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Berry-Esseen theorem for random walks conditioned to stay positive*

Wenming Hong† Mingyang Sun‡

Abstract

We establish a Berry-Esseen theorem for random walks conditioned to stay positive under $\mathbb{P}^+$ (the probability by Doob's h-transform), which quantifies the convergence rate in the Kolmogorov distance of the central limit theorem proved by Bryn-Jones and Doney (2006). Our approach is based on a recent analogous result by Grama and Xiao (2021) for random walks conditioned to stay positive over a finite time interval.

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1 Introduction and main results

We are interested in the asymptotic behavior of random walks conditioned to stay positive, which have been studied extensively in recent years. To our knowledge, the phrase "random walks conditioned to stay positive" has at least two different interpretations. Firstly, we consider the random walk conditioned to stay positive over a finite time interval; this is a discrete version of meander. It is well-known (see Iglehart [11], Bolthausen [2] and Doney [7]) that if the random walk is in the domain of attraction of a standard normal law, a suitably scaled version of this process converges weakly to a Brownian meander, which is the so-called Iglehart's invariance principle. Later, Caravenna [4], Vatutin and Wachtel [15] obtained the local limit theorem under conditions where the random walk is attracted to a normal law and stable law respectively. Recently, Grama and Xiao [9] proved the corresponding Berry-Esseen theorem which gives the convergence rate of Iglehart's result.

The second interpretation involves conditioning on the event that the random walk never goes negative, and so can be thought of as a discrete version of the Bessel process. When the random walk oscillates, the conditioning event has zero probability, and one can make sense of this conditioned process by means of Doob's h-transform (see Bertoin and Doney [1]). Bryn-Jones and Doney [3] proved that a suitably scaled version of this process converges weakly to a Bessel process if the random walk is in the domain of

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†School of Mathematical Sciences & Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. China. E-mail: wmhong@bnu.edu.cn
‡Corresponding author. School of Mathematical Sciences & Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. China. E-mail: sunmingyang@mail.bnu.edu.cn

attraction of a standard normal law. Later, Caravenna and Chaumont [5] extended this invariance principle to the stable case in a more straightforward way by exploiting the absolute continuity with the meander process. For the law of the iterated logarithm of this model, we refer the readers to Hambly, Kersting and Kyprianou [10]. Compared to meander case, the main purpose of this paper is to derive a Berry-Esseen theorem for random walks conditioned to stay (always) positive in the sense of h-transform.

Let $S = (S_n)_{n\geq 0}$ denote a random walk in $\mathbb{R}$ with starting point zero, that is, $S_0 = 0$, and for $n \geq 1$, $S_n = \sum_{i=1}^n X_i$, where $\{X_n : n \geq 1\}$ are i.i.d. random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}(X_1^2) =: \sigma^2 \in (0, \infty)$. The strict descending ladder process is defined recursively as follows:

$$\tau_0 = H_0 = 0, \ \ \tau_n = \inf\{\, k > \tau_{n-1} : \ S_k < S_{\tau_{n-1}} \,\}, \ \ H_n = -S_{\tau_n}, \quad n \geq 1,$$

and we write $\tau := \tau_1$ for convenience. Let $V(x)$ denote the renewal function associated with $(H_n)_{n\geq 0}$, which is a positive function defined by

$$V(x) = \sum_{n\geq 0} \mathbb{P}(H_n \leq x), \quad x \geq 0. \tag{1.1}$$

Note that $V(x)$ is the expected number of descending ladder heights which are $\leq x$. It is well-known that V is harmonic for the sub-Markov process obtained by killed $(S_n)_{n\geq 0}$ when entering the negative half-line (see Tanaka [14]), that is,

$$V(x) = \mathbb{E}\left[\, V(x + S_1); \ x + S_1 \geq 0 \,\right], \quad x \geq 0. \tag{1.2}$$

Next we introduce a change of measure $\mathbb{P}^+$ which is defined by the well-known Doob's h-transform: for any $n \in \mathbb{N}$ and $A \in \sigma(S_1, \cdots, S_n)$,

$$\mathbb{P}^+(A) := \mathbb{E}\left[\, V(S_n); \ A \cap \{\tau > n\} \,\right]. \tag{1.3}$$

According to Kolmogorov's extension theorem and the harmonic property of V, it is easy to see that $\mathbb{P}^+$ is well defined. The random walk $(S_n)_{n\geq 0}$ under the new probability $\mathbb{P}^+$ is called a random walk conditioned to stay positive, and this terminology is justified by the following weak convergence result (see Theorem 1 of Bertoin and Doney [1]):

$$\mathbb{P}^+(\cdot) = \lim_{n\to\infty} \mathbb{P}(\cdot|\tau > n). \tag{1.4}$$

Recently, Grama and Xiao (see Theorem 2.7 of [9]) establish a Berry-Esseen type theorem for random walks under the conditional probability $\mathbb{P}(\cdot|\tau > n)$, which quantifies the convergence rate in the Kolmogorov distance of Iglehart's central limit theorem.

Theorem A (Grama & Xiao, 2021)**.** Assume that $\mathbb{E}X_1 = 0$, $\mathbb{E}(X_1^2) =: \sigma^2 \in (0, \infty)$ and $\mathbb{E}(|X_1|^{2+\delta}) < \infty$ for some $\delta > 0$. Then there exists $\rho > 0$ such that as $n \to \infty$,

$$\sup_{x\geq 0} \left| \mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}} \leq x \,\middle|\, \tau > n \right) - \widetilde{\Phi}(x) \right| = O\left(n^{-\rho}\right), \tag{1.5}$$

where $\widetilde{\Phi}(x) = 1 - e^{-\frac{x^2}{2}}$, $x \geq 0$ is the Rayleigh distribution function.

Remark 1.1. Recently, the paper [9] has been accepted by AIHP, but the appendix contained the proof of (1.5) is omitted because of the limitation of the length. We refer the readers to the arXiv version for proofs. Furthermore, an upper bound for ρ is given by $\delta/(20 + 8\delta)$. Thus the error term in (1.5) is far from optimal compared with classical Berry-Esseen theorem for sums of i.i.d. variables, where $\rho = (\delta \wedge 1)/2$. At the moment giving the optimal error term under the current assumptions seems very delicate.

Our main result is a Berry-Esseen theorem for random walks conditioned to stay positive in the sense of h-transform, which quantifies the convergence rate in the Kolmogorov distance of the central limit theorem due to Bryn-Jones and Doney [3].

Theorem 1.2. *Assume that $\mathbb{E}X_1 = 0$, $\mathbb{E}(X_1^2) =: \sigma^2 \in (0,\infty)$ and $\mathbb{E}(|X_1|^{2+\delta}) < \infty$ for some $\delta > 0$. Then as $n \to \infty$,*

$$\sup_{x\geq 0}\left| \mathbb{P}^+\left(\frac{S_n}{\sigma\sqrt{n}} \leq x\right) - \Phi^+(x)\right| = O\left(n^{-\rho}\sqrt{\log n}\right), \tag{1.6}$$

where $\Phi^+(x) = \int_0^x \sqrt{\frac{2}{\pi}}\, y^2\, e^{-y^2/2}dy,\ x \geq 0$ and ρ is the constant in (1.5).

Remark 1.3. Note that (1.6) and (1.5) share almost the same convergence rate with a $\sqrt{\log n}$ gap, since the probability $\mathbb{P}^+$ and $\mathbb{P}(\cdot|\tau > n)$ are closely related to each other via an h-transform given by the harmonic function V, which is asymptotically linear and the error is under control. We will prove Theorem 1.2 by exploiting this absolute continuity relation and Grama and Xiao's result.

2 Preliminaries

The goal of this section is to give some auxiliary results which will be used to prove the Berry-Esseen theorem for random walks conditioned to stay positive in the sense of h-transform. We will denote the positive constants by $C_i,\ i \in \mathbb{N}$ which may change from line to line if a constant is not of our interest. Constants appearing in our claims and fixed throughout this paper will be denoted by c_i with $i \in \mathbb{N}$. To emphasize dependence on some variables, we put them in subscripts.

The following lemma gives the convergence rate of the elementary renewal theorem for the harmonic function $V(x)$, which plays an important role in the proof of our main theorem.

Lemma 2.1. *Assume that $\mathbb{E}X_1 = 0$ and $\mathbb{E}(|X_1|^{2+\delta}) < \infty$ for some $\delta \in (0,1]$. Then we have $\mathbb{E}H_1 \in (0,\infty)$, and there exists a constant $c_\delta > 0$ such that for any $x \geq 1$,*

$$\left|\frac{V(x)}{x} - \frac{1}{\mathbb{E}H_1}\right| \leq c_\delta\, x^{-\delta}. \tag{2.1}$$

Proof. According to Corollary 2 of Doney [6], we have $\mathbb{E}(H_1^{1+\delta}) < \infty$ under the moment condition $\mathbb{E}(|X_1|^{2+\delta}) < \infty$. Then by Corollary 4 of Rogozin [13], it follows that

$$\left|\frac{V(x)}{x} - \frac{1}{\mathbb{E}H_1}\right| = O\left(x^{-\delta}\right),\ \text{as } x \to \infty.$$

The desired result follows. □

We also need the following result that describes the large deviation probability for the random walk $(S_n)_{n\geq 0}$ conditioned on the event $\{\tau > n\}$.

Lemma 2.2. *Assume that $\mathbb{E}X_1 = 0$ and $\mathbb{E}(|X_1|^{2+\delta}) < \infty$ for some $\delta > 0$. Then there exist some constants $c_1, c_2 > 0$ such that for any $n \geq 1$ and $y \geq c_1\sqrt{n\log n}$,*

$$\mathbb{P}\left(S_n \geq y\,|\,\tau > n\right) \leq c_2\, n\, y^{-(2+\delta)}. \tag{2.2}$$

Proof. This relation, under the assumption that X_1 is in the domain of attraction of an α-stable law, was proved by Doney and Jones (see Lemma 1.2 of [8]). Note that the proof therein still works if we replace Proposition 0.1 of [8] with the following claim: there exists a constant $c_3 > 0$ such that for any $n \geq 1$ and $y \geq \sqrt{n}$,

$$\mathbb{P}(S_n \geq y) \leq c_3\, n\, y^{-(2+\delta)}. \tag{2.3}$$

In fact, by the moment condition $\mathbb{E}(|X_1|^{2+\delta}) < \infty$, it follows that

$$\mathbb{P}(X_1 > t) \le C\, t^{-(2+\delta)}, \quad \forall\, t \ge 0, \tag{2.4}$$

for some constant $C > 0$. Then applying the Fuk-Nagaev inequality for tail probabilities of sums of i.i.d. variables (see Corollary 1.7 of Nagaev [12]), we have for any $u, v > 0$ and $t \ge 2$,

$$\mathbb{P}(S_n \ge u) \le n\mathbb{P}(X_1 \ge v) + \exp\left\{-\frac{2u^2}{(t+2)^2 e^t \sigma^2}\right\} + \left\{\frac{(t+2)n\mathbb{E}|X_1|^t}{tuv^{t-1}}\right\}^{\frac{tu}{(t+2)v}}. \tag{2.5}$$

Putting here $u = y, v = \frac{(2+\delta)y}{4+\delta}, t = 2+\delta$ and combing (2.4), we get (2.3). The rest of the proof is in the same vein as Lemma 1.2 of Doney and Jones [8]. □

3 Proof of Theorem 1.2

Without loss of generality, we assume $\sigma^2 = 1$. Note that we need to show that (1.6) is valid uniformly for $x \ge 0$. To this aim, we split up the range of supremum into 3 parts:

$$\left\{0 \le x \le n^{-1/6}\right\}, \ \left\{n^{-1/6} < x \le A\sqrt{\log n}\right\} \ \text{and} \ \left\{x > A\sqrt{\log n}\right\},$$

for some constant $A \ge \max\{2, c_1\}$. We first consider the case $0 \le x \le n^{-1/6}$.

Lemma 3.1. *Assume that* $\mathbb{E}X_1 = 0$ *and* $\mathbb{E}(X_1^2) = 1$*. Then as* $n \to \infty$*,*

$$\sup_{0 \le x \le n^{-1/6}} \left| \mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} \le x\right) - \Phi^+(x) \right| = O\left(n^{-1/2}\right). \tag{3.1}$$

Proof. We first consider the lattice case and assume that X_1 is (h, a)-lattice, that is, the h is the maximal number such that the support of the distribution of X_1 is contained in the set $a + h\mathbb{Z}$ for some $a \in [0, h)$. According to Theorem 6 of Vatutin and Wachtel [15], it follows that uniformly in $y \in [0, o(\sqrt{n})] \cap (an + h\mathbb{Z})$,

$$\mathbb{P}(S_n = y, \tau > n) \le \frac{C_1(1+y)}{n^{3/2}}. \tag{3.2}$$

By the renewal theorem of V, we have $V(y) \sim y/\mathbb{E}H_1$ as $y \to \infty$, then there exist some positive constants c_4, c_5 such that

$$c_4(1+y) \le V(y) \le c_5(1+y), \ \text{ for any } y \ge 0. \tag{3.3}$$

Denote $A_n := [0, n^{1/3}] \cap (an + h\mathbb{Z})$, then by the definition of measure change $\mathbb{P}^+$, we have for any $x \le n^{-1/6}$,

$$\begin{aligned}
\mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} \le x\right) &= \mathbb{E}\left[V(S_n);\ \frac{S_n}{\sqrt{n}} \le x,\ \tau > n\right] = \int_0^{\sqrt{n}x} V(y)\, \mathbb{P}(S_n \in dy, \tau > n) \\
&\le c_5 \int_0^{n^{1/3}} (1+y)\, \mathbb{P}(S_n \in dy, \tau > n) \le C_2 \sum_{y \in A_n} (1+y)\, \mathbb{P}(S_n = y, \tau > n) \\
&\le C_1 C_2 \sum_{y \in A_n} \frac{(1+y)^2}{n^{3/2}} \le C_3\, n^{-1/2}.
\end{aligned} \tag{3.4}$$

On the other hand, applying Theorem 4 of Vatutin and Wachtel [15] in the non-lattice case, we obtain that uniformly in $y \in [0, o(\sqrt{n})]$,

$$\mathbb{P}(S_n = [y, y+1), \tau > n) \le \frac{C_4(1+y)}{n^{3/2}}. \tag{3.5}$$

Hence by the definition of measure change $\mathbb{P}^+$ and (3.3), we have for any $x \le n^{-1/6}$,

$$\begin{aligned} \mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} \le x\right) &= \int_0^{\sqrt{n}x} V(y)\,\mathbb{P}(S_n \in dy, \tau > n) \le c_5 \int_0^{n^{1/3}} (1+y)\,\mathbb{P}(S_n \in dy, \tau > n) \\ &\le C_5 \sum_{k=0}^{n^{1/3}} (2+k)\,\mathbb{P}\left(S_n = [k, k+1), \tau > n\right) \\ &\le C_4 C_5 \sum_{k=0}^{n^{1/3}} \frac{(2+k)^2}{n^{3/2}} \le C_6\, n^{-1/2}. \end{aligned} \tag{3.6}$$

Note that for any $x \le n^{-1/6}$, we have

$$\Phi^+(x) = \int_0^x \sqrt{\frac{2}{\pi}}\, y^2\, e^{-y^2/2} dy \le C_7\, x^3 \le C_7\, n^{-1/2}. \tag{3.7}$$

Therefore we conclude the proof of this lemma by combing (3.4), (3.6) and (3.7). □

Next we turn to the case of $n^{-1/6} < x \le A\sqrt{\log n}$, and show that it makes a major contribution to (1.6).

Lemma 3.2. *Assume that the conditions of Theorem 1.2 are valid. Then as $n \to \infty$,*

$$\sup_{n^{-1/6} < x \le A\sqrt{\log n}} \left| \mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} \le x\right) - \Phi^+(x) \right| = O\left(n^{-\rho}\sqrt{\log n}\right), \tag{3.8}$$

where ρ is the constant in (1.5).

Proof. Denote $\tilde{\delta} := \delta \wedge 1$, then by Lemma 2.1, we have

$$\left| V(y) - \frac{y}{\mathbb{E}H_1} \right| \le C_1 (1+y)^{1-\tilde{\delta}}, \quad \forall\, y \ge 0, \tag{3.9}$$

which implies that

$$\begin{aligned} &\left| \mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} \le x\right) - \int_0^{\sqrt{n}x} \frac{y}{\mathbb{E}H_1} \mathbb{P}(S_n \in dy, \tau > n) \right| \\ &\le \int_0^{\sqrt{n}x} \left| V(y) - \frac{y}{\mathbb{E}H_1} \right| \mathbb{P}(S_n \in dy, \tau > n) \\ &\le C_1 \int_0^{\infty} (1+y)^{1-\tilde{\delta}}\, \mathbb{P}(S_n \in dy, \tau > n). \end{aligned} \tag{3.10}$$

Next we split up the range of the above integral into two parts, and first show that

$$\int_0^{n^*} (1+y)^{1-\tilde{\delta}}\, \mathbb{P}(S_n \in dy, \tau > n) = O\left(n^{-\tilde{\delta}/2}\right), \tag{3.11}$$

where $n^* = \sqrt{n/\log n}$. In fact, by (3.5) we have in the non-lattice case,

$$\begin{aligned} &\int_0^{n^*} (1+y)^{1-\tilde{\delta}}\, \mathbb{P}(S_n \in dy, \tau > n) \\ &\le C_2 \sum_{k=0}^{n^*} (2+k)^{1-\tilde{\delta}}\, \mathbb{P}\left(S_n = [k, k+1), \tau > n\right) \\ &\le C_3 \sum_{k=0}^{n^*} \frac{(2+k)^{2-\tilde{\delta}}}{n^{3/2}} \le C_4\, n^{-\tilde{\delta}/2}. \end{aligned} \tag{3.12}$$

The lattice case is in the same vein, thus (3.11) holds true. Furthermore, by (3.3) we have for any $y \geq n^*$,

$$(1+y)^{1-\tilde{\delta}} \leq c_4^{-1}\, n^{-\tilde{\delta}/2}(\log n)^{\tilde{\delta}/2}\, V(y), \tag{3.13}$$

which implies that

$$\begin{aligned}
&\int_{n^*}^{\infty} (1+y)^{1-\tilde{\delta}}\, \mathbb{P}(S_n \in dy, \tau > n) \\
&\leq c_4^{-1}\, n^{-\tilde{\delta}/2}(\log n)^{\tilde{\delta}/2} \int_{n^*}^{\infty} V(y)\, \mathbb{P}(S_n \in dy, \tau > n) \\
&\leq c_4^{-1}\, n^{-\tilde{\delta}/2}(\log n)^{\tilde{\delta}/2}.
\end{aligned} \tag{3.14}$$

Implementing the bound (3.11) and (3.14) into (3.10), we get that

$$\left| \mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} \leq x\right) - \int_0^{\sqrt{n}x} \frac{y}{\mathbb{E}H_1} \mathbb{P}(S_n \in dy, \tau > n) \right| = O\left(n^{-\tilde{\delta}/2}(\log n)^{\tilde{\delta}/2}\right). \tag{3.15}$$

Now we turn to the estimate of

$$\begin{aligned}
&\int_0^{\sqrt{n}x} \frac{y}{\mathbb{E}H_1} \mathbb{P}(S_n \in dy, \tau > n) \\
&= \frac{\sqrt{n}x}{\mathbb{E}H_1}\, \mathbb{P}(S_n \leq \sqrt{n}x, \tau > n) - \frac{1}{\mathbb{E}H_1} \int_0^{\sqrt{n}x} \mathbb{P}(S_n \leq y, \tau > n) dy.
\end{aligned} \tag{3.16}$$

Applying Theorem 2.7 from Grama and Xiao [9], it follows that for any $x \geq 0$ and $n \geq 1$,

$$\left| \mathbb{P}\left(S_n \leq \sqrt{n}x, \tau > n\right) - \frac{\sqrt{2}\,\mathbb{E}H_1}{\sqrt{\pi n}}\, \widetilde{\Phi}(x) \right| \leq C_5\, n^{-(\rho+1/2)}. \tag{3.17}$$

For the first term of (3.16), using the above bound, we have for $n^{-1/6} < x \leq A\sqrt{\log n}$,

$$\begin{aligned}
&\frac{\sqrt{n}x}{\mathbb{E}H_1}\, \mathbb{P}(S_n \leq \sqrt{n}x, \tau > n) \\
&= \sqrt{\frac{2}{\pi}}\, x\, \widetilde{\Phi}(x) + \frac{\sqrt{n}x}{\mathbb{E}H_1} \left(\mathbb{P}(S_n \leq \sqrt{n}x, \tau > n) - \frac{\sqrt{2}\,\mathbb{E}H_1}{\sqrt{\pi n}}\, \widetilde{\Phi}(x) \right) \\
&= \sqrt{\frac{2}{\pi}}\, x\, \widetilde{\Phi}(x) + O\left(n^{-\rho}\sqrt{\log n}\right).
\end{aligned} \tag{3.18}$$

For the second term of (3.16), note that

$$\begin{aligned}
\frac{1}{\mathbb{E}H_1} \int_0^{\sqrt{n}x} \mathbb{P}(S_n \leq y, \tau > n) dy &= \int_0^{\sqrt{n}x} \frac{\sqrt{2}}{\sqrt{\pi n}}\, \widetilde{\Phi}\left(\frac{y}{\sqrt{n}}\right) dy \\
&\quad + \frac{1}{\mathbb{E}H_1} \int_0^{\sqrt{n}x} \left(\mathbb{P}(S_n \leq y, \tau > n) dy - \frac{\sqrt{2}\,\mathbb{E}H_1}{\sqrt{\pi n}}\, \widetilde{\Phi}\left(\frac{y}{\sqrt{n}}\right) \right) dy.
\end{aligned} \tag{3.19}$$

Using the bound (3.17) again, it follows that for $n^{-1/6} < x \leq A\sqrt{\log n}$,

$$\begin{aligned}
&\left| \frac{1}{\mathbb{E}H_1} \int_0^{\sqrt{n}x} \left(\mathbb{P}(S_n \leq y, \tau > n) dy - \frac{\sqrt{2}\,\mathbb{E}H_1}{\sqrt{\pi n}}\, \widetilde{\Phi}\left(\frac{y}{\sqrt{n}}\right) \right) dy \right| \\
&\leq \frac{1}{\mathbb{E}H_1} \int_0^{\sqrt{n}x} C_5\, n^{-(\rho+1/2)} dy \leq C_6\, n^{-\rho}\sqrt{\log n}.
\end{aligned} \tag{3.20}$$

On the other hand, changing the variables and integrating by parts, we obtain that

$$\begin{aligned}\int_0^{\sqrt{n}x} \frac{\sqrt{2}}{\sqrt{\pi n}}\,\widetilde{\Phi}\left(\frac{y}{\sqrt{n}}\right)dy &= \sqrt{\frac{2}{\pi}}\int_0^x \widetilde{\Phi}(z)dz \\ &= \sqrt{\frac{2}{\pi}}\,x\,\widetilde{\Phi}(x) - \int_0^x \sqrt{\frac{2}{\pi}}\;z^2\,e^{-z^2/2}dz \\ &= \sqrt{\frac{2}{\pi}}\,x\,\widetilde{\Phi}(x) - \Phi^+(x). \end{aligned} \tag{3.21}$$

Combing this and (3.19), (3.20), we get that

$$\frac{1}{\mathbb{E}H_1}\int_0^{\sqrt{n}x} \mathbb{P}(S_n \le y, \tau > n)dy = \sqrt{\frac{2}{\pi}}\,x\,\widetilde{\Phi}(x) - \Phi^+(x) + O\left(n^{-\rho}\sqrt{\log n}\right). \tag{3.22}$$

Then implementing the estimate (3.18) and (3.22) into (3.16), it follows that

$$\int_0^{\sqrt{n}x} \frac{y}{\mathbb{E}H_1}\mathbb{P}(S_n \in dy, \tau > n) = \Phi^+(x) + O\left(n^{-\rho}\sqrt{\log n}\right). \tag{3.23}$$

Therefore we conclude the proof of Lemma 3.2 by virtue of (3.15) and (3.23). □

Finally, we show that the supremum of (1.6) taking in $x > A\sqrt{\log n}$ can be ignored.

Lemma 3.3. *Assume that the conditions of Theorem 1.2 are valid. Denote $\tilde{\delta} := \delta \wedge 1$, then as $n \to \infty$,*

$$\sup_{x > A\sqrt{\log n}} \left| \mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} \le x\right) - \Phi^+(x) \right| = O\left(n^{-\tilde{\delta}/2}\right). \tag{3.24}$$

Proof. Recall that we fix the constant A such that $A \ge \max\{2, c_1\}$, then integrating by parts yields that for any $x > A\sqrt{\log n}$,

$$\begin{aligned} 1 - \Phi^+(x) &\le \int_{A\sqrt{\log n}}^{\infty} \sqrt{\frac{2}{\pi}}\;y^2\,e^{-y^2/2}dy \\ &= \sqrt{\frac{2}{\pi}}A\,n^{-A^2/2}\sqrt{\log n} + \int_{A\sqrt{\log n}}^{\infty} \sqrt{\frac{2}{\pi}}\;e^{-y^2/2}dy \\ &\le C_1\,n^{-A^2/2}\sqrt{\log n} \le \frac{C_2}{n}. \end{aligned} \tag{3.25}$$

Hence it is sufficient to show that

$$\sup_{x > A\sqrt{\log n}} \left| \mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} > x\right) \right| = O\left(n^{-\tilde{\delta}/2}\right). \tag{3.26}$$

By Lemma 2.2, we have for any $y \ge A\sqrt{n\log n}$,

$$\mathbb{P}\left(S_n > y, \tau > n\right) \le C_3\,\sqrt{n}\,y^{-(2+\delta)}. \tag{3.27}$$

Then integrating by parts and using (3.3), we obtain that for any $x > A\sqrt{\log n}$,

$$\begin{aligned} &\mathbb{P}^+\left(\frac{S_n}{\sqrt{n}} > x\right) \le c_5 \int_{\sqrt{n}x}^{\infty} (1+y)\,\mathbb{P}(S_n \in dy, \tau > n) \\ &= c_5(1+\sqrt{n}x)\,\mathbb{P}(S_n > \sqrt{n}x, \tau > n) + c_5\int_{\sqrt{n}x}^{\infty} \mathbb{P}(S_n > y, \tau > n)dy \\ &\le C_4(1+\sqrt{n}x)\sqrt{n}(\sqrt{n}x)^{-(2+\delta)} + C_5\int_{\sqrt{n}x}^{\infty} \sqrt{n}\,y^{-(2+\delta)}dy \\ &\le C_6\,n^{-\delta/2}, \end{aligned} \tag{3.28}$$

which concludes the proof of this lemma. □

Therefore, we conclude the proof of Theorem 1.2 by combing the above 3 lemmas.

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