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Critical wetting in the (2+1)D solid-on-solid model*

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Abstract

In this note, we study the low temperature $(2+1)$ D SOS interface above a hard floor with critical pinning potential $\lambda_w = \log(\frac{1}{1-e^{-4\beta}})$. At $\lambda < \lambda_w$ entropic repulsion causes the surface to delocalize and be rigid at height $\frac{1}{4\beta}\log n + O(1)$; at $\lambda > \lambda_w$ it is localized at some $O(1)$ height. We show that at $\lambda = \lambda_w$, there is delocalization, with rigidity now at height $\lfloor \frac{1}{6\beta}\log n + \frac{1}{3}\rfloor$, confirming a conjecture of Lacoin.

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1 Introduction

The Solid-On-Solid (SOS) model above a wall with a pinning force of $\lambda > 0$ along the wall, is the distribution over nonnegative height functions ϕ over $\Lambda_n = \{-\lfloor \frac{n}{2}\rfloor, \dots, \lfloor \frac{n}{2}\rfloor\}^2$:

$$\mathbb{P}(\phi) = \mathbb{P}_{\Lambda_n}(\phi) \propto \exp\Big(-\beta\sum_{x\sim y}|\phi_x - \phi_y| + \lambda\sum_x \mathbf{1}_{\phi_x=0}\Big)\,, \qquad \phi:\Lambda_n \to \mathbb{Z}_{\geq 0}\,. \tag{1.1}$$

A standard setting has 0 boundary condition on $\mathbb{Z}^2\setminus\Lambda_n$, which is incorporated into (1.1) by extending ϕ to be on all of $\mathbb{Z}^2$ but forced to take value 0 on $\mathbb{Z}^2\setminus\Lambda_n$.

When $\lambda = 0$ and β is large (low temperatures), there is a competition between rigidity of the interface, and entropic repulsion from the constraint that $\phi_v \geq 0$ for all v. Bricmont, El Mellouki and Fröhlich [2] showed that the typical height of the interface is of order $\log n$, and the works [3, 4] studied the typical height and the shapes of the level curves in detail. In particular, it was shown that the interface rises along the boundary, and is typically rigid about height $\lfloor \frac{1}{4\beta}\log n\rfloor$ (or possibly the preceding integer for certain n).

When $\lambda > 0$, the pinning potential competes with the entropic repulsion; Chalker [5] showed that for all large β, there is a critical $\lambda_w(\beta)$ separating a localized regime of $\lambda > \lambda_w$ in which the interface height at the origin is tight: $\phi_o = O_{\mathrm{p}}(1)$ as n grows (and ϕ_x is a certain constant $k(\lambda)$ for most $x \in \Lambda_n$), and a delocalized *wetting regime* of $\lambda < \lambda_w$ where $\phi_o \to \infty$ (as does ϕ_x for almost all $x \in \Lambda_n$). Since that work, the two regimes have

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been investigated in more detail, e.g., [11, 1], culminating in works of Lacoin [9, 10]. These works identified

$$\lambda_w = -\log(1 - e^{-4\beta})\,, \tag{1.2}$$

and showed that on the $\lambda > \lambda_w$ side, an infinite sequence of *layering* transitions occur as $\lambda \downarrow \lambda_w$. Namely, there is a sequence of $\lambda_1 > \lambda_2 > \dots$ exponentially converging to λ_w such that between λ_{i-1} and λ_i, the interface is rigid about height i (independent of n), and the λ_i mark discontinuous (first-order) jump points for the expected height above the origin, say. For more on this and related phenomena, see the surveys [12, 8].

Recently, Feldheim and Yang [6] showed that with probability $1 - o(1)$ (with high probability, or w.h.p.), the typical height remains $\lfloor \frac{1}{4\beta} \log n \rfloor + O(1)$, as is the case for $\lambda = 0$, throughout the wetting regime of $\lambda < \lambda_w$. The behavior exactly at the critical point $\lambda = \lambda_w$ remained open. Lacoin conjectured (as stated in [6]) that the interface at $\lambda = \lambda_w$ should still be delocalized, but it should be rigid about height $\frac{1}{6\beta} \log n$ rather than $\frac{1}{4\beta} \log n$. Some evidence towards a lower bound was offered in [6] (but only if one assumes an a priori bound of $O(n^{4/3})$ on the total number of zeros of the SOS function ϕ).

In this note, we prove this conjecture, showing that at the critical λ_w, all but an ε_β fraction of the sites have height exactly (up to a possible -1 for certain values of n)

$$h_n^* = \lfloor \tfrac{1}{6\beta} \log n + \tfrac{1}{3} \rfloor\,. \tag{1.3}$$

Theorem 1.1. *There is a constant C such that for all β large enough and $\lambda = \lambda_w$, w.h.p.,*

$$|\{x \in \Lambda_n : \phi_x \notin \{h_n^* - 1, h_n^*\}\}| \leq \frac{C}{\beta} n^2\,.$$

Remark 1.2. For "most" values of n (e.g., a subset of $\mathbb{N}$ with a natural density of at least 0.6), we can identify the single height $h \in \{h_n^* - 1, h_n^*\}$ such that, w.h.p., all but an ε_β-fraction of the sites have height h. For instance, if the fractional part of $\frac{1}{6\beta} \log n + \frac{1}{3}$ falls in the interval $[0.34, 1)$, then $h = h_n^*$ (see Theorem 3.4), and if it falls in the interval $[0, \frac{\log \beta}{8\beta})$ then $h = h_n^* - 1$ (see Remark 2.9).

Remark 1.3. Lemma 2.6 in fact shows that all connected sets of vertices of height at least $h_n^* + 1$ have exponential tails on their sizes, and by a union bound the largest one is $O(\log n)$ with high probability. One would expect, in analogy with the $\lambda = 0$ case [4], that there is a unique macroscopic connected component of the $\{x : \phi_x \in \{h_n^* - 1, h_n^*\}\}$, occupying all but some ε_β fraction of Λ_n, and whose scaling limit is a Wulff shape.

To understand where this height is coming from, let us recap the intuition given in [2] for the case $\lambda = 0$. Making the ansatz that at β large, the interface is rigid about a certain height $h(n)$, there is a tradeoff between $\exp(-4\beta n)$ for lifting the bulk of the interface up from height $h - 1$ to h, and $(1 + e^{-4\beta h})^{n^2}$ for the entropy gained by the newfound allowance of the interface to have downward oscillations of height h (the easiest way being a straight $1 \times 1 \times h$ "spike") and still remain nonnegative. These two terms are balanced at $\frac{1}{4\beta} \log n + O(1)$, where rigidity was established in [3] (and then refined to $\lfloor \frac{1}{4\beta} \log n \rfloor - 1, \lfloor \frac{1}{4\beta} \log n \rfloor$ in [4]). The work of [6] showed that this tradeoff is still governing the behavior as long as $\lambda < \lambda_w$.

At the critical $\lambda = \lambda_w$, as conveyed in [6, Section 1.6], one expects there to be a perfect cancellation of entropic repulsion via singleton ($1 \times 1 \times h$) downward spikes with the pinning potential, leading the only gain in entropy from lifting the interface to be from the lower order $2 \times 1 \times h$ downward spikes. This changes the above competition to be between $\exp(-4\beta n)$ and $(1 + e^{-6\beta h})^{n^2}$, which is balanced at $\frac{1}{6\beta} \log n + O(1)$. The true threshold we identify of (1.3) is actually governed by a minor refinement of the above expectation; namely, we show that the relevant entropic repulsion effect is by *toothlike-spikes* that consist of a $2\times 1\times (h-1)$ spike with a last block appended below one of its two bottom vertices. This is what causes the $\frac{1}{3}$ correction to $\lfloor \frac{1}{6\beta}\log n\rfloor$ in (1.3). For proving the sharp result of Theorem 1.1, it is important to have identified this mechanism in both the lower bound and upper bound on the typical interface height.

We conclude this section with a few necessary preliminaries on the model. Define the state space $\widetilde{\Omega}$ where the floor constraint is relaxed as

$$\widetilde{\Omega} = \{\varphi : \Lambda_n \to \mathbb{Z} \,:\, \nexists x \sim y \text{ with } \varphi_x \le -1 \text{ and } \varphi_y \le 0\}\,.$$

Roughly, that is saying all oscillations where negative heights are attained have singleton intersections with height zero. Note that $\widetilde{\Omega}$ is an increasing subset of $\{\varphi : \Lambda_n \to \mathbb{Z}\}$.

We let $\mathsf{Q}_{\ge 2}(\varphi)$ be the sites taking height zero with a neighbor taking height zero:

$$\mathsf{Q}_{\ge 2}(\varphi) = \{x : \varphi_x \le 0 \text{ and } \exists y \sim x \text{ with } \varphi_y \le 0\}\,. \tag{1.4}$$

Observe that in $\widetilde{\Omega}$, vertices of $\mathsf{Q}_{\ge 2}$ cannot take negative heights, and therefore must take height zero. Then we can define an SOS model $\widetilde{\mathbb{P}}$ on $\widetilde{\Omega}$, which when the positive part is taken, gives exactly the SOS model with pinning potential λ of (1.1) as follows.

$$\widetilde{\mathbb{P}}(\varphi) \propto \exp\Big(-\beta \sum_{x\sim y} |\varphi_x - \varphi_y| + \lambda |\mathsf{Q}_{\ge 2}(\varphi)|\Big)\,. \tag{1.5}$$

The following observation of [6] relating $\max\{0,\varphi\}$ for $\varphi \sim \widetilde{\mathbb{P}}$ to $\phi \sim \mathbb{P}$ will be very useful.

Observation 1.4. Fix any $\beta > 0$ and let $\lambda = \lambda_w$. Then for every $\phi : \Lambda_n \to \mathbb{Z}_{\ge 0}$,

$$\mathbb{P}(\phi) = \sum_{\varphi : \max\{0,\varphi\} = \phi} \widetilde{\mathbb{P}}(\varphi)\,, \quad \text{where } \max \text{ is taken pointwise.}$$

Lemma 3.1 of [9] also used a relaxation of the floor constraint, but that one was more complicated and for general λ. The case $\lambda = \lambda_w$ with the state space $\widetilde{\Omega}$ as suggested in [6, Section 4] is especially clean due to the equivalence between the ability in φ for a singleton at height ≤ 0 to take a geometric random variable with mean $e^{-4\beta}$ as a negative height, and the token of size $\frac{1}{1-e^{-4\beta}}$ that ϕ collects by being at height zero. Our proofs will go by identifying events for φ in terms of events in $\widetilde{\mathbb{P}}$ to which they correspond, and then bounding those events under $\widetilde{\mathbb{P}}$. We conclude below with the definition of up-contours and down-contours, applicable for both ϕ and φ, and refer the reader to [3, Sec. 3] for a full description of the contour representation of the SOS model.

Definition 1.5. *For each height h, let $\Gamma_h(\varphi)$ denote the set of all dual-edges that separate $v \sim w$ for $\varphi_v \ge h$ and $\varphi_w \le h-1$. To uniquely decompose it into a collection of loops, one must use a canonical splitting rule when four dual-edges are incident a dual-vertex. A standard choice is to split these along the SE-NW diagonal, producing a collection of disjoint contours $\gamma_1,\ldots,\gamma_m$ with interiors $\mathsf{Int}(\gamma_1),\ldots,\mathsf{Int}(\gamma_m)$ such that $\varphi_v \ge h$ on the interior vertex boundary and $\varphi_v \le h-1$ on the exterior vertex boundary for all i: the $\gamma_1,\ldots,\gamma_m$ are the up h-contours of φ. Analogously, define the down h-contours of φ where $\varphi_v \le h$ on the internal vertex boundary, and $\varphi \ge h+1$ on the external vertex boundary.*

We say $S \subset \Lambda_n$ is simply-connected if it can be the interior of a contour with the SE-NW splitting rule, and use ∂S for its bounding contour. Denote by $\mathbb{P}_S^h$ (respectively, $\widetilde{\mathbb{P}}_S^h$), the SOS measure (1.1) (resp. (1.5)) on S with boundary conditions h on S^c.

Throughout, β will be a large fixed constant, and all statements hold for all large n.

2 Upper bound

In this section we show the following upper bound on the typical height of φ.

Proposition 2.1. There exists $C > 0$ such that for all β large, we have

$$\widetilde{\mathbb{P}}\left(\#\{x : \varphi_x \geq h_n^* + 1\} \geq Ce^{-\beta} n^2\right) \leq e^{-n^{3/4}}.$$

The argument goes by showing that up-contours γ at height $h_n^* + 1$ have exponential tails. We show this by shifting down the interior of such a contour while remaining in the admissible space $\widetilde{\Omega}$; this requires control over downward oscillations reaching height 0.

2.1 A lifting map to gain entropy

We begin by showing that if there are too many non-isolated zeroes in a region, we can apply a lifting map and gain entropy outweighing the loss of tokens. Let S be a simply connected set, and let $S' \subseteq S$ be a subset to be thought of as the interior of a down-contour. Let $W \subseteq S'$ represent a set of points on which we want to keep φ fixed (ultimately, this will be a pair of adjacent vertices or the empty set).

For any φ, let A be an arbitrary subset of $\mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W$. For each such A, we want to consider the map which raises the height of φ inside $S' \setminus (W \cup A)$ by 1:

$$U_A\varphi = \begin{cases} (U_A\varphi)_z = \varphi_z & z \in (S')^c \cup W \cup A \\ (U_A\varphi)_z = \varphi_z + 1 & \text{else} \end{cases}.$$

For any vertex set V, we can define the (outer) edge boundary $\partial_e V$ as the set of edges (unordered pairs $u \sim v$) with one endpoint in V and the other not in V. Moreover, for a φ, define $\Delta S'$ as the change in energy attributed to the edge boundary of S':

$$\Delta S' = \Delta S'(\varphi) := |\{u \sim v : u \in S', v \notin S', \varphi_u \geq \varphi_v\}| - |\{u \sim v : u \in S', v \notin S', \varphi_u < \varphi_v\}|.$$

Recall that $\widetilde{\mathbb{P}}_S^h$ is the SOS measure $\widetilde{\mathbb{P}}$ from (1.5) on S with boundary conditions h.

Lemma 2.2. Let S be simply connected, $h \geq 0$, $S' \subseteq S$, and $W \subseteq S' \subseteq S$. Let φ be such that for all $v \in S$ in the exterior boundary of S' (w.r.t. S), we have $\varphi_v \geq 1$. Then,

$$\sum_{A \subseteq \mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W} \widetilde{\mathbb{P}}_S^h(U_A\varphi) \geq \widetilde{\mathbb{P}}_S^h(\varphi) e^{-\beta \Delta S' - \beta|\partial_e W| - \lambda|W|} (1 + \frac{1}{2} e^{-6\beta})^{|\mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W|/5}.$$

Proof. This proof follows the idea of the proof of [6, Thm 1.2], with some necessary modifications in order to generalize to the above setting. First of all, we have that

$$\widetilde{\mathbb{P}}_S^h(U_A\varphi) \geq \widetilde{\mathbb{P}}_S^h(\varphi) \exp\left(-\beta \Delta S' - \beta|\partial_e A| - \beta|\partial_e W| - \lambda(|\mathsf{Q}_{\geq 2}(\varphi)| - |\mathsf{Q}_{\geq 2}(U_A\varphi)|)\right). \quad (2.1)$$

Because $\varphi_v \geq 1$ for all $v \in S$ which are in the exterior boundary of S', the isolated zeroes of φ in $S \setminus S'$ remain the same no matter how we change φ inside of S'. Hence,

$$|\mathsf{Q}_{\geq 2}(\varphi)| - |\mathsf{Q}_{\geq 2}(U_A\varphi)| = |\mathsf{Q}_{\geq 2}(\varphi) \cap S'| - |\mathsf{Q}_{\geq 2}(U_A\varphi) \cap S'|.$$

Now consider a tiling $\mathcal{T} = \{T_i\}$ of the region $\mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W$ by the five shapes [6, Figure 1] (and their rotations). (It is a simple geometric fact that any finite subset of $\mathbb{Z}^2$ can be tiled in this way as long all its connected components have size at least 2.) Let $A_i = A \cap T_i$, and let $\mathsf{Q}_{\geq 2}(A_i) = \mathsf{Q}_{\geq 2}(U_A\varphi) \cap A_i$. Since $\mathsf{Q}_{\geq 2}(U_A\varphi) \subseteq \mathsf{Q}_{\geq 2}(\varphi)$, we can write

$$|\mathsf{Q}_{\geq 2}(U_A\varphi) \cap S'| = \sum_i |\mathsf{Q}_{\geq 2}(A_i)| + |\mathsf{Q}_{\geq 2}(U_A\varphi) \cap W| \geq \sum_i |\mathsf{Q}_{\geq 2}(A_i)|, \quad \text{and}$$

$$|\mathsf{Q}_{\geq 2}(\varphi) \cap S'| = \sum_i |T_i| + |\mathsf{Q}_{\geq 2}(\varphi) \cap W| \leq \sum_i |T_i| + |W|.$$

Combining the above, and summing (2.1) over $A \subseteq \mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W$, we then have

$$\sum_{A \subseteq \mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W} \widetilde{\mathbb{P}}^h_S(U_A \varphi) \geq \widetilde{\mathbb{P}}^h_S(\varphi) e^{-\beta \Delta S' - \beta |\partial_e W| - \lambda |W|} \prod_{i=1}^{|\mathcal{T}|} e^{-\lambda |T_i|} \sum_{A_i \subseteq T_i} e^{-\beta |\partial_e A_i| + \lambda |\mathsf{Q}_{\geq 2}(A_i)|}, \tag{2.2}$$

since summing over $A \subseteq \mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W$ is the same as summing over subsets of each of the covering tiles, and if $A = \bigcup_i A_i$, then $|\partial_e A| \leq \sum_i |\partial_e A_i|$. Furthermore, for each of the five possible choices of the tile T, it was calculated in [6, Lemma 4.1] that

$$e^{-\lambda |T|} \sum_{A \subseteq T} e^{-\beta |\partial_e A| + \lambda |\mathsf{Q}_{\geq 2}(A)|} \geq 1 + \frac{1}{2} e^{-6\beta}.$$

Plugging this bound into Eq. (2.2), and noting that the number of tiles in $\mathcal{T}$ is at least $|\mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W|/5$ (since the maximum number of vertices in a tile is 5), we have that

$$\sum_{A \subseteq \mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W} \widetilde{\mathbb{P}}^h_S(U_A \varphi) \geq \widetilde{\mathbb{P}}^h_S(\varphi) e^{-\beta \Delta S' - \beta |\partial_e W| - \lambda |W|} (1 + \frac{1}{2} e^{-6\beta})^{|\mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W|/5}. \qquad \square$$

Lemma 2.3. Let φ, φ', A, A' be such that φ and φ' are ≥ 1 on the exterior boundary of S', $A \subseteq \mathsf{Q}_{\geq 2}(\varphi) \cap S' \setminus W$, and $A' \subseteq \mathsf{Q}_{\geq 2}(\varphi') \cap S' \setminus W$. If $\varphi \neq \varphi'$ or $A \neq A'$ then $U_A \varphi \neq U_{A'} \varphi'$.

Proof. It suffices to show that we can recover φ from $U_A \varphi$. We first show how we can recover the set A given $U_A \varphi$. Let $\mathcal{Z}$ be the set of zeroes of $U_A \varphi$ inside $S' \setminus W$; note that $\mathcal{Z}$ is the disjoint union of A and $\{z \in \mathcal{Z} : \varphi_z = -1\}$. We claim that A is equal to the set

$$\mathcal{Z}_1 := \{z \in \mathcal{Z} : \exists v \in S' \setminus W, v \sim z, U_A \varphi_v \in \{0, 1\}\} \cup \{z \in \mathcal{Z} : \exists v \in W, v \sim z, U_A \varphi_v = 0\}.$$

To show $A \subseteq \mathcal{Z}_1$, note that for all $z \in A$, there must be some $v \sim z$ such that $\varphi_v = 0$ (since $A \subseteq \mathsf{Q}_{\geq 2}(\varphi)$). As φ is ≥ 1 on the exterior boundary of S', such v must be also be in S'. If $v \in W$, then we know $U_A \varphi_v = \varphi_v = 0$. If $v \notin W$, then the fact that $U_A \varphi$ either keeps the value of φ or increases it by 1 implies that $U_A \varphi_w \in \{0, 1\}$.

We show that $\mathcal{Z}_1 \subseteq A$ by showing that $\{z \in \mathcal{Z} : \varphi_z = -1\} \subseteq \mathcal{Z} \setminus \mathcal{Z}_1 =: \mathcal{Z}_2$. First note

$$\mathcal{Z}_2 = \{z \in \mathcal{Z} : \forall v \in S', v \sim z, v \notin W, U_A \varphi_v \geq 2\} \cap \{z \in \mathcal{Z} : \forall v, v \sim z, v \in W, U_A \varphi_v \neq 0\}.$$

Observe that for z such that $\varphi_z = -1$, any $v \sim z$ must be such that $\varphi_v \geq 1$ by definition of $\widetilde{\Omega}$. In particular, we know that $v \notin A$. By definition of $U_A \varphi$, in the case that $v \in S' \cap A^c \cap W^c$, then $U_A \varphi_v = \varphi_v + 1$, and so $U_A \varphi_v \geq 2$. In the case that $v \in W$, then $U_A \varphi_v = \varphi_v \geq 1$, so in particular $U_A \varphi_v \neq 0$. Since $\mathcal{Z}_1$ only depends on the values of $U_A \varphi$, this shows we can recover the set A. Once we have the set A, we can easily recover φ from $U_A \varphi$ by taking $\varphi_z = U_A \varphi_z$ if $z \in (S')^c \cup W \cup A$ and taking $\varphi_z = U_A \varphi_z - 1$ otherwise. $\square$

When $S = S' = \Lambda_n$, $h = 0$, and $W = \emptyset$, Lemma 2.2 corresponds to [6, Thm 1.2] that $\mathsf{Q}_{\geq 2}(\varphi) \leq C_\beta n$ (e.g., with $C_\beta = e^{7\beta}$). For self-containedness ($|\mathsf{Q}_{\geq 2}(\varphi)| = O(n)$ will be needed in Section 3), we show how to derive said bound as a corollary of the above.

Corollary 2.4 ([6, Thm. 1.2])**.** Let β be large enough, and fix $C_\beta > 80\beta e^{6\beta}$. Then

$$\widetilde{\mathbb{P}}(|\mathsf{Q}_{\geq 2}(\varphi)| \geq C_\beta n) \leq \exp\left(-n(\tfrac{C_\beta}{20} e^{-6\beta} - 4\beta)\right).$$

Proof. When $S = S' = \Lambda_n$, $h = 0$, $W = \emptyset$, we have $\Delta S' = 4n$ and $\widetilde{\mathbb{P}}^h_S = \widetilde{\mathbb{P}}$. By Lemma 2.2,

$$\sum_{A \subseteq \mathsf{Q}_{\geq 2}(\varphi)} \widetilde{\mathbb{P}}(U_A \varphi) \geq \widetilde{\mathbb{P}}(\varphi) e^{-4\beta n} (1 + \frac{1}{2} e^{-6\beta})^{|\mathsf{Q}_{\geq 2}(\varphi)|/5}.$$

Furthermore, when $S' = S$, the condition that φ is ≥ 1 on the exterior boundary of S' disappears. Summing over all φ such that $|\mathsf{Q}_{\geq 2}(\varphi)| \geq C_\beta n$, the left-hand side is at most 1 by Lemma 2.3, and thus using $1 + x \geq e^{\frac{1}{2}x}$ for $x \in [0,1]$,

$$\widetilde{\mathbb{P}}(|\mathsf{Q}_{\geq 2}(\varphi)| \geq C_\beta n) \leq e^{4\beta n - \frac{C_\beta}{20} n e^{-6\beta}} . \qquad \square$$

2.2 Identifying the rate for downward tooth-like oscillations

An implication of Lemma 2.2 is that down-contours have exponential tails. Using that, we can show that the rate for having $\varphi_x \leq 0, \varphi_y \leq 1$ is governed by the minimal weight way to generate this, the tooth-like spike which costs $e^{-6\beta h + 2\beta}$.

Lemma 2.5. There exists $\varepsilon_\beta > 0$ (going to zero as $\beta \to \infty$) such that for all simply connected S, all $h \geq 1$, and all pairs of adjacent $x, y \in S$,

$$\widetilde{\mathbb{P}}^h_S(\varphi_x \leq 0, \varphi_y \leq 1) \leq (1 + \varepsilon_\beta) e^{-6\beta h + 2\beta} .$$

Proof. It suffices to show

$$\widetilde{\mathbb{P}}^h_S(\varphi_x \leq 1, \varphi_y \leq 1) \leq (1 + \varepsilon_\beta) e^{-6\beta(h-1)} , \qquad \text{and} \tag{2.3}$$

$$\widetilde{\mathbb{P}}^h_S(\varphi_x \leq 0 \mid \varphi_x \leq 1, \varphi_y \leq 1) \leq (1 + \varepsilon_\beta) e^{-4\beta} . \tag{2.4}$$

To show Eq. (2.3), we first show that conditional on $\varphi_x \leq 1, \varphi_y \leq 1$, the outermost down-contour containing $\{x, y\}$ has no other points, with probability $1 - \varepsilon_\beta$. Fix a contour $C_{x,y}$ containing $\{x, y\}$. Let $\mathcal{C}_{x,y}(\varphi)$ denote the outermost down-contour containing $\{x, y\}$ in φ. Let $\mathcal{B}(C_{x,y})$ be the event that $\varphi_x \leq 1, \varphi_y \leq 1$, and $\mathcal{C}_{x,y}(\varphi) = C_{x,y}$. For every $\psi \in \mathcal{B}(C_{x,y})$, we know that ψ is $\geq h$ on the exterior boundary of $C_{x,y}$ by definition an outermost down-contour. Applying Lemma 2.2 with $W = \{x, y\}$ and $S' = \mathsf{Int}(C_{x,y})$, we get

$$\sum_{A \subseteq \mathsf{Q}_{\geq 2}(\psi) \cap \mathsf{Int}(C_{x,y}) \setminus \{x,y\}} \widetilde{\mathbb{P}}^h_S(U_A \psi) \geq \widetilde{\mathbb{P}}^h_S(\psi) e^{\beta |C_{x,y}| - 6\beta - 2\lambda} .$$

As $U_A\psi$ did not change the values of ψ on x, y, we still have $U_A\psi_x \leq 1, U_A\psi_y \leq 1$. Summing over $\psi \in \mathcal{B}(C_{x,y})$ above and applying Lemma 2.3,

$$\sum_{\psi \in \mathcal{B}(C_{x,y})} \widetilde{\mathbb{P}}^h_S(\psi) e^{\beta |C_{x,y}| - 6\beta - 2\lambda} \leq \sum_{\psi \in \mathcal{B}(C_{x,y})} \sum_{A \subseteq \mathsf{Q}_{\geq 2}(\psi) \cap \mathsf{Int}(C_{x,y}) \setminus \{x,y\}} \widetilde{\mathbb{P}}^h_S(U_A \psi) \leq \widetilde{\mathbb{P}}^h_S(\varphi_x \leq 1, \varphi_y \leq 1) ,$$

and in particular

$$\widetilde{\mathbb{P}}^h_S(\mathcal{B}(C_{x,y})) \leq e^{-\beta |C_{x,y}| + 6\beta + 2\lambda} \widetilde{\mathbb{P}}^h_S(\varphi_x \leq 1, \varphi_y \leq 1) .$$

Since the number of contours containing x, y with length l is at most C^l, we then have

$$\widetilde{\mathbb{P}}^h_S(\varphi_x \leq 1, \varphi_y \leq 1, |\mathcal{C}_{x,y}(\varphi)| > 6) = \sum_{l \geq 8} \sum_{C_{x,y} : |C_{x,y}| = l} \widetilde{\mathbb{P}}^h_S(\mathcal{B}(C_{x,y}))$$
$$\leq \sum_{l \geq 8} C^l e^{-\beta(l-6) + 2\lambda} \widetilde{\mathbb{P}}^h_S(\varphi_x \leq 1, \varphi_y \leq 1) \leq C' e^{-2\beta} \widetilde{\mathbb{P}}^h_S(\varphi_x \leq 1, \varphi_y \leq 1) ,$$

and in particular that

$$\widetilde{\mathbb{P}}^h_S(|\mathcal{C}_{x,y}(\varphi)| = 6 \mid \varphi_x \leq 1, \varphi_y \leq 1) \geq (1 - \varepsilon_\beta) . \tag{2.5}$$

Now, with the above in hand, we claim that

$$\widetilde{\mathbb{P}}^h_S(|\mathcal{C}_{x,y}(\varphi)|=6,\varphi_x\le 1,\varphi_y\le 1)\le(1+\varepsilon_\beta)e^{-6\beta(h-1)}\,. \tag{2.6}$$

First consider the case where $\varphi_x=\varphi_y=0,|\mathcal{C}_{x,y}(\varphi)|=6$. Consider the bijective map $T\varphi$ where $T\varphi_x=\varphi_x+6h$, $T\varphi_y=\varphi_y+6h$, and $T\varphi_w=\varphi_w$ for $w\notin\{x,y\}$. The fact that $|\mathcal{C}_{x,y}(\varphi)|=6$ implies that for any neighbors w of $\{x,y\}$, we have $\varphi_w\ge h$, so that $\widetilde{\mathbb{P}}^h_S(\varphi)\le\widetilde{\mathbb{P}}^h_S(T\varphi)e^{-6\beta h}e^{2\lambda}$. Summing over φ, we get

$$\widetilde{\mathbb{P}}^h_S(|\mathcal{C}_{x,y}(\varphi)|=6,\varphi_x=\varphi_y=0)\le e^{-6\beta h+2\lambda}.$$

Otherwise, if we have that at least one of φ_x or φ_y is not equal to 0, we consider the map where $T\varphi_x=\varphi_x+6(h-1),T\varphi_y=\varphi_y+6(h-1)$, and $T\varphi_w=\varphi_w$ for $w\notin\{x,y\}$. In this case, neither x nor y are in $\mathsf{Q}_{\ge 2}(\varphi)$, and so we have $\widetilde{\mathbb{P}}^h_S(\varphi)\le\widetilde{\mathbb{P}}^h_S(T\varphi)e^{-6\beta(h-1)}$. Summing over φ in this case proves that

$$\widetilde{\mathbb{P}}^h_S(|\mathcal{C}_{x,y}(\varphi)|=6,\varphi_x\le 1,\varphi_y\le 1,\varphi_x=0\text{ or }\varphi_y=0)\le e^{-6\beta(h-1)}.$$

The above displays prove Eq. (2.6), which combined with Eq. (2.5) proves Eq. (2.3). To prove Eq. (2.4), first note that the exact same proof of Eq. (2.5) yields

$$\widetilde{\mathbb{P}}^h_S(|\mathcal{C}_{x,y}(\varphi)|=6\mid\varphi_x\le 0,\varphi_y\le 1)\ge(1-\varepsilon_\beta)$$

(simply enforce the starting configurations ψ to satisfy $\psi_x\le 0,\psi_y\le 1$, and then apply the same maps $U_A\psi$). Thus, noting that $\widetilde{\mathbb{P}}^h_S(\varphi_x\le 0\mid\varphi_x\le 1,\varphi_y\le 1)$ is equal to

$$\begin{aligned}\frac{\widetilde{\mathbb{P}}^h_S(|\mathcal{C}_{x,y}(\varphi)|=6\mid\varphi_x\le 1,\varphi_y\le 1)}{\widetilde{\mathbb{P}}^h_S(|\mathcal{C}_{x,y}(\varphi)|=6\mid\varphi_x\le 0,\varphi_y\le 1)}&\widetilde{\mathbb{P}}^h_S(\varphi_x\le 0\mid\varphi_x\le 1,\varphi_y\le 1,|\mathcal{C}_{x,y}(\varphi)|=6)\\&\le(1+\varepsilon_\beta)\widetilde{\mathbb{P}}^h_S(\varphi_x\le 0\mid\varphi_x\le 1,\varphi_y\le 1,|\mathcal{C}_{x,y}(\varphi)|=6),\end{aligned}$$

it suffices to bound the right side from above by $(1+\varepsilon_\beta)e^{-4\beta}$. Let φ be such that $\varphi_x\le 0,\varphi_y\le 1,|\mathcal{C}_{x,y}(\varphi)|=6$. If $\varphi_x=\varphi_y=0$, then consider the map which sets $T\varphi_x=T\varphi_y=1$ and follow the same reasoning as before to obtain

$$\widetilde{\mathbb{P}}^h_S(\varphi_x=\varphi_y=0\mid\varphi_x\le 1,\varphi_y\le 1,|\mathcal{C}_{x,y}(\varphi)|=6)\le e^{-6\beta+2\lambda}. \tag{2.7}$$

If $\varphi_x=k\le -1$, then $\varphi_y=1$ by definition of $\widetilde{\Omega}$. Hence, it remains to consider when $\varphi_x\le 0,\varphi_y=1$. In this case, we can consider the map that sets $T\varphi_x=1$ and obtain that

$$\widetilde{\mathbb{P}}^h_S(\varphi_x=k\mid\varphi_x\le 1,\varphi_y\le 1,|\mathcal{C}_{x,y}(\varphi)|=6)\le e^{-4(1-k)\beta}\,.$$

Summing over $k\le 0$ and combining with Eq. (2.7) proves that the probability of $\varphi_x\le 0$ given $\varphi_x\le 1,\varphi_y\le 1$ and $|\mathcal{C}_{x,y}(\varphi)|=6$ is at most $(1+\varepsilon_\beta)e^{-4\beta}$. This concludes the proof of Eq. (2.4) and hence of the lemma. □

2.3 Tail bounds for up-contours

We can now show that above h_n^*, the gain from entropic repulsion is dominated by the boundary cost of an up-contour. Let $\mathcal{C}^{\uparrow}_{S,h}$ be the event that ∂S is an up h-contour.

Lemma 2.6. There exists $c_0>0$ such that for every simply connected set S,

$$\widetilde{\mathbb{P}}(\mathcal{C}^{\uparrow}_{S,h})\le\exp\big(-\beta|\partial S|+c_0e^{-6\beta h+2\beta}|S|\big)\,.$$

In particular, if $h=h_n^*+1$, then $\widetilde{\mathbb{P}}(\mathcal{C}^{\uparrow}_{S,h_n^*+1})\le\exp(-(\beta-\frac{c_0}{4})|\partial S|)$.

Proof. For a simply connected S (the interior of a contour), let $\mathcal{E}_S$ denote the event that there does not exist a pair of adjacent sites $x, y \in S$ such that $\varphi_x \le 0$ and $\varphi_y \le 1$. Observe that if $\varphi \in \mathcal{C}^{\uparrow}_{S,h} \cap \mathcal{E}_S$, then the configuration φ' obtained by shifting $\varphi \mapsto \varphi - 1$ in the interior of S keeps the configuration permissible (i.e., $\varphi' \in \widetilde{\Omega}$). We begin by establishing

$$\widetilde{\mathbb{P}}(\mathcal{E}_S \mid \mathcal{C}^{\uparrow}_{S,h}) \ge \exp\big(- c_0 e^{-6\beta h + 2\beta} |S| \big)\,, \tag{2.8}$$

as on $\mathcal{E}_S \cap \mathcal{C}^{\uparrow}_{S,h}$ the observation ensures we can shift down S and gain a factor of $e^{-\beta|\partial S|}$.

Associate to every $\varphi \in \widetilde{\Omega}$ in the state space of $\widetilde{\mathbb{P}}$, the SOS configuration $\phi = \max\{\varphi, 0\}$ in the state space of $\mathbb{P}$. For every $\varphi \in \mathcal{E}_S$, we see that ψ satisfies the same property (it does not contain any adjacent pair of sites x, y such that $\phi_x = 0$ and $\phi_y \in \{0, 1\}$). In addition, since $h \ge 1$, we have that $\varphi \in \mathcal{C}^{\uparrow}_{S,h}$ if and only if $\phi \in \mathcal{C}^{\uparrow}_{S,h}$ in its corresponding space. Thus, it suffices to bound from below $\mathbb{P}(\bigcap_{x \sim y} \{\phi_x \le 0\,,\, \phi_y \le 1\}^c \mid \mathcal{C}^{\uparrow}_{S,h})$.

By a routine reasoning, when conditioning on $\mathcal{C}^{\uparrow}_{S,h}$ we can modify the external boundary of S to h (domain Markov, using that the internal boundary is at least h), and then use monotonicity (see, e.g., [9, Sec. 4.1]; it is easy to check that the validity of Holley's lattice condition is unaffected by the λ tokens) to remove the conditioning that the internal boundary is at least h (as we will look to bound from below an increasing event), so it suffices to bound from below $\mathbb{P}^h_S(\bigcap_{x \sim y} \{\phi_x \le 0\,,\, \phi_y \le 1\}^c)$.

The proof is then concluded from Lemma 2.5 by FKG for $\mathbb{P}^h_S$ and the inequality $1 - x \ge \frac{1}{2} e^{-x}$ for $x \in [0, 1]$, using the equivalence of $\{\varphi_x \le 0, \varphi_y \le 1\}$ for $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ as mentioned above. (The constant c_0 comes from enumerating over ordered pairs (x, y) in S, and also absorbs the factor of $\frac{1}{2}$ in the inequality $1 - x \ge \frac{1}{2} e^{-x}$.)

Having shown (2.8), let us conclude the proof. We apply the map that shifts $\varphi \mapsto \varphi - 1$ in the interior of S for every $\varphi \in \mathcal{C}^{\uparrow}_{S,h} \cap \mathcal{E}_S$, and obtain that

$$\widetilde{\mathbb{P}}(\mathcal{C}^{\uparrow}_{S,h} \cap \mathcal{E}_S) \le e^{-\beta|\partial S|}\,. \tag{2.9}$$

Dividing this bound by $\widetilde{\mathbb{P}}(\mathcal{E}_s \mid \mathcal{C}^{\uparrow}_{S,h})$ and applying (2.8), we conclude the proof.

When $h = h^*_n + 1$, the bound follows from $e^{-6\beta(h^*_n+1)+2\beta} \le \frac{1}{n}$ and $|S| \le n|\partial S|/4$. ☐

By the same reasoning, we have the following more general version of Lemma 2.6.

Corollary 2.7. For any family of disjoint simply connected sets $S_1, \dots, S_m$,

$$\widetilde{\mathbb{P}}(\mathcal{C}^{\uparrow}_{S_1,h}, \dots, \mathcal{C}^{\uparrow}_{S_m,h}) \le \exp\Big(- \beta \sum_i |\partial S_i| + c_0 e^{-6\beta h + 2\beta} \sum_i |S_i| \Big)\,.$$

In particular, if $h = h^*_n + 1$, then $\widetilde{\mathbb{P}}(\bigcap_i \mathcal{C}^{\uparrow}_{S_i, h^*_n+1}) \le \exp(-(\beta - \frac{c_0}{4}) \sum_i |\partial S_i|)$.

Proof. By domain Markov, Eq. (2.8) holds even if we further condition on the values of φ outside of S. Since shifting $\varphi \mapsto \varphi - 1$ inside of S preserves φ outside of S, Eq. (2.9) also holds conditional on φ outside of S. Hence, Lemma 2.6 holds conditional on having a family of up-contours in the exterior of S, and Corollary 2.7 follows by induction. ☐

2.4 Proof of the upper bound

To conclude Proposition 2.1, by Lemma 2.6 there are no up $h^*_n + 1$ contours of size greater than $\log n$. The following lemma controls the total area in smaller contours; this type of estimate is fairly standard once one has the exponential tails of Corollary 2.7.

Lemma 2.8. There exist $C, c > 0$ such that with probability $1 - e^{-cn}$, the number of $x \in \Lambda_n$ interior to up-contours of height $h^*_n + 1$ and interior area at most $n^{1.9}$ is $Ce^{-\beta} n^2$.

Proof. Let Γ denote the set of all outermost h_n^*+1 up-contours having interior area at most $n^2/(\log n)^8$. We follow, e.g., the proof of [7, Lemma 4.7] considering the contributions from contours of dyadically increasing sizes. Partition the set of outermost h_n^*+1 up-contours into sets $\mathfrak{U}_1, \mathfrak{U}_2, \dots$ given by $\mathfrak{U}_k = \{\gamma \in \Gamma : 2^{k-1} \le |\mathsf{Int}(\gamma)| \le 2^k\}$.

We will show that for a suitable absolute constant $C_0 > 0$, for each $k = 1, \dots, \lceil \log_2 L_1 \rceil$,

$$\widetilde{\mathbb{P}}\Big(\sum_{\gamma \in \mathfrak{U}_k} |\mathsf{Int}(\gamma)| \ge (\varepsilon_{\beta,k} n)^2 \Big) \le \exp\Big(-(\beta - C)\frac{(\varepsilon_{\beta,k} n)^2}{2^{k/2}} \Big), \quad \text{for} \quad \varepsilon_{\beta,k} := \frac{C_0}{e^{\beta/2} k}. \tag{2.10}$$

A union bound over k implies the claimed bound.

Fix k. To show (2.10), suppose φ is such that $\sum_{\gamma \in \mathfrak{U}_k} |S_i| \ge (\varepsilon_{\beta,k} n)^2$ and suppose the elements of $\mathfrak{U}_k$ are $\gamma_1, \dots, \gamma_m$ with interiors $S_1, \dots, S_m$ respectively. By definition of $\mathfrak{U}_k$ and the isoperimetric inequality in $\mathbb{Z}^2$, it must be the case that

$$m \le \sum_{i=1}^m |S_i| 2^{1-k}, \qquad \text{and} \qquad \sum_{i=1}^m |\gamma_i| \ge 4 \sum_{i=1}^m |S_i| 2^{-k/2}. \tag{2.11}$$

By applying Corollary 2.7 to the S_i, we obtain

$$\widetilde{\mathbb{P}}\Big(\bigcap_i \mathcal{C}^{\uparrow}_{S_i, h_n^*+1} \Big) \le \exp\Big(-(\beta - c_0/4) \sum_i |\gamma_i| \Big).$$

We union bound over collections $\mathfrak{U}_k$ as follows: letting χ count $\frac{1}{n^2}\sum_{i=1}^m |S_i|$,

$$\begin{aligned}\widetilde{\mathbb{P}}\Big(\sum_{\gamma \in \mathfrak{U}_k} |\mathsf{Int}(\gamma)| \ge (\varepsilon_{\beta,k} n)^2 \Big) &\le \sum_{(\varepsilon_{\beta,k} n)^2 \le \chi n^2 \le n^2} \; \sum_{m \le \chi n^2 2^{1-k}} \binom{n^2}{m} \sum_{L \ge \chi n^2 2^{-\frac{k}{2}+2}} C^L e^{-(\beta - c_0/4)L} \\ &\le \sum_{\varepsilon_{\beta,k} n^2 \le \chi n^2 \le n^2} \; \sum_{m \le \chi n^2 2^{1-k}} \binom{n^2}{m} e^{-4(\beta - C)\chi n^2 2^{-\frac{k}{2}}}.\end{aligned}$$

The first line above used (2.11) to upper bound m and lower bound L, and once the root-points for the m distinct contours have been picked, there are at most C^L ways to generate m associated contours with total length L, for some absolute constant C. Using $\sum_{j \le \rho N} \binom{N}{j} \le \exp(\mathsf{H}(\rho) N)$ where $\mathsf{H}(\rho)$ is the binary entropy function, we get

$$\widetilde{\mathbb{P}}\Big(\sum_{\gamma \in \mathfrak{U}_k} |\mathsf{Int}(\gamma)| \ge (\varepsilon_{\beta,k} n)^2 \Big) \le \sum_{\varepsilon_{\beta,k} n^2 \le \chi n^2 \le n^2} \exp\Big(\Big(\mathsf{H}(\chi 2^{1-k}) - (\beta - C) 2^{2-\frac{k}{2}} \chi \Big) n^2 \Big).$$

It thus suffices to show for every $\chi > \varepsilon_{\beta,k}^2 = \frac{C_0^2}{e^{\beta} k^2}$, we have $\mathsf{H}(\chi 2^{1-k}) \le 3(\beta - C)\chi 2^{-k/2}$, to get (2.10) (absorbing the pre-factor of n^2 from the sum into the C in the exponent). To show this, using the bound $\mathsf{H}(\rho) \le \rho \log \frac{1}{\rho} + \rho$, and noting that $\chi 2^{1-k} \le (\beta - C)\chi 2^{-k/2}$ for all $k \ge 1$ and all β large, we just need to show

$$2^{1-k} \log \tfrac{2^{k-1}}{\chi} \le (\beta - C) 2^{1-\frac{k}{2}} \qquad \text{or} \qquad (k-1)(\log 2) + \log \chi^{-1} \le (\beta - C) 2^{k/2}.$$

By the lower bound on χ, we have $\log \chi^{-1} \le \beta + \log(k^2) - \log C_0^2$. At this point, C_0 can be taken large (independent of β because the β on the left is bounded by $2^{k/2}\beta$ on the right for all k) to only consider large values of k, and for those, the bound is evident. □

Proof of Proposition 2.1. Any vertex x having $\varphi_x \ge h_n^*+1$ must be contained in some up h_n^*+1 contour, so it suffices to bound the total area interior to outermost up h_n^*+1 contours. By application of Lemma 2.8, it suffices to bound the contribution from up h_n^*+1 contours with interior area at least $n^{1.9}$, which necessitates contour length at least

$n^{0.95}$. By the fact that there are at most C^ℓ many contours of length ℓ incident about a vertex x for a universal constant $C>0$, Lemma 2.6 and a union bound imply that

$$\widetilde{\mathbb{P}}\Big(\bigcup_{\ell\geq n^{0.75}}\bigcup_{\partial S: x\sim\partial S, |\partial S|=\ell}\mathcal{C}^{\uparrow}_{S,h_n^*+1}\Big)\leq\sum_{\ell\geq n^{0.75}}\sum_{\partial S: x\sim\partial S, |\partial S|=\ell}C^\ell e^{-(\beta-\frac{c_0}{4})\ell}\leq e^{-(\beta-C')n^{0.75}}\,.$$

We conclude by a union bound over the n^2 choices of x. □

Remark 2.9. Consider the fractional part $\xi_n := (\frac{1}{6\beta}\log n+\frac{1}{3})-\lfloor\frac{1}{6\beta}\log n+\frac{1}{3}\rfloor$, and notice that in uses of Lemma 2.6, we could take $h=h_n^*$ to in fact obtain

$$\widetilde{\mathbb{P}}\big(\mathcal{C}^{\uparrow}_{S,h_n^*}\big)\leq\exp\big(-\beta|\partial S|+c_0(|\partial S|/4)e^{6\beta\xi_n}\big)\,.$$

Whenever $\xi_n\leq\frac{\log\beta}{8\beta}$, this is at most $e^{-(\beta/2)|\partial S|}$ and the remainder of the proof of Proposition 2.1 goes through unchanged also for h_n^*.

3 Lower bound

We turn to showing the following lower bound on the typical height of φ. We will then combine it with Proposition 2.1 and moving back to $\mathbb{P}$ to deduce Theorem 1.1.

Proposition 3.1. There exists a constant $C>0$ such that for all β large,

$$\widetilde{\mathbb{P}}\left(\#\{x:\varphi_x\leq h_n^*-2\}\geq(C/\beta)n^2\right)\leq e^{-\beta n}\,.$$

The proof goes by examining the histogram of φ and finding a k such that the histogram has more faces than it should at height h_n^*-k. The map then lifts the interface up by k while injecting entropy through tooth-like spikes of depth k that are now permitted. We work on the event of $|\mathsf{Q}_{\geq 2}(\varphi)|\leq e^{7\beta}n$ which was proved to have high probability by [6, Thm. 1.2] (our Corollary 2.4 for completeness), ensuring that the loss of tokens from lifting $\mathsf{Q}_{\geq 2}$ up doesn't overwhelm the entropy gained.

Lemma 3.2. For every $k\geq 2$ define the set

$$X_k(\varphi):=\{x\sim y:\ \varphi_x=\varphi_y=h_n^*-k\,,\text{ and no } z \text{ adjacent to } x \text{ or } y \text{ has } \varphi_z\leq 0\}\,,$$

(where we counted unordered pairs of adjacent sites $x\sim y$). Then for every $k\geq 2$,

$$\widetilde{\mathbb{P}}\left(|X_k(\varphi)|\geq e^{8\beta-5\beta k}n^2\,,\ |\mathsf{Q}_{\geq 2}(\varphi)|\leq e^{7\beta}n\right)\leq\exp(-e^{\beta k}n)\,.$$

Proof. Fix $k\geq 2$, and let $\mathcal{B}_k$ be the bad event that $|X_k(\varphi)|\geq e^{8\beta-5\beta k}n^2$, and $|\mathsf{Q}_{\geq 2}(\varphi)|\leq e^{7\beta}n$. (Note that the choice of $5\beta k$ is somewhat arbitrary in that 5 could be any number strictly smaller than 6, since $e^{-6\beta k}n^2$ is, to first order, the expected number of sites at depth k below h_n^*.)

Let $X'_k(\varphi)$ be an arbitrary subset of disjoint ordered pairs of adjacent sites (x,y) among $X_k(\varphi)$, noting that we may collect at least $|X_k(\varphi)|/7$ such pairs greedily (e.g., by listing the edges xy of $X_k(\varphi)$, ordered $x<y$ lexicographically, proceeding sequentially and collecting each one that is not sharing a vertex with a previously selected edge).

For a prescribed subset of pairs $S\subset X'_k(\varphi)$, $S=\{(x_i,y_i)\}_i$, define the map $\varphi\mapsto T_S\varphi$ via $T_S\varphi_v$ is 0 if $v=x_i$ for some i, it is 1 if $v=y_i$ for some i, and it is φ_v+1 otherwise; i.e., it lifts φ by 1, then adjusts φ_x to 0 and φ_y to 1 for every $(x,y)\in S$ (noting these are all disjoint by construction). The fact that $T_S\varphi\in\widetilde{\Omega}$ follows from the fact that S is not adjacent to any v having $\varphi_v\leq 0$, and otherwise the constraint of $\widetilde{\Omega}$ is increasing, so lifting the rest of the configuration cannot take it outside $\widetilde{\Omega}$. Comparing probabilities,

$$\widetilde{\mathbb{P}}(T_S\varphi)\geq\widetilde{\mathbb{P}}(\varphi)\exp\big(-4\beta n-(6\beta(h_n^*-k+1)+2\beta)-\lambda|\mathsf{Q}_{\geq 2}(\varphi)|\big)\,. \tag{3.1}$$

We now claim that for fixed k, the sets of images $T_S\varphi$ across (φ, S) are all disjoint, so that we may sum the above expression. Note that if $(T_S\varphi_z, T_s\varphi_{z'}) = (0,1)$ for a pair $z \sim z'$, it could not be that $\varphi_z = -1, \varphi_{z'} = 0$ because that would violate $\varphi \in \widetilde{\Omega}$. Therefore, any such pair (z, z') must have both z, z' belonging to edges in S. Moreover, all vertices belonging to edges in S get either height 0 or 1 and have a neighbor in the other height. In this manner, the set S can be read off from $T_S\varphi$, and hence the interface φ can be read off from $T_S\varphi$ (recover S, then set everyone in S back to height $h_n^* - k$ and shift every other site's height down by 1). Summing over $\varphi \in \mathcal{B}_k$ and $S \subset X_k'(\varphi)$ and applying (3.1),

$$1 \geq \sum_{\varphi \in \mathcal{B}_k} \sum_{S \subset X_k'(\varphi)} e^{-6\beta(h_n^* - k + 1) + 2\beta} \exp(-4\beta n - \lambda |\mathsf{Q}_{\geq 2}(\varphi)|) \widetilde{\mathbb{P}}(\varphi) \,.$$

In turn, by binomial theorem, the definition of h_n^* (1.3), and the bounds of $|X_k'(\varphi)| \geq \frac{1}{7}|X_k(\varphi)| \geq \frac{1}{7} e^{8\beta - 5\beta k} n^2$ and $|\mathsf{Q}_{\geq 2}(\varphi)| \leq e^{7\beta} n$ on $\mathcal{B}_k$,

$$\begin{aligned} 1 &\geq \min_{\varphi \in \mathcal{B}_k} (1 + e^{-6\beta(h_n^* - k + 1) + 2\beta})^{|X_k'(\varphi)|} \exp(-4\beta n - \lambda |\mathsf{Q}_{\geq 2}(\varphi)|) \widetilde{\mathbb{P}}(\mathcal{B}_k) \\ &\geq \min_{\varphi \in \mathcal{B}_k} \exp\Big(\frac{1}{n} e^{6\beta(k-1)} |X_k'(\varphi)| - 4\beta n - \lambda |\mathsf{Q}_{\geq 2}(\varphi)|\Big) \widetilde{\mathbb{P}}(\mathcal{B}_k) \\ &\geq \exp\Big(\frac{1}{7} e^{\beta(k+2)} n - 4\beta n - \lambda e^{7\beta} n\Big) \widetilde{\mathbb{P}}(\mathcal{B}_k) \,. \end{aligned}$$

Thus, using that $\lambda \leq e^{-4\beta}/(1 + e^{-4\beta})$, for every $k \geq 2$ we have (for β large enough) that $(4\beta + \lambda e^{7\beta}) n \leq 2e^{3\beta} n \leq (\frac{1}{7} e^{2\beta} - 1) e^{\beta k} n$, and it follows that $\widetilde{\mathbb{P}}(\mathcal{B}_k) \leq \exp(-e^{\beta k} n)$. ☐

The following lemma will be used to show we aren't losing much of the histogram of φ by considering $X_k(\varphi)$ rather than $\varphi^{-1}(h_n^* - k)$.

Lemma 3.3. There exists an absolute constant $C > 0$ such that, if β is large enough then

$$\widetilde{\mathbb{P}}\Big(\sum_{x \sim y} |\varphi_x - \varphi_y| \geq (C/\beta) n^2\Big) \leq \exp(-n^2) \,.$$

Proof. By viewing the surface corresponding to a configuration φ as a subset of $\mathbb{Z}^3$, there are at most $e^{C_*(n^2 + k)}$ many possible φ having $\sum_{x \sim y} |\varphi_x - \varphi_y| = k$, for some universal constant C_*. Considering the trivial map which sends $\varphi \mapsto 0$ everywhere, and summing over φ with $k \geq (2C_*/\beta) n^2$, we obtain when $\beta > 3C^*$ that

$$\widetilde{\mathbb{P}}(\sum_{x \sim y} |\varphi_x - \varphi_y| \geq (2C_*/\beta) n^2) \leq \sum_{k \geq (2C_*/\beta) n^2} e^{C_*(n^2 + k) - \beta k} \leq 2 \exp(-\tfrac{1}{3} C_* n^2) \,.$$ ☐

Proof of Proposition 3.1. We can sum Lemma 3.2 for $k \geq 2$ and combine with Corollary 2.4 to see that the number of vertices in some pair of $\bigcup_{k \geq 2} X_k(\varphi)$ is at most $e^{-\beta} n^2$ with probability $1 - e^{-e^\beta n}$. The vertices not in $\bigcup_{k \geq 2} X_k(\varphi)$ are a subset of $\{x : \operatorname{dist}(x, A) \leq 2\}$ where A is the union of $\mathsf{Q}_{\geq 2}(\varphi)$ along with every x adjacent to a nonzero gradient of φ. By Corollary 2.4 and Lemma 3.3, we have $|A| \leq (C/\beta) n^2$ with probability $1 - O(\exp(-n^2))$. Noting $|\{x : \operatorname{dist}(x, A) \leq 2\}| \leq 13|A|$ concludes the proof. ☐

Remark 3.4. Recall the fractional part $\xi_n := (\frac{1}{6\beta} \log n + \frac{1}{3}) - \lfloor \frac{1}{6\beta} \log n + \frac{1}{3} \rfloor$, and note that in the proof of Lemma 3.2 the upper bound on $\widetilde{\mathbb{P}}(\mathcal{B}_k)$ is of the form

$$\widetilde{\mathbb{P}}(\mathcal{B}_k) \leq \max_{\varphi \in \mathcal{B}_k} \exp\big(4\beta n + \lambda |\mathsf{Q}_{\geq 2}(\varphi)| - \tfrac{1}{n} e^{6\beta(k-1) + 6\beta \xi_n} |X_k'(\varphi)|\big) \,.$$

A more careful application of Corollary 2.4 (using $C_\beta = 90\beta e^{6\beta}$ rather than $e^{7\beta}$ as we used in the proof of Lemma 3.2) allows us to only consider contribution from φ with

$4\beta n + \lambda|\mathsf{Q}_{\geq 2}(\varphi)| \leq 100\beta e^{2\beta} n$. Setting $k=1$, whenever we have $\xi_n \geq \frac{1}{3} + \delta_0$ for some absolute constant $\delta_0 > 0$, the right hand of our upper bound on $\widetilde{\mathbb{P}}(\mathcal{B}_1)$ is at most

$$\max_{\varphi \in \mathcal{B}_1} \exp\Big(- \big(\tfrac{1}{n^2} e^{(2+6\delta_0)\beta} |X_k'(\varphi)| - 100\beta e^{2\beta}\big) n \Big) .$$

Thus, for β large enough depending on δ_0, the event that $|X_k'(\varphi)| \geq e^{-\delta_0 \beta} n^2$ would have exponentially small probability in n, giving the same up to a factor of 7 for $|X_k(\varphi)|$.

Proof of Theorem 1.1. Under the identification $\phi_v = \max\{0, \varphi_v\}$ for all v, the sets of sites not at height $\{h_n^* - 1, h_n^*\}$ are fixed. Thus, by Observation 1.4,

$$\mathbb{P}(|\{x : \phi_x \notin \{h_n^* - 1, h_n^*\}\}| > \tfrac{C}{\beta} n^2) = \widetilde{\mathbb{P}}(|\{x : \varphi_x \notin \{h_n^* - 1, h_n^*\}\}| > \tfrac{C}{\beta} n^2) ,$$

and the result follows by combining Proposition 2.1 with Proposition 3.1. □

References

[1] Kenneth S. Alexander, François Dunlop, and Salvador Miracle-Solé, *Layering and wetting transitions for an SOS interface*, Journal of Statistical Physics **142** (2011), no. 3, 524–576. MR2771044

[2] J. Bricmont, A. El Mellouki, and J. Fröhlich, *Random surfaces in statistical mechanics: roughening, rounding, wetting, . . .* , J. Statist. Phys. **42** (1986), no. 5-6, 743–798. MR833220

[3] Pietro Caputo, Eyal Lubetzky, Fabio Martinelli, Allan Sly, and Fabio Lucio Toninelli, *Dynamics of $(2+1)$-dimensional SOS surfaces above a wall: Slow mixing induced by entropic repulsion*, Ann. Probab. **42** (2014), no. 4, 1516–1589. MR3262485

[4] Pietro Caputo, Eyal Lubetzky, Fabio Martinelli, Allan Sly, and Fabio Lucio Toninelli, *Scaling limit and cube-root fluctuations in SOS surfaces above a wall*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 5, 931–995. MR3500829

[5] J T Chalker, *The pinning of an interface by a planar defect*, Journal of Physics A: Mathematical and General **15** (1982), no. 9, L481. MR0673472

[6] Naomi Feldheim and Shangjie Yang, *Typical height of the (2+1)-D Solid-on-Solid surface with pinning above a wall in the delocalized phase*, Stochastic Processes and their Applications **165** (2023), 168–182. MR4637641

[7] Reza Gheissari and Eyal Lubetzky, *Entropic repulsion of 3D Ising interfaces conditioned to stay above a floor*, Electronic Journal of Probability **28** (2023), no. none, 1–44. MR4616514

[8] D. Ioffe and Y. Velenik, *Low-temperature interfaces: prewetting, layering, faceting and Ferrari-Spohn diffusions*, Markov Process. Related Fields **24** (2018), no. 3, 487–537. MR3821253

[9] Hubert Lacoin, *Wetting and layering for Solid-on-Solid I: Identification of the wetting point and critical behavior*, Communications in Mathematical Physics **362** (2018), no. 3, 1007–1048. MR3845294

[10] Hubert Lacoin, *Wetting and layering for Solid-on-Solid II: Layering transitions, Gibbs states, and regularity of the free energy*, Journal de l'École polytechnique — Mathématiques **7** (2020), 1–62 (en). MR4033749

[11] Salvador Miracle-Sole, *Layering and wetting transitions for an interface model*, 11th Granada Seminar (La Herradura (Granada), Spain) (P.L. Garrido, J. Marro, and F. de los Santos (Eds.), eds.), AIP Conference Proceedings, no. 1332, American Institute of Physics, Melville, NY, 2010, pp. 190–194.

[12] Yvan Velenik, *Localization and delocalization of random interfaces*, Probab. Surv. **3** (2006), 112–169. MR2216964

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