]$ . Then, for

$t \in [T_k, T_{k+1}]$ , by Lemma 4.4, we have

$$\begin{aligned} \frac{d}{dt} H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*) &= \int \frac{\delta H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*)}{\delta \bar{\mu}_t} \partial_t \bar{p}_t \, dz \\ &= \int \left\langle \nabla_z \left( \frac{\delta H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*)}{\delta \bar{\mu}_t} \right), v_{t,k} \right\rangle \bar{p}_t \, dz \end{aligned} \tag{4.2}$$

$$+ \int \left\langle \nabla_z \left( \frac{\delta H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*)}{\delta \bar{\mu}_t} \right), \int \bar{p}_{t|T_k}(z|z_k)(\hat{v}_{t,k} - v_{t,k})(z) \bar{p}_k(z_k) dz_k \right\rangle dz. \tag{4.3}$$

Computing (4.2) is very similar to Proposition 3.3 in continuous-time setting. And we mimic the proof of [25, Lemma 3] to compute (4.3). We summarize the result as follows, and the proof can be found in Appendix B.1.

Let  $M_k := \begin{pmatrix} 2I_d & 2I_d - \nabla_x^2 U \\ 2I_d - \nabla_x^2 U & (2 + \gamma(\varepsilon_k)\varepsilon_k) I_d - 2\nabla_x^2 U \end{pmatrix} \in \mathcal{M}_{2d \times 2d}(\mathbb{R})$ , and

$$A_t(z) := \frac{1}{\bar{p}_t(z)} \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k. \tag{4.4}$$

**Proposition 4.5.** *One has*

$$\begin{aligned} \frac{d}{dt} H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*) &= -4 \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \tilde{h}_t, M_k \nabla_z \log \tilde{h}_t \right\rangle \right] - 8\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z \nabla_y \log \tilde{h}_t \right\rangle_F \right] \\ &+ 4 \int \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z A_t(z) \right\rangle_F d\bar{\mu}_t(z) \end{aligned} \tag{4.5}$$

$$+ \int \left\langle \frac{4}{\varepsilon_k} \nabla_x \log \tilde{h}_t + \left( 2\gamma(\varepsilon_k) + \frac{4}{\varepsilon_k} \right) \nabla_y \log \tilde{h}_t, \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz, \tag{4.6}$$

By properly estimate terms (4.5) and (4.6) in above proposition, we could get the following estimates.

**Proposition 4.6.**

$$\begin{aligned} \frac{d}{dt} H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*) &\leq -\mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \frac{\bar{p}_t}{p_k^*}, \tilde{M}_k \nabla_z \log \frac{\bar{p}_t}{p_k^*} \right\rangle \right] + \frac{144d^{\frac{3}{2}}L^2}{\varepsilon_k} \Delta t_k^2 \\ &+ \left( \frac{(L')^2}{\varepsilon_k} + \frac{4L^2}{\varepsilon_k^2} \left( 2 + \frac{1}{2}\varepsilon_k\gamma(\varepsilon_k) \right) \right) \mathbb{E} \left[ |\bar{X}_t - \bar{X}_k|^2 \right], \end{aligned}$$

with

$$\tilde{M}_k := \begin{pmatrix} \frac{7}{4}I_d & 2I_d - \nabla_x^2 U \\ 2I_d - \nabla_x^2 U & \left( \frac{7}{4} + \frac{7}{8}\gamma(\varepsilon_k)\varepsilon_k \right) I_d - 2\nabla_x^2 U \end{pmatrix} \in \mathcal{M}_{2d \times 2d}(\mathbb{R}).$$

The main idea of the proof is borrowed from the proof of [25, Lemma 4]. We nevertheless sketch it in Appendix B.2 for completeness.

Recall that  $\rho_k$  is the constant in the LSI satisfied by  $\mu_k^*$  in Proposition 3.1. As the discrete time analogue to Lemma 3.4, we have the following inequality for  $\tilde{M}_k$ . The proof is omitted because it is similar to that of Lemma 3.4.

**Lemma 4.7.** *Let  $(\varepsilon_k)_{k \geq 0}$  be given as in Assumption 2.3 and  $\nabla_x U$  be  $L$ -Lipschitz. Choose  $\gamma(\varepsilon_k) := \frac{4}{\varepsilon_k} (1 + L^2)$  and  $c(\varepsilon_k) := \frac{1}{\gamma(\varepsilon_k)}$ . Then, for sufficiently large  $k \geq 0$ , we have  $\tilde{M}_k \succeq c(\varepsilon_k)\rho_k \left( S + \frac{\gamma(\varepsilon_k)}{2\rho_k} I_{2d} \right)$ .*

We finally present the main result in this subsection.

**Proposition 4.8.** *Let Assumptions 2.1, 2.3 hold. Suppose in addition that  $\nabla_x^2 U$  is  $L'$ -Lipschitz and  $\Delta t_k \leq \min\left\{\frac{\eta_*}{2}, \frac{1}{L}\right\}$  for sufficiently large  $k \geq 0$ . Further choose  $\gamma(\varepsilon_k) = \frac{1}{c(\varepsilon_k)} = \frac{4}{\varepsilon_k}(1 + L^2)$ . Then, there exist some constants  $C_1 > 0$  and  $C_2 > 0$  such that, for sufficiently large  $k \geq 0$  and for all  $\alpha > 0$ ,*

$$H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1}|\mu_k^*) \leq \left(1 - C_1 \Delta t_k T_k^{-\left(\frac{E_*}{E} + \alpha\right)}\right) H_{\gamma(\varepsilon_k)}(\bar{\mu}_k|\mu_k^*) + C_2 \Delta t_k^3 (\log T_k)^2.$$

*Proof.* First, for sufficiently large  $k \geq 0$ , it follows from Lemma 4.7 that

$$\begin{aligned} & - \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \frac{\bar{p}_t}{p_k^*}, \tilde{M}_k \nabla_z \log \frac{\bar{p}_t}{p_k^*} \right\rangle \right] \\ & \leq -c(\varepsilon_k) \rho_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \frac{\bar{p}_t}{p_k^*}, S \nabla_z \log \frac{\bar{p}_t}{p_k^*} \right\rangle + \frac{\gamma(\varepsilon_k)}{2\rho_k} \left| \nabla_z \log \frac{\bar{p}_t}{p_k^*} \right|^2 \right]. \end{aligned}$$

Using the LSI satisfied by  $\mu_k^*$  in Proposition 3.1, we obtain that

$$- \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \frac{\bar{p}_t}{p_k^*}, \tilde{M}_k \nabla_z \log \frac{\bar{p}_t}{p_k^*} \right\rangle \right] \leq -c(\varepsilon_k) \rho_k H_{\gamma(\varepsilon_k)}(\bar{\mu}_t|\mu_k^*). \tag{4.7}$$

Next, plugging (4.7) into the estimate of  $\frac{d}{dt} H_{\gamma(\varepsilon_k)}(\bar{\mu}_t|\mu_k^*)$  in Proposition 4.6 and recalling Proposition 4.3, we derive

$$\frac{d}{dt} H_{\gamma(\varepsilon_k)}(\bar{\mu}_t|\mu_k^*) \leq -c(\varepsilon_k) \rho_k H_{\gamma(\varepsilon_k)}(\bar{\mu}_t|\mu_k^*) + C \varepsilon_k^{-2} \Delta t_k^2.$$

Recall that  $\varepsilon_k = \frac{E}{\log T_k}$  with  $E > E_*$  for sufficiently large  $k \geq 0$  and that  $\rho_k = \chi(\varepsilon_k) e^{-\frac{E_*}{\varepsilon_k}}$ , where  $\varepsilon_k \log \chi(\varepsilon_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Then, for sufficiently large  $k \geq 0$  and for all  $\alpha > 0$ , we have

$$\frac{d}{dt} H_{\gamma(\varepsilon_k)}(\bar{\mu}_t|\mu_k^*) \leq -C T_k^{-\left(\frac{E_*}{E} + \alpha\right)} H_{\gamma(\varepsilon_k)}(\bar{\mu}_t|\mu_k^*) + C \Delta t_k^2 (\log T_k)^2.$$

By computing  $\frac{d}{dt} \left( e^{C t} T_k^{-\left(\frac{E_*}{E} + \alpha\right)} H_{\gamma(\varepsilon_k)}(\bar{\mu}_t|\mu_k^*) \right)$  and then integrating it from  $T_k$  to  $T_{k+1}$ , we obtain

$$\begin{aligned} & H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1}|\mu_k^*) \\ & \leq e^{-C \Delta t_k T_k^{-\left(\frac{E_*}{E} + \alpha\right)}} H_{\gamma(\varepsilon_k)}(\bar{\mu}_k|\mu_k^*) + C \Delta t_k^2 (\log T_k)^2 T_k^{\left(\frac{E_*}{E} + \alpha\right)} \left(1 - e^{-C \Delta t_k T_k^{-\left(\frac{E_*}{E} + \alpha\right)}}\right) \\ & \leq \left(1 - C_1 \Delta t_k T_k^{-\left(\frac{E_*}{E} + \alpha\right)}\right) H_{\gamma(\varepsilon_k)}(\bar{\mu}_k|\mu_k^*) + C_2 \Delta t_k^3 (\log T_k)^2, \end{aligned}$$

where  $C_1 > 0$  and  $C_2 > 0$  are some constants independent of  $k$ . The proof then completes.  $\square$

### 4.3 One-step entropy evolution with cooling temperature

Next, we compute the term in (2.15), which is the one-step change of  $H_{\gamma(\varepsilon)}(\bar{\mu}_{k+1}|\mu_{\varepsilon}^*)$  when the temperature  $\varepsilon$  in the discrete scheme (1.3) is updated from  $\varepsilon_k$  to  $\varepsilon_{k+1}$  in the  $k$ -th step.

**Lemma 4.9.** *Let Assumptions 2.1 and 2.3 hold. Then, there exists a constant  $C > 0$  such that for sufficiently large  $k \geq 0$  and for all  $\alpha > 0$ ,*

$$H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1}|\mu_{k+1}^*) \leq H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1}|\mu_k^*) + C \Delta t_k T_k^{-1+\alpha}.$$

*Proof.* The proof is a discrete time analogue of Lemma 3.7. We use  $C > 0$  to denote a generic constant whose value may change from line to line. First, use similar arguments to the ones in the proof of Lemma 3.7, we obtain

$$\begin{aligned} & \partial_\varepsilon \mathbb{E}_{\bar{\mu}_{k+1}} \left[ \left\langle \nabla_z \log \frac{\bar{p}_{k+1}}{p_\varepsilon^*}, S \nabla_z \log \frac{\bar{p}_{k+1}}{p_\varepsilon^*} \right\rangle \right] \\ & \leq \mathbb{E}_{\bar{\mu}_{k+1}} \left[ \left\langle \nabla_z \log \frac{\bar{p}_{k+1}}{p_\varepsilon^*}, S \nabla_z \log \frac{\bar{p}_{k+1}}{p_\varepsilon^*} \right\rangle \right] + C\varepsilon^{-4}(1 + \bar{C}_0) \end{aligned}$$

and

$$\partial_\varepsilon \left( \gamma(\varepsilon) \mathbb{E}_{\bar{\mu}_{k+1}} \left[ \log \frac{\bar{p}_{k+1}}{p_\varepsilon^*} \right] \right) \leq C\varepsilon^{-2} \mathbb{E}_{\bar{\mu}_{k+1}} \left[ \log \frac{\bar{p}_{k+1}}{p_\varepsilon^*} \right] + C\varepsilon^{-3}(1 + \bar{C}_0).$$

It follows that

$$\partial_\varepsilon H_{\gamma(\varepsilon)}(\bar{\mu}_{k+1} | \mu_\varepsilon^*) \leq (1 + \varepsilon^{-1}) H_{\gamma(\varepsilon)}(\bar{\mu}_{k+1} | \mu_\varepsilon^*) + C\varepsilon^{-4}(1 + \bar{C}_0).$$

Recall that  $\varepsilon_t = \frac{E}{\log t}$  with  $E > E_*$  for large  $t$ . Then, for any sufficiently large  $k$  and  $t \in [T_k, T_{k+1}]$ , we have  $\varepsilon_t \in [\varepsilon_{k+1}, \varepsilon_k]$  and

$$\partial_\varepsilon H_{\gamma(\varepsilon_t)}(\bar{\mu}_{k+1} | \mu_{\varepsilon_t}^*) \leq C \log T_{k+1} H_{\gamma(\varepsilon_t)}(\bar{\mu}_{k+1} | \mu_{\varepsilon_t}^*) + C(\log T_{k+1})^4(1 + \bar{C}_0).$$

It follows by Grönwall lemma that

$$H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1} | \mu_{\varepsilon_{k+1}}^*) \leq e^{C(\varepsilon_{k+1} - \varepsilon_k) \log T_{k+1}} H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1} | \mu_k^*) + C(\varepsilon_k - \varepsilon_{k+1})(\log T_{k+1})^4.$$

Notice that  $e^{C(\varepsilon_{k+1} - \varepsilon_k) \log T_{k+1}} \leq 1$  and  $\varepsilon_k - \varepsilon_{k+1} \leq C\Delta t_k T_k^{-1}(\log T_k)^{-2}$ . As a result, for any  $\alpha > 0$ , we have

$$\begin{aligned} H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1} | \mu_{\varepsilon_{k+1}}^*) & \leq H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1} | \mu_k^*) + C\Delta t_k T_k^{-1}(\log T_{k+1})^4(\log T_k)^{-2} \\ & \leq H_{\gamma(\varepsilon_k)}(\bar{\mu}_{k+1} | \mu_k^*) + C_3\Delta t_k T_k^{-1+\alpha}. \end{aligned}$$

□

#### 4.4 Proof of Theorem 2.8

For each  $k \geq 0$ , let  $\tilde{Z}_k = (\tilde{X}_k, \tilde{Y}_k)$  be a random variable with distribution  $\mu_k^*$  in the same probability space supporting  $(\bar{X}, \bar{Y})$ . Similar to (3.6), we can show that, for each  $\delta > 0$ ,

$$\mathbb{P}(U(\bar{X}_k) > \delta) \leq \mathbb{P}(U(\tilde{X}_k) > \delta) + \sqrt{2H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*)}. \quad (4.8)$$

The term  $\mathbb{P}(U(\tilde{X}_k) > \delta)$  can be handled by Lemma 3.9 to obtain the desired convergence rate. For the convergence of  $\sqrt{2H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*)}$ , we can apply Proposition 4.8 and Lemma 4.9 to show that there are some positive constants  $c_1, c'_1, c_2$ , and  $k_0 \geq 0$  such that

$$\begin{aligned} & H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1} | \mu_{\varepsilon_{k+1}}^*) - H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*) \\ & \leq -c_1\Delta t_k T_k^{-\left(\frac{E_*}{E} + \alpha\right)} H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*) + c'_1\Delta t_k^3(\log T_k)^2 + c_2\Delta t_k T_k^{-1+\alpha}, \end{aligned}$$

for all  $k \geq k_0$ . Because  $\limsup_{k \rightarrow \infty} \Delta t_k \sqrt{T_k} < \infty$ , we can find some positive constant  $c_3$  such that

$$H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1} | \mu_{\varepsilon_{k+1}}^*) \leq \left(1 - c_1\Delta t_k T_k^{-\left(\frac{E_*}{E} + \alpha\right)}\right) H_{\gamma(\varepsilon_k)}(\bar{\mu}_k | \mu_k^*) + c_3\Delta t_k T_k^{-1+\alpha}.$$

It is sufficient to consider  $\alpha < \frac{1}{2} (1 - \frac{E_*}{E})$ . Then,

$$H_{\gamma(\varepsilon_{k+1})}(\bar{\mu}_{k+1}|\mu_{k+1}^*) - \frac{2c_3}{c_1} T_{k+1}^{-(1-\frac{E_*}{E}-2\alpha)} \leq \left(1 - c_1 \Delta t_k T_k^{-\left(\frac{E_*}{E} + \alpha\right)}\right) \left(H_{\gamma(\varepsilon_k)}(\bar{\mu}_k|\mu_k^*) - \frac{2c_3}{c_1} T_k^{-(1-\frac{E_*}{E}-2\alpha)}\right),$$

which implies that, for  $k \geq k_0$ ,

$$H_{\gamma(\varepsilon_k)}(\bar{\mu}_k|\mu_k^*) \leq \frac{2c_3}{c_1} T_k^{-(1-\frac{E_*}{E}-2\alpha)} + \prod_{j=k_0}^{k-1} \left(1 - c_1 \Delta t_j T_j^{-\left(\frac{E_*}{E} + \alpha\right)}\right) \cdot \left(H_{\gamma(\varepsilon_{k_0})}(\bar{\mu}_{k_0}|\mu_{k_0}^*) - \frac{2c_3}{c_1} T_{k_0}^{-(1-\frac{E_*}{E}-2\alpha)}\right).$$

Further, for all  $k \geq k_0$ , by the definition of Riemann integral,

$$\int_{T_{k_0}}^{T_k} z^{-\left(\frac{E_*}{E} + \alpha\right)} dz < \sum_{j=k_0}^{T_k} \Delta t_j T_j^{-\left(\frac{E_*}{E} + \alpha\right)}.$$

Notice that  $\frac{E_*}{E} + \alpha < 1$ , then

$$\sum_{j=k_0}^{\infty} \Delta t_j T_j^{-\left(\frac{E_*}{E} + \alpha\right)} > \int_{T_{k_0}}^{\infty} z^{-\left(\frac{E_*}{E} + \alpha\right)} dz = \infty, \text{ and hence } \prod_{j=k_0}^{\infty} \left(1 - c_1 \Delta t_j T_j^{-\left(\frac{E_*}{E} + \alpha\right)}\right) = 0.$$

Finally, by Lemma 4.10 which will be presented and shown momentarily, we know that  $H_{\gamma(\varepsilon_{k_0})}(\bar{\mu}_{k_0}|\mu_{k_0}^*)$  is finite. Then, there exists a positive constant  $C > 0$  such that

$$H_{\gamma(\varepsilon_k)}(\bar{\mu}_k|\mu_k^*) \leq C T_k^{-(1-\frac{E_*}{E}-2\alpha)}.$$

This completes the proof. □

We finally provide a discrete time analogue of the result in Lemma 3.8, which is needed in the proof of Theorem 2.8.

**Lemma 4.10.** *For every  $k \geq 0$ , the distorted entropy  $H_{\gamma(\varepsilon_k)}(\bar{\mu}_k|\mu_k^*)$  is finite.*

*Proof.* As in the proof of Lemma 3.8, it suffices to prove that the Fisher information  $I(\bar{\mu}_k|\mu_k^*)$  is finite for every finite  $k \geq 0$ . Again, we use  $C$  to denote a generic positive constant whose value may change from line to line.

For  $t \in [T_k, T_{k+1}]$ , with  $\tilde{h}_t = \sqrt{\frac{\bar{p}_t}{p_k^*}}$ , recall that

$$I(\bar{\mu}_t|\mu_k^*) = \mathbb{E}_{\bar{\mu}_t} \left[ \left| \nabla_z \log \frac{\bar{p}_t}{p_k^*} \right|^2 \right] = 4 \mathbb{E}_{\bar{\mu}_t} \left[ \left| \nabla_z \log \tilde{h}_t \right|^2 \right].$$

Using a similar computation to the one in the calculation of  $H_{\gamma(\varepsilon_k)}(\bar{\mu}_t|\mu_k^*)$  in Section 4.2, we can obtain that

$$\begin{aligned} \frac{d}{dt} I(\bar{\mu}_t|\mu_k^*) &= \int \left\langle \nabla_z \left( \frac{\delta I(\bar{\mu}_t|\mu_k^*)}{\delta \bar{\mu}_t} \right), v_{t,k} \right\rangle \bar{p}_t(z) dz \\ &+ \int \left\langle \nabla_z \left( \frac{\delta I(\bar{\mu}_t|\mu_k^*)}{\delta \bar{\mu}_t} \right), \int \bar{p}_{t|T_k}(z|z_k) (\hat{v}_{t,k} - v_{t,k}) \bar{p}_k(z_k) dz_k \right\rangle dz, \end{aligned}$$

where

$$\frac{\delta I(\bar{\mu}_t|\mu_k^*)}{\delta \bar{\mu}_t} = \frac{4}{\tilde{h}_t} \tilde{\nabla}_z^* \nabla_z \tilde{h}_t.$$

Following the same steps as in (A.1), (A.2) and (A.3), we further obtain that

$$\begin{aligned} & \int \left\langle \nabla_z \left( \frac{\delta I(\bar{\mu}_t | \mu_k^*)}{\delta \bar{\mu}_t} \right), v_{t,k} \right\rangle \bar{p}_t(z) \, dz \tag{4.9} \\ &= -4 \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \tilde{h}_t, M_I \nabla_z \log \tilde{h}_t \right\rangle \right] - 8\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\| \nabla_z \nabla_y \log \tilde{h}_t \right\|_F \right], \end{aligned}$$

where  $M_I = \begin{pmatrix} 0 & I_d - \nabla_x^2 U \\ I_d - \nabla_x^2 U & 2I_d \end{pmatrix}$ . Further, by similar arguments to the ones in Lemma B.1, and with  $A_t(z)$  as given by (4.4), we have

$$\begin{aligned} & \int \left\langle \nabla_z \left( \frac{\delta I(\bar{\mu}_t | \mu_k^*)}{\delta \bar{\mu}_t} \right), \int \bar{p}_{t|T_k}(z|z_k)(\hat{v}_{t,k} - v_{t,k}) \bar{p}_k(z_k) dz_k \right\rangle dz \tag{4.10} \\ &= \frac{4}{\varepsilon_k} \int \left\langle \nabla_y \log \tilde{h}_t, \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ & \quad + 4 \int \left\langle \nabla_z \nabla_y \log \tilde{h}_t, \nabla_z A_t(z) \right\rangle_F d\bar{\mu}_t(z). \end{aligned}$$

Following the same arguments as in Lemma B.2 and B.3, we can further obtain the inequalities

$$\begin{aligned} & \frac{4}{\varepsilon_k} \int \left\langle \nabla_y \log \tilde{h}_t, \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ & \leq \mathbb{E}_{\bar{\mu}_t} \left[ \left\| \nabla_y \log \tilde{h}_t \right\|^2 \right] + \frac{4L^2}{\varepsilon_k^2} \mathbb{E} \left[ |\bar{X}_t - \bar{X}_k|^2 \right], \end{aligned}$$

and

$$\begin{aligned} & 4 \int \left\langle \nabla_z \nabla_y \log \tilde{h}_t, \nabla_z A_t(z) \right\rangle_F d\bar{\mu}_t(z) \\ & \leq 8\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\| \nabla_z \nabla_y \log \tilde{h}_t \right\|_F^2 \right] + \frac{(L')^2}{\varepsilon_k} \mathbb{E} \left[ |\bar{X}_t - \bar{X}_k|^2 \right] + \frac{72}{\varepsilon_k} dL^2 \Delta t_k^2. \end{aligned}$$

Next, by Proposition 4.3, (4.9) and (4.10), it follows that

$$\frac{d}{dt} I(\bar{\mu}_t | \mu_k^*) \leq -\mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \frac{\bar{p}_t}{p_k^*}, \tilde{M}_I \nabla_z \log \frac{\bar{p}_t}{p_k^*} \right\rangle \right] + C\varepsilon_k^{-2} \Delta t_k^2,$$

where  $\tilde{M}_I = \begin{pmatrix} 0 & I_d - \nabla_x^2 U \\ I_d - \nabla_x^2 U & \frac{7}{4} I_d \end{pmatrix} \succeq -(1+L)I_{2d}$  due to the  $L$ -Lipschitz of  $\nabla_x U$ . Then,

$$\frac{d}{dt} I(\bar{\mu}_t | \mu_k^*) \leq (1+L)I(\bar{\mu}_t | \mu_k^*) + C\varepsilon_k^{-2} \Delta t_k^2,$$

which implies that

$$I(\bar{\mu}_{k+1} | \mu_k^*) \leq e^{(1+L)\Delta t_k} I(\bar{\mu}_k | \mu_k^*) + C\Delta t_k^3. \tag{4.11}$$

Finally, following similar computation steps to the ones in Lemma 4.9, we obtain that, for all  $\varepsilon > 0$ ,

$$\partial_\varepsilon I(\bar{\mu}_{k+1} | \mu_\varepsilon^*) = \partial_\varepsilon \mathbb{E}_{\bar{\mu}_{k+1}} \left[ \left| \nabla_z \log \frac{\bar{p}_{k+1}}{p_\varepsilon^*} \right|^2 \right] \leq I(\bar{\mu}_{k+1} | \mu_\varepsilon^*) + C\varepsilon^{-4}.$$

The above, together with (4.11), implies that

$$\begin{aligned} I(\bar{\mu}_{k+1} | \mu_{\varepsilon_0}^*) & \leq e^{(\varepsilon_{k+1}-\varepsilon_k)} I(\bar{\mu}_{k+1} | \mu_k^*) + C e^{\varepsilon_{k+1}} \int_{\varepsilon_k}^{\varepsilon_{k+1}} \varepsilon^{-4} e^{-\varepsilon} d\varepsilon \\ & \leq e^{(1+L)\Delta t_k} I(\bar{\mu}_k | \mu_k^*) + C\Delta t_k^3 \\ & \leq e^{(1+L)T_{k+1}} I(\mu_0 | \mu_{\varepsilon_0}^*) + e^{(1+L)T_{k+1}} \sum_{i=0}^k \Delta t_i^3, \end{aligned}$$

where the r.h.s. of the last inequality in the above is clearly finite for every  $k \geq 0$ .  $\square$

## Appendix A Proof in Section 3

### A.1 Proof of Proposition 3.3

By Proposition 2.22 in [24], we have

$$\frac{\delta H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*)}{\delta \mu_t} = \frac{4}{h_t} \nabla_z^* S \nabla_z h_t + 2\gamma(\varepsilon_t) \log h_t + \gamma(\varepsilon_t).$$

Then using (3.3) and (3.4), a simple calculation leads to

$$\begin{aligned} & \partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) \\ &= \int \nabla_z \left( \frac{4}{h_t} \nabla_z^* S \nabla_z h_t + 2\gamma(\varepsilon_t) \log h_t \right) \cdot v_t p_t dz \\ &= -4\varepsilon_t \gamma(\varepsilon_t) \int \left\langle \nabla_z h_t, \begin{pmatrix} 0 & -I_d \\ I_d & I_d \end{pmatrix} \nabla_z h_t \right\rangle p_{\varepsilon_t}^* dz \end{aligned} \tag{A.1}$$

$$+ 8\varepsilon_t \int \left\langle \nabla_z h_t, \begin{pmatrix} 0 & -I_d \\ I_d & I_d \end{pmatrix} \nabla_z h_t \right\rangle \frac{\nabla_z^* S \nabla_z h_t}{h_t} p_{\varepsilon_t}^* dz \tag{A.2}$$

$$- 8\varepsilon_t \int \left\langle \nabla_z \nabla_z^* S \nabla_z h_t, \begin{pmatrix} 0 & -I_d \\ I_d & I_d \end{pmatrix} \nabla_z h_t \right\rangle p_{\varepsilon_t}^* dz. \tag{A.3}$$

For (A.1), it equals to

$$-4\varepsilon_t \gamma(\varepsilon_t) \mathbb{E}_{\mu_{\varepsilon_t}^*} [|\nabla_y h_t|^2] = -\varepsilon_t \gamma(\varepsilon_t) \mathbb{E}_{\mu_t} \left[ \left| \nabla_y \log \frac{p_t}{p_{\varepsilon_t}^*} \right|^2 \right]. \tag{A.4}$$

In addition, (A.2) can be simplified as

$$\begin{aligned} & 8\varepsilon_t \int \left\langle \nabla_z h_t, \begin{pmatrix} 0 & -I_d \\ I_d & I_d \end{pmatrix} \nabla_z h_t \right\rangle \frac{\nabla_z^* S \nabla_z h_t}{h_t} p_{\varepsilon_t}^* dz \\ &= 8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \frac{|\nabla_y h_t|^2}{h_t} \nabla_z^* S \nabla_z h_t \right] \\ &= 8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \left\langle \frac{\nabla_z |\nabla_y h_t|^2}{h_t}, S \nabla_z h_t \right\rangle \right] - 8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \left\langle \frac{|\nabla_y h_t|^2}{h_t^2} \nabla_z h_t, S \nabla_z h_t \right\rangle \right] \\ &= 16\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \left\langle \frac{\nabla_z h_t (\nabla_y h_t)^T}{h_t}, S \nabla_z \nabla_y h_t \right\rangle_F \right] \\ & \quad - 8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \left\langle \frac{\nabla_z h_t (\nabla_y h_t)^T}{h_t}, S \frac{\nabla_z h_t (\nabla_y h_t)^T}{h_t} \right\rangle_F \right]. \end{aligned} \tag{A.5}$$

For (A.3), we write it as the sum of the following three terms:

$$\begin{aligned} I_1 &:= -8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \left\langle \nabla_z (\nabla_x^* \nabla_y + \nabla_y^* \nabla_x) h_t, \begin{pmatrix} 0 & -I_d \\ I_d & I_d \end{pmatrix} \nabla_z h_t \right\rangle \right], \\ I_2 &:= -8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \left\langle \nabla_z \nabla_x^* \nabla_x h_t, \begin{pmatrix} 0 & -I_d \\ I_d & I_d \end{pmatrix} \nabla_z h_t \right\rangle \right], \\ I_3 &:= -8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \left\langle \nabla_z \nabla_y^* \nabla_y h_t, \begin{pmatrix} 0 & -I_d \\ I_d & I_d \end{pmatrix} \nabla_z h_t \right\rangle \right]. \end{aligned}$$



Further, by [25, Lemma 9], we have

$$\begin{aligned} I_1 &= 8 \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_y h_t, \nabla_x^2 U \nabla_y h_t \rangle - \langle \nabla_x h_t, \nabla_x h_t + \nabla_y h_t \rangle] \\ &\quad - 16\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_y \nabla_x h_t, \nabla_y^2 h_t \rangle_F], \\ I_2 &= 8 \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_x h_t, \nabla_x^2 U \nabla_y h_t \rangle] - 8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_x \nabla_y h_t, \nabla_x \nabla_y h_t \rangle_F], \\ I_3 &= -8 \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_x h_t + \nabla_y h_t, \nabla_y h_t \rangle + \varepsilon_t \langle \nabla_y^2 h_t, \nabla_y^2 h_t \rangle_F]. \end{aligned}$$

Consequently, we obtain that (A.3) equals to

$$\begin{aligned} &- 8 \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_x h_t, \nabla_x h_t + (2I_d - \nabla_x^2 U) \nabla_y h_t \rangle] \\ &\quad - 8 \mathbb{E}_{\mu_{\varepsilon_t}^*} [\varepsilon_t \langle \nabla_z \nabla_y h_t, S \nabla_z \nabla_y h_t \rangle_F + \langle \nabla_y h_t, (I_d - \nabla_x^2 U) \nabla_y h_t \rangle]. \end{aligned}$$

Combining the above with (A.4) and (A.5), we derive

$$\begin{aligned} &\partial_{\mu,t} H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) \\ &= -8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_z \nabla_y h_t - (\nabla_z \log h_t)(\nabla_y h_t)^T, S(\nabla_z \nabla_y h_t - (\nabla_z \log h_t)(\nabla_y h_t)^T) \rangle_F] \\ &\quad - 4 \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_z h_t, M_t \nabla_z h_t \rangle] \\ &= -8\varepsilon_t \mathbb{E}_{\mu_t} [\langle \nabla_z \nabla_y \log h_t, S \nabla_z \nabla_y \log h_t \rangle_F] - 4 \mathbb{E}_{\mu_t} [\langle \nabla_z \log h_t, M_t \nabla_z \log h_t \rangle], \end{aligned}$$

which is the desired result.

### A.2 Proof of Lemma 3.8

Recall  $H_{\gamma(\varepsilon_t)}(\mu_t | \mu_{\varepsilon_t}^*) = \mathbb{E}_{\mu_t} \left[ \left| \nabla_x \log \frac{d\mu_t}{d\mu_{\varepsilon_t}^*} + \nabla_y \log \frac{d\mu_t}{d\mu_{\varepsilon_t}^*} \right|^2 + \gamma(\varepsilon_t) \log \frac{d\mu_t}{d\mu_{\varepsilon_t}^*} \right]$ , and notice that

$$\mathbb{E}_{\mu_t} \left[ \left| \nabla_x \log \frac{d\mu_t}{d\mu_{\varepsilon_t}^*} + \nabla_y \log \frac{d\mu_t}{d\mu_{\varepsilon_t}^*} \right|^2 \right] \leq 2I(\mu_t | \mu_{\varepsilon_t}^*).$$

Then, by the log-Sobolev inequality, it suffices to prove that  $I(\mu_t | \mu_{\varepsilon_t}^*)$  is uniformly bounded on any finite horizon.

As in (2.8), we write

$$\frac{d}{dt} I(\mu_t | \mu_{\varepsilon_t}^*) = \partial_{\mu,t} I(\mu_t | \mu_{\varepsilon_t}^*) + \partial_{\varepsilon,t} I(\mu_t | \mu_{\varepsilon_t}^*), \tag{A.6}$$

where

$$\partial_{\mu,t} I(\mu_t | \mu_{\varepsilon_t}^*) := \frac{d}{dt} I(\mu_t | \nu) \Big|_{\nu=\mu_{\varepsilon_t}^*} \quad \text{and} \quad \partial_{\varepsilon,t} I(\mu_t | \mu_{\varepsilon_t}^*) := \frac{d}{dt} I(\nu | \mu_{\varepsilon_t}^*) \Big|_{\nu=\mu_t}.$$

Similar to the computation in Section 3.2, for  $h_t = \sqrt{p_t/p_{\varepsilon_t}^*}$ , we have

$$I(\mu_t | \mu_{\varepsilon_t}^*) = 4 \mathbb{E}_{\mu_t} [\langle \nabla_z \log h_t, \nabla_z \log h_t \rangle].$$

Then,

$$\begin{aligned} \partial_{\mu,t} I(\mu_t | \mu_{\varepsilon_t}^*) &= \int \nabla_z \left( \frac{4}{h_t} \nabla_z^* \nabla_z h_t \right) \cdot v_t p_t dz \\ &= -8\varepsilon_t \mathbb{E}_{\mu_{\varepsilon_t}^*} \left[ \left| \nabla_z \nabla_y h_t - (\nabla_z \log h_t)(\nabla_y h_t)^T \right|^2 \right] \\ &\quad - 4 \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_z h_t, \tilde{M} \nabla_z h_t \rangle] \\ &\leq -4 \mathbb{E}_{\mu_{\varepsilon_t}^*} [\langle \nabla_z h_t, \tilde{M} \nabla_z h_t \rangle], \end{aligned}$$

where  $v_t$  satisfies (3.3) and  $\tilde{M} = \begin{pmatrix} 0 & I_d - \nabla_x^2 U \\ I_d - \nabla_x^2 U & 2I_d \end{pmatrix}$ . Using the fact that  $\nabla U$  is  $L$ -Lipschitz, it is easy to show that  $\tilde{M} \succeq -(L+1)I_{2d}$  and thus

$$\partial_{\mu,t} I(\mu_t | \mu_{\varepsilon_t}^*) \leq (L+1)I(\mu_t | \mu_{\varepsilon_t}^*). \tag{A.7}$$

Further, a similar calculation to the one in the proof of Lemma 3.7 yields that, for some constant  $C > 0$ ,

$$\partial_\varepsilon I(\mu_t | \mu_\varepsilon^*) \leq I(\mu_t | \mu_\varepsilon^*) + C\varepsilon^{-4}(1 + C_0).$$

This inequality, together with Proposition 3.2, yields the existence of some sub-exponential  $\tilde{\omega}(\cdot)$ , such that

$$\partial_{\varepsilon,t} I(\mu_t | \mu_{\varepsilon_t}^*) \leq |\varepsilon_t'| \tilde{\omega}(\varepsilon_t)(1 + I(\mu_t | \mu_{\varepsilon_t}^*)). \tag{A.8}$$

Plugging (A.7) and (A.8) into (A.6), we obtain

$$\frac{d}{dt} I(\mu_t | \mu_{\varepsilon_t}^*) \leq (1 + L)I(\mu_t | \mu_{\varepsilon_t}^*) + |\varepsilon_t'| \tilde{\omega}(\varepsilon_t)(1 + I(\mu_t | \mu_{\varepsilon_t}^*)).$$

Because  $\tilde{\omega}(\varepsilon_t)$  and  $|\varepsilon_t'|$  are all bounded on any finite time interval, there exists  $C > 0$ , such that

$$\frac{d}{dt} I(\mu_t | \mu_{\varepsilon_t}^*) \leq C(I(\mu_t | \mu_{\varepsilon_t}^*) + 1).$$

Finally, by Assumption 2.3, we conclude  $I(\mu_0 | \mu_{\varepsilon_0}^*)$  is finite. The proof then completes.

## Appendix B Proof in Section 4

### B.1 Proof of Proposition 4.5

The first order variational derivative of  $H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*)$  is given by

$$\frac{\delta H_{\gamma(\varepsilon_k)}(\bar{\mu}_t | \mu_k^*)}{\delta \bar{\mu}_t} = \frac{4}{\tilde{h}_t} \tilde{\nabla}_z^* S \nabla_z \tilde{h}_t + 2\gamma(\varepsilon_k) \log \tilde{h}_t + \gamma(\varepsilon_k).$$

We then follow the same computation steps as in (A.1), (A.2) and (A.3) to obtain that the term (4.2) is equal to

$$-4 \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \log \tilde{h}_t, M_k \nabla_z \log \tilde{h}_t \right\rangle \right] - 8\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z \nabla_y \log \tilde{h}_t \right\rangle_F \right]. \tag{B.1}$$

Because  $\hat{v}_{t,k}(z) - v_{t,k}(z) = \begin{pmatrix} 0 \\ \nabla_x U(x) - \nabla_x U(x_k) \end{pmatrix}$ , term (4.3) can be directly computed as

$$\begin{aligned} & \int \left\langle \nabla_z \left( \frac{4}{\tilde{h}_t} \tilde{\nabla}_z^* S \nabla_z \tilde{h}_t \right) + 2\gamma(\varepsilon_k) \nabla_z \log \tilde{h}_t, \right. \\ & \quad \left. \int (\hat{v}_{t,k} - v_{t,k})(z) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ &= \int \left\langle 2\gamma(\varepsilon_k) \nabla_y \log \tilde{h}_t, \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ & \quad + \int \left\langle \nabla_y \left( \frac{4}{\tilde{h}_t} \tilde{\nabla}_z^* S \nabla_z \tilde{h}_t \right), \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz. \end{aligned} \tag{B.2}$$

Notice that  $\tilde{\nabla}_z^* S \nabla_z \tilde{h}_t = \tilde{\nabla}_x^* \nabla_x \tilde{h}_t + \tilde{\nabla}_y^* \nabla_y \tilde{h}_t + \tilde{\nabla}_x^* \nabla_x \tilde{h}_t + \tilde{\nabla}_y^* \nabla_y \tilde{h}_t$ , in the same spirit of [25, Lemma 3], we have the following results:

**Lemma B.1.** For  $\# \in \{x, y\}$ ,

$$\begin{aligned} & 4 \int \left\langle \nabla_y \left( \frac{\tilde{\nabla}_x^* \nabla_{\#} \tilde{h}_t}{\tilde{h}_t} \right), \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ = & 4 \int \left\langle \nabla_y \nabla_{\#} \log \tilde{h}_t, \nabla_x A_t(z) \right\rangle_F d\bar{\mu}_t(z) + 4 \int \left\langle \nabla_y \nabla_{\#} \tilde{h}_t, A_t(z) (\nabla_x \tilde{h}_t)^T \right\rangle_F d\mu_k^*(z) \\ & - 4 \int \left\langle \nabla_x \nabla_y \tilde{h}_t, A_t(z) (\nabla_{\#} \tilde{h}_t)^T \right\rangle_F d\mu_k^*(z), \end{aligned}$$

and

$$\begin{aligned} & 4 \int \left\langle \nabla_y \left( \frac{\tilde{\nabla}_y^* \nabla_{\#} \tilde{h}_t}{\tilde{h}_t} \right), \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ = & 4 \int \left\langle \nabla_y \nabla_{\#} \log \tilde{h}_t, \nabla_y A_t(z) \right\rangle_F d\bar{\mu}_t(z) + 4 \int \left\langle \nabla_y \nabla_{\#} \tilde{h}_t, A_t(z) (\nabla_y \tilde{h}_t)^T \right\rangle_F d\mu_k^*(z) \\ & + \frac{4}{\varepsilon_k} \int \left\langle \nabla_{\#} \log \tilde{h}_t, \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ & - 4 \int \left\langle \nabla_y^2 \tilde{h}_t, A_t(z) (\nabla_{\#} \tilde{h}_t)^T \right\rangle_F d\mu_k^*(z). \end{aligned}$$

Consequently, (4.3) is equal to

$$\begin{aligned} & 4 \int \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z A_t(z) \right\rangle_F d\bar{\mu}_t(z) + \int \left\langle \frac{4}{\varepsilon_k} \nabla_x \log \tilde{h}_t + \left( 2\gamma(\varepsilon_k) + \frac{4}{\varepsilon_k} \right) \nabla_y \log \tilde{h}_t, \right. \\ & \left. \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz. \end{aligned}$$

Now combine (B.1) and Lemma B.1, we complete the proof.

## B.2 Proof of Proposition 4.6

The Proposition 4.6 is proved as long as we have the following two lemmas.

**Lemma B.2.** It holds that

$$\begin{aligned} & \int \left\langle \frac{4}{\varepsilon_k} \nabla_x \log \tilde{h}_t + \left( 2\gamma(\varepsilon_k) + \frac{4}{\varepsilon_k} \right) \nabla_y \log \tilde{h}_t, \right. \\ & \left. \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ \leq & \mathbb{E}_{\bar{\mu}_t} \left[ \left| \nabla_x \log \tilde{h}_t \right|^2 \right] + \left( 1 + \frac{1}{2} \varepsilon_k \gamma(\varepsilon_k) \right) \mathbb{E}_{\bar{\mu}_t} \left[ \left| \nabla_y \log \tilde{h}_t \right|^2 \right] \\ & + \frac{4L^2}{\varepsilon_k^2} \left( 2 + \frac{1}{2} \varepsilon_k \gamma(\varepsilon_k) \right) \mathbb{E} \left[ \left| \bar{X}_t - \bar{X}_k \right|^2 \right]. \end{aligned}$$

**Lemma B.3.** Assume that  $\Delta t_k \leq 1/L$  for every  $k$  and  $\nabla_x^2 U$  is  $L'$ -Lipschitz. Then

$$\begin{aligned} & 4 \int \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z A_t(z) \right\rangle_F d\bar{\mu}_t(z) \tag{B.3} \\ \leq & 8\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z \nabla_y \log \tilde{h}_t \right\rangle_F \right] + \frac{(L')^2}{\varepsilon_k} \mathbb{E} \left[ \left| \bar{X}_t - \bar{X}_k \right|^2 \right] + \frac{144}{\varepsilon_k} d^{3/2} L^2 \Delta t_k^2. \end{aligned}$$

*Proof of Lemma B.2.* Using  $a \cdot b \leq \frac{\varepsilon_k}{4} |a|^2 + \frac{1}{\varepsilon_k} |b|^2$  and  $\bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) = \bar{p}_t(z) \bar{p}_{T_k|t}(z_k|z)$  where the backward conditional density function  $\bar{p}_{T_k|t}(z_k|z) := p(\bar{Z}_k = z_k | \bar{Z}_t = z)$ , and

by Jensen's inequality, we can show that

$$\begin{aligned} & \frac{4}{\varepsilon_k} \int \left\langle \nabla_x \log \tilde{h}_t, \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{t|T_k}(z|z_k) \bar{p}_k(z_k) dz_k \right\rangle dz \\ &= \frac{4}{\varepsilon_k} \int \left\langle \nabla_x \log \tilde{h}_t, \int (\nabla_x U(x) - \nabla_x U(x_k)) \bar{p}_{T_k|t}(z_k|z) dz_k \right\rangle \bar{p}_t(z) dz \\ &\leq \mathbb{E}_{\bar{\mu}_t} \left[ \left| \nabla_x \log \tilde{h}_t \right|^2 \right] + \frac{4}{\varepsilon_k^2} \mathbb{E} \left[ \left| \nabla_x U(\bar{X}_t) - \nabla_x U(\bar{X}_k) \right|^2 \right] \\ &\leq \mathbb{E}_{\bar{\mu}_t} \left[ \left| \nabla_x \log \tilde{h}_t \right|^2 \right] + \frac{4L^2}{\varepsilon_k^2} \mathbb{E} \left[ \left| \bar{X}_t - \bar{X}_k \right|^2 \right]. \end{aligned}$$

The  $\nabla_y \log \tilde{h}_t$  term can be treated similarly. The proof then completes. □

*Proof of Lemma B.3.* Without loss of generality, we assume  $L \geq 1$ . Using the same proof as the one of [25, Lemma 10], we can show that

$$\nabla_z A_t(z) = \int \left( \frac{\nabla_x^2 U(x) - \nabla_x^2 U(x_k) - \nabla_x^2 U(x_k) \left( (I_d + \varpi(t) \nabla_x^2 U(x_k))^{-1} - I_d \right)}{(e^{t-T_k} - 1) \nabla_x^2 U(x_k) (I_d + \varpi(t) \nabla_x^2 U(x_k))^{-1}} \right) \bar{p}_{T_k|t}(z_k|z) dz_k, \quad (\text{B.4})$$

where  $\varpi(t) = e^{t-T_k} - 1 - (t - T_k)$ . As a result,

$$\begin{aligned} & 4 \int \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z A_t(z) \right\rangle_F d\bar{\mu}_t(z) \\ &= 4 \mathbb{E} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \begin{pmatrix} \nabla_x^2 U(\bar{X}_t) - \nabla_x^2 U(\bar{X}_k) \\ 0 \end{pmatrix} \right\rangle_F \right] \\ &+ 4 \mathbb{E} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \begin{pmatrix} -\nabla_x^2 U(\bar{X}_k) \left( (I_d + \varpi(t) \nabla_x^2 U(\bar{X}_k))^{-1} - I_d \right) \\ (e^{t-T_k} - 1) \nabla_x^2 U(\bar{X}_k) (I_d + \varpi(t) \nabla_x^2 U(\bar{X}_k))^{-1} \end{pmatrix} \right\rangle_F \right]. \end{aligned}$$

On the one hand,

$$\begin{aligned} & 4 \mathbb{E} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \begin{pmatrix} \nabla_x^2 U(\bar{X}_t) - \nabla_x^2 U(\bar{X}_k) \\ 0 \end{pmatrix} \right\rangle_F \right] \\ &\leq 4\varepsilon_k \int \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z \nabla_y \log \tilde{h}_t \right\rangle_F d\bar{\mu}_t(z) \\ &+ \frac{1}{\varepsilon_k} \mathbb{E} \left[ \left\langle \begin{pmatrix} \nabla_x^2 U(\bar{X}_t) - \nabla_x^2 U(\bar{X}_k) \\ 0 \end{pmatrix}, S \begin{pmatrix} \nabla_x^2 U(\bar{X}_t) - \nabla_x^2 U(\bar{X}_k) \\ 0 \end{pmatrix} \right\rangle_F \right] \\ &= 4\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z \nabla_y \log \tilde{h}_t \right\rangle_F \right] + \frac{1}{\varepsilon_k} \mathbb{E} \left[ \left\| \nabla_x^2 U(\bar{X}_t) - \nabla_x^2 U(\bar{X}_k) \right\|_F^2 \right] \\ &\leq 4\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z \nabla_y \log \tilde{h}_t \right\rangle_F \right] + \frac{(L')^2}{\varepsilon_k} \mathbb{E} \left[ \left| \bar{X}_t - \bar{X}_k \right|^2 \right], \end{aligned}$$

where the last inequality is the case due to the  $L'$ -Lipschitz of  $\nabla_x^2 U$ . On the other hand, denoting  $J_t(\bar{X}_k) := \begin{pmatrix} -\nabla_x^2 U(\bar{X}_k) \left( (I_d + \varpi(t) \nabla_x^2 U(\bar{X}_k))^{-1} - I_d \right) \\ (e^{t-T_k} - 1) \nabla_x^2 U(\bar{X}_k) (I_d + \varpi(t) \nabla_x^2 U(\bar{X}_k))^{-1} \end{pmatrix}$ , we have

$$\begin{aligned} & 4 \mathbb{E} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \begin{pmatrix} -\nabla_x^2 U(\bar{X}_k) \left( (I_d + \varpi(t) \nabla_x^2 U(\bar{X}_k))^{-1} - I_d \right) \\ (e^{t-T_k} - 1) \nabla_x^2 U(\bar{X}_k) (I_d + \varpi(t) \nabla_x^2 U(\bar{X}_k))^{-1} \end{pmatrix} \right\rangle_F \right] \\ &\leq 4\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z \nabla_y \log \tilde{h}_t \right\rangle_F \right] + \frac{1}{\varepsilon_k} \mathbb{E}_{\bar{\mu}_k} \left[ \left\langle J_t(\bar{X}_k), S J_t(\bar{X}_k) \right\rangle_F \right]. \end{aligned}$$

As a result, we derive,

$$\begin{aligned} & 4 \int \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z A_t(z) \right\rangle_F d\bar{\mu}_t(z) \\ & \leq 8\varepsilon_k \mathbb{E}_{\bar{\mu}_t} \left[ \left\langle \nabla_z \nabla_y \log \tilde{h}_t, S \nabla_z \nabla_y \log \tilde{h}_t \right\rangle_F \right] + \frac{(L')^2}{\varepsilon_k} \mathbb{E} \left[ |\bar{X}_t - \bar{X}_k|^2 \right] \\ & \quad + \frac{1}{\varepsilon_k} \mathbb{E}_{\bar{\mu}_k} \left[ \left\langle J_t(\bar{X}_k), S J_t(\bar{X}_k) \right\rangle_F \right]. \end{aligned}$$

Comparing the above inequality with (B.3), to complete the proof of the lemma, we only need to prove

$$\mathbb{E}_{\bar{\mu}_k} \left[ \left\langle J_t(\bar{X}_k), S J_t(\bar{X}_k) \right\rangle_F \right] \leq 144d^{3/2}L^2\Delta t_k^2. \quad (\text{B.5})$$

By Cauchy-Schwarz inequality and matrix norm inequality, we have

$$\begin{aligned} \mathbb{E}_{\bar{\mu}_k} \left[ \left\langle J_t(\bar{X}_k), S J_t(\bar{X}_k) \right\rangle_F \right] & \leq \mathbb{E}_{\bar{\mu}_k} \left[ \|J_t(\bar{X}_k)\|_F \|S J_t(\bar{X}_k)\|_F \right] \\ & \leq \mathbb{E}_{\bar{\mu}_k} \left[ \|S\|_F \cdot \text{rank}(J_t(\bar{X}_k)) \|J_t(\bar{X}_k)\|_2^2 \right] \\ & \leq 2d^{\frac{3}{2}} \mathbb{E}_{\bar{\mu}_k} \left[ \|J_t(\bar{X}_k)\|_2^2 \right]. \end{aligned} \quad (\text{B.6})$$

It is straightforward to see that

$$\begin{aligned} \|J_t(\bar{X}_k)\|_2 & \leq \sqrt{2} \max \left\{ \|\nabla_x^2 U(\bar{X}_k) ((I_d + \varpi(t)\nabla_x^2 U(\bar{X}_k))^{-1} - I_d)\|_2, \right. \\ & \quad \left. \|(e^{t-T_k} - 1) \nabla_x^2 U(\bar{X}_k) (I_d + \varpi(t)\nabla_x^2 U(\bar{X}_k))^{-1}\|_2 \right\}. \end{aligned}$$

Because  $\|\nabla_x^2 U\|_2 \leq L$ , which is due to the  $L$ -Lipschitz of  $\nabla_x U$ , and because  $\Delta t_k \leq 1/L$ , we have  $\varpi(t) \leq \Delta t_k^2$  and  $(I_d + \varpi(t)\nabla_x^2 U(\bar{X}_k))^{-1} \preceq I_d - \varpi(t)\nabla_x^2 U(\bar{X}_k) + (\varpi(t)\nabla_x^2 U(\bar{X}_k))^2$ . Consequently,

$$\begin{aligned} & \|\nabla_x^2 U(\bar{X}_k) ((I_d + \varpi(t)\nabla_x^2 U(\bar{X}_k))^{-1} - I_d)\|_2 \\ & \leq \|\nabla_x^2 U(\bar{X}_k)\|_2 \cdot \left( \|\varpi(t)\nabla_x^2 U(\bar{X}_k)\|_2 + \|\varpi(t)\nabla_x^2 U(\bar{X}_k)\|_2^2 \right) \\ & \leq 2L^2\Delta t_k^2, \end{aligned}$$

and

$$\begin{aligned} & \left\| (e^{t-T_k} - 1) \nabla_x^2 U(\bar{X}_k) (I_d + \varpi(t)\nabla_x^2 U(\bar{X}_k))^{-1} \right\|_2 \\ & \leq 2\Delta t_k \|\nabla_x^2 U(\bar{X}_k)\|_2 \cdot \left\| (I_d + \varpi(t)\nabla_x^2 U(\bar{X}_k))^{-1} \right\|_2 \\ & \leq 2L\Delta t_k \left( \|I_d\|_2 + \|\varpi(t)\nabla_x^2 U(\bar{X}_k)\|_2 + \|\varpi(t)\nabla_x^2 U(\bar{X}_k)\|_2^2 \right) \\ & \leq 2L\Delta t_k (1 + L\Delta t_k^2 + L^2\Delta t_k^4) \leq 6L\Delta t_k. \end{aligned}$$

As a result,  $\|J_t(\bar{X}_k)\|_2 \leq \sqrt{2} \max \{2L^2\Delta t_k^2, 6L\Delta t_k\} = 6\sqrt{2}L\Delta t_k$ . This, together with (B.6), immediately leads to (B.5).  $\square$

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