Electron. Commun. Probab. **29** (2024), article no. 38, 1–8.
https://doi.org/10.1214/24-ECP609
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

The union of independent USFs on $\mathbb{Z}^d$ is transient*

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Abstract

We show that the union of two or more independent uniform spanning forests (USF) on $\mathbb{Z}^d$ with $d \geq 3$ almost surely forms a connected transient graph. In fact, this also holds when taking the union of a deterministic everywhere percolating set and an independent ε-Bernoulli percolation on a single USF sample.

Keywords: uniform spanning forest; percolation; stochastic domination.
MSC2020 subject classifications: 60K35.
Submitted to ECP on November 21, 2023, final version accepted on July 10, 2024.

1 Introduction

Given a finite connected graph G, the *uniform spanning tree* (UST) on G is a tree drawn uniformly at random from the finite set of spanning trees of G. The *wired uniform spanning forest* (WUSF) and *free uniform spanning forest* (FUSF) on an infinite, connected and locally finite graph G are the weak limits of USTs on an exhaustion of G with wired and free boundary conditions respectively. It was shown by Pemantle [14] that these do not depend on the choice of the exhaustion. Moreover, the two measures WUSF and FUSF coincide on $\mathbb{Z}^d$. We refer to [11, Chapter 4 and 10] for backgrounds on USTs and USFs.

A fundamental result of Morris [13] states that every component of the WUSF is almost surely recurrent on any graph. A natural question is whether this and other remarkable properties of USFs are stable under various perturbations. In this paper we will be concerned with taking unions of independent WUSFs. We say that a subgraph $\Lambda \subseteq G$ is **everywhere percolating** if every $x \in G$ is contained in an infinite connected component of Λ. Also, for any $\varepsilon > 0$ we write ε-WUSF for ε-Bernoulli percolation on the WUSF, that is, conditioned on the WUSF, independently retain each of its edges with probability ε or erase it otherwise.

Theorem 1.1. *Let $d \geq 3$ and let Λ be an everywhere percolating subgraph of $\mathbb{Z}^d$. Then, for any $\varepsilon > 0$ the union of Λ and ε-WUSF on $\mathbb{Z}^d$ is almost surely connected and transient.*

*This research is supported by the ERC consolidator grant 101001124 (UniversalMap) as well as ISF grants 1294/19 and 898/23.
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Benjamini and Tassion [4] proved that this result holds when replacing the ε-WUSF above with ε-Bernoulli percolation. Since the ε-Bernoulli percolation can be trivially coupled to contain an ε-WUSF, Theorem 1.1 is strictly stronger. The proof is an adaptation of a "spatial" version of the argument in [4] and follows arguments of [6]; see Section 3.

The above theorem applies when Λ itself is an independent WUSF and $\varepsilon = 1$, giving the following answer to a question posed to us by Peleg Michaeli.

Corollary 1.2. *The union of two independent samples of the* WUSF *on* $\mathbb{Z}^d$ *with* $d \geq 3$ *is almost surely connected and transient.*

Unions of independent spanning trees have origins in the computer science literature, where the union of k independent uniform spanning trees is called a *k-splicer* (see [7]). These were the first efficient constructions of *sparsifiers* – sparse graphs of an ambient graph which well approximate its spectrum. It is therefore of interest to understand what other properties are shared by the ambient graph and unions of WUSFs. Since the USF is a *critical* statistical physics model, the union of independent WUSF and ε-WUSF should be *supercritical* and hence "inherit" the structure of the ambient graph. Theorem 1.1 shows this is the case in $\mathbb{Z}^d$.

It is relatively easy to show that when G is a transitive unimodular nonamenable graph (see [11] for definitions) almost surely each component of the union of the WUSF with an independent ε-WUSF is transient. Indeed, one can readily repeat the proof of [2, Theorem 13.7] to show that when $\varepsilon > 0$ is small enough, this union, denoted here by W, has infinitely many components, and each has infinitely many ends. Therefore by Lemma 8.35 in [11] there is an invariant random subforest $\mathcal{F} \subset W$ such that almost surely each infinite cluster K of W contains a tree of $\mathcal{F}$ with infinitely many ends. By Corollary 8.20 in [11], any such tree has $p_c < 1$ and in particular it is transient by Theorem 3.5 in [11]. By Rayleigh's monotonicity principle, each connected component of W is also transient. We conjecture that this behavior holds in general.

Conjecture 1.3. *On any transitive transient graph and for any $\varepsilon > 0$, every connected component of the union of the* WUSF *and an independent* ε-WUSF *is almost surely transient.*

It is not hard to directly argue that on any bounded degree graph and any $p < 1$ there exists an integer $k \geq 1$ such that the union of k independent WUSFs dominates p-Bernoulli percolation on the graph (one can also use [10]). Hence by [9] we deduce that for any transitive transient graph there exists an integer k such that almost surely each component of the union of k independent WUSFs is transient and hence Conjecture 1.3 asks whether the same holds for "$k = 1 + \varepsilon$". It is the analogue of [3, Conjecture 1.7] stating that in the same setup of Conjecture 1.3, for any $p > p_c$ all infinite p-Bernoulli percolation clusters are transient almost surely.

We conclude this section with two open questions related to Theorem 1.1 that are again based on the intuition that such unions should be "supercritical". In the case of supercritical Bernoulli percolation on $\mathbb{Z}^d$ it is shown in [1] that the spectral dimension of the infinite cluster equals d almost surely, and in [5, 12, 15] it is shown that almost surely the cluster is such that the simple random walk on it diffuses to Brownian motion. It is thus natural to ask the following.

Question 1.4. Consider the union of independent WUSF and ε-WUSF on $\mathbb{Z}^d$.

1. Is the spectral dimension of the union equal to d almost surely?
2. Does the simple random walk diffuse to d-dimensional Brownian motion almost surely?

1.1 Box percolation

In the proof of Theorem 1.1 we will consider a slightly more general structure, for which we need the notion of *box percolation*.

Definition 1.5 ((k,ε)-box percolation)**.** *Let $d \geq 1$. Given $k \geq 1$ and $\varepsilon \in (0,1)$ we sample a percolation configuration $\varphi_k(\varepsilon)$ as follows. Firstly, for each $k \geq 1$ we write*

$$B_k = [-k,k]^d$$

and for each $z = (z_1,\ldots,z_d) \in (2k\mathbb{Z})^d$ write

$$B_k^z = [-k,k]^d + z,$$

and let $E(B_k^z)$ denote its induced edge set. We also let Q_k^z denote the subgraph of B_k^z with the edgeset

$$\{(x,y) \subset E(B_k^z) : \nexists\, \ell \in \{1,\ldots,d\} \text{ such that } x_\ell = y_\ell = z_\ell - k\}.$$

Then, independently for every $z \in (2k\mathbb{Z})^d$, choose a uniform edge from Q_k^z and declare it open with probability ε. Let $\varphi_k(\varepsilon)$ be the set of open edges.

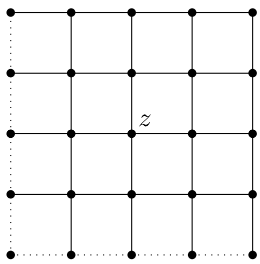

Figure 1: Q_2^z in $\mathbb{Z}^2$, dotted edges are not in Q_2^z.

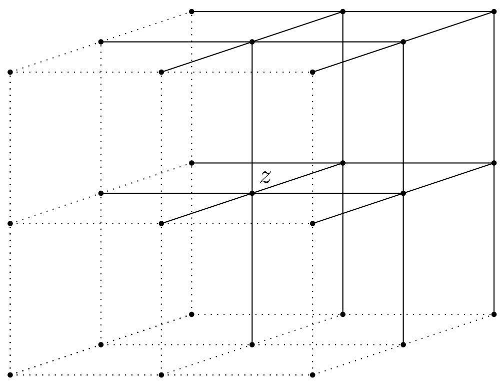

Figure 2: Q_1^z in $\mathbb{Z}^3$, dotted edges are not in Q_1^z.

The rather odd choice of edgeset in Definition 1.5 is made to guarantee two properties: (i) the sets $\{E(Q_k^z)\}_{z\in(2k\mathbb{Z})^d}$ form a partition of the edges of $\mathbb{Z}^d$ and (ii) for any cycle in $\mathbb{Z}^d$ there exists $z \in (2k\mathbb{Z})^d$ such that at least two edges of the cycle belong to $E(Q_k^z)$. This is proved in Claim 2.2 and is used in Lemma 2.3 to show that for all $\varepsilon > 0$ and all $k \geq 1$ the ε-WUSF stochastically dominates $\varphi_k(\varepsilon/2d)$. As a result, to prove Theorem 1.1 it will in fact be sufficient to prove the following theorem.

Theorem 1.6. *Let $d \geq 3$. Take any $k \geq 1$ and any $\varepsilon \in (0,1]$. Let Λ be an everywhere percolating subgraph of $\mathbb{Z}^d$. Then, almost surely, $\Lambda \cup \varphi_k(\varepsilon)$ is connected and transient.*

Actually, we remark that for the proof of Theorem 1.1, proving that $\Lambda \cup \varphi_1(\varepsilon)$ is connected and transient instead of showing it for every $k \geq 1$ would suffice.

1.2 Organization

We first show in Section 2 how Theorem 1.1 can be formally deduced from Theorem 1.6. In Section 3 we prove Theorem 1.6.

2 Proof of Theorem 1.1 given Theorem 1.6

For the rest of this section we denote by $\mathcal{F}$ a sample of the WUSF and by $\mathcal{F}_\varepsilon$ a sample of the ε-WUSF. Also recall the definition of $\varphi_k(\varepsilon)$ in Definition 1.5. To show that Theorem 1.1 follows from Theorem 1.6, we will show that $\mathcal{F}$ stochastically dominates $\varphi_k(1/2d)$. This will follow by the two claims below.

Claim 2.1. *Let H be a (possibly infinite) set of edges of $\mathbb{Z}^d$ that does not contain a cycle. Then, there is an ordering $\{e_n\}_{n\geq 1}$ of H such that for all $n \geq 1$ and any partition of $\{e_1, \dots, e_{n-1}\}$ into two sets A and B we have that*

$$\mathbb{P}(e_n \in \mathcal{F} \mid A \subseteq \mathcal{F}, B \cap \mathcal{F} = \emptyset) \geq \frac{1}{2d}.$$

Proof. Note that since the edges H do not contain a cycle, the edge-set H forms a forest. Hence we can order H by first ordering the edges in every tree T of H in such a way that at each step we discover a new vertex, and then concatenate the orderings for all the trees arbitrarily. Write $\{e_n\}_{n\geq 1}$ for this ordering of H. Then for every $n \geq 1$ and A and B partitioning $\{e_i\}_{i=1}^{n-1}$, the edge e_n has at least one endpoint which is not an endpoint of any of the edges in $A \cup B$. Denote this endpoint by v_n and the other one by u_n.

Then, by Kirchhoff's formula [11, Section 4.2 and Section 10.2] and the spatial Markov property [2, Proposition 4.2], we have for all $1 \leq k \leq n$ that

$$\mathbb{P}(e_n \in \mathcal{F} \mid A \subset \mathcal{F}, B \cap \mathcal{F} = \emptyset) = R_{\text{eff}}^{(\mathbb{Z}^d/A)\setminus B}(e_n),$$

where the right hand side denotes the wired effective resistance between the endpoints of e_n in the network $(\mathbb{Z}^d/A) \setminus B$ which is obtained from a copy of $\mathbb{Z}^d$ in which each of the edges in A have been contracted and each of the edges in B have been removed. Note that $\deg(v_n) = 2d$ in this graph so by the Nash-Williams inequality [11, (2.13)] this resistance is at least $\frac{1}{2d}$. ☐

Claim 2.2. *Fix $k \geq 1$. For any edge $e \in \mathbb{Z}^d$ there exists a unique $z \in (2k\mathbb{Z})^d$, which we denote $g(e)$, such that e is contained in Q_k^z. Furthermore, any cycle $(e_1, \dots, e_n)$ in $\mathbb{Z}^d$ contains at least two edges e_i, e_j with $g(e_i) = g(e_j)$.*

Proof. For any vertex $u \in \mathbb{Z}^d$ write r_u for the number of coordinate of u that are of the form $k + (2k)\mathbb{Z}$. Let $e = (u, v)$ be an edge and assume without loss of generality that $r_u \leq r_v$. For any coordinate $\ell \in \{1, \dots, d\}$ there exists a unique $n_\ell \in \mathbb{Z}$ such that $u_\ell - 2kn_\ell \in (-k, k]$, we claim that $z = (2kn_1, \dots, 2kn_\ell)$ is the unique desired z. Indeed, for any coordinate ℓ, if $|z_\ell - u_\ell| < k$, then $|z_\ell - v_\ell| \leq k$ since u and v are neighbors. If $|z_\ell - u_\ell| = k$ (i.e., if $u_\ell = z_\ell + k$) we must have that $u_\ell = v_\ell$ since $r_u \leq r_v$. Hence $(u, v) \subseteq B_k^z$. By our choice of z we have that $u_\ell > z_\ell - k$ for all ℓ and therefore $(u, v) \in Q_k^z$. Uniqueness of z follows immediately since the sets $\{Q_k^z\}_{z\in(2k\mathbb{Z})^d}$ are disjoint.

For the second assertion write $v_0, \dots, v_{n-1}, v_n = v_0$ for the vertices of the cycle, so that $e_i = (v_{i-1}, v_i)$ (we consider the coordinates mod n). For every $j \in \{0, \dots, n\}$, write $r(j)$ for r_{v_j}. Note that $|r(j) - r(j+1)| \leq 1$ for all j and that $r(0) = r(n)$. Let i be a global weak minimum of $r : \{0, \dots, n\} \to \{0, 1, \dots, \}$, so that $r(i) \leq r(i+1) \wedge r(i-1)$. From our description of z in the previous paragraph we obtain that $g(e_i) = g(e_{i+1})$. ☐

We now combine the previous two claims to obtain the desired stochastic domination.

Lemma 2.3. *Let $d \geq 1$ and $k \geq 1$. Then for any $\varepsilon > 0$ the forest $\mathcal{F}_\varepsilon$ stochastically dominates $\varphi_k(\frac{\varepsilon}{2d})$.*

Proof. It suffices to prove for $\varepsilon = 1$; the general case follows by applying the same ε-Bernoulli percolation to $\mathcal{F}$ and $\varphi_k(\frac{1}{2d})$. To sample $\varphi_k(\frac{1}{2d})$, for each $z \in (2k\mathbb{Z})^d$ we sample an independent uniform edge from Q_k^z and denote this collection of edges by H. Note that H does not contain any cycles by Claim 2.2. Therefore, by revealing the edges of $\mathcal{F}\cap H$ edge by edge according to the ordering obtained from Claim 2.1 we deduce that $\mathcal{F}\cap H$ dominates $\frac{1}{2d}$-Bernoulli percolation on H, and the latter is precisely $\varphi_k(\frac{1}{2d})$. □

3 Proof of Theorem 1.6

We start with some notation. As in Definition 1.5, for $n \geq 1$ and $z \in \mathbb{Z}^d$ we let B_n denote the box $[-n,n]^d$ and let $B_n^z = B_n + z$. In addition, when $m \leq n$ we let $A_{m,n}$ denote the annulus $B_n \setminus B_m$. For a subset of vertices $K \subset \mathbb{Z}^d$, the *edge boundary* or simply the *boundary* $\partial_E K$ of K is defined to be the set of edges that connect K to its complement $\mathbb{Z}^d \setminus K$. Finally, for $z \in \mathbb{Z}^d$ and $k \geq 1$ we take Q_k^z as in Definition 1.5.

As explained in the introduction, the proof of Theorem 1.6 follows an existing proof of Contreras, Martineau and Tassion [6], relying on techniques from Benjamini-Tassion [4], for the case when $\varphi_k(\varepsilon)$ is replaced by an ε-Bernoulli percolation. By standard renormalization techniques, it is in fact sufficient to prove that there exist constants $c > 0$ and $C < \infty$ such that for all Λ and all sufficiently large n (cf. [4, Lemma 1.1])

$$\mathbb{P}(\forall x, y \in B_n, x \text{ is connected to } y \text{ in } (\Lambda \cup \varphi_k(\varepsilon)) \cap B_{2n}) \geq 1 - C\exp(-c\sqrt{n}). \tag{3.1}$$

To establish (3.1) in the case of Bernoulli percolation, Benjamini and Tassion apply a technique known as *sprinkling* to the component graph of the connected clusters of Λ inside a large box; more specifically, they consider the effect of adding an ε-percolation to this component graph by instead adding $4d$ independent $\frac{\varepsilon}{4d}$-percolations. At each step, they showed that there is a high probability that the number of connected components decreases by a factor of $n^{1/4}$. We cannot directly apply the same strategy since we cannot decompose $\varphi_k(\varepsilon)$ into independent copies of $\varphi_k(\frac{\varepsilon}{4d})$; however we can instead use a form of "spatial sprinkling" by considering $\varphi_k(\varepsilon)$ on a sequence of roughly $\log(n)$ disjoint annuli, decreasing the number of connected components by a factor of $1/2$ when we sprinkle on each annulus (similarly to [6, Section 8]).

Their proof strategy relies on the fact that there are many edges which could merge components when adding percolation and the fact that the percolation on each of these edges are independent. This is no longer the case for box percolation, but this can also be easily overcome using the geometry of $\mathbb{Z}^d$: since the components must reach infinity this means that their boundary is large, and hence must contain many edges in distinct Q_k^z.

3.1 Proof of Theorem 1.6 assuming (3.1)

Write $\vec{e_i}$ for the i-th unit vector in $\mathbb{Z}^d$ and consider the following random field $(X_s^\Lambda)_{s\in\mathbb{Z}^d}$. For each $s \in \mathbb{Z}^d$ let $X_s^\Lambda = 1$ if and only if for each $y \in \{ns \pm n\vec{e_i}, i = 1,\dots,d\}$ there is a finite path in $\Lambda \cup \varphi_k(\varepsilon)$ connecting ns and y that also lies entirely in the box B_{2n}^{ns}. Let $p = p^\Lambda(n) = \inf_{s\in\mathbb{Z}^d} \mathbb{P}\big(X_s^\Lambda = 1\big)$. By (3.1), we have that $p^\Lambda(n) \to 1$ as $n \to \infty$ (in fact uniformly over choices of Λ) and moreover, provided that $n > 2k$, we have that X_s^Λ and X_t^Λ are independent whenever $||s-t||_\infty > 4$.

Hence by [10, Corollary 1.4] we deduce that $(X_s^\Lambda)_{s\in\mathbb{Z}^d}$ dominates a supercritical Bernoulli site percolation for all sufficiently large n. Since the infinite cluster of a supercritical Bernoulli site percolation on $\mathbb{Z}^d$ is transient for $d \geq 3$ [8], it follows that there is a transient connected subgraph formed by the open sites (open means that $X_s^\Lambda = 1$) of the random field $(X_s^\Lambda)_{s\in\mathbb{Z}^d}$. By the definition of the random field, this subgraph is roughly equivalent to a subgraph of $\Lambda \cup \varphi_k(\varepsilon)$ (see definition above [11, Theorem 2.17]. Since transience is preserved under rough equivalences [11, Theorem 2.17], we deduce that this subgraph of $\Lambda \cup \varphi_k(\varepsilon)$ is transient and hence by Rayleigh's monotonicity principle the connected graph $\Lambda \cup \varphi_k(\varepsilon)$ is also transient. □

3.2 Proof of (3.1)

We will present a proof of (3.1) which is based on the adaptation of the argument in [4] made in [6]. To this end, we will look at our everywhere percolating graph Λ, restricted to B_{2n}. Let $c > 0$ be a constant (depending only on d) that we will choose later, and write $m = \lfloor c \log(n) \rfloor$ and $b_j = n + j \cdot \frac{n}{c \log n}$ for every $j \in \{0, \dots, m\}$. Furthermore, define the annuli $A_j = A(b_j, b_{j+1} - k)$ for every $j \in \{0, \dots, m-1\}$. Note that the $\{\varphi_k \cap A_j\}$ are independent. We will sample them one by one, starting from $Y_{m-1}(\varepsilon) := A_{m-1} \cap \varphi_k(\varepsilon)$ to $Y_0(\varepsilon) := A_0 \cap \varphi_k(\varepsilon)$. Finally, for every $r < 2n$ and every graph (not necessarily related to Λ) $H \subseteq B_{2n}$, we will define $N_r(H)$ to be the set of connected components in H intersecting both B_r and ∂B_{2n}. We note that as we usually deal with everywhere percolating graphs Λ, every component of $\Lambda \cap B_{2n}$ intersects the boundary of B_{2n}). Then, for every $j \in \{0, \dots, m-1\}$, set

$$M_j(\varepsilon) = N_{b_j}\left((\Lambda \cap B_{2n}) \bigcup \left(\cup_{r=j}^{m-1} Y_j(\varepsilon) \right) \right).$$

That is, $M_j(\varepsilon)$ is the set of connected components intersecting B_{b_j} after adding the box-percolation in the annuli $A_j, \dots, A_{m-1}$.

The proof is based on the following lemma:

Lemma 3.1. *Let $d \in \mathbb{N}$, let $k \in \mathbb{N}$, let $n \in \mathbb{N}$ and let $\varepsilon > 0$. Let m and $M_j(\varepsilon)$ be as in the previous paragraph for every $j \in \{0, \dots, m-1\}$. Then, for every $j \in \{0, \dots, m-1\}$.*

$$\mathbb{P}\left(|M_j(\varepsilon)| > \max\left\{ 1, \frac{|M_{j+1}(\varepsilon)|}{2} \right\} \right) \leq (2n)^d \exp\left(-\frac{\varepsilon\sqrt{n}}{d^2(2k)^{2d}} \right). \tag{3.2}$$

Proof. For a given graph H, note that $N_{b_j}(H) \subseteq N_{b_{j+1}}(H)$. We will start by taking

$$H = \left((\Lambda \cap B_{2n}) \bigcup \left(\cup_{r=j+1}^{m-1} Y_r(\varepsilon) \right) \right)$$

and look at components of $N_{b_j}(H)$. If $N_{b_j}(H)$ contains only one component, we have nothing to prove (as $M_j(\varepsilon) = N_{b_j}(H \cup Y_j(\varepsilon))$). Else, we claim that with high probability, every component C of $N_{b_j}(H)$ is merged when adding the box-percolation on A_j. More precisely, we will show that when adding $Y_j(\varepsilon)$, every such C is connected to at least one more $C' \in N_{b_{j+1}}(H) = M_{j+1}(\varepsilon)$ (note that we used here b_{j+1} and not b_j).

So, let us assume that C is a connected component in $N_{b_j}(H)$ and that it is not trivial, that is, there is at least one more component in $N_{b_j}(H)$. As C crosses the annulus A_j, there must be at least one edge in every level r in the annulus A_j between C and its complement. In particular, the size of C's edge boundary intersected with A_j is at least the radius of A_j, which is at least $n/c \log(n) - k$. For n large enough, this is larger than, say, $\sqrt{n}$. As every box of the form Q_k^z has at most $d(2k)^d$ edges, we can find a subset of at least $\sqrt{n}/(d(2k)^d)$ boundary edges, each belonging to distinct Q_k^z. If one of these edges are open when adding $Y_j(\varepsilon)$, then there is an edge in $Y_j(\varepsilon)$ between C and another component in $N_{b_{j+1}}(H)$ and hence C is merged. It follows that the probability that C does not merge when adding $Y_j(\varepsilon)$ is bounded from above by the probability that all these $\sqrt{n}/(d(2k)^d)$ edges are closed. This probability is bounded by

$$\mathbb{P}(C \text{ does not merge}) \leq \left(1 - \frac{\varepsilon}{d(2k)^d} \right)^{\frac{\sqrt{n}}{d(2k)^d}} \leq \exp\left(-\frac{\varepsilon\sqrt{n}}{d^2(2k)^{2d}} \right).$$

Therefore, as the number of components is bounded by the number of vertices in B_{b_j}, we have that

$$\mathbb{P}\big(\exists C \in N_{b_j}(H) \text{ such that } C \text{ does not merge}\big) \le (2n)^d \exp\left(-\frac{\varepsilon\sqrt{n}}{d^2(2k)^{2d}}\right). \tag{3.3}$$

Finally, if indeed $N_{b_j}(H)$ has more than one component, we observe that under the event that all components merge we have

$$|M_j(\varepsilon)| \le \frac{|M_{j+1}(\varepsilon)|}{2}. \tag{3.4}$$

Indeed, if we take some component C in $M_j(\varepsilon)$, we have that there exists some $C_1 \in N_{b_j}(H) \subseteq N_{b_{j+1}}(H)$ such that $C_1 \subseteq C$. Also, since this C_1 was merged, there exists $C_2 \in N_{b_{j+1}}(H)$ (where $C_2 \neq C_1$) that it was merged with, that is, $C_2 \subseteq C$. Hence, C contains at least two components of $N_{b_{j+1}}(H)$. This means that every component in $M_j(\varepsilon)$ contains at least two components of $M_{j+1}(\varepsilon)$, yielding (3.4). □

Corollary 3.2. *Let $k, d \in \mathbb{N}$ and let $\varepsilon > 0$. Then, there exist $c, C > 0$ such that for all $n \in \mathbb{N}$ we have that*

$$\mathbb{P}(\forall x, y \in B_n, x \textit{ is connected to } y \textit{ in } (\Lambda \cup \varphi_k(\varepsilon)) \cap B_{2n}) \ge 1 - C\exp(-c\sqrt{n}).$$

Proof. Let $k, d \in \mathbb{N}$ and let $\varepsilon > 0$. Take n large enough such that Lemma 3.1 holds and apply it iteratively for $j = m-1$ until $j = 0$. The probability that there exists some j such that the event in (3.2) does not hold is bounded by $c\log(n)(2n)^d \cdot \exp\left(-\frac{\varepsilon\sqrt{n}}{d^2(2k)^{2d}}\right)$. Moreover, when the event holds for all $j \in \{0, \dots, m-1\}$, we have that the number of components in $N_{b_0}(\varphi_k(\varepsilon) \cup \Lambda)$ is at most 2^{-m} the number of components in $N_{b_{m-1}}(\Lambda)$. As the original number of components is bounded by the size of B_{2n} which is bounded by n^d, we can choose c large enough such that $2^{-c\log(n)} < n^d$, and hence we obtain that $N_{b_0}(\varphi_k(\varepsilon) \cup \Lambda)$ contains only one component. This in turn means that every two vertices in B_n are in the same connected component in $\Lambda \cup \varphi_k(\varepsilon)$, finishing the proof. □

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Acknowledgments. We would like to thank Peleg Michaeli for posing the question of Theorem 1.2. We also thank him, Matan Harel and Ofir Karin for useful discussions, and the anonymous referee for a careful reading and comments which greatly simplified the proof.