

Yang-Baxter maps and independence preserving property*

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Abstract

We raise a question of whether there might be deep mathematical connections between two properties for bijective functions $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ for a set \mathcal{X} which are introduced from very different backgrounds, and study the supporting evidence that suggests this. One of the property is that F is a Yang-Baxter map, namely it satisfies the “set-theoretical” Yang-Baxter equation, and the other property is the independence preserving property, which means that there exist independent (non-constant) \mathcal{X} -valued random variables X, Y such that U, V are also independent with $(U, V) = F(X, Y)$. Recently in the study of invariant measures for a discrete integrable system, a class of functions having these two properties were found. Motivated by this, we analyze a relationship between the Yang-Baxter maps and the independence preserving property, which has never been studied as far as we are aware. We focus on the case $\mathcal{X} = \mathbb{R}_+$. Our first main result is that *all* quadrirational Yang-Baxter maps $F : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \times \mathbb{R}_+$ in an important subclass have the independence preserving property. In particular, we found new classes of bijections having the independence preserving property. Our second main result is that these newly introduced bijections are fundamental in the class of (known) bijections with the independence preserving property, in the sense that many known bijections having the independence preserving property are derived from these maps by taking special parameters or performing some limiting procedure. This reveals that the independence preserving property, which has been investigated for specific functions individually, can be understood in a more unified manner.

Keywords: Yang-Baxter map; independence preserving property; Matsumoto-Yor property; quadrirational map; characterization of distributions.

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1 Introduction

1.1 Overview

In this paper, we consider the question of whether there is any connection between two properties for bijective functions $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ for a set \mathcal{X} which are introduced from very different backgrounds and seemingly unrelated. One of the properties is that F is a Yang-Baxter map, namely, it satisfies the “set-theoretical” Yang-Baxter equation

$$F_{12} \circ F_{13} \circ F_{23} = F_{23} \circ F_{13} \circ F_{12}$$

where F_{ij} acts on the i -th and j -th factors of the product $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$. Yang-Baxter maps are also called set-theoretical solutions to the quantum Yang-Baxter equation. The other is the independence preserving property (IP property for short), which means that there is a class of quadruplets of (non-Dirac) probability distributions $\mu, \nu, \tilde{\mu}, \tilde{\nu}$ satisfying $F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}$. In other words, there exist independent (non-constant) \mathcal{X} -valued random variables X, Y such that U, V are also independent with $(U, V) := F(X, Y)$. The IP property has been mostly studied for explicit functions F with $\mathcal{X} = \mathbb{R}$ or an open interval of \mathbb{R} and used to characterize special probability distributions such as normal, gamma, exponential, inverse-Gaussian, beta and so on (more details are discussed below). Such property is also called Matsumoto-Yor property if F is given in a special form.

Recently, these characterization results with the IP property were also used to study stationary solutions of stochastic integrable models [6, 10] and invariant measures of discrete integrable systems [8]. In this context, it was pointed out that the maps

$$F_{\text{GIG}}^{\alpha, \beta}(x, y) = \left(y \frac{1 + \beta xy}{1 + \alpha xy}, x \frac{1 + \alpha xy}{1 + \beta xy} \right)$$

on $\mathbb{R}_+ \times \mathbb{R}_+$ into itself with parameters $\alpha, \beta \geq 0$, which originate from the discrete KdV equation, have these two properties, namely, they are (parameter-dependent) Yang-Baxter maps and also have the IP property for the generalized inverse Gaussian (GIG) distributions in [8]. This motivates us to study the relationship between the Yang-Baxter maps and the IP property, which has never been studied as far as we are aware. We note that later in [4, 31], it is proved that U and V given by $(U, V) = F_{\text{GIG}}^{\alpha, \beta}(X, Y)$ are independent only when X and Y have GIG distributions with proper parameters, namely the IP property for this map characterizes the GIG distribution.

Besides this specific example, recent developments on the study of probabilistic properties of discrete integrable systems lead us to expect a deep relationship between the IP property of a map and the integrability of dynamical systems associated with the map. Actually, one important consequence of the IP property is that the dynamics on the two-dimensional lattice defined by the map F , which typically turns out to be a discrete integrable system for such map, has a spatially independent and identically/alternately distributed invariant measure (See Theorem 2.1 of [8]. Note that the IP property is called the detailed balance condition there). Hence, one may expect that there is a connection between the IP property and the integrability, while the Yang-Baxter equation is one of the typical ways to characterize integrability.

As a first study in this direction, we focus on the case $\mathcal{X} = \mathbb{R}_+$ in this paper, but we do not lose that much generality as discussed in Subsection 3.1. There are two main contributions of our paper to connect these two properties. First, we prove that all quadrirational Yang-Baxter maps $F : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \times \mathbb{R}_+$ in the subclass $[2 : 2]$ (whose definition is given in Section 2), which was introduced in [1] and also studied in [41], that are $F_{\text{GIG}} = (F_{\text{GIG}}^{\alpha, \beta})_{\alpha, \beta}$, $H_I^+ = (H_I^{+, \alpha, \beta})_{\alpha, \beta}$, $H_{II}^+ = (H_{II}^{+, \alpha, \beta})_{\alpha, \beta}$ and $H_{III, A} = (H_{III, A}^{\alpha, \beta})_{\alpha, \beta}$

given as

$$\begin{aligned}
 H_I^{+, \alpha, \beta}(x, y) &= \left(\frac{y}{\alpha} \frac{\beta + \alpha x + \beta y + \alpha \beta xy}{1 + x + y + \beta xy}, \frac{x}{\beta} \frac{\alpha + \alpha x + \beta y + \alpha \beta xy}{1 + x + y + \alpha xy} \right), \\
 H_{II}^{+, \alpha, \beta}(x, y) &= \left(\frac{y}{\alpha} \frac{\beta + \alpha x + \beta y}{1 + x + y}, \frac{x}{\beta} \frac{\alpha + \alpha x + \beta y}{1 + x + y} \right), \\
 H_{III, A}^{\alpha, \beta}(x, y) &= \left(\frac{y}{\alpha} \frac{\alpha x + \beta y}{x + y}, \frac{x}{\beta} \frac{\alpha x + \beta y}{x + y} \right)
 \end{aligned}$$

with parameters $\alpha, \beta > 0$, have the IP property as stated in the next theorem. In [41], F_{GIG} is named $H_{III, B}$. We give more background of these functions in Section 2. In the following, for any $p, q > 0$, $Be'(\lambda, a, b; p, q)$, $K(\lambda, a, b; p, q)$ and $GIG(\lambda, a, b; p, q)$ are some generalizations of beta prime, Kummer of Type 2 and GIG distributions respectively. The explicit density function of each probability distribution is given in Subsection 1.3. The IP property for $F_{GIG} = H_{III, B}$ was obtained in [8] so we omit it in the theorem.

Theorem 1.1. *Let $\alpha, \beta > 0$. If X and Y are independent random variables with the following marginal distributions, then the random variables U, V given by $(U, V) = F(X, Y)$ for each map are independent and have the following marginal distributions.*

(i) For $F = H_I^{+, \alpha, \beta}$:

$$\begin{aligned}
 X &\sim Be'(\lambda, a, b; \alpha, 1), & Y &\sim Be'(-\lambda, a, b; \beta, 1), \\
 U &\sim Be'(-\lambda, a, b; \alpha, 1), & V &\sim Be'(\lambda, a, b; \beta, 1)
 \end{aligned}$$

where $\lambda \in \mathbb{R}$, $a, b > 0$, $-\min\{a, b\} < \frac{\lambda}{2} < \min\{a, b\}$.

(ii) For $F = H_{II}^{+, \alpha, \beta}$:

$$\begin{aligned}
 X &\sim K(\lambda, a, b; \alpha, 1), & Y &\sim K(-\lambda, a, b; \beta, 1), \\
 U &\sim K(-\lambda, a, b; \alpha, 1), & V &\sim K(\lambda, a, b; \beta, 1)
 \end{aligned}$$

where $\lambda \in \mathbb{R}$, $a, b > 0$, $-b < \frac{\lambda}{2} < b$.

(iii) For $F = H_{III, A}^{\alpha, \beta}$:

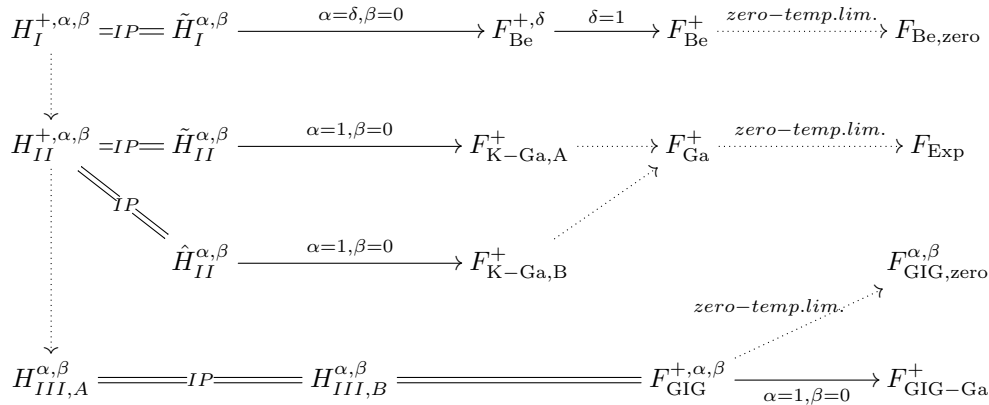
$$\begin{aligned}
 X &\sim GIG(\lambda, a, b; \alpha, 1), & Y &\sim GIG(-\lambda, a, b; \beta, 1), \\
 U &\sim GIG(-\lambda, a, b; \alpha, 1), & V &\sim GIG(\lambda, a, b; \beta, 1)
 \end{aligned}$$

where $\lambda \in \mathbb{R}$, $a, b > 0$.

By change of variables, the IP property for $H_{III, A}^{\alpha, \beta}$ is easily reduced to the property for $F_{GIG}^{\alpha, \beta}$, but the results for $H_I^{+, \alpha, \beta}$ and $H_{II}^{+, \alpha, \beta}$ are not reduced to any known cases (except for special parameters), hence we obtain new classes having the IP property. We note that Koudou and Wesolowski also found the IP property for $H_{II}^{+, \alpha, \beta}$ independently [28]. We conjecture that these maps characterize each probability distribution, which was actually proved in the literature for some special cases such as $F_{GIG}^{\alpha, \beta}$ (and so true for $H_{III, A}^{\alpha, \beta}$). The characterization for $H_{II}^{+, \alpha, \beta}$ was proved by Koudou and Wesolowski in [28], which was announced just after the first version of the present paper was announced. As seen from the explicit expressions, there are beautiful similarities between these maps and the parameters of distributions, and in fact, we can prove the claims (ii) and (iii) of Theorem 1.1 by a certain limiting procedure from the result (i), besides the direct computation. This limiting procedure is studied in Section 4.

Our second contribution is that, as Theorem 5.1 in Section 5, we reveal that many of the known functions F on a product of open intervals of \mathbb{R} having the IP property are derived from the quadrirational Yang-Baxter maps H_I^+ , H_{II}^+ or $H_{III, A}$ by taking a

special parameter or performing a singular limit with an appropriate coordinate-wise change of variables. These relations are summarized in the following figure, which is the consequence of Lemma 2.3 and Theorem 5.1.



The dotted lines represent singular scaling limits, the equality with label *IP* means two functions are IP-equivalent, whose definition is introduced in Section 3, and the arrows represent special parameter selections. The explicit expressions of bijections in the diagram are given in Section 3. This excellently unifies many of the known results for the IP property, which is summarized in Corollary 5.2.

To summarize these two results in short, for the case $\mathcal{X} = \mathbb{R}_+$, all quadrirational Yang-Baxter maps in the subclass $[2 : 2]$ (whose definition is given in Section 2) have the IP property, and many functions having the IP property are obtained from a Yang-Baxter map, which was a big surprise for us.

Finally, we discuss possible future developments. First, since H_I^+, H_{II}^+ and $H_{III,A}$ are all subtraction-free, as already pointed out in several contexts [25, 41, 8, 10], there is a zero-temperature version (also called a tropical version as well as an ultra-discretized version) of them where the $(+, \times)$ -algebra is replaced by $(\min, +)$ -algebra. For $F_{GIG}^{\alpha,\beta}$, such generalization was already studied in [8, 4]. These zero-temperature versions also satisfy the Yang-Baxter property as well as the IP property. However, since the IP property may also hold with some discrete distributions for such zero-temperature versions, the characterization of distributions may become more complicated. Another promising generalization is the positive definite matrix versions of H_I^+, H_{II}^+ and $H_{III,A}$. Actually, for most of the bijections F having the IP property discussed in Section 3, its positive definite matrix version has been introduced and shown that they also have the IP property (cf. [32, 19, 29, 26, 31]). The matrix version of $F_{GIG}^{\alpha,\beta}$ was introduced in [31] and its IP property was already shown there. A very natural and interesting question is whether these matrix versions are Yang-Baxter maps. Recently, the matrix versions of stochastic integrable systems, such as random polymers and interacting diffusions are introduced [39, 2] and the matrix versions of the IP property (for a certain function) play an essential role. Of course, the most fundamental question is to figure out why there is a relationship between the Yang-Baxter maps and the IP property. Also, using the connection between the Yang-Baxter maps and the IP property, it is worth trying to find a more direct connection between quantum, stochastic and classical integrable systems and their stationary distributions in general.

1.2 Outline of the remaining part of the paper

In Section 2, we explain a brief background on the Yang-Baxter maps and review the result on the characterization of the quadrirational Yang-Baxter maps on $\mathbb{C}P^1$, the

one-dimensional complex projective space, as well as the origin of H_I^+ , H_{II}^+ and $H_{III,A}$. In Section 3, we review known results on the IP property for bijections on a product of open intervals of \mathbb{R} and classify them into some classes. Then, with a few exceptions of bijections, we introduce normalized versions of them by coordinate-wise change of variables. Here, the normalized version means that it is a birational map on $\mathbb{R}_+ \times \mathbb{R}_+$ into itself. In Section 4, we prove Theorem 1.1, which is essentially a consequence of direct computation, and also discuss several relations between the maps H_I^+ , H_{II}^+ and $H_{III,A}$ as well as the probability distributions which appeared in Theorem 1.1. Finally, in Section 5, we give explicit relations between the functions discussed in Section 3, which are already known in the literature, and newly introduced maps H_I^+ , H_{II}^+ and $H_{III,A}$ to derive known IP properties from Theorem 1.1.

1.3 Notation and probability distributions

Here, we list frequently used bijections:

- $I : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $I(x) = \frac{1}{x}$, $I^{-1} = I$,
- For $\alpha > 0$, $\theta_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\theta_\alpha(x) = \alpha x$, $(\theta_\alpha)^{-1} = \theta_{\alpha^{-1}}$,
- $\pi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$, $\pi(x, y) = (y, x)$, $\pi^{-1} = \pi$.

We also introduce three classes of probability distributions with two positive parameters $p, q > 0$. In our parametrization, unnecessary extra parameters have been introduced for the Kummer distribution of Type 2 and the Generalized inverse Gaussian distribution, but this is deliberately used to provide a unified notation for how the three distributions relate concerning scale transformation and weak convergence. These properties are discussed in detail in Lemmas 4.1 and 4.2. The use of such a parameterization makes the statement of Theorem 1.1 very unified and is also the key to understanding the relations between the known results on the IP property summarized in Corollary 5.2.

Generalized beta prime distribution (p, q) For $\lambda, a, b \in \mathbb{R}$, $-b < \frac{\lambda}{2} < a$, the *Generalized beta prime* distribution with parameters $(\lambda, a, b; p, q)$, which we denote $\text{Be}'(\lambda, a, b; p, q)$, has density

$$\frac{1}{Z} x^{\lambda-1} (1+px)^{-a-\frac{\lambda}{2}} (1+qx^{-1})^{-b+\frac{\lambda}{2}}, \quad x \in \mathbb{R}_+,$$

where Z is a normalizing constant.

Kummer distribution of Type 2 (p, q) For $\lambda, b \in \mathbb{R}$, $a > 0$, $-b < \frac{\lambda}{2}$, the *Kummer* distribution of Type 2 with parameters $(\lambda, a, b; p, q)$, which we denote $\text{K}(\lambda, a, b; p, q)$, has density

$$\frac{1}{Z} x^{\lambda-1} e^{-apx} (1+qx^{-1})^{-b+\frac{\lambda}{2}}, \quad x \in \mathbb{R}_+,$$

where Z is a normalizing constant.

Generalized inverse Gaussian distribution (p, q) For $\lambda \in \mathbb{R}$, $a, b > 0$, the *generalized inverse Gaussian* distribution with parameters $(\lambda, a, b; p, q)$, which we denote $\text{GIG}(\lambda, a, b; p, q)$, has density

$$\frac{1}{Z} x^{\lambda-1} e^{-apx-bqx^{-1}}, \quad x \in \mathbb{R}_+,$$

where Z is a normalizing constant.

We also list special classes of them, which are more common in literature. In the following, $\text{Be}'(\lambda, a, b; p, q)$, $\text{K}(\lambda, a, b; p, q)$ and $\text{GIG}(\lambda, a, b; p, q)$ are extended properly for $p, q \in \{0, \infty\}$ in a natural way.

Generalized beta prime distribution (δ) For $a, b, \delta > 0$ and $c \in \mathbb{R}$, the *Generalized beta prime* distribution with parameters $(a, b, c; \delta)$, which we denote $\text{Be}'_{\delta}(a, b, c)$, has density

$$\frac{1}{Z} x^{a-1} (1+x)^{-a-b} \left(\frac{1+\delta x}{1+x} \right)^c, \quad x \in \mathbb{R}_+,$$

where Z is a normalizing constant. Note that $\text{Be}'(\lambda, a, b; \delta, 1) = \text{Be}'_{\delta}(b + \frac{\lambda}{2}, a - \frac{\lambda}{2}, -a - \frac{\lambda}{2})$.

Beta prime distribution For $a, b > 0$, the *Beta prime* distribution with parameters (a, b) , which we denote $\text{Be}'(a, b)$, has density

$$\frac{1}{Z} x^{a-1} (1+x)^{-a-b}, \quad x \in \mathbb{R}_+,$$

where Z is a normalizing constant. Note that $\text{Be}'(\lambda, a, b; 1, 0) = \text{Be}'(\lambda, a - \frac{\lambda}{2})$ for $\lambda > 0$, $\text{Be}'(\lambda, a, b; 0, 1) = \text{Be}'(b + \frac{\lambda}{2}, -\lambda)$ for $\lambda < 0$, $\text{K}(\lambda, a, b; 0, 1) = \text{Be}'(b + \frac{\lambda}{2}, -\lambda)$ for $\lambda < 0$, and $\text{Be}'_1(a, b, c) = \text{Be}'(a, b)$.

Kummer distribution of Type 2 For $a, c > 0$ and $b \in \mathbb{R}$, the *Kummer* distribution with parameters (a, b, c) , which we denote $\text{K}^{(2)}(a, b, c)$, has density

$$\frac{1}{Z} x^{a-1} (1+x)^{-a-b} e^{-cx}, \quad x \in \mathbb{R}_+,$$

where Z is a normalizing constant. Note that $\text{K}(\lambda, a, b; p, 1) = \text{K}^{(2)}(b + \frac{\lambda}{2}, -\lambda, ap)$.

Gamma distribution For $\lambda, a > 0$, the *Gamma* distribution with parameters (λ, a) , which we denote $\text{Ga}(\lambda, a)$, has density

$$\frac{1}{Z} x^{\lambda-1} e^{-ax}, \quad x \in \mathbb{R}_+,$$

where Z is a normalizing constant. Note that $\text{K}(\lambda, a, b; p, 0) = \text{Ga}(\lambda, ap)$ for $\lambda > 0$, $\text{K}(\lambda, a, b; p, \infty) = \text{Ga}(b + \frac{\lambda}{2}, ap)$ and $\text{GIG}(\lambda, a, b; p, 0) = \text{Ga}(\lambda, ap)$ for $\lambda > 0$.

Generalized inverse Gaussian distribution For $\lambda \in \mathbb{R}$ and $a, b > 0$, the *generalized inverse Gaussian* with parameters (λ, a, b) , which we denote $\text{GIG}(\lambda, a, b)$, has density

$$\frac{1}{Z} x^{\lambda-1} e^{-ax-bx^{-1}}, \quad x \in \mathbb{R}_+,$$

where Z is a normalizing constant. Note that $\text{GIG}(\lambda, a, b; p, q) = \text{GIG}(\lambda, ap, bq)$.

We also note that the class of distributions $\text{K}(\lambda, a, b; p, q)$ is called Whittaker distribution in [40].

2 Yang-Baxter maps on \mathbb{CP}^1 and \mathbb{R}_+

In this section, we briefly review the background on the Yang-Baxter maps and known results on the classification of Yang-Baxter maps on $\mathbb{CP}^1 \times \mathbb{CP}^1$ in a special class of functions, called quadrirational functions. Then, we discuss when we can restrict these Yang-Baxter maps to the domain $\mathbb{R}_+ \times \mathbb{R}_+$, which is the case we are interested in.

The Yang-Baxter maps, whose notion was introduced in [12] and the term was proposed in [46], are bijective maps of a Cartesian product

$$F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$$

satisfying the “set-theoretical” Yang-Baxter equation

$$F_{12} \circ F_{13} \circ F_{23} = F_{23} \circ F_{13} \circ F_{12}.$$

Here, F_{ij} are maps on the product of three sets $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ into itself and act as F on the i -th and j -th factors and as the identity on the other. For a set of bijective maps $F(\alpha, \beta) : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ with parameters α, β in a certain set of parameters Θ , we also say that they are Yang-Baxter maps if

$$F_{12}(\lambda_1, \lambda_2) \circ F_{13}(\lambda_1, \lambda_3) \circ F_{23}(\lambda_2, \lambda_3) = F_{23}(\lambda_2, \lambda_3) \circ F_{13}(\lambda_1, \lambda_3) \circ F_{12}(\lambda_1, \lambda_2) \quad (2.1)$$

holds for any parameters λ_1, λ_2 and $\lambda_3 \in \Theta$. Actually, by replacing \mathcal{X} with $\mathcal{X} \times \Theta$ and considering

$$\tilde{F}((x, \alpha), (y, \beta)) := F(\alpha, \beta)(x, y),$$

we obtain a (parameter-independent) Yang-Baxter map \tilde{F} .

Many important examples of the Yang-Baxter maps were found in the literature and it is not possible to enumerate them all, but some examples are in [21, 13] and some classification results for the case when \mathcal{X} is a finite set were obtained in [33, 14]. For more background on the Yang-Baxter maps and their transfer dynamics, see the nice review paper [47].

In [1], a classification of quadrirational maps on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is given and it is shown that any quadrirational map is equivalent, by some Möbius transformations acting independently on each variable, to some Yang-Baxter map. To state their result more precisely, recall that a bijection

$$F : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1, \quad (x, y) \mapsto (u(x, y), v(x, y))$$

is said to be birational if F and F^{-1} are both rational functions. The authors of [1] defined that a map $F : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ is said to be *quadrirational* if F and \bar{F} , which is called the companion map of F , satisfying $\bar{F}(u, y) = (x, v)$ for $(u, v) = F(x, y)$ is a well-defined bijection on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and moreover, F and \bar{F} are both birational functions. In other words, F is quadrirational if F, F^{-1}, \bar{F} and \bar{F}^{-1} are well-defined rational functions. They proved that quadrirational maps $F(x, y) = (u(x, y), v(x, y))$ have the form:

$$u(x, y) = \frac{a(y)x + b(y)}{c(y)x + d(y)}, \quad v(x, y) = \frac{A(x)y + B(x)}{C(x)y + D(x)}$$

where $a(y), \dots, d(y)$ are polynomials in y and $A(x), \dots, D(x)$ are polynomials in x , whose degrees are all less than or equal to two. Hence, there exist three subclasses of such maps, which are denoted by a pair of numbers as $[1 : 1], [1 : 2]$ and $[2 : 2]$ depending on the highest degrees of the coefficients of the polynomials for x and y . The most rich subclass is $[2 : 2]$, and so the case is studied in detail in [1] and also in [41].

To classify quadrirational maps, the authors of [1] introduce the equivalence with respect to Möbius transformations acting independently on each variable x, y, u, v :

Definition 2.1. *Quadrirational maps F and \tilde{F} are (Möb)-equivalent if there exist Möbius transformations g_1, g_2, h_1 and h_2 on \mathbb{CP}^1 such that*

$$\tilde{F} = (h_1 \times h_2) \circ F \circ (g_1 \times g_2),$$

namely

$$\tilde{F}(x, y) = (h_1(u(g_1(x), g_2(y))), h_2(v(g_1(x), g_2(y))))$$

where $F(x, y) = (u(x, y), v(x, y))$.

Theorem 1 of [1] states that in the subclass [2 : 2] of quadrirational maps, up to this (Möb)-equivalence, there are only five families of quadrirational maps $F_I = (F_I^{\alpha,\beta}), F_{II} = (F_{II}^{\alpha,\beta}), \dots, F_V = (F_V^{\alpha,\beta})$ where each of them has two complex parameters α, β . Remarkably, they found that all of these five canonical representative maps are Yang-Baxter maps and moreover involutions, and coincide with their companion maps.

However, [41] pointed out that not all quadrirational maps satisfy the Yang-Baxter relation, since the Möbius transformations on each variable, in general, destroy the Yang-Baxter property. Hence, the classification result was further refined in [41], considering the following equivalence.

Definition 2.2. Families of parameter-dependent quadrirational maps $F = (F^{\alpha,\beta})_{\alpha,\beta}$ and $\tilde{F} = (\tilde{F}^{\alpha,\beta})_{\alpha,\beta}$ are YB-equivalent if there exists a family of bijections $\phi(\alpha) : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ such that

$$\tilde{F}^{\alpha,\beta} = (\phi(\alpha)^{-1} \times \phi(\beta)^{-1}) \circ F^{\alpha,\beta} \circ (\phi(\alpha) \times \phi(\beta)).$$

Proposition 1 of [41] showed that if $F = (F^{\alpha,\beta})_{\alpha,\beta}$ and $\tilde{F} = (\tilde{F}^{\alpha,\beta})_{\alpha,\beta}$ are YB-equivalent, then F satisfies the relation (2.1) if and only if \tilde{F} satisfies it. Hence, this is a natural equivalence among the parameter-dependent Yang-Baxter maps. Theorem 2 of [41] showed that, up to this YB-equivalence, there are ten families of quadrirational Yang-Baxter maps of subclass [2 : 2]: the five families F_I, F_{II}, \dots, F_V obtained in [1] and additional five families named $H_I, H_{II}, H_{III}^A, H_{III}^B$ and H_V , where all of them also have two complex parameters α, β . For $* \in \{I, II, V\}$, F_* and H_* are (Möb)-equivalent but not YB-equivalent respectively, and $F_{III}, H_{III}^A, H_{III}^B$ are (Möb)-equivalent but not YB-equivalent.

A rational function is said to be subtraction-free if it is expressed as a ratio of two polynomials with positive real coefficients. As noted in [41], H_I and H_{II} have convenient subtraction-free representatives, which are H_I^+ and H_{II}^+ given in Introduction, and $H_{III,A}, H_{III,B}$ are originally subtraction-free when parameters α, β are positive. If a rational function on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is subtraction-free, then the image of \mathbb{R}_+^2 is a subset of \mathbb{R}_+^2 . Note that is is not the case for \mathbb{R}^2 . In particular, since the maps $H_I^+, H_{II}^+, H_{III,A}$ and $H_{III,B}$ are involutions on $\mathbb{CP}^1 \times \mathbb{CP}^1$, they are also involutions on \mathbb{R}_+^2 . Hence, we can consider them as Yang-Baxter maps on \mathbb{R}_+ by restricting the domain and the codomain to $\mathbb{R}_+ \times \mathbb{R}_+$ and parameters α, β to be positive. Namely, these four families are natural candidates to study the IP property with probability measures supported on \mathbb{R}_+ .

Since $H_{III,B} = F_{GIG}$, the IP property for $H_{III,B}$ was already shown in [8]. Moreover, the (Möb)-equivalence preserves the IP property which means that H_{III}^A would also have the IP property. In fact, by a direct computation, we have

$$H_{III,B} = ((I \circ \theta_\alpha) \times I_d) \circ H_{III,A} \circ (I_d \times (I \circ \theta_\beta))$$

where I_d is the identity map, and by this relation, we can easily conclude the claim of Theorem 1.1 (iii). Hence, the essential novelty of Theorem 1.1 is in the claims for H_I^+ and H_{II}^+ .

In [41], it is also mentioned that there is an obvious limiting procedure starting from H_I^+ to obtain H_{II}^+ and H_{III}^A . This is useful for understanding relations between probability measures which appeared in Theorem 1.1, so we state this relation more explicitly as follows:

Lemma 2.3. The following scaling limits hold in the sense of the compact convergence, namely, the sequence converges uniformly on any compact set $K \subset \mathbb{R}_+^2$.

- (i) $\lim_{\epsilon \downarrow 0} H_I^{+,\epsilon\alpha,\epsilon\beta} = H_{II}^{+,\alpha,\beta}$.
- (ii) $\lim_{\epsilon \downarrow 0} (\theta_\epsilon \times \theta_\epsilon) \circ H_{II}^{+,\alpha,\beta} \circ (\theta_\epsilon \times \theta_\epsilon)^{-1} = H_{III,A}^{\alpha,\beta}$.

Remark 2.4. For the quadrirational Yang-Baxter maps F_{IV}, F_V and H_V , since there does not seem to be a suitable subtraction-free representative, we should either consider probability measures on \mathbb{CP}^1 and see if the IP property holds, or we consider them as maps from \mathbb{R}^2 to \mathbb{R}^2 defined except on a set of Lebesgue measure 0 and see if there is a class of probability measures on \mathbb{R} with the IP property holds. It is not yet known whether they have the IP property by considering the problem in any of these two ways.

3 Independence preserving property

In this section, we introduce a setting in which we study the independence preserving property. Then, we list specific functions that are already known to have this property, in the historical order with a brief background. We have not found any literature that discusses functions having the IP property in a unified manner, and this section is the first such attempt.

3.1 The independence preserving property

Let \mathfrak{F} be the set of measurable bijections between two products of open intervals of \mathbb{R} :

$$\mathfrak{F} := \{F : I_1 \times I_2 \rightarrow J_1 \times J_2 \mid I_1, I_2, J_1, J_2 : \text{open intervals of } \mathbb{R}, F : \text{measurable bijection}\}.$$

For $F \in \mathfrak{F}$, we are interested in the existence and the characterization of independent random variables X, Y taking values in I_1, I_2 respectively such that U, V are also independent with $(U, V) := F(X, Y)$. If there exists such a pair of non-constant random variables X, Y , we say that F has the independence preserving property. Studying this property is equivalent to studying the existence and the characterization of quadruplets of probability measures $\mu, \nu, \tilde{\mu}, \tilde{\nu}$ with suitable supports, which are not delta measures, satisfying $\tilde{\mu} \times \tilde{\nu} = F(\mu \times \nu)$. In [8], the relation $\tilde{\mu} \times \tilde{\nu} = F(\mu \times \nu)$ is called the detailed balance equation, and the quadruplets of probability measures $\mu, \nu, \tilde{\mu}, \tilde{\nu}$ are called solutions of the detailed balance equation, so we follow this terminology. Solutions of the detailed balance equation are also called solutions for short in the following.

To study this question, it is natural to introduce an equivalence relation among such bijections.

Definition 3.1. For $F : I_1 \times I_2 \rightarrow J_1 \times J_2$ and $\tilde{F} : \tilde{I}_1 \times \tilde{I}_2 \rightarrow \tilde{J}_1 \times \tilde{J}_2$ in \mathfrak{F} , we denote $F \sim \tilde{F}$ if one of the following holds:

(a) There exist four measurable bijections g_1, g_2, h_1, h_2 such that

$$g_i : \tilde{I}_i \rightarrow I_i, \quad h_i : J_i \rightarrow \tilde{J}_i, \quad i = 1, 2,$$

and

$$\tilde{F} = (h_1 \times h_2) \circ F \circ (g_1 \times g_2).$$

(b) $\tilde{F} = F^{-1}$.

(c) $\tilde{F} = F \circ \pi$ where $\pi(x, y) = (y, x)$.

Moreover, F and $\tilde{F} \in \mathfrak{F}$ are IP-equivalent if there exists a finite sequence $F_0 = F, F_1, F_2, \dots, F_n = \tilde{F} \in \mathfrak{F}$ such that $F_i \sim F_{i+1}$ for $i = 0, 1, \dots, n - 1$.

The condition (a) means that \tilde{F} is obtained from F by a certain coordinate-wise change of variables, which is similar to the (Möb)-equivalence. It is obvious that if F and \tilde{F} are IP-equivalent, then F has the IP property if and only if \tilde{F} has the IP property. In particular, without loss of generality, we can only consider functions $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ or $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, or fix any intervals as domains and codomains. However, it should be noted that by the change of variables, the probability distributions satisfying the IP property,

namely the solutions of the detailed balance equation, also change. In particular, up to this IP equivalence, in general (more precisely the case where μ and ν are continuous distributions with supports I_1 and I_2 respectively), we can change the solution to be the product of uniform probability measures on $[0, 1]$ by using the distribution functions of μ and ν for the coordinate-wise change of variables. Hence, when we apply the IP property to characterize some probability distributions, the choice of representative F plays an essential role. As we see in the next subsection, all of examples we consider have birational representatives or piecewise linear representatives and such representatives characterize important probability distributions, such as normal, gamma, exponential, beta and so on.

Remark 3.2. The IP property is also known to hold for $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ where \mathcal{X} are discrete sets (cf. [9, 8]) or a set of positive definite matrices with size r (cf. [32, 31]). There are also examples of the IP property in a more abstract setting where \mathcal{X} is an abstract topological group (cf. [15]). In this paper, we focus on the case where we can reduce the problem to $\mathcal{X} = \mathbb{R}_+$.

3.2 History of the study of bijections with the IP property

The IP property was first discovered for the bijection

$$F_N(x, y) = (x + y, x - y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

The solution of the detailed balance equation for this bijection was characterized independently by Kac [23] (1939) and Bernstein [5] (1941) as $X \sim N(a, \sigma), Y \sim N(b, \sigma)$ for any $a, b \in \mathbb{R}$ and $\sigma > 0$ where $N(a, \sigma)$ is the normal distribution with the mean a and the variance σ . The original characterization results were given under some assumption on random variables, but later the characterization was given in full generality (cf. [24]).

After this impressive discovery, several bijections having the IP property have been introduced in different contexts. Most classical ones were found in the context of characterization of important probability distributions. Such examples are

$$F_{Ga}(x, y) = \left(x + y, \frac{x}{y}\right) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2, \quad F_{Exp}(x, y) = (\min\{x, y\}, x - y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

The solutions for F_{Ga} were characterized by Lukacs [34] (1955) as $X \sim Ga(a, \lambda), Y \sim Ga(b, \lambda)$ for any $a, b, \lambda > 0$. For F_{Exp} , the solutions were characterized by Ferguson [17, 16] (1964, 1965) under some assumptions on random variables and then by Crawford [7] (1966) in full generality as pairs of (possibly shifted) exponential distributions or (possibly shifted) geometric distributions with certain parameters. Hence, the IP property for each bijection characterizes the gamma, exponential and geometric distributions. Note that F_{Exp} can be considered as a zero-temperature limit of F_{Ga} , namely, F_{Exp} is obtained from F_{Ga} by replacing $(+, \times)$ -algebra with $(\min, +)$ -algebra. Consistent with this, exponential distributions can be understood as a zero-temperature limit of gamma distributions as discussed in [8, 10]. In particular, as discussed in [8, 10], the IP property is typically inherited to the zero-temperature limit, and the zero-temperature limit version has both continuous and discrete solutions.

As a slightly different example, the function

$$F_C(x, y) = \left(y, \frac{x + y}{1 - xy}\right) : \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \mid xy = 1\} \rightarrow \mathbb{R}^2$$

is known to have the IP property which characterizes the Cauchy distribution. Precisely, though F_C is not well-defined on \mathbb{R}^2 , when X, Y are independent absolutely continuous random variables, (U, V) are well-defined almost surely, and under this condition, U and

V are independent if and only if X has the Cauchy distribution with a specific parameter, which was shown in [3] (1979). In this case, Y can be any absolutely continuous random variable. Moreover, by change of variables $x \rightarrow \arctan x$ and $y \rightarrow \arctan y$, the IP property for the equivalent function is also known to characterize the uniform distribution.

After a blank period of around twenty years, another bijection

$$F_{\text{Be}}(x, y) = \left(\frac{1-y}{1-xy}, 1-xy \right) : (0, 1)^2 \rightarrow (0, 1)^2$$

was found to have the IP property in [48] (2003) and the solutions were completely characterized in [43] as $X \sim \text{Be}(a, b)$, $Y \sim \text{Be}(a + b, c)$ for any $a, b, c > 0$ where $\text{Be}(a, b)$ is the beta distribution with shape parameters a, b .

From a very different context, which was a study of an exponential-version of Pitman's transform for geometric Brownian motions, Matsumoto and Yor considered the involution

$$F_{\text{GIG-Ga}}(x, y) = \left(\frac{1}{x+y}, \frac{1}{x} - \frac{1}{x+y} \right) = \left(\frac{1}{x+y}, \frac{y}{x(x+y)} \right) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$$

in [37] (see also [35, 36]) (2001) and found that this function has the IP property with $X \sim \text{GIG}(-\lambda, a, b)$ and $Y \sim \text{Ga}(\lambda, a)$ for $\lambda, a, b > 0$. The complete characterization of the solutions was done in [32] (2000). Then, to generalize this result, Koudou and Vallois [27] (2012) considered the class of bijections of the form

$$F(x, y) = (f(x+y), f(x) - f(x+y)) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2 \tag{3.1}$$

given by a decreasing three times differentiable bijection $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Remarkably, they completely characterized bijections F in this class having the IP property for some probability distributions with positive and twice differentiable densities. Up to the IP-equivalence we introduced, they proved that other than the case $f(x) = \frac{1}{x}$ which leads to $F_{\text{GIG-Ga}}$, there are only two cases having the IP property: $f(x) = \log\left(\frac{1+x}{x}\right)$ or $f(x) = \log\left(\frac{e^x + \delta - 1}{e^x - 1}\right)$ with a parameter $\delta > 0$. By introducing a certain coordinate-wise change of variables, they showed that $f(x) = \log\left(\frac{1+x}{x}\right)$ leads to the bijection

$$F_{\text{K-Ga,A}}(x, y) = \left(x+y, \frac{1 + \frac{1}{x+y}}{1 + \frac{1}{x}} \right) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \times (0, 1)$$

and $f(x) = \log\left(\frac{e^x + \delta - 1}{e^x - 1}\right)$ leads to the involution

$$F_{\text{Be}}^\delta(x, y) = \left(\frac{1-xy}{1+(\delta-1)xy}, \frac{1-x}{1+(\delta-1)x} \frac{1+(\delta-1)xy}{1-xy} \right) : (0, 1)^2 \rightarrow (0, 1)^2$$

with a parameter $\delta > 0$. When $\delta = 1$, F_{Be}^1 is IP-equivalent to F_{Be} since $F_{\text{Be}} = \pi \circ F_{\text{Be}}^1 \circ \pi$. The solutions for $F_{\text{K-Ga,A}}$ are given as $X \sim \text{K}^{(2)}(a, b, c)$ and $Y \sim \text{Ga}(b, c)$ for $a, b, c > 0$ while those for F_{Be}^δ are given as $X \sim \text{Be}_\delta(a + b, \lambda, -\lambda - b)$ and $Y \sim \text{Be}(a, b)$ for $a, b, \lambda > 0$ where Be_δ is related by the change of variable $x \rightarrow \frac{x}{1-x} : (0, 1) \rightarrow \mathbb{R}_+$ to Be'_δ . Note that F_{Ga} is IP-equivalent to a bijection of the form (3.1) with $f(x) = \log x$, but since this f takes negative values, it was not included in the framework of [27]. The functions of the form (3.1) having the IP property are also said to have Matsumoto-Yor property.

In [18] (2015), the authors introduced another bijection with the IP property

$$F_{\text{K-Ga,B}}(x, y) = \left(\frac{y}{1+x}, \frac{x(1+x+y)}{1+x} \right) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$$

which also involves the Kummer distribution and gamma distribution with a motivation to give a closed identity satisfied by the Kummer distribution. The characterizations of

solutions without any assumption on distributions for $F_{K-Ga,A}$ and $F_{K-Ga,B}$ are given in [42] (2018).

Most recently, in the study of invariant measures for the discrete KdV equation, which is a well-known discrete integrable system, one of the authors of the present paper found, with the coauthor of [8], that for the class of involutions $F_{GIG}^{\alpha,\beta}$, the IP property holds with $X \sim GIG(\lambda, a\alpha, b)$ and $Y \sim GIG(\lambda, b\beta, a)$ for $\lambda \in \mathbb{R}, a, b > 0$. Moreover, the special case $\alpha = 1, \beta = 0$ is IP-equivalent to F_{GIG-Ga} since

$$F_{GIG-Ga} = (I_d \times I) \circ F_{GIG}^{1,0} \circ (I_d \times I).$$

The zero-temperature version of $F_{GIG}^{\alpha,\beta}$, which is related to the ultra-discrete KdV equation, and the zero-temperature version of F_{Be} are also introduced in the study of discrete integrable systems and stochastic integrable systems, and shown to have the IP property [8, 10]:

$$\begin{aligned} F_{GIG,zero}^{\alpha,\beta}(x, y) &= (x + \min\{y, \alpha - x\} - \min\{x, \beta - y\}, y - \min\{y, \alpha - x\} + \min\{x, \beta - y\}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ F_{Be,zero}(x, y) &= (\min\{x, 0\} - y, \min\{x, y, 0\} - x - y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2. \end{aligned}$$

The characterizations of solutions for them are studied in [8, 4], but not fully solved yet, even for the special case $(\alpha, \beta) = (0, \infty)$:

$$F_{GIG,zero}^{0,\infty}(x, y) = (\min\{y, -x\}, y + x - \min\{y, -x\}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

which is IP-equivalent to the zero-temperature version of F_{GIG-Ga} .

In the next subsection, we discuss common properties of them and give a classification.

Remark 3.3. An anonymous reviewer gave us the following two examples having the IP property, which we were unaware of. The first example is

$$F(x, y) = \left(x + y, \frac{1}{x} + \frac{1}{y} - \frac{4}{x + y} \right) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \times [0, \infty)$$

for independent X and Y having inverse Gaussian distributions, which was first shown in [45] and the characterization of solutions was given in [30]. The second example is

$$F(x, y) = (x, x + y - \mathbb{1}_{x+y>1}) : [0, 1]^2 \rightarrow [0, 1]^2$$

for independent X and Y having uniform distributions on $[0, 1]$, which goes back to [44], see also [11, 20]. We note that the first example is not a bijection since the function is symmetric for x and y . It remains to be seen if these examples can also be understood in some unified way.

3.3 Normalized bijections having the IP property in literature and its classification

The bijections introduced in the last subsection are simply classified into two classes: the zero-temperature version of some other bijections and the rest of them. All of the zero-temperature versions, namely $F_{Exp}, F_{GIG,zero}^{\alpha,\beta}$ and $F_{Be,zero}$ are piecewise linear functions and involve the “min” function. Since the IP property of these zero-temperature versions can be understood from the same property of the original (namely, the corresponding “positive-temperature”) bijection, from now on, we focus on the rest of them, namely, $F_N, F_{Ga}, F_C, F_{Be}, F_{GIG-Ga}, F_{K-Ga,A}, F_{Be}^\delta, F_{K-Ga,B}, F_{GIG}^{\alpha,\beta}$ and consider the relation between them.

Remark 3.4. As mentioned in [41] and already discussed in Introduction, there should be natural zero-temperature versions of H_I^+ and H_{II}^+ . Applying Theorem 1.1, we will be able to show that the zero-temperature versions of H_I^+ and H_{II}^+ have the IP property and their solutions should be the zero-temperature version of generalized beta prime distributions and Kummer distributions of Type 2. Specifically, we should be able to do it the way described in Subsection 5.2 of [10]. This is not technically difficult but requires the introduction of a new family of associated probability distributions. We do not pursue this topic further here to simplify the paper.

From the explicit expressions of these bijections, we can find that they are all birational and naturally extended to the birational functions on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Hence, we may consider coordinate-wise Möbius transformations to normalize these functions to compare. Namely, instead of considering the equivalence with respect to (a) of Definition 3.1, consider the (Möb)-equivalence. Then, any open interval $I \subsetneq \mathbb{R}$ can be mapped to \mathbb{R}_+ by a Möbius transformation, but \mathbb{R} cannot. Hence, in this sense, F_N and F_C are exceptional, which are not (Möb)-equivalent to a birational function which can be restricted to the one having domains and codomains \mathbb{R}_+^2 . In other words, F_{Ga} , F_{Be} , F_{GIG-Ga} , $F_{K-Ga,A}$, F_{Be}^δ , $F_{K-Ga,B}$, $F_{GIG}^{\alpha,\beta}$ share the common property that they are (Möb)-equivalent to a subtraction-free birational function with domains and codomains \mathbb{R}_+^2 . Hence, to study relations between these bijections, we give such representatives for them, which are denoted by F_*^+ , explicitly by applying the change of variable $x \rightarrow \frac{x}{1-x} : (0, 1) \rightarrow \mathbb{R}_+$ for each coordinate in F_{Be}^δ and the second component of the output of $F_{K-Ga,A}$:

- $F_{Ga}^+(x, y) = F_{Ga}(x, y) = \left(x + y, \frac{x}{y}\right)$, $(F_{Ga}^+)^{-1}(x, y) = \left(\frac{xy}{1+y}, \frac{x}{1+y}\right)$,
- $F_{Be}^{+\delta}(x, y) = \left(\frac{1+x+y}{\delta xy}, \frac{1+x+y+\delta xy}{x(\delta+\delta x)}\right)$, $(F_{Be}^{+\delta})^{-1} = F_{Be}^{+\delta}$,
- $F_{K-Ga,A}^+(x, y) = \left(x + y, \frac{x(x+y+1)}{y}\right)$, $(F_{K-Ga,A}^+)^{-1}(x, y) = \left(\frac{xy}{1+x+y}, \frac{x(1+x)}{1+x+y}\right)$,
- $F_{K-Ga,B}^+(x, y) = F_{K-Ga,B}(x, y) = \left(\frac{y}{1+x}, \frac{x(1+x+y)}{1+x}\right)$, $(F_{K-Ga,B}^+)^{-1} = F_{K-Ga,B}^+$,
- $F_{GIG}^{+\alpha,\beta}(x, y) = F_{GIG}^{\alpha,\beta}(x, y) = \left(y \frac{1+\beta xy}{1+\alpha xy}, x \frac{1+\alpha xy}{1+\beta xy}\right)$, $(F_{GIG}^{+\alpha,\beta})^{-1} = F_{GIG}^{+\alpha,\beta}$.

For F_{Be} and F_{GIG-Ga} , since we know they are (Möb)-equivalent to F_{Be}^1 and $F_{GIG}^{1,0}$ respectively, we choose representatives as

- $F_{Be}^+(x, y) = F_{Be}^{+,1}(x, y) = \left(\frac{1+x+y}{xy}, \frac{1+y}{x}\right)$,
- $F_{GIG-Ga}^+ = F_{GIG}^{+,1,0}(x, y) = \left(\frac{y}{1+xy}, x(1+xy)\right)$.

Other than these two simple equivalence, any relation between these bijections has not been discussed in the literature. From these explicit expressions, we can easily see that some of them are obtained by a singular limit procedure from others by introducing scaling parameters. Such procedure typically conserves the IP property, and so the IP property of them are also related. Surprisingly, all of these bijections are obtained from H_I^+ , H_{II}^+ and $H_{III,A}$ by such procedure and/or taking special values of parameters together with a proper coordinate-wise change of variables, which is our second main result stated in Theorem 5.1 in Section 5.

4 Proof of Theorem 1.1 and properties of probability distributions

In this section, we discuss proofs of Theorem 1.1. In the first subsection, we give a proof by a direct computation for the most complicated case, namely for Claim (i).

In the second subsection, we study some useful properties of probability distributions appearing in Theorem 1.1, and explain a way to obtain Theorem 1.1 (ii) and (iii) from Theorem 1.1 (i).

4.1 Proof of Theorem 1.1

In this subsection, we give a proof of Theorem 1.1. Actually, the result is given by a direct computation, but for clarity, we give some key formulas.

To prove the claims of Theorem 1.1, we only need to prove that

$$p_X(x)p_Y(y) = J_F(x, y)p_U(u(x, y))p_V(v(x, y))$$

holds where $F(x, y) = (u(x, y), v(x, y))$ and $J_F(x, y)$ is the Jacobian of F , namely

$$J_F(x, y) = \left| \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right|$$

and p_X, p_Y, p_U, p_V are probability density functions for the distributions with parameters given in each claim.

To check the relation for (i), let $u(x, y) = \frac{y \sigma_1(x, y)}{\alpha \sigma_2(x, y)}$ and $v(x, y) = \frac{x \sigma_3(x, y)}{\beta \sigma_4(x, y)}$, namely

$$\begin{aligned} \sigma_1(x, y) &= \beta + \alpha x + \beta y + \alpha \beta xy, & \sigma_2(x, y) &= 1 + x + y + \beta xy, \\ \sigma_3(x, y) &= \alpha + \alpha x + \beta y + \alpha \beta xy, & \sigma_4(x, y) &= 1 + x + y + \alpha xy. \end{aligned}$$

Then, by simple calculations, we have

$$\begin{aligned} y\sigma_1 + \sigma_2 &= \sigma_4(1 + \beta y), & y\sigma_1 + \alpha\sigma_2 &= \sigma_3(1 + y), \\ x\sigma_3 + \sigma_4 &= \sigma_2(1 + \alpha x), & x\sigma_3 + \beta\sigma_4 &= \sigma_1(1 + x), \\ J_F(x, y) &= \frac{\sigma_1\sigma_3}{\alpha\beta\sigma_2\sigma_4} \end{aligned}$$

where we simply denote $\sigma_i(x, y)$ by σ_i for $i = 1, \dots, 4$. Hence, if $U \sim \text{Be}'(-\lambda, a, b; \alpha, 1)$, we have

$$\begin{aligned} p_U(u(x, y)) &= \frac{1}{Z} \left(\frac{y \sigma_1}{\alpha \sigma_2} \right)^{-\lambda-1} \left(1 + y \frac{\sigma_1}{\sigma_2} \right)^{-a+\frac{\lambda}{2}} \left(1 + \frac{\alpha \sigma_2}{y \sigma_1} \right)^{-b-\frac{\lambda}{2}} \\ &= \frac{1}{Z} y^{-\lambda-1} \frac{\sigma_1^{-\lambda-1}}{\sigma_2^{-\lambda-1}} \left(\frac{\sigma_2 + y\sigma_1}{\sigma_2} \right)^{-a+\frac{\lambda}{2}} \left(\frac{y\sigma_1 + \alpha\sigma_2}{y\sigma_1} \right)^{-b-\frac{\lambda}{2}} \\ &= \frac{1}{Z} y^{-\lambda-1} \sigma_1^{-\frac{\lambda}{2}-1+b} \sigma_2^{\frac{\lambda}{2}+1+a} (\sigma_4(1 + \beta y))^{-a+\frac{\lambda}{2}} (\sigma_3(1 + y))^{-b-\frac{\lambda}{2}} \end{aligned}$$

where Z is a normalizing constant which may change line by line. In the same way, if $V \sim \text{Be}'(\lambda, a, b; \beta, 1)$, we have

$$\begin{aligned} p_V(v(x, y)) &= \frac{1}{Z} \left(\frac{x \sigma_3}{\beta \sigma_4} \right)^{\lambda-1} \left(1 + x \frac{\sigma_3}{\sigma_4} \right)^{-a-\frac{\lambda}{2}} \left(1 + \frac{\beta \sigma_4}{x \sigma_3} \right)^{-b+\frac{\lambda}{2}} \\ &= \frac{1}{Z} x^{\lambda-1} \frac{\sigma_3^{\lambda-1}}{\sigma_4^{\lambda-1}} \left(\frac{\sigma_4 + x\sigma_3}{\sigma_4} \right)^{-a-\frac{\lambda}{2}} \left(\frac{x\sigma_3 + \beta\sigma_4}{x\sigma_3} \right)^{-b+\frac{\lambda}{2}} \\ &= \frac{1}{Z} x^{\lambda-1} \sigma_3^{\frac{\lambda}{2}-1+b} \sigma_4^{-\frac{\lambda}{2}+1+a} (\sigma_2(1 + \alpha x))^{-a-\frac{\lambda}{2}} (\sigma_1(1 + x))^{-b+\frac{\lambda}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} &J_F(x, y)p_U(u(x, y))p_V(v(x, y)) \\ &= \frac{1}{Z} y^{-\lambda-1} (1 + \beta y)^{-a+\frac{\lambda}{2}} (1 + y)^{-b-\frac{\lambda}{2}} x^{\lambda-1} (1 + \alpha x)^{-a-\frac{\lambda}{2}} (1 + x)^{-b+\frac{\lambda}{2}}, \end{aligned}$$

which is equal to $p_X(x)p_Y(y)$ if $X \sim \text{Be}'(\lambda, a, b; \alpha, 1)$ and $Y \sim \text{Be}'(-\lambda, a, b; \beta, 1)$. Therefore we conclude that the claim (i) holds.

For Claims (ii) and (iii), we can prove in a similar way. In the next subsection, we give another proof by applying a singular limit procedure.

4.2 Some properties of probability distributions under Möbius transformations and scaling limits

We have introduced probability distributions $\text{Be}'(\lambda, a, b; p, q)$, $K(\lambda, a, b; p, q)$ and $\text{GIG}(\lambda, a, b; p, q)$ with two auxiliary parameters $p, q > 0$ because these classes are closed under the change of scaling, and also two of them are closed under the map I as follows.

Lemma 4.1. *For the maps I and θ_α for any $\alpha > 0$, the following relations hold.*

(i) *Suppose $X \sim \text{Be}'(\lambda, a, b; p, q)$. Then,*

$$I(X) = X^{-1} \sim \text{Be}'(-\lambda, b, a; q, p), \quad \theta_\alpha(X) = \alpha X \sim \text{Be}'(\lambda, a, b; \frac{p}{\alpha}, \alpha q).$$

(ii) *Suppose $X \sim K(\lambda, a, b; p, q)$. Then,*

$$\theta_\alpha(X) = \alpha X \sim K(\lambda, a, b; \frac{p}{\alpha}, \alpha q).$$

(iii) *Suppose $X \sim \text{GIG}(\lambda, a, b; p, q)$. Then,*

$$I(X) = X^{-1} \sim \text{GIG}(-\lambda, b, a; q, p), \quad \theta_\alpha(X) = \alpha X \sim \text{GIG}(\lambda, a, b; \frac{p}{\alpha}, \alpha q).$$

This lemma is proved by direct computations.

Moreover, these probability measures are connected by the following scaling limits.

Lemma 4.2. *The following scaling limits hold as the weak limit of probability measures.*

(i) *For $a > 0$ and $b, \lambda \in \mathbb{R}$ such that $-b < \frac{\lambda}{2}$,*

$$\lim_{\epsilon \downarrow 0} \text{Be}'(\lambda, \frac{a}{\epsilon}, b; \epsilon p, q) = K(\lambda, a, b; p, q).$$

(ii) *For $a, b > 0$ and $\lambda \in \mathbb{R}$,*

$$\lim_{\epsilon \downarrow 0} K(\lambda, a, \frac{b}{\epsilon}; p, \epsilon q) = \text{GIG}(\lambda, a, b; p, q).$$

This lemma is also proved by the explicit forms of probability density functions, by noting

$$\lim_{\epsilon \downarrow 0} (1 + \epsilon p x)^{-\frac{a}{\epsilon} - \frac{\lambda}{2}} = e^{-apx}.$$

To obtain a proof of (ii) and (iii) of Theorem 1.1 by the scaling limits given in Lemmas 2.3 and 4.2, we prove a general result concerning the weak convergence of push-forward of weakly converging probability measures by compactly converging functions.

Lemma 4.3. *Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of probability measures on \mathbb{R}_+^2 and $(f_n)_{n \in \mathbb{N}}$ be a sequence of continuous functions from \mathbb{R}_+^2 to \mathbb{R}_+^2 . Assume that there exist a probability measure μ on \mathbb{R}_+^2 and a continuous function $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ such that*

$$\mu_n \rightarrow \mu \text{ weakly, } f_n \rightarrow f \text{ uniformly on any compact set } K \subset \mathbb{R}_+^2$$

as n goes to ∞ . Then, we have $\lim_{n \rightarrow \infty} f_n \circ \mu_n = f \circ \mu$ weakly.

Proof. We prove that for any bounded continuous function $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} E[g(f_n(X_n))] = E[g(f(X))]$$

if $X_n \sim \mu_n$ and $X \sim \mu$. Since $\mu_n \rightarrow \mu$ weakly, we have

$$\lim_{n \rightarrow \infty} E[g(f(X_n))] = E[g(f(X))].$$

Hence, we only need to prove that

$$\lim_{n \rightarrow \infty} |E[g(f(X_n))] - E[g(f_n(X_n))]| = 0. \tag{4.1}$$

Now, we prove that $g \circ f_n \rightarrow g \circ f$ uniformly on any compact set $K \subset \mathbb{R}_+^2$. Fix a compact set $K \subset \mathbb{R}_+^2$ and suppose that $g \circ f_n|_K$ does not converge uniformly to $g \circ f|_K$. Then, there exist $\epsilon > 0$, a sequence $(x_k)_{k \in \mathbb{N}} \subset K$ and an increasing sequence $n_k \rightarrow \infty$ ($k \rightarrow \infty$) such that

$$|g(f_{n_k}(x_k)) - g(f(x_k))| > \epsilon$$

for any $k \in \mathbb{N}$. Since K is compact, we can even assume that x_k converges to some $x_\infty \in K$ by passing to a subsequence if necessary. Then, since f_n converges to f uniformly on K , $\lim_{k \rightarrow \infty} f_{n_k}(x_k) = f(x_\infty)$ and so

$$\lim_{k \rightarrow \infty} (g(f_{n_k}(x_k)) - g(f(x_k))) = g(f(x_\infty)) - g(f(x_\infty)) = 0$$

gives the contradiction.

Having this uniform convergence $g \circ f_n \rightarrow g \circ f$ on compact sets, it is straightforward to prove (4.1). In fact, assuming $\|g\|_\infty \neq 0$, for any $\epsilon > 0$, since $\mu_n \rightarrow \mu$ weakly, there exists a compact set $K_\epsilon \subset \mathbb{R}_+^2$ such that $\sup_n P(X_n \notin K_\epsilon) < \frac{\epsilon}{4\|g\|_\infty}$. Also, there exists n_ϵ such that for any $n \geq n_\epsilon$ and $x \in K_\epsilon$, $|g(f_n(x)) - g(f(x))| < \frac{\epsilon}{2}$. Hence,

$$\begin{aligned} |E[g(f_n(X_n))] - E[g(f(X_n))]| &\leq E[|g(f_n(X_n)) - g(f(X_n))|] \\ &\leq E[2\|g\|_\infty \mathbb{1}_{X_n \notin K_\epsilon}] + E[|g(f_n(X_n)) - g(f(X_n))| \mathbb{1}_{X_n \in K_\epsilon}] \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, the lemma follows. □

From the claim (i) of Theorem 1.1, for any $a > 0$ and $b, \lambda \in \mathbb{R}$ such that $-b < \frac{\lambda}{2}$, we have the IP property holds for $H_I^{+, \epsilon\alpha, \epsilon\beta}$ with

$$\begin{aligned} X &\sim \text{Be}'(\lambda, \frac{a}{\epsilon}, b; \epsilon\alpha, 1), \quad Y \sim \text{Be}'(-\lambda, \frac{a}{\epsilon}, b; \epsilon\beta, 1), \\ U &\sim \text{Be}'(-\lambda, \frac{a}{\epsilon}, b; \epsilon\alpha, 1), \quad V \sim \text{Be}'(\lambda, \frac{a}{\epsilon}, b; \epsilon\beta, 1) \end{aligned}$$

with sufficiently small $\epsilon > 0$. Hence, applying Lemmas 2.3 (i), 4.2 (i) and 4.3, we conclude the claim (ii) of Theorem 1.1.

By this claim (ii) of Theorem 1.1 and Lemma 4.1 (ii), for $(\theta_\epsilon \times \theta_\epsilon) \circ H_{II}^{+, \alpha, \beta} \circ (\theta_\epsilon \times \theta_\epsilon)^{-1}$, the IP property holds with

$$\begin{aligned} X &\sim \text{K}(\lambda, a, b; \frac{\alpha}{\epsilon}, \epsilon), \quad Y \sim \text{K}(-\lambda, a, b; \frac{\alpha}{\epsilon}, \epsilon), \\ U &\sim \text{K}(-\lambda, a, b; \frac{\alpha}{\epsilon}, \epsilon), \quad V \sim \text{K}(\lambda, a, b; \frac{\alpha}{\epsilon}, \epsilon). \end{aligned}$$

By changing parameters as $a \rightarrow a\epsilon$ and $b \rightarrow \frac{b}{\epsilon}$, the IP property holds for the same function with

$$X \sim K(\lambda, a\epsilon, \frac{b}{\epsilon}; \frac{\alpha}{\epsilon}, \epsilon) = K(\lambda, a, \frac{b}{\epsilon}; \alpha, \epsilon), \quad Y \sim K(-\lambda, a\epsilon, \frac{b}{\epsilon}; \frac{\alpha}{\epsilon}, \epsilon) = K(-\lambda, a, \frac{b}{\epsilon}; \alpha, \epsilon),$$

$$U \sim K(-\lambda, a\epsilon, \frac{b}{\epsilon}; \frac{\alpha}{\epsilon}, \epsilon) = K(-\lambda, a, \frac{b}{\epsilon}; \alpha, \epsilon), \quad V \sim K(\lambda, a\epsilon, \frac{b}{\epsilon}; \frac{\alpha}{\epsilon}, \epsilon) = K(\lambda, a, \frac{b}{\epsilon}; \beta, \epsilon)$$

for sufficiently small $\epsilon > 0$. Then, applying Lemmas 2.3 (ii), 4.2 (ii) and 4.3, we conclude the claim (iii) of Theorem 1.1.

5 Reduction of known IP properties from Theorem 1.1

In this section, we recover the IP property for bijections discussed in Section 3 from Theorem 1.1. For this, we give our second main result which connects bijections having a subtraction-free representation to the newly introduced bijections H_I^+, H_{II}^+ and $H_{III,A}$.

Theorem 5.1. *The bijections $F_{Be}^{+, \delta}$, $F_{K-Ga,A}^+$, $F_{K-Ga,B}^+$, F_{Ga}^+ and $F_{GIG}^{+, \alpha, \beta}$ are obtained from one of $H_I^{+, \alpha, \beta}$, $H_{II}^{+, \alpha, \beta}$ and $H_{III,A}^{+, \alpha, \beta}$ by Möbius transformations and singular limits as follows.*

(i) $F_{Be}^{+, \delta} = \tilde{H}_I^{\delta, 0}$ where

$$\tilde{H}_I^{\alpha, \beta} = ((I \circ \theta_\alpha) \times (I \circ \theta_\beta)) \circ H_I^{+, \alpha, \beta}.$$

(ii) $F_{K-Ga,A}^+ = \tilde{H}_{II}^{1, 0}$ where

$$\tilde{H}_{II}^{\alpha, \beta} = H_{II}^{+, \alpha, \beta} \circ (\theta_{\alpha^{-1}} \times \theta_{\beta^{-1}}).$$

(iii) $F_{K-Ga,B}^+ = \hat{H}_{II}^{1, 0}$ where

$$\hat{H}_{II}^{\alpha, \beta} = (\theta_{\alpha^{-1}} \times \theta_{\beta^{-1}}) \circ H_{II}^{+, \alpha^{-1}, \beta^{-1}} \circ (\theta_\alpha \times \theta_\beta).$$

(iv) $F_{Ga}^+ = \tilde{F}_{K-Ga,A}^0$ where

$$\tilde{F}_{K-Ga,A}^\epsilon = (\theta_{\epsilon^{-1}} \times I_d) \circ F_{K-Ga,A}^+ \circ (\theta_\epsilon \times \theta_\epsilon)$$

and $\tilde{F}_{K-Ga,A}^0 = \lim_{\epsilon \downarrow 0} \tilde{F}_{K-Ga,A}^\epsilon$. Note that the sequence converges compactly.

(v) $F_{Ga}^+ = \tilde{F}_{K-Ga,B}^0$ where

$$\tilde{F}_{K-Ga,B}^\epsilon = \pi \circ (I \times \theta_\epsilon) \circ F_{K-Ga,B}^+ \circ (\theta_{\epsilon^{-1}} \times \theta_{\epsilon^{-1}})$$

and $\tilde{F}_{K-Ga,B}^0 = \lim_{\epsilon \downarrow 0} \tilde{F}_{K-Ga,B}^\epsilon$. Note that the sequence converges compactly.

(vi) $F_{GIG}^{+, \alpha, \beta} = ((I \circ \theta_\alpha) \times I_d) \circ H_{III,A}^{+, \alpha, \beta} \circ (I_d \times (I \circ \theta_\beta))$.

The theorem is proved by a direct computation.

Applying these explicit relations and Lemmas 4.1 and 4.3 to Theorem 1.1, we have the solutions of the detailed balance equations for all of the bijections that appeared in the last theorem in a systematic way. Note that when parameters p, q take singular values 0 or ∞ , the range of parameters (λ, a, b) may narrow depending on the distribution.

Corollary 5.2. *Let $\alpha, \beta > 0$. If X and Y are independent random variables with the following marginal distributions, then the random variables U, V given by $(U, V) = F(X, Y)$ for each map are independent and have the following marginal distributions.*

(i) For $F = \tilde{H}_I^{\alpha, \beta}$,

$$X \sim Be'(\lambda, a, b; \alpha, 1), \quad Y \sim Be'(-\lambda, a, b; \beta, 1),$$

$$U \sim Be'(\lambda, b, a; \alpha, 1), \quad V \sim Be'(-\lambda, b, a; \beta, 1)$$

and for $F = F_{\text{Be}}^{+, \delta}$,

$$\begin{aligned} X &\sim \text{Be}'(\lambda, a, b; \delta, 1) = \text{Be}'_{\delta} \left(b + \frac{\lambda}{2}, a - \frac{\lambda}{2}, -a - \frac{\lambda}{2} \right), \\ Y &\sim \text{Be}'(-\lambda, a, b; 0, 1) = \text{Be}' \left(b - \frac{\lambda}{2}, \lambda \right) \\ U &\sim \text{Be}'(\lambda, b, a; \delta, 1) = \text{Be}'_{\delta} \left(a + \frac{\lambda}{2}, b - \frac{\lambda}{2}, -b - \frac{\lambda}{2} \right), \\ V &\sim \text{Be}'(-\lambda, b, a; 0, 1) = \text{Be}' \left(a - \frac{\lambda}{2}, \lambda \right) \end{aligned}$$

where $\lambda \in \mathbb{R}$, $a, b > 0$, $-\min\{a, b\} < \frac{\lambda}{2} < \min\{a, b\}$.

(ii) For $F = \tilde{H}_{II}^{\alpha, \beta}$,

$$\begin{aligned} X &\sim \text{K}(\lambda, a, b; 1, \alpha), \quad Y \sim \text{K}(-\lambda, a, b; 1, \beta), \\ U &\sim \text{K}(-\lambda, a, b; \alpha, 1), \quad V \sim \text{K}(\lambda, a, b; \beta, 1) \end{aligned}$$

where $\lambda \in \mathbb{R}$, $a, b > 0$, $-b < \frac{\lambda}{2} < b$ and for $F = F_{\text{K-Ga}, A}^+$

$$\begin{aligned} X &\sim \text{K}(\lambda, a, b; 1, 1) = \text{K}^{(2)} \left(b + \frac{\lambda}{2}, -\lambda, a \right), \quad Y \sim \text{K}(-\lambda, a, b; 1, 0) = \text{Ga}(-\lambda, a), \\ U &\sim \text{K}(-\lambda, a, b; 1, 1) = \text{K}^{(2)} \left(b - \frac{\lambda}{2}, \lambda, a \right), \quad V \sim \text{K}(\lambda, a, b; 0, 1) = \text{Be}' \left(b + \frac{\lambda}{2}, -\lambda \right) \end{aligned}$$

where $\lambda < 0$, $a, b > 0$, $-b < \frac{\lambda}{2}$.

(iii) For $F = \hat{H}_{II}^{\alpha, \beta}$,

$$\begin{aligned} X &\sim \text{K}(\lambda, a, b; 1, \alpha^{-1}), \quad Y \sim \text{K}(-\lambda, a, b; 1, \beta^{-1}), \\ U &\sim \text{K}(-\lambda, a, b; 1, \alpha^{-1}), \quad V \sim \text{K}(\lambda, a, b; 1, \beta^{-1}) \end{aligned}$$

and for $F = F_{\text{K-Ga}, B}^+$

$$\begin{aligned} X &\sim \text{K}(\lambda, a, b; 1, 1) = \text{K}^{(2)} \left(b + \frac{\lambda}{2}, -\lambda, a \right), \quad Y \sim \text{K}(-\lambda, a, b; 1, \infty) = \text{Ga} \left(b - \frac{\lambda}{2}, a \right), \\ U &\sim \text{K}(-\lambda, a, b; 1, 1) = \text{K}^{(2)} \left(b - \frac{\lambda}{2}, \lambda, a \right), \quad V \sim \text{K}(\lambda, a, b; 1, \infty) = \text{Ga} \left(b + \frac{\lambda}{2}, a \right) \end{aligned}$$

where $\lambda \in \mathbb{R}$, $a, b > 0$, $-b < \frac{\lambda}{2} < b$.

(iv) For $F = \tilde{F}_{\text{K-Ga}, A}^{\epsilon}$

$$\begin{aligned} X &\sim \text{K}(\lambda, a, b; \epsilon, \epsilon^{-1}) = \text{K}(\lambda, a\epsilon, b; 1, \epsilon^{-1}), \quad Y \sim \text{K}(-\lambda, a, b; \epsilon, 0) = \text{K}(-\lambda, a\epsilon, b; 1, 0), \\ U &\sim \text{K}(-\lambda, a, b; \epsilon, \epsilon^{-1}) = \text{K}(-\lambda, a\epsilon, b; 1, \epsilon^{-1}), \quad V \sim \text{K}(\lambda, a, b; 0, 1) = \text{Be}' \left(b + \frac{\lambda}{2}, -\lambda \right) \end{aligned}$$

where $\lambda < 0$, $a, b > 0$, $-b < \frac{\lambda}{2}$. By changing the parameter as $a \rightarrow \epsilon^{-1}a$ and taking $\epsilon \downarrow 0$, for $F = F_{\text{Ga}}^+$

$$\begin{aligned} X &\sim \text{K}(\lambda, a, b; 1, \infty) = \text{Ga} \left(b + \frac{\lambda}{2}, a \right), \quad Y \sim \text{K}(-\lambda, a, b; 1, 0) = \text{Ga}(-\lambda, a), \\ U &\sim \text{K}(-\lambda, a, b; 1, \infty) = \text{Ga} \left(b - \frac{\lambda}{2}, a \right), \quad V \sim \text{K}(\lambda, a, b; 0, 1) = \text{Be}' \left(b + \frac{\lambda}{2}, -\lambda \right) \end{aligned}$$

where $\lambda < 0$, $a, b > 0$, $-b < \frac{\lambda}{2}$.

(v) For $F = F_{\text{GIG}}^{+, \alpha, \beta}$,

$$\begin{aligned} X &\sim \text{GIG}(\lambda, a, b; \alpha, 1) = \text{GIG}(\lambda, a\alpha, b), & Y &\sim \text{GIG}(\lambda, b, a; \beta, 1) = \text{GIG}(\lambda, b\beta, a), \\ U &\sim \text{GIG}(\lambda, b, a; \alpha, 1) = \text{GIG}(\lambda, b\alpha, b), & V &\sim \text{GIG}(\lambda, a, b; \beta, 1) = \text{GIG}(\lambda, a\beta, b) \end{aligned}$$

where $\lambda \in \mathbb{R}$, $a, b > 0$.

This recovers all results we consider in this paper on the IP property except for F_N and F_C and the zero-temperature versions. Though we do not discuss this in detail in this paper, the zero-temperature limits of the above subtraction-free bijections and their (continuous) solutions are also systematically obtained. Hence, we can conclude that the IP property for H_I^+ is quite fundamental and all other examples considered in this paper except F_N and F_C are derived by changing variables, taking special parameters and performing some limiting procedures from that for H_I^+ . This explains why the solutions have (at least) three parameters for all bijections in Corollary 5.2, which was a mystery until now.

Remark 5.3. In Corollary 5.2, we did not apply the relation (v) of Theorem 5.1 since the distribution $K(\lambda, a, b; p, q)$ is not closed for the bijection $I(x) = x^{-1}$. If we introduce the inverse Kummer distribution properly, we can also derive the IP property for F_{Ga}^+ using the relation (v) of Theorem 5.1.

Remark 5.4. After finishing the manuscript of this paper, we found that there is another IP property result for beta distribution obtained in [22]. The bijection studied in the paper is

$$F(x, y) = \left(\frac{xy}{1 - y + xy}, 1 - y + xy \right) : (0, 1)^2 \rightarrow (0, 1)^2$$

and for independent X and Y having beta distributions, the function has the IP property. By applying the change of variable $x \rightarrow \frac{x}{1-x} : (0, 1) \rightarrow \mathbb{R}_+$ for each coordinate of the input and the output of F , the bijection is IP-equivalent to

$$F^+(x, y) = \left(\frac{xy}{1+x}, \frac{1+x+xy}{y} \right) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2.$$

This function is obtained from $H_I^{+, \alpha, \beta}$ as

$$F^+(x, y) = \lim_{\alpha \uparrow \infty} H_I^{+, \alpha, 1}.$$

In this way, the IP property obtained in [22] is also recovered from the IP property of $H_I^{+, \alpha, \beta}$. We also note that the matrix variate version of this IP property has been also studied recently in [38].

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⁴BS: Bernoulli Society <http://www.bernoulli-society.org/>

⁵Project Euclid: <https://projecteuclid.org/>

⁶IMS Open Access Fund: <https://imstat.org/shop/donation/>