Electronic Journal of Probability

Electron. J. Probab. **29** (2024), article no. 66, 1–82.
ISSN: 1083-6489 https://doi.org/10.1214/24-EJP1123

Reflections on BSDEs*

Dylan Possamaï† Marco Rodrigues‡

Abstract

We prove well-posedness results for backward stochastic differential equations (BSDEs) and reflected BSDEs with an optional obstacle process in the case of appropriately weighted $\mathbb{L}^2$-data when the generator is integrated with respect to a possibly purely discontinuous process. This leads to a unified treatment of discrete-time and continuous-time (reflected) BSDEs. We compare our well-posedness results with the current literature and highlight that our results are sharp and cannot be improved within the framework presented here. Finally, we provide sufficient conditions for a comparison principle.

Keywords: BSDEs; reflections; general filtrations.
MSC2020 subject classifications: 60H20.
Submitted to EJP on July 25, 2023, final version accepted on April 10, 2024.

1 Introduction

The primary motivation for this work is to develop a unified approach to discrete-time and continuous-time (reflected) BSDEs with jumps. The formulation we employ here permits the integration of the generator with a potentially purely discontinuous process. Informally, for the time being, the (reflected) BSDE studied here is of the form

$$\begin{cases} Y_t = \xi_T + \displaystyle\int_t^T f_s\big(Y_s, Y_{s-}, Z_s, U_s(\cdot)\big)\mathrm{d}C_s \\ \qquad\qquad - \displaystyle\int_t^T Z_s \mathrm{d}X_s - \int_t^T\int_E U_s(x)(\mu-\nu)(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}N_s + \int_t^T \mathrm{d}K_s, \\ Y \geq \xi, \\ \displaystyle\int_0^T (Y_{s-}-\xi_{s-})\mathrm{d}K^r_s + \int_0^T (Y_s - \xi_s)\mathrm{d}K^\ell_s = 0, \text{ where} \\ \qquad\qquad K^\ell := \displaystyle\sum_{s\in[0,\cdot]} (K_{s+}-K_s) \text{ and } K^r := K - K^\ell_-. \end{cases}$$

*The first author gratefully acknowledges the support of the SNF project MINT 205121-219818.
†ETH Zürich, Switzerland. E-mail: dylan.possamai@math.ethz.ch
‡ETH Zürich, Switzerland. E-mail: marco.rodrigues@math.ethz.ch

Here, T is a stopping time, the process ξ is the obstacle, f is the generator with stochastic Lipschitz coefficient α, X is the driving martingale, μ the driving random measure, and C is a non-decreasing, predictable process that dominates the predictable quadratic variation $\langle X\rangle$ of X and the predictable compensator ν of μ. Note that the above system reduces to a BSDE in case $\xi = -\infty$ on $[0, T)$. We are then seeking to find a class of processes within which there exists a unique collection (Y, Z, U, N, K) of adapted processes that solve the above system. In this work, we prove well-posedness of these BSDEs and reflected BSDEs under appropriately integrable $\mathbb{L}^2$-data (ξ, f) in case $\alpha^2\Delta C \leq \Phi \in (0, 1)$.

Although not yet strictly referred to as BSDEs, the first appearance of these types of objects in case the generator f is linear was in the works of Bismut [14; 15], and even earlier by Davis and Varaiya [39]. In the former, BSDEs were introduced as adjoint equations that originated from an application of the Pontryagin stochastic maximum principle, and in the latter, they were used to study stochastic control problems with drift control. What has been recognised already in [39] is that value process of certain stochastic control problems can be recasted as the Y-component of a BSDE, which through comparison principles allows one to deduce characterisations of optimal control responses to stochastic systems. The first systematic study of BSDEs in the non-linear setting with Lipschitz generator was carried out by Pardoux and Peng [117], and the same authors later connected BSDEs to quasi-linear PDEs through Feynman–Kac–type formulas in [118]. The seminal survey article by El Karoui, Peng, and Quenez [55] collected properties of BSDEs and showed how they may be applied to solve problems in mathematical finance. Reflected BSDEs were introduced later by El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [53] directly in a Lipschitz setting, where the immediate connection to optimal stopping problems and obstacle problems for parabolic PDEs was drawn. Concomitantly to these early contributions, reflected BSDEs have also been applied to hedging problems of American options by El Karoui and Quenez [51], and El Karoui, Pardoux, and Quenez [54], and to an optimal control problem with consumption by El Karoui and Jeanblanc-Picqué [49], see also Bally, Caballero, Fernandez, and El Karoui [9].

With time, BSDEs and reflected BSDEs were applied in many areas. In finance, as mentioned above, for pricing of financial derivatives, see El Karoui, Peng, and Quenez [55] and [51; 54], or for utility maximisation problems in Hu, Imkeller, and Müller [74]. One can also use BSDEs to construct recursive utilities as in Duffie and Epstein [43; 38]. There have been works on applications to zero-sum games by Hamadène and Lepeltier [69] and to Dynkin games by Cvitanić and Karatzas [35]. Recently, based on some new backward propagation of chaos techniques appearing in Laurière and Tangpi [94], BSDEs have also been applied to mean-field games by Possamaï and Tangpi [128] to deduce convergence rates of the N-player game to its mean-field counterpart.

Ever since the seminal works [117] and [53], the theory of BSDEs and reflected BSDEs has expanded rapidly, and there have been various forms of generalisation of well-posedness results for these systems. Kobylanski [91] and Tevzadze [139] studied well-posedness of BSDEs with generators that are quadratic in the z-variable, see also the works of Jackson and Žitković [75], Zheng, Zhang, and Meng [143] and the references therein, and Kazi-Tani, Possamaï, and Zhou [79; 82] as well as El Karoui, Matoussi, and Ngoupeyou [56], Jeanblanc, Matoussi, and Ngoupeyou [78], or Matoussi and Salhi [109] for BSDEs with jumps in the quadratic case. Reflected BSDEs with quadratic growth have also been considered by Kobylanski, Lepeltier, Quenez, and Torres [93], Lepeltier and Xu [98], and Bayraktar and Yao [12]. Another possible direction of generalisation is to consider BSDEs and reflected BSDEs on random time horizon. Here the first well-posedness result for BSDEs was established by Peng [119], and then extended by Darling and Pardoux [37]. Other works on BSDEs with random terminal time include Briand and Hu [21] and Royer [134]. More recently, Lin, Ren, Touzi, and Yang [106] complemented this theory and proved well-posedness of random horizon BSDEs, 2BSDEs and reflected BSDEs with $\mathbb{L}^p$-data to which we also refer the interested reader for more references on the subject of random horizon systems. Other than the aforementioned works, there have also been works by Rozkosz and Słomiński [136], Klimsiak [83] and Klimsiak, Rzymowski, and Słomiński [89] on reflected BSDEs with $\mathbb{L}^p$-data and by Alsheyab and Choulli [4] on random horizon reflected BSDEs. Furthermore, El Asri and Ourkiya [45], Li and Liu [102] and Qian [131] study multidimensional reflected BSDEs.

What most of the above references have in common is that they consider one driving Brownian motion of the system. Works that considered also a driving Poisson random measure include Barles, Buckdahn, and Pardoux [11], Royer [135], Quenez and Sulem [132], Becherer, Büttner, and Kentia [13] in the BSDE case, and Hamadène and Ouknine [70], Crépey and Matoussi [34], Quenez and Sulem [133], Perninge [123; 124] in the reflected BSDE case, to name but a few. On the other hand, there are very few results that go beyond the case of Brownian motion to more general martingales. In the BSDE case there is work by Buckdahn [25], El Karoui and Huang [48], Carbone, Ferrario, and Santacroce [26], Confortola, Fuhrman, and Jacod [33], Bandini [10], Cohen and Elliott [31] and the more recent work by Papapantoleon, Possamaï, and Saplaouras [113], to which we also refer the reader interested in further history for BSDEs. The results obtained in [113] have been applied to well-posedness results for backward stochastic Volterra integral equations with jumps by Popier [126], to an optimal reinsurance problem by Brachetta, Callegaro, Ceci, and Sgarra [20], and to a stochastic control problem involving Lévy processes by di Nunno [42]. Reflected BSDEs with jumps have also been considered by Nie and Rutkowski [111], see also the references therein. However, the well-posedness result in [111] relies on assumptions that are too restrictive for the applications we have in mind since, for example, the integrator C in [111] is assumed to be continuous. This immediately excludes piecewise constant integrators, which we want to cover to some extent at least. In other recent contributions, Aksamit, Li, and Rutkowski [3] and Li, Liu, and Rutkowski [104] study 'generalised' BSDEs and reflected BSDEs with a view towards applications to pricing of vulnerable options. Additionally, Gu, Lin, and Xu [63; 64] and Lin and Xu [105] study reflected BSDEs driven by a marked point process.

In the reflected BSDE case, the regularity of the obstacle has also been lifted over the years. There are works considering obstacles that are càdlàg, see Hamadène [65], Lepeltier and Xu [97], Hamadène and Ouknine [70], or merely measurable, see Peng and Xu [120; 121], Klimsiak [84; 85] and Klimsiak, Rzymowski, and Słomiński [89]. See also the works of Kobylanski and Quenez [92] on a general approach to optimal stopping problems. Recently, in a series of two inspiring papers, Grigorova, Imkeller, Offen, Ouknine, and Quenez [60] and Grigorova, Imkeller, Ouknine, and Quenez [62] considered reflected BSDEs driven by Brownian motion and a Poisson random measure whose obstacle is merely an optional process. Other than proving well-posedness of the corresponding reflected BSDE, the aforementioned reference also draws the connection to the corresponding optimal stopping problem with respect to the induced f-expectation. Results related to the aforementioned works were obtained by Baadi and Ouknine [7; 8], Akdim, Haddadi, and Ouknine [2] and Bouhadou, Hilbert, and Ouknine [19]. The case of predictable obstacles has also been studied by Bouhadou and Ouknine [17; 18].

Although not entirely related to what we study here, we would like to mention that there have also been works on doubly reflected BSDEs, where the BSDEs are constrained to stay within an upper and lower obstacle process. Here the first well-posedness study was carried out by Cvitanić and Karatzas [35], where they also connected the

Y-component of the corresponding doubly reflected BSDE to the value of a Dynkin game. Other results were then obtained by Hamadène, Lepeltier, and Matoussi [71], Lepeltier and San Martín [96], Hamadène and Hassani [67; 68], Hamadène [66], Crépey and Matoussi [34], Pham and Zhang [125], Essaky and Hassani [57], Dumitrescu, Quenez, and Sulem [44], Grigorova, Imkeller, Ouknine, and Quenez [61], Nie and Rutkowski [111], Klimsiak [84; 85; 86], Klimsiak, Rzymowski, and Słomiński [90], Klimsiak and Rzymowski [87; 88], Arharas and Ouknine [5], Baadi [6], Li and Ning [103] and Li [101].

Our main contributions are two well-posedness results, one for BSDEs and another for reflected BSDEs. These well-posedness results, as presented, turn out to be sharp within the framework we lay out and cannot be improved with our methods, due to counterexamples to existence and uniqueness in Confortola, Fuhrman, and Jacod [33]. This also refines the BSDE results in [113]. The method of proof we use is based on a fixed-point argument and *a priori* estimates that we establish by techniques that are inspired by [48], although we do not use exponential weights like in [113], but stochastic exponential weights like in [10; 31]. With the techniques in [48], we can exploit the $\mathbb{L}^2$-structure of our problem, allowing us to circumvent a reliance on Itô's formula. Our BSDE results thus extend straightforwardly to BSDEs with a multi-dimensional generator and terminal condition. In the reflected BSDE case, we will also use a fixed-point argument and *a priori* estimates. Classically, these estimates are derived by an application of Itô's formula. However, in the generality we are aiming for, this would necessitate imposing additional assumptions on the integrator C. We thus approach this problem differently, and apply it in combination with the methods in [48] and [62] to deduce the desired *a priori* estimates. To the best of our knowledge, the well-posedness result we will present in the reflected BSDE case is the first of its kind. Let us mention here that after completing the first version of this manuscript, we became aware that Papapantoleon, Saplaouras, and Theodorakopoulos [116] independently obtained similar results for BSDEs.

The link between discrete-time BSDEs and control theory has already been mentioned in Cohen and Elliott [29; 30; 31]. In continuous-time, BSDEs and reflected BSDEs are intimately connected to control problems with drift control only, as the dynamic programming equation in this context is semi-linear (thus quasi-linear). However, for stochastic control problems with drift and volatility control, the dynamic programming equation is fully non-linear and can thus not be analysed by classical BSDEs. This fact was the starting point for the new notion of second-order BSDEs (2BSDEs) introduced by Cheridito, Soner, Touzi, and Victoir [28] and Soner, Touzi, and Zhang [137; 138]. Here, the Y-component of these 2BSDE corresponds to the classical value process of a control problem with drift and volatility control. This fact has been recently applied to principal–agent problems in [36]. For an excellent and comprehensive introduction to BSDEs, we can refer the interested reader to the books by Touzi [140] and Zhang [142]. The latter reference also covers 2BSDEs. In the previously mentioned seminal works [137; 138] on 2BSDEs the terminal random variable ξ_T had to satisfy strong regularity conditions for existence and uniqueness to hold. This assumption has been lifted by Possamaï, Tan, and Zhou [129]. In future work, we seek to combine the results of this work and the techniques in [129] to show well-posedness of 2BSDEs with jumps that go beyond the case of Poisson random measures in Kazi-Tani, Possamaï, and Zhou [80; 81].

The remainder of this paper is structured as follows: in Section 2, we recall preliminaries on (vector) stochastic integration and orthogonal decompositions of martingales. We also fix the data and formulate the BSDE and reflected BSDE. In Section 3, we formulate our main results, separately, for the reflected BSDE first, and then for the BSDE. We also compare our assumptions with the current literature. In Section 4, we revisit the Snell envelope and optimal stopping theory, with which we solve the reflected

BSDE in case of a generator not depending on the solution. In Section 5, we establish the *a priori* estimates, which we will use in the contraction argument, separately in the reflected BSDE case first, and then in the BSDE case. We then establish the two well-posedness results in Section 6. Finally, we prove a comparison result for our BSDE in Section 7. The appendices contain proofs of technical results and some auxiliary results we make use of throughout this work.

Notations: throughout this work, we fix a positive integer m. Let $\mathbb{N}$ and $\mathbb{R}$ denote the non-negative integers and real numbers, respectively. For $(a,b)\in[-\infty,\infty]^2$, we write $a\vee b:=\max\{a,b\}$, $a\wedge b:=\min\{a,b\}$ and $a^+:=a\vee 0=\max\{a,0\}$. We write $|a|$ for the modulus of $a\in[-\infty,\infty]$, and for $b\in\mathbb{R}^m$, we write $\|b\|$ for the Euclidean norm of $b\in\mathbb{R}^m$. For a finite-dimensional matrix M, we denote by $M^\top$ its transpose. For a subset V of a Hilbert space H, we write $V^\perp$ for its orthogonal in H; for two subspaces W and W' of H with $W\cap W'=\{0_H\}$, we write $W\oplus W'$ for the internal direct sum of W and W' in H. For a set Ω and $A\subseteq\Omega$, we denote by $\mathbf{1}_A$ its indicator function defined on Ω. We abuse notation and sometimes also write $\mathbf{1}_{\{x\in A\}}$ for $\mathbf{1}_A(x)$. For a nonempty set $\mathcal{Z}$, we denote the Dirac-measure at $z\in\mathcal{Z}$ by $\boldsymbol{\delta}_z$. For two measurable spaces $(\Omega,\mathcal{F})$ and $(\Omega',\mathcal{F}')$, we denote by $\mathcal{F}\otimes\mathcal{F}'$ the product-σ-algebra on the product space $\Omega\times\Omega'$. For $t\in(0,\infty]$, a limit of the form $s\uparrow\uparrow t$ means that $s\to t$ along $s<t$. Analogously, for $t\in[0,\infty)$, a limit of the form $s\downarrow\downarrow t$ means that $s\to t$ along $s>t$. For a stochastic process Y indexed by $[0,\infty)$ or $[0,\infty]$, we let $Y^T:=Y_{\cdot\wedge T}$. Let $\mathsf{y}:[0,\infty)\to\mathbb{R}$ be a làdlàg function, that is, y admits limits from the right on $[0,\infty)$ and from the left on $(0,\infty)$. We define $\mathsf{y}_{t-}:=\lim_{s\uparrow\uparrow t}\mathsf{y}_s$, $t\in(0,\infty)$, and analogously $\mathsf{y}_{t+}:=\lim_{s\downarrow\downarrow t}\mathsf{y}_s$, $t\in[0,\infty)$. Then $\Delta\mathsf{y}:[0,\infty)\to\mathbb{R}$ is defined by $\Delta\mathsf{y}_t:=\mathsf{y}_t-\mathsf{y}_{t-}$ if $t\in(0,\infty)$ and $\Delta\mathsf{y}_0=0$. Similarly, we define $\Delta^+\mathsf{y}:[0,\infty)\to\mathbb{R}$ by $\Delta^+\mathsf{y}_t:=\mathsf{y}_{t+}-\mathsf{y}_t$ for $t\in[0,\infty)$. If y is additionally right-continuous (thus càdlàg) and non-decreasing, the functions y^c and y^d denote the continuous part and purely discontinuous part of y, respectively. They are defined through the formulas $\mathsf{y}^d_t:=\sum_{s\in(0,t]}\Delta\mathsf{y}_s$ and $\mathsf{y}^c:=\mathsf{y}-\mathsf{y}^d$. Note that $\mathsf{y}^c_0=\mathsf{y}_0$. In case y is defined on $[0,\infty]$, we additionally define $\mathsf{y}_{\infty-}:=\lim_{s\uparrow\uparrow\infty}\mathsf{y}_s$, $\Delta\mathsf{y}_\infty=\mathsf{y}_\infty-\mathsf{y}_{\infty-}$, $\mathsf{y}^d_\infty=\sum_{s\in(0,\infty]}\Delta\mathsf{y}_s$ and $\mathsf{y}^c_\infty:=\mathsf{y}_\infty-\mathsf{y}^d_\infty$.

2 Preliminaries and formulation of the reflected BSDE

This section will lay the foundations for the analysis that follows. We recall the construction and some properties of the vector stochastic integral and compensated stochastic integral with respect to an integer-valued random measure. We then present the assumptions we impose on the data and the formulation of our reflected BSDE.

2.1 Stochastic basis

We fix once and for all a probability space $(\Omega,\mathcal{G},\mathbb{P})$ and a right-continuous filtration $\mathbb{G}=(\mathcal{G}_t)_{t\in[0,\infty)}$. We denote by $\mathcal{G}_\infty:=\mathcal{G}_{\infty-}$ the σ-algebra on Ω generated by the sets in $\cup_{t\in[0,\infty)}\mathcal{G}_t$. We denote the $\mathbb{P}$-augmentation of $\mathbb{G}$ by $\mathbb{G}^\mathbb{P}=(\mathcal{G}^\mathbb{P}_t)_{t\in[0,\infty)}$, that is, each $\mathcal{G}^\mathbb{P}_t$, $t\in[0,\infty)$, is generated by $\mathcal{G}_t\vee\mathcal{N}^\mathbb{P}$, where $\mathcal{N}^\mathbb{P}$ is the collection of $(\mathbb{P},\mathcal{G})$–null sets. The universal completion of a σ-algebra $\mathcal{A}$ is the σ-algebra $\mathcal{A}^U:=\cap_{\mathbb{P}'\in\mathcal{P}(\Omega,\mathcal{A})}\mathcal{A}^{\mathbb{P}'}$, where the intersection is over the set $\mathcal{P}(\Omega,\mathcal{A})$ of all probability measures $\mathbb{P}'$ on $(\Omega,\mathcal{A})$. We will also assume for simplicity that $\mathcal{G}^U_\infty\subseteq\mathcal{G}$. Unless stated otherwise, probabilistic notions requiring a filtration or a probability measure will implicitly refer to $\mathbb{G}$ or $\mathbb{P}$, respectively.

Remark 2.1. The reason we suppose that $\mathcal{G}^U_\infty$ is included in $\mathcal{G}$ is that this ensures the $\mathcal{G}$-measurability of $\sup_{s\in[0,\infty]}\xi_s$ for a product-measurable process $\xi:\Omega\times[0,\infty]\longrightarrow[-\infty,\infty]$, see [52, Proposition 2.21].

For two stopping times S and T, we denote by $\mathcal{T}_{S,T}$ the collection of all stopping times τ satisfying $\mathbb{P}[S\le\tau\le T]=1$. Note that $\mathcal{T}_{S,T}$ is empty if $\mathbb{P}[S>T]>0$. We denote by $\mathcal{G}_T$ the σ-algebra of all $A \in \mathcal{G}_\infty$ for which $A \cap \{T \leq t\} \in \mathcal{G}_t$ for all $t \in [0,\infty)$, and by $\mathcal{G}_{T-}$ the σ-algebra generated by $\mathcal{G}_0$ and and all sets of the form $A \cap \{t < T\}$ for $A \in \mathcal{G}_t$ and $t \in [0,\infty)$. If $\mathcal{C}$ is a collection of processes indexed by $[0,\infty)$, we define $\mathcal{C}_{\text{loc}}$ as the collection of processes $X = (X_t)_{t\in[0,\infty)}$ for which there is a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ that is $\mathbb{P}$–a.s. increasing to infinity, with $X_{\cdot\wedge\tau_n} \in \mathcal{C}$ for each $n \in \mathbb{N}$.

A real-valued martingale indexed by $[0,\infty)$ on our filtered probability space always has a $\mathbb{P}$-modification for which *all* its paths are real-valued and right-continuous, and $\mathbb{P}$–*almost all* its paths have left-limits on $(0,\infty)$ (see von Weizsäcker and Winkler [141, Theorem 3.2.6]). We will always choose such a modification of a martingale. We denote by $\mathcal{M}$ the space of uniformly integrable martingales and by $\mathcal{H}^2$ the space of square-integrable martingales indexed by $[0,\infty)$. We denote by $\mathcal{H}^2_0$ the space of all $M \in \mathcal{H}^2$ for which $M_0 = 0$, $\mathbb{P}$–almost surely. Note that $\mathcal{H}^2_0$ is a closed subspace of $\mathcal{H}^2$. For $(M,N) \in \mathcal{H}^2_{\text{loc}} \times \mathcal{H}^2_{\text{loc}}$, we denote by $\langle M,N\rangle$ the predictable quadratic covariation between M and N in the sense of [77, Theorem I.4.2] and let $\langle M\rangle := \langle M,M\rangle$. We endow $\mathcal{H}^2$ with the scalar product $(M,N)_{\mathcal{H}^2} := \mathbb{E}[M_0N_0] + \mathbb{E}[\langle M,N\rangle_\infty] = \mathbb{E}[M_\infty N_\infty]$, which turns it into a Hilbert space by identifying processes whose paths agree $\mathbb{P}$–almost surely. We denote the norm associated to $(\cdot,\cdot)_{\mathcal{H}^2}$ by $\|\cdot\|_{\mathcal{H}^2}$. A stable subspace of $\mathcal{H}^2$ is a closed linear subspace $\mathcal{X}$ of $\mathcal{H}^2$ such that $\mathbf{1}_A M_{\cdot\wedge T} \in \mathcal{X}$ for each $T \in \mathcal{T}_{0,\infty}$, each $A \in \mathcal{G}_0$ and each $M \in \mathcal{X}$. A very thorough study of stable subspaces can be found in Cohen and Elliott [32], Jacod [76], and Protter [130]. Let us note, however, that according to [76, Proposition 4.26], if $M \in \mathcal{H}^2$ is orthogonal to a stable subspace $\mathcal{X}$ for the scalar product $(\cdot,\cdot)_{\mathcal{H}^2}$, then $\langle M,N\rangle = 0$ for all $N \in \mathcal{X}$.

An element $M \in \mathcal{M}_{\text{loc}}$ has by [77, Theorem I.4.18] a, up to $\mathbb{P}$-evanescence, unique decomposition $M = M_0 + M^c + M^d$, where $M^c \in \mathcal{H}^2_{\text{loc}}$ has $\mathbb{P}$–a.s. continuous paths and satisfies $M^c_0 = 0$, while $M^d \in \mathcal{M}_{\text{loc}}$ is purely discontinuous in the sense that $M^d_0 = 0$ and $M^dN \in \mathcal{M}_{\text{loc}}$ for every $N \in \mathcal{H}^2_{\text{loc}}$ with $\mathbb{P}$–a.s. continuous paths. The processes M^c and M^d are referred to as the continuous and purely discontinuous local martingale parts of M, respectively. It is immediate that if $M \in \mathcal{H}^2_{\text{loc}}$, we also have that $M^d \in \mathcal{H}^2_{\text{loc}}$, and that we can write $\langle M\rangle = \langle M^c\rangle + \langle M^d\rangle$. Thus $(M^c, M^d) \in \mathcal{H}^2 \times \mathcal{H}^2$ if $M \in \mathcal{H}^2$ by [76, Théorème 2.34]. Lastly, we denote by $[X,Y]$ the optional quadratic covariation between semi-martingales X and Y in the sense of [77, Definition I.4.45]. In particular, if $M \in \mathcal{H}^2$, then $[M]_\infty = [M,M]_\infty$ is integrable, $[M] - \langle M\rangle$ is a uniformly integrable martingale, and $[M^c] = \langle M^c\rangle$ since $[M]_t = \langle M^c\rangle_t + \sum_{s\in(0,t]}(\Delta M_s)^2,\ t \in [0,\infty)$, $\mathbb{P}$–a.s. Note that this implies $[M] = [M^c] + [M^d] = \langle M^c\rangle + [M^d]$ since $[M^d] = \sum_{s\in(0,\cdot]}(\Delta M_s)^2$.

We now introduce the optional and predictable σ-algebras induced by our filtration $\mathbb{G}$. The optional σ-algebra $\overline{\mathcal{O}}(\mathbb{G})$ on $\Omega \times [0,\infty]$ is generated by all $\mathbb{G}$-adapted processes $\xi : \Omega \times [0,\infty] \longrightarrow \mathbb{R}$ that are right-continuous on $[0,\infty)$ and admit left-hand limits on $(0,\infty]$. The optional σ-algebra on $\Omega \times [0,\infty)$ is given by the trace–σ-algebra $\mathcal{O}(\mathbb{G}) := \overline{\mathcal{O}}(\mathbb{G}) \cap (\Omega \times [0,\infty))$. We have that $\mathcal{O}(\mathbb{G})$ is generated by $\mathbb{G}$-adapted, real-valued, càdlàg processes defined on $\Omega\times[0,\infty)$. The predictable σ-algebra $\overline{\mathcal{P}}(\mathbb{G})$ on $\Omega\times[0,\infty]$ is generated by all $\mathbb{G}$-adapted processes $\xi : \Omega \times [0,\infty] \longrightarrow \mathbb{R}$ that are continuous on $[0,\infty]$. The predictable σ-algebra on $\Omega \times [0,\infty)$ is the trace-σ-algebra $\mathcal{P}(\mathbb{G}) := \overline{\mathcal{P}}(\mathbb{G}) \cap (\Omega \times [0,\infty))$, which is also generated by all $\mathbb{G}$-adapted, real-valued, continuous processes defined on $\Omega \times [0,\infty)$. If no confusion may arise, we simply write $\mathcal{P} := \mathcal{P}(\mathbb{G})$, $\overline{\mathcal{P}} := \overline{\mathcal{P}}(\mathbb{G})$, $\mathcal{O} := \mathcal{O}(\mathbb{G})$ and $\overline{\mathcal{O}} := \overline{\mathcal{O}}(\mathbb{G})$. We agree to use the following convention: if not stated otherwise, a process indexed by $[0,\infty]$ is optional (resp. predictable), if it is measurable with respect to $\overline{\mathcal{O}}$ (resp. $\overline{\mathcal{P}}$), and a process indexed by $[0,\infty)$ is optional (resp. predictable), if it is measurable with respect to $\mathcal{O}$ (resp. $\mathcal{P}$). Finally, a stopping time T is predictable, if $[\![0,T[\![:= \{(\omega,t) \in \Omega\times[0,\infty) : t < T(\omega)\}$ is in $\mathcal{P}$, and we denote the collection of predictable stopping times by $\mathcal{T}^p_{0,\infty}$.

The following result appears in [76, Proposition 1.1] and [77, Lemma I.1.19 and

Lemma I.2.17]. It will allow us to use results proved under the usual conditions in our setting where the filtration is not assumed to be $\mathbb{P}$-complete. Of course, a similar result holds for the optional or predictable σ-algebra on $\Omega \times [0, \infty]$.

Lemma 2.2. (i) *Suppose that* T *is a* $\mathbb{G}^{\mathbb{P}}$*-stopping time* (*resp.* $\mathbb{G}^{\mathbb{P}}$*-predictable stopping time*). *There exists a* $\mathbb{G}$*-stopping time* (*resp.* $\mathbb{G}$*-predictable stopping time*) T' *such that* $T = T'$, $\mathbb{P}$*–almost surely.*

(ii) *Suppose that* X *is* $\mathcal{O}(\mathbb{G}^{\mathbb{P}})$*-measurable* (*resp.* $\mathcal{P}(\mathbb{G}^{\mathbb{P}})$*-measurable*), *then there exists an* $\mathcal{O}(\mathbb{G})$*-measurable* (*resp.* $\mathcal{P}(\mathbb{G})$*-measurable*) *process* X' *such that* $X = X'$ *up to* $\mathbb{P}$*-indistinguishability.*

2.2 A lot of integrals

This part is purely for completeness as we recall the construction and some results of (stochastic) integration theory. The integrals in this work are always constructed on $[0, \infty)$, and the corresponding value at infinity of the (stochastic) integrals are determined by taking the limit $[0, \infty) \ni t \uparrow\uparrow \infty$. This allows us to consider them as optional processes indexed by $[0, \infty]$ which are additionally left-continuous at infinity.

2.2.1 Lebesgue–Stieltjes integral

We have collected some results on Lebesgue–Stieltjes integrals in Proposition C.2.[1] Suppose that $C = (C_t)_{t\in[0,\infty)}$ is an optional process for which $\mathbb{P}$–almost every path is right-continuous, non-decreasing and $[0, \infty)$–valued. Let $f = (f_u)_{s\in[0,\infty)}$ be an optional process with values in $[0, \infty]$, or with values in $[-\infty, \infty]$ and satisfying $\int_{[0,t]} |f_u| \mathrm{d}C_u < \infty$, $\mathbb{P}$–a.s., $t \in [0, \infty)$. Here, the measure $\mathrm{d}C_u$ charges $\{0\}$ with mass C_0. We denote by $\int_0^{\cdot} f_u \mathrm{d}C_u = (\int_0^t f_u \mathrm{d}C_u)_{t\in[0,\infty)}$ the, up to $\mathbb{P}$–indistinguishability, unique, optional process with $\mathbb{P}$-a.s. right-continuous paths satisfying $\int_0^t f_u \mathrm{d}C_u = \int_{[0,t]} f_u \mathrm{d}C_u$, $t \in [0, \infty)$, $\mathbb{P}$–a.s. Note that $\int_0^{t-} f_u \mathrm{d}C_u = \int_{[0,t)} f_u \mathrm{d}C_u$, $\mathbb{P}$–a.s. We denote by $\int_0^\infty f_u \mathrm{d}C_u$ the $\mathbb{P}$–a.s. unique $\mathcal{G}_\infty$-measurable random variable satisfying $\int_0^\infty f_u \mathrm{d}C_u = \lim_{t\uparrow\uparrow\infty} \int_0^t f_u \mathrm{d}C_u = \int_{[0,\infty)} f_u \mathrm{d}C_u$, $\mathbb{P}$–a.s., in case f is non-negative or $\int_{[0,\infty)} |f_u| \mathrm{d}C_u < \infty$, $\mathbb{P}$–a.s. For two stopping times S and T, we then use the convention $\int_S^T f_u \mathrm{d}C_u \coloneqq \int_0^T f_u \mathrm{d}C_u - \int_0^{S\wedge T} f_u \mathrm{d}C_u = \int_{[0,\infty)} \mathbf{1}_{(S,T]}(u) f_u \mathrm{d}C_u$, $\mathbb{P}$–a.s. We also note that in case $C_0 = 0$, $\mathbb{P}$–a.s., we have $\int_{[0,t]} f_u \mathrm{d}C_u = \int_{(0,t]} f_u \mathrm{d}C_u$. This then implies $\int_S^T f_u \mathrm{d}C_u = \int_{(0,\infty)} \mathbf{1}_{(S,T]}(u) f_u \mathrm{d}C_u$, $\mathbb{P}$–a.s. In case C and f are both predictable, the integral process $\int_0^{\cdot} f_u \mathrm{d}C_u$ can be chosen to be predictable as well. Finally, even when $C = (C_t)_{t\in[0,\infty]}$ has a well-defined value C_∞ at infinity, which does not necessarily correspond to its left limit $C_{\infty-}$, we never include ∞ in the domain of the integration. So we always have $\int_S^T f_u \mathrm{d}C_u = \int_{[0,\infty)} \mathbf{1}_{(S,T]}(u) f_u \mathrm{d}C_u$ and $\int_{S-}^T f_u \mathrm{d}C_u = \int_{[0,\infty)} \mathbf{1}_{[S,T]}(u) f_u \mathrm{d}C_u$ up to a $\mathbb{P}$–null set.

2.2.2 Vector stochastic integral

In this part, we recall the $\mathbb{L}^2$-theory of the vector stochastic integral and refer the reader to [77, Section III.6] or [76, Section IV.2] for details. Let $X = (X)_{t\in[0,\infty)}$ be an $\mathbb{R}^m$-valued process with components in $\mathcal{H}^2_{\mathrm{loc}}$ with $X_0 = 0$, $\mathbb{P}$–a.s., and denote by $\mathbb{H}^{2,0}(X)$ the linear space of $\mathbb{R}^m$-valued, predictable processes $Z = (Z_t)_{t\in[0,\infty)}$ for which each component Z^i satisfies

$$\mathbb{E}\bigg[\int_{(0,\infty)} |Z^i_s|^2 \mathrm{d}\langle X^i, X^i\rangle_s\bigg] < \infty,\ i \in \{1, \ldots, m\}.$$

[1]While we do not rely on Proposition C.2 in this work, we have included it in order to correct [113, Lemma B.1].

Let $\mathcal{L}^{2,0}(X)$ denote the linear subspace of $\mathcal{H}^2$ consisting of the component-wise stochastic integrals $\sum_{i=1}^m \int_{(0,\cdot]} Z_s^i \mathrm{d}X_s^i$ for $Z \in \mathbb{H}^{2,0}(X)$. We endow $\mathbb{H}^{2,0}(X)$ with the semi-norm

$$\|Z\|^2_{\mathbb{H}^{2,0}(X)} \coloneqq \mathbb{E}\bigg[\sum_{i=1}^m\sum_{j=1}^m \int_{(0,\infty)} Z_s^i Z_s^j \mathrm{d}\langle X^i, X^j\rangle_s\bigg],$$

which is finite on $\mathbb{H}^{2,0}(X)$ by the Kunita–Watanabe inequality. The map

$$\mathbb{H}^{2,0}(X) \ni Z \longmapsto \sum_{i=1}^m \int_{(0,\cdot]} Z_s^i \mathrm{d}X_s^i \in \mathcal{H}^2, \tag{2.1}$$

is an isometry between the semi-normed space $\mathbb{H}^{2,0}(X)$ and the Hilbert space $\mathcal{H}^2$ since

$$\bigg\langle \sum_{i=1}^m \int_{(0,\cdot]} Z_s^i \mathrm{d}X_s^i, \sum_{j=1}^m \int_{(0,\cdot]} Z_s^j \mathrm{d}X_s^j \bigg\rangle = \sum_{i=1}^m\sum_{j=1}^m \int_{(0,\cdot]} Z_s^i Z_s^j \mathrm{d}\langle X^i, X^j\rangle_s.$$

Note that for $(Z, Z') \in \mathbb{H}^{2,0}(X) \times \mathbb{H}^{2,0}(X)$ with $\|Z - Z'\|_{\mathbb{H}^{2,0}} = 0$, we have $\int_{(0,\cdot]} Z_s \mathrm{d}X_s = \int_{(0,\cdot]} Z'_s \mathrm{d}X_s$, up to $\mathbb{P}$-indistinguishability. In general, the space $\mathbb{H}^{2,0}(X)$ is not complete, and therefore $\mathcal{L}^{2,0}(X)$ is not closed in $\mathcal{H}^2$. In what follows, we will construct the completion of $\mathbb{H}^{2,0}(X)$, which leads to the notion of the vector stochastic integral whose collection thereof is a stable subspace of $\mathcal{H}^2$, and thus, in particular, closed.

Let $C = (C_t)_{t\in[0,\infty)}$ be a predictable process which is $\mathbb{P}$–a.s. right-continuous, non-decreasing and starts from zero. Consider a predictable process $c = (c^{i,j})_{(i,j)\in\{1,\dots,m\}^2}$ with values in the space of positive semi-definite matrices that satisfies

$$\langle X^i, X^j\rangle_\cdot = \int_{(0,\cdot]} c_s^{i,j} \mathrm{d}C_s,\ \mathbb{P}\text{–a.s.} \tag{2.2}$$

Remark 2.3. We borrow the following construction of the pair (c, C) from [112]. Consider $C \coloneqq \sum_{i=1}^m \langle X^i, X^i\rangle$, and let

$$c_t^{i,j} \coloneqq \hat{c}_t^{i,j} \mathbf{1}_{\{\hat{c}_s^{i,j} \in \mathbb{S}_+^m\}},\ \text{where}\ \hat{c}_t^{i,j} \coloneqq \limsup_{n\to\infty} \frac{\langle X^i, X^j\rangle_t - \langle X^i, X^j\rangle_{(t-1/n)\vee 0}}{C_t - C_{(t-1/n)\vee 0}},$$

and $\mathbb{S}_+^m$ is the space of positive semi-definite, real-valued $m \times m$ matrices, and where we used the convention $0 \coloneqq 0/0$.

The linear space of all predictable processes $Z = (Z_t)_{t\in[0,\infty)}$ with values in $\mathbb{R}^m$ satisfying

$$\|Z\|^2_{\mathbb{H}^2(X)} \coloneqq \mathbb{E}\bigg[\int_{(0,\infty)} \sum_{i=1}^m\sum_{j=1}^m Z_s^i c_s^{i,j} Z_s^j \mathrm{d}C_s\bigg] < \infty,$$

is denoted by $\mathbb{H}^2(X)$. Note that $\mathbb{H}^2(X)$ does not depend on the choice of the pair (c, C) for which (2.2) holds and that $\|\cdot\|_{\mathbb{H}^2(X)}$ is a semi-norm which coincides with $\|\cdot\|_{\mathbb{H}^{2,0}}$ on $\mathbb{H}^{2,0}(X)$. The space $\mathbb{H}^2(X)$ together with $\|\cdot\|_{\mathbb{H}^2(X)}$ is the semi-norm completion of $\mathbb{H}^{2,0}(X)$ by [76, Théorème 4.35].[2] The isometry in (2.1) thus extends uniquely to an isometry

$$\mathbb{H}^2(X) \ni Z \longmapsto \int_{(0,\cdot]} Z_s \mathrm{d}X_s \in \mathcal{H}^2, \tag{2.3}$$

between the semi-normed space $\mathbb{H}^2(X)$ and the Hilbert space $\mathcal{H}^2$. Since by continuity, for two elements Z and Z' in $\mathbb{H}^2(X)$ satisfying $\|Z - Z'\|_{\mathbb{H}^2(X)} = 0$, we have $\int_{(0,\cdot]} Z_s \mathrm{d}X_s = \int_{(0,\cdot]} Z'_s \mathrm{d}X_s$, we can turn (2.3) into an isometry between Banach spaces

[2] To be precise, $\mathbb{H}^2(X)$, together with $\|\cdot\|_{\mathbb{H}^2(X)}$, forms a complete semi-normed space containing $\mathbb{H}^{2,0}(X)$ as a dense subset.

after identifying processes in $\mathbb{H}^2(X)$ with $\|Z-Z'\|_{\mathbb{H}^2(X)}=0$. For each Z in $\mathbb{H}^2(X)$, $\int_{(0,\cdot]} Z_s\mathrm{d}X_s = (\int_{(0,t]} Z_s\mathrm{d}X_s)_{t\in[0,\infty)}$, is the *vector stochastic integral* of Z with respect to X. It is the, up to $\mathbb{P}$-indistinguishability, unique process in $\mathcal{H}^2$ that starts from zero, $\mathbb{P}$–a.s., and satisfies

$$\bigg\langle \int_{(0,\cdot]} Z_s\mathrm{d}X_s, N\bigg\rangle_{\cdot} = \int_{(0,\cdot]}\bigg(\sum_{i=1}^m H^i_s c^{N,i}_s\bigg)\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.},$$

where $c^{N,i}$ is a predictable process with $\langle N,X^i\rangle_\cdot = \int_{(0,\cdot]} c^{N,i}_s\mathrm{d}C_s$. Similarly to the stochastic integral of one-dimensional processes, for $Z\in\mathbb{H}^2(X)$ and $T\in\mathcal{T}_{0,\infty}$, the predictable process $\Omega\times[0,\infty)\ni(\omega,t)\longmapsto Z_s(\omega)\mathbf{1}_{[0,T(\omega)]}(s)\in\mathbb{R}^m$ is in $\mathbb{H}^2(X)$ and $\int_{(0,\cdot]} Z_s\mathbf{1}_{[0,T]}(s)\mathrm{d}X_s = \int_{(0,\cdot\wedge T]} Z_s\mathrm{d}X_s$, $\mathbb{P}$–a.s. Moreover, for W in $\mathbb{H}^2\big(\int_{(0,\cdot]} Z_s\mathrm{d}X_s\big)$, the product WZ is in $\mathbb{H}^2(X)$ and

$$\int_{(0,\cdot]} W_sZ_s\mathrm{d}X_s = \int_{(0,\cdot]} W_s\mathrm{d}\bigg(\int_{(0,\cdot]} Z_u\mathrm{d}X_u\bigg)_s.$$

As the space of vector stochastic integrals $\mathcal{L}^2(X) \coloneqq \big\{\int_{(0,\cdot]} Z_s\mathrm{d}X_s : Z\in\mathbb{H}^2(X)\big\}$ is the image of an isometry defined on a Banach space, it is a closed subspace of $\mathcal{H}^2$. Additionally, it is also a stable subspace of $\mathcal{H}^2$ (see also [76, Definition 4.4 and Theorem 4.35]). Let us stress that for $Z\in\mathbb{H}^2(X)$, we have the following equalities

$$\|Z\|^2_{\mathbb{H}^2(X)} = \mathbb{E}\bigg[\int_{(0,\infty)}\sum_{i=1}^m\sum_{j=1}^m Z^i_s c^{i,j}_s Z^j_s\mathrm{d}C_s\bigg] = \bigg\|\int_{(0,\cdot]} Z_s\mathrm{d}X_s\bigg\|^2_{\mathcal{H}^2}.$$

We close this part by agreeing on adopting a useful convention that will ease the notation in what follows. First, we agree to write $\int_0^t Z_s\mathrm{d}X_s \coloneqq \int_{(0,t]} Z_s\mathrm{d}X_s,\ t\in[0,\infty)$. Since the vector stochastic integral is in $\mathcal{H}^2$, it will have a $\mathbb{P}$–a.s. unique $\mathcal{G}_\infty$-measurable, real-valued, square-integrable limit at infinity which we denote by $\int_0^\infty Z_s\mathrm{d}X_s$. For two stopping times S and T, we use the convention $\int_S^T Z_u\mathrm{d}X_u \coloneqq \int_0^T Z_u\mathrm{d}X_u - \int_0^{S\wedge T} Z_u\mathrm{d}X_u$.

2.2.3 Stochastic integral with respect to a compensated integer-valued random measure

Here, we recall the construction of the stochastic integral with respect to compensated integer-valued random measures in the sense of [32; 76; 77]. Before doing so, we need to introduce some terminology and notation. The *graph* of a stopping time T is the set $[\![T]\!] \coloneqq \{(\omega,t)\in\Omega\times[0,\infty) : T(\omega)=t\}$, and an optional set $D\subseteq\Omega\times[0,\infty)$ is *thin* if there exists a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ such that $D=\cup_{n\in\mathbb{N}}[\![T_n]\!]$. If $[\![T_m]\!]\cap[\![T_n]\!]=\varnothing$ for $m\neq n$, then $(T_n)_{n\in\mathbb{N}}$ is referred to as an *exhausting sequence* of stopping times for D. Note that every thin set admits an exhausting sequence of stopping times (see [77, Lemma I.1.31]). Recall from Section 2.1 that a *predictable stopping time* is a stopping time T for which $[\![0,T[\![$ is a predictable subset of $\Omega\times[0,\infty)$.

We now turn to random measures. Let $(E,\mathcal{E})$ be a Blackwell space in the sense of Dellacherie and Meyer [40, Definition III.24] and let $\widetilde{\Omega}\coloneqq\Omega\times[0,\infty)\times E$.[3] We consider two σ-algebrae on $\widetilde{\Omega}$, the predictable one given by $\widetilde{\mathcal{P}}\coloneqq\mathcal{P}\otimes\mathcal{E}$ and the optional one given by $\widetilde{\mathcal{O}}\coloneqq\mathcal{O}\otimes\mathcal{E}$. Let $\mu=\{\mu(\omega;\mathrm{d}t,\mathrm{d}x):\omega\in\Omega\}$ be a random measure on $\mathcal{B}([0,\infty))\otimes\mathcal{E}$. For an $\mathcal{G}\otimes\mathcal{B}([0,\infty))\otimes\mathcal{E}$-measurable function $U:\widetilde{\Omega}\longrightarrow\mathbb{R}$, we define the process $U\star\mu=(U\star\mu_t)_{t\in[0,\infty)}$ by

$$U\star\mu_t(\omega)\coloneqq\begin{cases}\displaystyle\int_{(0,t]\times E} U_s(\omega;x)\mu(\omega;\mathrm{d}s,\mathrm{d}x),\ \text{if}\ \int_{(0,t]\times E}|U_s(\omega;x)|\mu(\omega;\mathrm{d}s,\mathrm{d}x)<\infty,\\ \infty,\ \text{otherwise.}\end{cases} \tag{2.4}$$

[3] One can think for simplicity of E being a Polish space together with its Borel-σ-algebra $\mathcal{E}=\mathcal{B}(E)$.

We suppose that μ is an integer-valued random measure in the sense of [77], that is, there exists a $\widetilde{\mathcal{P}}$-measurable function $V > 0$ satisfying $\mathbb{E}[V \star \mu_\infty] < \infty$, an $\mathcal{O}$–measurable, E–valued process $\varrho = (\varrho)_{t\in[0,\infty)}$, and a thin set D with an exhausting sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ such that

$$\mu(\omega; \mathrm{d}t, \mathrm{d}x) = \sum_{(\omega,s)\in D} \boldsymbol{\delta}_{(s,\varrho_s(\omega))}(\mathrm{d}t, \mathrm{d}x) = \sum_{n\in\mathbb{N}} \mathbf{1}_{\{T_n<\infty\}}(\omega)\boldsymbol{\delta}_{(T_n(\omega),\varrho_{T_n(\omega)}(\omega))}(\mathrm{d}t, \mathrm{d}x).$$

Here $\boldsymbol{\delta}_{(s,z)}(\mathrm{d}t, \mathrm{d}x)$ denotes the Dirac measure at (s, z). Note that $U \star \mu$ is an optional processes for any $\widetilde{\mathcal{O}}$-measurable function $U : \widetilde{\Omega} \longrightarrow \mathbb{R}$.

Example 2.4. An example of an integer-valued random measure is the jump measure of an adapted, càdlàg process X

$$\mu(\omega; \mathrm{d}t, \mathrm{d}x) = \sum_{s\in(0,\infty)} \mathbf{1}_{\{\Delta X_s(\omega)\neq 0\}}\boldsymbol{\delta}_{(s,\Delta X_s(\omega))}(\mathrm{d}t, \mathrm{d}x).$$

The predictable compensator of μ is the random measure $\nu = \{\nu(\omega; \mathrm{d}t, \mathrm{d}x) : \omega \in \Omega\}$, which is (up to $\mathbb{P}$–null sets) uniquely characterised by the following: $\nu(\omega; \{0\} \times E) = 0$ for each $\omega \in \Omega$, the process $U \star \nu$ is $\mathcal{P}$-measurable and satisfies $\mathbb{E}[U \star \mu_\infty] = \mathbb{E}[U \star \nu_\infty]$, for each non-negative, $\widetilde{\mathcal{P}}$-measurable function U (see [77, Proposition II.1.28]). Moreover, we choose a version of ν that satisfies $\nu(\omega; \{t\} \times E) \leq 1$ for every $(\omega, t) \in \Omega \times [0, \infty)$ and such that $\{(\omega, t) \in \Omega \times [0, \infty) : \nu(\omega; \{t\} \times E) > 0\}$ can be exhausted by a sequence of predictable stopping times (see [77, Proposition II.1.17]). Next, there exists (see [77, Theorem II.1.8] together with [141, Lemma 6.5.10]) a right-continuous, $\mathbb{P}$–a.s. non-decreasing, predictable process $C = (C_t)_{t\in[0,\infty)}$ and a transition kernel K from $(\Omega \times [0, \infty), \mathcal{P})$ to $(E, \mathcal{E})$ such that

$$\nu(\omega; \mathrm{d}s, \mathrm{d}x) = K_s(\omega; \mathrm{d}x)\mathrm{d}C_s(\omega),\ \mathbb{P}\text{–a.e.} \tag{2.5}$$

For any $\mathcal{G} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{E}$-measurable function $U : \widetilde{\Omega} \longrightarrow \mathbb{R}$, we define the process $\widehat{U} = (\widehat{U}_t)_{t\in[0,\infty)}$ by

$$\widehat{U}_t(\omega) := \begin{cases} \displaystyle\int_{\{t\}\times E} U_t(\omega; x)\nu(\omega; \mathrm{d}t, \mathrm{d}x), \text{ if } \int_{\{t\}\times E} |U_t(\omega; x)|\nu(\omega; \mathrm{d}t, \mathrm{d}x) < \infty, \\ \infty, \text{ otherwise,} \end{cases}$$

and $\widetilde{U} = (\widetilde{U}_t)_{t\in[0,\infty)}$ by $\widetilde{U}_t(\omega) := U_t(\omega, \varrho_t(\omega))\mathbf{1}_D(\omega, t) - \widehat{U}_t(\omega)$. Note that $\widetilde{U}$ is an optional process and that for each $\omega \in \Omega$, the collection $\{t \in [0, \infty) : \widetilde{U}_t(\omega) \neq 0\}$ is at most countable, and thus $\{(\omega, t) \in \Omega \times [0, \infty) : \widetilde{U}_t(\omega) \neq 0\}$ admits an exhausting sequence of stopping times (see [40, Theorem IV.88]). The sum $\sum_{s\in(0,\infty)} |\widetilde{U}_s|^2$ is thus well-defined and $\mathcal{G}$-measurable.

We now turn to the construction of the compensated stochastic integral with respect to μ. We denote by $\mathbb{H}^2(\mu)$ the linear space of $\widetilde{\mathcal{P}}$-measurable functions $U : \widetilde{\Omega} \longrightarrow \mathbb{R}$ satisfying $\mathbb{E}[\sum_{s\in(0,\infty)} |\widetilde{U}_s|^2] < \infty$. For each $U \in \mathbb{H}^2(\mu)$, there exists a, up to $\mathbb{P}$–indistinguishability, unique purely discontinuous $U \star \tilde{\mu} \in \mathcal{M}_{\mathrm{loc}}$ whose jumps are given by $\Delta(U \star \tilde{\mu}) = \widetilde{U}$ up to $\mathbb{P}$-evanescence (see [77, Theorem I.4.56]). Note that $(U + U') \star \tilde{\mu} = U \star \tilde{\mu} + U' \star \tilde{\mu}$, $\mathbb{P}$–a.s., for $(U, U') \in \mathbb{H}^2(\mu) \times \mathbb{H}^2(\mu)$. Since

$$[U \star \tilde{\mu}]_\cdot = \sum_{s\in(0,\cdot]} |\Delta(U \star \tilde{\mu})_s|^2 = \sum_{s\in(0,\cdot]} |\widetilde{U}_s|^2,\ \mathbb{P}\text{–a.s.},$$

we find that $[U \star \tilde{\mu}]_\infty \in \mathbb{L}^1(\mathcal{G}_\infty)$ and thus $U \star \tilde{\mu} \in \mathcal{H}^2$ (see [77, Proposition I.4.50]). By [77, Theorem II.1.33], the predictable quadratic variation of the process $U \star \tilde{\mu}$ is given by

$$\langle U \star \tilde{\mu}\rangle_t(\omega) = (U - \widehat{U})^2 \star \nu_t(\omega) + \sum_{s\in(0,t]} \big(1 - \zeta_s(\omega)\big)|\widehat{U}_s(\omega)|^2$$

$$
\begin{aligned}
&= \int_{(0,\cdot]} \int_E \big(U_s(\omega;x) - \widehat{U}_s(\omega)\big)^2 K_s(\omega;\mathrm{d}x)\mathrm{d}C_s(\omega) \\
&\qquad + \int_{(0,t]} \big(1-\zeta_s(\omega)\big)\bigg(\int_E U_s(\omega;x)K_s(\omega;\mathrm{d}x)\bigg)^2 \Delta C_s(\omega)\mathrm{d}C_s(\omega) \\
&= \int_{(0,\cdot]} \Bigg(\int_E \big(U_s(\omega;x) - \widehat{U}_s(\omega)\big)^2 K_s(\omega;\mathrm{d}x) \\
&\qquad + \big(1-\zeta_s(\omega)\big)\bigg(\int_E U_s(\omega;x)K_s(\omega;\mathrm{d}x)\bigg)^2 \Delta C_s(\omega)\Bigg)\mathrm{d}C_s(\omega) \\
&=: \int_{(0,\cdot]} \big(|||U_s(\omega;\cdot)|||_s(\omega)\big)^2 \mathrm{d}C_s(\omega),\ t \in [0,\infty),\ \mathbb{P}\text{–a.e. } \omega \in \Omega,
\end{aligned}
$$

where $\zeta = (\zeta_s)_{s\in[0,\infty)}$ is the predictable process defined by $\zeta_s(\omega) \coloneqq \nu(\omega;\{s\}\times E) \in [0,1]$. Note that we can choose a predictable version of ΔC that is $[0,\infty)$-valued, and agrees with the jump process of C up to a $\mathbb{P}$–null set. Indeed, we define

$$
\Delta C_t \coloneqq \Delta C'_t \mathbf{1}_{\{0\leq \Delta C'_t<\infty\}},\ \text{where } \Delta C'_t(\omega) \coloneqq \limsup_{n\uparrow\uparrow\infty} \big(C_t - C_{(t-1/n)\vee 0}\big).
$$

With this construction of the jump process of C, the map

$$
\Omega \times [0,\infty) \ni (\omega,s) \longmapsto |||U_s(\omega;\cdot)|||_s(\omega) \in [0,\infty]
$$

becomes predictable. Conversely, if we start with a $\widetilde{\mathcal{P}}$-measurable function U such that $\|U\|^2_{\mathbb{H}^2(\mu)} \coloneqq \mathbb{E}[\int_{(0,\infty)} |||U_s(\cdot)|||^2_s \mathrm{d}C_s] < \infty$, then $U \in \mathbb{H}^2(\mu)$ and thus $U \star \tilde{\mu} \in \mathcal{H}^2$ (see [77, Theorem II.1.33]). Note that for $(U,V) \in \mathbb{H}^2(\mu)\times\mathbb{H}^2(\mu)$ satisfying $\|U-V\|_{\mathbb{H}^2(\mu)} = 0$, we have $U\star\tilde{\mu} = V\star\tilde{\mu}$. We therefore identify U und V in $\mathbb{H}^2(\mu)$ in this case, which turns $\mathbb{H}^2(\mu)$ into a normed space. The space of compensated stochastic integrals $\mathcal{K}^2(\mu) \coloneqq \{U\star\tilde{\mu} : U \in \mathbb{H}^2(\mu)\}$ is a stable subspace of $\mathcal{H}^2$ by [76, Proposition 3.71 and Theorem 4.46] and thus closed in $\mathcal{H}^2$. We end up with the following result, whose proof we defer to Appendix A.

Proposition 2.5. *The space $\mathbb{H}^2(\mu)$ endowed with the norm $\|\cdot\|_{\mathbb{H}^2(\mu)}$ is a Banach space. Moreover, for each $U \in \mathbb{H}^2(\mu)$,*

$$
\|U\|^2_{\mathbb{H}^2(\mu)} = \mathbb{E}\bigg[\int_{(0,\infty)} \big(|||U_s(\cdot)|||_s\big)^2 \mathrm{d}C_s\bigg] = \mathbb{E}\big[\langle U\star\tilde{\mu}\rangle_\infty\big] = \mathbb{E}\big[[U\star\tilde{\mu}]_\infty\big].
$$

For $(\omega,t) \in \Omega\times[0,\infty)$, let $\mathfrak{H}_{\omega,t}$ denote the collection of $\mathcal{E}$-measurable maps $\mathcal{U} : E \longrightarrow \mathbb{R}$ satisfying $|||\mathcal{U}(\cdot)|||_t(\omega) < \infty$. Define $\mathfrak{H}$ as the collection of $\widetilde{\mathcal{P}}$-measurable functions $U : \widetilde{\Omega} \longrightarrow \mathbb{R}$ such that $U_t(\omega;\cdot) \in \mathfrak{H}_{\omega,t}$ for each $(\omega,t) \in \Omega\times[0,\infty)$. Since for $U \in \mathbb{H}^2(\mu)$, we have $|||U_t(\omega;\cdot)|||_t(\omega) < \infty$ for $\mathrm{d}\mathbb{P}\times\mathrm{d}C$–a.e. $(\omega,t) \in \Omega\times[0,\infty)$, we can define $U'_t(\omega;x) \coloneqq U_t(\omega;x)\mathbf{1}_N(\omega,t)$, where $N = \{(\omega,t) \in \Omega\times[0,\infty) : |||U_t(\omega;\cdot)|||_t(\omega) = \infty\} \in \mathcal{P}$, which yields a version of U in $\mathbb{H}^2(\mu)$ which is also in $\mathfrak{H}$. We thus always choose a version of $U \in \mathbb{H}^2(\mu)$ that is also in $\mathfrak{H}$. The space $\mathfrak{H}$ will be fundamental in our formulation of reflected BSDEs in Section 2.5.

Let us close this part by a agreeing on a convention similar to the one we made about vector stochastic integrals. Note first that since $U\star\tilde{\mu} \in \mathcal{H}^2$ for $U \in \mathbb{H}^2(X)$, there's a well-defined limit at infinity $U\star\mu_\infty$. In the sequel, we will denote the process $U\star\tilde{\mu}$ by $\int_0^t\int_E U_s(x)\mathrm{d}\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) \coloneqq U\star\tilde{\mu}_t$, $t\in[0,\infty]$, and for two stopping times S and T, we write

$$
\begin{aligned}
\int_S^T \int_E U_s(x)\mathrm{d}\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) &\coloneqq \int_0^T \int_E U_s(x)\mathrm{d}\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) - \int_0^{S\wedge T}\int_E U_s(x)\mathrm{d}\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) \\
&= U\star\tilde{\mu}_T - U\star\tilde{\mu}_{S\wedge T}.
\end{aligned}
$$

2.3 Orthogonal decomposition

Let X be the $\mathbb{R}^m$-valued process with components in $\mathcal{H}^2_{\mathrm{loc}}$ from Section 2.2.2 and μ be the integer-valued random measure from Section 2.2.3. We are not working under the assumption of martingale representation, and thus want to find conditions on X and μ that allow us to decompose a square-integrable martingale uniquely along X, μ and another square-integrable martingale N appropriately orthogonal to X and μ. We mentioned in Section 2.2.2 and Section 2.2.3 that the spaces $\mathcal{L}^2(X)$ and $\mathcal{K}^2(\mu)$ of stochastic integrals with respect to X and $\mu-\nu$, respectively, are stable (and thus closed) subspaces of $\mathcal{H}^2$. We give sufficient conditions on X and μ under which $\mathcal{L}^2(X)\cap\mathcal{K}^2(\mu)$ is the null space in $\mathcal{H}^2$. This allows us to write

$$\mathcal{H}^2=\mathcal{L}^2(X)\oplus\mathcal{K}^2(\mu)\oplus\mathcal{H}^{2,\perp}(X,\mu) \text{ with } \mathcal{H}^{2,\perp}(X,\mu):=\big(\mathcal{L}^2(X)\oplus\mathcal{K}^2(\mu)\big)^{\perp}.$$

This part is based on [113] and we borrow notations from [77]. Let M_μ be the Doléans measure defined by

$$M_\mu[W]:=\int W(\omega,s,x)M_\mu(\mathrm{d}\omega,\mathrm{d}s,\mathrm{d}x):=\mathbb{E}[W\star\mu_\infty],$$

for each $\mathcal{G}\otimes\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{E}$-measurable function $W:\widetilde{\Omega}\longrightarrow[0,\infty]$. Recall that there exists a $\widetilde{\mathcal{P}}$ measurable function $V>0$ with $\mathbb{E}[V\star\mu_\infty]<\infty$. Thus the restriction of M_μ to $\widetilde{\mathcal{P}}$ is a σ-finite measure. By the Radon–Nikodým theorem, there exists for every $\mathcal{G}\otimes\mathcal{B}([0,\infty))\otimes\mathcal{E}$-measurable function $W:\widetilde{\Omega}\longrightarrow[0,\infty]$, a M_μ–a.e. unique, $\widetilde{\mathcal{P}}$-measurable function $M_\mu[W|\widetilde{\mathcal{P}}]:=W':\widetilde{\Omega}\longrightarrow[0,\infty]$ satisfying $M_\mu[W'U]=M_\mu[WU]$, for each $\widetilde{\mathcal{P}}$-measurable function $U:\widetilde{\Omega}\longrightarrow[0,\infty]$. For a general measurable function $W:\widetilde{\Omega}\longrightarrow[-\infty,\infty]$, we use the same convention as in [77] to define $M_\mu[W|\widetilde{\mathcal{P}}]$, namely

$$M_\mu\big[W\big|\widetilde{\mathcal{P}}\big]:=\begin{cases}M_\mu\big[\max\{W,0\}\big|\widetilde{\mathcal{P}}\big]-M_\mu\big[\max\{-W,0\}\big|\widetilde{\mathcal{P}}\big],\ \text{on } \{M_\mu\big[|W|\big|\widetilde{\mathcal{P}}\big]<\infty\},\\ \infty,\ \text{otherwise.}\end{cases}$$

The following is the main result about orthogonal decompositions along X and μ, whose proof we defer to Appendix A.

Proposition 2.6. *Suppose that $M_\mu[\Delta X^i|\widetilde{\mathcal{P}}]=0$ for every $i\in\{1,\dots,m\}$. For each $M\in\mathcal{H}^2$, there exists a unique pair $(Z,U)\in\mathbb{H}^2(X)\times\mathbb{H}^2(\mu)$ such that $N=(N_t)_{t\in[0,\infty)}\in\mathcal{H}^2$ defined by*

$$N_t:=M_t-\int_0^t Z_s\mathrm{d}X_s-\int_0^t\int_E U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x),$$

satisfies $\langle N,X^i\rangle=0$ for each $i\in\{1,\dots,m\}$ and $M_\mu[\Delta N|\widetilde{\mathcal{P}}]=0$.

Corollary 2.7. *Under the assumptions of* Proposition 2.6*, we have $\mathcal{H}^2=\mathcal{L}^2(X)\oplus\mathcal{K}^2(\mu)\oplus\mathcal{H}^{2,\perp}(X,\mu)$, where $\mathcal{H}^{2,\perp}(X,\mu):=\big(\mathcal{L}^2(X)\oplus\mathcal{K}^2(\mu)\big)^{\perp}$. In particular, $\mathcal{H}^{2,\perp}(X,\mu)=\big\{N\in\mathcal{H}^2:M_\mu[\Delta N|\widetilde{\mathcal{P}}]=0,\ \langle N,X^i\rangle=0 \text{ for } i\in\{1,\dots,m\}\big\}$.*

2.4 Data and the corresponding weighted spaces

In this section we fix the data of the reflected BSDE and the weighted spaces in which we will construct its solution. The obstacle and terminal condition are described by a single optional process ξ as in [60; 62]. From Remark 2.10, this is without loss of generality. Throughout this work, we fix once and for all the data $(X,\mu,\mathbb{G},T,\xi,f,C)$, where

(D1) $\mathbb{G}$ is the filtration in the stochastic basis;

(D2) $X = (X_t)_{t\in[0,\infty)}$ is an $\mathbb{R}^m$-valued process whose components are in $\mathcal{H}^2_{\mathrm{loc}}$ with $X_0 = 0$, $\mathbb{P}$–a.s., μ is an integer-valued random measure on $\mathbb{R}_+ \times E$, where $(E, \mathcal{E})$ is some Blackwell space, and $M_\mu[\Delta X^i|\widetilde{\mathcal{P}}] = 0$ for each $i \in \{1, \dots, m\}$;

(D3) $C = (C_t)_{t\in[0,\infty)}$ is a real-valued, predictable process with $\mathbb{P}$–a.s. right-continuous and non-decreasing paths starting from zero that satisfies

$$\mathrm{d}\langle X^i, X^j\rangle_s(\omega) = c^{i,j}_s \mathrm{d}C_s(\omega), \text{ and } \nu(\omega; \mathrm{d}s, \mathrm{d}x) = K_s(\omega, \mathrm{d}x)\mathrm{d}C_s(\omega),$$

for $\mathbb{P}$–a.e. $\omega \in \Omega$, for each $(i,j) \in \{1, \dots, d\}^2$, where each $c^{i,j} = (c^{i,j}_t)_{t\in[0,\infty)}$ is a predictable process with values in the set of positive, semi-definite, symmetric matrices, and K is a transition kernel from $(\Omega \times [0,\infty), \mathcal{P})$ into $(E, \mathcal{E})$;

(D4) T is a $\mathbb{G}$–stopping time;

(D5) $\xi = (\xi_t)_{t\in[0,\infty]}$ is a $[-\infty, \infty)$-valued, optional process satisfying[4]

$$\mathbb{E}[|\xi_T|^2] + \mathbb{E}\bigg[\sup_{s\in[0,T)} |\xi^+_s|^2\bigg] < \infty;$$

(D6) $f : \bigsqcup_{(\omega,t)\in\Omega\times[0,\infty)} \big(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathfrak{H}_{\omega,t}\big) \longrightarrow \mathbb{R}$ is such that for each $(y, \mathrm{y}, z, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathfrak{H}$, the map[5]

$$\Omega \times [0,\infty) \ni (\omega, t) \longmapsto f_t(\omega, y, \mathrm{y}, z, u_t(\omega; \cdot)) \in \mathbb{R},$$

is optional and f is $(r, \theta^X, \theta^\mu)$–Lipschitz continuous on $[\![0, T]\!] := \{(\omega, t) \in \Omega \times [0,\infty) : t \le T(\omega)\}$ in the sense that

$$\begin{aligned}&\big|f_t(\omega, y, \mathrm{y}, z, u_t(\omega; \cdot)) - f_t(\omega, y', \mathrm{y}', z', u'_t(\omega; \cdot))\big|^2 \\ &\quad\le r_t(\omega)|y - y'|^2 + \mathrm{r}_t(\omega)|\mathrm{y} - \mathrm{y}'|^2 + \theta^X_t(\omega)\|c^{1/2}_t(\omega)(z - z')\|^2 \\ &\qquad + \theta^\mu_t(\omega)\Big(|\!|\!| u_t(\omega; \cdot) - u'_t(\omega; \cdot)|\!|\!|_t(\omega)\Big)^2,\end{aligned}$$

for $\mathbb{P}\otimes\mathrm{d}C$–a.e. $(\omega, t) \in [\![0, T]\!]$, where $r = (r_t)_{t\in[0,\infty)}$, $\mathrm{r} = (\mathrm{r}_t)_{t\in[0,\infty)}$, $\theta^X = (\theta^X_t)_{t\in[0,\infty)}$ and $\theta^\mu = (\theta^\mu_t)_{t\in[0,\infty)}$ are $[0,\infty)$-valued, predictable processes, and $c^{1/2}$ is the unique square-root matrix-valued process of c;[6]

(D7) the optional process $f_\cdot(0, 0, 0, \mathbf{0})$ satisfies[7]

$$\mathbb{E}\bigg[\bigg(\int_0^T |f_s(0, 0, 0, \mathbf{0})|\mathrm{d}C_s\bigg)^2\bigg] < \infty;$$

(D8) the non-negative, predictable process $\alpha = (\alpha_t)_{t\in[0,\infty)}$ defined through

$$\alpha^2_t = \max\{\sqrt{r_t}, \sqrt{\mathrm{r}_t}, \theta^X_t, \theta^\mu_t\}$$

satisfies $\alpha_t(\omega) > 0$ for $\mathbb{P} \otimes \mathrm{d}C$–a.e. $(\omega, t) \in [\![0, T]\!]$, and the predictable process $A = (A_t)_{t\in[0,\infty)}$ defined by $A_t := \int_0^{t\wedge T} \alpha^2_s \mathrm{d}C_s$ is real-valued and satisfies $\Delta A \le \Phi$, up to $\mathbb{P}$-evanescence, for some $\Phi \in [0,\infty)$.

[4] Recall from Remark 2.1 that the expectation is well-defined as $\sup_{s\in[0,T)} \xi^+_s = \sup_{s\in[0,\infty]} \xi^+_s \mathbf{1}_{\{s<T\}}$.

[5] The symbol $\bigsqcup$ denotes the disjoint union, and therefore each $f_t(\omega, \cdot, \cdot, \cdot, \cdot)$ is a map from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathfrak{H}_{\omega,t}$ into $\mathbb{R}$.

[6] See Horn and Johnson [73, page 439].

[7] Here $\mathbf{0}$ denotes the zero element of the space $\mathbb{H}^2(\mu)$.

Remark 2.8. (i) If we start with a process $\tilde{\xi}$ that is only defined on $\{(\omega,t)\in\Omega\times[0,\infty]:t\leq T(\omega)\}$, then we ask here that the process $\xi=(\xi_t)_{t\in[0,\infty]}$ defined by $\xi_t(\omega):=\tilde{\xi}_t(\omega)\mathbf{1}_{[0,T(\omega)]}(t)+\tilde{\xi}_{T(\omega)}(\omega)\mathbf{1}_{(T(\omega),\infty]}(t)$ is optional. The process $\tilde{\xi}$ is still the lower barrier and $\tilde{\xi}_T$ is the terminal condition of the reflected BSDE.

(ii) Note that **(D2)** allows us to decompose each $M\in\mathcal{H}^2$ uniquely into

$$M_t=M_0+\int_0^t Z_s\mathrm{d}X_s+\int_0^t\int_E U_s(x)\mu(\mathrm{d}s,\mathrm{d}x)+N_t,\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.},$$

for $(Z,U,N)\in\mathbb{H}^2(X)\times\mathbb{H}^2(\mu)\times\mathcal{H}^{2,\perp}(X,\mu)$ by Proposition 2.6.

The process $\mathcal{E}(\beta A)$ denotes the stochastic exponential of βA, that is, $\mathcal{E}(\beta A)=\big(\mathcal{E}(\beta A)_t\big)_{t\in[0,\infty)}$ is the unique right-continuous, adapted process satisfying

$$\mathcal{E}(\beta A)_t=1+\int_0^t\mathcal{E}(\beta A)_{s-}\beta\mathrm{d}A_s,\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.}$$

As A is $\mathbb{P}$–a.s. non-decreasing, it follows from [77, Theorem I.4.61] that

$$\mathcal{E}(\beta A)_t=\mathrm{e}^{\beta A_t}\prod_{s\in(0,t]}(1+\gamma\Delta A_s)\mathrm{e}^{-\gamma\Delta A_s},\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.}$$

Therefore $\mathcal{E}(\beta A)$ is $\mathbb{P}$–a.s. non-decreasing as well and satisfies $1\leq\mathcal{E}(\beta A)\leq\mathrm{e}^{\beta A}$ up to $\mathbb{P}$–indistinguishability. We now introduce the (weighted) classical spaces in which we will construct the solution to the reflected BSDE. Although these spaces depend on (α,C,T), we will suppress the dependence on (α,C) to ease the notation. For $\beta\in[0,\infty)$

- $\mathbb{L}^p_\beta(\mathcal{F})$, for $p\in[1,\infty)$ and a sub–σ-algebra $\mathcal{F}\subseteq\mathcal{G}$, denotes the space of real-valued, $\mathcal{F}$-measurable random variables ζ satisfying

$$\|\zeta\|^2_{\mathbb{L}^2_\beta}:=\mathbb{E}\big[\big|\mathcal{E}(\beta A)_T^{1/2}\zeta\big|^2\big]<\infty;$$

- $\mathcal{H}^2_{T,\beta}$ denotes the Banach space of real-valued martingales $M=(M_t)_{t\in[0,\infty)}$ in $\mathcal{H}^2$ satisfying $M=M_{\cdot\wedge T}$ and

$$\|M\|^2_{\mathcal{H}^2_{T,\beta}}:=\mathbb{E}[M_0^2]+\mathbb{E}\bigg[\int_{(0,\infty)}\mathcal{E}(\beta A)_s\mathrm{d}\langle M^T\rangle_s\bigg]=\mathbb{E}[M_0^2]+\mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_s\mathrm{d}\langle M\rangle_s\bigg]<\infty;$$

- $\mathcal{S}^2_{T,\beta}$ denotes the Banach space[8] of real-valued, optional processes $Y=(Y_t)_{t\in[0,\infty]}$ satisfying $Y=Y_{\cdot\wedge T}$ and

$$\|Y\|^2_{\mathcal{S}^2_{T,\beta}}:=\mathbb{E}\bigg[\sup_{s\in[0,T]}\big|\mathcal{E}(\beta A)_s^{1/2}Y_s\big|^2\bigg]<\infty;$$

- $\mathbb{H}^2_{T,\beta}$ denotes the Banach space of real-valued, optional processes $\phi=(\phi_t)_{t\in[0,\infty]}$ satisfying $\phi=\phi_{\cdot\wedge T}$ and

$$\|\phi\|^2_{\mathbb{H}^2_{T,\beta}}:=\mathbb{E}\bigg[\int_{(0,\infty)}\mathcal{E}(\beta A)_s|\phi_s|^2\mathrm{d}C^T_s\bigg]=\mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_s|\phi_s|^2\mathrm{d}C_s\bigg]<\infty;$$

- $\mathbb{H}^2_{T,\beta}(X)$ denotes the Banach space of $\mathbb{R}^m$-valued, predictable processes Z in $\mathbb{H}^2(X)$ satisfying $Z=Z\mathbf{1}_{[\![0,T]\!]}$ and

$$\|Z\|^2_{\mathbb{H}^2_{T,\beta}(X)}:=\big\|\mathcal{E}(\beta A)^{1/2}Z\mathbf{1}_{[\![0,T]\!]}\big\|^2_{\mathbb{H}^2(X)}=\mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_s\sum_{i=1}^m\sum_{j=1}^m Z^i_s c^{i,j}_s Z^j_s\mathrm{d}C_s\bigg]<\infty;$$

[8] That $\mathcal{S}^2_{T,\beta}$ is a Banach space follows with the arguments described in [41, IV.21, pp. 82–83].

- $\mathbb{H}^2_{T,\beta}(\mu)$ denotes the Banach space of real-valued, $\widetilde{\mathcal{P}}$-predictable processes U in $\mathbb{H}^2(\mu)$ satisfying $U = U\mathbf{1}_{[\![0,T]\!]}$ and

$$\|U\|^2_{\mathbb{H}^2_{T,\beta}(\mu)} \coloneqq \big\|\mathcal{E}(\beta A)^{1/2}U\mathbf{1}_{[\![0,T]\!]}\big\|^2_{\mathbb{H}^2(\mu)} = \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}\langle U \star \tilde{\mu}\rangle_s\bigg] < \infty;$$

- $\mathcal{H}^{2,\perp}_{0,T,\beta}(X,\mu)$ denotes the closed subspace of real-valued martingales $N = (N_t)_{t\in[0,\infty)}$ in $\mathcal{H}^2_{T,\beta}$ with $N \in \mathcal{H}^2_0 \cap \mathcal{H}^{2,\perp}(X,\mu)$;

- $\mathcal{I}^2_{T,\beta}$ denotes the space of $[0,\infty)$-valued, optional processes $K = (K_t)_{t\in[0,\infty]}$ whose paths are $\mathbb{P}$–a.s. right-continuous and non-decreasing, satisfying $K = K_{\cdot\wedge T}$, $\mathbb{E}[K_T^2] < \infty$ and

$$\|K\|^2_{\mathcal{I}^2_{T,\beta}} \coloneqq \mathbb{E}\bigg[\bigg(\int_{[0,\infty)} \mathcal{E}(\beta A)^{1/2}_s \mathrm{d}K^T_s\bigg)^2\bigg] = \mathbb{E}\bigg[\bigg(\int_0^T \mathcal{E}(\beta A)^{1/2}_s \mathrm{d}K_s\bigg)^2\bigg] < \infty,$$

  with convention $K_{0-} \coloneqq 0$.

Finally, for $\beta = 0$, we simply write $\mathbb{L}^p \coloneqq \mathbb{L}^p_0(\mathcal{G})$, $\mathbb{L}^p(\mathcal{F}) \coloneqq \mathbb{L}^p_0(\mathcal{F})$, $\mathcal{H}^2_T \coloneqq \mathcal{H}^2_{T,0}$, $\mathcal{S}^2_T \coloneqq \mathcal{S}^2_{T,0}$, $\mathbb{H}^2_T \coloneqq \mathbb{H}^2_{T,0}$, $\mathbb{H}^2_T(X) \coloneqq \mathbb{H}^2_{T,0}(X)$, $\mathbb{H}^2_T(\mu) \coloneqq \mathbb{H}^2_{T,0}(\mu)$, $\mathcal{H}^{2,\perp}_{0,T}(X,\mu) \coloneqq \mathcal{H}^{2,\perp}_{0,T,0}(X,\mu)$ and $\mathcal{I}^2_T \coloneqq \mathcal{I}^2_{T,0}$.

2.5 Formulation of the reflected BSDE

In this work, we consider reflected BSDEs driven by càdlàg martingales and integer-valued random measures on a possibly unbounded time horizon. It turns out that for the analysis that follows, it is convenient to construct the solution directly on $[0,\infty]$, although the driving martingales and integer-valued random measures are only defined on $[0,\infty)$. As we have seen before, we can and will assign a value to the (stochastic) integrals at infinity by taking the limit $t \uparrow\uparrow \infty$, whenever this makes sense. Inspired by the work of Grigorova, Imkeller, Ouknine, and Quenez [62], we will not suppose any regularity on the paths of the obstacle process ξ. We thus have to consider its left-limit process $\overline{\xi} = (\overline{\xi}_t)_{t\in[0,\infty]}$ defined by

$$\overline{\xi}_t \coloneqq \xi_0\mathbf{1}_{\{t=0\}} + \limsup_{s\uparrow\uparrow t} \xi_s\mathbf{1}_{\{t\in(0,\infty]\}},$$

which is $(\mathcal{G}^U_t)_{t\in[0,\infty]}$-predictable by Proposition C.6. Note that this process is $\mathbb{P}$–indistinguishable from a $(\mathcal{G}_t)_{t\in[0,\infty]}$-predictable process by Lemma 2.2 or [76, Proposition 1.1]. The solution to the reflected BSDE with generator f and obstacle process ξ is a collection of processes (Y,Z,U,N,K^r,K^ℓ) satisfying the following conditions

(R1) $(Z,U,N) \in \mathbb{H}^2_T(X) \times \mathbb{H}^2_T(\mu) \times \mathcal{H}^{2,\perp}_{0,T}(X,\mu)$, and $Y = (Y_t)_{t\in[0,\infty]}$ is optional with $\mathbb{P}$–a.s. làdlàg paths;

(R2) $(K^r,K^\ell) \in \mathcal{I}^2_T \times \mathcal{I}^2_T$ with $K^r_0 = 0$, $\mathbb{P}$–a.s.;

(R3) (Y,Z,U,N,K^r,K^ℓ) satisfies[9]

$$\mathbb{E}\bigg[\int_0^T \big|f_s\big(Y_s,Y_{s-},Z_s,U_s(\cdot)\big)\big|\mathrm{d}C_s\bigg] < \infty,$$

and $\mathbb{P}$–a.s., for each $t \in [0,\infty]$,

[9] We use a predictable version Y_- of the left-limit process of Y here (see for example Equation 2.3.3 and Lemma 6.1.3 in [141]).

$$Y_t = \xi_T + \int_t^T f_s\big(Y_s, Y_{s-}, Z_s, U_s(\cdot)\big)\mathrm{d}C_s - \int_t^T Z_s \mathrm{d}X_s \\ - \int_t^T \int_E U_s(x)\tilde{\mu}(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}N_s + K^r_T - K^r_t + K^\ell_{T-} - K^\ell_{t-};$$

(R4) $Y_T = \xi_T$, and $Y = Y_{\cdot\wedge T}$;

(R5) $Y_{\cdot\wedge T} \geq \xi_{\cdot\wedge T}$;

(R6) K^r is predictable, K^ℓ satisfies $K^\ell_T = K^\ell_{T-}$, $\mathbb{P}$–a.s., and is purely discontinuous, that is, $K^\ell_t = K^\ell_0 + \sum_{s\in(0,t]} \Delta K^\ell_s$, $t \in [0,\infty]$, $\mathbb{P}$–a.s., and the following Skorokhod condition holds, with $K^\ell_{0-} \coloneqq 0$,

$$\big(Y_{T-} - \overline{\xi}_T\big)\Delta K^r_T + \int_{(0,T)} \big(Y_{s-} - \overline{\xi}_s\big)\mathrm{d}K^r_s + \int_{[0,T)} (Y_s - \xi_s)\mathrm{d}K^\ell_s = 0, \ \mathbb{P}\text{–a.s.};$$

(R7) $Y_S = \operatorname*{ess\,sup}^{\mathcal{G}_S}_{\tau\in\mathcal{T}_{S,T}} \mathbb{E}\bigg[\xi_\tau + \int_S^\tau f_s\big(Y_s, Y_{s-}, Z_s, U_s(\cdot)\big)\mathrm{d}C_s \bigg| \mathcal{G}_S\bigg]$, $\mathbb{P}$–a.s., $S \in \mathcal{T}_{0,T}$.

Let us comment on the form of the generator.

Remark 2.9. To the best of our knowledge, and except in [113], the literature only considers the case where the generator depends on Y_{s-} and not on Y_s. When the integrator C does not jump and is thus continuous, the dependence on Y_s or Y_{s-} does not matter as $\{s : Y_s \neq Y_{s-}\}$ will be at most countable and thus of $\mathrm{d}C_s$-measure zero. When the process C can jump, the dependence matters, and we include both cases for the following reasons:

(i) a dependence on Y_{s-} in the generator has been considered in numerical schemes, see among others Briand, Delyon, and Mémin [22], Briand, Delyon, and Mémin [23], Briand, Geiss, Geiss, and Labart [24], Cheridito and Stadje [27], Madan, Pistorius, and Stadje [107], Possamaï and Tan [127], and Papapantoleon, Possamaï, and Saplaouras [114; 115];

(ii) a linear BSDE only seems to allow for an explicit representation of its Y-component as a conditional expectation if the linearity of the generator depends not on Y_{s-} but on Y_s. To see this, one can adapt the techniques of [26, Lemma 2.2] to our setup. Similarly, in the reflected BSDE case, a Y_s-dependence in the generator already appears in the following seemingly simple example. Consider the optimal stopping problem $\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}[\xi_\tau/D_\tau]$, where the discounting process is given by $D = \mathcal{E}(\int_0^\cdot r_s \mathrm{d}C_s)$ for some predictable process $r = (r_s)_{s\in[0,\infty)}$ that is C-integrable. The value process $V_t = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[\xi_\tau D_t/D_\tau]$ of this optimal stopping problem, after applying some standard transformations and then Mertens's decomposition, is the Y-component of a reflected BSDE with obstacle ξ and generator of the form

$$f_s(Y_s) = \frac{r_s Y_s}{1 + r_s \Delta C_s}.$$

As our main motivation is to develop (reflected) BSDEs to analyse certain discrete- and continuous-time problems in a unified manner, we stress that the dependence on Y_s in the generator is thus crucial.

We now comment on the formulation of the reflected BSDE.

Remark 2.10. (i) That ξ plays the role of the lower barrier and terminal condition is without loss of generality since for any lower barrier $L=(L_t)_{t\in[0,\infty)}$ and terminal condition ζ, we can define the obstacle ξ as $\xi_t \coloneqq L_t\mathbf{1}_{[0,T)}(t)+\zeta\mathbf{1}_{[T,\infty]}(t),\ t\geq 0$. With our formulation, it is clear that on $[0,T)$, ξ is the lower barrier L and ξ_T is the terminal condition ζ.

(ii) With the conventions we agreed upon in Section 2.2, the integral processes appearing in **(R3)** and **(R7)** never include the points 0 or ∞ in the domain of integration.

(iii) Note that the forward dynamics of Y are, $\mathbb{P}$ –a.s.,

$$Y_t = Y_0 - \int_0^{t\wedge T} f_s\big(Y_s,Y_{s-},Z_s,U_s(\cdot)\big)\mathrm{d}C_s + \int_0^{t\wedge T} Z_s\mathrm{d}X_s + \int_0^{t\wedge T}\int_E U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) + N_{t\wedge T} - K^r_{t\wedge T} - K^\ell_{(t\wedge T)-},\ t\in[0,\infty].$$

In particular, Y is a $\mathbb{P}$ –a.s. làdlàg optional semimartingale.

(iv) Conditions **(R4)**, **(R5)** and **(R7)** are equivalent to

$$Y_S = \underset{\tau\in\mathcal{T}_{S,\infty}}{\operatorname{ess\,sup}}^{\mathcal{G}_S}\mathbb{E}\bigg[\xi_{\tau\wedge T} + \int_S^{\tau\wedge T} f_s\big(Y_{s-},Y_s,Z_s,U_s(\cdot)\big)\mathrm{d}C_s\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}_{0,\infty}.$$

(v) We will see in Lemma 6.2 that if $(\alpha Y,\alpha Y_-,Z,U,N)\in\mathbb{H}^2_{T,\hat{\beta}}\times\mathbb{H}^2_{T,\hat{\beta}}\times\mathbb{H}^2_{T,\hat{\beta}}(X)\times\mathbb{H}^2_{T,\hat{\beta}}(\mu)$, then **(R1)** up to **(R6)** imply **(R7)**.

(vi) If ξ is $\mathbb{P}$–a.s. left–upper semicontinuous along stopping times, then K^r is continuous. This is similar to [62, Remark 2.4].

(vii) If ξ is $\mathbb{P}$ –a.s. right-continuous, then so is Y. Indeed, let $Y^{(+)}$ be the right-continuous, optional process that $\mathbb{P}$–a.s. agrees with the process of right-hand side limits of Y (see [41, Appendix I, Remark 5(b)]). For $\tau\in\mathcal{T}_{0,T}$, we have $Y_{\tau+}-Y_\tau = -(K^\ell_{\tau\wedge T}-K^\ell_{\tau\wedge T-})\leq 0$, $\mathbb{P}$–a.s., since K^ℓ is $\mathbb{P}$–a.s. non-decreasing, and therefore $Y_{\tau+}\leq Y_\tau$ up to a $\mathbb{P}$–null set. This then also implies $Y_\tau\geq Y_{\tau+}\geq\xi_{\tau+}=\xi_\tau$ up to a $\mathbb{P}$–null set. Following [60, Remark 3.3], we then find with **(R5)** and **(R6)** that

$$\begin{aligned}1=\mathbb{P}[Y_\tau\geq\xi_\tau] &= \mathbb{P}[Y_\tau>\xi_\tau]+\mathbb{P}[Y_\tau=\xi_\tau]\\ &= \mathbb{P}[\Delta K^\ell_\tau=0,Y_\tau>\xi_\tau]+\mathbb{P}[Y_\tau=Y_{\tau+},Y_\tau=\xi_\tau]\\ &= \mathbb{P}[Y_{\tau+}=Y_\tau,Y_\tau>\xi_\tau]+\mathbb{P}[Y_{\tau+}=Y_\tau,Y_\tau=\xi_\tau]=\mathbb{P}[Y_\tau=Y_{\tau+}].\end{aligned}$$

Hence $Y_\tau=Y_{\tau+}=Y^{(+)}_\tau$, $\mathbb{P}$ –a.s., which implies that $Y=Y_{\cdot\wedge T}$ is $\mathbb{P}$–indistinguishable from $Y^{(+)}_{\cdot\wedge T}$ by the optional cross-section theorem in [40, Theorem IV.84, p. 137] (or see Proposition C.3). Incidentally, since K^ℓ is purely discontinuous and the source of right-hand side jumps of Y, this implies $K^\ell=0$ up to a $\mathbb{P}$–null set.

$(viii)$ Instead of considering two predictable processes (K^r,K^ℓ) in the above formulation of the reflected BSDE, we could simply consider a single $\mathbb{P}$–a.s. non-decreasing, predictable process K satisfying $K_0=0$ and $K=K_{\cdot\wedge T}$, $\mathbb{P}$–a.s., defined by $K=K^r+K^\ell_-$ up to $\mathbb{P}$–indistinguishability. The processes (K,K^r,K^ℓ) are then related to each other as follows

$$K^r_t = K_t - K^\ell_{t-},\ \text{where}\ K^\ell_t=\sum_{s\in[0,t]}(K_{s+}-K_s),\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.},\ \text{with}\ K^\ell_{0-}=0.$$

3 Main results

This section contains the main results of our work. We present them first in the reflected BSDE and then in the (non-reflected) BSDE setting. Although the formulation we chose for reflected BSDEs includes BSDEs as special cases, it turns out that the *a priori* estimates in Section 5 can be improved. We therefore report the results separately.

3.1 Existence and uniqueness for reflected BSDEs

Before stating our main well-posedness result for reflected BSDEs, we introduce some notation to ease the presentation. For $(\Psi,\beta)\in[0,\infty)\times(0,\infty)$, let

$$\mathfrak{f}^{\Psi}(\beta) \coloneqq \inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\beta\Psi)}{\gamma(\beta-\gamma)}\bigg\} = \frac{4(1+\beta\Psi)}{\beta^2},$$
$$\mathfrak{g}^{\Psi}(\beta) \coloneqq \inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\gamma\Psi)}{\gamma(\beta-\gamma)}\bigg\} = \frac{4}{\beta^2}\mathbf{1}_{\{\Psi=0\}} + \frac{\Psi^2\sqrt{1+\beta\Psi}}{\big(1+\beta\Psi-\sqrt{1+\beta\Psi}\big)\big(\sqrt{1+\beta\Psi}-1\big)}\mathbf{1}_{\{\Psi>0\}}.$$

Here the equalities follow from Lemma B.1. We define

$$M_1^{\Psi}(\beta) \coloneqq \mathfrak{f}^{\Psi}(\beta) + \frac{4}{\beta} + \max\bigg\{1,\frac{(1+\beta\Psi)}{\beta}\bigg\}\bigg(\frac{5}{\beta}+\frac{4}{\beta}(1+\beta\Psi)^{1/2}+\beta\mathfrak{g}^{\Psi}(\beta)\bigg),$$
$$M_2^{\Psi}(\beta) \coloneqq \mathfrak{f}^{\Psi}(\beta) + \bigg(\frac{5}{\beta}+\frac{4}{\beta}(1+\beta\Psi)^{1/2}+\beta\mathfrak{g}^{\Psi}(\beta)\bigg),$$
$$M_3^{\Psi}(\beta) \coloneqq \frac{4}{\beta} + \max\bigg\{1,\frac{(1+\beta\Psi)}{\beta}\bigg\}\bigg(\frac{5}{\beta}+\frac{4}{\beta}(1+\beta\Psi)^{1/2}+\beta\mathfrak{g}^{\Psi}(\beta)\bigg),\ \beta\in(0,\infty).$$

The constants $M_1^{\Phi}(\beta)$, $M_2^{\Phi}(\beta)$ and $M_3^{\Phi}(\beta)$ will appear when we construct the contraction mappings on the weighted solution spaces of the reflected BSDE. Being able to keep them strictly less than one will allow us to use a fixed-point argument to deduce well-posedness. We now turn to the integrability conditions we need to impose on the obstacle ξ and the generator f to make our method of proof work, in particular, to ensure that the contraction mappings will be well-defined. Let ${}^*\xi=({}^*\xi_t)_{t\in[0,\infty]}$ be the process defined by

$$ {}^*\xi_0 \coloneqq 0,\ \text{and}\ {}^*\xi_t \coloneqq \lim_{t'\uparrow\uparrow t}\bigg\{\sup_{s\in[t',\infty]}|\xi_s^+\mathbf{1}_{\{s<T\}}|\bigg\},\ t\in(0,\infty], \tag{3.1}$$

which is $\mathcal{G}^U_\infty\otimes\mathcal{B}([0,\infty])$-measurable by [52, Proposition 2.21]. Although this process depends on the stopping time T, we suppress this to ease the notation. The following definition contains the main integrability condition that we impose.

Definition 3.1. *The collection $(X,\mu,\mathbb{G},T,\xi,f,C)$ is* standard data for $\hat\beta\in[0,\infty)$*, if the pair (ξ,f) satisfies*

$$\|\xi_T\|_{\mathbb{L}^2_{\hat\beta}} + \|\alpha\,{}^*\xi\|_{\mathbb{H}^2_{T,\hat\beta}} + \bigg\|\frac{f(0,0,0,\mathbf{0})}{\alpha}\bigg\|_{\mathbb{H}^2_{T,\hat\beta}} < \infty.$$

Remark 3.2. Our integrability assumption on ξ is slightly different than the one imposed in [60; 62], which reads

$$\mathbb{E}\bigg[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}{}^{\mathcal{G}_\tau}|\xi_\tau|^2\bigg]<\infty,$$

in a bounded horizon, Brownian–Poisson framework. Our assumption is not stronger than the one above, if we set ourselves into their framework. On the contrary, we actually only need to consider the positive part of ξ on $[0,T)$, which is more general than the integrability condition in [60; 62]. We will have a more thorough comparison with the literature in Section 3.3.

Let us mention here a sufficient condition to have $\|\alpha\,{}^*\xi\|_{\mathbb{H}^2_{T,\hat\beta}}$ finite. We defer the proof to Appendix A.

Lemma 3.3. *Let $(\hat\beta,\beta^\star)\in[0,\infty)^2$ with $\hat\beta<\beta^\star$. Then*

$$\|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\hat\beta}} \leq \frac{(1+\beta^\star\Phi)(1+\hat\beta\Phi)}{(\beta^\star-\hat\beta)}\|\xi^+_\cdot\mathbf{1}_{\{\cdot<T\}}\|^2_{\mathcal{S}^2_{T,\beta^\star}}.$$

In particular, if $\|\xi^+_\cdot\mathbf{1}_{\{\cdot<T\}}\|_{\mathcal{S}^2_{T,\beta^\star}}$ is finite, then so is $\|\alpha\,{}^*\xi\|_{\mathbb{H}^2_{T,\hat\beta}}$.

We now turn to our main well-posedness result for reflected BSDEs. It covers the case where the generator depends, additionally to Z and $U(\cdot)$, on both Y_s and Y_{s-}, just on Y_s or just on Y_{s-}. The proof is deferred to Section 6 as it is based on the optimal stopping and Snell envelope theory we revisit in Section 4 and the *a priori* estimates we establish in Section 5.

Theorem 3.4. *Suppose that $(X, \mathbb{G}, T, \xi, f, C)$ is standard data for some $\hat{\beta} \in (0, \infty)$.*

(i) *If $M_1^{\Phi}(\hat{\beta}) < 1$, there exists a solution $(Y, Z, U, N, K^r, K^\ell)$ to the reflected* BSDE *satisfying* **(R1)** *up to* **(R6)** *such that $(Y, \alpha Y, \alpha Y_-, Z, U, N)$ is in $\mathcal{S}_T^2 \times \mathbb{H}_{T,\hat{\beta}}^2 \times \mathbb{H}_{T,\hat{\beta}}^2 \times \mathbb{H}_{T,\hat{\beta}}^2(X) \times \mathbb{H}_{T,\hat{\beta}}^2(\mu) \times \mathcal{H}_{0,T,\hat{\beta}}^{2,\perp}(X, \mu)$.*

(ii) *If $M_2^{\Phi}(\hat{\beta}) < 1$ and f does not depend on the component Y_{s-}, then there exists a solution $(Y, Z, U, N, K^r, K^\ell)$ to the reflected* BSDE *satisfying* **(R1)** *up to* **(R6)** *such that $(\alpha Y, Z, U, N)$ is in $\mathbb{H}_{T,\hat{\beta}}^2 \times \mathbb{H}_{T,\hat{\beta}}^2(X) \times \mathbb{H}_{T,\hat{\beta}}^2(\mu) \times \mathcal{H}_{0,T,\hat{\beta}}^{2,\perp}(X, \mu)$. Moreover, Y is in $\mathcal{S}_T^2$.*

(iii) *If $M_3^{\Phi}(\hat{\beta}) < 1$ and f does not depend on the component Y_s, then there exists a solution $(Y, Z, U, N, K^r, K^\ell)$ to the reflected* BSDE *satisfying* **(R1)** *up to* **(R6)** *such that $(Y, \alpha Y_-, Z, U, N)$ is in $\mathcal{S}_T^2 \times \mathbb{H}_{T,\hat{\beta}}^2 \times \mathbb{H}_{T,\hat{\beta}}^2(X) \times \mathbb{H}_{T,\hat{\beta}}^2(\mu) \times \mathcal{H}_{0,T,\hat{\beta}}^{2,\perp}(X, \mu)$.*

In all three cases, the triple (Y, K^r, K^ℓ) is unique up to $\mathbb{P}$–indistinguishability and (Z, U, N) is unique in $\mathbb{H}_{T,\hat{\beta}}^2(X) \times \mathbb{H}_{T,\hat{\beta}}^2(\mu) \times \mathcal{H}_{0,T,\hat{\beta}}^{2,\perp}(X, \mu)$. Furthermore, **(R7)** *holds. If, additionally, $\xi^+ \mathbf{1}_{[0,T)} \in \mathcal{S}_{T,\beta}^2$ for some $\beta \in (0, \hat{\beta})$, then $(K^r, K^\ell) \in \mathcal{I}_{T,\beta}^2 \times \mathcal{I}_{T,\beta}^2$.*

Remark 3.5. The fixed-point argument used in the proof of Theorem 3.4 relies on the usage of the $\mathcal{S}_T^2$–norm in cases (i) and (iii). However, one can use the following alternative norm[10]

$$\|Y\|_{\mathcal{T}_T^2}^2 := \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\big[|Y_\tau|^2\big] \tag{3.2}$$

in the contraction argument. The statements of Theorem 3.4 and Corollary 3.6 remain unchanged, except that we would be able to replace the constants $M_1^{\Phi}(\hat{\beta})$ and $M_3^{\Phi}(\hat{\beta})$ by

$$M_1^{\Phi}(\hat{\beta}) = \mathfrak{f}^{\Phi}(\hat{\beta}) + \frac{1}{\hat{\beta}} + \max\left\{1, \frac{(1 + \hat{\beta}\Phi)}{\hat{\beta}}\right\}\left(\frac{5}{\hat{\beta}} + \frac{4}{\hat{\beta}}(1 + \hat{\beta}\Phi)^{1/2} + \hat{\beta}\mathfrak{g}^{\Phi}(\hat{\beta})\right)$$

and

$$M_3^{\Phi}(\hat{\beta}) = \frac{1}{\hat{\beta}} + \max\left\{1, \frac{(1 + \hat{\beta}\Phi)}{\hat{\beta}}\right\}\left(\frac{5}{\hat{\beta}} + \frac{4}{\hat{\beta}}(1 + \hat{\beta}\Phi)^{1/2} + \hat{\beta}\mathfrak{g}^{\Phi}(\hat{\beta})\right),$$

respectively. For the BSDE case, see Remark 3.8. We refer to Remark 5.5 and 6.4 for more details.

Theorem 3.4 and the analysis of the contraction constants in Lemma B.2 yield the following immediate result.

Corollary 3.6. *Suppose that $\Phi < 1$. For each $i \in \{1, 2, 3\}$, there exists $\beta_i^\star \in (0, \infty)$ such that $M_i^{\Phi}(\hat{\beta}) < 1$ for every $\hat{\beta} \in (\beta_i^\star, \infty)$. Moreover*

(i) *if $(X, \mathbb{G}, T, \xi, f, C)$ is standard data for $\hat{\beta} \in (\beta_1^\star, \infty)$, then* Theorem 3.4. (i) *holds;*

(ii) *if $(X, \mathbb{G}, T, \xi, f, C)$ is standard data for $\hat{\beta} \in (\beta_2^\star, \infty)$, then* Theorem 3.4. (ii) *holds;*

(iii) *if $(X, \mathbb{G}, T, \xi, f, C)$ is standard data for $\hat{\beta} \in (\beta_3^\star, \infty)$, then* Theorem 3.4. (iii) *holds.*

[10] For more details on this norm, see [41, IV.21, pp. 82–83].

3.2 Existence and uniqueness for BSDEs

We now discuss the existence and uniqueness of the non-reflected BSDE with generator f and terminal condition ξ_T. More precisely, we look for a unique quadruple (Y,Z,U,N) within a class of processes satisfying

(B1) $(Z,U,N) \in \mathbb{H}^2_T(X) \times \mathbb{H}^2_T(\mu) \times \mathcal{H}^{2,\perp}_{0,T}(X,\mu)$;

(B2) $Y = (Y_t)_{t\in[0,\infty]}$ is optional with $\mathbb{P}$–a.s. làdlàg paths,[11]

$$\mathbb{E}\bigg[\int_0^T \big|f_s\big(Y_s, Y_{s-}, Z_s, U_s(\cdot)\big)\big|\mathrm{d}C_s\bigg] < \infty,$$

and

$$Y_t = \xi_T + \int_t^T f_s\big(Y_s, Y_{s-}, Z_s, U_s(\cdot)\big)\mathrm{d}C_s - \int_t^T Z_s \mathrm{d}X_s \\ - \int_t^T \int_E U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}N_s,\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.}$$

To deduce existence and uniqueness within a class of processes of the BSDE above, one could just redefine the obstacle ξ to be $-\infty$ on $[0,T)$, note that then $K^r = 0$ and $K^\ell_- = 0$, the representation **(R7)** turns into

$$Y_S = \mathbb{E}\bigg[\zeta + \int_S^T f_s\big(Y_s, Y_{s-}, Z_s, U_s(\cdot)\big)\mathrm{d}C_s\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty},$$

and then refer to Theorem 3.4 for the conditions that provide existence and uniqueness in case there exists $\hat{\beta} \in (0,\infty)$ with

$$\|\xi_T\|_{\mathbb{L}^2_{\hat{\beta}}} + \bigg\|\frac{f(0,0,0,\mathbf{0})}{\alpha}\bigg\|_{\mathbb{H}^2_{T,\hat{\beta}}} < \infty. \tag{3.3}$$

However, it is worthwhile to redo the *a priori* estimates in Section 5 in this case as the contraction constants improve significantly. Let us also emphasise here that the techniques we employ to establish well-posedness for BSDEs do not depend on Itô's formula, and extending them to BSDEs with a multi-dimensional generator and terminal condition is straightforward. The constants we want to control in the contraction argument to prove well-posedness of **(B1)–(B2)** become

$$\widetilde{M}^\Psi_1(\beta) \coloneqq \mathfrak{f}^\Psi(\beta) + \frac{4}{\beta} + \max\bigg\{1, \frac{(1+\beta\Psi)}{\beta}\bigg\}\bigg(\frac{1}{\beta} + \beta\mathfrak{g}^\Psi(\beta)\bigg),$$

$$\widetilde{M}^\Psi_2(\beta) \coloneqq \mathfrak{f}^\Psi(\beta) + \bigg(\frac{1}{\beta} + \beta\mathfrak{g}^\Psi(\beta)\bigg),$$
$$\widetilde{M}^\Psi_3(\beta) \coloneqq \frac{4}{\beta} + \max\bigg\{1, \frac{(1+\beta\Psi)}{\beta}\bigg\}\bigg(\frac{1}{\beta} + \beta\mathfrak{g}^\Psi(\beta)\bigg).$$

The following is our main well-posedness result for BSDEs. We defer its proof to the end of Section 6.

Theorem 3.7. *Suppose that $(X,\mathbb{G},T,\xi,f,C)$ is standard data for some $\hat{\beta} \in (0,\infty)$ and $\xi = -\infty$ on $[0,T)$.*

[11] Note that Y is then *a fortiori* $\mathbb{P}$–a.s. càdlàg.

(i) *If* $\widetilde{M}_1^\Phi(\hat\beta) < 1$*, then there exists a solution* (Y,Z,U,N) *to the* BSDE **(B1)–(B2)** *such that* $(Y,\alpha Y,\alpha Y_-,Z,U,N)$ *is in* $\mathcal{S}_T^2 \times \mathbb{H}^2_{T,\hat\beta} \times \mathbb{H}^2_{T,\hat\beta} \times \mathbb{H}^2_{T,\hat\beta}(X) \times \mathbb{H}^2_{T,\hat\beta}(\mu) \times \mathcal{H}^{2,\perp}_{0,T,\hat\beta}(X,\mu)$.

(ii) *If* $\widetilde{M}_2^\Phi(\hat\beta) < 1$ *and the generator* f *does not depend on* Y_{s-}*, then there exists a solution* (Y,Z,U,N) *to the* BSDE **(B1)–(B2)** *such that* $(\alpha Y,Z,U,N)$ *is in* $\mathbb{H}^2_{T,\hat\beta} \times \mathbb{H}^2_{T,\hat\beta}(X) \times \mathbb{H}^2_{T,\hat\beta}(\mu) \times \mathcal{H}^{2,\perp}_{0,T,\hat\beta}(X,\mu)$.

(iii) *If* $\widetilde{M}_3^\Phi(\hat\beta) < 1$ *and the generator* f *does not depend on* Y_s*, then there exists a solution* (Y,Z,U,N) *to the* BSDE **(B1)–(B2)** *such that* $(Y,\alpha Y_-,Z,U,N)$ *is in* $\mathcal{S}_T^2 \times \mathbb{H}^2_{T,\hat\beta} \times \mathbb{H}^2_{T,\hat\beta}(X) \times \mathbb{H}^2_{T,\hat\beta}(\mu) \times \mathcal{H}^{2,\perp}_{0,T,\hat\beta}(X,\mu)$.

In all three cases, Y *is in* $\mathcal{S}^2_{T,\hat\beta}$ *and unique up to* $\mathbb{P}$*–indistinguishability, and* (Z,U,N) *is unique in* $\mathbb{H}^2_{T,\hat\beta}(X) \times \mathbb{H}^2_{T,\hat\beta}(\mu) \times \mathcal{H}^{2,\perp}_{0,T,\hat\beta}(X,\mu)$.

Remark 3.8. Similar to the discussion in Remark 3.5, we can also replace here in the contraction argument the norm $\|\cdot\|_{\mathcal{S}^2_T}$ by the norm $\|\cdot\|_{\mathcal{T}^2_T}$ introduced in (3.2). Incidentally, we would be able to replace the constants $\widetilde{M}_1^\Phi(\hat\beta)$ and $\widetilde{M}_3^\Phi(\hat\beta)$ in the statement of Theorem 3.7 and Corollary 3.9 by

$$\widetilde{M}_1^\Phi(\hat\beta) = \mathfrak{f}^\Phi(\hat\beta) + \frac{1}{\hat\beta} + \max\left\{1, \frac{(1+\hat\beta\Phi)}{\hat\beta}\right\}\left(\frac{1}{\hat\beta} + \hat\beta\mathfrak{g}^\Phi(\hat\beta)\right)$$

and

$$\widetilde{M}_3^\Phi(\hat\beta) = \frac{1}{\hat\beta} + \max\left\{1, \frac{(1+\hat\beta\Phi)}{\hat\beta}\right\}\left(\frac{1}{\hat\beta} + \hat\beta\mathfrak{g}^\Phi(\hat\beta)\right),$$

respectively. We refer to Remark 5.5 and 6.4 for more details.

Combining Theorem 3.7 with the analysis of the contraction constants in Lemma B.3, we find the following.

Corollary 3.9. *Suppose that* $\Phi < 1$ *and that* $\xi = -\infty$ *on* $[0,T)$*. For each* $i \in \{1,2,3\}$*, there exists* $\beta_i^\star \in (0,\infty)$ *such that* $\widetilde{M}_i^\Phi(\hat\beta) < 1$ *for every* $\hat\beta \in (\beta_i^\star,\infty)$*. Moreover*

(i) *if* $(X,\mathbb{G},T,\xi,f,C)$ *is standard data for* $\hat\beta \in (\beta_1^\star,\infty)$*, then* Theorem 3.7. (i) *holds;*

(ii) *if* $(X,\mathbb{G},T,\xi,f,C)$ *is standard data for* $\hat\beta \in (\beta_2^\star,\infty)$*, then* Theorem 3.7. (ii) *holds;*

(iii) *if* $(X,\mathbb{G},T,\xi,f,C)$ *is standard data for* $\hat\beta \in (\beta_3^\star,\infty)$*, then* Theorem 3.7. (iii) *holds.*

Remark 3.10. (i) Here, the condition we need to impose on Φ to get well-posedness for sufficiently integrable data is weaker than the claimed condition $\Phi < 1/(18\mathrm{e})$ in [113, Corollary 3.6] in case the generator depends on $\big(Y_s, Z_s, U_s(\cdot)\big)$. There is actually a slight issue that occurs when deriving the *a priori* estimates in [113]. By correcting the issue appearing in the derivation of the *a priori* estimates, it turns out that the constant Φ needs to be even lower than $1/(18\mathrm{e})$ for the contraction argument to go through. We will come back to this in Remark 5.6.

(ii) The second well-posedness result in [113, Theorem 3.23] relies on the generator depending on $\big(Y_{s-}, Z_s, U_s(\cdot)\big)$. The proof is also based on a fixed-point argument, but the *a priori* estimates are derived by an application of Itô's formula and not by the more direct approach as in the first part of [113]. Although the integrability condition **(H4)** imposed on f in [113] under which well-posedness is established is not comparable to our integrability condition (3.3), their result relies on the more restrictive conditions **(H5)** and **(H6)** imposed on the integrator C and the driving martingale X in [113].

(iii) It turns out that within the framework we are working in, the condition $\Phi < 1$ to establish well-posedness of *BSDEs* with jumps is not only sufficient, but also necessary in the following sense: if we ask ourselves whether a condition of the form $\Phi < a$ for some $a \in (1,\infty)$ would still allow us to have a general well-posedness result, then the is answer is no as can be seen by the counterexamples to existence and uniqueness established in Confortola, Fuhrman, and Jacod [33, Section 4.3]. See also the discussion in Papapantoleon, Possamaï, and Saplaouras [113, Section 3.3.1].

(iv) An extension to d-dimensional BSDEs, for $d \in \mathbb{N}$, is straightforward as we will never use Itô's formula to derive the *a priori* estimates in the BSDE case. In this case, the generator $f = (f^1,\ldots,f^d)$ is $\mathbb{R}^d$-valued and the system of BSDEs will then take the form

$$Y_t^i = \xi_T^i + \int_t^T f_s^i\big(Y_s, Y_{s-}, Z_s, U_s(\cdot)\big)\mathrm{d}C_s - \int_t^T Z_s^i \mathrm{d}X_s \\ - \int_t^T \int_E U_s^i(x)(\mu-\mu^p)(\mathrm{d}s,\mathrm{d}x) - \int_t^T N_s^i,\ i \in \{1,\ldots,d\},$$

where $Y = (Y^1,\ldots,Y^d)^\top$, $Z = (Z^1,\ldots,Z^d)^\top$, $U = (U^1,\ldots,U^d)^\top$ and $N = (N^1,\ldots,N^d)^\top$. To adapt our method of proof, we need to replace in **(D6)**, in the definition of the weighted norms of Section 2.4, and in the proof of Theorem 3.7 the absolute value $|\cdot|$ by the Euclidean norm $\|\cdot\|_{\mathbb{R}^d}$. All computations that we will carry out in Section 5 in the BSDE case and in the proof of Theorem 3.7 will still go through.

3.3 Comparison with the literature and some consequences

In this part we compare our well-posedness results with other results in the literature. We are mostly interested in a comparison of the integrability conditions imposed on the data, and, in case the integrator C jumps, whether some a condition similar to our condition $\alpha\Delta C \leq \Phi \in [0,1)$ is needed to ensure well-posedness. However, we restrict ourselves to works which are closest to ours, although there are far more well-posedness results out there. In particular, all well-posedness results we mention in this part, except one, consider $\mathbb{L}^2$-data and Lipschitz-continuous generators.

3.3.1 When the obstacle is predictable

The case of a predictable obstacle process ξ together with a notion of 'predictable reflected BSDE' was studied by Bouhadou and Ouknine [17]. While we do allow, of course, for an obstacle ξ that is merely predictable, the solution to the reflected BSDE in [17] consists of predictable processes. Specifically, the processes Y in [17] is predictable. Their study closely relies on the theory of predictable strong supermartingales by Meyer [110, page 388] and the corresponding predictable Snell envelopes by El Karoui [46]. Although we do not cover the well-posedness result of [17] by simply taking predictable projections of our solution processes, it would be intriguing to explore whether this can be achieved by our techniques in this work in combination with the results in [46] and [110].

3.3.2 When Wiener meets Poisson

We start with comparing our results in the reflected BSDE case to Grigorova, Imkeller, Offen, Ouknine, and Quenez [60] and Grigorova, Imkeller, Ouknine, and Quenez [62], in which they show well-posedness of bounded horizon reflected BSDEs in a Brownian–Poisson framework whose obstacle, as in our case, is merely an optional process. Let us translate their setup into ours to see that we cover their well-posedness result. Let $\mathrm{d}C_s = \mathrm{d}s$, let T be a deterministic and finite time horizon, let X be a Brownian motion and let μ be a Poisson random measure, that is, the predictable compensator ν of μ disintegrates as $\nu(\mathrm{d}s,\mathrm{d}x) = F(\mathrm{d}x)\mathrm{d}s$ for some σ-finite measure $F : (E,\mathcal{E}) \longrightarrow [0,\infty]$, see also [76; 77]. Since C is continuous, the dependence on Y_s or Y_{s-} in the generator does not matter, so we drop one of the arguments in the generator that involves Y. Suppose that f has a universal and deterministic Lipschitz coefficient $\alpha \in (0,\infty)$. The integrability condition on the obstacle ξ and the generator f in [60; 62] that ensures well-posedness is

$$\mathbb{E}\bigg[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}{}^{\mathcal{G}_T}|\xi_\tau|^2\bigg] + \mathbb{E}\bigg[\int_0^T |f_s(0,0,\mathbf{0})|^2\mathrm{d}s\bigg] < \infty.$$

We now show that our integrability condition in Definition 3.1 is satisfied for any $\hat\beta \in (0,\infty)$, although it looks rather different at first sight. Note that we can choose $\Phi = 0$ since the integrator C, and thus A, never jumps. Moreover, $\mathcal{E}(\hat\beta A) = \mathrm{e}^{\hat\beta A}$, and since $1 \leq \mathrm{e}^{\hat\beta A} \leq \mathrm{e}^{\hat\beta\alpha^2 T}$, we immediately find

$$\mathbb{E}\bigg[\int_0^T \mathrm{e}^{\hat\beta A_s}\frac{|f_s(0,0,\mathbf{0})|^2}{\alpha_s^2}\mathrm{d}C_s\bigg] \leq \frac{\mathrm{e}^{\hat\beta\alpha^2 T}}{\alpha^2}\mathbb{E}\bigg[\int_0^T |f_s(0,0,\mathbf{0})|^2\mathrm{d}s\bigg] < \infty,$$

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T \mathrm{e}^{\hat\beta A_s}|{}^*\xi_s|^2\mathrm{d}A_s\bigg] &\leq \alpha^2\mathrm{e}^{\hat\beta\alpha^2 T}\mathbb{E}\bigg[\int_0^T \sup_{u\in[s,T]}|\xi_u|^2\mathrm{d}s\bigg]\\ &\leq \alpha^2 T\mathrm{e}^{\hat\beta\alpha^2 T}\mathbb{E}\bigg[\sup_{u\in[0,T]}|\xi_u|^2\bigg] \leq 4\alpha^2 T\mathrm{e}^{\hat\beta\alpha^2 T}\mathbb{E}\bigg[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}{}^{\mathcal{G}_T}|\xi_\tau|^2\bigg] < \infty,\end{aligned}$$

for any $\hat\beta \in (0,\infty)$. The last inequality follows from an application of Proposition C.7. Furthermore, $\mathbb{H}^2_T = \mathbb{H}^2_{T,\hat\beta}$, $\mathbb{H}^2_T(X) = \mathbb{H}^2_{T,\hat\beta}(X)$, $\mathbb{H}^2_T(\mu) = \mathbb{H}^2_{T,\hat\beta}(\mu)$ and $\mathcal{H}^2_T(X,\mu) = \mathcal{H}^{2,\perp}_{0,T,\hat\beta}(X,\mu)$ since $\mathrm{e}^{\hat\beta A}$ is bounded. For $\hat\beta$ large enough, we have $M^\Phi_i(\hat\beta) < 1$ for $i \in \{1,2,3\}$, and thus our Theorem 3.4 provides well-posedness of the reflected BSDE considered in [60; 62]. However, we want to mention that the class of processes in which uniqueness holds in [62] is not completely clear, at least to us. We will discuss this further in Remark 6.3.

3.3.3 When the random measure is a marked point process

We would like to draw attention to the recent work of Foresta [58] on reflected BSDEs driven by Brownian motion X and a marked point process μ, that is, μ is an integer-valued measure such that

$$\mu(\omega;\mathrm{d}t,\mathrm{d}x) = \sum_{n\in\mathbb{N}}\mathbf{1}_{\{T_n<\infty\}}(\omega)\boldsymbol{\delta}_{(T_n(\omega),\varrho_{T_n(\omega)}(\omega))}(\mathrm{d}t,\mathrm{d}x),$$

for a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ that satisfies $T_n \leq T_{n+1}$, $\mathbb{P}$–a.s., and $T_n < T_{n+1}$, $\mathbb{P}$–a.s. on $\{T_n < \infty\}$. The well-posedness result of the reflected BSDE considered in [58, Theorem 4.1] can be covered by our Theorem 3.4 in case of sufficiently integrable data. We start by translating their setup into our notation. Let $C_t = t + C'_t$, where C' is some continuous process for which we can write the predictable compensator ν of μ as $\nu(\mathrm{d}s,\mathrm{d}x) = K'_s(\omega;\mathrm{d}x)\mathrm{d}C'_s$. Since A is continuous, we can choose $\Phi = 0$, which then implies that $M^\Phi(\beta) \longrightarrow 0$ for $\beta \longrightarrow \infty$. Moreover, the stochastic exponential weight $\mathcal{E}(\hat\beta A)$ reduces to $\mathrm{e}^{\hat\beta A}$. Suppose that the generator is of the form

$$f_s\big(\omega,y,\mathrm{y},z,u_s(\omega;\cdot)\big) = f^1_s(\omega,y,z) + f^2_s\big(\omega,y,u_s(\omega;\cdot)\big)\bigg(1-\frac{\mathrm{d}s}{\mathrm{d}C_s}\bigg),$$

with deterministic and time-independent Lipschitz coefficients, so $\alpha \in (0,\infty)$. The integrability condition in [58] reads

$$\mathbb{E}\Big[\mathrm{e}^{\beta C'_T}|\xi_T|^2\Big] + \mathbb{E}\bigg[\sup_{s\in[0,T)}\mathrm{e}^{(\beta+\delta)C'_s}|\xi_s|^2\bigg]$$

$$+\mathbb{E}\bigg[\int_0^T \mathrm{e}^{\beta C'_s}|f^1_s(0,0)|^2\mathrm{d}s\bigg]+\mathbb{E}\bigg[\int_0^T \mathrm{e}^{\beta C'_s}|f^2_s(0,\mathbf{0})|^2\mathrm{d}C'_s\bigg]<\infty. \quad (3.4)$$

It is then straightforward to check that for $\hat{\beta}=\beta/\alpha^2$, which then also satisfies $\hat{\beta}A=\hat{\beta}\alpha^2C=\beta C$, we have $\|\xi_T\|_{\mathbb{L}^2_{\hat{\beta}}}+\big\|\frac{f(0,0,0,0,\mathbf{0})}{\alpha}\big\|_{\mathbb{H}^2_{T,\hat{\beta}}}<\infty$. That $\|\alpha\,{}^*\xi\|_{\mathbb{H}^2_{T,\hat{\beta}}}$ is finite follows from Lemma 3.3. We thus conclude that if (3.4) holds for β, our Theorem 3.4 provides well-posedness of the reflected BSDE considered in [58].

3.3.4 When the horizon is finite but random

Inspired by applications to random horizon principal–agent problems, Lin, Ren, Touzi, and Yang [106] proved well-posedness of random horizon BSDEs, 2BSDEs and reflected BSDEs. Although we cannot cover their results in general as they work with $\mathbb{L}^p$-data for $p>1$ and the novel norms they use do not fit with our setup, we can compare to some extent their well-posedness result in the reflected BSDE case for $p=2$. In our notation, the setup studied in [106] is the following: there is no integer-valued random measure μ, the process X is a Brownian motion, the obstacle process ξ is optional and càdlàg, the stopping time T is finite, the integrator C satisfies $\mathrm{d}C_s=\mathrm{d}s$, the Lipschitz-continuous generator f has deterministic and time-independent Lipschitz coefficients, thus $\alpha\in(0,\infty)$, and moreover, f is monotone in the y-variable, see [106, Assumption 3.1. (ii)]. As before, we drop one of the arguments in the generator that depends on Y. Note that we can choose $\Phi=0$, so that $M^\Phi_2(\beta)\longrightarrow 0$ as $\beta\longrightarrow\infty$. This ensures that we can provide well-posedness for data that is sufficiently integrable. We now turn to the integrability condition in [106] that provides well-posedness in an $\mathbb{L}^2$-setting. Let $\mathcal{Q}_\alpha(\mathbb{P})$ be the collection of probability measures $\mathbb{Q}^\lambda$ on $(\Omega,\mathcal{G})$ that satisfies

$$\frac{\mathrm{d}\mathbb{Q}^\lambda|_{\mathcal{G}_t}}{\mathrm{d}\mathbb{P}|_{\mathcal{G}_t}}=\mathcal{E}\bigg(\int_0^\cdot \lambda_s\mathrm{d}X_s\bigg)_t,\ t\in[0,\infty),$$

for some predictable process $\lambda=(\lambda_s)_{s\in[0,\infty)}$ with $|\lambda_s|\leq\alpha$. Suppose now that there exists $\varepsilon\in(0,\infty)$ and $\hat{\beta}\in(0,\infty)$ sufficiently large such that

$$\sup_{\mathbb{Q}\in\mathcal{Q}_\alpha(\mathbb{P})}\mathbb{E}^{\mathbb{Q}}\Big[|\mathrm{e}^{\hat{\beta}\alpha^2T/2}\xi_T|^{2+\varepsilon}\Big]+\sup_{\mathbb{Q}\in\mathcal{Q}_\alpha(\mathbb{P})}\mathbb{E}^{\mathbb{Q}}\bigg[\sup_{s\in[0,\infty)}|\mathrm{e}^{\hat{\beta}\alpha^2T/2}\xi^+_{s\wedge T}|^{2+\varepsilon}\bigg]$$
$$+\sup_{\mathbb{Q}\in\mathcal{Q}_\alpha(\mathbb{P})}\mathbb{E}^{\mathbb{Q}}\bigg[\bigg(\int_0^T \mathrm{e}^{\hat{\beta}\alpha^2 s}\frac{|f_s(0,0)|^2}{\alpha^2}\mathrm{d}s\bigg)^{(2+\varepsilon)/2}\bigg]<\infty.$$

An argument similar to the one in the previous paragraph shows that $\|\alpha\,{}^*\xi\|_{\mathbb{H}^2_{T,\hat{\beta}}}<\infty$ as $\mathbb{P}\in\mathcal{Q}_\alpha(\mathbb{P})$. Suppose now that $\hat{\beta}$ is large enough, so that the conditions in [106, Theorem 3.9] and in our Theorem 3.7. (ii) are satisfied. Fix $\beta<\hat{\beta}$ such that the difference $\hat{\beta}-\beta$ is small. By Theorem 3.7. (ii), there exists a unique solution (Y,Z,N,K^r,K^ℓ) to **(R1)–(R7)** such that $(\alpha Y,Z,N)\in\mathbb{H}^2_{T,\beta}\times\mathbb{H}^2_{T,\beta}(X)\times\mathcal{H}^{2,\perp}_{T,\beta}(X)$. Recall here that due to the obstacle ξ having càdlàg paths, the process K^ℓ vanishes, see Remark 2.10. (vii), we thus write $K\coloneqq K^r$. Let us now compare our solution to the one constructed in [106]. The solution (Y',Z',N',K') constructed using Theorem 3.4 in [106] satisfies **(R1)–(R6)**. Moreover, we also have $(Y',Z',N')\in\mathcal{S}^2_T\times\mathbb{H}^2_{T,\beta}(X)\times\mathcal{H}^{2,\perp}_{T,\beta}(X)$, and also $Y'\in\mathcal{S}^2_{T,\beta^\dagger}$ for each $\beta^\dagger\in(\beta,\hat{\beta})$. The latter property implies that $\alpha Y'\in\mathbb{H}^2_{T,\beta}$ since

$$\|\alpha Y'\|^2_{T,\beta}=\mathbb{E}\bigg[\int_0^T \mathrm{e}^{\beta A_s}|Y'_s|^2\mathrm{d}A_s\bigg]=\mathbb{E}\bigg[\sup_{s\in[0,T]}|\mathrm{e}^{\beta^\dagger A_s/2}Y'_s|^2\int_0^T \mathrm{e}^{(\beta-\beta^\dagger)A_s}\mathrm{d}A_s\bigg]$$
$$\leq\mathbb{E}\bigg[\sup_{s\in[0,T]}|\mathrm{e}^{\beta^\dagger A_s/2}Y'_s|^2\int_0^\infty \mathrm{e}^{(\beta-\beta^\dagger)s}\mathrm{d}s\bigg]=\frac{1}{(\beta-\beta^\dagger)}\|Y'\|^2_{\mathcal{S}^2_{T,\beta^\dagger}}.$$

With our uniqueness statement in Theorem 3.4. (ii), we conclude that $(Y,K)=(Y',K')$, up to $\mathbb{P}$–indistinguishability, and that $(Z,N)=(Z',N')$ in $\mathbb{H}^2_{T,\beta}(X)\times\mathcal{H}^{2,\perp}_{T,\beta}(X)$. Since each $\mathbb{Q}\in\mathcal{Q}_\alpha(\mathbb{P})$ is locally absolutely continuous with respect to $\mathbb{P}$, it is straightforward to check that our solution (Y,Z,N,K) also coincides with (Y',Z',N',K') with respect to the norm used in [106]. We thus conclude that our solution is in their solution space.

3.3.5 When the generator has stochastic Lipschitz coefficients

Perninge [122] also studies reflected BSDEs in a Brownian setting on an infinite horizon, and with stochastic Lipschitz coefficients, where the Y-component will converge to zero at infinity. The stochasticity in the Lipschitz coefficient actually only appears in Z-component of the generator. Here the integrability conditions imposed on the data are not comparable to ours. However, the range of applications seem to be more restrictive than in our setup for the following reason: in our notation, the process X is a Brownian motion, and the stochastic Lipschitz coefficient θ^X in [122] is supposed to be an adapted and continuous process and should satisfy $\mathbb{E}[\mathcal{E}(\int_0^\cdot \zeta_s \mathrm{d}X_s)_t]=1$ for each $t\in[0,\infty)$ and optional process ζ satisfying $|\zeta|^2\leq\theta^X$. In particular, the process $\sqrt{\theta^X}$ itself should satisfy this condition *a fortiori*. Thus the well-posedness result in [122] is not for arbitrary Lipschitz generators with stochastic Lipschitz coefficients.

3.3.6 When the non-reflected BSDE is driven by arbitrary martingales

Let us close this section by a comparison of our BSDE results to the works of Bandini [10], Cohen and Elliott [31] and Papapantoleon, Possamaï, and Saplaouras [113], as we feel that these works are closest to the BSDE formulation we chose here. In [10], the integrability condition imposed on the data is not comparable to ours. We can thus not cover the well-posedness result in [10] in general. What is surprising nonetheless is that by translating [10] into our notation, we see that the condition $\sqrt{r_s}\Delta C_s\leq\Phi\in[0,1/\sqrt{2})$, which only involves the Lipschitz coefficient of f with respect to the Y-component, is sufficient for the contraction argument to go through in case of sufficiently integrable data, see [10, Theorem 4.1]. However, the setup in [10] is simpler than the one we study here, as the only driving force in the BSDE is a random measure μ with finite activity, that is, $\{t\in[0,t']:\mu(\omega;\{t\}\times E)=1\}$ is finite for each $(\omega,t')\in\Omega\times[0,\infty)$.

The BSDE considered in [31] is rather different from ours. We initially fix a driving martingale X and an integrator C so that $\mathrm{d}\langle X\rangle_s$ is absolutely continuous with respect to $\mathrm{d}C_s$. Contrary to our case, in [31] the integrator C is fixed in the beginning, and the driving martingales of the BSDE are a sequence of orthogonal martingales that are constructed from a general martingale representation theorem relying on the assumption that the underlying probability space is separable. In [31], the integrator C may have no relation at all to the the predictable quadratic variations of the driving martingales. We are thus not able to link our well-posedness result to theirs. However, they suppose that C is deterministic and strictly increasing, which immediately excludes piecewise constant integrators. The condition ensuring well-posedness in case of sufficiently integrable data is similar to [10], namely, in our notation, $\sqrt{r_s}\Delta C_s\leq\Phi\in[0,1)$ is enough for the arguments to go through, see [31, Lemma 5.5 and Theorem 6.1]. This again only involves the Lipschitz coefficient with respect to the Y-component of the generator.

Lastly, the formulation of BSDEs we chose in this work already appeared in [113], with the slight difference that we allow our driving martingale X to have components in $\mathcal{H}^2_{\mathrm{loc}}$ and not in $\mathcal{H}^2$, thus enabling us to apply our results in case X is a Brownian motion, and the random measure μ in this work is a general integer-valued random measure in the sense of [76; 77], which is not necessarily induced by the jump measure of a process in $\mathcal{H}^2$ as in [113]. Moreover, we allow the generator to depend on both Y_s and its left-limit Y_{s-}. Our integrability condition on the data is also weaker, as we use stochastic exponential weights $\mathcal{E}(\beta A)$, while the well-posedness result in [113] relies on exponential weights $\mathrm{e}^{\beta A}$. Interestingly, this change of the weights allows us to build a contraction map under the condition $\alpha_s^2 \Delta C_s \leq \Phi \in [0,1)$, while in [113] the more restrictive condition $\alpha_s^2 \Delta C_s \leq \Phi < 1/(18\mathrm{e})$ is needed. We thus cover [113, Theorem 3.5]. As already mentioned in Remark 3.10. (i), there is a small issue in the proof of the *a priori* estimates in [113] to which we will come back in Remark 5.6.

4 Optimal stopping and Mertens' decomposition

In this section, we solve the reflected BSDE in case the generator does not depend on (y, y, z, u), which we assume throughout. We imposed the following integrability condition on f and ξ[12]

$$\mathbb{E}[|\xi_T|^2] + \mathbb{E}\bigg[\sup_{u\in[0,T)} |\xi_u^+|^2\bigg] + \mathbb{E}\bigg[\bigg(\int_0^T |f_u| \mathrm{d}C_u\bigg)^2\bigg] < \infty. \tag{4.1}$$

From **(R7)**, it is clear that the first component of the solution is related to the optimal stopping problem

$$V(S) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S} \mathbb{E}\bigg[\xi_{\tau\wedge T} + \int_0^{\tau\wedge T} f_u \mathrm{d}C_u \bigg| \mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}. \tag{4.2}$$

Note that the conditional expectations are well-defined in $[-\infty, \infty)$. The fact that we can actually find a solution to **(R1)–(R7)** is *a priori* not clear, since we cannot directly employ the classical results of optimal stopping and the Snell envelope theory, as the gains process is not necessarily non-negative (see [47; 108]) or in $\mathcal{S}_T^2$ (see [60; 62]). We thus need to go through a series of technical lemmata to modify our optimal stopping problem first. Since we do not need their proofs for the analysis that follows, we defer them to Appendix B.

For the following lemma, we fix a martingale $M = (M_t)_{t\in[0,\infty]}$ satisfying

$$M_S = \mathbb{E}\bigg[\xi_T + \int_0^T f_s \mathrm{d}C_s \bigg| \mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}.$$

Note that $V(S) \geq M_S$, $\mathbb{P}$–a.s., $S \in \mathcal{T}_{0,\infty}$. Let $J = (J_t)_{t\in[0,\infty]}$ and $L = (L_t)_{t\in[0,\infty]}$ be the optional processes defined by

$$J_t := \xi_{t\wedge T} + \int_0^{t\wedge T} f_s \mathrm{d}C_s,\ \text{and}\ L_t = J_t \vee \big(M_t - \mathbf{1}_{\{t<T\}}\big),\ t \in [0, \infty].$$

We now rewrite the optimal stopping problem (4.2) using the auxiliary process L. This idea stems from the proof of [142, Proposition 6.3.2].

Lemma 4.1. *The process L is in $\mathcal{S}_T^2$, satisfies $L_\cdot = L_{\cdot\wedge T}$, up to $\mathbb{P}$–indistinguishability, and*

$$V(S) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S} \mathbb{E}[L_\tau|\mathcal{G}_S],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}.$$

The previous lemma now allows us to deduce the following.

Lemma 4.2. *The family $(V(S))_{S\in\mathcal{T}_{0,\infty}}$ satisfies the following properties*

(i) $V(S) \in \mathbb{L}^2(\mathcal{G}_S)$ *and* $V(S) \geq \mathbb{E}[V(U)|\mathcal{G}_S]$, $\mathbb{P}$ –a.s., *for all* $(S, U) \in \mathcal{T}_{0,\infty} \times \mathcal{T}_{0,\infty}$ *with* $\mathbb{P}[S \leq U] = 1$.

[12] Recall that $\xi^+ = \max\{\xi, 0\}$.

(ii) $V(S) = V(U)$, $\mathbb{P}$ –a.s. *on* $\{U = S\}$, *for all* $(S, U) \in \mathcal{T}_{0,\infty} \times \mathcal{T}_{0,\infty}$.

The next result shows that we can aggregate the family $(V(S))_{S\in\mathcal{T}_{..}}$ into a process.

Lemma 4.3. *There exists a, up to* $\mathbb{P}$*–indistinguishability, unique optional process* $V = (V_t)_{t\in[0,\infty]} \in \mathcal{S}^2_T$ *satisfying* $V_S = V(S)$, $\mathbb{P}$–a.s., *for each* $S \in \mathcal{T}_{0,\infty}$. *Moreover,* $V_\cdot = V_{\cdot\wedge T}$, *up to* $\mathbb{P}$*–indistinguishability.*

By Lemma 4.2, the process $V = (V_t)_{t\in[0,\infty]}$ constructed in Lemma 4.3 is a strong optional supermartingale in the sense of [41, Appendix I], and thus $\mathbb{P}$–almost all its paths are làdlàg (see [41, Appendix I, Theorem 4]). The next lemma allows us to deduce the Skorokhod condition using our modified optimal stopping problem.

Lemma 4.4. (i) *For each* $S \in \mathcal{T}_{0,\infty}$, *we have* $\mathbf{1}_{\{V_S = L_S\}} = \mathbf{1}_{\{V_S = J_S\}}$, $\mathbb{P}$ –a.s.

(ii) *For each* $S \in \mathcal{T}^p_{0,\infty}$, *we have* $\mathbf{1}_{\{V_{S-} = \bar{L}_S\}} = \mathbf{1}_{\{V_{S-} = \bar{J}_S\}}$, $\mathbb{P}$ –a.s.[13]

We have now established the necessary technical results that allow us to apply the arguments laid out in [62] to construct the solution to the reflected BSDE for the generator f that does not depend on (y, z, u). First, define $Y = (Y_t)_{t\in[0,\infty]}$ by $Y_t \coloneqq V_t - \int_0^{t\wedge T} f_s \mathrm{d}C_s$ $t \geq 0$. Then $Y_\cdot = Y_{\cdot\wedge T}$, up to $\mathbb{P}$-indistinguishability, and

$$Y_S = V_S - \int_0^{S\wedge T} f_s \mathrm{d}C_s = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S} \mathbb{E}\bigg[\xi_{\tau\wedge T} + \int_S^{\tau\wedge T} f_s \mathrm{d}C_s \bigg| \mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}.$$

We know by now that V is a strong optional supermartingale in the sense of [41, Appendix I]. We can therefore apply Mertens' decomposition to construct the solution to the reflected BSDE.

Proposition 4.5. *There exists a unique triple* $(Z, U, N) \in \mathbb{H}^2_T(X) \times \mathbb{H}^2_T(\mu) \times \mathcal{H}^{2,\perp}_{0,T}(X, \mu)$ *and a, up to* $\mathbb{P}$*–indistinguishability, unique pair* $(K^r, K^\ell) \in \mathcal{I}^2_T \times \mathcal{I}^2_T$ *such that* K^r *is predictable and starts* $\mathbb{P}$–a.s. *from zero,* K^ℓ *has* $\mathbb{P}$–a.s. *purely discontinuous paths and satisfies* $K^\ell_T = K^\ell_{T-}$, $\mathbb{P}$–a.s., *and*

$$\begin{aligned} Y_t = \xi_T + \int_t^T f_s \mathrm{d}C_s - \int_t^T Z_s \mathrm{d}X_s - \int_t^T \int_E U_s(x) \tilde{\mu}(\mathrm{d}s, \mathrm{d}x) \\ - \int_t^T \mathrm{d}N_s + K^r_T - K^r_t + K^\ell_{T-} - K^\ell_{t-},\ t \in [0, \infty],\ \mathbb{P}\text{ –a.s.}, \end{aligned}$$

holds with convention $K^\ell_{0-} \coloneqq 0$. *Moreover,*

$$\big(Y_{T-} - \overline{\xi}_T\big)\Delta K^r_T + \int_{(0,T)} \big(Y_{s-} - \overline{\xi}_s\big)\mathrm{d}K^r_s + \int_{[0,T)} (Y_s - \xi_s)\mathrm{d}K^\ell_s = 0,\ \mathbb{P}\text{–a.s.}$$

Proof. We use $V_\cdot = V_{\cdot\wedge T}$, up to $\mathbb{P}$–indistinguishability, and [62, Lemma 3.2][14] (which is based on Mertens' unique decomposition of a strong supermartingale) to find a martingale $M = (M_t)_{t\in[0,\infty]}$ with $M = M_{\cdot\wedge T}$, $M_T \in \mathbb{L}^2(\mathcal{G}_T)$ and $M_0 = 0$, $\mathbb{P}$–a.s., and two processes $(K^r, K^\ell) \in \mathcal{I}^2_T \times \mathcal{I}^2_T$ such that K^r is predictable and satisfies $K^r_0 = 0$, $\mathbb{P}$–a.s., K^ℓ has $\mathbb{P}$–a.s. purely discontinuous paths and satisfies $K^\ell_T = K^\ell_{T-}$, $\mathbb{P}$–a.s., with convention $K^\ell_{0-} \coloneqq 0$,

$$Y_t + \int_0^{t\wedge T} f_s \mathrm{d}C_s = V_t = V_0 + M_t - K^r_t - K^\ell_{t-},\ t \in [0, \infty],\ \mathbb{P}\text{–a.s.},$$

[13] Recall that $\mathcal{T}^p_{0,\infty}$ is the collection of predictable stopping times and $\overline{L}$ is defined by $\overline{L}_0 \coloneqq L_0$ and $\overline{L}_t \coloneqq \limsup_{s\uparrow\uparrow\infty} L_s$ for $t \in (0, \infty]$. The process $\overline{J}$ is defined analogously.

[14] Their results still applies to our infinite, but right-closed, horizon $[0, \infty]$.

and $\Delta K^\ell_\tau = \mathbf{1}_{\{V_\tau = L_\tau\}}\Delta K^\ell_\tau$, $\mathbb{P}$–a.s., for each $\tau \in \mathcal{T}_{0,\infty}$, and $\Delta K^r_\tau = \mathbf{1}_{\{V_{\tau-} = \overline{L}_\tau\}}\Delta K^r_\tau$, $\mathbb{P}$–a.s., for each $\tau \in \mathcal{T}^p_{0,\infty}$. Moreover, [62, Lemma 3.3] implies

$$\int_{(0,T)} \mathbf{1}_{\{V_{s-} > \overline{L}_s\}}\mathrm{d}K^{r,c}_s = 0,\ \mathbb{P}\text{–a.s., where } K^{r,c} = K^r - \sum_{s\in(0,\cdot]}\Delta K^r_s.$$

Therefore, by Lemma 4.4,

$$(Y_{T-} - \overline{\xi}_T)\Delta K^r_T + \int_{(0,T)}(Y_{s-} - \overline{\xi}_s)\mathrm{d}K^r_s + \int_{[0,T)}(Y_s - \xi_s)\mathrm{d}K^\ell_s = 0,\ \mathbb{P}\text{–a.s.}$$

Since M_∞ is $\mathcal{G}_\infty$-measurable, and because $\mathcal{G}_\infty = \mathcal{G}_{\infty-} = \sigma(\cup_{t\in[0,\infty)}\mathcal{G}_t)$, the martingale M is $\mathbb{P}$–a.s. left-continuous at infinity. We can therefore decompose $(M_t)_{t\in[0,\infty)}$ as

$$M_t = \int_0^t Z_s\mathrm{d}X_s + \int_0^t\int_E U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) + N_t,\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.,} \tag{4.3}$$

for unique $(Z,U,N) \in \mathbb{H}^2_T(X)\times\mathbb{H}^2_T(\mu)\times\mathcal{H}^{2,\perp}_{0,T}(X,\mu)$ with $N_0 = 0$, $\mathbb{P}$–a.s., such that the value M_∞ is the $\mathbb{P}$–a.s. limit at infinity of the processes on the right-hand-side of (4.3), that is,

$$M_\infty = \int_0^\infty Z_s\mathrm{d}X_s + \int_0^\infty\int_E U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) + N_\infty,\ \mathbb{P}\text{–a.s.}$$

It remains to write the dynamics of Y in backward form.

We now present the argument that implies uniqueness separately as it might seem odd that we do not impose $K^\ell_0 = 0$, $\mathbb{P}$–almost surely. Suppose that $(M', K^{r,\prime}, K^{\ell,\prime})$ is a triplet satisfying the same properties as (M, K^r, K^ℓ), and such that

$$Y_t + \int_0^{t\wedge T} f_s\mathrm{d}C_s = V_t = V_0 + M'_t - K^{r,\prime}_t - K^{\ell,\prime}_{t-},\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.}$$

It follows that $M_t - K^r_t - K^\ell_{t-} = M'_t - K^{r,\prime}_t - K^{\ell,\prime}_{t-},\ t\in[0,\infty],\ \mathbb{P}$–a.s., hence $M - M'$ is a martingale that is $\mathbb{P}$–indistinguishable from a predictable process with $\mathbb{P}$–a.s. locally finite variation paths. Thus $M_t = M'_t$, $t\in[0,\infty]$, $\mathbb{P}$–a.s., by [77, Corollary I.3.16], and therefore

$$K^r_t + K^\ell_{t-} = K^{r,\prime}_t + K^{\ell,\prime}_{t-},\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.}$$

Since K^ℓ and $K^{\ell,\prime}$ are $\mathbb{P}$–a.s. purely discontinuous, we can write

$$K^r_t - K^{r,\prime}_t = K^\ell_{t-} - K^{\ell,\prime}_{t-} = K^\ell_0 - K^{\ell,\prime}_0 + \sum_{s\in(0,t)}(\Delta K^\ell_s - \Delta K^{\ell,\prime}_s),\ t\in(0,\infty],\ \mathbb{P}\text{–a.s.}$$

As $K^r - K^{r,\prime}$ is $\mathbb{P}$–a.s. right-continuous, it follows by taking limits from the right that

$$K^r_t - K^{r,\prime}_t = K^\ell_0 - K^{\ell,\prime}_0 + \sum_{s\in(0,t]}(\Delta K^\ell_s - \Delta K^{\ell,\prime}_s),\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.}$$

Hence, $0 = K^r_0 - K^{r,\prime}_0 = K^\ell_0 - K^{\ell,\prime}_0$ and $\Delta K^\ell_s = \Delta K^{\ell,\prime}_s,\ s\in(0,\infty],\ \mathbb{P}$–a.s., which implies $K^\ell = K^{\ell,\prime}$, up to $\mathbb{P}$–indistinguishability, and thus also $K^r = K^{r,\prime}$, up to $\mathbb{P}$–indistinguishability. This completes the proof. □

5 *A priori* estimates

We now devote ourselves to *a priori* estimates, which will allow us to construct a contraction mapping on the weighted normed spaces introduced in Section 2.4. Classically, these sort of estimates are derived by an application of Itô's formula to the square of the (weighted) difference of two solutions to the (reflected) BSDE (see [16; 60; 62]). In our generality, we were not able to deduce appropriate estimates solely using this tool, and thus we will approach the estimates differently. We first derive bounds on conditional expectations of the martingale processes using both Itô's formula and a pathwise version of Doob's martingale inequality (Lemma C.4). Then we employ the ideas of [50, page 27] to get the desired weighted norm estimates. For $i \in \{1,2\}$, let $f^i = (f^i_u)_{u\in[0,\infty)}$ be an optional processes satisfying

$$\mathbb{E}\bigg[\bigg(\int_0^T |f^i_u|\mathrm{d}C_u\bigg)^2\bigg] < \infty,$$

and suppose that we are given an optional process $y^i = (y^i_t)_{t\in[0,\infty]}$ satisfying

$$\begin{cases} y^i_t = \xi_T + \displaystyle\int_t^T f^i_s \mathrm{d}C_s - \int_t^T \mathrm{d}\eta^i_s + k^{r,i}_T - k^{r,i}_t + k^{\ell,i}_{T-} - k^{\ell,i}_{t-},\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.}, & (5.1a)\\ y^i = y^i_{\cdot\wedge T} \geq \xi_{\cdot\wedge T},\ \mathbb{P}\text{–a.s.}, & (5.1b)\\ y^i_S = \underset{\tau\in\mathcal{T}_{S,\infty}}{\operatorname{ess\,sup}}^{\mathcal{G}_S}\mathbb{E}\bigg[\xi_{\tau\wedge T} + \displaystyle\int_S^{\tau\wedge T} f^i_u \mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}_{0,\infty}, & (5.1c)\\ \big(y^i_{T-} - \overline{\xi}_T\big)\Delta k^{r,i}_T + \displaystyle\int_{(0,T)}\big(y^i_{s-} - \overline{\xi}_s\big)\mathrm{d}k^{r,i}_s + \int_{[0,T)}(y^i_s - \xi_s)\mathrm{d}k^{\ell,i}_s = 0,\ \mathbb{P}\text{–a.s.}, & (5.1d)\end{cases}$$

for some $\eta^i \in \mathcal{H}^2_T$ and $(k^{r,i}, k^{\ell,i}) \in \mathcal{I}^2_T \times \mathcal{I}^2_T$ with $k^{r,i}$ predictable and starting $\mathbb{P}$–a.s. from zero, and $k^{\ell,i}_T = k^{\ell,i}_{T-}$ up to a $\mathbb{P}$–null set. Here we use the convention $k^{\ell,i}_{0-} \coloneqq 0$. Let

$$\delta y \coloneqq y^1 - y^2,\ \delta\eta \coloneqq \eta^1 - \eta^2,\ \delta k^r \coloneqq k^{r,1} - k^{r,2},\ \delta k^\ell \coloneqq k^{\ell,1} - k^{\ell,2},\ \text{and}\ \delta f \coloneqq f^1 - f^2.$$

To ease the notation, we denote by L the infimum of the function ℓ defined on $\{(\varepsilon,\kappa)\in(0,\infty)^2 : 0 < 1-4\kappa \leq 1\}$ by

$$\ell(\varepsilon,\kappa) \coloneqq \frac{\max\{1+\varepsilon+4\kappa+12/\varepsilon,\,12/\varepsilon+2/\kappa\}}{1-4\kappa}.$$

Recall from **(D5)** that the obstacle ξ satisfies $\mathbb{E}[|\xi_T|^2] + \mathbb{E}\big[\sup_{s\in[0,T)}|\xi^+_s|^2\big] < \infty$. The following is the main result of this section.

Proposition 5.1. *Let $\beta \in (0,\infty)$. Then*[15]

$$\|\delta y\|^2_{\mathcal{S}^2_T} \leq \frac{4}{\beta}\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},\ \|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} \leq \mathfrak{f}^\Phi(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},\ \|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} \leq (1+\beta\Phi)\mathfrak{g}^\Phi(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},$$

$$\begin{aligned}\mathbb{E}\big[(\delta y_0)^2\big] + \frac{\beta}{(1+\beta\Phi)}\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}[\delta k^r]_s\bigg]&\\ + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}[\delta k^\ell]_s\bigg] \leq \bigg(\frac{5}{\beta} + \frac{4}{\beta}(1+\beta\Phi)^{1/2} + \beta\mathfrak{g}^\Phi(\beta)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}&,\end{aligned}$$

and

$$\|\delta y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} + \|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq M^\Phi_1(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},$$

$$\|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq M^\Phi_2(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},$$

[15] Recall the definition of $\mathfrak{f}^\Phi$, $\mathfrak{g}^\Phi$, M^Φ_1, M^Φ_2 and M^Φ_3 at the beginning of Section 3.1.

$$\|\delta y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq M_3^\Phi(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.$$

Moreover, for each $i \in \{1,2\}$,

$$\|y^i\|^2_{\mathcal{S}^2_T} \leq 12\left(\|\xi_T\|^2_{\mathbb{L}^2} + \Big\|\sup_{u\in[0,T)}\xi^+_u\Big\|^2_{\mathbb{L}^2} + \frac{1}{\beta}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\right),$$

$$\|\alpha y^i\|^2_{\mathbb{H}^2_{T,\beta}} \leq 3\left(\frac{(1+\beta\Phi)}{\beta}\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + \mathfrak{f}^\Phi(\beta)\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\right),$$

$$\|\alpha y^i_-\|^2_{\mathbb{H}^2_{T,\beta}} \leq 3\left(\frac{(1+\beta\Phi)}{\beta}\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + (1+\beta\Phi)\mathfrak{g}^\Phi(\beta)\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\right),$$

and

$$\begin{aligned}\mathbb{E}\big[|y^i_0|^2\big] + \frac{\beta}{(1+\beta\Phi)}\|\alpha y^i_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\eta^i\|^2_{\mathcal{H}^2_{T,\beta}}&\\ + \mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_s\mathrm{d}[k^{r,i}]_s\bigg] + \mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_s\mathrm{d}[k^{\ell,i}]_s\bigg]&\\ \leq L\Bigg(\|\xi_T\|^2_{\mathbb{L}^2} + \|\xi^+\mathbf{1}_{[\![0,T)\!)}\|^2_{\mathcal{S}^2_T} + \frac{1}{\beta}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}&\\ + \beta\bigg(\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + \mathfrak{g}^\Phi(\beta)\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg)\Bigg)&.\end{aligned}$$

The proof of the preceding proposition will be based on the two following lemmata whose proofs we defer to Section 5.1. Here we use the convention $\zeta_{0-} := 0$ for a process $\zeta = (\zeta_t)_{t\in[0,\infty]}$, and we recall from Section 2.2.1 that we never include the point ∞ in the domain of integration.

Lemma 5.2. *The following inequalities hold:*

$$|\delta y_S| \leq \mathbb{E}\bigg[\int_S^T|\delta f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ |y^i_S| \leq \mathbb{E}\bigg[|\xi_T| + \sup_{u\in[S,\infty]}|\xi^+_u\mathbf{1}_{\{u<T\}}| + \int_S^T|f^i_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg], \tag{5.2}$$

$\mathbb{P}$–a.s., for $S \in \mathcal{T}_{0,\infty}$ and $i \in \{1,2\}$, and

$$|\delta y_{S-}| \leq \mathbb{E}\bigg[\int_{S-}^T|\delta f_u|\mathrm{d}C_u\bigg|\mathcal{G}_{S-}\bigg],\ |y^i_{S-}| \leq \mathbb{E}\bigg[|\xi_T| + {}^*\xi_S + \int_{S-}^T|f^i_u|\mathrm{d}C_u\bigg|\mathcal{G}_{S-}\bigg], \tag{5.3}$$

$\mathbb{P}$–a.s., for $S \in \mathcal{T}^p_{0,\infty}$ and $i \in \{1,2\}$.[16] Moreover,

$$\|\delta y\|^2_{\mathcal{S}^2_T} \leq 4\mathbb{E}\bigg[\bigg(\int_0^T|\delta f_u|\mathrm{d}C_u\bigg)^2\bigg] < \infty,$$

$$\|y^i\|^2_{\mathcal{S}^2_T} \leq 12\mathbb{E}\bigg[|\xi_T|^2 + \sup_{u\in[0,T)}|\xi^+_u|^2 + \bigg(\int_0^T|f^i_u|\mathrm{d}C_u\bigg)^2\bigg] < \infty,\ i\in\{1,2\}. \tag{5.4}$$

Lemma 5.3. *The following inequalities hold:*

$$|\delta y_S|^2 + \mathbb{E}\bigg[\int_S^T\mathrm{d}\langle\delta\eta\rangle_u\bigg|\mathcal{G}_S\bigg] + \mathbb{E}\bigg[\int_S^T\mathrm{d}[\delta k^r]_u\bigg|\mathcal{G}_S\bigg] + \mathbb{E}\bigg[\int_{S-}^T\mathrm{d}[\delta k^\ell]_u\bigg|\mathcal{G}_S\bigg]$$

[16] Recall from Section 2.1 that $\mathcal{T}^p_{0,\infty}$ denotes the collection of predictable stopping times.

$$\leq 2\mathbb{E}\bigg[\int_S^T \delta y_u \delta f_u \mathrm{d}C_u\bigg|\mathcal{G}_S\bigg] + \mathbb{E}\bigg[\int_S^T (\delta f_u)^2 \mathrm{d}[C]_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}_{0,\infty}, \quad (5.5)$$

$$|\delta y_{S-}|^2 + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}\langle\delta\eta\rangle_u\bigg|\mathcal{G}_{S-}\bigg] + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[\delta k^r]_u\bigg|\mathcal{G}_{S-}\bigg] + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[\delta k^\ell]_u\bigg|\mathcal{G}_{S-}\bigg]$$
$$\leq 2\mathbb{E}\bigg[\int_{S-}^T \delta y_u \delta f_u \mathrm{d}C_u\bigg|\mathcal{G}_{S-}\bigg] + \mathbb{E}\bigg[\int_{S-}^T (\delta f_u)^2 \mathrm{d}[C]_u\bigg|\mathcal{G}_{S-}\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}^p_{0,\infty}. \quad (5.6)$$

Moreover, for each $i\in\{1,2\}$,

$$|y^i_S|^2 + \mathbb{E}\bigg[\int_S^T \mathrm{d}\langle\eta^i\rangle_u\bigg|\mathcal{G}_S\bigg] + \mathbb{E}\bigg[\int_S^T \mathrm{d}[k^{r,i}]_u\bigg|\mathcal{G}_S\bigg] + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[k^{\ell,i}]_u\bigg|\mathcal{G}_S\bigg]$$
$$\leq L\mathbb{E}\bigg[|\xi_T|^2 + \sup_{u\in[S,\infty]}|\xi^+_u \mathbf{1}_{\{u<T\}}|^2 + \bigg(\int_S^T |f^i_u|\mathrm{d}C_u\bigg)^2\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}_{0,\infty}, \quad (5.7)$$

$$|y^i_{S-}|^2 + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}\langle\eta^i\rangle_u\bigg|\mathcal{G}_{S-}\bigg] + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[k^{r,i}]_u\bigg|\mathcal{G}_{S-}\bigg] + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[k^{\ell,i}]_u\bigg|\mathcal{G}_{S-}\bigg]$$
$$\leq L\mathbb{E}\bigg[|\xi_T|^2 + |{}^*\xi_S|^2 + \bigg(\int_{S-}^T |f^i_u|\mathrm{d}C_u\bigg)^2\bigg|\mathcal{G}_{S-}\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}^p_{0,\infty}. \quad (5.8)$$

We turn to the proof of the *a priori* estimates.

Proof of Proposition 5.1. To ease the presentation, let us abuse notation sligthly in this proof and denote by $\mathbb{E}[W_\cdot|\mathcal{G}_\cdot]$ and $\mathbb{E}[W_\cdot|\mathcal{G}_{\cdot-}]$ the optional and predictable projections, respectively, of a non-negative, product-measurable process $W=(W_t)_{t\in[0,\infty]}$. We refer to [41, Section VI.2 and Appendix I] for their existence and properties.

We suppose, without loss of generality, that $\|\frac{f^1}{\alpha}\|_{\mathbb{H}^2_{T,\beta}}$, $\|\frac{f^2}{\alpha}\|_{\mathbb{H}^2_{T,\beta}}$ and $\|\frac{\delta f}{\alpha}\|_{\mathbb{H}^2_{T,\beta}}$ are all finite; otherwise the stated inequalities trivially hold. We start with some introductory calculations. First, let $(\gamma,\beta)\in(0,\infty)^2$ with $\gamma<\beta$, and recall that the stochastic exponential $\mathcal{E}(\gamma A)$ of the non-decreasing, predictable process γA satisfies

$$\mathcal{E}(\gamma A)_t = 1 + \int_0^t \mathcal{E}(\gamma A)_{s-}\mathrm{d}(\gamma A)_s = 1 + \int_0^t \mathcal{E}(\gamma A)_{s-}\gamma \mathrm{d}A_s,\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.} \quad (5.9)$$

In particular, $\mathcal{E}(\gamma A)$ is predictable and (see [77, page 134])

$$\mathcal{E}(\gamma A)_t = \mathrm{e}^{\gamma A_t}\prod_{s\in(0,t]}(1+\gamma\Delta A_s)\mathrm{e}^{-\gamma\Delta A_s},\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.} \quad (5.10)$$

Since A is $\mathbb{P}$–a.s. non-decreasing, so is $\mathcal{E}(\gamma A)$ by (5.9) and (5.10). Note also that

$$\begin{aligned}\mathcal{E}(\gamma A)_t &= \mathcal{E}(\gamma A)_{t-} + \Delta\mathcal{E}(\gamma A)_t\\ &= \mathcal{E}(\gamma A)_{t-} + \mathcal{E}(\gamma A)_{t-}\gamma\Delta A_t = \mathcal{E}(\gamma A)_{t-}(1+\gamma\Delta A_t),\ t\in(0,\infty),\ \mathbb{P}\text{–a.s.}\end{aligned} \quad (5.11)$$

By [31, Lemma 4.4] or Lemma C.1. (i), we have $\mathcal{E}(\gamma A)^{-1}=\mathcal{E}(-\overline{\gamma A})$, $\mathbb{P}$–a.s., where $\overline{\gamma A}$ is the predictable process satisfying

$$\overline{\gamma A} = \gamma A - \sum_{s\in(0,\cdot]}\frac{(\gamma\Delta A_s)^2}{1+\gamma\Delta A_s},\ \mathbb{P}\text{–a.s.},$$
$$(\overline{\gamma A})^c = (\gamma A)^c = \gamma A^c,\ \Delta\overline{\gamma A} = \gamma\Delta A - \frac{(\gamma\Delta A)^2}{1+\gamma\Delta A} = \frac{\gamma\Delta A}{1+\gamma\Delta A},\ \mathbb{P}\text{–a.s.} \quad (5.12)$$

In particular, $\overline{\gamma A}$ is $\mathbb{P}$–a.s. non-decreasing. Let $F = \big(F(t)\big)_{t\in[0,\infty]}$ be the process defined by

$$\begin{aligned} F(t) &:= \int_t^T |\delta f_s| \mathrm{d}C_s \\ &:= \int_0^T |\delta f_s|\mathrm{d}C_s - \int_0^{t\wedge T} |\delta f_s|\mathrm{d}C_s = \int_{(0,\infty)} |\delta f_s|\mathbf{1}_{(t,T]}(s)\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.} \end{aligned} \tag{5.13}$$

For $t \in (0,\infty]$, we have $F(t-) = \int_{(0,\infty)} |\delta f_s|\mathbf{1}_{[t,T]}(s)\mathrm{d}C_s$, $\mathbb{P}$–a.s., and

$$|F(t-)|^2 \le \int_{(0,\infty)} \frac{1}{\mathcal{E}(\gamma A)_s}\mathbf{1}_{[t,T]}(s)\mathrm{d}A_s \int_{(0,\infty)} \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathbf{1}_{[t,T]}(s)\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.}, \tag{5.14}$$

by the Cauchy–Schwarz inequality. The first integral on the right-hand side above can be bounded as follows

$$\begin{aligned} \int_{(0,\infty)} \frac{1}{\mathcal{E}(\gamma A)_s}\mathbf{1}_{[t,T]}(s)\mathrm{d}A_s &= \lim_{t'\uparrow\uparrow\infty} \int_{(0,\infty)} \frac{1}{\mathcal{E}(\gamma A)_s}\mathbf{1}_{[t,t'\wedge T]}(s)\mathrm{d}A_s \\ &= \lim_{t'\uparrow\uparrow\infty} \int_{(0,\infty)} \frac{1}{\mathcal{E}(\gamma A)_{s-}}\frac{1}{(1+\gamma\Delta A_s)}\mathbf{1}_{[t,t'\wedge T]}(s)\mathrm{d}A_s \\ &= \lim_{t'\uparrow\uparrow\infty} \int_{(0,\infty)} \mathcal{E}(-\overline{\gamma A})_{s-}\frac{1}{(1+\gamma\Delta A_s)}\mathbf{1}_{[t,t'\wedge T]}(s)\mathrm{d}A_s \\ &= \lim_{t'\uparrow\uparrow\infty} \int_{(0,\infty)} \mathcal{E}(-\overline{\gamma A})_{s-}\frac{1}{(1+\gamma\Delta A_s)}\mathbf{1}_{[t,t'\wedge T]}(s)\mathrm{d}A^c_s \\ &\quad + \sum_{s\in[t,\infty)} \mathcal{E}(-\overline{\gamma A})_{s-}\mathbf{1}_{\{s\le t'\wedge T\}}\frac{\Delta A_s}{(1+\gamma\Delta A_s)} \\ &= \lim_{t'\uparrow\uparrow\infty} \frac{1}{\gamma}\int_{(0,\infty)} \mathcal{E}(-\overline{\gamma A})_{s-}\mathbf{1}_{[t,t'\wedge T]}(s)\mathrm{d}(\overline{\gamma A})^c_s \\ &\quad + \frac{1}{\gamma}\sum_{s\in[t,\infty)} \mathcal{E}(-\overline{\gamma A})_{s-}\mathbf{1}_{\{s\le t'\wedge T\}}\Delta\overline{\gamma A}_s \\ &= \lim_{t'\uparrow\uparrow\infty} \frac{1}{\gamma}\int_{(0,\infty)} \mathcal{E}(-\overline{\gamma A})_{s-}\mathbf{1}_{[t,t'\wedge T]}(s)\mathrm{d}\overline{\gamma A}_s \\ &= -\frac{1}{\gamma}\lim_{t'\uparrow\uparrow\infty}\Big(\mathcal{E}(-\overline{\gamma A})_{t'\wedge T} - \mathcal{E}(-\overline{\gamma A})_{t\wedge T-}\Big) \\ &\le \frac{1}{\gamma}\frac{1}{\mathcal{E}(\gamma A)_{t\wedge T-}}. \end{aligned} \tag{5.15}$$

Here, the fourth line follows from (5.12) and

$$\frac{1}{(1+\gamma\Delta A)}\mathrm{d}A^c = \mathrm{d}A^c \tag{5.16}$$

since A^c is continuous, and thus $\mathrm{d}A^c$ does not charge any points on $(0,\infty)$. Thus

$$|F(t-)|^2 \le \frac{1}{\gamma}\frac{1}{\mathcal{E}(\gamma A)_{t\wedge T-}}\int_{(0,\infty)} \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathbf{1}_{[t,T]}(s)\mathrm{d}C_s,\ t\in(0,\infty),\ \mathbb{P}\text{–a.s.},$$

and therefore

$$\begin{aligned} &\int_0^T \mathcal{E}(\beta A)_{t-}|F(t-)|^2\mathrm{d}A_t \\ &\quad = \int_{(0,\infty)} \mathbf{1}_{(0,T]}(t)\mathcal{E}(\beta A)_{t-}|F(t-)|^2\mathrm{d}A_t \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\gamma}\int_{(0,\infty)} \mathbf{1}_{(0,T]}(t)\mathcal{E}(\beta A)_{t-}\frac{1}{\mathcal{E}(\gamma A)_{t-}}\int_{(0,\infty)}\mathcal{E}(\gamma A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathbf{1}_{[t,T]}(s)\mathrm{d}C_s\mathrm{d}A_t\\
&= \frac{1}{\gamma}\int_{(0,\infty)}\int_{(0,\infty)} \mathbf{1}_{\{0<t\leq s\leq T\}}\mathcal{E}(\beta A)_{t-}\frac{1}{\mathcal{E}(\gamma A)_{t-}}\mathcal{E}(\gamma A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s\mathrm{d}A_t\\
&= \frac{1}{\gamma}\int_{(0,\infty)}\int_{(0,\infty)} \mathbf{1}_{\{0<t\leq s\leq T\}}\mathcal{E}(\beta A)_{t-}\frac{1}{\mathcal{E}(\gamma A)_{t-}}\mathcal{E}(\gamma A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}A_t\mathrm{d}C_s\\
&= \frac{1}{\gamma}\int_{(0,\infty)}\mathcal{E}(\gamma A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\int_{(0,\infty)} \mathbf{1}_{\{0<t\leq s\leq T\}}\mathcal{E}(\beta A)_{t-}\frac{1}{\mathcal{E}(\gamma A)_{t-}}\mathrm{d}A_t\mathrm{d}C_s\\
&= \frac{1}{\gamma}\int_0^T\mathcal{E}(\gamma A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\int_0^s \mathcal{E}(\beta A)_{t-}\frac{1}{\mathcal{E}(\gamma A)_{t-}}\mathrm{d}A_t\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.}
\end{aligned}
\tag{5.17}
$$

Here, the fifth line follows from Tonelli's theorem. Lemma C.1. (ii) yields

$$
\int_0^s \mathcal{E}(\beta A)_{t-}\frac{1}{\mathcal{E}(\gamma A)_{t-}}\mathrm{d}A_t = \int_0^s \mathcal{E}(\widehat{A}^{\beta,\gamma})_{t-}\mathrm{d}A_t,\ s\in[0,\infty),\ \mathbb{P}\text{–a.s.},
$$

where $\widehat{A}^{\beta,\gamma} = (\widehat{A}^{\beta,\gamma}_t)_{t\in[0,\infty)}$ is the predictable process satisfying

$$
\widehat{A}^{\beta,\gamma} = (\beta-\gamma)A^c + \sum_{s\in(0,\cdot]}(\beta-\gamma)\frac{\Delta A_s}{1+\gamma\Delta A_s},\ \mathbb{P}\text{–a.s.}
\tag{5.18}
$$

Since $\beta-\gamma>0$, the process $\widehat{A}^{\beta,\gamma}$ is $\mathbb{P}$–a.s. non-decreasing. Similar to the derivation of (5.15), we find

$$
\begin{aligned}
\int_0^s \mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_{t-}\mathrm{d}A_t &= \int_{(0,s]}\mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_{t-}\mathrm{d}A^c_t + \sum_{t\in(0,s]}\mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_{t-}\Delta A_t\\
&= \frac{1}{(\beta-\gamma)}\int_{(0,s]}\mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_{t-}(1+\gamma\Delta A_t)\mathrm{d}\big(\widehat{A}^{\beta,\gamma}\big)^c_t\\
&\quad + \frac{1}{(\beta-\gamma)}\sum_{t\in(0,s]}\mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_{t-}(1+\gamma\Delta A_t)\Delta\widehat{A}^{\beta,\gamma}_t\\
&= \frac{1}{(\beta-\gamma)}\int_{(0,s]}\mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_{t-}(1+\gamma\Delta A_t)\mathrm{d}\widehat{A}^{\beta,\gamma}_t\\
&\leq \frac{(1+\gamma\Phi)}{(\beta-\gamma)}\int_{(0,s]}\mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_{t-}\mathrm{d}\widehat{A}^{\beta,\gamma}_t\\
&= \frac{(1+\gamma\Phi)}{(\beta-\gamma)}\big(\mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_s-1\big)\\
&\leq \frac{(1+\gamma\Phi)}{(\beta-\gamma)}\mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_s\\
&= \frac{(1+\gamma\Phi)}{(\beta-\gamma)}\mathcal{E}(\beta A)_s\frac{1}{\mathcal{E}(\gamma A)_s},\ s\in(0,\infty),\ \mathbb{P}\text{–a.s.}
\end{aligned}
\tag{5.19}
$$

Here, we used $(1+\gamma\Delta A)\mathrm{d}A^c = \mathrm{d}A^c$ and (5.18) in the second line, and the definition of $\widehat{A}^{\beta,\gamma}$ in the last equality. By substituting (5.19) into (5.17), we obtain

$$
\int_0^T \mathcal{E}(\beta A)_{t-}|F(t-)|^2\mathrm{d}A_t \leq \frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.}
$$

Consequently, this implies

$$
\begin{aligned}
\int_0^T \mathcal{E}(\beta A)_t|F(t-)|^2\mathrm{d}A_t &= \int_0^T \mathcal{E}(\beta A)_{t-}(1+\beta\Delta A_t)|F(t-)|^2\mathrm{d}A_t\\
&\leq (1+\beta\Phi)\int_0^T \mathcal{E}(\beta A)_{t-}|F(t-)|^2\mathrm{d}A_t
\end{aligned}
$$

$$\leq \frac{(1+\beta\Phi)(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.}$$

Next, we note that we also have

$$|F(t)|^2 \leq \int_{(0,\infty)} \frac{1}{\mathcal{E}(\gamma A)_s}\mathbf{1}_{(t,T]}(s)\mathrm{d}A_s \int_{(0,\infty)} \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathbf{1}_{(t,T]}(s)\mathrm{d}C_s,\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.}, \tag{5.20}$$

and similarly to (5.15), we find

$$\int_{(0,\infty)} \frac{1}{\mathcal{E}(\gamma A)_s}\mathbf{1}_{(t,T]}(s)\mathrm{d}A_s \leq \frac{1}{\gamma}\frac{1}{\mathcal{E}(\gamma A)_{t\wedge T}},\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.} \tag{5.21}$$

This also follows by taking right-hand limits along t in (5.15). We insert (5.21) into (5.20) and find

$$|F(t)|^2 \leq \frac{1}{\gamma}\frac{1}{\mathcal{E}(\gamma A)_{t\wedge T}}\int_{(0,\infty)} \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathbf{1}_{(t,T]}(s)\mathrm{d}C_s,\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.} \tag{5.22}$$

This yields

$$\begin{aligned}
&\int_0^T \mathcal{E}(\beta A)_t |F(t)|^2 \mathrm{d}A_t \\
&= \int_{(0,\infty)} \mathcal{E}(\beta A)_t |F(t)|^2 \mathbf{1}_{(0,T]}(t)\mathrm{d}A_t \\
&\leq \frac{1}{\gamma}\int_{(0,\infty)} \mathcal{E}(\beta A)_t \frac{1}{\mathcal{E}(\gamma A)_t}\mathbf{1}_{(0,T]}(t)\int_{(0,\infty)} \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathbf{1}_{(t,T]}(s)\mathrm{d}C_s\mathrm{d}A_t \\
&= \frac{1}{\gamma}\int_{(0,\infty)}\int_{(0,\infty)} \mathbf{1}_{\{0<t<s\leq T\}}\mathcal{E}(\beta A)_t \frac{1}{\mathcal{E}(\gamma A)_t}\mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s\mathrm{d}A_t \\
&= \frac{1}{\gamma}\int_{(0,\infty)}\int_{(0,\infty)} \mathbf{1}_{\{0<t<s\leq T\}}\mathcal{E}(\beta A)_t \frac{1}{\mathcal{E}(\gamma A)_t}\mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}A_t\mathrm{d}C_s \\
&= \frac{1}{\gamma}\int_{(0,\infty)} \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\int_{(0,\infty)} \mathbf{1}_{\{0<t<s\leq T\}}\mathcal{E}(\beta A)_t \frac{1}{\mathcal{E}(\gamma A)_t}\mathrm{d}A_t\mathrm{d}C_s \\
&= \frac{1}{\gamma}\int_0^T \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\int_0^{s-} \mathcal{E}(\beta A)_t \frac{1}{\mathcal{E}(\gamma A)_t}\mathrm{d}A_t\mathrm{d}C_s \\
&= \frac{1}{\gamma}\int_0^T \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\int_0^{s-} \frac{\mathcal{E}(\beta A)_{t-}}{\mathcal{E}(\gamma A)_{t-}}\frac{(1+\beta\Delta A_t)}{(1+\gamma\Delta A_t)}\mathrm{d}A_t\mathrm{d}C_s \\
&\leq \frac{1}{\gamma}\frac{(1+\beta\Phi)}{(1+\gamma\Phi)}\int_0^T \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\int_0^{s-} \frac{\mathcal{E}(\beta A)_{t-}}{\mathcal{E}(\gamma A)_{t-}}\mathrm{d}A_t\mathrm{d}C_s \\
&= \frac{1}{\gamma}\frac{(1+\beta\Phi)}{(1+\gamma\Phi)}\int_0^T \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\int_0^{s-} \mathcal{E}\big(\widehat{A}^{\beta,\gamma}\big)_{t-}\mathrm{d}A_t\mathrm{d}C_s \\
&\leq \frac{(1+\beta\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\gamma A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathcal{E}(\beta A)_{s-}\frac{1}{\mathcal{E}(\gamma A)_{s-}}\mathrm{d}C_s \\
&= \frac{(1+\beta\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_{s-}\frac{|\delta f_s|^2}{\alpha_s^2}\frac{\mathcal{E}(\gamma A)_{s-}(1+\gamma\Delta A_s)}{\mathcal{E}(\gamma A)_{s-}}\mathrm{d}C_s \\
&\leq \frac{(1+\beta\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_{s-}(1+\beta\Delta A_s)\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s \\
&= \frac{(1+\beta\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_s \frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.}
\end{aligned}$$

Here, we used Tonelli's theorem in the fourth line, (5.11) in the seventh line, the fact that $(1+\beta\Delta A)/(1+\gamma\Delta A) \leq (1+\beta\Phi)/(1+\gamma\Phi)$ since $x \longmapsto (1+\beta x)/(1+\gamma x)$ is increasing on $[0,\Phi]$ in the eighth line, Equation (5.19) in the tenth line, and Equation (5.11) again in the third-to-last and last line. Consequently, this implies

$$\begin{aligned}\int_0^T \mathcal{E}(\beta A)_{t-}|F(t)|^2\mathrm{d}A_t &\leq \int_0^T \mathcal{E}(\beta A)_{t-}|F(t-)|^2\mathrm{d}A_t\\ &\leq \frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.},\end{aligned}$$

since $F(t) \leq F(t-)$ for $t \in (0,\infty]$. To summarise, we found

$$\int_0^T \mathcal{E}(\beta A)_t|F(t)|^2\mathrm{d}A_t \leq \frac{(1+\beta\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s,$$

$$\int_0^T \mathcal{E}(\beta A)_{t-}|F(t)|^2\mathrm{d}A_t \leq \frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s,$$

$$\int_0^T \mathcal{E}(\beta A)_t|F(t-)|^2\mathrm{d}A_t \leq \frac{(1+\beta\Phi)(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s,$$

$$\int_0^T \mathcal{E}(\beta A)_{t-}|F(t-)|^2\mathrm{d}A_t \leq \frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\int_0^T \mathcal{E}(\beta A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s,\ \mathbb{P}\text{–a.s.} \tag{5.23}$$

We turn to the stated bounds. The $\mathcal{S}^2_T$-bounds

$$\|\delta y\|^2_{\mathcal{S}^2_T} \leq \frac{4}{\beta}\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}} \text{ and } \|y^i\|^2_{\mathcal{S}^2_T} \leq 12\left(\|\xi_T\|^2_{\mathbb{L}^2} + \Big\|\sup_{u\in[0,T)}\xi^+_u\Big\|^2_{\mathbb{L}^2} + \frac{1}{\beta}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\right) \tag{5.24}$$

follow from Lemma 5.2 together with

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_0^T|\delta f_s|\mathrm{d}C_s\bigg)^2\bigg] &\leq \mathbb{E}\bigg[\bigg(\int_0^T\frac{1}{\mathcal{E}(\beta A)_s}\mathrm{d}A_s\bigg)\bigg(\int_0^T\mathcal{E}(\beta A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s\bigg)\bigg]\\ &\leq \frac{1}{\beta}\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},\end{aligned} \tag{5.25}$$

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_0^T|f^i_s|\mathrm{d}C_s\bigg)^2\bigg] &\leq \mathbb{E}\bigg[\bigg(\int_0^T\frac{1}{\mathcal{E}(\beta A)_s}\mathrm{d}A_s\bigg)\bigg(\int_0^T\mathcal{E}(\beta A)_s\frac{|f^i_s|^2}{\alpha_s^2}\mathrm{d}C_s\bigg)\bigg]\\ &\leq \frac{1}{\beta}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.\end{aligned} \tag{5.26}$$

Here we first use the Cauchy–Schwarz inequality and then (5.21). The $\mathbb{H}^2$-bound for δy follows from

$$\begin{aligned}\|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} &= \mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_t|\delta y_t|^2\alpha_t^2\mathrm{d}C_t\bigg] = \mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_t|\delta y_t|^2\mathrm{d}A_t\bigg]\\ &\leq \mathbb{E}\bigg[\int_0^T\mathbb{E}\big[\mathcal{E}(\beta A)_t|F(t)|^2\big|\mathcal{G}_t\big]\mathrm{d}A_t\bigg] = \mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_t|F(t)|^2\mathrm{d}A_t\bigg]\\ &\leq \frac{(1+\beta\Phi)}{\gamma(\beta-\gamma)}\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},\end{aligned}$$

where the first inequality follows from optional projection together with Lemma 5.2, the step from the first to the second line follows from [41, Theorem VI.57], and the last inequality follows from (5.23). Hence,

$$\|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} \leq \inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\beta\Phi)}{\gamma(\beta-\gamma)}\bigg\}\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}} = \frac{4(1+\beta\Phi)}{\beta^2}\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}} = \mathfrak{f}^\Phi(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}. \quad (5.27)$$

We similarly find

$$\begin{aligned}\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} &= \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|\delta y_{t-}|^2\mathrm{d}A_t\bigg]\\
&\leq (1+\beta\Phi)\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}|\delta y_{t-}|^2\mathrm{d}A_t\bigg]\\
&= (1+\beta\Phi)\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}|F(t-)|^2\mathrm{d}A_t\bigg]\\
&\leq (1+\beta\Phi)\inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\bigg\}\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}} = (1+\beta\Phi)\mathfrak{g}^\Phi(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}, \quad (5.28)\end{aligned}$$

where we now use the predictable projection together with Lemma 5.2 and [41, Theorem VI.57]. Assuming for the moment that $f^2 = 0$, we find analogously

$$\begin{aligned}\|\alpha y^1\|^2_{\mathbb{H}^2_{T,\beta}} &= \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|y^1_t|^2\alpha_t^2\mathrm{d}C_t\bigg] = \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|y^1_t|^2\mathrm{d}A_t\bigg]\\
&\leq 3\mathbb{E}\bigg[\int_0^T \mathbb{E}\bigg[\mathcal{E}(\beta A)_t|\xi_T|^2 + \mathcal{E}(\beta A)_t\sup_{u\in[t,\infty]}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2 + \mathcal{E}(\beta A)_t|F(t)|^2\bigg|\mathcal{G}_t\bigg]\mathrm{d}A_t\bigg]\\
&= 3\bigg(\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|\xi_T|^2\mathrm{d}A_t\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t\sup_{u\in[t,\infty]}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2\mathrm{d}A_t\bigg]\\
&\quad + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|F(t)|^2\mathrm{d}A_t\bigg]\bigg)\\
&= \frac{3}{\beta}\mathbb{E}\bigg[|\xi_T|^2\int_0^T \mathcal{E}(\beta A)_{t-}(1+\beta\Delta A_t)\mathrm{d}(\beta A)_t\bigg]\\
&\quad + 3\bigg(\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t\sup_{u\in[t,\infty]}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2\mathrm{d}A_t\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|F(t)|^2\mathrm{d}A_t\bigg]\bigg)\\
&\leq 3\frac{(1+\beta\Phi)}{\beta}\mathbb{E}\bigg[|\xi_T|^2\int_0^T \mathcal{E}(\beta A)_{t-}\mathrm{d}(\beta A)_t\bigg]\\
&\quad + 3\bigg(\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t\sup_{u\in[t,\infty]}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2\mathrm{d}A_t\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|F(t)|^2\mathrm{d}A_t\bigg]\bigg)\\
&\leq 3\bigg(\frac{(1+\beta\Phi)}{\beta}\mathbb{E}\Big[\mathcal{E}(\beta A)_T|\xi_T|^2\Big] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t\sup_{u\in[t,\infty]}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2\mathrm{d}A_t\bigg]\\
&\quad + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|F(t)|^2\mathrm{d}A_t\bigg]\bigg)\\
&\leq 3\bigg(\frac{(1+\beta\Phi)}{\beta}\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + \inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\beta\Phi)}{\gamma(\beta-\gamma)}\bigg\}\bigg\|\frac{f^1}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg)\\
&= 3\bigg(\frac{(1+\beta\Phi)}{\beta}\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + \mathfrak{f}^\Phi(\beta)\bigg\|\frac{f^1}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg),\end{aligned}$$

where the second line follows from the optional projection together with Lemma 5.2 and $(a^2+b^2+c^3) \leq 3(a^2+b^2+c^2)$, the third line follows from [41, Theorem VI.57], and the last line follows from (5.23). An analogous argument leads to the $\mathbb{H}^2$-bound on y^2. By using predictable projection, we find

$$\begin{aligned}
\|\alpha y^1_-\|^2_{\mathbb{H}^2_{T,\beta}} &= \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t |y^1_{t-}|^2 \alpha_t^2 \mathrm{d}C_t\bigg] = \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t |y^1_{t-}|^2 \mathrm{d}A_t\bigg] \\
&\leq 3\mathbb{E}\bigg[\int_0^T \mathbb{E}\bigg[\mathcal{E}(\beta A)_t\bigg(|\xi_T|^2 + \lim_{t'\uparrow\uparrow t}\sup_{u\in[t',\infty]}|\xi^+_u \mathbf{1}_{\{u<T\}}|^2 + |F(t-)|^2\bigg)\bigg|\mathcal{G}_{t-}\bigg]\mathrm{d}A_t\bigg] \\
&= 3\bigg(\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|\xi_T|^2\mathrm{d}A_t\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t \lim_{t'\uparrow\uparrow t}\sup_{u\in[t',\infty]}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2\mathrm{d}A_t\bigg] \\
&\quad + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_t|F(t-)|^2\mathrm{d}A_t\bigg]\bigg) \\
&\leq 3\bigg(\frac{(1+\beta\Phi)}{\beta}\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + (1+\beta\Phi)\inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\bigg\}\bigg\|\frac{f^1}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg) \\
&= 3\bigg(\frac{(1+\beta\Phi)}{\beta}\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + (1+\beta\Phi)\mathfrak{g}^\Phi(\beta)\bigg\|\frac{f^1}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg).
\end{aligned}$$

An analogous argument leads to the $\mathbb{H}^2$-bound on y^2_-.

We turn to the bounds on the martingale $\delta\eta$ and first note that for a $\mathbb{P}$–a.s. càdlàg process $V=(V_t)_{t\in[0,\infty)}$ with $\mathbb{P}$–a.s. non-decreasing paths starting from zero, we have

$$\begin{aligned}
\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}V_s &= \int_0^T \bigg(1+\int_0^s \mathcal{E}(\beta A)_{t-}\mathrm{d}(\beta A)_t\bigg)\mathrm{d}V_s \\
&= V_T + \int_0^T\int_0^s \mathcal{E}(\beta A)_{t-}\beta \mathrm{d}A_t \mathrm{d}V_s \\
&= V_T + \beta\int_{(0,\infty)}\int_{(0,\infty)} \mathbf{1}_{\{0<t\leq s\leq T\}}\mathcal{E}(\beta A)_{t-}\mathrm{d}A_t\mathrm{d}V_s \\
&= V_T + \beta\int_{(0,\infty)}\int_{(0,\infty)} \mathbf{1}_{\{0<t\leq s\leq T\}}\mathcal{E}(\beta A)_{t-}\mathrm{d}V_s\mathrm{d}A_t \\
&= V_T + \beta\int_{(0,\infty)} \mathbf{1}_{\{0<t\leq T\}}\mathcal{E}(\beta A)_{t-}\int_{(0,\infty)}\mathbf{1}_{\{t\leq s\leq T\}}\mathrm{d}V_s\mathrm{d}A_t \\
&= V_T + \beta\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T \mathrm{d}V_s\mathrm{d}A_t \\
&= V_T + \beta\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T \mathrm{d}V_s\mathrm{d}A_t,\ \mathbb{P}\text{–a.s.}
\end{aligned} \tag{5.29}$$

Here the fourth line follows from Tonelli's theorem. Thus, by letting $V := \langle\delta\eta\rangle + [\delta k^r] + [\delta k^\ell]$, we get

$$\begin{aligned}
&\|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[\delta k^r]_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[\delta k^\ell]_s\bigg] \\
&\quad= \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}\langle\delta\eta\rangle_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[\delta k^r]_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[\delta k^\ell]_s\bigg] \\
&\quad= \mathbb{E}\big[\langle\delta\eta\rangle_T + [\delta k^r]_T + [\delta k^\ell]_T\big] + \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T \mathrm{d}(\langle\delta\eta\rangle + [\delta k^r] + [\delta k^\ell])_s\mathrm{d}A_t\bigg].
\end{aligned} \tag{5.30}$$

We now apply [41, Theorem VI.57], the predictable projection together with Lemma 5.3, and find

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T \mathrm{d}(\langle\delta\eta\rangle+[\delta k^r]+[\delta k^\ell])_s\mathrm{d}A_t\bigg]\\
&\quad\le 2\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T |\delta y_s\delta f_s|\mathrm{d}C_s\mathrm{d}A_t\bigg]\\
&\qquad+\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T (\delta f_s)^2\mathrm{d}[C]_s\mathrm{d}A_t\bigg]-\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}|\delta y_{t-}|^2\mathrm{d}A_t\bigg].
\end{aligned}$$

We insert this inequality into (5.30) and find, after a rearrangement of the terms, that

$$\begin{aligned}
&\beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}|\delta y_{t-}|^2\mathrm{d}A_t\bigg]+\|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}}+\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[\delta k^r]_s\bigg]\\
&\qquad+\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[\delta k^\ell]_s\bigg]\\
&\quad\le \mathbb{E}\big[\langle\delta\eta\rangle_T+[\delta k^r]_T+[\delta k^\ell]_T\big]+2\beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T |\delta y_s\delta f_s|\mathrm{d}C_s\mathrm{d}A_t\bigg]\\
&\qquad+\beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T (\delta f_s)^2\mathrm{d}[C]_s\mathrm{d}A_t\bigg].
\end{aligned}\tag{5.31}$$

We now bound the second expectation on the second line. By the Cauchy–Schwartz-inequality and since $A=\int_0^\cdot \alpha_s^2\mathrm{d}C_s$, we find

$$\begin{aligned}
\int_{t-}^T|\delta y_s\delta f_s|\mathrm{d}C_s&=\int_{t-}^T\big(\mathcal{E}(\gamma A)_s^{-1/2}|\delta y_s|\alpha_s\big)\bigg(\mathcal{E}(\gamma A)_s^{1/2}\frac{|\delta f_s|}{\alpha_s}\bigg)\mathrm{d}C_s\\
&\le\bigg(\int_{t-}^T\mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2\mathrm{d}A_s\bigg)^{1/2}\bigg(\int_{t-}^T\mathcal{E}(\gamma A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s\bigg)^{1/2}.
\end{aligned}$$

Therefore, for $\gamma\in(0,\beta)$, we find

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T |\delta y_s\delta f_s|\mathrm{d}C_s\mathrm{d}A_t\bigg]\\
&\le\mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_{t-}^{1/2}\mathcal{E}(\gamma A)_{t-}^{1/2}\bigg(\int_{t-}^T\mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2\mathrm{d}A_s\bigg)^{1/2}\\
&\qquad\mathcal{E}(\beta A)_{t-}^{1/2}\frac{1}{\mathcal{E}(\gamma A)_{t-}^{1/2}}\bigg(\int_{t-}^T\mathcal{E}(\gamma A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s\bigg)^{1/2}\mathrm{d}A_t\bigg]\\
&\le\mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_{t-}\mathcal{E}(\gamma A)_{t-}\int_{t-}^T\mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2\mathrm{d}A_s\mathrm{d}A_t\bigg]^{1/2}\\
&\qquad\mathbb{E}\bigg[\gamma\frac{1}{\gamma}\int_0^T\mathcal{E}(\beta A)_{t-}\frac{1}{\mathcal{E}(\gamma A)_{t-}}\int_{t-}^T\mathcal{E}(\gamma A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s\mathrm{d}A_t\bigg]^{1/2}\\
&\le\mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_{t-}\mathcal{E}(\gamma A)_{t-}\int_{t-}^T\mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2\mathrm{d}A_s\mathrm{d}A_t\bigg]^{1/2}\\
&\qquad\mathbb{E}\bigg[\gamma\frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\int_0^T\mathcal{E}(\beta A)_s\frac{|\delta f_s|^2}{\alpha_s^2}\mathrm{d}C_s\bigg]^{1/2}\\
&=\mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_{t-}\mathcal{E}(\gamma A)_{t-}\int_{t-}^T\mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2\mathrm{d}A_s\mathrm{d}A_t\bigg]^{1/2}\bigg(\frac{(1+\gamma\Phi)}{(\beta-\gamma)}\bigg)^{1/2}\bigg\|\frac{\delta f}{\alpha}\bigg\|_{\mathbb{H}^2_{T,\beta}}.
\end{aligned}\tag{5.32}$$

Here, the second inequality follows from the Cauchy–Schwarz inequality, and the third inequality follows from the equalities starting on the third line of (5.17) together with (5.19). Next, note that

$$
\begin{aligned}
&\int_0^T \mathcal{E}(\beta A)_{t-}\mathcal{E}(\gamma A)_{t-}\int_{t-}^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \mathrm{d}A_s\mathrm{d}A_t \\
&= \int_{(0,\infty)}\int_{(0,\infty)} \mathbf{1}_{\{t\le s\le T\}}\mathcal{E}(\beta A)_{t-}\mathcal{E}(\gamma A)_{t-}\mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \mathrm{d}A_s\mathrm{d}A_t \\
&= \int_{(0,\infty)}\int_{(0,\infty)} \mathbf{1}_{\{t\le s\le T\}}\mathcal{E}(\beta A)_{t-}\mathcal{E}(\gamma A)_{t-}\mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \mathrm{d}A_t\mathrm{d}A_s \\
&= \int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \int_0^s \mathcal{E}(\beta A)_{t-}\mathcal{E}(\gamma A)_{t-}\mathrm{d}A_t\mathrm{d}A_s \\
&= \int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \int_0^s \mathcal{E}\big(\underbrace{(\beta+\gamma)A+\beta\gamma[A]}_{\overline{A}:=}\big)_{t-}\mathrm{d}A_t\mathrm{d}A_s \\
&= \int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \bigg(\int_0^s \mathcal{E}(\overline{A})_{t-}\mathrm{d}A_t^c + \sum_{0<t\le s}\mathcal{E}(\overline{A})_{t-}\Delta A_t\bigg)\mathrm{d}A_s \\
&= \int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \bigg(\int_0^s \mathcal{E}(\overline{A})_{t-}\frac{1}{(\beta+\gamma)}\mathrm{d}\overline{A}_t^c \\
&\quad + \sum_{0<t\le s}\mathcal{E}(\overline{A})_{t-}\frac{1}{(\beta+\gamma+\beta\gamma\Delta A_t)}\Delta\overline{A}_t\bigg)\mathrm{d}A_s \\
&= \int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \bigg(\int_0^s \mathcal{E}(\overline{A})_{t-}\frac{1}{(\beta+\gamma+\beta\gamma\Delta A_t)}\mathrm{d}\overline{A}_t^c \\
&\quad + \sum_{0<t\le s}\mathcal{E}(\overline{A})_{t-}\frac{1}{(\beta+\gamma+\beta\gamma\Delta A_t)}\Delta\overline{A}_t\bigg)\mathrm{d}A_s \\
&= \int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \int_0^s \mathcal{E}(\overline{A})_{t-}\frac{1}{(\beta+\gamma+\beta\gamma\Delta A_t)}\mathrm{d}\overline{A}_t\mathrm{d}A_s \\
&\le \int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2 \frac{1}{(\beta+\gamma)}\int_0^s \mathcal{E}(\overline{A})_{t-}\mathrm{d}\overline{A}_t\mathrm{d}A_s \\
&\le \frac{1}{\beta+\gamma}\int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2\mathcal{E}(\overline{A})_s\mathrm{d}A_s \\
&= \frac{1}{\beta+\gamma}\int_0^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2\mathcal{E}(\beta A)_s\mathcal{E}(\gamma A)_s\mathrm{d}A_s \\
&= \frac{1}{\beta+\gamma}\int_0^T \mathcal{E}(\beta A)_s|\delta y_s|^2\mathrm{d}A_s. \qquad (5.33)
\end{aligned}
$$

Here the seventh line follows from $\overline{A}^c = (\beta+\gamma)A^c$ and $\Delta\overline{A} = (\beta+\gamma)\Delta A + \beta\gamma(\Delta A)^2 = (\beta+\gamma+\beta\gamma\Delta A)\Delta A$, the eigth line follows from the fact that $\mathrm{d}\overline{A}^c$ puts no mass on the points $s\in(0,\infty)$ where $\Delta A_s \neq 0$, and the tenth line follows from $\beta+\gamma\le\beta+\gamma+\beta\gamma\Delta A$. This then yields

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\mathcal{E}(\gamma A)_{t-}\int_{t-}^T \mathcal{E}(\gamma A)_s^{-1}|\delta y_s|^2\mathrm{d}A_s\mathrm{d}A_t\bigg]^{1/2} \\
&\quad \le \bigg(\frac{1}{\beta+\gamma}\bigg)^{1/2}\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s|\delta y_s|^2\mathrm{d}A_s\bigg]^{1/2} = \bigg(\frac{1}{\beta+\gamma}\bigg)^{1/2}\|\alpha\delta y\|_{\mathbb{H}^2_{T,\beta}}
\end{aligned}
$$

$$\leq \left(\frac{1}{\beta+\gamma}\right)^{1/2}\big(\mathfrak{f}^{\Phi}(\beta)\big)^{1/2}\left\|\frac{\delta f}{\alpha}\right\|_{\mathbb{H}^2_{T,\beta}} = \left(\frac{1}{\beta+\gamma}\right)^{1/2}\left(\frac{4(1+\beta\Phi)}{\beta^2}\right)^{1/2}\left\|\frac{\delta f}{\alpha}\right\|_{\mathbb{H}^2_{T,\beta}}. \quad (5.34)$$

Here the inequality on the third line follows from (5.27). We now combine Equation (5.32) and Equation (5.34) and find that

$$\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T |\delta y_s \delta f_s|\mathrm{d}C_s\mathrm{d}A_t\bigg] \leq \left(\frac{1}{\beta+\gamma}\right)^{1/2}\left(\frac{4(1+\beta\Phi)}{\beta^2}\right)^{1/2}\left(\frac{(1+\gamma\Phi)}{(\beta-\gamma)}\right)^{1/2}\left\|\frac{\delta f}{\alpha}\right\|^2_{\mathbb{H}^2_{T,\beta}}.$$

Since $\gamma\in(0,\beta)$ was arbitrary, we find

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T |\delta y_s \delta f_s|\mathrm{d}C_s\mathrm{d}A_t\bigg] &\leq \left(\inf_{\gamma\in(0,\beta)}\left\{\frac{(1+\gamma\Phi)}{(\beta^2-\gamma^2)}\right\}\frac{4(1+\beta\Phi)}{\beta^2}\right)^{1/2}\left\|\frac{\delta f}{\alpha}\right\|^2_{\mathbb{H}^2_{T,\beta}}\\ &= \left(\frac{1}{\beta^2}\frac{4(1+\beta\Phi)}{\beta^2}\right)^{1/2}\left\|\frac{\delta f}{\alpha}\right\|^2_{\mathbb{H}^2_{T,\beta}}\\ &= \frac{2}{\beta^2}(1+\beta\Phi)^{1/2}\left\|\frac{\delta f}{\alpha}\right\|^2_{\mathbb{H}^2_{T,\beta}}. \quad (5.35)\end{aligned}$$

We now bound the last summand in (5.31). Since

$$\begin{aligned}\int_{t-}^T(\delta f_s)^2\mathrm{d}[C]_s &= \sum_{t\leq s\leq T}\mathbf{1}_{\{s<\infty\}}(\delta f_s)^2(\Delta C_s)^2 = \sum_{t\leq s\leq T}\mathbf{1}_{\{s<\infty\}}(|\delta f_s|\Delta C_s)^2\\ &\leq \bigg(\sum_{t\leq s\leq T}\mathbf{1}_{\{s<\infty\}}|\delta f_s|\Delta C_s\bigg)^2 \leq \bigg(\int_{t-}^T|\delta f_s|\mathrm{d}C_s\bigg)^2 = |F(t-)|^2,\end{aligned}$$

it follows from (5.23) that

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T(\delta f_s)^2\mathrm{d}[C]_s\mathrm{d}A_t\bigg] &\leq \mathbb{E}\bigg[\int_0^T\mathcal{E}(\beta A)_{t-}|F(t-)|^2\mathrm{d}A_t\bigg]\\ &\leq \inf_{\gamma\in(0,\beta)}\left\{\frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\right\}\left\|\frac{\delta f}{\alpha}\right\|^2_{\mathbb{H}^2_{T,\beta}} \quad (5.36)\end{aligned}$$

by arbitrariness of $\gamma\in(0,\beta)$. We now turn to the first expectation on the second line in (5.31). We find

$$\begin{aligned}&\mathbb{E}\big[\langle\delta\eta\rangle_T+[\delta k^r]_T+[\delta k^\ell]_T\big]\\ &\leq -\mathbb{E}\big[(\delta y_0)^2\big]+2\mathbb{E}\bigg[\int_0^T\delta y_s\delta f_s\mathrm{d}C_s\bigg]+\mathbb{E}\bigg[\int_0^T(\delta f_s)^2\mathrm{d}[C]_s\bigg]\\ &\leq -\mathbb{E}\big[(\delta y_0)^2\big]+2\mathbb{E}\bigg[\sup_{s\in[0,T]}|\delta y_s|\int_0^T|\delta f_s|\mathrm{d}C_s\bigg]+\mathbb{E}\bigg[\bigg(\int_0^T|\delta f_s|\mathrm{d}C_s\bigg)^2\bigg]\\ &\leq -\mathbb{E}\big[(\delta y_0)^2\big]+2\mathbb{E}\bigg[\sup_{s\in[0,T]}|\delta y_s|^2\bigg]^{1/2}\mathbb{E}\bigg[\bigg(\int_0^T|\delta f_s|\mathrm{d}C_s\bigg)^2\bigg]^{1/2}+\mathbb{E}\bigg[\bigg(\int_0^T|\delta f_s|\mathrm{d}C_s\bigg)^2\bigg]\\ &\leq -\mathbb{E}\big[(\delta y_0)^2\big]+2\frac{2}{\beta^{1/2}}\left\|\frac{\delta f}{\alpha}\right\|_{\mathbb{H}^2_{T,\beta}}\frac{1}{\beta^{1/2}}\left\|\frac{\delta f}{\alpha}\right\|_{\mathbb{H}^2_{T,\beta}}+\frac{1}{\beta}\left\|\frac{\delta f}{\alpha}\right\|^2_{\mathbb{H}^2_{T,\beta}}\\ &= -\mathbb{E}\big[(\delta y_0)^2\big]+\frac{5}{\beta}\left\|\frac{\delta f}{\alpha}\right\|^2_{\mathbb{H}^2_{T,\beta}} \quad (5.37)\end{aligned}$$

Here, the second-to-last line follows from (5.24). We substitute (5.37), (5.36) and (5.35) into (5.31) and find

$$\begin{aligned}
&\mathbb{E}\big[(\delta y_0)^2\big] + \frac{\beta}{(1+\beta\Phi)}\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \\
&\quad + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}[\delta k^r]_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}[\delta k^\ell]_s\bigg] \\
&\leq \mathbb{E}\big[(\delta y_0)^2\big] + \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}|\delta y_{s-}|^2 \mathrm{d}A_s\bigg] + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \\
&\quad + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}[\delta k^r]_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}[\delta k^\ell]_s\bigg] \\
&\leq \bigg(\frac{5}{\beta} + (2\beta)\frac{2}{\beta^2}(1+\beta\Phi)^{1/2} + \beta \inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\bigg\}\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}} \\
&= \bigg(\frac{5}{\beta} + \frac{4}{\beta}(1+\beta\Phi)^{1/2} + \beta\mathfrak{g}^\Phi(\beta)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.
\end{aligned}$$

In particular, this implies

$$\begin{aligned}
&\min\bigg\{1, \frac{\beta}{(1+\beta\Phi)}\bigg\}\Big(\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}}\Big) \\
&\leq \frac{\beta}{(1+\beta\Phi)}\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \bigg(\frac{5}{\beta} + \frac{4}{\beta}(1+\beta\Phi)^{1/2} + \beta\mathfrak{g}^\Phi(\beta)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\|\delta y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \\
&\quad \leq \bigg(\frac{4}{\beta} + \max\bigg\{1, \frac{(1+\beta\Phi)}{\beta}\bigg\}\bigg(\frac{5}{\beta} + \frac{4}{\beta}(1+\beta\Phi)^{1/2} + \beta\mathfrak{g}^\Phi(\beta)\bigg)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}} \\
&\quad = M_3^\Phi(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.
\end{aligned}$$

The inequalities involving $M_1^\Phi(\beta)$ or $M_2^\Phi(\beta)$ follow immediately.

Finally, we turn to the inequality involving η^i. Similar to before, we choose $V = \langle\delta\eta^i\rangle + [k^{r,i}] + [k^{r,\ell}]$ in (5.29) and find

$$\begin{aligned}
&\|\eta^i\|^2_{\mathcal{H}^2_{T,\beta}} + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}[k^{r,i}]_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}[k^{\ell,i}]_s\bigg] \\
&= \mathbb{E}\big[\langle\eta^i\rangle_T + [k^{r,i}]_T + [k^{\ell,i}]_T\big] + \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T \mathrm{d}(\langle\eta^i\rangle + [k^{r,i}] + [k^{\ell,i}])_s \mathrm{d}A_t\bigg].
\end{aligned} \tag{5.38}$$

To bound the first expectation on the last line, we apply (5.7) and find

$$\begin{aligned}
&\mathbb{E}\big[\langle\eta^i\rangle_T + [k^{r,i}]_T + [k^{\ell,i}]_T\big] \\
&\qquad \leq -\mathbb{E}\big[|y_0^i|^2\big] + L\mathbb{E}\bigg[|\xi_T|^2 + \sup_{u\in[0,\infty]}|\xi_u^+\mathbf{1}_{\{u<T\}}|^2 + \bigg(\int_0^T |f_u^i|\mathrm{d}C_u\bigg)^2\bigg] \\
&\qquad \leq -\mathbb{E}\big[|y_0^i|^2\big] + L\|\xi_T\|^2_{\mathbb{L}^2} + L\|\xi^+\mathbf{1}_{[\![0,T[\![}}\|^2_{\mathcal{S}^2_T} + \frac{L}{\beta}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},
\end{aligned}$$

where the last inequality follows from (5.26). To bound the second expectation on the last line of (5.38), we apply (5.8) and then (5.23) and find

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\int_{t-}^T \mathrm{d}(\langle \eta^i\rangle + [k^{r,i}] + [k^{\ell,i}])_s \mathrm{d}A_t\bigg]\\
&\leq -\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}|y^i_{t-}|^2\mathrm{d}A_t\bigg]\\
&\quad + L\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}\bigg(|\xi_T|^2 + |{}^*\xi_t|^2 + \bigg(\int_{t-}^T |f^i_s|\mathrm{d}C_s\bigg)^2\bigg)\mathrm{d}A_t\bigg]\\
&\leq -\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{t-}|y^i_{t-}|^2\mathrm{d}A_t\bigg]\\
&\quad + L\|\xi_T\|^2_{\mathbb{L}^2_\beta} + L\|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + L\inf_{\gamma\in(0,\infty)}\bigg\{\frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\bigg\}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.
\end{aligned}$$

We now substitute this back into (5.38) and find after a rearrangement of the terms that

$$\begin{aligned}
&\mathbb{E}\big[|y^i_0|^2\big] + \frac{\beta}{(1+\beta\Phi)}\|\alpha y^i_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\eta^i\|^2_{\mathcal{H}^2_{T,\beta}}\\
&\quad + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[k^{r,i}]_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[k^{\ell,i}]_s\bigg]\\
&\leq \mathbb{E}\big[|y^i_0|^2\big] + \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}|y^i_{s-}|^2\mathrm{d}A_s\bigg] + \|\eta^i\|^2_{\mathcal{H}^2_{T,\beta}}\\
&\quad + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[k^{r,i}]_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s\mathrm{d}[k^{\ell,i}]_s\bigg]\\
&\leq L\bigg(\|\xi_T\|^2_{\mathbb{L}^2} + \|\xi^+\mathbf{1}_{[\![0,T)\!)}\|^2_{\mathcal{S}^2_T} + \frac{1}{\beta}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\\
&\quad + \beta\bigg(\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + \inf_{\gamma\in(0,\infty)}\bigg\{\frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\bigg\}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg)\bigg)\\
&= L\bigg(\|\xi_T\|^2_{\mathbb{L}^2} + \|\xi^+\mathbf{1}_{[\![0,T)\!)}\|^2_{\mathcal{S}^2_T} + \frac{1}{\beta}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\\
&\quad + \beta\bigg(\|\xi_T\|^2_{\mathbb{L}^2_\beta} + \|\alpha\,{}^*\xi\|^2_{\mathbb{H}^2_{T,\beta}} + \mathfrak{g}^\Phi(\beta)\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg)\bigg).
\end{aligned}$$

This completes the proof. □

Although the *a priori* estimates in Proposition 5.1 also hold for non-reflected BSDEs, we obtain sharper bounds by redoing them in this special case. Furthermore, as previously indicated in Remark 3.10. (iv), the techniques we employ below do not rely on Itô's formula, and an extension to BSDEs with multi-dimensional generator and terminal condition is straightforward. As before, we use the convention $\zeta_{0-} := 0$ for a process $\zeta = (\zeta_t)_{t\in[0,\infty]}$.

Proposition 5.4. *Suppose that* $\xi = -\infty$ *on* $[0,T)$. *Then*

$$|\delta y_S|^2 + \mathbb{E}\bigg[\int_S^T \mathrm{d}\langle\delta\eta\rangle_u\bigg|\mathcal{G}_S\bigg] = \mathbb{E}\bigg[\bigg(\int_S^T |\delta f_u|\mathrm{d}C_u\bigg)^2\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}_{0,\infty},$$
$$|\delta y_{S-}|^2 + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}\langle\delta\eta\rangle_u\bigg|\mathcal{G}_{S-}\bigg] = \mathbb{E}\bigg[\bigg(\int_{S-}^T |\delta f_u|\mathrm{d}C_u\bigg)^2\bigg|\mathcal{G}_{S-}\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}^p_{0,\infty}.$$

Moreover, for $\beta \in (0, \infty)$*,*

$$\frac{\beta}{(1+\beta\Phi)}\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \bigg(\frac{1}{\beta} + \beta\mathfrak{g}^\Phi(\beta)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},$$

and thus[17]

$$\|\delta y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} + \|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \widetilde{M}^\Phi_1(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},$$

$$\|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \widetilde{M}^\Phi_2(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}$$

$$\|\delta y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \widetilde{M}^\Phi_3(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.$$

Proof. As in the proof of Proposition 5.1, we slightly abuse notation and denote by $\mathbb{E}[W_\cdot|\mathcal{G}_\cdot]$ and $\mathbb{E}[W_\cdot|\mathcal{G}_{\cdot-}]$ the optional and predictable projection, respectively, of a non-negative, measurable process $W = (W_t)_{t\in[0,\infty]}$.

Note first that $\int_t^T \mathrm{d}(\delta\eta)_s = -\delta y_t + \int_t^T \delta f_s \mathrm{d}C_s = -\mathbb{E}\big[\int_t^T \delta f_s \mathrm{d}C_s\big|\mathcal{G}_t\big] + \int_t^T \delta f_s \mathrm{d}C_s$, which implies

$$\begin{aligned}
\mathbb{E}\bigg[\int_t^T \mathrm{d}\langle\delta\eta\rangle_s\bigg|\mathcal{G}_t\bigg] &= \mathbb{E}\bigg[\bigg(\int_t^T \mathrm{d}(\delta\eta)_s\bigg)^2\bigg|\mathcal{G}_t\bigg] \\
&= \mathbb{E}\bigg[\bigg(\int_t^T \delta f_s \mathrm{d}C_s\bigg)^2\bigg|\mathcal{G}_t\bigg] - \bigg(\mathbb{E}\bigg[\int_t^T \delta f_s \mathrm{d}C_s\bigg|\mathcal{G}_t\bigg]\bigg)^2 \\
&= \mathbb{E}\bigg[\bigg(\int_t^T \delta f_s \mathrm{d}C_s\bigg)^2\bigg|\mathcal{G}_t\bigg] - (\delta y_t)^2,\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.}
\end{aligned} \tag{5.39}$$

A similar argument, but now using the predictable projection, implies

$$(\delta y_{t-})^2 + \mathbb{E}\bigg[\int_{t-}^T \mathrm{d}\langle\delta\eta\rangle_s\bigg|\mathcal{G}_{t-}\bigg] = \mathbb{E}\bigg[\bigg(\int_{t-}^T |\delta f_s|\mathrm{d}C_s\bigg)^2\bigg|\mathcal{G}_{t-}\bigg],\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.} \tag{5.40}$$

As in the proof of Proposition 5.1, we have

$$\begin{aligned}
\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}\langle\delta\eta\rangle_s\bigg] &= \mathbb{E}[\langle\delta\eta\rangle_T] + \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}\int_{t-}^T \mathrm{d}\langle\delta\eta\rangle_s \mathrm{d}A_t\bigg] \\
&= \mathbb{E}[\langle\delta\eta\rangle_T] + \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}\bigg(\int_{t-}^T |\delta f_s|\mathrm{d}C_s\bigg)^2 \mathrm{d}A_t\bigg] \\
&\quad - \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}|\delta y_{t-}|^2 \mathrm{d}A_t\bigg].
\end{aligned}$$

We now rearrange the terms and find for $\gamma \in (0,\beta)$ arbitrary that

$$\begin{aligned}
&\beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}|\delta y_{s-}|^2 \mathrm{d}A_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}\langle\delta\eta\rangle_s\bigg] \\
&\quad = \mathbb{E}[\langle\delta\eta\rangle_T] + \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}\bigg(\int_{t-}^T |\delta f_s|\mathrm{d}C_s\bigg)^2 \mathrm{d}A_t\bigg] \leq \bigg(\frac{1}{\beta} + \beta\frac{(1+\gamma\Phi)}{\gamma(\beta-\gamma)}\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.
\end{aligned}$$

[17] Recall the definition of $\widetilde{M}^\Phi_1$, $\widetilde{M}^\Phi_2$ and $\widetilde{M}^\Phi_3$ from Section 3.2.

Here the inequality follows from (5.22), (5.23), (5.39) and (5.40). Hence

$$\beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}|\delta y_{s-}|^2 \mathrm{d}A_s\bigg] + \mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s \mathrm{d}\langle \delta\eta\rangle_s\bigg] \leq \bigg(\frac{1}{\beta} + \beta\mathfrak{g}^{\Phi}(\beta)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.$$

Since

$$\begin{aligned}\frac{\beta}{(1+\beta\Phi)}\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} &= \frac{\beta}{(1+\beta\Phi)}\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_s |\delta y_{t-}|^2 \mathrm{d}A_t\bigg] \\ &= \frac{\beta}{(1+\beta\Phi)}\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-}(1+\beta\Delta A_s) |\delta y_{t-}|^2 \mathrm{d}A_t\bigg] \\ &\leq \beta\mathbb{E}\bigg[\int_0^T \mathcal{E}(\beta A)_{s-} |\delta y_{t-}|^2 \mathrm{d}A_t\bigg],\end{aligned}$$

we thus have

$$\begin{aligned}\min\bigg\{1, \frac{\beta}{(1+\beta\Phi)}\bigg\}\Big(\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}}\Big) &\leq \frac{\beta}{(1+\beta\Phi)}\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \\ &\leq \bigg(\frac{1}{\beta} + \beta\mathfrak{g}^{\Phi}(\beta)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},\end{aligned}$$

which implies

$$\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \max\bigg\{1, \frac{(1+\beta\Phi)}{\beta}\bigg\}\bigg(\frac{1}{\beta} + \beta\mathfrak{g}^{\Phi}(\beta)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}$$

Together with $\|\delta y\|^2_{\mathcal{S}^2_T} \leq \frac{4}{\beta}\big\|\frac{\delta f}{\alpha}\big\|^2_{\mathbb{H}^2_{T,\beta}}$ from Proposition 5.1, we find

$$\begin{aligned}\|\delta y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} &\leq \bigg(\frac{4}{\beta} + \max\bigg\{1, \frac{(1+\beta\Phi)}{\beta}\bigg\}\bigg(\frac{1}{\beta} + \beta\mathfrak{g}^{\Phi}(\beta)\bigg)\bigg)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}} \\ &= \widetilde{M}^{\Phi}_3(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.\end{aligned}$$

Finally, since by Proposition 5.1 $\|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} \leq \mathfrak{f}^{\Phi}(\beta)\big\|\frac{\delta f}{\alpha}\big\|^2_{\mathbb{H}^2_{T,\beta}}$, we find the remaining two bounds

$$\|\delta y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} + \|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \widetilde{M}^{\Phi}_1(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}},$$

$$\|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\beta}} + \|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \widetilde{M}^{\Phi}_2(\beta)\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}.$$

This completes the proof. □

Remark 5.5. We note here that with (5.2), (5.25) and (5.26), we find

$$\|\delta y\|^2_{\mathcal{T}^2_T} \leq \frac{1}{\beta}\bigg\|\frac{\delta f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}} \text{ and } \|y^i\|^2_{\mathcal{T}^2_T} \leq 3\Bigg(\|\xi_T\|^2_{\mathbb{L}^2} + \bigg\|\sup_{u\in[0,T)}\xi^+_u\bigg\|^2_{\mathbb{L}^2} + \frac{1}{\beta}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\Bigg),\ i\in\{1,2\}.$$

Hence, we could substitute the $\mathcal{S}^2_T$–norm $\|\cdot\|_{\mathcal{S}^2_T}$ with the $\mathcal{T}^2_T$–norm[18] $\|\cdot\|_{\mathcal{T}^2_T}$ in both Proposition 5.1 and 5.4. This would involve adjusting the constants $M^{\Phi}_1(\beta)$, $M^{\Phi}_3(\beta)$, $\widetilde{M}^{\Phi}_1(\beta)$ and $\widetilde{M}^{\Phi}_3(\beta)$ in a similar manner as discussed in Remark 3.5 and 3.8.

[18] We refer to Remark 3.5 for the definition of $\|\cdot\|_{\mathcal{T}^2_T}$.

Remark 5.6. Since our existence and uniqueness result is in spirit similar to the one presented in [113], we want to comment here on the constant appearing in their contraction argument. Note that the weights used in [113] are exponential functions and not stochastic exponentials as in our setting. Inequality (3.24) in [113] reads

$$\|\delta\eta\|^2_{\mathcal{H}^2_{T,\beta}} \leq \beta\mathbb{E}\bigg[\int_0^T \mathrm{e}^{\beta A_t}\int_t^T \mathrm{d}\langle\delta\eta\rangle_s \mathrm{d}A_t\bigg] + \mathbb{E}[\langle\delta\eta\rangle_T].$$

This differs from our Inequality (5.30) since $\{t\}$ is not included in the domain of integration of the innermost integral. However, the above inequality is derived by applying Tonelli's theorem to

$$\int_0^T \mathrm{e}^{\beta A_s}\mathrm{d}\langle\delta\eta\rangle_s \leq \beta\int_0^T\int_0^s \mathrm{e}^{\beta A_t}\mathrm{d}A_t\mathrm{d}\langle\delta\eta\rangle_s + \langle\delta\eta\rangle_T,$$

and as we saw in the proofs of this section, changing the order of integration necessitates including $\{t\}$ in the domain of integration of the innermost integral. Consequently, a weighted bound on $F(t-)$ rather than on $F(t)$ (see (5.13)) is required. Hence, an additional term of the form $\mathrm{e}^{(\gamma\vee\beta)\Phi}$ should appear in the contraction constant $M^\Phi(\hat\beta)$ in [113]. It now seems that no closed-form expression for this contraction constant can be derived. Therefore, one naturally has to resort to employing numerical schemes.

For the sake completeness, we close this section with the following weighted bound on the increasing processes (k^r, k^ℓ). We have refrained from including this estimate in Proposition 5.4 as we will not need it in the contraction argument. To state the bound, we define the function $\mathrm{j}:[0,\infty)^3 \longrightarrow [0,\infty)$ by the formula

$$\begin{aligned}\mathrm{j}(\gamma,\beta,\Psi) &\coloneqq \max\bigg\{\frac{\gamma}{\beta-\gamma}, \frac{(\sqrt{1+\gamma\Psi}-1)\sqrt{1+\beta\Psi}}{\sqrt{1+\beta\Psi}-\sqrt{1+\gamma\Psi}}\bigg\}\\ &= \max\bigg\{\frac{\gamma}{\beta-\gamma}, \frac{(\sqrt{1+\gamma\Psi}-1)\sqrt{1+\beta\Psi}\big(\sqrt{1+\beta\Psi}+\sqrt{1+\gamma\Psi}\big)}{(\beta-\gamma)\Psi}\bigg\}.\end{aligned}$$

Here we use the convention $0 \coloneqq 0/0$.

Proposition 5.7. *Let $(\gamma,\beta)\in(0,\infty)^2$ with $\gamma<\beta$. For each $i\in\{1,2\}$,*

$$\begin{aligned}\|k^{r,i}\|^2_{\mathcal{I}^2_{T,\gamma}} + \|k^{\ell,i}\|^2_{\mathcal{I}^2_{T,\gamma}} \leq 3\bigg(&\|\eta^i\|^2_{\mathcal{H}^2_{T,\gamma}} + 108\max\{1,\mathrm{j}(\gamma,\beta,\Phi)\}^2\Big(\|\xi_T\|^2_{\mathbb{L}^2_\gamma} + \|\xi^+_\cdot\mathbf{1}_{\{\cdot<T\}}\|^2_{\mathcal{S}^2_{T,\gamma}}\Big)\\ &+ \big(108\max\{1,\mathrm{j}(\gamma,\beta,\Phi)\}^2+1\big)\frac{(1+\gamma\Phi)}{\beta-\gamma}\bigg\|\frac{f^i}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg).\end{aligned} \tag{5.41}$$

Remark 5.8. To formulate a bound on $\|k^{r,i}\|_{\mathcal{I}^2_{T,\gamma}}$ and $\|k^{\ell,i}\|_{\mathcal{I}^2_{T,\gamma}}$ solely by terms involving f^i and ξ^i, one can combine the previous result with the bound on $\|\eta^i\|_{\mathcal{H}^2_{T,\gamma}}$ from Proposition 5.1.

Proof. We fix $i\in\{1,2\}$ and thus drop the superscript i for ease of notation. Moreover, we suppose without loss of generality that the right side of (5.41) is finite. In a first step, we fix $t\in[0,\infty)$ and apply Itô's formula for optional semi-martingales to the function $f(x,z)=xz$ for $(x,z)\in\mathbb{R}^2$, see [60, Theorem A.3] or [95, page 538], and find

$$\begin{aligned}&\mathcal{E}(\gamma A)^{1/2}_t y_t\\ &= \mathcal{E}(\gamma A)^{1/2}_0 y_0 + \int_{(0,t]} y_{s-}\mathrm{d}\mathcal{E}(\gamma A)^{1/2}_s - \int_{(0,t]}\mathcal{E}(\gamma A)^{1/2}_{s-} f_s \mathrm{d}C_s\end{aligned}$$

$$
\begin{aligned}
&+\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}\mathrm{d}\eta_s-\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}^{1/2}\mathrm{d}k_s^r-\int_{[0,t)}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^\ell\\
&+\sum_{s\in(0,t]}\Big(\mathcal{E}(\gamma A)_s^{1/2}y_s-\mathcal{E}(\gamma A)_{s-}^{1/2}y_{s-}-y_{s-}\Delta(\mathcal{E}(\gamma A)^{1/2})_s-\mathcal{E}(\gamma A)_{s-}^{1/2}\Delta y_s\Big)\\
&+\sum_{s\in[0,t)}\Big(\mathcal{E}(\gamma A)_{s+}^{1/2}y_{s+}-\mathcal{E}(\gamma A)_s^{1/2}y_s-y_s\big(\mathcal{E}(\gamma A)_{s+}^{1/2}-\mathcal{E}(\gamma A)_s^{1/2}\big)-\mathcal{E}(\gamma A)_s^{1/2}(y_{s+}-y_s)\Big)\\
=&\;\mathcal{E}(\gamma A)_0^{1/2}y_0+\int_{(0,t]}y_{s-}\mathrm{d}\mathcal{E}(\gamma A)_s^{1/2}-\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}^{1/2}f_s\mathrm{d}C_s\\
&+\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}\mathrm{d}\eta_s-\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}^{1/2}\mathrm{d}k_s^r-\int_{[0,t)}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^\ell\\
&+\sum_{s\in(0,t]}\Big(y_s(\mathcal{E}(\gamma A)_s^{1/2}-\mathcal{E}(\gamma A)_{s-}^{1/2})-y_{s-}(\mathcal{E}(\gamma A)_s^{1/2}-\mathcal{E}(\gamma A)_{s-}^{1/2})\Big)\\
=&\;\mathcal{E}(\gamma A)_0^{1/2}y_0+\int_{(0,t]}y_{s-}\mathrm{d}\mathcal{E}(\gamma A)_s^{1/2}-\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}^{1/2}f_s\mathrm{d}C_s\\
&+\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}\mathrm{d}\eta_s-\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}^{1/2}\mathrm{d}k_s^r-\int_{[0,t)}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^\ell\\
&+\sum_{s\in(0,t]}\Big((y_s-y_{s-})(\mathcal{E}(\gamma A)_s^{1/2}-\mathcal{E}(\gamma A)_{s-}^{1/2})\Big)\\
=&\;\mathcal{E}(\gamma A)_0^{1/2}y_0+\int_{(0,t]}y_{s-}\mathrm{d}\mathcal{E}(\gamma A)_s^{1/2}-\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}^{1/2}f_s\mathrm{d}C_s\\
&+\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}\mathrm{d}\eta_s-\int_{(0,t]}\mathcal{E}(\gamma A)_{s-}^{1/2}\mathrm{d}k_s^r-\int_{[0,t)}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^\ell\\
&+\sum_{s\in(0,t]}\Big(-f_s\Delta\mathcal{E}(\gamma A)_s^{1/2}\Delta C_s+\Delta\mathcal{E}(\gamma A)_s^{1/2}\Delta\eta_s-\Delta\mathcal{E}(\gamma A)_s^{1/2}\Delta k_s^r\Big)\\
=&\;\mathcal{E}(\gamma A)_0^{1/2}y_0+\int_{(0,t]}y_{s-}\mathrm{d}\mathcal{E}(\gamma A)_s^{1/2}-\int_{(0,t]}\mathcal{E}(\gamma A)_s^{1/2}f_s\mathrm{d}C_s\\
&+\int_{(0,t]}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}\eta_s-\int_{(0,t]}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^r-\int_{[0,t)}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^\ell,\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.}
\end{aligned}\tag{5.42}
$$

Here we used the fact that $\mathcal{E}(\gamma A)^{1/2}$ is predictable and non-decreasing, thus locally bounded. We now analyze the terms in (5.42) one by one. First, Lemma C.1. (iii) implies that $\mathcal{E}(\gamma A)^{1/2}=\mathcal{E}(D^\gamma)$, and $\mathcal{E}(\beta A)^{1/2}=\mathcal{E}(D^\beta)$, where $D^\gamma=(D^\gamma)_{t\in[0,\infty)}$ and $D^\beta=(D^\beta)_{t\in[0,\infty)}$ are the predictable processes satisfying

$$
D_t^\gamma=\frac{\gamma}{2}A_t^c+\sum_{s\in(0,t]}\big(\sqrt{1+\gamma\Delta A_s}-1\big),\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.},
$$

and

$$
D_t^\beta=\frac{\beta}{2}A_t^c+\sum_{s\in(0,t]}\big(\sqrt{1+\beta\Delta A_s}-1\big),\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.}
$$

Recall that A^c denotes the continuous part of the process A. We bound the second term on the right of (5.42) as follows

$$
\begin{aligned}
\int_{(0,t]}y_{s-}\mathrm{d}\mathcal{E}(\gamma A)_s^{1/2}&=\int_{(0,t]}\mathcal{E}(\beta A)_{s-}^{1/2}y_{s-}\frac{1}{\mathcal{E}(\beta A)_{s-}^{1/2}}\mathrm{d}\mathcal{E}(\gamma A)_s^{1/2}\\
&\le\sup_{s\in(0,t)}\big\{\mathcal{E}(\beta A)_s^{1/2}|y_s|\big\}\int_{(0,t]}\frac{1}{\mathcal{E}(\beta A)_{s-}^{1/2}}\mathrm{d}\mathcal{E}(\gamma A)_s^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{s\in(0,t)}\big\{\mathcal{E}(\beta A)_s^{1/2}|y_s|\big\}\int_{(0,t]}\frac{1}{\mathcal{E}(D^\beta)_{s-}}\mathrm{d}\mathcal{E}(D^\gamma)_s\\
&=\sup_{s\in(0,t)}\big\{\mathcal{E}(\beta A)_s^{1/2}|y_s|\big\}\int_{(0,t]}\frac{1}{\mathcal{E}(D^\beta)_{s-}}\mathcal{E}(D^\gamma)_{s-}\mathrm{d}D^\gamma_s,\ \mathbb{P}\text{–a.s.}
\end{aligned}\tag{5.43}
$$

By Lemma C.1. (ii), we can write $\mathcal{E}(D^\beta)^{-1}\mathcal{E}(D^\gamma)=\mathcal{E}(\widehat{D}^{\gamma,\beta})$, where $\widehat{D}^{\gamma,\beta}=(\widehat{D}^{\gamma,\beta}_t)_{t\in[0,\infty)}$ is the predictable process satisfying

$$
\begin{aligned}
\widehat{D}^{\gamma,\beta}_t&=D^{\gamma,c}_t-D^{\beta,c}_t+\sum_{s\in(0,t]}\frac{\Delta D^\gamma_s-\Delta D^\beta_s}{1+\Delta D^\beta_s}\\
&=-\frac{(\beta-\gamma)}{2}A^c_t-\sum_{s\in(0,t]}\frac{\sqrt{1+\beta\Delta A_s}-\sqrt{1+\gamma\Delta A_s}}{\sqrt{1+\beta\Delta A_s}},\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.}
\end{aligned}
$$

Note that $\Delta\widehat{D}^{\gamma,\beta}\geq-1$ although $\widehat{D}^{\gamma,\beta}$ is non-increasing. Then

$$
\begin{aligned}
\int_{(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\mathrm{d}D^\gamma_s&=\int_{(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\mathrm{d}D^{\gamma,c}_s+\sum_{s\in(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\Delta D^\gamma_s\\
&=\frac{\gamma}{2}\int_{(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\mathrm{d}A^c_s+\sum_{s\in(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\big(\sqrt{1+\gamma\Delta A_s}-1\big)\\
&=\frac{\gamma}{\beta-\gamma}\int_{(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\frac{(\beta-\gamma)}{2}\mathrm{d}A^c_s\\
&\quad+\sum_{s\in(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\frac{(\sqrt{1+\gamma\Delta A_s}-1)\sqrt{1+\beta\Delta A_s}}{\sqrt{1+\beta\Delta A_s}-\sqrt{1+\gamma\Delta A_s}}\Delta(-\widehat{D}^{\gamma,\beta})_s\\
&=\frac{\gamma}{\beta-\gamma}\int_{(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\mathrm{d}(-\widehat{D}^{\gamma,c})_s\\
&\quad+\sum_{s\in(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\frac{(\sqrt{1+\gamma\Delta A_s}-1)\sqrt{1+\beta\Delta A_s}}{\sqrt{1+\beta\Delta A_s}-\sqrt{1+\gamma\Delta A_s}}\Delta(-\widehat{D}^{\gamma,\beta})_s\\
&\leq\mathfrak{l}(\gamma,\beta)\int_{(0,t]}\mathcal{E}(\widehat{D}^{\gamma,\beta})_{s-}\mathrm{d}(-\widehat{D}^{\gamma})_s\\
&=\mathfrak{l}(\gamma,\beta)\bigg(1-\frac{\mathcal{E}(D^\gamma)_t}{\mathcal{E}(D^\beta)_t}\bigg)\leq\mathfrak{l}(\gamma,\beta),\ \mathbb{P}\text{–a.s.},
\end{aligned}\tag{5.44}
$$

where the last inequality follows from $1\leq\mathcal{E}(D^\gamma)\leq\mathcal{E}(D^\beta)$, and where

$$
\mathfrak{l}(\gamma,\beta)\coloneqq\max\bigg\{\frac{\gamma}{\beta-\gamma},\sup_{s\in[0,\infty)}\frac{(\sqrt{1+\gamma\Delta A_s}-1)\sqrt{1+\beta\Delta A_s}}{\sqrt{1+\beta\Delta A_s}-\sqrt{1+\gamma\Delta A_s}}\bigg\}.
$$

Let $[0,\infty)\ni x\longmapsto\mathfrak{k}(\gamma,\beta,x)\in[0,\infty)$ be defined by

$$
\mathfrak{k}(\gamma,\beta,x)\coloneqq\frac{(\sqrt{1+\gamma x}-1)\sqrt{1+\beta x}}{\sqrt{1+\beta x}-\sqrt{1+\gamma x}}=\frac{(\sqrt{1+\gamma x}-1)\sqrt{1+\beta x}(\sqrt{1+\beta x}+\sqrt{1+\gamma x})}{(\beta-\gamma)x}.
$$

Then

$$
\begin{aligned}
\frac{\partial}{\partial x}\mathfrak{k}(\gamma,\beta,x)&=\frac{\beta\gamma x^2(\sqrt{1+\beta x}+\sqrt{1+\gamma x})}{2x^2(\beta-\gamma)\sqrt{1+\beta x}\sqrt{1+\gamma x}}\\
&\quad+\frac{x(\beta+\gamma-\beta\sqrt{1+\gamma x}-\gamma\sqrt{1+\beta x})+2(\sqrt{1+\beta x}-1)(\sqrt{1+\gamma x}-1)}{2x^2(\beta-\gamma)\sqrt{1+\beta x}\sqrt{1+\gamma x}},
\end{aligned}
$$

and by using $\sqrt{1+x} \leq 1 + x/2$, for $x \geq -1$, we deduce $\frac{\partial}{\partial x}\mathfrak{k}(\gamma,\beta,x) \geq 0$. This implies that $x \longmapsto \mathfrak{k}(\gamma,\beta,x)$ is non-decreasing, and therefore

$$\mathfrak{l}(\gamma,\beta) \leq \max\left\{\frac{\gamma}{\beta-\gamma}, \frac{(\sqrt{1+\gamma\Phi}-1)\sqrt{1+\beta\Phi}}{\sqrt{1+\beta\Phi}-\sqrt{1+\gamma\Phi}}\right\} = \mathfrak{j}(\gamma,\beta,\Phi),\ \mathbb{P}\text{–a.s.} \tag{5.45}$$

We deduce from (5.45), (5.44) and (5.43) that

$$\int_{(0,t]} y_{s-}\mathrm{d}\mathcal{E}(\gamma A)_s^{1/2} \leq \mathfrak{j}(\gamma,\beta,\Phi)\sup_{s\in(0,t)}\big\{\mathcal{E}(\beta A)_s^{1/2}|y_s|\big\},\ \mathbb{P}\text{–a.s.} \tag{5.46}$$

We turn to the third term on the right of (5.42). Cauchy–Schwarz's inequality and Lemma C.1. (ii) yields

$$\begin{aligned}
&\bigg(\int_{(0,t]}\mathcal{E}(\gamma A)_s^{1/2}|f_s|\mathrm{d}C_s\bigg)^2 && (5.47)\\
&\quad\leq \bigg(\int_{(0,t]}\frac{1}{\mathcal{E}(\beta A)_s}\mathcal{E}(\gamma A)_s\mathrm{d}A_s\bigg)\bigg(\int_{(0,t]}\mathcal{E}(\beta A)_s\frac{|f_s|^2}{\alpha_s^2}\mathrm{d}C_s\bigg)\\
&\quad= \bigg(\int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s\mathrm{d}A_s\bigg)\bigg(\int_{(0,t]}\mathcal{E}(\beta A)_s\frac{|f_s|^2}{\alpha_s^2}\mathrm{d}C_s\bigg),\ \mathbb{P}\text{–a.s.}, && (5.48)
\end{aligned}$$

where $\widehat{A}^{\gamma,\beta}$ is the predictable process satisfying

$$\widehat{A}^{\gamma,\beta}_t = -(\beta-\gamma)A^c_t - \sum_{s\in(0,t]}(\beta-\gamma)\frac{\Delta A_s}{1+\beta\Delta A_s},\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.}$$

We now explicitly bound the integral

$$\begin{aligned}
\int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s\mathrm{d}A_s &= \int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s\mathrm{d}A^c_s + \sum_{s\in(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s\Delta A_s\\
&= \frac{1}{(\gamma-\beta)}\int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s(1+\beta\Delta A_s)(\gamma-\beta)\mathrm{d}A^c_s\\
&\quad + \frac{1}{(\gamma-\beta)}\sum_{s\in(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s(1+\beta\Delta A_s)(\gamma-\beta)\frac{\Delta A_s}{1+\beta\Delta A_s}\\
&= \frac{1}{(\gamma-\beta)}\int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s(1+\beta\Delta A_s)\mathrm{d}\big(\widehat{A}^{\gamma,\beta}\big)^c_s\\
&\quad + \frac{1}{(\gamma-\beta)}\sum_{s\in(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s(1+\beta\Delta A_s)\Delta\widehat{A}^{\gamma,\beta}_s\\
&= \frac{1}{(\gamma-\beta)}\int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_s(1+\beta\Delta A_s)\mathrm{d}\widehat{A}^{\gamma,\beta}_s\\
&= \frac{1}{(\gamma-\beta)}\int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_{s-}(1+\Delta\widehat{A}^{\gamma,\beta}_s)(1+\beta\Delta A_s)\mathrm{d}\widehat{A}^{\gamma,\beta}_s\\
&= \frac{1}{(\gamma-\beta)}\int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_{s-}(1+\gamma\Delta A_s)\mathrm{d}\widehat{A}^{\gamma,\beta}_s\\
&\leq \frac{(1+\gamma\Phi)}{(\gamma-\beta)}\int_{(0,t]}\mathcal{E}(\widehat{A}^{\gamma,\beta})_{s-}\mathrm{d}\widehat{A}^{\gamma,\beta}_s\\
&= \frac{(1+\gamma\Phi)}{(\gamma-\beta)}\big(\mathcal{E}(\widehat{A}^{\gamma,\beta})_t-1\big)\\
&= \frac{(1+\gamma\Phi)}{(\beta-\gamma)}\bigg(1-\mathcal{E}(\gamma A)_t\frac{1}{\mathcal{E}(\beta A)_t}\bigg) \leq \frac{(1+\gamma\Phi)}{(\beta-\gamma)},\ \mathbb{P}\text{–a.s.} \qquad (5.49)
\end{aligned}$$

Here the third-to-last line follows from $\mathcal{E}(\widehat{A}^{\gamma,\beta}) = \mathcal{E}(\gamma A)/\mathcal{E}(\beta A) > 0$ and from the fact that $\widehat{A}^{\gamma,\beta}/(\gamma-\beta)$ is non-decreasing since $\gamma-\beta<0$ and since $\widehat{A}^{\gamma,\beta}$ is non-increasing, and the last inequality follows from $0<\mathcal{E}(\gamma A)\leq\mathcal{E}(\beta A)$. Combining (5.49) and (5.47) yields

$$\bigg(\int_{(0,t]}\mathcal{E}(\gamma A)_s^{1/2}|f_s|\mathrm{d}C_s\bigg)^2 \leq \frac{(1+\gamma\Phi)}{(\beta-\gamma)}\bigg(\int_{(0,t]}\mathcal{E}(\beta A)_s\frac{|f_s|^2}{\alpha_s^2}\mathrm{d}C_s\bigg),\ \mathbb{P}\text{–a.s.} \tag{5.50}$$

Combining (5.46) and (5.42), then rearranging the terms and applying $(a+b+c)^2\leq 3(a^2+b^2+c^2)$ yields

$$\begin{aligned}
&\bigg(\int_{(0,t]}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^r+\int_{[0,t)}\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^\ell\bigg)^2\\
&\leq 3\Bigg(9\max\{1,\mathrm{j}(\gamma,\beta,\Phi)\}^2\sup_{s\in[0,t]}\{\mathcal{E}(\gamma A)|y_t|^2\}\\
&\quad+\bigg(\int_0^t\mathcal{E}(\gamma A)_s^{1/2}|f_s|\mathrm{d}C_s\bigg)^2+\bigg(\int_0^t\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}\eta_s\bigg)^2\Bigg),\ \mathbb{P}\text{–a.s.}
\end{aligned} \tag{5.51}$$

We now plug Equation (5.50) into Equation (5.51), let $t\uparrow\uparrow\infty$ in the resulting inequality and then take the expectation and find[19]

$$\begin{aligned}
\|k^r\|^2_{\mathcal{I}^2_{T,\gamma}}+\|k^\ell\|^2_{\mathcal{I}^2_{T,\gamma}} &\leq \mathbb{E}\Bigg[\bigg(\int_0^T\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^r+\int_0^T\mathcal{E}(\gamma A)_s^{1/2}\mathrm{d}k_s^\ell\bigg)^2\Bigg]\\
&\leq 3\bigg(9\max\{1,\mathrm{j}(\gamma,\beta,\Phi)\}^2\|y\|^2_{\mathcal{S}^2_{T,\gamma}}+\|\eta\|^2_{\mathcal{H}^2_{T,\gamma}}+\frac{(1+\gamma\Phi)}{(\beta-\gamma)}\bigg\|\frac{f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg).
\end{aligned}$$

It remains to bound $\|y\|^2_{\mathcal{S}^2_{T,\gamma}}$ by terms involving f and ξ. Lemma 5.2 implies

$$\begin{aligned}
&\mathcal{E}(\gamma A)_S^{1/2}|y_S|\\
&\leq\mathbb{E}\bigg[|\mathcal{E}(\gamma A)_T^{1/2}\xi_T|+\sup_{u\in[0,\infty]}|\mathcal{E}(\gamma A)_u^{1/2}\xi_u^+\mathbf{1}_{\{u<T\}}|+\int_0^T\mathcal{E}(\gamma A)_u^{1/2}|f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg]\\
&\leq\sqrt{3}\mathbb{E}\Bigg[\sqrt{|\mathcal{E}(\gamma A)_T^{1/2}\xi_T|^2+\sup_{u\in[0,\infty]}|\mathcal{E}(\gamma A)_u^{1/2}\xi_u^+\mathbf{1}_{\{u<T\}}|^2+\bigg(\int_0^T\mathcal{E}(\gamma A)_u^{1/2}|f_u|\mathrm{d}C_u\bigg)^2}\Bigg|\mathcal{G}_S\Bigg],
\end{aligned}$$

$\mathbb{P}$–a.s., for $S\in\mathcal{T}_{0,T}$, and Doob's $\mathbb{L}^2$-inequality for martingales leads to

$$\|y\|^2_{\mathcal{S}^2_{T,\gamma}}\leq 12\bigg(\|\xi_T\|^2_{\mathbb{L}^2_\gamma}+\|\xi^+_\cdot\mathbf{1}_{\{\cdot<T\}}\|^2_{\mathcal{S}^2_{T,\gamma}}+\frac{(1+\gamma\Phi)}{(\beta-\gamma)}\bigg\|\frac{f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg).$$

This yields

$$\begin{aligned}
\|k^r\|^2_{\mathcal{I}^2_{T,\gamma}}+\|k^\ell\|^2_{\mathcal{I}^2_{T,\gamma}}\leq 3\bigg(&\|\eta\|^2_{\mathcal{H}^2_{T,\gamma}}+108\max\{1,\mathrm{j}(\gamma,\beta,\Phi)\}^2\Big(\|\xi_T\|^2_{\mathbb{L}^2_\gamma}+\|\xi^+_\cdot\mathbf{1}_{\{\cdot<T\}}\|^2_{\mathcal{S}^2_{T,\gamma}}\Big)\\
&+\big(1+108\max\{1,\mathrm{j}(\gamma,\beta,\Phi)\}^2\big)\frac{(1+\gamma\Phi)}{\beta-\gamma}\bigg\|\frac{f}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\beta}}\bigg),
\end{aligned}$$

which completes the proof. □

[19] We refer to Section 2.5 for the conventions we agreed upon when writing an integral of the form $\int_0^T$.

5.1 Proof of Lemma 5.2 and Lemma 5.3

Proof of Lemma 5.2. Let us first discuss how to deduce (5.3) from (5.2) and (5.4). Suppose that $V = (V_t)_{t\in[0,\infty]}$ and $V' = (V'_t)_{t\in[0,\infty]}$ are two product-measurable processes whose $\mathbb{P}$–almost all paths admit limits from the left on $(0,\infty]$, such that $\mathbb{E}[\sup_{u\in[0,\infty]}|V_u|]+\mathbb{E}[\sup_{u\in[0,\infty]}|V'_u|] < \infty$ and $\mathbb{E}[V_t|\mathcal{G}_t] \leq \mathbb{E}[V'_t|\mathcal{G}_t]$, $\mathbb{P}$–a.s., $t \in [0,\infty]$. An application of Lemma C.5 yields $\mathbb{E}[V_{t-}|\mathcal{G}_{t-}] \leq \mathbb{E}[V'_{t-}|\mathcal{G}_{t-}]$, $\mathbb{P}$–a.s., $t \in [0,\infty]$. Here we use the conventions $V_{0-} := 0$, $V'_{0-} := 0$ and $\mathcal{G}_{0-} := \mathcal{G}_0$. Let us denote by $W = (W_t)_{t\in[0,\infty]}$ and $W' = (W'_t)_{t\in[0,\infty]}$ the processes $W_t := V_{t-}$ and $W'_t := V'_{t-}$, respectively. We write ${}^pW$ and ${}^pW'$ for the predictable projections of W and W'. Since the predictable projection of a process with $\mathbb{P}$–a.s. left-continuous paths also has $\mathbb{P}$–a.s. left-continuous paths (see [41, Theorem VI.47 and Remark VI.50.(f)]), we find from

$$ {}^pW_t = \mathbb{E}[V_{t-}|\mathcal{G}_{t-}] \leq \mathbb{E}[V'_{t-}|\mathcal{G}_{t-}] = {}^pW'_t,\ \mathbb{P}\text{–a.s.},\ t \in [0,\infty], $$

that ${}^pW_t \leq {}^pW'_t$, $t \in [0,\infty]$, $\mathbb{P}$–almost surely. Sampling ${}^pW$ and ${}^pW'$ at a predictable stopping time $S \in \mathcal{T}_{0,\infty}$ thus yields

$$ \mathbb{E}[V_{S-}|\mathcal{G}_{S-}] = \mathbb{E}[W_S|\mathcal{G}_{S-}] = {}^pW_S \leq {}^pW'_S = \mathbb{E}[W'_S|\mathcal{G}_{S-}] = \mathbb{E}[V'_{S-}|\mathcal{G}_{S-}],\ \mathbb{P}\text{–a.s.} $$

We now turn to the proof of (5.2) and (5.4). Note first that

$$ \begin{aligned} y^1_S &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S}\mathbb{E}\bigg[\xi_{\tau\wedge T} + \int_S^{\tau\wedge T} f^2_s \mathrm{d}C_s + \int_S^{\tau\wedge T}\delta f_u \mathrm{d}C_u\bigg|\mathcal{G}_S\bigg] \\ &\leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S}\mathbb{E}\bigg[\xi_{\tau\wedge T} + \int_S^{\tau\wedge T} f^2_s \mathrm{d}C_s\bigg|\mathcal{G}_S\bigg] + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S}\mathbb{E}\bigg[\int_S^{\tau\wedge T}|\delta f_u| \mathrm{d}C_u\bigg|\mathcal{G}_S\bigg] \\ &\leq y^2_S + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S}\mathbb{E}\bigg[\int_S^{\tau\wedge T}|\delta f_u| \mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}, \end{aligned} $$

which, since $y^i_S \in \mathbb{L}^2$ by Lemma 4.2 and Lemma 4.3, leads by symmetry to

$$ \begin{aligned} |\delta y_S| = |y^1_S - y^2_S| &\leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S}\mathbb{E}\bigg[\int_S^{\tau\wedge T}|\delta f_u| \mathrm{d}C_u\bigg|\mathcal{G}_S\bigg] \\ &\leq \mathbb{E}\bigg[\int_S^{T}|\delta f_u| \mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}. \end{aligned} $$

We now have $|\delta y_S| \leq M_S$, $\mathbb{P}$–a.s., $S \in \mathcal{T}_{0,\infty}$, where $M = (M_t)_{t\in[0,\infty]}$ is the martingale satisfying

$$ M_S = \mathbb{E}\bigg[\int_0^{T}|\delta f_u| \mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}. $$

By Proposition C.3, we find

$$ \mathbb{E}\bigg[\sup_{s\in[0,\infty]}|\delta y|^2\bigg] \leq \mathbb{E}\bigg[\sup_{s\in[0,\infty]}|M_s|^2\bigg] \leq 4\mathbb{E}[|M_\infty|^2] = 4\mathbb{E}\bigg[\bigg(\int_0^T|\delta f_u|\mathrm{d}C_u\bigg)^2\bigg]. $$

We turn to the inequalities containing only y^i for $i \in \{1,2\}$, and drop the superscripts from now on due to the symmetry of the problem. As in the proof of Lemma 4.2, we find

$$ \begin{aligned} &-\mathbb{E}\big[|\xi_T|\big|\mathcal{G}_S\big] - \mathbb{E}\bigg[\int_S^T|f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg] \\ &\quad\leq y_S \leq \mathbb{E}\bigg[\sup_{u\in[S,\infty]}|\xi^+_{u\wedge T}| + \int_S^T|f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}, \end{aligned} $$

and thus

$$|y_S| \leq \mathbb{E}\bigg[|\xi_T| + \sup_{u\in[S,\infty]} |\xi^+_u \mathbf{1}_{\{u<T\}}| + \int_S^T |f_u| \mathrm{d}C_u \bigg| \mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}.$$

By abusing notation, let $M = (M_t)_{t\in[0,\infty]}$ now be the martingale satisfying

$$M_S = \mathbb{E}\bigg[|\xi_T| + \sup_{u\in[0,T)} |\xi^+_u| + \int_0^T |f_u| \mathrm{d}C_u \bigg| \mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}.$$

We derive similarly to before that

$$\begin{aligned}
\mathbb{E}\bigg[\sup_{s\in[0,\infty]} |y_s|^2\bigg] \leq \mathbb{E}\bigg[\sup_{s\in[0,\infty]} |M_s|^2\bigg] &\leq 4\mathbb{E}[|M_\infty|^2] \\
&= 4\mathbb{E}\bigg[\bigg(|\xi_T| + \sup_{u\in[0,T)} |\xi^+_u| + \int_0^T |f_u| \mathrm{d}C_u\bigg)^2\bigg] \\
&\leq 12\mathbb{E}\bigg[|\xi_T|^2 + \sup_{u\in[0,T)} |\xi^+_u|^2 + \bigg(\int_0^T |f_u| \mathrm{d}C_u\bigg)^2\bigg],
\end{aligned}$$

where in the last inequality we used $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. This completes the proof. □

Proof of Lemma 5.3. As in the proof of Lemma 5.2, it suffices to prove (5.5) and (5.7). We start with (5.5). Although we fix $(t,t') \in [0,\infty)$ with $t \leq t'$ to ease the notation, the equalities and inequalities that follow should be read as holding, up to a $\mathbb{P}$–null set, for each pair $(t,t') \in [0,\infty)$ with $t \leq t'$ unless stated otherwise. Note that the processes under consideration are all constant after time T apart from C and f. From (5.1a), we see that δy satisfies

$$\delta y_t = \delta y_0 - \int_0^t \delta f_s \mathbf{1}_{[0,T]}(s) \mathrm{d}C_s + \delta\eta_t - \delta k^r_t - \delta k^\ell_{t-},\ t \in [0,\infty],\ \mathbb{P}\text{–a.s.}$$

To ease the notation and without loss of generality, we suppose that $\delta f_s = \delta f_s \mathbf{1}_{[0,T]}(s)$ and $\xi_s = \xi_{s\wedge T}$. By an application of the Gal'chouk–Itô–Lenglart formula (see [60, Theorem A.3 and Corollary A.2] or [59, Theorem 8.2]) on $(t,t']$, we find the (optional) semimartingale decomposition of $|\delta y|^2$ to be

$$\begin{aligned}
|\delta y_t|^2 = |\delta y_{t'}|^2 &+ 2\int_{(t,t']} \delta y_{s-} \delta f_s \mathrm{d}C_s + 2\int_{(t,t']} \delta y_{s-} \mathrm{d}(\delta k^r)_s \\
&- 2\int_{(t,t']} \delta y_{s-} \mathrm{d}(\delta\eta)_s - \int_{(t,t']} \mathrm{d}[\delta\eta^c]_s - \sum_{s\in(t,t']} (\delta y_s - \delta y_{s-})^2 \\
&+ 2\int_{[t,t')} \delta y_s \mathrm{d}(\delta k^\ell)_s - \sum_{s\in[t,t')} (\delta y_{s+} - \delta y_s)^2.
\end{aligned} \tag{5.52}$$

We decided here to write the integral bounds more clearly, as it is crucial whether one takes left-open or right-open intervals. Let us analyse the terms in the above decomposition one by one. First, note that the last term $\sum_{s\in[t,t')} (\delta y_{s+} - \delta y_s)^2$ is non-negative, and by adding zero to the last term, we find

$$\begin{aligned}
&- \sum_{s\in(t,t']} (\delta y_s - \delta y_{s-})^2 \\
&= - \sum_{s\in(t,t']} (\delta y_s - \delta y_{s-})^2 - 2\sum_{s\in(t,t']} (\delta y_s - \delta y_{s-})(\delta f_s \Delta C_s) - \sum_{s\in(t,t']} (\delta f_s \Delta C_s)^2
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{s\in(t,t']}(\delta y_s-\delta y_{s-})(\delta f_s\Delta C_s)+\sum_{s\in(t,t']}(\delta f_s\Delta C_s)^2\\
&=-\sum_{s\in(t,t']}\big(\delta y_s-\delta y_{s-}+\delta f_s\Delta C_s\big)^2+2\sum_{s\in(t,t']}(\delta y_s-\delta y_{s-})(\delta f_s\Delta C_s)+\sum_{s\in(t,t']}(\delta f_s\Delta C_s)^2\\
&=-\sum_{s\in(t,t']}\big(\Delta\delta\eta_s-\Delta\delta k^r_s\big)^2+2\sum_{s\in(t,t']}(\delta y_s-\delta y_{s-})(\delta f_s\Delta C_s)+\sum_{s\in(t,t']}(\delta f_s\Delta C_s)^2\\
&=-\sum_{s\in(t,t']}\big(\Delta\delta\eta_s-\Delta\delta k^r_s\big)^2+2\int_{(t,t']}(\delta y_s-\delta y_{s-})\delta f_s\mathrm{d}C_s+\sum_{s\in(t,t']}(\delta f_s\Delta C_s)^2. \qquad (5.53)
\end{aligned}
$$

By substituting this back into (5.52), rearranging the terms, and using $\delta y_{s+}-\delta y_s=-(\delta k^\ell_s-\delta k^\ell_{s-})$ and

$$
\begin{aligned}
\int_{(t,t']}\mathrm{d}[\delta\eta]_s-2\int_{(t,t']}\mathrm{d}[\delta\eta,\delta k^r]_s+\int_{(t,t']}\mathrm{d}[\delta k^r]_s&=\int_{(t,t']}\mathrm{d}[\delta\eta-\delta k^r]_s\\
&=\int_{(t,t']}\mathrm{d}[\delta\eta^c]_s+\sum_{s\in(t,t']}\big(\Delta\delta\eta_s-\Delta\delta k^r_s\big)^2,
\end{aligned}
$$

we find

$$
\begin{aligned}
|\delta y_t|^2&+\int_{(t,t']}\mathrm{d}[\delta\eta]_s+\int_{(t,t']}\mathrm{d}[\delta k^r]_s+\int_{[t,t')}\mathrm{d}[\delta k^\ell]_s-2\int_{(t,t']}\mathrm{d}[\delta\eta,\delta k^r]_s\\
&=|\delta y_{t'}|^2+2\int_{(t,t']}\delta y_s\delta f_s\mathrm{d}C_s-2\int_{(t,t']}\delta y_{s-}\mathrm{d}(\delta\eta)_s\\
&\quad+\sum_{s\in(t,t']}(\delta f_s\Delta C_s)^2+2\int_{(t,t']}\delta y_{s-}\mathrm{d}(\delta k^r)_s+2\int_{[t,t')}\delta y_s\mathrm{d}(\delta k^\ell)_s. \qquad (5.54)
\end{aligned}
$$

The Skorokhod condition (5.1d) implies

$$
\int_{(t,t']}\delta y_{s-}\mathrm{d}\delta k^r_s\le 0,\ \text{and}\ \int_{[t,t')}\delta y_s\mathrm{d}\delta k^\ell_s\le 0, \qquad (5.55)
$$

which then yields

$$
\begin{aligned}
|\delta y_t|^2&+\int_{(t,t']}\mathrm{d}[\delta\eta]_s+\int_{(t,t']}\mathrm{d}[\delta k^r]_s+\int_{[t,t')}\mathrm{d}[\delta k^\ell]_s-2\int_{(t,t']}\mathrm{d}[\delta\eta,\delta k^r]_s\\
&\le|\delta y_{t'}|^2+2\int_{(t,t']}\delta y_s\delta f_s\mathrm{d}C_s-2\int_{(t,t']}\delta y_{s-}\mathrm{d}(\delta\eta)_s+\sum_{s\in(t,t']}(\delta f_s\Delta C_s)^2. \qquad (5.56)
\end{aligned}
$$

Note that $\int_{(0,\cdot]}\delta y_{s-}\mathrm{d}(\delta\eta)_s$ and $\int_{(0,\cdot]}\mathrm{d}[\delta\eta,\delta k^r]$ are uniformly integrable martingales since $\delta y\in\mathcal{S}^2_T$ by Lemma 5.2, $\delta\eta\in\mathcal{H}^2_T$ by assumption, $|\delta k^r|\le k^{r,1}_T+k^{r,2}_T\in\mathbb{L}^2$, and $[\delta\eta,\delta k^r]=\int_{(0,\cdot]}\Delta(\delta k^r)_s\mathrm{d}(\delta\eta)_s$ by [77, Proposition I.4.49]. Indeed, since $(k^{r,1},k^{r,2})\in\mathcal{I}^2_T\times\mathcal{I}^2_T$ and $\delta\eta\in\mathcal{H}^2_T$, and thus

$$
\begin{aligned}
\sqrt{\Big\langle\int_{(0,\cdot]}\Delta(\delta k^r)_s\mathrm{d}\delta\eta_s\Big\rangle_{\infty-}}&=\sqrt{\int_{(0,\infty)}\big(\Delta(\delta k^r)_s\big)^2\mathrm{d}\langle\delta\eta\rangle_s}\\
&\le\sqrt{2\Big(\big(k^{r,1}_T\big)^2+\big(k^{r,2}_T\big)^2\Big)\int_{(0,\infty)}\mathrm{d}\langle\delta\eta\rangle_s}\\
&\le\frac{1}{\sqrt{2}}\bigg(\big(k^{r,1}_T\big)^2+\big(k^{r,2}_T\big)^2+\int_{(0,\infty)}\mathrm{d}\langle\delta\eta\rangle_s\bigg),
\end{aligned}
$$

with an integrable right-hand side, the Burkholder–Davis–Gundy inequality implies that $\int_{(0,\cdot]} \Delta(\delta k^r)_s \mathrm{d}(\delta\eta)_s$ is bounded by an integrable random variable, and thus it is a uniformly integrable martingale. A similar argument, together with Lemma 5.2, implies that $\int_{(0,\cdot]} \delta y_{s-} \mathrm{d}(\delta\eta)_s$ is a uniformly integrable martingale. Since by Lemma 5.2

$$|\delta y_{t'}| \leq \mathbb{E}\bigg[\int_{t'}^T |\delta f_s| \mathrm{d}C_s \bigg| \mathcal{G}_{t'}\bigg] = \mathbb{E}\bigg[\int_0^T |\delta f_s| \mathrm{d}C_s \bigg| \mathcal{G}_{t'}\bigg] - \int_0^{t'\wedge T} |\delta f_s| \mathrm{d}C_s,$$

and since the right-hand side converges $\mathbb{P}$–a.s. to zero as t' tends to infinity, we deduce from (5.56) that

$$\begin{aligned} &|\delta y_t|^2 + \int_{(t,\infty)} \mathrm{d}[\delta\eta]_s + \int_{(t,\infty)} \mathrm{d}[\delta k^r]_s + \int_{[t,\infty)} \mathrm{d}[\delta k^\ell]_s - 2\int_{(t,\infty)} \mathrm{d}[\delta\eta, \delta k^r]_s \\ &\leq 2\int_{(t,\infty)} \delta y_s \delta f_s \mathrm{d}C_s - 2\int_{(t,\infty)} \delta y_{s-} \mathrm{d}(\delta\eta)_s + \sum_{s\in(t,\infty)} (\delta f_s \Delta C_s)^2,\ t \in [0,\infty],\ \mathbb{P}\text{–a.s.} \end{aligned} \tag{5.57}$$

Since

$$\sum_{s\in(t,\infty)} (\delta f_s \Delta C_s)^2 = \int_{(t,\infty)} |\delta f_s|^2 \mathrm{d}[C]_s,\ t \in [0,\infty],\ \mathbb{P}\text{–a.s.},$$

we find for any stopping time $S \in \mathcal{T}_{0,\infty}$ and by taking conditional expectation in (5.57) that

$$\begin{aligned} |\delta y_S|^2 &+ \mathbb{E}\bigg[\int_S^T \mathrm{d}[\delta\eta]_s \bigg| \mathcal{G}_S\bigg] + \mathbb{E}\bigg[\int_S^T \mathrm{d}[\delta k^r]_s \bigg| \mathcal{G}_S\bigg] + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[\delta k^\ell]_s \bigg| \mathcal{G}_S\bigg] \\ &\leq 2\mathbb{E}\bigg[\int_S^T \delta y_s \delta f_s \mathrm{d}C_s \bigg| \mathcal{G}_S\bigg] + \mathbb{E}\bigg[\int_S^T (\delta f_s)^2 \mathrm{d}[C]_s \bigg| \mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.} \end{aligned}$$

Analogously, in case $S \in \mathcal{T}^p_{0,\infty}$, we find by taking left-hand limits in (5.57) that

$$\begin{aligned} |\delta y_{S-}|^2 &+ \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[\delta\eta]_s \bigg| \mathcal{G}_{S-}\bigg] + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[\delta k^r]_s \bigg| \mathcal{G}_{S-}\bigg] + \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[\delta k^\ell]_s \bigg| \mathcal{G}_{S-}\bigg] \\ &\leq 2\mathbb{E}\bigg[\int_{S-}^T \delta y_s \delta f_s \mathrm{d}C_s \bigg| \mathcal{G}_{S-}\bigg] + \mathbb{E}\bigg[\int_{S-}^T (\delta f_s)^2 \mathrm{d}[C]_s \bigg| \mathcal{G}_{S-}\bigg],\ \mathbb{P}\text{–a.s.} \end{aligned}$$

This yields (5.5) and (5.6) since

$$\mathbb{E}\bigg[\int_S^T \mathrm{d}\langle\delta\eta\rangle_u \bigg| \mathcal{G}_S\bigg] = \mathbb{E}\bigg[\int_S^T \mathrm{d}[\delta\eta]_u \bigg| \mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},$$

and

$$\mathbb{E}\bigg[\int_{S-}^T \mathrm{d}\langle\delta\eta\rangle_u \bigg| \mathcal{G}_{S-}\bigg] = \mathbb{E}\bigg[\int_{S-}^T \mathrm{d}[\delta\eta]_u \bigg| \mathcal{G}_{S-}\bigg],\ \mathbb{P}\text{–a.s.}$$

Before turning to the remaining inequalities, it is worth noting the following. With Lemma 5.2 and Lemma C.4, we find that

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{u\in[t',\infty]} |y_u|^2 \bigg| \mathcal{G}_{t'}\bigg] \\ &\leq 12\mathbb{E}\bigg[|\xi_T|^2 + \sup_{s\in[t',\infty]} |\xi^+_s \mathbf{1}_{\{s<T\}}|^2 + \bigg(\int_{(t',\infty)} |f_s| \mathrm{d}C_s\bigg)^2 \bigg| \mathcal{G}_{t'}\bigg],\ \mathbb{P}\text{–a.s.},\ t' \in (0,\infty). \end{aligned}$$

By taking the conditional expectation with respect to $\mathcal{G}_t$ for $t \in [0,t')$, and then letting t' tend to t, we find

$$\mathbb{E}\bigg[\sup_{u\in(t,\infty]}|y_u|^2\bigg|\mathcal{G}_t\bigg]$$
$$\leq 12\mathbb{E}\bigg[|\xi_T|^2+\sup_{s\in(t,\infty]}|\xi^+_s\mathbf{1}_{\{s<T\}}|^2+\bigg(\int_{(t,\infty)}|f_s|\mathrm{d}C_s\bigg)^2\bigg|\mathcal{G}_t\bigg],\ \mathbb{P}\text{–a.s.},\ t\in[0,\infty].$$

As before, the processes within the conditional expectations are $\mathbb{P}$–a.s. right-continuous, and therefore, by the $\mathbb{P}$–a.s. right-continuity of their respective optional projections, we even have

$$\mathbb{E}\bigg[\sup_{u\in(S,\infty]}|y_u|^2\bigg|\mathcal{G}_S\bigg]$$
$$\leq 12\mathbb{E}\bigg[|\xi_T|^2+\sup_{u\in(S,\infty]}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2+\bigg(\int_{(S,\infty)}|f_u|\mathrm{d}C_u\bigg)^2\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S\in\mathcal{T}_{0,\infty}. \tag{5.58}$$

We now turn to (5.7). It is enough to show the bound for $i=1$, and we thus also drop the superscript in what follows. An analogous argument to the one which lead to (5.54) yields by letting t' tend to infinity that

$$\begin{aligned}
&|y_t|^2+\int_{(t,\infty)}\mathrm{d}[\eta]_s+\int_{(t,\infty)}\mathrm{d}[k^r]_s+\int_{[t,\infty)}\mathrm{d}[k^\ell]_s-2\int_{(t,\infty)}\mathrm{d}[\eta,k^r]_s\\
&\quad=|y_{\infty-}|^2+2\int_{(t,\infty)}y_sf_s\mathrm{d}C_s-2\int_{(t,\infty)}y_{s-}\mathrm{d}\eta_s\\
&\qquad+\sum_{s\in(t,\infty)}(f_s\Delta C_s)^2+2\int_{(t,\infty)}y_{s-}\mathrm{d}k^r_s+2\int_{[t,\infty)}y_s\mathrm{d}k^\ell_s\\
&\quad\leq|\xi_T|^2+2\int_{(t,\infty)}y_sf_s\mathrm{d}C_s-2\int_{(t,\infty)}y_{s-}\mathrm{d}\eta_s\\
&\qquad+\sum_{s\in(t,\infty)}(f_s\Delta C_s)^2+2\int_{(t,\infty)}y_{s-}\mathrm{d}k^r_s+2\int_{[t,\infty)}y_s\mathrm{d}k^\ell_s,\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.}
\end{aligned}\tag{5.59}$$

Here the inequality follows from (5.3). Now the Skorokhod condition implies that

$$\begin{aligned}
\int_{(t,\infty)}y_{s-}\mathrm{d}k^r_s&=\int_{(t,\infty)}y_{s-}\mathbf{1}_{\{y_{s-}=\bar{\xi}_s\}}\mathbf{1}_{\{s\leq T\}}\mathrm{d}k^r_s\\
&=\int_{(t,\infty)}\bar{\xi}_s\mathbf{1}_{\{y_{s-}=\bar{\xi}_s\}}\mathbf{1}_{\{s\leq T\}}\mathrm{d}k^r_s=\int_{(t,\infty)}\bar{\xi}_s\mathbf{1}_{\{s\leq T\}}\mathrm{d}k^r_s
\end{aligned}$$

and

$$\begin{aligned}
\int_{[t,\infty)}y_s\mathrm{d}k^\ell_s&=\int_{[t,\infty)}y_s\mathbf{1}_{\{y_s=\xi_s\}}\mathbf{1}_{\{s<T\}}\mathrm{d}k^\ell_s\\
&=\int_{[t,\infty)}\xi_s\mathbf{1}_{\{y_s=\xi_s\}}\mathbf{1}_{\{s<T\}}\mathrm{d}k^\ell_s=\int_{[t,\infty)}\xi_s\mathbf{1}_{\{s<T\}}\mathrm{d}k^\ell_s.
\end{aligned}$$

Thus

$$\begin{aligned}
&2\int_{(t,\infty)}y_{s-}\mathrm{d}k^r_s=2\int_{(t,\infty)}\bar{\xi}_s\mathbf{1}_{\{s\leq T\}}\mathrm{d}k^r_s\\
&\quad\leq 2\sup_{s\in(t,\infty)}\big\{\xi^+_s\mathbf{1}_{\{s<T\}}\big\}\int_{(t,\infty)}\mathrm{d}k^r_s\leq\frac{1}{\kappa}\sup_{s\in(t,\infty)}|\xi^+_s\mathbf{1}_{\{s<T\}}|^2+\kappa\bigg(\int_{(t,\infty)}\mathrm{d}k^r_s\bigg)^2,
\end{aligned}\tag{5.60}$$

$$
\begin{aligned}
2\int_{[t,\infty)} y_s \mathrm{d}k^\ell_s &= 2\int_{[t,\infty)} \xi_s \mathbf{1}_{\{s<T\}} \mathrm{d}k^\ell_s \\
&\leq 2 \sup_{s\in[t,\infty)} \big\{\xi^+_s \mathbf{1}_{\{s<T\}}\big\} \int_{[t,\infty)} \mathrm{d}k^\ell_s \leq \frac{1}{\kappa} \sup_{s\in[t,\infty)} |\xi^+_s \mathbf{1}_{\{s<T\}}|^2 + \kappa\bigg(\int_{[t,\infty)} \mathrm{d}k^\ell_s\bigg)^2,
\end{aligned} \tag{5.61}
$$

for every $\kappa \in (0,\infty)$. Similarly, we find

$$
2\int_{(t,\infty)} y_s f_s \mathrm{d}C_s \leq \frac{1}{\varepsilon} \sup_{s\in(t,\infty)} |y_s|^2 + \varepsilon\bigg(\int_{(t,\infty)} |f_s| \mathrm{d}C_s\bigg)^2, \tag{5.62}
$$

for every $\varepsilon \in (0,\infty)$. Since

$$
\int_{(t,\infty)} \mathrm{d}k^r_s + \int_{[t,\infty)} \mathrm{d}k^\ell_s = y_t - y_{\infty-} - \int_{(t,\infty)} f_s \mathrm{d}C_s + \int_{(t,\infty)} \mathrm{d}\eta_s,\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.},
$$

we also have

$$
\begin{aligned}
&\bigg(\int_{(t,\infty)} \mathrm{d}k^r_s + \int_{[t,\infty)} \mathrm{d}k^\ell_s\bigg)^2 \\
&\quad\leq 4\bigg(|y_t|^2 + |y_{\infty-}|^2 + \bigg(\int_{(t,\infty)} f_s \mathrm{d}C_s\bigg)^2 + \bigg(\int_{(t,\infty)} \mathrm{d}\eta_s\bigg)^2\bigg),\ t\in[0,\infty),\ \mathbb{P}\text{–a.s.}
\end{aligned} \tag{5.63}
$$

Combining (5.59) with Equations (5.60) to (5.63) yields

$$
\begin{aligned}
&|y_t|^2 + \int_{(t,\infty)} \mathrm{d}[\eta]_s + \int_{(t,\infty)} \mathrm{d}[k^r]_s + \int_{[t,\infty)} \mathrm{d}[k^\ell]_s - 2\int_{(t,\infty)} \mathrm{d}[\eta,k^r]_s \\
&\leq |\xi_T|^2 + \frac{1}{\varepsilon} \sup_{s\in(t,\infty)} |y_s|^2 + \varepsilon\bigg(\int_{(t,\infty)} |f_s| \mathrm{d}C_s\bigg)^2 - 2\int_{(t,\infty)} y_{s-} \mathrm{d}\eta_s + \sum_{s\in(t,\infty)} (f_s \Delta C_s)^2 \\
&\quad + \frac{1}{\kappa} \sup_{s\in(t,\infty)} |\xi^+_s \mathbf{1}_{\{s<T\}}|^2 + \kappa\bigg(\int_{(t,\infty)} \mathrm{d}k^r_s\bigg)^2 + \frac{1}{\kappa} \sup_{s\in[t,\infty)} |\xi^+_s \mathbf{1}_{\{s<T\}}|^2 + \kappa\bigg(\int_{[t,\infty)} \mathrm{d}k^\ell_s\bigg)^2 \\
&\leq |\xi_T|^2 + \frac{1}{\varepsilon} \sup_{s\in(t,\infty)} |y_s|^2 + (\varepsilon + 4\kappa)\bigg(\int_{(t,\infty)} |f_s| \mathrm{d}C_s\bigg)^2 - 2\int_{(t,\infty)} y_{s-} \mathrm{d}\eta_s + \sum_{s\in(t,\infty)} (f_s \Delta C_s)^2 \\
&\quad + \frac{2}{\kappa} \sup_{s\in[t,\infty)} |\xi^+_s \mathbf{1}_{\{s<T\}}|^2 + 4\kappa\bigg(|y_t|^2 + |y_{\infty-}|^2 + \bigg(\int_{(t,\infty)} \mathrm{d}\eta_s\bigg)^2\bigg) \\
&\leq |\xi_T|^2 + \frac{1}{\varepsilon} \sup_{s\in(t,\infty)} |y_s|^2 + (\varepsilon + 4\kappa)\bigg(\int_{(t,\infty)} |f_s| \mathrm{d}C_s\bigg)^2 - 2\int_{(t,\infty)} y_{s-} \mathrm{d}\eta_s + \sum_{s\in(t,\infty)} (f_s \Delta C_s)^2 \\
&\quad + \frac{2}{\kappa} \sup_{s\in[t,\infty)} |\xi^+_s \mathbf{1}_{\{s<T\}}|^2 + 4\kappa\bigg(|y_t|^2 + |\xi_T|^2 + \bigg(\int_{(t,\infty)} \mathrm{d}\eta_s\bigg)^2\bigg) \\
&\leq (1+4\kappa)|\xi_T|^2 + \frac{1}{\varepsilon} \sup_{s\in(t,\infty)} |y_s|^2 + (\varepsilon + 4\kappa)\bigg(\int_{(t,\infty)} |f_s| \mathrm{d}C_s\bigg)^2 \\
&\quad - 2\int_{(t,\infty)} y_{s-} \mathrm{d}\eta_s + \sum_{s\in(t,\infty)} (f_s \Delta C_s)^2 + \frac{2}{\kappa} \sup_{s\in[t,\infty)} |\xi^+_s \mathbf{1}_{\{s<T\}}|^2 \\
&\quad + 4\kappa\bigg(|y_t|^2 + \bigg(\int_{(t,\infty)} \mathrm{d}\eta_s\bigg)^2\bigg),\ t\in[0,\infty],\ \mathbb{P}\text{–a.s.}
\end{aligned} \tag{5.64}
$$

Let $\kappa \in (0,\infty)$ be such that $0 < 1 - 4\kappa \leq 1$, and let $S \in \mathcal{T}_{0,\infty}$. By taking conditional expectation, rearranging the terms and using $\mathbb{E}[(\int_{(S,\infty)} \mathrm{d}\eta_u)^2|\mathcal{G}_S] = \mathbb{E}[\int_{(S,\infty)} \mathrm{d}[\eta]_u|\mathcal{G}_S]$

and $\sum_{s\in(S,\infty)}(f_s\Delta C_s)^2 \leq (\int_S^T |f_s|\mathrm{d}C_s)^2$, we find

$$\begin{aligned}
&(1-4\kappa)\bigg(|y_S|^2+\mathbb{E}\bigg[\int_{(S,\infty)}\mathrm{d}[\eta]_u\bigg|\mathcal{G}_S\bigg]\bigg)+\mathbb{E}\bigg[\int_{(S,\infty)}\mathrm{d}[k^r]_u\bigg|\mathcal{G}_S\bigg]+\mathbb{E}\bigg[\int_{[S,\infty)}\mathrm{d}[k^\ell]_u\bigg|\mathcal{G}_S\bigg]\\
&\leq(1+4\kappa)\mathbb{E}\big[|\xi_T|^2\big|\mathcal{G}_S\big]+\frac{1}{\varepsilon}\mathbb{E}\bigg[\sup_{u\in(S,\infty)}|y_u|^2\bigg|\mathcal{G}_S\bigg]+(1+\varepsilon+4\kappa)\mathbb{E}\bigg[\bigg(\int_{(S,\infty)}|f_u|\mathrm{d}C_u\bigg)^2\bigg|\mathcal{G}_S\bigg]\\
&\quad+\frac{2}{\kappa}\mathbb{E}\bigg[\sup_{u\in[S,\infty)}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2\bigg|\mathcal{G}_S\bigg]\\
&\leq(1+4\kappa+12/\varepsilon)\mathbb{E}\big[|\xi_T|^2\big|\mathcal{G}_S\big]+(1+\varepsilon+4\kappa+12/\varepsilon)\mathbb{E}\bigg[\bigg(\int_{(S,\infty)}|f_u|\mathrm{d}C_u\bigg)^2\bigg|\mathcal{G}_S\bigg]\\
&\quad+\bigg(\frac{12}{\varepsilon}+\frac{2}{\kappa}\bigg)\mathbb{E}\bigg[\sup_{u\in[S,\infty)}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2\bigg|\mathcal{G}_S\bigg].
\end{aligned}$$

Here the second inequality follows from (5.58). Since $0<(1-4\kappa)\leq 1$, we can thus divide both sides by $(1-4\kappa)$ and find

$$\begin{aligned}
&|y_S|^2+\mathbb{E}\bigg[\int_{(S,\infty)}\mathrm{d}[\eta]_u\bigg|\mathcal{G}_S\bigg]+\mathbb{E}\bigg[\int_{(S,\infty)}\mathrm{d}[k^r]_u\bigg|\mathcal{G}_S\bigg]+\mathbb{E}\bigg[\int_{[S,\infty)}\mathrm{d}[k^\ell]_u\bigg|\mathcal{G}_S\bigg]\\
&\quad\leq\frac{\max\{1+\varepsilon+4\kappa+12/\varepsilon,12/\varepsilon+2/\kappa\}}{1-4\kappa}\Bigg(\mathbb{E}\big[|\xi_T|^2\big|\mathcal{G}_S\big]+\mathbb{E}\bigg[\bigg(\int_{(S,\infty)}|f_u|\mathrm{d}C_u\bigg)^2\bigg|\mathcal{G}_S\bigg]\\
&\qquad+\mathbb{E}\bigg[\sup_{u\in[S,\infty)}|\xi^+_u\mathbf{1}_{\{u<T\}}|^2\bigg|\mathcal{G}_S\bigg]\Bigg).
\end{aligned}$$

Finally, as explained at the beginning of the proof of Lemma 5.2, the inequality (5.8) follows from (5.7), which completes the proof. □

6 Proofs of the main results

We are now in a position to prove Theorem 3.4 and Theorem 3.7. The proofs are based on the optimal stopping theory we revisited in Section 4 and on the *a priori* estimates we established in Section 5. The existence and uniqueness of the BSDE and the reflected BSDE are based on defining a contraction map on the weighted spaces of Section 2.4. Therefore, we first need to show that such a contraction map is well-defined, meaning that it maps its domain into itself.

Proposition 6.1. *Suppose that f does not depend on (y,y,z,u). There exists a unique triple $(Z,U,N)\in\mathbb{H}^2_T(X)\times\mathbb{H}^2_T(\mu)\times\mathcal{H}^{2,\perp}_{0,T}(X,\mu)$ and a, up to $\mathbb{P}$–indistinguishability, unique triple (Y,K^r,K^ℓ) such that the collection (Y,Z,U,N,K^r,K^ℓ) satisfies* **(R1)** *up to* **(R7)**. *Moreover, $Y\in\mathcal{S}^2_T$. If, in addition, $(X,\mathbb{G},T,\xi,f,C)$ is standard data for some $\hat{\beta}\in(0,\infty)$, then $(\alpha Y,\alpha Y_-,Z,U,N)\in\mathbb{H}^2_{T,\hat{\beta}}\times\mathbb{H}^2_{T,\hat{\beta}}\times\mathbb{H}^2_{T,\hat{\beta}}(X)\times\mathbb{H}^2_{T,\hat{\beta}}(\mu)\times\mathcal{H}^{2,\perp}_{T,\hat{\beta}}(X,\mu)$.*

Proof. Let (Y,Z,U,N,K^ℓ,K^r) be the collection of processes constructed in Proposition 4.5, which clearly is the unique collection of processes satisfying **(R1)** up to **(R7)**. That $Y\in\mathcal{S}^2_T$ follows from Lemma 5.2.

That $(\alpha Y,\alpha Y_-,Z,U,N)\in\mathbb{H}^2_{T,\hat{\beta}}\times\mathbb{H}^2_{T,\hat{\beta}}\times\mathbb{H}^2_{T,\hat{\beta}}(X)\times\mathbb{H}^2_{T,\hat{\beta}}(\mu)\times\mathcal{H}^{2,\perp}_{T,\hat{\beta}}(X,\mu)$ in case $(X,\mathbb{G},T,\xi,f,C)$ is standard data under some $\hat{\beta}\in(0,\infty)$ follows from Proposition 5.1. This completes the proof. □

Before proving the next result, recall from Remark 2.10 that the first component of the reflected BSDE is an optional semimartingale indexed by $[0,\infty]$ since $Y=Y_0+M+A$,

$\mathbb{P}$–a.s., where

$$M_t \coloneqq \int_0^t Z_s \mathrm{d}X_s + \int_0^t \int_E U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) + \int_0^t \mathrm{d}N_t,\ t \in [0,\infty]$$

and

$$A_t \coloneqq -\int_0^t f_s\big(Y_s, Y_{s-}, Z_s, U_s(\cdot)\big)\mathrm{d}C_s - K^r_t - K^\ell_{t-},\ t \in [0,\infty].$$

The integrals above do not include ∞ in their domain of integration, yet their values at infinity are determined by letting $t \uparrow\uparrow \infty$.

Lemma 6.2. *Let $\hat{\beta} \in (0,\infty)$. Suppose that $(Y, Z, U, N, K^r, K^\ell)$ satisfy* **(R1)** *up to* **(R6)** *and one of the following conditions holds:*

(i) $(\alpha Y, \alpha Y_-, Z, U, N) \in \mathbb{H}^2_{T,\hat{\beta}} \times \mathbb{H}^2_{T,\hat{\beta}} \times \mathbb{H}^2_{T,\hat{\beta}}(X) \times \mathbb{H}^2_{T,\hat{\beta}}(\mu)$;

(ii) the generator f does not depend on Y_{s-} and $(\alpha Y, Z, U, N) \in \mathbb{H}^2_{T,\hat{\beta}} \times \mathbb{H}^2_{T,\hat{\beta}}(X) \times \mathbb{H}^2_{T,\hat{\beta}}(\mu)$;

(iii) the generator f does not depend on Y_s and $(\alpha Y_-, Z, U, N) \in \mathbb{H}^2_{T,\hat{\beta}} \times \mathbb{H}^2_{T,\hat{\beta}}(X) \times \mathbb{H}^2_{T,\hat{\beta}}(\mu)$.

Then $Y \in \mathcal{S}^2_T$ and **(R7)** *holds.*

Proof. We prove this result under the assumption that (ii) holds. The other cases follow analogously. Note first that

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_0^T |f_s\big(Y_s, Z_s, U_s(\cdot)\big)|\mathrm{d}C_s\bigg)^2\bigg] \\
&\qquad \le \mathbb{E}\bigg[\bigg(\int_0^T \frac{1}{\mathcal{E}(\hat{\beta}A)_s}\mathrm{d}A_s\bigg)\bigg(\int_0^T \mathcal{E}(\hat{\beta}A)_s \frac{|f_s\big(Y_s, Z_s, U_s(\cdot)\big)|^2}{\alpha_s^2}\mathrm{d}C_s\bigg)\bigg] \\
&\qquad \le \frac{1}{\hat{\beta}}\mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat{\beta}A)_s \frac{|f_s\big(Y_s, Z_s, U_s(\cdot)\big)|^2}{\alpha_s^2}\mathrm{d}C_s\bigg] < \infty.
\end{aligned}$$

Here we first used the Cauchy–Schwarz inequality, then (5.15), and finally that $(Y, Z, U) \in \mathbb{H}^2_{T,\hat{\beta}} \times \mathbb{H}^2_{T,\hat{\beta}}(X) \times \mathbb{H}^2_{T,\hat{\beta}}(\mu)$ together with the Lipschitz property of f and **(D7)**. We now show that $Y \in \mathcal{S}^2_T$. Let

$$\begin{aligned}
J_t &\coloneqq Y_{t\wedge T} - Y_0 + \int_0^{t\wedge T} f_s\big(Y_s, Z_s, U_s(\cdot)\big)\mathrm{d}C_s + K^r_{t\wedge T} + K^\ell_{(t\wedge T)-} \\
&= \int_0^{t\wedge T} Z_s \mathrm{d}X_s + \int_0^{t\wedge T}\int_E U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) + N_{t\wedge T},\ t \in [0,\infty),\ \mathbb{P}\text{–a.s.}
\end{aligned}$$

Then $J \in \mathcal{S}^2_T$ since

$$\begin{aligned}
\mathbb{E}\bigg[\sup_{t\in[0,T]}|J_t|^2\bigg] &= \mathbb{E}\bigg[\sup_{t\in[0,\infty]}|J_t|^2\bigg] = \mathbb{E}\bigg[\sup_{t\in[0,\infty)}|J_t|^2\bigg] \\
&\le 4\mathbb{E}\big[|J_\infty|^2\big] = 4\Big(\|Z\|^2_{\mathbb{H}^2_{T,0}(X)} + \|U\|^2_{\mathbb{H}^2_{T,0}(\mu)} + \|N\|^2_{\mathcal{H}^2_{T,0}}\Big) < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\|Y - J\|_{\mathcal{S}^2_T} \\
&\quad \le \|Y_0\|_{\mathbb{L}^2} + \mathbb{E}\bigg[\bigg(\int_0^T |f_s\big(Y_s, Z_s, U_s(\cdot)\big)|\mathrm{d}C_s\bigg)^2\bigg]^{1/2} + \|K^r_T + K^\ell_T\|_{\mathbb{L}^2} \\
&\quad \le \|\xi_T - J_T\|_{\mathbb{L}^2} + 2\mathbb{E}\bigg[\bigg(\int_0^T |f_s\big(Y_s, Z_s, U_s(\cdot)\big)|\mathrm{d}C_s\bigg)^2\bigg]^{1/2} + 2\|K^r_T + K^\ell_T\|_{\mathbb{L}^2}
\end{aligned}$$

$$\leq \|\xi_T\|_{\mathbb{L}^2} + \|J_T\|_{\mathbb{L}^2} + 2\mathbb{E}\bigg[\bigg(\int_0^T |f_s\big(Y_s, Z_s, U_s(\cdot)\big)|\mathrm{d}C_s\bigg)^2\bigg]^{1/2} + 2\|K^r_T + K^\ell_T\|_{\mathbb{L}^2} < \infty.$$

Here we used $Y_T = \xi_T$ in the second-to-last line. This yields

$$\|Y\|_{\mathcal{S}^2_T} \leq \|Y - J\|_{\mathcal{S}^2_T} + \|J\|_{\mathcal{S}^2_T} < \infty.$$

We turn to the proof of **(R7)**. Let $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{N}, \tilde{K}^r, \tilde{K}^\ell)$ be the solution to the reflected BSDE satisfying **(R1)** up to **(R7)** with $\tilde{Y} \in \mathcal{S}^2_T$ and generator $f(Y, Z, U)$ given by Proposition 6.1. Applying the Gal'chouk–Itô–Lenglart formula to $|Y - \tilde{Y}|^2$ yields

$$\begin{aligned}
|Y_t - \tilde{Y}_t|^2 = {} & |Y_{t'} - \tilde{Y}_{t'}|^2 - \int_{(t,t']} \mathrm{d}[M^c - \tilde{M}^c]_s - 2\int_{(t,t']} (Y_{s-} - \tilde{Y}_{s-})\mathrm{d}(M - \tilde{M})_s \\
& + 2\int_{(t,t']} (Y_{s-} - \tilde{Y}_{s-})\mathrm{d}(K^r - \tilde{K}^r)_s + 2\int_{[t,t')} (Y_s - \tilde{Y}_s)\mathrm{d}(K^\ell - \tilde{K}^\ell)_s \\
& - \sum_{s\in(t,t']} \big(Y_s - \tilde{Y}_s - (Y_{s-} - \tilde{Y}_{s-})\big)^2 - \sum_{s\in[t,t')} \big(Y_{s+} - \tilde{Y}_{s+} - (Y_s - \tilde{Y}_s)\big)^2 \\
\leq {} & |Y_{t'} - \tilde{Y}_{t'}|^2 + 2\int_{(t,t']} \big(Y_{s-} - \overline{\xi}_s - (\tilde{Y}_{s-} - \overline{\xi}_s)\big)\mathrm{d}(K^r - \tilde{K}^r)_s \\
& - 2\int_{(t,t']} (Y_{s-} - \tilde{Y}_{s-})\mathrm{d}(M - \tilde{M})_s + 2\int_{[t,t')} \big(Y_s - \xi_s - (\tilde{Y}_s - \xi_s)\big)\mathrm{d}(K^\ell - \tilde{K}^\ell)_s \\
\leq {} & |Y_{t'} - \tilde{Y}_{t'}|^2 - 2\int_{(t,t']} (Y_{s-} - \tilde{Y}_{s-})\mathrm{d}(M - \tilde{M})_s,\ 0 \leq t \leq t' < \infty,\ \mathbb{P}\text{–a.s.,}
\end{aligned} \tag{6.1}$$

where

$$M_t := \int_0^t Z_s \mathrm{d}X_s + \int_0^t \int_E U_s(x)\tilde{\mu}(\mathrm{d}s, \mathrm{d}x) + N_t,\ t \in [0, \infty],$$

and

$$\tilde{M}_t := \int_0^t \tilde{Z}_s \mathrm{d}X_s + \int_0^t \int_E \tilde{U}_s(x)\tilde{\mu}(\mathrm{d}s, \mathrm{d}x) + \tilde{N}_t,\ t \in [0, \infty].$$

Note that we have $M_\infty = \lim_{t\uparrow\uparrow\infty} M_t$ and $\tilde{M}_\infty = \lim_{t\uparrow\uparrow\infty} M_t$ up to a $\mathbb{P}$–null set. Since $Y - \tilde{Y} \in \mathcal{S}^2_T$, the local martingale

$$\int_0^\cdot (Y_{s-} - \tilde{Y}_{s-})\mathrm{d}(M - \tilde{M})_s = \int_0^{\cdot\wedge T} (Y_{s-} - \tilde{Y}_{s-})\mathrm{d}(M - \tilde{M})_s,$$

is a uniformly integrable martingale by the Burkholder–Davis–Gundy inequality. Taking the conditional expectation in (6.1) yields $|Y_S - \tilde{Y}_S|^2 \leq \mathbb{E}[|Y_{S'} - \tilde{Y}_{S'}|^2|\mathcal{G}_S]$, $\mathbb{P}$–a.s., for two finite stopping times S and S' with $S \leq S'$. In particular, since $(Y, \tilde{Y}) \in \mathcal{S}^2_T$, and by choosing $S' = S \vee n$ and then letting n tend to infinity, this yields

$$|Y_S - \tilde{Y}_S|^2 \leq \mathbb{E}[|Y_{\infty-} - \tilde{Y}_{\infty-}|^2|\mathcal{G}_S],\ \mathbb{P}\text{–a.s.}$$

Suppose for the moment that $\mathbb{P}[A] = 0$, where $A = \{T = \infty\} \cap \{|Y_{\infty-} - \tilde{Y}_{\infty-}| > 0\}$. Then $\mathbb{E}[|Y_{\infty-} - \tilde{Y}_{\infty-}|^2|\mathcal{G}_S] = 0$, $\mathbb{P}$–almost surely. Proposition C.3 together with $Y_T = \xi_T = \tilde{Y}_T$, $\mathbb{P}$–a.s., implies that $Y = \tilde{Y}$ up to $\mathbb{P}$–indistinguishability. Hence

$$Y_S = \tilde{Y}_S = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S} \mathbb{E}\bigg[\int_S^{\tau\wedge T} f_s(Y_s, Z_s, U_s(\cdot))\mathrm{d}C_s + \xi_{\tau\wedge T}\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}.$$

It remains to prove $\mathbb{P}[A] = 0$, and we suppose, for the sake of reaching a contradiction, that $\mathbb{P}[A] > 0$. On $B := A \cap \{\Delta Y_\infty = Y_\infty - Y_{\infty-} = 0\}$, we have $-\Delta Y_\infty = \Delta K^r_\infty =$

$K^r_\infty - K^r_{\infty-} = 0$ and $\overline{\xi}_\infty \leq Y_{\infty-} = Y_\infty = \xi_\infty$, $\mathbb{P}$–a.s. on B. This implies $|\tilde{Y}_\infty - \tilde{Y}_{\infty-}| = |\xi_\infty - \tilde{Y}_{\infty-}| = |Y_{\infty-} - \tilde{Y}_{\infty-}| > 0$, $\mathbb{P}$–a.s. on B, therefore $-\Delta\tilde{Y}_\infty = \Delta\tilde{K}^r_\infty > 0$, $\mathbb{P}$–a.s. on B, and thus $\tilde{Y}_{\infty-} = \overline{\xi}_\infty$, $\mathbb{P}$–a.s. on B. However, this also yields $|Y_{\infty-} - \overline{\xi}_\infty| = |Y_{\infty-} - \tilde{Y}_{\infty-}| > 0$, $\mathbb{P}$–a.s. on B, which now implies $\tilde{Y}_{\infty-} = \overline{\xi}_\infty < Y_{\infty-} = Y_\infty = \xi_\infty = \tilde{Y}_\infty$, $\mathbb{P}$–a.s. on B. Therefore, we must have $0 > -\Delta\tilde{K}^r_\infty = \Delta\tilde{Y}_\infty > 0$, $\mathbb{P}$–a.s. on B, which now yields $\mathbb{P}[B] = 0$. Let $B' \coloneqq A \cap \{\Delta Y_\infty \neq 0\}$. Then $Y_\infty - Y_{\infty-} = \Delta Y_\infty = -\Delta K^r_\infty < 0$ and thus $Y_{\infty-} = \overline{\xi}_\infty$, $\mathbb{P}$–a.s. on B'. Since $|\overline{\xi}_\infty - \tilde{Y}_{\infty-}| = |Y_{\infty-} - \tilde{Y}_{\infty-}| > 0$, $\mathbb{P}$–a.s. on B', this implies $\Delta\tilde{K}^r_\infty = 0$ and therefore $\Delta\tilde{Y}_\infty = 0$, $\mathbb{P}$–a.s. on B'. Hence $Y_{\infty-} > Y_\infty = \xi_\infty = \tilde{Y}_\infty = \tilde{Y}_{\infty-} > \overline{\xi}_\infty = Y_{\infty-}$, $\mathbb{P}$–a.s. on B', and therefore $\mathbb{P}[B'] = 0$. This now yields $\mathbb{P}[A] = \mathbb{P}[B] + \mathbb{P}[B'] = 0$ and completes the proof. □

We now turn to the proof of our main result.

Proof of Theorem 3.4. We prove the theorem under assumption (ii). Let $\mathcal{L}^{(2)}_{T,\hat{\beta}}$ be the collection of processes (Y, Z, U, N) for which $Y = Y_{\cdot\wedge T}$ is optional, $(Z, U, N) \in \mathbb{H}^2_T(X) \times \mathbb{H}^2_T(\mu) \times \mathcal{H}^{2,\perp}_{0,T}(X, \mu)$, and

$$\|(Y,Z,U,N)\|^2_{\mathcal{L}^{(2)}_{T,\hat{\beta}}} \coloneqq \|\alpha Y\|^2_{\mathbb{H}^2_{T,\hat{\beta}}} + \|Z\|^2_{\mathbb{H}^2_{T,\hat{\beta}}(X)} + \|U\|^2_{\mathbb{H}^2_{T,\hat{\beta}}(\mu)} + \|N\|^2_{\mathcal{H}^2_{T,\hat{\beta}}} < \infty.$$

Then $\mathcal{L}^{(2)}_{T,\hat{\beta}}$ together with the semi-norm $\|\cdot\|_{\mathcal{L}^{(2)}_{T,\hat{\beta}}}$ is a Banach space after identifying processes (Y, Z, U, N) and (Y', Z', U', N') for which $\|(Y, Z, U, N) - (Y', Z', U', N')\|_{\mathcal{L}^{(2)}_{T,\hat{\beta}}} = 0$ holds. Let $(y, z, u, n) \in \mathcal{L}^{(2)}_{T,\hat{\beta}}$, and note that

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat{\beta}A)_s \frac{|f_s\big(y_s, z_s, u_s(\cdot)\big)|^2}{\alpha_s^2}\mathrm{d}C_s\bigg] \\
&\quad\leq 2\mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat{\beta}A)_s \frac{|f_s\big(y_s, z_s, u_s(\cdot)\big) - f_s(0,0,\mathbf{0})|^2}{\alpha_s^2}\mathrm{d}C_s + \int_0^T \mathcal{E}(\hat{\beta}A)_s \frac{|f_s(0,0,\mathbf{0})|^2}{\alpha_s^2}\mathrm{d}C_s\bigg] \\
&\quad\leq 2\left(\|\alpha y\|^2_{\mathbb{H}^2_{T,\hat{\beta}}} + \|z\|^2_{\mathbb{H}^2_{T,\hat{\beta}}(X)} + \|u\|^2_{\mathbb{H}^2_{T,\hat{\beta}}(\mu)} + \left\|\frac{f(0,0,\mathbf{0})}{\alpha}\right\|^2_{\mathbb{H}^2_{T,\hat{\beta}}}\right) < \infty.
\end{aligned}$$

We denote by $(Y, Z, U, N, K^r, K^\ell)$ the collection satisfying **(R1)** up to **(R7)** with generator $f_s\big(y_s, z_s, u_s(\cdot)\big)$ constructed in Proposition 4.5 (or in Proposition 6.1). Then $(Y, Z, U, N) \in \mathcal{L}^{(2)}_{T,\hat{\beta}}$ by the bounds of Proposition 5.1. The map $\Upsilon_2 : \mathcal{L}^{(2)}_{T,\hat{\beta}} \longrightarrow \mathcal{L}^{(2)}_{T,\hat{\beta}}$ given by $\Upsilon_2(y, z, u, n) \coloneqq (Y, Z, U, N)$ is thus well-defined.

We prove that Υ_2 is a contraction. For $i \in \{1, 2\}$, let $(y^i, z^i, u^i, n^i) \in \mathcal{L}^{(2)}_{T,\hat{\beta}}$ and let $(Y^i, Z^i, U^i, N^i) \coloneqq \Upsilon_2(y^i, z^i, u^i, n^i)$. Let $\delta y \coloneqq y^1 - y^2$, $\delta z = z^1 - z^2$, $\delta u = u^1 - u^2$, and $\delta n = n^1 - n^2$, and define δY, δZ, δU and δN similarly. Denote by $\psi = (\psi_t)_{t\in[0,\infty)}$ the process $\psi_t \coloneqq f_t\big(y^1_t, z^1_t, u^1_t(\cdot)\big) - f_t\big(y^2_t, z^2_t, u^2_t(\cdot)\big)$. With Proposition 5.1, we find

$$\begin{aligned}
&\big\|\Upsilon_2(y^1, z^1, u^1, n^1) - \Upsilon_2(y^2, z^2, u^2, n^2)\big\|^2_{\mathcal{L}^{(2)}_{T,\hat{\beta}}} \\
&\qquad = \|\alpha\delta Y\|^2_{\mathbb{H}^2_{T,\hat{\beta}}} + \|\delta Z\|^2_{\mathbb{H}^2_{T,\hat{\beta}}(X)} + \|\delta U\|^2_{\mathbb{H}^2_{T,\hat{\beta}}(\mu)} + \|\delta N\|^2_{\mathcal{H}^2_{T,\hat{\beta}}} \\
&\qquad \leq M^\Phi_2(\hat{\beta})\Big\|\frac{\psi}{\alpha}\Big\|^2_{\mathbb{H}^2_{T,\hat{\beta}}} \\
&\qquad \leq M^\Phi_2(\hat{\beta})\Big(\|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\hat{\beta}}} + \|\delta z\|^2_{\mathbb{H}^2_{T,\hat{\beta}}(X)} + \|\delta u\|^2_{\mathbb{H}^2_{T,\hat{\beta}}(\mu)}\Big) \\
&\qquad \leq M^\Phi_2(\hat{\beta})\big\|(y^1, z^1, u^1, n^1) - (y^2, z^2, u^2, n^2)\big\|^2_{\mathcal{L}^{(2)}_{T,\hat{\beta}}}.
\end{aligned}$$

Here, we used the Lipschitz-continuity of the generator f in the fourth line. Since $M_2^\Phi(\hat\beta) < 1$, the map Υ_2 is indeed a contraction on $\mathcal{L}^{(2)}_{T,\hat\beta}$. By Banach's fixed-point theorem, there exists a unique fixed-point of Υ_2, which we denote by (Y,Z,U,N). Denoting by (K^r, K^ℓ) the two corresponding non-decreasing processes coming from the decomposition of the Snell envelope $Y_\cdot + \int_0^{\cdot\wedge T} f_s(Y_s,Z_s,U_s(\cdot))\mathrm{d}C_s$, we see that (Y,Z,U,N,K^r,K^ℓ) satisfies **(R1)** up to **(R7)**.

Suppose that $(Y',Z',U',N',K'^{,r},K'^{,\ell})$ is a solution satisfying **(R1)** up to **(R6)** such that $(\alpha Y',Z',U',N')$ is in $\mathbb{H}^2_{T,\hat\beta} \times \mathbb{H}^2_{T,\hat\beta}(X) \times \mathbb{H}^2_{T,\hat\beta}(\mu) \times \mathcal{H}^{2,\perp}_{0,T,\hat\beta}(X,\mu)$. Then **(R7)** holds by Lemma 6.2 and thus (Y',Z',U',N') is the fixed-point of Υ_2. Hence $(Y',Z',U',N') = (Y,Z,U,N)$ in $\mathcal{L}^{(2)}_{T,\hat\beta}$, and $Y = Y'$ up to $\mathbb{P}$–indistinguishability by Proposition 5.1. That $(K'^{,r},K'^{,\ell}) = (K^r,K^\ell)$ up to indistinguishability follows from Proposition 6.1. This implies the stated uniqueness. That Y is in $\mathcal{S}^2_T$ follows from Lemma 5.2 since

$$\mathbb{E}\bigg[\bigg(\int_0^T |f_s\big(Y_s,Z_s,U_s(\cdot)\big)|\mathrm{d}C_s\bigg)^2\bigg] \leq \frac{1}{\hat\beta}\mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat\beta A)_s \frac{|f_s\big(Y_s,Z_s,U_s(\cdot)\big)|^2}{\alpha_s^2}\mathrm{d}C_s\bigg] < \infty$$

by the Cauchy–Schwarz inequality and (5.15). This completes the proof of (ii).

For (i) and (iii) we define $\mathcal{L}^{(1)}_{T,\hat\beta}$ and $\mathcal{L}^{(3)}_{T,\hat\beta}$ as the spaces of processes (Y,Z,U,N) for which $(Y,Z,U,N) \in \mathcal{S}^2_T \times \mathbb{H}^2_T(X) \times \mathbb{H}^2_T(\mu) \times \mathcal{H}^{2,\perp}_{0,T}(X,\mu)$, $\mathbb{P}$–a.e. path of Y is làdlàg, and

$$\|(Y,Z,U,N)\|^2_{\mathcal{L}^{(1)}_{T,\hat\beta}} := \|Y\|^2_{\mathcal{S}^2_T} + \|\alpha Y\|^2_{\mathbb{H}^2_{T,\hat\beta}} + \|\alpha Y_-\|^2_{\mathbb{H}^2_{T,\hat\beta}} + \|Z\|^2_{\mathbb{H}^2_{T,\hat\beta}(X)} + \|U\|^2_{\mathbb{H}^2_{T,\hat\beta}(\mu)} + \|N\|^2_{\mathcal{H}^2_{T,\hat\beta}} < \infty,$$

$$\|(Y,Z,U,N)\|^2_{\mathcal{L}^{(3)}_{T,\hat\beta}} := \|Y\|^2_{\mathcal{S}^2_T} + \|\alpha Y_-\|^2_{\mathbb{H}^2_{T,\hat\beta}} + \|Z\|^2_{\mathbb{H}^2_{T,\hat\beta}(X)} + \|U\|^2_{\mathbb{H}^2_{T,\hat\beta}(\mu)} + \|N\|^2_{\mathcal{H}^2_{T,\hat\beta}} < \infty,$$

respectively. We turn both $\mathcal{L}^{(1)}_{T,\hat\beta}$ and $\mathcal{L}^{(3)}_{T,\hat\beta}$ into Banach spaces by identifying (Y,Z,U,N) and (Y',Z',U',N') for which

$$\|(Y,Z,U,N) - (Y',Z',U',N')\|_{\mathcal{L}^{(1)}_{T,\hat\beta}} = 0, \text{ and } \|(Y,Z,U,N) - (Y',Z',U',N')\|_{\mathcal{L}^{(3)}_{T,\hat\beta}} = 0,$$

respectively. The approach to deduce the existence of a unique fixed-point is then analogous to our previous argument. For (i) and (iii) we define the maps $\Upsilon_1 : \mathcal{L}^{(1)}_{T,\hat\beta} \longrightarrow \mathcal{L}^{(1)}_{T,\hat\beta}$ and $\Upsilon_3 : \mathcal{L}^{(3)}_{T,\hat\beta} \longrightarrow \mathcal{L}^{(3)}_{T,\hat\beta}$ analogously to Υ_2, and then note that by Proposition 5.1 the maps are well-defined. By the Lipschitz property of f, the *a priori* estimates of Proposition 5.1, we find

$$\begin{aligned}
&\big\|\Upsilon_1(y^1,z^1,u^1,n^1) - \Upsilon_1(y^2,z^2,u^2,n^2)\big\|^2_{\mathcal{L}^{(1)}_{T,\hat\beta}} \\
&\quad = \|\delta Y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta Y\|^2_{\mathbb{H}^2_{T,\hat\beta}} + \|\delta Z\|^2_{\mathbb{H}^2_{T,\hat\beta}(X)} + \|\delta U\|^2_{\mathbb{H}^2_{T,\hat\beta}(\mu)} + \|\delta N\|^2_{\mathcal{H}^2_{T,\hat\beta}} \\
&\quad \leq M_1^\Phi(\hat\beta)\Big\|\frac{\psi}{\alpha}\Big\|^2_{\mathbb{H}^2_{T,\hat\beta}} \\
&\quad \leq M_1^\Phi(\hat\beta)\Big(\|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\hat\beta}} + \|\delta z\|^2_{\mathbb{H}^2_{T,\hat\beta}(X)} + \|\delta u\|^2_{\mathbb{H}^2_{T,\hat\beta}(\mu)}\Big) \\
&\quad \leq M_1^\Phi(\hat\beta)\Big(\|\delta y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta y\|^2_{\mathbb{H}^2_{T,\hat\beta}} + \|\delta z\|^2_{\mathbb{H}^2_{T,\hat\beta}(X)} + \|\delta u\|^2_{\mathbb{H}^2_{T,\hat\beta}(\mu)}\Big) \\
&\quad \leq M_1^\Phi(\hat\beta)\big\|(y^1,z^1,u^1,n^1) - (y^2,z^2,u^2,n^2)\big\|^2_{\mathcal{L}^{(1)}_{T,\hat\beta}},
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\Upsilon_3(y^1,z^1,u^1,n^1) - \Upsilon_3(y^2,z^2,u^2,n^2)\big\|^2_{\mathcal{L}^{(3)}_{T,\hat\beta}} \\
&\quad = \|\delta Y\|^2_{\mathcal{S}^2_T} + \|\alpha\delta Y_-\|^2_{\mathbb{H}^2_{T,\hat\beta}} + \|\delta Z\|^2_{\mathbb{H}^2_{T,\hat\beta}(X)} + \|\delta U\|^2_{\mathbb{H}^2_{T,\hat\beta}(\mu)} + \|\delta N\|^2_{\mathcal{H}^2_{T,\hat\beta}}
\end{aligned}$$

$$\begin{aligned}
&\leq M_3^{\Phi}(\hat\beta)\Big\|\frac{\psi}{\alpha}\Big\|^2_{\mathbb{H}^2_{T,\hat\beta}}\\
&\leq M_3^{\Phi}(\hat\beta)\Big(\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\hat\beta}}+\|\delta z\|^2_{\mathbb{H}^2_{T,\hat\beta}(X)}+\|\delta u\|^2_{\mathbb{H}^2_{T,\hat\beta}(\mu)}\Big)\\
&\leq M_3^{\Phi}(\hat\beta)\Big(\|\delta y\|^2_{\mathcal{S}^2_T}+\|\alpha\delta y_-\|^2_{\mathbb{H}^2_{T,\hat\beta}}+\|\delta z\|^2_{\mathbb{H}^2_{T,\hat\beta}(X)}+\|\delta u\|^2_{\mathbb{H}^2_{T,\hat\beta}(\mu)}\Big)\\
&\leq M_3^{\Phi}(\hat\beta)\big\|(y^1,z^1,u^1,n^1)-(y^2,z^2,u^2,n^2)\big\|^2_{\mathcal{L}^{(3)}_{T,\hat\beta}}.
\end{aligned}$$

Thus Υ_1 (resp. Υ_3) is a contraction if (i) (resp. (iii)) holds. For both (i) and (iii) the representation **(R7)** is immediate by Lemma 6.2. The stated uniqueness can be deduced similarly to before, we thus omit the details.

Finally, if, in addition, $\xi^+\mathbf{1}_{[0,T)}\in\mathcal{S}^2_{T,\beta}$ for some $\beta\in(0,\hat\beta)$, then $(K^r,K^\ell)\in\mathcal{I}^2_{T,\beta}\times\mathcal{I}^2_{T,\beta}$ in (i), (ii) and (iii) by Proposition 5.7. This completes the proof. □

Remark 6.3. As we saw in the previous proof, the Y-component of the unique fixed-point of Υ_2 necessarily has to satisfy **(R7)**. So the class in which uniqueness can be deduced from the fixed-point property of Υ_2 is necessarily defined by **(R7)** as well. It turns out that the proof of existence and uniqueness in [62] of their reflected BSDE overlooks this intricate point, as an argument like our Lemma 6.2 is missing.

Proof of Theorem 3.7. The proof of this result is analogous to the proof of Theorem 3.4. The main difference is that we do not use the *a priori* estimates of Proposition 5.1, but instead use Proposition 5.4 to deduce the existence of a unique fixed-point in the three cases (i), (ii) and (iii). Let us show that in all three cases (i), (ii) and (iii), the component Y is in $\mathcal{S}^2_{T,\hat\beta}$. Note first that from $Y_S=\mathbb{E}\big[\xi_T+\int_S^T f_s\big(Y_s,Y_{s-},Z_s,U_s(\cdot)\big)\mathrm{d}C_s\big|\mathcal{G}_S\big]$, it follows that

$$\begin{aligned}
\mathcal{E}(\hat\beta A)_S^{1/2}|Y_S|&\leq\sqrt2\,\mathbb{E}\Bigg[\sqrt{\mathcal{E}(\hat\beta A)_S|\xi_T|^2+\mathcal{E}(\hat\beta A)_S\bigg(\int_S^T\big|f_s\big(Y_s,Y_{s-},Z_s,U_s(\cdot)\big)\big|\mathrm{d}C_s\bigg)^2}\,\Bigg|\mathcal{G}_S\Bigg]\\
&\leq\sqrt2\,\mathbb{E}\Bigg[\sqrt{\mathcal{E}(\hat\beta A)_S|\xi_T|^2+\frac{1}{\hat\beta}\int_S^T\mathcal{E}(\hat\beta A)_s\frac{\big|f_s\big(Y_s,Y_{s-},Z_s,U_s(\cdot)\big)\big|^2}{\alpha_s^2}\mathrm{d}C_s}\,\Bigg|\mathcal{G}_S\Bigg]\\
&\leq\sqrt2\,\mathbb{E}\Bigg[\sqrt{\mathcal{E}(\hat\beta A)_T|\xi_T|^2+\frac{1}{\hat\beta}\int_0^T\mathcal{E}(\hat\beta A)_s\frac{\big|f_s\big(Y_s,Y_{s-},Z_s,U_s(\cdot)\big)\big|^2}{\alpha_s^2}\mathrm{d}C_s}\,\Bigg|\mathcal{G}_S\Bigg].
\end{aligned}$$

Here the second line follows from the same arguments we used to deduce (5.22). Thus, by Lemma C.4, it follows that

$$\|Y\|^2_{\mathcal{S}^2_{T,\hat\beta}}=\mathbb{E}\bigg[\sup_{s\in[0,T]}\big|\mathcal{E}(\hat\beta A)_s^{1/2}Y_s\big|^2\bigg]\leq8\|\xi_T\|^2_{\mathbb{L}^2_{\hat\beta}}+\frac{8}{\hat\beta}\bigg\|\frac{f\big(Y,Y_-,Z,U(\cdot)\big)}{\alpha}\bigg\|^2_{\mathbb{H}^2_{T,\hat\beta}}.$$

Since $\|\xi_T\|_{\mathbb{L}^2_{\hat\beta}}<\infty$ and

$$\begin{aligned}
&\bigg\|\frac{f\big(Y,Y_-,Z,U(\cdot)\big)}{\alpha}\bigg\|_{\mathbb{H}^2_{T,\hat\beta}}\\
&\quad\leq\bigg\|\frac{f\big(Y,Y_-,Z,U(\cdot)\big)-f(0,0,0,\mathbf{0})}{\alpha}\bigg\|_{\mathbb{H}^2_{T,\hat\beta}}+\bigg\|\frac{f(0,0,0,\mathbf{0})}{\alpha}\bigg\|_{\mathbb{H}^2_{T,\hat\beta}}\\
&\quad=\|\alpha Y\|_{\mathbb{H}^2_{T,\hat\beta}}+\|\alpha Y_-\|_{\mathbb{H}^2_{T,\hat\beta}}+\|Z\|_{\mathbb{H}^2_{T,\hat\beta}(X)}+\|U\|_{\mathbb{H}^2_{T,\hat\beta}(\mu)}+\bigg\|\frac{f(0,0,0,\mathbf{0})}{\alpha}\bigg\|_{\mathbb{H}^2_{T,\hat\beta}}<\infty,
\end{aligned}$$

we deduce that $\|Y\|_{\mathcal{S}^2_{T,\hat\beta}}<\infty$. This completes the proof. □

Remark 6.4. (i) In the proof of Theorem 3.4 and 3.7, we could substitute the $\mathcal{S}^2_T$–norm $\|\cdot\|_{\mathcal{S}^2_T}$ by the $\mathcal{T}^2_T$–norm $\|\cdot\|_{\mathcal{T}^2_T}$ introduced in Remark 3.5. We merely need to keep in mind the changes to the *a priori* estimates in both Proposition 5.1 and 5.4 described in Remark 5.5. Therefore, modifying the contraction constants as described in Remark 3.5 and 3.8 would still ensure the desired well-posedness of our reflected BSDE and BSDE, as stated in Theorem 3.4 and 3.7, respectively.

(ii) In [113, Theorem 3.5] and its corresponding proof, the claim is that a weighted $\mathcal{S}^2$-type norm for the Y-component is also sufficient to construct a contraction map in the *BSDE* case. This corresponds to the norm $\|\cdot\|_{\star,\hat{\beta}}$ in their notation. It does not seem as though this is actually possible since by the Lipschitz property of the generator, we are forced to use an $\mathbb{H}^2$-type norm on Y in a fixed-point argument. Although in the classical cases this is possible since weighted $\mathcal{S}^2$-norms and $\mathbb{H}^2$-norms on Y are comparable, in their and our generality, the norms are not comparable.

7 A comparison principle for BSDEs

Comparison principles for BSDEs play a crucial role in the study of stochastic control problems as they give rise to necessary conditions optimisers ought to satisfy. In this section, we prove a comparison principle for our BSDEs. Given the several counterexamples for BSDEs with jumps in Barles, Buckdahn, and Pardoux [11], Royer [135] and Quenez and Sulem [132], we are forced to impose stronger assumptions on the generator. The conditions we lay out in this section will allow us to conclude that the operator which maps ξ_T to the first component $Y(\xi_T)$ of the solution to the BSDE with generator f and terminal condition ξ_T is monotone, that is, if $\xi_T \leq \xi'_T$, $\mathbb{P}$–a.s., then $Y(\xi_T) \leq Y(\xi'_T)$, $\mathbb{P}$–a.s. The method of proof we use is the classical linearisation and change of measure argument. Here, we suppose that we are given another generator $f' : \bigsqcup_{(\omega,t)\in\Omega\times[0,\infty)} \big(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m\times\mathfrak{H}_{\omega,t}\big) \longrightarrow \mathbb{R}$ such that $\Omega\times[0,\infty) \ni (\omega,t) \longmapsto f'_t\big(Y_t(\omega), Y_{t-}(\omega), Z_t(\omega), U_t(\omega;\cdot)\big) \in \mathbb{R}$, is optional for each $(Y,Z,U) \in \mathcal{S}^2_T\times\mathbb{H}^2_T(X)\times\mathbb{H}^2_T(\mu)$. The Lipschitz property of f then allows us to write

$$
\begin{aligned}
&f_s\big(\omega,y,\mathrm{y},z,u_s(\omega;\cdot)\big) - f'_s\big(\omega,y',\mathrm{y}',z',u'_s(\omega;\cdot)\big)\\
&\qquad\geq \lambda^{y,y'}_s(\omega)(y-y') + \widehat{\lambda}^{\mathrm{y},\mathrm{y}'}_s(\omega)(\mathrm{y}-\mathrm{y}') + \eta^{z,z',\top}_s(\omega)c_s(\omega)(z-z')\\
&\qquad\quad + f_s\big(\omega,y',\mathrm{y}',z',u_s(\omega;\cdot)\big) - f_s\big(\omega,y',\mathrm{y}',z',u'_s(\omega;\cdot)\big)\\
&\qquad\quad + f_s\big(\omega,y',\mathrm{y}',z',u'_s(\omega;\cdot)\big) - f'_s\big(\omega,y',\mathrm{y}',z',u'_s(\omega;\cdot)\big),
\end{aligned}
$$

where

$$
\lambda^{y,y'}_s(\omega) := -\sqrt{r_s(\omega)}\,\mathrm{sgn}(y-y'),\ \widehat{\lambda}^{\mathrm{y},\mathrm{y}'}_s(\omega) := -\sqrt{\mathrm{r}_s(\omega)}\,\mathrm{sgn}(\mathrm{y}-\mathrm{y}'),
$$
$$
\eta^{z,z'}_s(\omega) := -\sqrt{\theta^X_s(\omega)}\frac{(z-z')}{\big\|c^{1/2}_s(\omega)(z-z')\big\|}\mathbf{1}_{\{c^{1/2}_s(z-z')\neq 0\}}(\omega).
$$
[20]

The comparison result we present will be based on the following assumption.

Assumption 7.1. *The following conditions hold:*

(i) $\Phi < 1$*, and the non-negative random variables* $\int_0^T \sqrt{r_s}\mathrm{d}C_s$*,* $\int_0^T \sqrt{\mathrm{r}_s}\mathrm{d}C_s$ *and* $\int_0^T \theta^X_s\mathrm{d}C_s$ *are* $\mathbb{P}$–a.s. *bounded;*

(ii) *For each* $\mathbb{P}$–a.s. *làdlàg process* $Y \in \mathcal{S}^2_T$*, and* $(Z,Z',U,U') \in \mathbb{H}^2_T(X)\times\mathbb{H}^2_T(X)\times\mathbb{H}^2_T(\mu)\times\mathbb{H}^2_T(\mu)$*, there exists* $\rho = \rho^{Y,Z,U,U'} \in \mathbb{H}^2_T(\mu)$ *such that*[21] $\eta^{Z,Z'}_t\Delta X_{t\wedge T} + \Delta(\rho\,\star$

[20] Recall from Section 2.4 that $c^{1/2}$ is the unique (predictable) square-root matrix-valued process of c.

[21] Here, $\eta^{Z,Z'} = (\eta^{Z,Z'}_t)_{t\in[0,\infty)}$ denotes the predictable process defined by $\eta^{Z,Z'}_t(\omega) := \eta^{Z_t(\omega),Z'_t(\omega)}_t(\omega)$.

$\tilde{\mu})_{t\wedge T} > -1$, $\mathbb{P}$–a.s., $t \in [0,\infty)$, *the random variable* $\langle \rho\star\tilde{\mu}\rangle_T$ *is bounded,* $\mathbb{P}$–a.s., *and*

$$f_s\big(Y_s,Y_{s-},Z_s,U_s(\cdot)\big)-f_s\big(Y_s,Y_{s-},Z_s,U_s'(\cdot)\big)\geq\frac{\mathrm{d}\langle\rho\star\tilde{\mu},(U-U')\star\tilde{\mu}\rangle_s}{\mathrm{d}C_s},\ \mathbb{P}\times\mathrm{d}C_s\text{–a.e.} \tag{7.1}$$

Remark 7.2. (i) In the standard case, where there is only a Brownian motion X, the stopping time T is deterministic and finite, C satisfies $\mathrm{d}C_s = \mathrm{d}s$, and the Lipschitz-coefficients of the generator f are bounded, Assumption 7.1 is clearly satisfied. However, if additionally there is an integer-valued random measure μ such that its compensator can be written in the form $\nu(\mathrm{d}s,\mathrm{d}x) = F(\mathrm{d}x)\mathrm{d}t$, then (7.1) turns into

$$\begin{aligned}f_s\big(Y_s,Y_{s-},Z_s,U_s(\cdot)\big)&-f_s\big(Y_s,Y_{s-},Z_s,U_s'(\cdot)\big)\\&\geq\frac{\mathrm{d}\langle\rho\star\tilde{\mu},(U-U')\star\tilde{\mu}\rangle_s}{\mathrm{d}C_s}=\int_E\rho_s(x)U_s(x)F(\mathrm{d}x),\ \mathbb{P}\times\mathrm{d}s\text{–a.e.}\end{aligned}$$

This is now reminiscent of the classical (A_γ)–condition in [135, Section 2.2, p. 1362] or the assumption in [132, Theorem 4.1].

(ii) The conditions in Assumption 7.1 are simple enough to check in practice, but are not necessary, as some of them can be weakened by carefully redoing the proof of the comparison principle in Proposition 7.3. We will discuss this in Remark 7.5.

(iii) If the condition $\eta^{Z,Z}\Delta X_{\cdot\wedge T}+\Delta(\rho\star\tilde{\mu})_{\cdot\wedge T} > -1$ in Assumption 7.1. (ii) fails to hold, then the comparison principle for our BSDE is false in general. See [132, Example 3.1] for a counterexample in a Brownian–Poisson setting.

The following comparison principle is the main result of this section.

Proposition 7.3. *Suppose that* Assumption 7.1 *holds. Let* $(\xi_T,\xi_T')\in(\mathbb{L}^2(\mathcal{G}_T))^2$, *and suppose that* (Y,Z,U,N) *and* (Y',Z',U',N') *are solutions in* $\mathcal{S}^2_T\times\mathbb{H}^2_T(X)\times\mathbb{H}^2_T(\mu)\times\mathcal{H}^{2,\perp}_T(X,\mu)$ *to the* BSDEs *with generator* f *and terminal condition* ξ_T *and generator* f' *and terminal condition* ξ_T', *respectively. If* $\xi_T'\leq\xi_T$, $\mathbb{P}$ –a.s., *and*

$$f_s'\big(Y_s',Y_{s-}',Z_s',U_s'(\cdot)\big)\leq f_s\big(Y_s',Y_{s-}',Z_s',U_s'(\cdot)\big),\ \mathbb{P}\otimes\mathrm{d}C_s\text{–a.e.},$$

then $Y'\leq Y$ *up to* $\mathbb{P}$*–indistinguishability.*

The proof of this result is based on the following lemma.

Lemma 7.4. *Suppose that* $M\in\mathcal{M}_{\mathrm{loc}}$ *with* $M_0=0$ *is such that* $\langle M\rangle$ *is bounded, where* $\langle M\rangle$ *denotes the compensator of the optional quadratic variation* $[M]$*. Then* $\mathcal{E}(M)\in\mathcal{H}^2$.

Proof. Although this result follows from [76, Proposition 8.27], we would like to present another argument by following the proof of [99, Théorème II.3]. By [100, Proposition II.1], we can write

$$|\mathcal{E}(M)|^2=\mathcal{E}(M)\mathcal{E}(M)=\mathcal{E}(2M+[M])=\mathcal{E}(2M+[M]-\langle M\rangle+\langle M\rangle)=\mathcal{E}(\widetilde{N})\mathcal{E}(\langle M\rangle),$$

for some $\widetilde{N}\in\mathcal{M}_{\mathrm{loc}}$. Since $\langle M\rangle$ is non-decreasing, we have that $\mathcal{E}(\langle M\rangle)\leq\mathrm{e}^{\langle M\rangle}$. In particular, the local martingale satisfies $\mathcal{E}(\widetilde{N})\geq 0$ and is thus a supermartingale. Since $\langle M\rangle$ is bounded, say by $a\in(0,\infty)$, we can write $|\mathcal{E}(M)|^2\leq\mathcal{E}(\widetilde{N})\mathrm{e}^a$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence such that each stopped process $\mathcal{E}(M)_{\cdot\wedge\tau_n}$ is a uniformly integrable martingale, and fix $t\in(0,\infty)$. We find

$$\mathbb{E}\bigg[\sup_{s\in[0,t]}|\mathcal{E}(M)_{s\wedge\tau_n}|^2\bigg]\leq4\mathbb{E}[|\mathcal{E}(M)_{t\wedge\tau_n}|^2]\leq4\mathrm{e}^a\mathbb{E}[\mathcal{E}(\widetilde{N})_{t\wedge\tau_n}]\leq4\mathrm{e}^a\mathbb{E}[\mathcal{E}(\widetilde{N})_0]=4\mathrm{e}^a.$$

Here the first inequality follows from Doob's $\mathbb{L}^2$-inequality, and the third inequality follows from the optional stopping theorem (see [141, Theorem 3.2.7]) together with the fact that $\mathcal{E}(\widetilde{N})$ is a supermartingale. By applying Fatou's lemma twice, we deduce that $\mathbb{E}\big[\sup_{s\in[0,\infty)}|\mathcal{E}(M)_s|^2\big]\leq 4\mathrm{e}^a$. In particular, the stochastic exponential $\mathcal{E}(M)$ is a uniformly integrable martingale and is in $\mathcal{H}^2$. This completes the proof. □

Proof of Proposition 7.3. First, let us write

$$\delta Y\coloneqq Y-Y',\ \delta Z\coloneqq Z-Z',\ \delta U\coloneqq U-U',\ \delta N\coloneqq N-N',\ \delta\xi_T\coloneqq\xi_T-\xi'_T,\\ \delta f\coloneqq f\big(Y,Y_-,Z,U(\cdot)\big)-f'\big(Y',Y'_-,Z',U'(\cdot)\big),$$

and

$$\lambda_s(\omega)\coloneqq\lambda_s^{Y_s(\omega),Y'_s(\omega)}(\omega),\ \widehat{\lambda}_s(\omega)\coloneqq\widehat{\lambda}_s^{Y_{s-}(\omega),Y'_{s-}(\omega)}(\omega),\ \eta_s(\omega)\coloneqq\eta_s^{Z_s(\omega),Z'_s(\omega)}(\omega),\\ \rho_s(\omega;x)=\rho_s^{Y',Z',U,U'}(\omega;x),$$

for simplicity. Now consider $v\coloneqq\int_0^{\cdot\wedge T}\gamma_s\mathrm{d}C_s$, where $\gamma\coloneqq\frac{\widehat{\lambda}}{1-\widehat{\lambda}\Delta C}$. Here the process γ is predictable and the integral is well-defined by Assumption 7.1. (i) since $\widehat{\lambda}_s\Delta C_s\leq|\widehat{\lambda}_s|\Delta C_s\leq\sqrt{\mathrm{r}_s}\Delta C_s\leq\Phi<1$ and thus

$$|\gamma|\leq\frac{|\widehat{\lambda}|}{1-\Phi}\leq\frac{\sqrt{\mathrm{r}_s}}{1-\Phi}.$$

Moreover, $\Delta v=\gamma\Delta C=\widehat{\lambda}\Delta C/(1-\widehat{\lambda}\Delta C)>-1$. Thus, the stochastic exponential $\mathcal{E}(v)$ is positive and because v is predictable and of finite variation, the stochastic exponential $\mathcal{E}(v)$ is predictable and satisfies $0<\mathcal{E}(v)\leq\mathrm{e}^v$. In particular, $\mathcal{E}(v)$ is bounded by Assumption 7.1. (i). With the integration by parts formula, we derive

$$\begin{aligned}\mathrm{d}\big(\mathcal{E}(v)\delta Y\big)_s&=\mathcal{E}(v)_{s-}\mathrm{d}(\delta Y)_s+\delta Y_{s-}\mathrm{d}\mathcal{E}(v)_s+\mathrm{d}[\mathcal{E}(v),\delta Y]_s\\&=-\mathcal{E}(v)_{s-}\delta f_s\mathrm{d}C_s+\mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X)_s+\mathcal{E}(v)_{s-}\mathrm{d}(\delta U\star\tilde{\mu})_s+\mathcal{E}(v)_{s-}\mathrm{d}\delta N_s\\&\quad+\mathcal{E}(v)_{s-}\delta Y_{s-}\gamma_s\mathrm{d}C_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta Y]_s\\&=-\mathcal{E}(v)_{s-}\delta f_s\mathrm{d}C_s+\mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X)_s+\mathcal{E}(v)_{s-}\mathrm{d}(\delta U\star\tilde{\mu})_s+\mathcal{E}(v)_{s-}\mathrm{d}\delta N_s\\&\quad+\mathcal{E}(v)_{s-}\delta Y_{s-}\gamma_s\mathrm{d}C_s-\mathcal{E}(v)_{s-}\gamma_s\delta f_s\Delta C_s\mathrm{d}C_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta Z\boldsymbol{\cdot}X]_s\\&\quad+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta U\star\tilde{\mu}]_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta N]_s\\&=-\mathcal{E}(v)_{s-}\big(\delta f_s(1+\gamma_s\Delta C_s)-\gamma_s\delta Y_{s-}\big)\mathrm{d}C_s\\&\quad+\mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X)_s+\mathcal{E}(v)_{s-}\mathrm{d}(\delta U\star\tilde{\mu})_s+\mathcal{E}(v)_{s-}\mathrm{d}\delta N_s\\&\quad+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta Z\boldsymbol{\cdot}X]_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta U\star\tilde{\mu}]_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta N]_s.\end{aligned}\tag{7.2}$$

Note that the processes on the last line are local martingales by [77, Proposition I.4.49.(c)] since C is predictable. We now define the probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{G})$ through the density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\coloneqq\mathcal{E}(L)_\infty\coloneqq\mathcal{E}\bigg(\int_0^{\cdot\wedge T}\eta_s\mathrm{d}X_s+\rho\star\tilde{\mu}_{\cdot\wedge T}\bigg)_\infty.\tag{7.3}$$

Assumption 7.1 implies that $\eta\in\mathbb{H}^2(X^T)$, that $\mathcal{E}(L)$ is non-negative since $\Delta L=\eta\Delta X+\Delta(\rho\star\tilde{\mu})>-1$ and that $\langle L\rangle$ is bounded since

$$\langle L\rangle_T=\int_0^T\eta_s^\top c_s\eta_s\mathrm{d}C_s+\langle\rho\star\tilde{\mu}\rangle_T=\int_0^T\theta_s^X\mathrm{d}C_s+\langle\rho\star\tilde{\mu}\rangle_T.$$

Therefore, $\mathcal{E}(L)\in\mathcal{H}^2$ by Lemma 7.4, and $\mathcal{E}(L)_\infty=\mathcal{E}(L)_T\in\mathbb{L}^2(\mathcal{G}_T;\mathbb{P})$. We now rewrite the $\mathbb{P}$–local martingales appearing in (7.2) as $\mathbb{Q}$–semimartingales. By an application of Girsanov's theorem [76, Proposition 7.25 and 7.26], we find the $\mathbb{Q}$–semimartingale decompositions

$$\begin{aligned}
\mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X)_s &= \mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X-\langle\delta Z\boldsymbol{\cdot}X,L\rangle)_s+\mathcal{E}(v)_{s-}\mathrm{d}\langle\delta Z\boldsymbol{\cdot}X,L\rangle_s\\
&= \mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X-\langle\delta Z\boldsymbol{\cdot}X,L\rangle)_s+\mathcal{E}(v)_{s-}\mathrm{d}\langle\delta Z\boldsymbol{\cdot}X,\eta\boldsymbol{\cdot}X\rangle_s\\
&= \mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X-\langle\delta Z\boldsymbol{\cdot}X,L\rangle)_s+\mathcal{E}(v)_{s-}\eta_s^\top c_s\delta Z_s\mathrm{d}C_s,\\
\mathcal{E}(v)_{s-}\mathrm{d}(\delta U\star\tilde{\mu})_s &= \mathcal{E}(v)_{s-}\mathrm{d}\big(\delta U\star\tilde{\mu}-\langle\delta U\star\tilde{\mu},L\rangle\big)_s+\mathcal{E}(v)_{s-}\mathrm{d}\langle\delta U\star\tilde{\mu},L\rangle_s\\
&= \mathcal{E}(v)_{s-}\mathrm{d}\big(\delta U\star\tilde{\mu}-\langle\delta U\star\tilde{\mu},L\big)_s+\mathcal{E}(v)_{s-}\mathrm{d}\langle\delta U\star\tilde{\mu},\rho\star\tilde{\mu}\rangle_s,\\
\mathcal{E}(v)_{s-}\mathrm{d}\delta N_s &= \mathcal{E}(v)_{s-}\mathrm{d}(\delta N-\langle\delta N,L\rangle_s)+\mathcal{E}(v)_{s-}\mathrm{d}\langle\delta N,L\rangle_s\\
&= \mathcal{E}(v)_{s-}\mathrm{d}\delta N_s,
\end{aligned}$$

where the last equality follows from the $\mathbb{P}$–orthogonality of N with respect to X and μ from **(D2)**,

$$\begin{aligned}
&\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta Z\boldsymbol{\cdot}X]_s\\
&\quad= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta Z\boldsymbol{\cdot}X]-\big\langle[C,\delta Z\boldsymbol{\cdot}X],L\big\rangle\big)_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big\langle[C,\delta Z\boldsymbol{\cdot}X],L\big\rangle_s\\
&\quad= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta Z\boldsymbol{\cdot}X]-\big\langle[C,\delta Z\boldsymbol{\cdot}X],L\big\rangle\big)_s+\mathcal{E}(v)_{s-}\gamma_s\Delta C_s\mathrm{d}\langle\delta Z\boldsymbol{\cdot}X,L\rangle_s\\
&\quad= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta Z\boldsymbol{\cdot}X]-\big\langle[C,\delta Z\boldsymbol{\cdot}X],L\big\rangle\big)_s+\mathcal{E}(v)_{s-}\gamma_s\Delta C_s\eta_s^\top c_s\delta Z_s\mathrm{d}C_s,
\end{aligned}$$

here the second-to-last equality follows from $\mathrm{d}[C,\delta Z\boldsymbol{\cdot}X]_s=\Delta C\mathrm{d}(\delta Z\boldsymbol{\cdot}X)_s$, see [77, Proposition I.4.49], and similarly,

$$\begin{aligned}
\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta U\star\tilde{\mu}]_s &= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta U\star\tilde{\mu}]-\big\langle[C,\delta U\star\tilde{\mu}],L\big\rangle\big)_s\\
&\quad+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big\langle[C,\delta U\star\tilde{\mu}],L\big\rangle_s\\
&= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta U\star\tilde{\mu}]-\big\langle[C,\delta U\star\tilde{\mu}],L\big\rangle\big)_s\\
&\quad+\mathcal{E}(v)_{s-}\gamma_s\Delta C_s\mathrm{d}\langle\delta U\star\tilde{\mu},\rho\star\tilde{\mu}\rangle_s,\\
\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,N]_s &= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta N]-\big\langle[C,\delta N],L\big\rangle\big)_s+\mathcal{E}(v)_{s-}\mathrm{d}\big\langle[C,\delta N],L\big\rangle_s\\
&= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta N]-\big\langle[C,\delta N],L\big\rangle\big)_s+\mathcal{E}(v)_{s-}\Delta C_s\mathrm{d}\big\langle\delta N,L\big\rangle_s\\
&= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta N]-\big\langle[C,\delta N],L\big\rangle\big)_s\\
&= \mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta N]_s.
\end{aligned}$$

Inserting the previous identities into (7.2) yields

$$\begin{aligned}
\mathrm{d}\big(\mathcal{E}(v)\delta Y\big)_s &= -\mathcal{E}(v)_{s-}\big(\delta f_s(1+\gamma_s\Delta C_s)-\gamma_s\delta Y_{s-}\big)\mathrm{d}C_s+\mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X)_s\\
&\quad+\mathcal{E}(v)_{s-}\mathrm{d}(\delta U\star\tilde{\mu})_s+\mathcal{E}(v)_{s-}\mathrm{d}\delta N_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta Z\boldsymbol{\cdot}X]_s\\
&\quad+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta U\star\tilde{\mu}]_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta N]_s\\
&= -\mathcal{E}(v)_{s-}\big(\delta f_s(1+\gamma_s\Delta C_s)-\gamma_s\delta Y_{s-}\big)\mathrm{d}C_s+\mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X_s-\langle\delta Z\boldsymbol{\cdot}X,L\rangle)_s\\
&\quad+\mathcal{E}(v)_{s-}\eta_s^\top c_s\delta Z_s\mathrm{d}C_s+\mathcal{E}(v)_{s-}\mathrm{d}\big(\delta U\star\tilde{\mu}-\langle\delta U\star\tilde{\mu},L\rangle\big)_s\\
&\quad+\mathcal{E}(v)_{s-}\mathrm{d}\langle\delta U\star\tilde{\mu},\rho\star\tilde{\mu}\rangle_s+\mathcal{E}(v)_{s-}\mathrm{d}\delta N_s\\
&\quad+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta Z\boldsymbol{\cdot}X]-\big\langle[C,\delta Z\boldsymbol{\cdot}X],L\big\rangle\big)_s+\mathcal{E}(v)_{s-}\eta_s^\top c_s\delta Z_s\gamma_s\Delta C_s\mathrm{d}C_s\\
&\quad+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta U\star\tilde{\mu}]-\big\langle[C,\delta U\star\tilde{\mu}],L\big\rangle\big)_s\\
&\quad+\mathcal{E}(v)_{s-}\gamma_s\Delta C_s\mathrm{d}\langle\delta U\star\tilde{\mu},\rho\star\tilde{\mu}\rangle_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta N]_s\\
&= -\mathcal{E}(v)_{s-}\big(\delta f_s(1+\gamma_s\Delta C_s)-\gamma_s\delta Y_{s-}\big)\mathrm{d}C_s+\mathcal{E}(v)_{s-}\mathrm{d}(\delta Z\boldsymbol{\cdot}X_s-\langle\delta Z\boldsymbol{\cdot}X,L\rangle)_s\\
&\quad+\mathcal{E}(v)_{s-}\eta_s^\top c_s\delta Z_s(1+\gamma_s\Delta C_s)\mathrm{d}C_s+\mathcal{E}(v)_{s-}\mathrm{d}\big(\delta U\star\tilde{\mu}-\langle\delta U\star\tilde{\mu},L\rangle\big)_s\\
&\quad+\mathcal{E}(v)_{s-}(1+\gamma_s\Delta C_s)\mathrm{d}\langle\delta U\star\tilde{\mu},\rho\star\tilde{\mu}\rangle_s+\mathcal{E}(v)_{s-}\mathrm{d}\delta N_s
\end{aligned}$$

$$+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta Z\boldsymbol{\cdot}X]-\big\langle[C,\delta Z\boldsymbol{\cdot}X],L\big\rangle\big)_s$$
$$+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}\big([C,\delta U\star\tilde{\mu}]-\big\langle[C,\delta U\star\tilde{\mu}],L\big\rangle\big)_s+\mathcal{E}(v)_{s-}\gamma_s\mathrm{d}[C,\delta N]_s. \tag{7.4}$$

Note now that $\mathcal{E}(v)_-(1+\gamma\Delta C)=\mathcal{E}(v)>0$, and thus

$$\begin{aligned}\mathcal{E}(v)_{s-}&\delta f_s(1+\gamma_s\Delta C_s)\\&\geq\mathcal{E}(v)_{s-}\bigg(\lambda_s\delta Y_s+\widehat{\lambda}_s\delta Y_{s-}+\eta_s^\top c_s\delta Z_s+\frac{\mathrm{d}\langle\delta U\star\tilde{\mu},\rho\star\tilde{\mu}\rangle}{\mathrm{d}C_s}\bigg)(1+\gamma_s\Delta C_s)\\&=\mathcal{E}(v)_{s-}\bigg(\lambda_s\delta Y_s+\eta_s^\top c_s\delta Z_s+\frac{\mathrm{d}\langle\delta U\star\tilde{\mu},\rho\star\tilde{\mu}\rangle}{\mathrm{d}C_s}\bigg)(1+\gamma_s\Delta C_s)+\mathcal{E}(v)_{s-}\gamma_s\delta Y_{s-}.\end{aligned}$$

Therefore

$$\mathrm{d}\big(\mathcal{E}(v)\delta Y\big)_s\leq-\mathcal{E}(v)_{s-}\lambda_s\delta Y_s(1+\gamma_s\Delta C_s)\mathrm{d}C_s+\mathrm{d}M_s^{\mathbb{Q}}=-\mathcal{E}(v)_s\lambda_s\delta Y_s\mathrm{d}C_s+\mathrm{d}M_s^{\mathbb{Q}}.$$

Here we use $\mathcal{E}(v)=\mathcal{E}(v)_-(1+\gamma\Delta C)$ in the last line, and we denote by $M^{\mathbb{Q}}$ the sum of the $\mathbb{Q}$–local martingales appearing in (7.4). Consider now $w:=\int_0^{\cdot\wedge T}\lambda_s\mathrm{d}C_s$. Since $|\lambda_s\Delta C_s|=|\lambda_s|\Delta C_s\leq\sqrt{r_s}\Delta C_s\leq\Phi<1$, we have $\mathcal{E}(w)>0$. Moreover, since $|\lambda_s|\leq\sqrt{r_s}$, Assumption 7.1. (i) ensures that w is bounded. As before, this also implies that $\mathcal{E}(w)$ is bounded since $\mathcal{E}(w)\leq\mathrm{e}^w$. Another application of the integration by parts formula yields

$$\begin{aligned}\mathrm{d}(\mathcal{E}(w)\mathcal{E}(v)\delta Y)_s&=\mathcal{E}(w)_{s-}\mathrm{d}(\mathcal{E}(v)\delta Y)_s+(\mathcal{E}(v)\delta Y)_{s-}\mathrm{d}\mathcal{E}(w)_s+\mathrm{d}[\mathcal{E}(w),\mathcal{E}(v)\delta Y]\\&=\mathcal{E}(w)_{s-}\mathrm{d}(\mathcal{E}(v)\delta Y)_s+(\mathcal{E}(v)\delta Y)_s\mathrm{d}\mathcal{E}(w)_s\\&=\mathcal{E}(w)_{s-}\mathrm{d}(\mathcal{E}(v)\delta Y)_s+\mathcal{E}(w)_{s-}\mathcal{E}(v)_s\lambda_s\delta Y_s\mathrm{d}C_s\\&\leq-\mathcal{E}(w)_{s-}\mathcal{E}(v)_s\lambda_s\delta Y_s\mathrm{d}C_s+\mathcal{E}(w)_{s-}\mathrm{d}M_s^{\mathbb{Q}}+\mathcal{E}(w)_{s-}\mathcal{E}(v)_s\lambda_s\delta Y_s\mathrm{d}C_s\\&=\mathcal{E}(w)_{s-}\mathrm{d}M_s^{\mathbb{Q}}.\end{aligned}$$

This implies that, $\mathbb{P}$–a.s., for each $(t,t')\in[0,\infty)$ with $t\leq t'$, we have

$$\mathcal{E}(w)_{t\wedge T}\mathcal{E}(v)_{t\wedge T}\delta Y_{t\wedge T}\geq\mathcal{E}(w)_{t'\wedge T}\mathcal{E}(v)_{t'\wedge T}\delta Y_{t'\wedge T}-\int_{t\wedge T}^{t'\wedge T}\mathcal{E}(w)_{s-}\mathrm{d}M_s^{\mathbb{Q}}. \tag{7.5}$$

Let $(\tau_n)_{n\in\mathbb{N}}$ be a $\mathbb{Q}$–localising sequence such that for each $n\in\mathbb{N}$, the stopped process $\int_0^{\cdot\wedge\tau_n}\mathcal{E}(w)_{s-}\mathrm{d}M_s^{\mathbb{Q}}$, is a uniformly integrable martingale under $\mathbb{Q}$. By taking the conditional expectation in (7.5), we find

$$\mathcal{E}(w)_{t\wedge\tau_n\wedge T}\mathcal{E}(v)_{t\wedge\tau_n\wedge T}\delta Y_{t\wedge\tau_n\wedge T}\geq\mathbb{E}^{\mathbb{Q}}\bigg[\mathcal{E}(w)_{t'\wedge\tau_n\wedge T}\mathcal{E}(v)_{t'\wedge\tau_n\wedge T}\delta Y_{t'\wedge\tau_n\wedge T}\bigg|\mathcal{G}_t\bigg]. \tag{7.6}$$

Since $\mathcal{E}(v)$ and $\mathcal{E}(w)$ are bounded, $\delta Y\in\mathcal{S}_T^2$ and $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\in\mathbb{L}^2(\mathcal{G},\mathbb{P})$, we can apply dominated convergence with $n\longrightarrow\infty$ and find

$$\mathcal{E}(w)_{t\wedge T}\mathcal{E}(v)_{t\wedge T}\delta Y_{t\wedge\tau_n\wedge T}\geq\mathbb{E}^{\mathbb{Q}}\bigg[\mathcal{E}(w)_{t'\wedge T}\mathcal{E}(v)_{t'\wedge T}\delta Y_{t'\wedge T}\bigg|\mathcal{G}_t\bigg]. \tag{7.7}$$

Applying dominated convergence again, now letting $t'\uparrow\uparrow\infty$, yields

$$\mathcal{E}(w)_{t\wedge T}\mathcal{E}(v)_{t\wedge T}\delta Y_{t\wedge T}\geq\mathbb{E}^{\mathbb{Q}}\bigg[\mathcal{E}(w)_{\infty-}^T\mathcal{E}(v)_{\infty-}^T\delta Y_{\infty-}^T\bigg|\mathcal{G}_t\bigg]. \tag{7.8}$$

Since $\mathcal{G}_\infty=\mathcal{G}_{\infty-}=\bigvee_{t\in[0,\infty)}\mathcal{G}_t$, we deduce from

$$\int_0^t\delta f_s\mathbf{1}_{[0,T]}\mathrm{d}C_s+\delta Y_t^T=\mathbb{E}\bigg[\delta\xi_T+\int_0^T\delta f_s\mathrm{d}C_s\bigg|\mathcal{G}_t\bigg]=:M_t,\ t\in[0,\infty),$$

that

$$\int_0^T \delta f_s \mathrm{d}C_s + \delta Y^T_{\infty-} = M_{\infty-} = \delta\xi_T + \int_0^T \delta f_s \mathrm{d}C_s,\ \mathbb{P}\text{–a.s.},$$

and thus $\delta Y^T_{\infty-} = \delta\xi_T = \xi_T - \xi'_T \geq 0$, $\mathbb{P}$–a.s. Then (7.8) implies

$$\mathcal{E}(w)_{t\wedge T}\mathcal{E}(v)_{t\wedge T}\delta Y_{t\wedge T} \geq 0,\ \mathbb{Q}\text{–a.s.} \tag{7.9}$$

Because $\mathcal{E}(w)$ and $\mathcal{E}(v)$ are positive under $\mathbb{P}$ and thus under $\mathbb{Q}$, this implies that $\delta Y_{t\wedge T} \geq 0$, $\mathbb{Q}$–a.s. Since $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures on each $(\Omega, \mathcal{G}_t)$, $t \in [0,\infty)$, we also have $\delta Y_{t\wedge T} \geq 0$, $\mathbb{P}$–a.s., and therefore $Y \geq Y'$, $\mathbb{P}$–a.s. by ($\mathbb{P}$–a.s.) right-continuity of Y and Y'. This completes the proof. □

Remark 7.5. As we mentioned in Remark 7.2, one could weaken Assumption 7.1. For the arguments laid out in the proof of Proposition 7.3 to go through, we have to ensure the following:

(i) the probability measure $\mathbb{Q}$ defined in (7.3) has to be locally equivalent to $\mathbb{P}$, that is, the restriction $\mathbb{Q}_t$ of $\mathbb{Q}$ to $\mathcal{G}_t$ is equivalent to the restriction $\mathbb{P}_t$ of $\mathbb{P}$ to $\mathcal{G}_t$ for each $t \in [0,\infty)$;

(ii) we need to be able to let $n \to \infty$ and $t' \uparrow\uparrow \infty$ inside the conditional expectations of (7.6) and (7.7);

(iii) the stochastic exponentials $\mathcal{E}(w)$ and $\mathcal{E}(v)$ have to be positive, so that we can divide both sides of (7.9) by $\mathcal{E}(w)\mathcal{E}(v)$.

A Proofs of Section 2 and 3

Proof of Proposition 2.5. That $\|\cdot\|_{\mathbb{H}^2(\mu)}$ is a norm and that the stated equalities hold is clear from our preceding discussion. It remains to show completeness. Let $(U^k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{H}^2(\mu)$. Then, in particular,

$$\lim_{(k,\ell)\to\infty} \|U^k - U^\ell\|^2_{\mathbb{H}^2(\mu)} = \lim_{(k,\ell)\to\infty} \mathbb{E}\big[\langle (U^k - U^\ell)\star\tilde{\mu}\rangle_\infty\big] = \lim_{(k,\ell)\to\infty} \mathbb{E}\big[\langle U^k\star\tilde{\mu} - U^\ell\star\tilde{\mu}\rangle_\infty\big] = 0.$$

Hence the sequence $(U^k \star \tilde{\mu})_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{K}^2(\mu) \subseteq \mathcal{H}^2$. Since $\mathcal{K}^2(\mu)$ is closed in $\mathcal{H}^2$, there exists $U \star \tilde{\mu} \in \mathcal{K}^2(\mu)$ with

$$\lim_{k\to\infty} \mathbb{E}\big[\langle U^k \star \tilde{\mu} - U \star \tilde{\mu}\rangle_\infty\big] = 0.$$

Therefore $\lim_{k\to\infty} \|U^k - U\|^2_{\mathbb{H}^2(\mu)} = \lim_{k\to\infty} \mathbb{E}\big[\langle (U^k - U)\star\tilde{\mu}\rangle_\infty\big] = \lim_{k\to\infty} \mathbb{E}\big[\langle U^k\star\tilde{\mu} - U\star\tilde{\mu}\rangle_\infty\big] = 0$. □

Proof of Proposition 2.6. Let us start by showing that $\mathcal{L}^2(X)\cap\mathcal{K}^2(\mu)$ is the null subspace of $\mathcal{H}^2$. For $M \in \mathcal{L}^2(X)\cap\mathcal{K}^2(\mu)$, we can write

$$M = \int_{(0,\cdot]} Z_s \mathrm{d}X_s = \int_{(0,\cdot]\times E} U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x).$$

Therefore

$$\langle M, M\rangle = \left\langle M, \int_{(0,\cdot]\times E} U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x)\right\rangle = 0,$$

by [32, Theorem 13.3.16] or [76, Lemme 7.39] since $M_\mu\big[\Delta M\big|\widetilde{\mathcal{P}}\big] = M_\mu\big[Z^\top\Delta X\big|\widetilde{\mathcal{P}}\big] = \sum_{i=1}^n Z^i M_\mu\big[\Delta X^i\big|\widetilde{\mathcal{P}}\big] = 0$, which implies that $M = M_0 = 0$. We now define $\mathcal{H}^{2,\perp}(X,\mu) = \big(\mathcal{L}^2(X)\oplus\mathcal{K}^2(\mu)\big)^\perp$, which by the previous considerations lead to a decomposition $\mathcal{H}^2 =$

$\mathcal{L}^2(X) \oplus \mathcal{K}^2(\mu) \oplus \mathcal{H}^{2,\perp}(X,\mu)$. We now fix $M \in \mathcal{H}^2$. By the previous considerations, we can decompose M uniquely as

$$M = \int_{(0,\cdot]} Z_s \mathrm{d}X_s + \int_{(0,\cdot]\times E} U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) + N,$$

where uniqueness is meant in the spaces $\mathcal{L}^2(X)$, $\mathcal{K}^2(\mu)$ and $\mathcal{H}^{2,\perp}(X,\mu)$, respectively. In particular, since $X^i \in \mathcal{L}^2(X)$, we have that $\langle X^i, N\rangle = 0$. Moreover, by [77, Theorem III.4.20], we have $M_\mu[\Delta N|\widetilde{\mathcal{P}}] = 0$.

We now turn to the claimed uniqueness. Suppose that there exists $(Z', U', N') \in \mathbb{H}^2(X) \times \mathbb{H}^2(\mu) \times \mathcal{H}^{2,\perp}(X,\mu)$ satisfying

$$M = \int_{(0,\cdot]} Z'_s \mathrm{d}X_s + \int_{(0,\cdot]\times E} U'_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) + N',\ \langle N', X^i\rangle = 0,\ \text{for each } i \in \{1,\dots,n\},$$

and $M_\mu\big[\Delta N'\big|\widetilde{\mathcal{P}}\big] = 0$. We now show that $N' \in \mathcal{H}^{2,\perp}$. Since for any $H \in \mathbb{H}^2(X)$,

$$\left\langle N', \int_{(0,\cdot]} H_s \mathrm{d}X_s \right\rangle = \int_{(0,\cdot]} \left(\sum_{i=1}^m H^i_s c^{N',i}_s\right)\mathrm{d}C_s,$$

for a predictable process $c^{N,i}$ satisfying $0 = \langle N', X^i\rangle_\cdot = \int_{(0,\cdot]} c^{N',i}_s \mathrm{d}C_s$, we see that $c^{N,i}_s = 0$, $\mathrm{d}C_s$–a.e., $\mathbb{P}$–a.s., which then implies

$$\left\langle N', \int_{(0,\cdot]} H_s \mathrm{d}X_s \right\rangle = 0.$$

Hence $N' \in \big(\mathcal{L}^2(X)\big)^\perp$. Next, since $M_\mu[\Delta N'|\widetilde{\mathcal{P}}] = 0$, we have again by [32, Theorem 13.3.16] or [76, Lemme 7.39] that $\langle N', V \star \tilde{\mu}\rangle = 0$ for each $V \star \tilde{\mu}$, and therefore $N' \in (\mathcal{K}^2(\mu))^\perp$, so $N \in \mathcal{H}^{2,\perp}$. This implies that

$$M = \int_{(0,\cdot]} Z'_s \mathrm{d}X_s + \int_{(0,\cdot]\times E} U'_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) + N',$$

is a decomposition of M in $\mathcal{L}^2(X) \oplus \mathcal{K}^2(\mu) \oplus \mathcal{H}^{2,\perp}$. We therefore have $(Z, U, N) = (Z', U', N')$ in $\mathbb{H}^2(X) \times \mathbb{H}^2(\mu) \times \mathcal{H}^{2,\perp}$ since

$$\int_{(0,\cdot]} Z_s \mathrm{d}X_s = \int_{(0,\cdot]} Z'_s \mathrm{d}X_s,\ \text{and}\ \int_{(0,\cdot]\times E} U_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) = \int_{(0,\cdot]\times E} U'_s(x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x),$$

implies that $\|Z - Z'\|_{\mathbb{H}^2(X)} = 0$ and $\|U - U'\|_{\mathbb{H}^2(\mu)} = 0$, which then also implies $N = N'$. This completes the proof. □

Proof of Lemma 3.3. Note first that

$$\begin{aligned}
&\|\alpha^*\xi\|^2_{\mathbb{H}^2_{T,\hat\beta}} \\
&= \mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat\beta A)_s \lim_{s'\uparrow\uparrow s}\Big\{\sup_{t\in[s',\infty]}\big|\xi^+_s \mathbf{1}_{\{s<T\}}\big|^2\Big\}\mathrm{d}A_s\bigg] \\
&= \mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat\beta A)_s \lim_{s'\uparrow\uparrow s}\Big\{\sup_{t\in[s',\infty]}\big|\mathcal{E}(\beta^\star A)_s^{-1/2}\mathcal{E}(\beta^\star A)_s^{1/2}\xi^+_s \mathbf{1}_{\{s<T\}}\big|^2\Big\}\mathrm{d}A_s\bigg] \\
&\leq \mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat\beta A)_s \lim_{s'\uparrow\uparrow s}\mathcal{E}(\beta^\star A)_{s'}^{-1}\Big\{\sup_{t\in[s',\infty]}\big|\mathcal{E}(\beta^\star A)_s^{1/2}\xi^+_s \mathbf{1}_{\{s<T\}}\big|^2\Big\}\mathrm{d}A_s\bigg] \\
&= \mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat\beta A)_s\mathcal{E}(\beta^\star A)_{s-}^{-1} \lim_{s'\uparrow\uparrow s}\Big\{\sup_{t\in[s',\infty]}\big|\mathcal{E}(\beta^\star A)_s^{1/2}\xi^+_s \mathbf{1}_{\{s<T\}}\big|^2\Big\}\mathrm{d}A_s\bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\bigg[\int_0^T \mathcal{E}(\hat\beta A)_{s-}(1+\hat\beta\Delta A_s)\mathcal{E}(\beta^\star A)_{s-}^{-1}\lim_{s'\uparrow\uparrow s}\bigg\{\sup_{t\in[s',\infty]}\big|\mathcal{E}(\beta^\star A)_s^{1/2}\xi_s^+\mathbf{1}_{\{s<T\}}\big|^2\bigg\}\mathrm{d}A_s\bigg]\\
&\leq (1+\hat\beta\Phi)\mathbb{E}\bigg[\sup_{s\in[0,\infty]}\big|\mathcal{E}(\beta^\star A)_s^{1/2}\xi_s^+\mathbf{1}_{\{s<T\}}\big|^2\int_0^T \mathcal{E}(\hat\beta A)_{s-}\mathcal{E}(\beta^\star A)_{s-}^{-1}\mathrm{d}A_s\bigg]\\
&= (1+\hat\beta\Phi)\mathbb{E}\bigg[\sup_{s\in[0,\infty]}\big|\mathcal{E}(\beta^\star A)_s^{1/2}\xi_s^+\mathbf{1}_{\{s<T\}}\big|^2\int_0^T \mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\mathrm{d}A_s\bigg],
\end{aligned}
$$

where $A^{\hat\beta,\beta^\star} = -(\beta^\star-\hat\beta)A^c - \sum_{s\in(0,\cdot]}\frac{(\beta^\star-\hat\beta)\Delta A_s}{1+\beta^\star\Delta A_s}$, by Lemma C.1. (ii) and A^c denotes the continuous part of A. Next,

$$
\begin{aligned}
&\int_0^{t\wedge T}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\mathrm{d}A_s\\
&= \int_0^{t\wedge T}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\mathrm{d}A^c_s + \sum_{s\in(0,t\wedge T]}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\Delta A_s\\
&= -\frac{1}{(\beta^\star-\hat\beta)}\int_0^{t\wedge T}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\mathrm{d}(A^{\hat\beta,\beta^\star})^c_s\\
&\quad - \sum_{s\in(0,t\wedge T]}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\frac{(1+\beta^\star\Delta A_s)}{(\beta^\star-\hat\beta)}\Delta(A^{\hat\beta,\beta^\star})_s\\
&= \int_0^{t\wedge T}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\frac{(1+\beta^\star\Delta A_s)}{(\beta^\star-\hat\beta)}\mathrm{d}(-A^{\hat\beta,\beta^\star})^c_s\\
&\quad + \sum_{s\in(0,t\wedge T]}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\frac{(1+\beta^\star\Delta A_s)}{(\beta^\star-\hat\beta)}\Delta(-A^{\hat\beta,\beta^\star})_s.
\end{aligned}
$$

Since $-A^{\hat\beta,\beta^\star}$ is non-decreasing, the stochastic exponential $\mathcal{E}(A^{\hat\beta,\beta^\star}) = \mathcal{E}(\hat\beta A)/\mathcal{E}(\beta^\star A)$ is non-negative, and $\beta^\star-\hat\beta>0$, we deduce that

$$
\begin{aligned}
\int_0^{t\wedge T}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\mathrm{d}A_s &\leq \frac{(1+\beta^\star\Phi)}{(\beta^\star-\hat\beta)}\int_0^{t\wedge T}\mathcal{E}(A^{\hat\beta,\beta^\star})_{s-}\mathrm{d}(-A^{\hat\beta,\beta^\star})_s\\
&= \frac{(1+\beta^\star\Phi)}{(\beta^\star-\hat\beta)}\Big(1-\mathcal{E}(A^{\hat\beta,\beta^\star})_{t\wedge T}\Big) \leq \frac{(1+\beta^\star\Phi)}{(\beta^\star-\hat\beta)},
\end{aligned}
$$

where in the last inequality we used that $0<\mathcal{E}(A^{\hat\beta,\beta^\star})\leq 1$ since $1\leq\mathcal{E}(\hat\beta A)\leq\mathcal{E}(\beta^\star A)$. We thus conclude that

$$
\begin{aligned}
\|\alpha^*\xi\|^2_{\mathbb{H}^2_{T,\hat\beta}} &\leq \frac{(1+\beta^\star\Phi)(1+\hat\beta\Phi)}{(\beta^\star-\hat\beta)}\mathbb{E}\bigg[\sup_{s\in[0,\infty]}\big|\mathcal{E}(\beta^\star A)_s^{1/2}\xi_s^+\mathbf{1}_{\{s<T\}}\big|^2\bigg]\\
&= \frac{(1+\beta^\star\Phi)(1+\hat\beta\Phi)}{(\beta^\star-\hat\beta)}\|\xi^+_\cdot\mathbf{1}_{\{\cdot<T\}}\|^2_{\mathcal{S}^2_{T,\beta^\star}}.
\end{aligned}
$$

This completes the proof. □

B Proofs of technical lemmata

Lemma B.1. *Let $\Psi\in[0,\infty)$, and let $(\mathfrak{f}^\Psi,\mathfrak{g}^\Psi):(0,\infty)\to\mathbb{R}\times\mathbb{R}$ be defined by*

$$
\mathfrak{f}^\Psi(\beta) := \inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\beta\Psi)}{\gamma(\beta-\gamma)}\bigg\} \text{ and } \mathfrak{g}^\Psi(\beta) := \inf_{\gamma\in(0,\beta)}\bigg\{\frac{(1+\gamma\Psi)}{\gamma(\beta-\gamma)}\bigg\}.
$$

Then, for $\beta\in(0,\infty)$,

$$
\mathfrak{f}^\Psi(\beta) = \frac{4(1+\beta\Psi)}{\beta^2} \text{ and } \mathfrak{g}^\Psi(\beta) = \frac{4}{\beta^2}\mathbf{1}_{\{\Psi=0\}} + \frac{\Psi^2\sqrt{1+\beta\Psi}}{\big(1+\beta\Psi-\sqrt{1+\beta\Psi}\big)\big(\sqrt{1+\beta\Psi}-1\big)}\mathbf{1}_{\{\Psi>0\}}.
$$

Proof. Fix $\beta \in (0,\infty)$, and let $f:(0,\beta)\to\mathbb{R}$ be given by $f(\gamma) \coloneqq \frac{(1+\beta\Psi)}{\gamma(\beta-\gamma)}$, so that $\mathfrak{f}^\Psi(\beta) = \inf f$. Note that f is strictly convex since

$$\frac{\partial^2 f}{\partial\gamma^2} = \frac{2(\beta^2 - 3\gamma\beta + 3\gamma^2)(1+\beta\Psi)}{\gamma^3(\beta-\gamma)^3},$$

because

$$\begin{aligned}\beta^2 - 3\gamma\beta + 3\gamma^2 &= \beta^2 - 2\frac{3}{2}\gamma\beta + \frac{9}{4}\gamma^2 - \frac{9}{4}\gamma^2 + 3\gamma^2\\ &= \left(\beta - \frac{3}{2}\gamma\right)^2 - \frac{9}{4}\gamma^2 + \frac{12}{4}\gamma^2 = \left(\beta - \frac{3}{2}\gamma\right)^2 + \frac{3}{4}\gamma^2 > 0,\end{aligned} \tag{B.1}$$

and as f tends to infinity at the boundaries of the interval $(0,\beta)$, there is a unique critical point $\gamma^\star$ in $(0,\beta)$ at which f attains its minimum. The point $\gamma^\star$ satisfies

$$\frac{\partial f}{\partial\gamma}(\gamma^\star) = \frac{(2\gamma^\star-\beta)(1+\beta\Psi)}{(\gamma^\star)^2(\beta-\gamma^\star)^2} = 0,$$

and therefore $\gamma^\star = \beta/2$. This implies $\mathfrak{f}^\Psi(\beta) = \inf f = f(\gamma^\star) = \frac{4(1+\beta\Psi)}{\beta^2}$.

We turn to $\mathfrak{g}^\Psi$. Let $g:(0,\beta)\longrightarrow\mathbb{R}$ be given by $g(\gamma) \coloneqq \frac{(1+\gamma\Psi)}{\gamma(\beta-\gamma)}$. Then $\mathfrak{g}^\Psi(\beta) = \inf g$. We also note here that g tends to infinity at the boundary of $(0,\beta)$. Similar to before, g is strictly convex since

$$\frac{\partial^2 g}{\partial\gamma^2} = \frac{2(\beta^2 - 3\gamma\beta + 3\gamma^2 + \gamma^3\Psi)}{\gamma^3(\beta-\gamma)^3} > 0,$$

where the strict inequality follows from (B.1). As before, we now only need to find the unique critical point $\gamma^\star$ of g in $(0,\beta)$. Suppose first that $\Psi\in(0,\infty)$. From

$$\frac{\partial g}{\partial\gamma}(\gamma^\star) = \frac{2\gamma - \beta + \gamma^2\Psi}{\gamma^2(\beta-\gamma)^2} = 0,$$

we find that $\gamma^\star_{1,2} = \frac{-1\pm\sqrt{1+\beta\Psi}}{\Psi}$, and thus the critical point we are looking for is $\gamma^\star = (-1+\sqrt{1+\beta\Psi})/\Psi$. This implies

$$\mathfrak{g}^\Psi(\beta) = \inf g = g(\gamma^\star) = \frac{\Psi^2\sqrt{1+\beta\Psi}}{\big(1+\beta\Psi - \sqrt{1+\beta\Psi}\big)\big(\sqrt{1+\beta\Psi}-1\big)}.$$

In case $\Psi = 0$, we have $\mathfrak{g}^0(\beta) = \mathfrak{f}^0(\beta) = 4/\beta^2$. This completes the proof. $\square$

Lemma B.2. *Let* $\Psi\in[0,\infty)$. *Then*

$$\lim_{\beta\uparrow\uparrow\infty}\mathfrak{f}^\Psi(\beta) = 0 \text{ and } \lim_{\beta\uparrow\uparrow\infty}\beta\mathfrak{g}^\Psi(\beta) = \Psi.$$

Furthermore, for each $i\in\{1,2,3\}$, *the constant* $M_i^\Psi(\beta)$ *is decreasing in* β *and*

$$\lim_{\beta\uparrow\uparrow\infty} M_1^\Psi(\beta) = \max\{1,\Psi\}\Psi,\ \lim_{\beta\uparrow\uparrow\infty} M_2^\Psi(\beta) = \Psi \textit{ and } \lim_{\beta\uparrow\uparrow\infty} M_3^\Psi(\beta) = \max\{1,\Psi\}\Psi.$$

Proof. It is clear that $(1+\beta\Psi)/\beta$, $\mathfrak{f}^\Psi(\beta)$ and $\mathfrak{g}^\Psi(\beta)$ are decreasing in β. Moreover, $(1+\beta\Psi)/\beta$ converges to Ψ, $\mathfrak{f}^\Psi(\beta)$ converges to zero and $\beta\mathfrak{g}^\Psi(\beta)$ converges to Ψ by Lemma B.1 as β tends to infinity. The stated limits of M_1^Ψ, M_2^Ψ and M_3^Ψ thus follow immediately. $\square$

The following result can be deduced similarly, we thus omit its proof.

Lemma B.3. *For each* $i\in\{1,2,3\}$, *the constant* $\widetilde{M}^\Psi(\beta)$ *is decreasing in* β, *and*

$$\lim_{\beta\to\infty}\widetilde{M}_1^\Psi(\beta) = \max\{1,\Psi\}\Psi,\ \lim_{\beta\to\infty}\widetilde{M}_2^\Psi(\beta) = \Psi, \text{ and } \lim_{\beta\to\infty}\widetilde{M}_3^\Psi(\beta) = \max\{1,\Psi\}\Psi.$$

We now prove the technical lemmata of Section 4.

Proof of Lemma 4.1. We clearly have $L = L_{\cdot\wedge T}$, $\mathbb{P}$–a.s., and since $-|M| - 1 \leq L \leq \xi^+_{\cdot\wedge T} + \int_0^T |f_u|\mathrm{d}C_u + |M| + 1$, up to $\mathbb{P}$–indistinguishability, we find using (4.1) and Doob's $\mathbb{L}^2$-inequality that

$$\mathbb{E}\bigg[\sup_{s\in[0,T]} |L_s|^2\bigg] = \mathbb{E}\bigg[\sup_{s\in[0,\infty]} |L_s|^2\bigg] < \infty.$$

Next, it is clear that $L \geq J$ and thus $V(S) \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{S,\infty}}{}^{\mathcal{G}_S}\mathbb{E}[L_\tau|\mathcal{G}_S]$, $\mathbb{P}$–a.s., $S \in \mathcal{T}_{0,\infty}$. We turn to the converse inequality. Fix $S \in \mathcal{T}_{0,\infty}$, let $\tau \in \mathcal{T}_{S,\infty}$ and let $\hat{\tau} := \tau\mathbf{1}_{\{J_\tau\geq L_\tau\}} + \infty\mathbf{1}_{\{J_\tau<L_\tau\}} \in \mathcal{T}_{S,\infty}$. Then

$$\begin{aligned}
\mathbb{E}[L_\tau|\mathcal{G}_S] &= \mathbb{E}[L_\tau\mathbf{1}_{\{J_\tau\geq L_\tau\}} + \big(M_\tau - \mathbf{1}_{\{\tau<T\}}\big)\mathbf{1}_{\{J_\tau<L_\tau\}}|\mathcal{G}_S] \\
&\leq \mathbb{E}[L_\tau\mathbf{1}_{\{J_\tau\geq L_\tau\}} + M_\tau\mathbf{1}_{\{J_\tau<L_\tau\}}|\mathcal{G}_S] \\
&\leq \mathbb{E}[J_\tau\mathbf{1}_{\{J_\tau\geq L_\tau\}} + \mathbb{E}[J_\infty|\mathcal{G}_\tau]\mathbf{1}_{\{J_\tau<L_\tau\}}|\mathcal{G}_S] \\
&= \mathbb{E}[J_\tau\mathbf{1}_{\{J_\tau\geq L_\tau\}} + J_\infty\mathbf{1}_{\{J_\tau<L_\tau\}}|\mathcal{G}_S] = \mathbb{E}[J_{\hat{\tau}}|\mathcal{G}_S] \leq V(S),\ \mathbb{P}\text{–a.s.},
\end{aligned}$$

and therefore $\operatorname{ess\,sup}^{\mathcal{G}_S}_{\tau\in\mathcal{T}_{S,\infty}}\mathbb{E}\big[L_\tau\big|\mathcal{G}_S\big] \leq V(S)$, $\mathbb{P}$–a.s.$, S \in \mathcal{T}_{0,\infty}$, which completes the proof. □

Proof of Lemma 4.2. We only show that $V(S) \in \mathbb{L}^2(\mathcal{G}_S)$, for each $S \in \mathcal{T}_{0,\infty}$, as the rest follows from Proposition 1.3 and Proposition 1.5 in [92][22] together with the argument in [62, Footnote 4]. Fix $S \in \mathcal{T}_{0,\infty}$, and note that

$$-\mathbb{E}\big[|\xi_T|\big|\mathcal{G}_S\big] - \mathbb{E}\bigg[\int_0^T |f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg] \leq V(S) \leq \mathbb{E}\bigg[|\xi_T| + \sup_{u\in[0,T)}\xi^+_u + \int_0^T |f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},$$

implies

$$|V(S)| \leq \mathbb{E}\bigg[|\xi_T| + \sup_{u\in[0,T)}\xi^+_u + \int_0^T |f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},$$

from which we deduce that $V(S) \in \mathbb{L}^2(\mathcal{G}_S)$ by (4.1). This completes the proof. □

Proof of Lemma 4.3. The existence of the process $V = (V_t)_{t\in[0,\infty]}$ is a mere application Lemma 4.2 and [41, Appendix I, Remark 23(b)]. Next, let $S \in \mathcal{T}_{0,\infty}$. Since $V_T = V(T) = V(S \vee T) = V_{S\vee T}$, $\mathbb{P}$–a.s., we find

$$V_{S\wedge T} = V_S\mathbf{1}_{\{S\leq T\}} + V_T\mathbf{1}_{\{S>T\}} = V_S\mathbf{1}_{\{S\leq T\}} + V_{S\vee T}\mathbf{1}_{\{S>T\}} = V_S,\ \mathbb{P}\text{–a.s.}$$

In particular, $V_\cdot = V_{\cdot\wedge T}$ up to $\mathbb{P}$–indistinguishability by Proposition C.3.

The fact that V is in $\mathcal{S}^2_T$ can be argued as follows. Let $N = (N_t)_{t\in[0,\infty]}$ be the martingale satisfying

$$N_S = \mathbb{E}\bigg[|\xi_T| + \sup_{u\in[0,T)}|\xi^+_u| + \int_0^T |f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg],\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty}.$$

Note that N is square-integrable by (4.1). As in the proof of Lemma 4.2, we find that

$$|V_S| \leq \mathbb{E}\bigg[|\xi_T| + \sup_{u\in[0,T)}|\xi^+_u| + \int_0^T |f_u|\mathrm{d}C_u\bigg|\mathcal{G}_S\bigg] = N_S,\ \mathbb{P}\text{–a.s.},\ S \in \mathcal{T}_{0,\infty},$$

[22] The fact that the filtration is complete and the horizon is finite in [92] is not relevant for the proof of their Proposition 1.3 and Proposition 1.5.

Thus $|V| \leq N$ up to $\mathbb{P}$–indistinguishability by Proposition C.3, and with Doob's $\mathbb{L}^2$-inequality for martingales, we find

$$\mathbb{E}\bigg[\sup_{s\in[0,\infty]}|V_{s\wedge T}|^2\bigg] \leq \mathbb{E}\bigg[\sup_{s\in[0,\infty]}|N_s|^2\bigg] \leq 4\mathbb{E}\big[|N_\infty|^2\big] < \infty,$$

which completes the proof. □

Proof of Lemma 4.4. We start with (i). Let $S \in \mathcal{T}_{0,\infty}$. Note that on $\{S < T\}$, we have $\mathbf{1}_{\{V_S = L_S\}} = \mathbf{1}_{\{V_S = J_S\}}$, $\mathbb{P}$–a.s., since $V_S \geq M_S > M_S - 1$, $\mathbb{P}$–almost surely. On $\{S \geq T\}$, we have that $V_S = V_T = J_T$, $\mathbb{P}$–a.s., and therefore $\mathbf{1}_{\{V_S = L_S\}} = \mathbf{1}_{\{V_S = J_S\}}$, $\mathbb{P}$–almost surely.

We turn to (ii). Let $S \in \mathcal{T}^p_{0,\infty}$. On $\{S \leq T\}$, we have $\mathbf{1}_{\{V_{S-} = \bar{L}_S\}} = \mathbf{1}_{\{V_{S-} = \bar{J}_S\}}$ since $V_{S-} \geq M_{S-} > M_{S-} - 1$, $\mathbb{P}$–almost surely. On $\{S > T\}$, we have $V_{S-} = V_T = J_T = J_{S-}$, $\mathbb{P}$–a.s., and therefore $\mathbf{1}_{\{V_{S-} = \bar{L}_S\}} = \mathbf{1}_{\{V_{S-} = \bar{J}_S\}}$, $\mathbb{P}$–a.s., which completes the proof. □

C Miscellaneous

Lemma C.1. *Let A and B be two optional, real-valued processes with $\mathbb{P}$–a.s. right-continuous and non-decreasing paths such that $A_0 = B_0 = 0$, $\mathbb{P}$–almost surely. Then the following holds:*

(i) $\mathcal{E}(A)^{-1} = \mathcal{E}(-\overline{A})$, *where* $\overline{A} = A - \sum_{s\in(0,\cdot]} \frac{(\Delta A_s)^2}{1+\Delta A_s}$;

(ii) $\mathcal{E}(A)^{-1}\mathcal{E}(B) = \mathcal{E}(C)$, *where* $C = B^c - A^c + \sum_{s\in(0,\cdot]} \frac{\Delta B_s - \Delta A_s}{1+\Delta A_s}$;

(iii) $\mathcal{E}(A)^{1/2} = \mathcal{E}(D)$, *where* $D = \frac{1}{2}A^c + \sum_{s\in(0,\cdot]}\big(\sqrt{1+\Delta A_s} - 1\big)$.

Proof. Assertion (i) follows from [31, Lemma 4.4]. We prove (ii). The product formula for stochastic exponentials yields $\mathcal{E}(A)^{-1}\mathcal{E}(B) = \mathcal{E}(C)$, where $C = -\overline{A} + B + -[\overline{A}, B]$. By differentiating the continuous part C^c of C from the purely discontinuous part C^d, we can explicitly write

$$\begin{aligned}
C &= -\overline{A} + B - [\overline{A}, B] \\
&= -A^c + B^c + \sum_{s\in(0,\cdot]}\bigg(-\Delta A_s + \frac{(\Delta A_s)^2}{1+\Delta A_s} + \Delta B_s - \bigg(\Delta A_s - \frac{(\Delta A_s)^2}{1+\Delta A_s}\bigg)\Delta B_s\bigg) \\
&= B^c - A^c + \sum_{s\in(0,\cdot]}\bigg(-\frac{\Delta A_s}{1+\Delta A_s} + \frac{\Delta B_s + \Delta A_s\Delta B_s}{1+\Delta A_s} - \frac{\Delta A_s \Delta B_s}{1+\Delta A_s}\bigg) \\
&= B^c - A^c + \sum_{s\in(0,\cdot]} \frac{\Delta B_s - \Delta A_s}{1+\Delta A_s}.
\end{aligned}$$

Assertion (iii) follows by squaring $\mathcal{E}(D)$, using the product formula for stochastic exponentials, and then checking that this coincides with $\mathcal{E}(A)$. This completes the proof. □

Proposition C.2. *Let A be a non-decreasing, $[0,\infty]$-valued function with the conventions $A_{0-} := 0$, $A_{\infty-} := \lim_{t\uparrow\uparrow\infty} A_t$ and $A_\infty := \infty$. Let $R = (R_t)_{t\in[0,\infty]}$ and $L = (L_t)_{t\in[0,\infty]}$ be defined by*

$$R_t := \begin{cases} \inf\{s \in [0,\infty) : A_s > t\},\ t \in [0,\infty), \\ \infty,\ t = \infty, \end{cases}$$

with the conventions $\inf\varnothing = \infty$ *and* $R_{0-} = 0$, *and* $L_t := R_{t-}$, $t \in [0,\infty]$. *For any* $t \in [0,\infty]$

(i) $t \leq A_{L_t} = A_{R_{t-}} \leq A_{R_t}$;

(ii) $A_{L_t-} = A_{R_{t-}-} \leq A_{R_t-} \leq t$;

(iii) *for any* $s \in [0,\infty]$, $t \leq A_s$ *if and only if* $L_t \leq s$;

(iv) *for any* $s \in [0,\infty]$, *if* $t < A_s$, *then* $R_t \leq s$ (*the other direction is false in general, and equality is possible*);

(v) *for any function* f *on* $[0,\infty)$ *which is non-negative and Borel-measurable, if* A *is finite on* $[0,\infty)$, *then*

$$\int_{[0,\infty)} f(s)\mathrm{d}A_s = \int_{[0,\infty)} f(L_s)\mathbf{1}_{\{L<\infty\}}(s)\mathrm{d}s = \int_{[0,\infty)} f(R_s)\mathbf{1}_{\{R<\infty\}}(s)\mathrm{d}s;$$

(vi) *for any non-decreasing, Borel-measurable,* ℓ*–sub-multiplicative*[23] $g : [0,\infty) \longrightarrow [0,\infty)$, *and if* A *is finite on* $[0,\infty)$, *then*

$$\int_{(0,t]} g(A_s)\mathrm{d}A_s \leq \ell g\bigg(\max_{\{s:L_s<\infty\}} \Delta A_{L_s}\bigg) \int_{(A_0,A_t]} g(s)\mathrm{d}s.$$

Proof. For (i) through (v) see [41, page 119–120] or [72, page 21–22], for (vi) see the proof of Lemma 2.14 in [113]. □

Since we have not been able to locate a reference for the following result, we prove it for the convenience of the reader.

Proposition C.3. *Suppose that* $X = (Y_t)_{t\in[0,\infty]}$ *and* $Y = (Y_t)_{t\in[0,\infty]}$ *are two optional processes for which* $X_S \leq Y_S$, $\mathbb{P}$ –a.s., *holds for each* $S \in \mathcal{T}_{0,\infty}$. *Then* $X \leq Y$ *holds up to* $\mathbb{P}$*-evanescence.*

Proof. Let $\varepsilon > 0$, and let $A := \{(\omega,t) \in \Omega\times[0,\infty) : X_s(\omega) > Y_s(\omega)\}$, which is an optional subset of $\Omega\times[0,\infty)$. By [40, Theorem IV.84], there is a $\mathbb{G}$–stopping time S such that for all $\omega \in \Omega$ with $S(\omega) < \infty$, we have $(\omega, S(\omega)) \in A$ and $\mathbb{P}[S<\infty] \geq \mathbb{P}[\pi_\Omega(A)] - \varepsilon$. Here, $\pi_\Omega(A)$ is the projection of A onto Ω, which is $\mathcal{G}^U_\infty$-measurable. Since $\mathbb{P}[S<\infty] \leq \mathbb{P}[S<\infty, X_S > Y_S] = 0$ and $\varepsilon > 0$ was arbitrary, it follows that $\mathbb{P}[\pi_\Omega(A)] = 0$. This, with $\mathbb{P}[X_\infty > Y_\infty] = 0$, implies the claim. □

Lemma C.4. *Suppose that* $M = (M_t)_{t\in[0,\infty]}$ *is a non-negative,* $\mathbb{P}$ –a.s. *right-continuous,* $(\mathbb{F},\mathbb{P})$*-martingale with* $M_\infty \in \mathbb{L}^2(\mathcal{G}_\infty)$. *Then*

$$\mathbb{E}\bigg[\sup_{s\in[t,\infty]} M_s^2 \bigg| \mathcal{G}_t\bigg] \leq 4\mathbb{E}[M_\infty^2|\mathcal{G}_t],\ \mathbb{P}\text{–a.s.},\ t\in[0,\infty]$$

Proof. First, the condition that $M_\infty \in \mathbb{L}^2(\mathcal{G}_\infty)$ implies already that $M_t \in \mathbb{L}^2(\mathcal{G}_t)$ for each $t \in [0,\infty)$. Moreover, the result is trivial for $t=\infty$. So we fix $t \in [0,\infty)$, and consider a subdivision $t = t_0 < t_1 < \cdots < t_N = \infty$. By Proposition 2.1 in [1], we have that

$$\max_{k\in\{0,\ldots,N\}} M_{t_k}^2 \leq -4\sum_{\ell=0}^{N-1} \max_{k\in\{0,\ldots,\ell\}} M_{t_k}\big(M_{t_{\ell+1}} - M_{t_\ell}\big) - 2M_{t_0}^2 + 4M_{t_N}^2.$$

For each $\ell \in \{0,1,\ldots,N-1\}$,

$$\bigg|\max_{k\in\{0,\ldots,\ell\}} M_{t_k}\big(M_{t_{\ell+1}} - M_{t_\ell}\big)\bigg| \leq \frac{1}{2}\max_{k\in\{0,\ldots,\ell\}} M_{t_k}^2 + \frac{1}{2}\big(M_{t_{\ell+1}} - M_{t_\ell}\big)^2,$$

[23] This means that $\ell \in (0,\infty)$ and

$$g(x+y) \leq \ell g(x)g(y),\ (x,y)\in[0,\infty)^2.$$

and the right-hand side in the previous inequality is integrable. Therefore, by the martingale property of M, we deduce that

$$\begin{aligned}
&\mathbb{E}\bigg[\max_{k\in\{0,\dots,N\}} M_{t_k}^2\bigg|\mathcal{G}_{t_0}\bigg] \\
&\quad\leq \mathbb{E}\bigg[-4\sum_{\ell=0}^{N-1}\max_{k\in\{0,\dots,\ell\}} M_{t_k}\big(M_{t_{\ell+1}}-M_{t_\ell}\big)\bigg|\mathcal{G}_{t_0}\bigg] - \mathbb{E}\big[2M_{t_0}^2\big|\mathcal{G}_{t_0}\big] + 4\mathbb{E}\big[M_{t_N}^2\big|\mathcal{G}_{t_0}\big] \\
&\quad\leq 4\mathbb{E}\big[M_{t_N}^2\big|\mathcal{G}_{t_0}\big].
\end{aligned}$$

The claim then follows by approximating the supremum of M on $[t,\infty]$ and then using Fatou's lemma for conditional expectations, which completes the proof. ☐

Lemma C.5. *Let $L=(L_t)_{t\in[0,\infty]}$ and $J=(J_t)_{t\in[0,\infty]}$ be two product-measurable processes whose $\mathbb{P}$–almost all paths admit limits from the left on $(0,\infty]$. Suppose that $\mathbb{E}[\sup_{t\in[0,\infty]}|L_t|]+\mathbb{E}[\sup_{t\in[0,\infty]}|J_t|]<\infty$ and that $\mathbb{E}[L_t|\mathcal{G}_t]\leq\mathbb{E}[J_t|\mathcal{G}_t]$, $\mathbb{P}$–a.s., $t\in[0,\infty]$. Then $\mathbb{E}[L_{t-}|\mathcal{G}_{t-}]\leq\mathbb{E}[J_{t-}|\mathcal{G}_{t-}]$, $\mathbb{P}$–a.s., $t\in[0,\infty]$, where $L_{0-}:=J_{0-}:=0$ and $\mathcal{G}_{0-}:=\mathcal{G}_0$.*

Proof. The assertion trivially holds for $t=0$. So suppose that $t\in(0,\infty]$. Let $(t_n)_{n\in\mathbb{N}}\subseteq[0,t)$ be a sequence that converges increasingly to t. We have

$$\lim_{n\to\infty}\mathbb{E}\big[|\mathbb{E}[L_{t_n}-L_{t-}|\mathcal{G}_{t_n}]|\big]\leq\lim_{n\to\infty}\mathbb{E}\big[|L_{t_n}-L_{t-}|\big]=0.$$

Here the last equality follows by dominated convergence. Therefore, upon choosing a suitable subsequence of $(t_n)_{n\in\mathbb{N}}$ if necessary, we have

$$\lim_{n\to\infty}\mathbb{E}[L_{t_n}|\mathcal{G}_{t_n}]=\mathbb{E}[L_{t-}|\mathcal{G}_{t-}],\ \mathbb{P}\text{–a.s.}$$

The same argument applied to $\mathbb{E}[J_{t_n}|\mathcal{G}_{t_n}]$, $n\in\mathbb{N}$, and upon choosing a further subsequence if necessary, then yields

$$\mathbb{E}[L_{t-}|\mathcal{G}_{t-}]=\lim_{n\to\infty}\mathbb{E}[L_{t_n}|\mathcal{G}_{t_n}]\leq\lim_{n\to\infty}\mathbb{E}[J_{t_n}|\mathcal{G}_{t_n}]=\mathbb{E}[J_{t-}|\mathcal{G}_{t-}],\ \mathbb{P}\text{–a.s.},$$

which completes the proof. ☐

Proposition C.6. *Let $\xi=(\xi_t)_{t\in[0,\infty]}$ be $(\mathcal{G}_t)_{t\in[0,\infty]}$-predictable. Then the process $\overline{\xi}=(\overline{\xi}_t)_{t\in[0,\infty]}$ defined by*

$$\overline{\xi}_0:=\xi_0,\ \text{and}\ \overline{\xi}_t:=\limsup_{s\uparrow\uparrow t}\xi_s\ \text{for}\ t\in(0,\infty],$$

is $(\mathcal{G}_t^U)_{t\in[0,\infty]}$-predictable.

Proof. For each $n\in\mathbb{N}$, define $\xi^n=(\xi_t^n)_{t\in[0,\infty]}$ by $\xi_0^n:=\xi_0$ and for $t\in(0,\infty]$ by

$$\xi_t^n:=\bigg(\sup_{s\in[n,t)}\xi_s\bigg)\mathbf{1}_{(n,\infty]}(t)+\sum_{k=0}^{n2^n-1}\bigg(\sup_{s\in[k2^{-n},t)}\xi_s\bigg)\mathbf{1}_{(k2^{-n},(k+1)2^{-n}]}(t).$$

Each ξ^n is $(\mathcal{G}_t^U)_{t\in[0.\infty]}$-adapted by [52, Proposition 2.21] and left-continuous on $(0,\infty]$. The claim now follows from the fact that $\overline{\xi}$ is the limit of the sequence $(\xi^n)_{n\in\mathbb{N}}$, which completes the proof. ☐

Proposition C.7. *Let $\xi=(\xi_t)_{t\in[0,\infty]}$ be a real-valued, optional process, T be a stopping time. Suppose that $\xi_\cdot=\xi_{\cdot\wedge T}$. Then $\sup_{s\in[0,T]}|\xi_s|$ is $\mathcal{G}_\infty^U$-measurable and for each $p\in(1,\infty)$*

$$\mathbb{E}\bigg[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}{}^{\mathcal{G}_\infty}|\xi_\tau|^p\bigg]\leq\mathbb{E}\bigg[\sup_{s\in[0,T]}|\xi_s|^p\bigg]\leq\bigg(\frac{p}{p-1}\bigg)^p\mathbb{E}\bigg[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}{}^{\mathcal{G}_\infty}|\xi_\tau|^p\bigg]\tag{C.1}$$

Proof. Fix $p \in (1, \infty)$. Since $\xi_{\cdot \wedge T}$ is optional, and thus $\mathcal{B}([0,\infty]) \otimes \mathcal{G}_\infty$-measurable, we have that $\sup_{s\in[0,T]} |\xi_s| = \sup_{s\in[0,\infty]} |\xi_{s\wedge T}|$ is $\mathcal{G}^U_\infty$-measurable by an application of [52, Proposition 2.21]. The first claimed inequality follows from

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}{}^{\mathcal{G}_\infty} |\xi_\tau|^p \leq \sup_{s\in[0,T]} |\xi_s|^p, \ \mathbb{P}\text{–a.s.}$$

Next, we suppose without loss of generality, that the right hand side in (C.1) is finite, otherwise the second inequality trivially holds. Let $M = (M_t)_{t\in[0,\infty]}$ be the non-negative martingale satisfying

$$M_S = \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}{}^{\mathcal{G}_\infty} |\xi_\tau| \middle| \mathcal{G}_S\right], \ \mathbb{P}\text{–a.s.}, \ S \in \mathcal{T}_{0,\infty}.$$

Note that $|\xi_S| \leq M_S$, $\mathbb{P}$–a.s., for each $S \in \mathcal{T}_{0,\infty}$ and thus in particular $\sup_{s\in[0,T]} |\xi_s| = \sup_{s\in[0,\infty]} |\xi_{s\wedge T}| \leq \sup_{s\in[0,\infty]} |M_s|$ by Proposition C.3. By using Doob's inequality for martingales, we thus find

$$\mathbb{E}\left[\sup_{s\in[0,T]} |\xi_s|^p\right] \leq \mathbb{E}\left[\sup_{s\in[0,\infty]} M_s^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[M_\infty^p] = \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}{}^{\mathcal{G}_\infty} |\xi_\tau|^p\right],$$

which concludes the proof. □

References

[1] B. Acciaio, M. Beiglböck, F. Penkner, W. Schachermayer, and J. Temme. A trajectorial interpretation of Doob's martingale inequalities. *The Annals of Applied Probability*, 23(4):1494–1505, 2013. MR3098440

[2] K. Akdim, M. Haddadi, and Y. Ouknine. Strong Snell envelopes and RBSDEs with regulated trajectories when the barrier is a semimartingale. *Stochastics: An International Journal of Probability and Stochastic Processes*, 92(3):335–355, 2020. MR4085755

[3] A. Aksamit, L. Li, and M. Rutkowski. Generalized BSDE and reflected BSDE with random time horizon. *Electronic Journal of Probability*, 28(40):1–41, 2023. MR4557762

[4] S. Alsheyab and T. Choulli. Optimal stopping problem under random horizon: mathematical structures and linear RBSDEs. *ArXiv preprint arXiv:2301.09836*, 2023.

[5] I. Arharas and Y. Ouknine. Reflected and Doubly Reflected Backward Stochastic Differential Equations with Irregular Obstacles and a Large set of Stopping Strategies. *Journal of Theoretical Probability*, 2024.

[6] B. Baadi. Reflected BSDEs with two completely separated barriers and regulated trajectories in general filtration. *Random Operators and Stochastic Equations*, 30(2):97–111, 2022. MR4431364

[7] B. Baadi and Y. Ouknine. Reflected BSDEs when the obstacle is not right-continuous in a general filtration. *ALEA*, 14:201–218, 2017. MR3655127

[8] B. Baadi and Y. Ouknine. Reflected BSDEs with optional barrier in a general filtration. *Afrika Matematika*, 29(7–8):1049–1064, 2018. MR3869547

[9] V. Bally, M.E. Caballero, B. Fernandez, and N. El Karoui. Reflected BSDE's, PDE's and variational inequalities. Rapport de recherche 4455, Unité de recherche INRIA Rocquencourt, 2002.

[10] E. Bandini. Existence and uniqueness for backward stochastic differential equations driven by a random measure, possibly non quasi–left continuous. *Electronic Communications in Probability*, 20(71):1–13, 2015. MR3407215

[11] G. Barles, R. Buckdahn, and É. Pardoux. Backward stochastic differential equations and integral–partial differential equations. *Stochastics: An International Journal of Probability and Stochastic Processes*, 60(1–2):57–83, 1997. MR1436432

[12] E. Bayraktar and S. Yao. Quadratic reflected BSDEs with unbounded obstacles. *Stochastic Processes and their Applications*, 122(4):1155–1203, 2012. MR2914748

[13] D. Becherer, M. Büttner, and K. Kentia. On the monotone stability approach to BSDEs with jumps: extensions, concrete criteria and examples. In *Frontiers in stochastic analysis–BSDEs, SPDEs and their applications, Edinburgh, July 2017, selected, revised and extended contributions*, volume 289 of *Springer proceedings in mathematics & statistics*, pages 1–41, 2019. MR4008339

[14] J.-M. Bismut. *Analyse convexe et probabilités*. PhD thesis, Faculté des sciences de Paris, 1973. MR0324734

[15] J.-M. Bismut. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*, 44(2):384–404, 1973. MR0329726

[16] B. Bouchard, D. Possamaï, X. Tan, and C. Zhou. A unified approach to *a priori* estimates for supersolutions of BSDEs in general filtrations. *Annales de l'institut Henri Poincaré, Probabilités et Statistiques* (B), 54(1):154–172, 2018. MR3765884

[17] S. Bouhadou and Y. Ouknine. Non linear optimal stopping problem and reflected BSDEs in the predictable setting. *ArXiv preprint arXiv:1811.00695*, 2018.

[18] S. Bouhadou and Y. Ouknine. Optimal stopping in general predictable framework. *ArXiv preprint arXiv:1812.01759*, 2018.

[19] S. Bouhadou, A. Hilbert, and Y. Ouknine. RBSDEs with optional barriers: monotone approximation. *Probability, Uncertainty and Quantitative Risk*, 7(2):67–84, 2022. MR4488347

[20] M. Brachetta, G. Callegaro, C. Ceci, and C. Sgarra. Optimal reinsurance via BSDEs in a partially observable model with jump clusters. *ArXiv preprint arXiv:2207.05489*, 2022. MR4725239

[21] P. Briand and Y. Hu. Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs. *Journal of Functional Analysis*, 155(2):455–494, 1998. MR1624569

[22] P. Briand, B. Delyon, and J. Mémin. Donsker-type theorem for BSDEs. *Electronic Communications in Probability*, 6:1–14, 2001. MR1817885

[23] P. Briand, B. Delyon, and J. Mémin. On the robustness of backward stochastic differential equations. *Stochastic Processes and their Applications*, 97(2):229–253, 2002. MR1875334

[24] P. Briand, C. Geiss, S. Geiss, and C. Labart. Donsker-type theorem for BSDEs: rate of convergence. *Bernoulli*, 27(2):899–929, 2021. MR4255220

[25] R. Buckdahn. Backward stochastic differential equations driven by a martingale. Prépublication 93–05, URA 225 Université de Provence, Marseille, 1993.

[26] R. Carbone, B. Ferrario, and M. Santacroce. Backward stochastic differential equations driven by càdlàg martingales. *Theory of Probability & Its Applications*, 52(2):304–314, 2008. MR2742510

[27] P. Cheridito and M. Stadje. BS Δ Es and BSDEs with non-Lipschitz drivers: comparison, convergence and robustness. *Bernoulli*, 19(3):1047–1085, 2013. MR3079306

[28] P. Cheridito, H.M. Soner, N. Touzi, and N. Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Communications on Pure and Applied Mathematics*, 60(7):1081–1110, 2007. MR2319056

[29] S. Cohen and R. Elliott. A general theory of finite state backward stochastic difference equations. *Stochastic Processes and their Applications*, 120(4):442–466, 2010. MR2594366

[30] S.N. Cohen and R.J. Elliott. Backward stochastic difference equations and nearly time-consistent nonlinear expectations. *SIAM Journal on Control and Optimization*, 49(1):125–139, 2011. MR2765659

[31] S.N. Cohen and R.J. Elliott. Existence, uniqueness and comparisons for BSDEs in general spaces. *The Annals of Probability*, 40(5):2264–2297, 2012. MR3025717

[32] S.N. Cohen and R.J. Elliott. *Stochastic calculus and applications*. Probability and its applications. Springer New York, 2015. MR3443368

[33] F. Confortola, M. Fuhrman, and J. Jacod. Backward stochastic differential equation driven by a marked point process: an elementary approach with an application to optimal control. *The Annals of Applied Probability*, 26(3):1743–1773, 2016. MR3513605

[34] S. Crépey and A. Matoussi. Reflected and doubly reflected BSDEs with jumps: *a priori* estimates and comparison. *The Annals of Applied Probability*, 18(5):2041–2069, 2008. MR2462558

[35] J. Cvitanić and I. Karatzas. Backward stochastic differential equations with reflection and Dynkin games. *The Annals of Probability*, 24(4):2024–2056, 1996. MR1415239

[36] J. Cvitanić, D. Possamaï, and N. Touzi. Dynamic programming approach to principal–agent problems. *Finance and Stochastics*, 22(1):1–37, 2018. MR3738664

[37] R.W.R. Darling and É. Pardoux. Backwards SDE with random terminal time and applications to semilinear elliptic PDE. *The Annals of Probability*, 25(3):1135–1159, 1997. MR1457614

[38] D. Darrell and L.G. Epstein. Stochastic differential utility. *Econometrica*, 60(2):353–394, 1992. MR1162620

[39] M.H.A. Davis and P. Varaiya. Dynamic programming conditions for partially observable stochastic systems. *SIAM Journal on Control and Optimization*, 11(2):226–261, 1973. MR0319642

[40] C. Dellacherie and P.-A. Meyer. *Probabilities and potential*, volume 29 of *Mathematics studies*. North-Holland, 1978. MR0521810

[41] C. Dellacherie and P.-A. Meyer. *Probabilities and potential B: theory of martingales*. North-Holland Mathematics Studies. Elsevier Science, 1982. MR0745449

[42] G. di Nunno. On stochastic control for time changed Lévy dynamics. *SeMA Journal*, 79(3):529–547, 2022. MR4476289

[43] D. Duffie and L.G. Epstein. Asset pricing with stochastic differential utility. *The Review of Financial Studies*, 5(3):411–436, 1992. MR1162620

[44] R. Dumitrescu, M.-C. Quenez, and A. Sulem. Generalized Dynkin games and doubly reflected BSDEs with jumps. *Electronic Journal of Probability*, 21(64):1–32, 2016. MR3580030

[45] B. El Asri and N. Ourkiya. Infinite horizon multi-dimensional BSDE with oblique reflection and switching problem. *Stochastics and Dynamics*, 23(5):2350035, 2023. MR4663996

[46] N. El Karoui. Arrêt optimal prévisible. In G. Kallianpur and D. Kölzow, editors, *Measure theory. Applications to stochastic analysis. Proceedings, Oberwolfach conference, Germany, July 3–9, 1977*, volume 695 of *Lecture notes in mathematics*, pages 1–11. Springer Berlin, Heidelberg, 1978. MR0527069

[47] N. El Karoui. Les aspects probabilistes du contrôle stochastique. In *École d'été de probabilités de Saint-Flour IX–1979*, volume 876 of *Lecture notes in mathematics*, pages 73–238. Springer, 1981. MR0637471

[48] N. El Karoui and S.-J. Huang. A general result of existence and uniqueness of backward stochastic differential equations. In N. El Karoui and L. Mazliak, editors, *Backward stochastic differential equations*, volume 364 of *Pitman research notes in mathematics*, pages 27–36. Longman, 1997. MR1752673

[49] N. El Karoui and M. Jeanblanc-Picqué. Optimization of consumption with labor income. *Finance and Stochastics*, 2(4):409–440, 1998.

[50] N. El Karoui and L. Mazliak, editors. *Backward stochastic differential equations*, volume 364 of *Pitman research notes in mathematics*. Longman, 1997. MR1752672

[51] N. El Karoui and M.-C. Quenez. Non-linear pricing theory and backward stochastic differential equations. In W.J. Runggaldier, editor, *Financial mathematics. Lectures given at the 3rd session of the centro internazionale matematico estivo (C.I.M.E.) held in Bressanone, Italy, July 8–13, 1996*, volume 1656 of *Lecture notes in mathematics*, pages 191–246. Springer, 1997. MR1478198

[52] N. El Karoui and X. Tan. Capacities, measurable selection and dynamic programming part I: abstract framework. Technical report, École Polytechnique and université Paris–Dauphine, 2013.

[53] N. El Karoui, C. Kapoudjian, É. Pardoux, S. Peng, and M.-C. Quenez. Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *The Annals of Probability*, 25(2):702–737, 1997. MR1434123

[54] N. El Karoui, É. Pardoux, and M.-C. Quenez. Reflected backward SDEs and American options. In L.C.G. Rogers and D. Talay, editors, *Numerical methods in finance*, pages 215–231. Cambridge University Press, 1997. MR1470516

[55] N. El Karoui, S. Peng, and M.-C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997. MR1434407

[56] N. El Karoui, A. Matoussi, and A. Ngoupeyou. Quadratic exponential semimartingales and application to BSDEs with jumps. Technical report, Université Pierre & Marie Curie, Université du Maine, and banque des États de l'Afrique centrale, 2016.

[57] E.H. Essaky and M. Hassani. Generalized BSDE with 2-reflecting barriers and stochastic quadratic growth. *Journal of Differential Equations*, 254(3):1500–1528, 2013. MR2997380

[58] N. Foresta. Optimal stopping of marked point processes and reflected backward stochastic differential equations. *Applied Mathematics & Optimization*, 83(3):1219–1245, 2021. MR4261257

[59] L.I. Gal'čuk. Optional martingales. *Mathematics of the USSR–Sbornik*, 40(4):435–468, 1981. MR0587036

[60] M. Grigorova, P. Imkeller, E. Offen, Y. Ouknine, and M.-C. Quenez. Reflected BSDEs when the obstacle is not right-continuous and optimal stopping. *The Annals of Applied Probability*, 27(5):3153–3188, 2017. MR3719955

[61] M. Grigorova, P. Imkeller, Y. Ouknine, and M.-C. Quenez. Doubly reflected BSDEs and $\mathcal{E}^f$-Dynkin games: beyond the right-continuous case. *Electronic Journal of Probability*, 23(122):1–38, 2018. MR3896859

[62] M. Grigorova, P. Imkeller, Y. Ouknine, and M.-C. Quenez. Optimal stopping with f-expectations: the irregular case. *Stochastic Processes and their Applications*, 130(3):1258–1288, 2020. MR4058273

[63] Z. Gu, Y. Lin, and K. Xu. Mean reflected BSDE driven by a marked point process and application in insurance risk management. *ArXiv preprint arXiv:2310.15203*, 2023.

[64] Z. Gu, Y. Lin, and K. Xu. Reflected BSDE driven by a marked point process with a convex/concave generator. *ArXiv preprint arXiv:2310.20361*, 2023.

[65] S. Hamadène. Reflected BSDE's with discontinuous barrier and application. *Stochastics: An International Journal of Probability and Stochastic Processes*, 74(3–4):571–596, 2002. MR1943580

[66] S. Hamadène. Mixed zero-sum stochastic differential game and American game options. *SIAM Journal on Control and Optimization*, 45(2):496–518, 2006. MR2246087

[67] S. Hamadène and M. Hassani. BSDEs with two reflecting barriers: the general result. *Probab. Theory Related Fields*, 132(2):237–264, 2005. MR2199292

[68] S. Hamadène and M. Hassani. BSDEs with two reflecting barriers driven by a Brownian and a Poisson noise and related Dynkin game. *Electron. J. Probab.*, 11:no. 5, 121–145, 2006. MR2217812

[69] S. Hamadène and J.-P. Lepeltier. Zero-sum stochastic differential games and backward equations. *Systems & Control Letters*, 24(4):259–263, 1995. MR1321134

[70] S. Hamadène and Y. Ouknine. Reflected backward stochastic differential equation with jumps and random obstacle. *Electronic Journal of Probability*, 8(2):1–20, 2003. MR1961164

[71] S. Hamadène, J.-P. Lepeltier, and A. Matoussi. Double barrier backward SDEs with continuous coefficient. In N. El Karoui and L. Mazliak, editors, *Backward stochastic differential equations*, volume 364 of *Pitman research notes in mathematics*, pages 161–175. Longman, 1997. MR1752681

[72] S. He, J. Wang, and J.A. Yan. *Semimartingale theory and stochastic calculus*. Science Press, 1992. MR1219534

[73] R.A. Horn and C.R. Johnson. *Matrix analysis*. Cambridge University Press, second edition, 2013. MR2978290

[74] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005. MR2152241

[75] J. Jackson and G. Žitković. A characterization of solutions of quadratic BSDEs and a new approach to existence. *Stochastic Processes and their Applications*, 147:210–225, 2022. MR4379381

[76] J. Jacod. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture notes in mathematics*. Springer, 1979. MR0542115

[77] J. Jacod and A.N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der mathematischen Wissenschaften*. Springer–Verlag Berlin Heidelberg, 2003. MR1943877

[78] M. Jeanblanc, A. Matoussi, and A. Ngoupeyou. Robust utility maximization problem in model with jumps and unbounded claim. Technical report, Université d'Évry-Val-d'Essonne, université du Maine, and banque des États de l'Afrique centrale, 2012.

[79] N. Kazi-Tani, D. Possamaï, and C. Zhou. Quadratic BSDEs with jumps: a fixed-point approach. *Electronic Journal of Probability*, 20(66):1–28, 2015. MR3361254

[80] N. Kazi-Tani, D. Possamaï, and C. Zhou. Second-order BSDEs with jumps: formulation and uniqueness. *The Annals of Applied Probability*, 25(5):2867–2908, 2015. MR3375890

[81] N. Kazi-Tani, D. Possamaï, and C. Zhou. Second order BSDEs with jumps: existence and probabilistic representation for fully-nonlinear PIDEs. *Electronic Journal of Probability*, 20(65):1–31, 2015. MR3361253

[82] N. Kazi-Tani, D. Possamaï, and C. Zhou. Quadratic BSDEs with jumps: related nonlinear expectations. *Stochastics and Dynamics*, 16(04):1650012, 2016. MR3494687

[83] T. Klimsiak. Reflected BSDEs with monotone generator. *Electronic Journal of Probability*, 17(107):1–25, 2012. MR3015691

[84] T. Klimsiak. BSDEs with monotone generator and two irregular reflecting barriers. *Bulletin des Sciences Mathématiques*, 137(3):268–321, 2013. MR3043077

[85] T. Klimsiak. Reflected BSDEs on filtered probability spaces. *Stochastic Processes and their Applications*, 125(11):4204–4241, 2015. MR3385601

[86] T. Klimsiak. Non-semimartingale solutions of reflected BSDEs and applications to Dynkin games. *Stochastic Processes and their Applications*, 134:208–239, 2021. MR4206845

[87] T. Klimsiak and M. Rzymowski. Reflected BSDEs with two optional barriers and monotone coefficient on general filtered space. *Electronic Journal of Probability*, 26(91):1–24, 2021. MR4278602

[88] T. Klimsiak and M. Rzymowski. Nonlinear BSDEs with two optional Doob's class barriers satisfying weak Mokobodzki's condition and extended Dynkin games. *ArXiv preprint arXiv:2205.06222*, 2022. MR4643869

[89] T. Klimsiak, M. Rzymowski, and L. Słomiński. Reflected BSDEs with regulated trajectories. *Stochastic Processes and their Applications*, 129(4):1153–1184, 2019. MR3926552

[90] T. Klimsiak, M. Rzymowski, and L. Słomiński. Reflected backward stochastic differential equations with two optional barriers. *Bulletin des Sciences Mathématiques*, 158:102820, 2020. MR4030244

[91] M. Kobylanski. Résultats d'existence et d'unicité pour des équations différentielles stochastiques rétrogrades avec des générateurs à croissance quadratique. *Comptes Rendus de l'Académie des Sciences–Series I–Mathematics*, 324(1):81–86, 1997. MR1435592

[92] M. Kobylanski and M.-C. Quenez. Optimal stopping time problem in a general framework. *Electronic Journal of Probability*, 17(72):1–28, 2012. MR2968679

[93] M. Kobylanski, J.-P. Lepeltier, M.-C. Quenez, and S. Torres. Reflected BSDE with superlinear quadratic coefficient. *Probability and Mathematical Statistics*, 22(1):51–83, 2002. MR1944142

[94] M. Laurière and L. Tangpi. Backward propagation of chaos. *ArXiv preprint arXiv:1911.06835*, 2019. MR4434358

[95] É. Lenglart. Tribus de Meyer et théorie des processus. *Séminaire de probabilités de Strasbourg*, XIV:500–546, 1980. MR0580151

[96] J.-P. Lepeltier and J. San Martín. Backward SDEs with two barriers and continuous coefficient: an existence result. *Journal of Applied Probability*, 41(1):162–175, 2004. MR2036279

[97] J.-P. Lepeltier and M. Xu. Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier. *Statistics & Probability Letters*, 75(1):58–66, 2005. MR2185610

[98] J.-P. Lepeltier and M. Xu. Reflected BSDE with quadratic growth and unbounded terminal value. Technical report, Université du Maine and Fudan University, 2007.

[99] D. Lépingle and J. Mémin. Intégrabilité uniforme et dans L^r des martingales exponentielles. *Publications des séminaires de mathématiques et informatique de Rennes*, 1:1–14, 1978. MR0602524

[100] D. Lépingle and J. Mémin. Sur l'intégrabilité uniforme des martingales exponentielles. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 42(3):175–203, 1978. MR0489492

[101] H. Li. Backward stochastic differential equations with double mean reflections. *ArXiv preprint arXiv:2307.05947*, 2023.

[102] H. Li and G. Liu. Multi-dimensional reflected BSDEs driven by G–Brownian motion with diagonal generators. *ArXiv preprint arXiv:2108.09012*, 2023.

[103] H. Li and N. Ning. Propagation of chaos for doubly mean reflected BSDEs. *ArXiv preprint arXiv:2401.16617*, 2024.

[104] L. Li, R. Liu, and M. Rutkowski. Well-posedness and penalization schemes for generalized BSDEs and reflected generalized BSDEs. *ArXiv preprint arXiv:2212.12854*, 2022.

[105] Y. Lin and K. Xu. Mean-field reflected BSDEs driven by a marked point process. *ArXiv preprint arXiv:2401.07723*, 2024.

[106] Y. Lin, Z. Ren, N. Touzi, and J. Yang. Second order backward SDE with random terminal time. *Electronic Journal of Probability*, 25(99):1–43, 2020.

[107] D. Madan, M. Pistorius, and M. Stadje. Convergence of BS Δ Es driven by random walks to BSDEs: the case of (in)finite activity jumps with general driver. *Stochastic Processes and their Applications*, 126(5):1553–1584, 2016. MR3473105

[108] M.A. Maingueneau. Temps d'arrêt optimaux et théorie générale. *Séminaire de probabilités de Strasbourg*, XII:457–467, 1978. MR0520020

[109] A. Matoussi and R. Salhi. Exponential quadratic BSDEs with infinite activity jumps. *ArXiv preprint arXiv:1904.08666*, 2019.

[110] P.-A. Meyer. Un cours sur les intégrales stochastiques. *Séminaire de probabilités de Strasbourg*, X:245–400, 1976. MR0501332

[111] T. Nie and M. Rutkowski. Reflected and doubly reflected BSDEs driven by RCLL martingales. *Stochastics and Dynamics*, 22(05):2250012, 2022. MR4486232

[112] M. Nutz. Robust superhedging with jumps and diffusion. *Stochastic Processes and their Applications*, 125(12):4543–4555, 2015. MR3406595

[113] A. Papapantoleon, D. Possamaï, and A. Saplaouras. Existence and uniqueness for BSDEs with jumps: the whole nine yards. *Electronic Journal of Probability*, 23(121):1–68, 2018. MR3896858

[114] A. Papapantoleon, D. Possamaï, and A. Saplaouras. Stability results for martingale representations: the general case. *Transactions of the American Mathematical Society*, 372(8):5891–5946, 2019. MR4014298

[115] A. Papapantoleon, D. Possamaï, and A. Saplaouras. Stability of backward stochastic differential equations: the general Lipschitz case. *Electronic Journal of Probability*, 28(51):1–56, 2023. MR4571185

[116] A. Papapantoleon, A. Saplaouras, and S. Theodorakopoulos. Existence, uniqueness and propagation of chaos for general McKean–Vlasov and mean-field BSDEs. *In preparation*, 2024.

[117] É. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *System and Control Letters*, 14(1):55–61, 1990. MR1037747

[118] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In B.L. Rozovskii and R.B. Sowers, editors, *Stochastic partial differential equations and their applications. Proceedings of IFIP WG 7/1 international conference University of North Carolina at Charlotte, NC June 6–8, 1991*, volume 176 of *Lecture notes in control and information sciences*, pages 200–217. Springer, 1992. MR1176785

[119] S. Peng. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics and Stochastic Reports*, 37(1–2):61–74, 1991. MR1149116

[120] S. Peng and M. Xu. The smallest g-supermartingale and reflected BSDE with single and double L^2 obstacles. *Annales de l'institut Henri Poincaré, Probabilités et Statistiques* (B), 41(3):605–630, 2005. MR2139035

[121] S. Peng and M. Xu. Reflected BSDE with a constraint and its applications in an incomplete market. *Bernoulli*, 16(3):614–640, 2010. MR2730642

[122] M. Perninge. A note on reflected BSDEs in infinite horizon with stochastic Lipschitz coefficients. *ArXiv preprint arXiv:2103.05603*, 2021.

[123] M. Perninge. Optimal stopping of BSDEs with constrained jumps and related zero-sum games. *Stochastic Processes and their Applications*, 173:104355, 2024. MR4732936

[124] M. Perninge. Optimal stopping of BSDEs with constrained jumps and related double obstacle PDEs. *ArXiv preprint arXiv:2402.17541*, 2024.

[125] T. Pham and J. Zhang. Some norm estimates for semimartingales. *Electronic Journal of Probability*, 18(109):1–25, 2013. MR3151729

[126] A. Popier. Backward stochastic Volterra integral equations with jumps in a general filtration. *ESAIM: Probability and Statistics*, 25:133–203, 2021. MR4234131

[127] D. Possamaï and X. Tan. Weak approximation of second-order BSDEs. *The Annals of Applied Probability*, 25(5):2535–2562, 2015. MR3375883

[128] D. Possamaï and L. Tangpi. Non-asymptotic convergence rates for mean-field games: weak formulation and McKean–Vlasov bsdes. *ArXiv preprint arXiv:2105.00484*, 2021.

[129] D. Possamaï, X. Tan, and C. Zhou. Stochastic control for a class of nonlinear kernels and applications. *The Annals of Probability*, 46(1):551–603, 2018. MR3758737

[130] P.E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic modelling and applied probability*. Springer–Verlag Berlin Heidelberg, 2nd edition, 2005. MR2273672

[131] R. Qian. Multi-dimensional reflected McKean–Vlasov BSDEs with the obstacle depending on both the first unknown and its distribution. *ArXiv preprint arXiv:2309.10427*, 2023.

[132] M.-C. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Processes and their Applications*, 123(8):3328–3357, 2013. MR3062447

[133] M.-C. Quenez and A. Sulem. Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps. *Stochastic Processes and their Applications*, 124(9):3031–3054, 2014. MR3217432

[134] M. Royer. BSDEs with a random terminal time driven by a monotone generator and their links with PDEs. *Stochastics and Stochastic Reports*, 76(4):281–307, 2004. MR2075474

[135] M. Royer. Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic Processes and their Applications*, 116(10):1358–1376, 2006. MR2260739

[136] A. Rozkosz and L. Słomiński. $\mathbb{L}^p$ solutions of reflected BSDEs under monotonicity condition. *Stochastic Processes and their Applications*, 122(12):3875–3900, 2012. MR2971718

[137] H.M. Soner, N. Touzi, and J. Zhang. Quasi-sure stochastic analysis through aggregation. *Electronic Journal of Probability*, 16(2):1844–1879, 2011. MR2842089

[138] H.M. Soner, N. Touzi, and J. Zhang. Wellposedness of second order backward SDEs. *Probability Theory and Related Fields*, 153(1–2):149–190, 2012. MR2925572

[139] R. Tevzadze. Solvability of backward stochastic differential equations with quadratic growth. *Stochastic Processes and their Applications*, 118(3):503–515, 2008. MR2389055

[140] N. Touzi. *Optimal stochastic control, stochastic target problems, and backward SDE*, volume 29 of *Fields institute monographs*. Springer–Verlag New York, 2013. MR2976505

[141] H. von Weizsäcker and G. Winkler. *Stochastic integrals*. Advanced lectures in mathematics. Vieweg+Teubner Verlag Wiesbaden, 1990. MR1062600

[142] J. Zhang. *Backward stochastic differential equations—from linear to fully nonlinear theory*, volume 86 of *Probability theory and stochastic modelling*. Springer–Verlag New York, 2017. MR3699487

[143] S. Zheng, L. Zhang, and X. Meng. One-dimensional reflected BSDEs and BSDEs with quadratic growth generators. *ArXiv preprint arXiv:2110.06907*, 2023.