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A complete characterization of a correlated Bernoulli process*

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Abstract

We present a complete characterization of the asymptotic behaviour of a correlated Bernoulli sequence that depends on the parameter $\theta \in [0,1)$. A martingale theory based approach allows us to prove versions of the law of large numbers, quadratic strong law, law of iterated logarithm, almost sure central limit theorem and functional central limit theorem, in the case $\theta \leq 1/2$. For $\theta > 1/2$, we obtain a strong convergence to a non-degenerated random variable, including a central limit theorem and a law of iterated logarithm for the fluctuations.

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1 Introduction

The asymptotics for the success rate of a Bernoulli sequence is a pretty well known subject matter in probability and statistics. Its intuitive character and great applicability made it quite popular in the scientific community and rather useful for modelling problems on different areas.

Since the independent case has already been well studied, many works have considered some dependence structure on the source. A particularly interesting case arises when the probability of success depends on the number (or rate) of previous successes. In this sense, we refer to the earliest works [16, 18], in which the authors deal with the probability distribution of random variables with this kind of structure.

By following the approach of looking to the previous success rate to determine the next step's probability distribution, the present paper considers a generalization of the binomial distribution proposed in [6]. In this case, for $n \geq 0$, the associated random variable S_n counts the number of successes in a correlated Bernoulli sequence, denoted

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by $\{X_n, n \geq 1\}$ and with conditional probabilities given, for some parameter $0 \leq \theta < 1$, by

$$\mathbb{P}(X_{n+1} = 1 | S_n) = (1-\theta)p + \theta \frac{S_n}{n}, \tag{1.1}$$

where $S_n = X_1 + X_2 + \cdots + X_n$ at time $n \geq 1$, and X_1 is distributed as a Bernoulli random variable with parameter $0 < \alpha < 1$. In this sense, we may imagine that S_n represents the number of infected individuals in a population and each person is infected with a probability that depends on previous contagious; or $\{X_n, n \geq 1\}$ represents binary opinions and each individual can be influenced by previous opinions. In addition, from a technical point of view, success probabilities at time $n \geq 0$ are a weighting of the previous rate of successes and the fixed probability p.

In the case $\alpha = p$, it is possible (as we can see in [13]) to show that $\mathbb{E}(X_n) = p$ and $\mathbb{E}(S_n) = np$, for any $n \geq 1$, and the limiting behaviour of S_n depends on the parameter θ, in the sense that we could have under or over dispersion, as follows:

$$\mathbb{E}(S_n - np)^2 \sim \begin{cases} \dfrac{p(1-p)n}{1-2\theta} & , \text{ if } \theta < 1/2, \\ p(1-p)n\log(n) & , \text{ if } \theta = 1/2, \\ \dfrac{p(1-p)n^{2\theta}}{(2\theta-1)\Gamma(\theta)} & , \text{ if } \theta > 1/2. \end{cases} \tag{1.2}$$

At this same work, author also proves versions of the central limit theorem in the regions $\theta \leq 1/2$. Later, versions of the law of iterated logarithm were demonstrated in [14]. Moreover, it was proven in [13] that in the case $\theta > 1/2$, the number of successes S_n, properly centered and normalized, converges almost surely to a non-degenerated random variable L, for which it is possible to compute the first and second moments. In fact, the papers [5, 13] discussed some evidence that L must be not normally distributed. Also at this region, for finite values of n, the random variable S_n could be bimodal, as showed in [5].

In this paper we propose a detailed asymptotic analysis of the random variable S_n, for which is possible to prove the functional central limit theorem, the almost sure central limit theorem, the law of iterated logarithm and the convergence of moments in the regions $\theta \leq 1/2$. In addition, for $\theta > 1/2$, we obtain a strong convergence to a non-degenerated random variable, showing a dependence of its moments on the initial probability α. Moreover, we include a central limit theorem and a law of iterated logarithm for the fluctuations.

It should be emphasized that this correlated Bernoulli sequence is related to the minimal random walk (MRW) introduced in [15], which is a unidirectional random walk (moving one unit forward or staying in the same position at each time). In such model, the distribution of the current movement of the walker depends conditionally on the observed movement at a uniformly chosen time from the past. If at such time there was no movement then the conditional distribution is Bernoulli with parameter $q \in (0,1)$. On the other hand, if there was a movement, the conditional distribution is also Bernoulli but with parameter $r \in (0,1)$. Consequently, the position of the MRW is governed by the conditional probabilities in (1.1), with $p = \frac{q}{1-r+q}$ and $\theta = r - q$.

The rest of the paper is organized as follows. The next section states the main results. We finish the paper in Section 3, which is dedicated to the proofs.

2 Main results

This section presents the main results of this work. Namely we state a collection of limiting theorems for the number of successes (S_n) on the correlated Bernoulli process conducted by (1.1).

Our first result deals with the whole range of parameter θ, and shows the almost sure convergence of the successes rate to the fixed probability p.

Theorem 2.1. For all $\theta \in [0,1)$,

$$\lim_{n\to\infty} \frac{S_n}{n} = p \ \text{ a.s.} \tag{2.1}$$

In what follows, we define a quantity that will appear a number of times along the paper. Namely

$$\sigma^2_{p,\theta} = \frac{p(1-p)}{1-2\theta}\ .$$

At the $\theta < 1/2$ setting, we provide the following limit theorem:

Theorem 2.2. If $0 \le \theta < 1/2$, then, as $n \to \infty$

i) We have the distributional convergence in $D([0,\infty[)$ the Skorokhod space of right-continuous functions with left-hand limits,

$$\Big(\sqrt{n}\Big(\frac{S_{\lfloor nt\rfloor}}{\lfloor nt\rfloor} - p\Big), t \ge 0\Big) \Longrightarrow \big(W_t, t \ge 0\big), \tag{2.2}$$

where $\Longrightarrow$ stands for convergence in distribution, and $\big(W_t, t \ge 0\big)$ is a real-valued centered Gaussian process starting at the origin with covariance given by $\mathbb{E}[W_s W_t] = \frac{1}{s}\sigma^2_{p,\theta}\Big(\frac{t}{s}\Big)^{\theta-1}$, for all $0 < s \le t$. In particular,

$$\sqrt{n}\Big(\frac{S_n}{n} - p\Big) \Longrightarrow N\left(0, \sigma^2_{p,\theta}\right).$$

ii) We have the following almost sure convergence of empirical measures

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\delta_{\left\{\sqrt{k}\left(\frac{S_k}{k}-p\right)\right\}} \Rightarrow G \text{ a.s. }, \tag{2.3}$$

where δ is the Dirac measure, and G is the Gaussian measure $N\Big(0, \sigma^2_{p,\theta}\Big)$.

iii) We obtain the law of iterated logarithm

$$\limsup_{n\to\infty} \pm \frac{S_n - np}{\sqrt{2\sigma^2_{p,\theta} n \log\log(n)}} = 1 \ \text{ a.s.} \tag{2.4}$$

iv) For all integer $m \ge 1$, we have the following almost sure convergence

$$\frac{1}{\log n}\sum_{k=1}^{n} k^{m-1}\left(\frac{S_k}{k} - p\right)^{2m} \to \frac{\sigma^{2m}_{p,\theta}(2m)!}{2^m m!} \ \text{ a.s.} \tag{2.5}$$

We continue by displaying the corresponding asymptotic analysis at the critical value $\theta = 1/2$.

Theorem 2.3. If $\theta = 1/2$, then, as $n \to \infty$, we have

i) The distributional convergence in $D([0,\infty[)$,

$$\left(\sqrt{\frac{n^t}{\log n}}\Big(\frac{S_{\lfloor n^t\rfloor}}{\lfloor n^t\rfloor} - p\Big), t \ge 0\right) \Longrightarrow \left(p(1-p)B_t, t \ge 0\right), \tag{2.6}$$

where $\big(B_t, t \ge 0\big)$ is a standard Brownian motion. In particular, $\sqrt{\frac{n}{\log n}}\left(\frac{S_n}{n} - p\right) \Longrightarrow N\left(0, p(1-p)\right)$.

ii) The almost sure central limit theorem

$$\frac{1}{\log\log n}\sum_{k=2}^{n}\frac{1}{k\log k}\delta_{\left\{\sqrt{\frac{k}{\log k}}\left(\frac{S_k}{k}-p\right)\right\}} \Rightarrow G \quad \text{a.s.,} \tag{2.7}$$

where G is the Gaussian measure $N(0, p(1-p))$.

iii) The law of iterated logarithm

$$\limsup_{n\to\infty} \pm \frac{S_n - np}{\sqrt{2p(1-p)\cdot n\cdot \log n\cdot \log\log(\log n)}} = 1 \text{ a.s.} \tag{2.8}$$

iv) For all integer $m \geq 1$, we have the almost sure convergence

$$\frac{1}{\log\log n}\sum_{k=2}^{n}\left(\frac{1}{k\log k}\right)^{m+1} k^{m-1}\left(\frac{S_k}{k}-p\right)^{2m} \to \frac{[p(1-p)]^m (2m)!}{2^m m!} \text{ a.s.} \tag{2.9}$$

Finally, at the last setting we have the following limit results, whose deductions are technically more complex.

Theorem 2.4. If $\theta > 1/2$, as $n \to \infty$

i) We have the almost sure convergence,

$$\left(n^{1-\theta}\left(\frac{S_{\lfloor nt\rfloor}}{\lfloor nt\rfloor}-p\right), t>0\right) \longrightarrow \left(\frac{1}{t^{1-\theta}}L, t>0\right). \tag{2.10}$$

where L is a non-degenerated random variable such that

$$\mathbb{E}[L] = \frac{\alpha - p}{\Gamma(\theta+1)} \text{ and } \mathbb{E}[L^2] = \frac{\alpha + (\alpha-p)(1-4p) + p\left(\frac{1-2\theta p}{2\theta-1}\right)}{\Gamma(2\theta+1)}, \tag{2.11}$$

ii) The Gaussian fluctuations hold

$$\sqrt{n^{2\theta-1}}\left(n^{1-\theta}\left(\frac{S_n}{n}-p\right)-L\right) \Longrightarrow N\left(0, \frac{p(1-p)}{2\theta-1}\right) \tag{2.12}$$

iii) We obtain the law of iterated logarithms for fluctuations

$$\limsup_{n\to\infty} \pm \frac{\sqrt{n^{2\theta-1}}\left(n^{1-\theta}\left(\frac{S_n}{n}-p\right)-L\right)}{\sqrt{\log\log n}} = \sqrt{\frac{2p(1-p)}{2\theta-1}} \text{ a.s.} \tag{2.13}$$

3 Proofs

The proofs are based on the papers [3, 9, 10, 12]. First of all, we construct a martingale associated to S_n. Denote by $(\mathcal{F}_n)$ the increasing sequence of σ-algebras $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. We have from (1.1) that

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \theta\frac{S_n}{n} + \omega \quad \text{a.s.,} \tag{3.1}$$

where $\omega := (1-\theta)p$. Hence

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \gamma_n S_n + \omega \quad \text{a.s.,} \tag{3.2}$$

where $\gamma_n = 1 + \frac{\theta}{n}$. Which leads us to base the asymptotic analysis of (S_n) on the sequence (M_n), given by $M_0 = 0$ and for $n \geq 1$ by

$$M_n = a_n S_n - \omega A_n, \tag{3.3}$$

where the sequence (a_n) is given by $a_1 = 1$ and for $n \geq 2$ as

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(n)\Gamma(\theta+1)}{\Gamma(n+\theta)} \sim \frac{\Gamma(1+\theta)}{n^\theta}, \tag{3.4}$$

where Γ stands for the Euler gamma function. Moreover, the sequence (A_n) is given by $A_0 = 0$ and for $n \geq 1$ as $A_n = \sum_{k=1}^n a_k$. Additionally, we observe from (3.2) and (3.3) that almost surely

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = a_{n+1}(\gamma_n S_n + \omega) - \omega A_{n+1} = a_n S_n - \omega A_n = M_n.$$

Thus, (M_n) is a discrete time martingale with respect to the filtration $(\mathcal{F}_n)$. From the definition of the proposed martingale given in (3.3), we observe that

$$\Delta M_n = M_n - M_{n-1} = a_n \left(S_n - \gamma_{n-1} S_{n-1} - \omega\right) = a_n \xi_n, \tag{3.5}$$

where $\xi_1 = X_1 - \mathbb{E}[X_1] = X_1 - \alpha$ and by (3.2) we get that, for all $n \geq 1$, $\xi_{n+1} = S_{n+1} - \mathbb{E}[S_{n+1}|\mathcal{F}_n] = S_{n+1} - (\omega + \gamma_n S_n)$, which leads to

$$M_n = \sum_{k=1}^n a_k \xi_k. \tag{3.6}$$

Furthermore, note that almost surely $\xi_{n+1} = X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{F}_n]$ and $\mathbb{E}[X_{n+1}^k|\mathcal{F}_n] = \mathbb{E}[X_{n+1}|\mathcal{F}_n]$, for all $k \geq 1$. From this and (3.1), we find for $k \geq 2$ that,

$$\mathbb{E}[\xi_{n+1}^k|\mathcal{F}_n] = \sum_{j=0}^{k-2} \binom{k}{j} (-1)^j \left(\omega + \theta \frac{S_n}{n}\right)^{j+1} + (-1)^{k-1}(k-1)\left(\omega + \theta\frac{S_n}{n}\right)^k \quad \text{a.s.},$$

that implies that

$$\sup_{n\geq 0} \mathbb{E}\left[\xi_{n+1}^k|\mathcal{F}_n\right] < \infty, \ \ \text{a.s.}, \tag{3.7}$$

because $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq 1$, almost surely. In particular,

$$\mathbb{E}\left[\xi_{n+1}^2|\mathcal{F}_n\right] = \left(\theta\frac{S_n}{n} + \omega\right)\left(1 - \left(\theta\frac{S_n}{n} + \omega\right)\right), \text{ a.s.} \tag{3.8}$$

Hence, by analysing the polynomial $p(x) := x - x^2$ we have that

$$\sup_{n\geq 0} \mathbb{E}\left[\xi_{n+1}^2|\mathcal{F}_n\right] \leq 1/4, \text{ a.s.} \tag{3.9}$$

On the same line,

$$\mathbb{E}[\xi_{n+1}^4|\mathcal{F}_n] = \left(\omega + \theta\frac{S_n}{n}\right) - 4\left(\omega + \theta\frac{S_n}{n}\right)^2 + 6\left(\omega + \theta\frac{S_n}{n}\right)^3 - 3\left(\omega + \theta\frac{S_n}{n}\right)^4, \text{ a.s.}$$

By maximizing the polynomial $p(x) = x - 4x^2 + 6x^3 - 3x^4$, we conclude that

$$\sup_{n\geq 0} \mathbb{E}\left[\xi_{n+1}^4|\mathcal{F}_n\right] \leq 1/12, \text{ a.s.} \tag{3.10}$$

Moreover, the predictable quadratic variation of (M_n) satisfies, for all $n \geq 1$, that

$$\langle M\rangle_n = \sum_{k=1}^n \mathbb{E}[\Delta M_k^2|\mathcal{F}_{k-1}] = O(v_n), \tag{3.11}$$

where $v_n = \sum_{k=1}^n a_k^2$. Then via standard results on the asymptotic of the gamma function, we conclude that, as n goes to infinity, it holds that

1. If $\theta < 1/2$ then
$$\lim_{n\to\infty} \frac{v_n}{n^{1-2\theta}} = \frac{\Gamma^2(\theta+1)}{1-2\theta}. \tag{3.12}$$
2. If $\theta = 1/2$ then
$$\lim_{n\to\infty} \frac{v_n}{\log n} = \frac{\pi}{4}. \tag{3.13}$$
3. If $\theta > 1/2$ then, from (3.4) it is possible to deduce that (v_n) converges into a finite value, more precisely
$$\lim_{n\to\infty} v_n = \sum_{k=0}^{\infty} \left(\frac{\Gamma(\theta+1)\Gamma(k+1)}{\Gamma(k+\theta+1)} \right)^2 = {}_3F_2 \left(\begin{matrix} 1 & 1 & 1 \\ \theta+1 & \theta+1 & \end{matrix} ; 1 \right), \tag{3.14}$$
where the above limit is the generalized hypergeometric function.

3.1 Proof of Theorem 2.1

First, note that
$$\frac{1}{na_n} = \frac{1}{n} \prod_{k=1}^{n-1} \left(1 + \frac{\theta}{k}\right) = \prod_{k=1}^{n-1} \left(\frac{k+\theta}{k+1} \right) = \prod_{k=1}^{n-1} \left(1 - \frac{1-\theta}{k+1} \right). \tag{3.15}$$
Moreover since $0 \le \theta < 1$, we clearly have that that $\displaystyle\sum_{k=1}^{\infty} \frac{1-\theta}{k+1} = \infty$, which implies that $\lim_{n\to\infty} \frac{1}{na_n} = 0$. It follows that $(\frac{1}{na_n})$ is non increasing. Now, note that, for all $n \ge 1$,
$$\xi_{n+1} = S_n + X_{n+1} - \mathbb{E}[S_n|\mathcal{F}_n] - \mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{F}_n].$$
Since $X_n \in \{0,1\}$, we have that $|\xi_{n+1}| \le 1$. Additionally, let (N_n) be the sequence defined, for all $n \ge 1$, by $N_n = \frac{\Delta M_n}{na_n}$. Since (M_n) is a martingale, it is straightforward that (N_n) is a martingale difference sequence. Furthermore, (3.9) implies that $\displaystyle\sum_{j=1}^{\infty} \mathbb{E}[N_j^2|\mathcal{F}_{j-1}] \le \sum_{j=1}^{\infty} \frac{1}{4j^2} < \infty$. From Theorem 2.17 in [11], $\displaystyle\sum_{j=1}^{\infty} N_j$ converges almost surely. An application of Kronecker's lemma together with the fact that $na_n \to \infty$ let us find that
$$\frac{1}{na_n} \sum_{j=1}^{n} \Delta M_j = \frac{1}{na_n} M_n \to 0 \text{ a.s.}$$
Since $\left| \frac{A_n}{na_n} - \frac{1}{1-\theta} \right| \sim \frac{1}{(1-\theta)\Gamma(\theta)} \frac{1}{n^{1-\theta}}$, for $\theta \in [0,1)$, we get that
$$\lim_{n\to\infty} \frac{A_n}{na_n} = \frac{1}{1-\theta} \ .$$
Finally, by remembering that $w = (1-\theta)p$, we conclude that
$$\lim_{n\to\infty} \frac{1}{na_n} (a_n S_n - w A_n) = \lim_{n\to\infty} \left(\frac{S_n}{n} - p \right) = 0 \ \text{ a.s.}$$

3.2 Proof of Theorem 2.2

To prove (i), from (3.8) and the definition in (3.11), we get that
$$\langle M \rangle_n = \alpha(1-\alpha) + \sum_{k=1}^{n-1} a_{k+1}^2 \left[w(1-w) + \theta \frac{S_k}{k}(1-2w) - \theta^2 \frac{S_k^2}{k^2} \right].$$

Now, recalling that $w=(1-\theta)p$, we use Toeplitz lemma to find that

$$\begin{aligned}\lim_{n\to\infty}\frac{\langle M\rangle_n}{n^{1-2\theta}} &= \frac{\Gamma^2(1+\theta)}{1-2\theta}\left(w(1-w)+\theta(1-2w)p-\theta^2p^2\right)\\ &= p(1-p)\frac{\Gamma^2(1+\theta)}{1-2\theta} \text{ a.s.}\end{aligned}$$

Therefore, by applying the functional central limit theorem for martingales, given in Theorem 2.5 of [8], we obtain that

$$\lim_{n\to\infty}\frac{1}{n^{1-2\theta}}\langle M\rangle_{\lfloor nt\rfloor}=p(1-p)\frac{\Gamma^2(1+\theta)}{1-2\theta}t^{1-2\theta} \text{ a.s.}$$

In order to prove Lindeberg's condition, from (3.10) note that for any $\varepsilon>0$ that

$$\begin{aligned}\frac{1}{n^{1-2\theta}}\sum_{k=1}^{n}\mathbb{E}[\Delta M_k^2\mathbb{I}_{\{|\Delta M_k|>\varepsilon\sqrt{n^{1-2\theta}}\}}|\mathcal{F}_{k-1}] &\leq \frac{1}{n^{2(1-2\theta)}\varepsilon^2}\sum_{k=1}^{n}\mathbb{E}[\Delta M_k^4|\mathcal{F}_{k-1}]\\ \leq \frac{1}{n^{2(1-2\theta)}\varepsilon^2}\sum_{k=1}^{n}a_k^4\mathbb{E}[\xi_k^4|\mathcal{F}_{k-1}] &\leq \frac{1}{12n^{2(1-2\theta)}\varepsilon^2}\sum_{k=1}^{n}a_k^4.\end{aligned}$$

Therefore, since $\frac{n^2a_n^4}{v_n^2}\to(1-2\theta)^2$ as $n\to\infty$ implies that $\frac{1}{n^{1-4\theta}}\sum_{k=1}^n a_k^4$ converges to $\frac{\Gamma(\theta+1)^4}{1-4\theta}$, we obtain that

$$\frac{1}{n^{1-2\theta}}\sum_{k=1}^{n}\mathbb{E}[\Delta M_k^2\mathbb{I}_{\{|\Delta M_k|>\varepsilon\sqrt{n^{1-2\theta}}\}}|\mathcal{F}_{k-1}]\to 0 \text{ as } n\to\infty \text{ in probability.}$$

Then, we conclude that for all $t\geq 0$ and for any $\varepsilon>0$,

$$\frac{1}{n^{1-2\theta}}\sum_{k=1}^{\lfloor nt\rfloor}\mathbb{E}[\Delta M_k^2\mathbb{I}_{\{|\Delta M_k|>\varepsilon\sqrt{n^{1-2\theta}}\}}|\mathcal{F}_{k-1}]\to 0, \tag{3.16}$$

as $n\to\infty$ in probability. In addition, note that $\lim_{n\to\infty}\frac{\lfloor nt\rfloor a_{\lfloor nt\rfloor}}{n^{1-2\theta}}=t^{1-\theta}\Gamma(\theta+1)$, the definition of (3.4) and the fact that

$$\frac{A_n}{na_n}=\frac{1}{\theta-1}\left(\frac{\Gamma(n+\theta)}{\Gamma(n+1)\Gamma(\theta)}-1\right)=\frac{1}{\theta-1}\left(\frac{\theta}{na_n}-1\right), \tag{3.17}$$

(by Lemma $B.1$ of [2]) imply that

$$\frac{M_{\lfloor nt\rfloor}}{\sqrt{n^{1-2\theta}}}=\frac{\lfloor nt\rfloor a_{\lfloor nt\rfloor}}{\sqrt{n^{1-2\theta}}}\left(\frac{S_{\lfloor nt\rfloor}}{\lfloor nt\rfloor}-p\right)+\frac{p\theta}{\sqrt{n^{1-2\theta}}} \qquad \text{a.s.}, \tag{3.18}$$

we conclude via Theorem 2.5 of [8] that $\left(\sqrt{n}\left(\frac{S_{\lfloor nt\rfloor}}{\lfloor nt\rfloor}-p\right),t\geq 0\right)\Longrightarrow\left(W_t,t\geq 0\right)$, where $W_t=B_t/(t^{1-\theta}\Gamma(\theta+1))$, which completes the proof of part (i) of Theorem 2.2.

Lets prove (ii). As previously seen, by using Lemma 4.1 in [10] we get

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{1}{v_k}\mathbb{E}\left[|\Delta M_k|^2\mathbb{I}_{\{|\Delta M_k|\geq\varepsilon\sqrt{v_k}\}}|\mathcal{F}_{k-1}\right]\leq\frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{v_k^2}\mathbb{E}\left[|\Delta M_k|^4|\mathcal{F}_{k-1}\right]\\ &\leq\sup_{k\geq 1}\mathbb{E}[\xi_k^4|\mathcal{F}_{k-1}]\frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{a_k^4}{v_k^2}\leq\frac{1}{12\varepsilon^2}\sum_{k=1}^{\infty}\frac{a_k^4}{v_k^2}\sim\frac{1}{12\varepsilon^2}\sum_{k=1}^{\infty}\frac{(1-2\theta)^2}{k^2}<\infty,\end{aligned}$$

again by using (3.10). The second condition of Lemma 4.1 in [10] is analogously proved by using $a = 2$. Therefore we get that

$$\frac{1}{\log v_n} \sum_{k=1}^{n} \left(\frac{v_k - v_{k-1}}{v_k} \right) \delta_{\{M_k/\sqrt{v_{k-1}}\}} \Longrightarrow G^* \text{ a.s.},$$

where G^* is the Gaussian measure $N(0, p(1-p))$. Now, since the explosion coefficient is given by $f_k = \frac{v_k - v_{k-1}}{v_k} = \frac{a_k^2}{v_k} \sim \frac{1-2\theta}{k}$, and by observing that $\log v_n \sim (1-2\theta)\log n$ we obtain that $\frac{M_k}{\sqrt{v_{k-1}}} \sim \sqrt{\frac{1-2\theta}{k}} \left(S_k - \frac{kw}{1-\theta} \right)$, which leads us to complete the proof of part (ii) in Theorem 2.2.

To prove (iii), we first remark that Theorem 2.1 together with (3.8) implies the following almost sure convergence

$$\lim_{n\to\infty} \mathbb{E}\left[\xi_{n+1}^2 | \mathcal{F}_n\right] = p(1-p), \tag{3.19}$$

which jointly with

$$\sum_{k=1}^{\infty} \frac{a_k^4}{v_n^2} = \frac{[\pi(1-2\theta)]^2}{6},$$

and the law of iterated logarithm for martingales (see [17] for instance), we get that

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{2\log\log n}} = -\liminf_{n\to\infty} \frac{M_n}{\sqrt{2\log\log n}} = \sqrt{p(1-p)} \qquad \text{a.s.},$$

which implies that

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n} \right)^{1/2} \left(\frac{S_n}{n} - \frac{wA_n}{na_n} \right) = \sqrt{\sigma_{p,\theta}^2} \qquad \text{a.s.}$$

Therefore, (3.17) conduces us to

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n} \right)^{1/2} \left(\frac{S_n}{n} - p + \frac{\theta p}{na_n} \right) = \sqrt{\sigma_{p,\theta}^2} \qquad \text{a.s.}$$

We remark that, since $\lim_{n\to\infty} \left(\frac{n}{2\log\log n} \right)^{1/2} \frac{1}{n^{1+\theta}} = 0$, we get that the last term vanishes as n diverges, which completes the proof.

Finally, we focus our attention on the last part (iv). The proof is based on Lemma 4.2 from [10]. Notice that we have already seen that $f_n \to 0$, as $n \to \infty$ and that $\frac{M_k}{\sqrt{v_{k-1}}} \sim \sqrt{(1-2\theta)k} \left(\frac{S_k}{k} - p \right)$ which together with (3.7) complete the proof.

3.3 Proof of Theorem 2.3

We will proceed in a similar way as in the proof of Theorem 2.2. First, due to (3.4) and (3.13) we obtain that

$$\frac{a_k^4}{v_k^2} \sim \left(\frac{1}{n\log n} \right)^2, \tag{3.20}$$

which implies that

$$\sum_{k=1}^{\infty} \frac{a_k^4}{v_k^2} < \infty. \tag{3.21}$$

Now, in order to demonstrate (i) we note that Lindeberg condition holds from (3.21). Then, from the functional central limit theorem for martingales [8], the definition of (M_n), convergence (3.13), and relation equation (3.20) we complete the proof of part i).

In the sequel, we focus in the proof of (ii). Note that the conditions of Lemma 4.1 in [10] follows from (3.21). In addition, it may be found from the definition of M_n, the fact that $\left|\frac{A_n}{na_n} - 2\right| \sim \frac{2}{\sqrt{n\pi}}$, (3.13) and (3.20) that $\frac{M_n}{\sqrt{v_{n-1}}} \sim \sqrt{\frac{n}{\log n}}\left(\frac{S_n}{n} - p\right)$, which leads to

$$\frac{1}{\log\log n}\sum_{k=1}^{n}\frac{1}{k\log k}\delta_{\left\{\sqrt{\frac{k}{\log k}}\left(\frac{S_k}{k}-p\right)\right\}} \Rightarrow G \text{ a.s.},$$

where G stands for the $N(0, p(1-p))$ distribution.

To prove (iii), we use similar arguments as in the proof of Theorem 2.2, based on (3.21).

For (iv), we notice that condition of Lemma 4.2 in [10] holds in the same manner than in the $\theta < 1/2$ regime. From (3.19) and (3.20), we obtain that f_n converges to zero as $n \to \infty$. Hence, we may conclude (2.9) from the definition of (M_n).

3.4 Proof of Theorem 2.4

For proving i), we recall that $\sup_{n\geq 0}\mathbb{E}[\xi_{n+1}^2|\mathcal{F}_n] \leq 1/4$ almost surely and (v_n) is a non increasing sequence. Then, it follows from (3.14) that

$$\sup_{n\geq 1}\mathbb{E}[M_n^2] \leq \frac{1}{4}\cdot {}_3F_2\begin{pmatrix} 1 & 1 & 1 \\ & \theta+1 & \theta+1 \end{pmatrix}; 1\Big) < \infty.$$

That is to say, martingale (M_n) is bounded in L^2. Thus, it converges in L^2 and almost surely to the random variable $M = \sum_{k=1}^{\infty} a_k\xi_k$. Now, from (3.17) we have that

$$\lim_{n\to\infty} na_n\left(\frac{S_n}{n} - p\right) = M - p\theta \text{ a.s.},$$

then

$$\lim_{n\to\infty} n^{1-\theta}\left(\frac{S_n}{n} - p\right) = L := \frac{M - p\theta}{\Gamma(1+\theta)} \text{ a.s.}$$

Moreover, given that (M_n) converges to M in L^2 we have that (3.17) and definition of the limit random variable L guide us to

$$\lim_{n\to\infty}\mathbb{E}\left[\left(n^{1-\theta}\left(\frac{S_n}{n} - p\right) - L\right)^2\right] = 0.$$

We will find now the first two moments of the limiting random variable L given in (2.10). For this, let us note that $\mathbb{E}[X_1] = \alpha$. In addition, from (3.2) we have for $n = 1, 2, \ldots$, that

$$\mathbb{E}[S_n] = \frac{1}{a_n}\left(\alpha + \omega\cdot\sum_{l=1}^{n-1} a_{l+1}\right) = \frac{1}{a_n}\left(\alpha + \omega\cdot(A_n - 1)\right), \tag{3.22}$$

that directly lets us see that $\mathbb{E}[M_n] = \alpha - \omega$, which implies that $\lim_{n\to\infty}\mathbb{E}[M_n] = \mathbb{E}[M] = \alpha - \omega$, that leads us to

$$\mathbb{E}[L] = \frac{1}{\Gamma(\theta+1)}\left(\mathbb{E}(M) - p\theta\right) = \frac{1}{\Gamma(\theta+1)}(\alpha - \omega - p\theta) = \frac{\alpha - p}{\Gamma(\theta+1)}.$$

We will find the second moment of the limiting random variable L. For this, note that

$$\begin{aligned}\mathbb{E}[M_n^2] &= a_n^2\mathbb{E}[S_n^2] - 2\omega a_n A_n\mathbb{E}[S_n] + \omega^2 A_n^2 \\ &= a_n^2\mathbb{E}[S_n^2] - 2\omega A_n(\alpha - \omega) - \omega^2 A_n^2. \end{aligned} \tag{3.23}$$

Moreover, relations (3.1) and (3.2) combined with the fact that $S_{n+1} = S_n + X_{n+1}$ imply that $\mathbb{E}[S_{n+1}^2] = g_n \mathbb{E}[S_n^2] + h_n$, where, $g_n := 1 + \frac{2\theta}{n}$, and $h_n := \left(2\omega + \frac{\theta}{n}\right)\mathbb{E}[S_n] + \omega$, for $n \geq 1$. Hence, it may be found recursively for $n \geq 1$, that

$$\mathbb{E}[S_n^2] = \frac{\Gamma(n+2\theta)}{\Gamma(n)}\Big(\frac{\alpha}{\Gamma(2\theta+1)} + \sum_{k=1}^{n-1} h_k \frac{\Gamma(k+1)}{\Gamma(k+1+2\theta)}\Big). \tag{3.24}$$

However, from (3.17) and (3.22) we may see that

$$h_n = p(1+2n\omega) + (\alpha - p)\frac{(2n\omega + \theta)}{na_n}.$$

In addition, (3.24) and a repeated application of Lemma $B.1$ of [2], that is, for $b \neq a+1$,

$$\sum_{k=1}^{n-1} \frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{\Gamma(a+1)}{(b-a-1)\Gamma(b)}\left(1 - \frac{\Gamma(n+a)\Gamma(b)}{\Gamma(n+b-1)\Gamma(a+1)}\right),$$

lets us to observe that

$$\begin{aligned} \mathbb{E}[S_n^2] &= \alpha\frac{\Gamma(n+2\theta)}{\Gamma(n)\Gamma(2\theta+1)} \\ &+ \frac{p}{\Gamma(n)}\left[\frac{(1-2\omega)}{2\theta-1}\left(\frac{\Gamma(n+2\theta)}{\Gamma(1+2\theta)} - \Gamma(n+1)\right) - p\left(2\frac{\Gamma(n+2\theta)}{\Gamma(1+2\theta)} - \Gamma(n+2)\right)\right] \\ &+ \frac{(\alpha-p)}{\Gamma(n)}\left[(1-2\omega)\left(\frac{\Gamma(n+2\theta)}{\Gamma(1+2\theta)} - \frac{\Gamma(n+\theta)}{\Gamma(1+\theta)}\right) - 2p\left((\theta+1)\frac{\Gamma(n+2\theta)}{\Gamma(1+2\theta)} - \frac{\Gamma(n+\theta+1)}{\Gamma(1+\theta)}\right)\right]. \end{aligned}$$

Hence, (3.4) and (3.23) lead us to

$$\lim_{n\to\infty} \mathbb{E}[M_n^2] = \frac{\Gamma^2(\theta+1)}{\Gamma(2\theta+1)}\left(\alpha + (\alpha-p)(1-4p) + p\left(\frac{1-2\theta p}{2\theta-1}\right)\right) + \theta p(2(\alpha-p)+\theta p), \tag{3.25}$$

which finally implies that

$$\mathbb{E}[L^2] = \frac{1}{\Gamma^2(\theta+1)}\Big(\mathbb{E}[M^2] - 2p\theta\mathbb{E}[M] + p^2\theta^2\Big) = \frac{\alpha + (\alpha-p)(1-4p) + p\left(\frac{1-2\theta p}{2\theta-1}\right)}{\Gamma(2\theta+1)}.$$

In what follows, we will demonstrate items ii) and iii). This proof will be based on Lemma 4.3 from [10], which in turn is a consequence of Theorem 1 and Corollaries 1 and 2 from [12]. In this sense, note that (3.19) and the bounded convergence theorem imply that $\sum_{k=1}^{\infty} \mathbb{E}[(\Delta M_k)^2] \sim p(1-p)\Gamma(\theta+1)^2 \sum_{k=1}^{\infty} \frac{1}{k^{2\theta}}$. Then, since $\theta > 1/2$, we have that $\sum_{k=1}^{\infty} \mathbb{E}[(\Delta M_k)^2] < \infty$. Now, given (3.19), we obtain that

$$\begin{aligned} \sum_{k=n}^{\infty} \mathbb{E}[(\Delta M_k)^2 | \mathcal{F}_{k-1}] &\sim \sum_{k=n}^{\infty} p(1-p)a_k^2 \sim p(1-p)\Gamma(\theta+1)^2 \sum_{k=n}^{\infty} \frac{1}{k^{2\theta}} \\ &\sim \frac{p(1-p)\Gamma(\theta+1)^2}{(2\theta-1)n^{2\theta-1}} \sim \frac{p(1-p)}{(2\theta-1)} n a_n^2 \quad \text{a.s.} \end{aligned}$$

By using the bounded convergence theorem, it follows that

$$r_n^2 := \sum_{k=n}^{\infty} \mathbb{E}[(\Delta M_k)^2] \sim \frac{p(1-p)}{(2\theta-1)} n a_n^2 \quad \text{a.s.} \tag{3.26}$$

Then, conditions a) and a') of Lemma 4.3 in [10] are satisfied. Hence, (3.4), (3.5) and the expectation of (3.10) guide us to obtain an upper bound for $\mathbb{E}\left[|\Delta M_{k+1}|^4\right]$, then, (3.26) and (3.4) let us conclude that

$$\frac{1}{r_n^2}\sum_{k=n}^{\infty}\mathbb{E}\left[(\Delta M_{k+1})^2\mathbb{I}_{\{|\Delta M_{k+1}|\geq\varepsilon r_n\}}\right]\leq\frac{1}{\varepsilon^2 r_n^4}\sum_{k=n}^{\infty}\mathbb{E}\left[|\Delta M_{k+1}|^4\right]$$

$$\leq\frac{1}{12\varepsilon^2 r_n^4}\sum_{k=n}^{\infty}a_k^4\sim\frac{1}{r_n^4}\sum_{k=n}^{\infty}\frac{1}{k^{4\theta}}\sim n^{4\theta-2}n^{1-4\theta},$$

which implies condition b) of Lemma 4.3 in [10]. Then, by noticing that $M_n - M = a_n\left(S_n - np - n^{\theta}L\right)$, and using (3.26), the convergence (2.12) holds. Additionally, for $\varepsilon > 0$, and from similar arguments than above, we have that

$$\frac{1}{r_k}\mathbb{E}\left[|\Delta M_{k+1}|\mathbb{I}_{\{|\Delta M_{k+1}|\geq\varepsilon r_k\}}\right]\leq\frac{1}{r_k}\frac{1}{\varepsilon^3 r_k^3}\mathbb{E}\left[|\Delta M_{k+1}|^4\right]\leq\frac{a_k^4}{12\varepsilon^3 r_k^4}\sim\frac{1}{k^2},$$

which implies that condition c) of the same Lemma is satisfied. In addition, given that $\sum_{k=1}^{\infty}\frac{1}{r_k^4}\mathbb{E}[(\Delta M_k)^4]<\infty$, we obtain condition d). Finally, let us denote the martingale difference $d_k := \frac{1}{r_k^2}\left((\Delta M_k)^2 - \mathbb{E}[(\Delta M_k)^2|\mathcal{F}_{k-1}]\right)$, and observe that

$$\sum_{k=1}^{\infty}\mathbb{E}\left[d_k^2|\mathcal{F}_{k-1}\right]=\sum_{k=1}^{\infty}\frac{1}{r_k^4}\left(\mathbb{E}[(\Delta M_k)^4|\mathcal{F}_{k-1}]-\mathbb{E}^2[(\Delta M_k)^2|\mathcal{F}_{k-1}]\right)$$

$$\leq\sum_{k=1}^{\infty}\frac{1}{r_k^4}\mathbb{E}[(\Delta M_k)^4|\mathcal{F}_{k-1}]\leq\frac{1}{12}\sum_{k=1}^{\infty}\frac{a_k^4}{r_k^4}\leq\frac{1}{12}\left(\frac{2\theta-1}{p(1-p)}\right)^2\sum_{k=1}^{\infty}\frac{1}{k^2}<+\infty.$$

Then, as a consequence of Theorem 2.15 in [11], we can use Corollary 2 from [12], and therefore (2.13) holds.

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