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Probabilistic representations of fragmentation equations

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Abstract: In this survey article, we present an overview of a large class of probabilistic representations of the fragmentation equation, and we develop and study the interconnections in between these representations. We focus on the stochastic process which represents the evolution of the mass of a typical particle which undergoes fragmentation in time. These probabilistic representations range from Markov chains to stochastic differential equations with jumps, and we aim at constructing how they are inter-related. In particular, we show how these representations can be used to develop easy to implement numerical methods.

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Contents

1. Introduction

Fragmentation plays a fundamental role in numerous applications arising both in natural science and in industrial processes. Most often, in these phenomena one needs to describe how the clusters present in the process evolve in time and more precisely how do they break into smaller clusters. This is an important topic in physics (engine reactions), astrophysics (formation of asteroids), in geophysics (landslide, avalanches), in the copper industry, etc.

For answering this question several issues can be concerned. We study in this paper the fragmentation equation which describes the time evolution of a particle system which is characterized by particle masses. We will consider here binary fragmentation procedure, that means that at some times a particle can

split into two particles of smaller masses. We suppose in our model that the described phenomenon preserves the global mass of the system.

Denote by $c(t,x)$ the concentration of particles of mass x in the system at time t. The fragmentation equation is the following deterministic integro-differential equation, for all $x \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$:

$$\begin{cases} \dfrac{\partial c(t,x)}{\partial t} = -F(x)c(t,x) + \displaystyle\int_0^{+\infty} c(t,x+y)F(x+y)\mathsf{G}(x+y,\mathrm{d}y) \\ c(0,x) = c_0(x), \end{cases} \tag{1.1}$$

where F is a function, called the *rate*, and $\mathsf{G}(x,\cdot)$ is a probability kernel with support in $(0,x)$.

In the first line of equation (1.1), the first term in the right hand side accounts for the disappearance of particles of mass x after a breakage and the second term specifies the mass of a randomly chosen fragment after the breakage.

A huge literature deals with this equation. Both deterministic and stochastic interpretations have been considered. In some of these papers the coagulation-fragmentation phenomenon is also treated. Existence results are obtained in [33, 63, 64, 71, 74]. The analytic treatment of fragmentation equations is usually dealt using the theory of semi-groups, which is sometimes difficult to handle due to the variety of behaviors of fragmentation kernels. An extensive account may be found in the book [9].

The main objective of this paper is to discuss and describe the various stochastic approaches of these equations and to highlight possible links between the stochastic tools that can be used. The stochastic approach is namely based on the conservation of mass observed in the physical system which induces naturally the construction of a random process with a distribution of the form $xc(x,t)\,\mathrm{d}x$. This representation leads to a natural interpretation for both discrete and continuous masses. Such analysis was started by A.N. Kolmogorov [60].

In particular, our probabilistic representations are derived one from the other, starting from the simplest one using Continuous Time Markov Chain as in [40] to end up with Stochastic Differential Equations with Jumps as in [44] and passing through Random Point Processes. As all these representations are equivalent, one may choose the most appropriate framework according to the problem to solve, as each representation comes with different tools and viewpoints.

We focus on generic results in the sense that we do not consider particular shapes of the kernels and their related properties. This is a wide subject. To keep things simple and to focus on applied results, we left aside probabilistic representations in terms of clouds of particles or in branching processes where the states space encodes all the particles' mass. Such representations require higher levels of abstraction. The book of J. Bertoin [12] (see also [13]) presents a detailed panorama on Continuous Time Markov Chain. A recent paper gives an interpretation of the fragmentation in terms of branching processes [16]. More precisely, the paper constructs a continuous time Markov branching process which is associated to the fragmentation equation and which lives in the state space $S^{\downarrow}$ of all fragmentation sizes, introduced by J. Bertoin in [12]. This

interpretation is applied further to a particular class of kernels (non smooth), connected to the avalanches and new results are obtained for this context [17]. A numerical algorithm, based on these stochastic interpretations, is developped in [18].

Coagulation phenomenon is usually studied with fragmentation. Yet, it leads to more intricate results as the coagulation induces some non-linear behaviors.

Related problems

Many other approaches are connected and developed for the fragmentation equation. We focus here on some directions without being exhaustive as the literature on this topic is very large and in particular we emphasize the probabilistic approach for these issues.

Our study concerns the integro-differential equation (1.1) and highlights classes of stochastic processes that can be linked to it.

This is completely different with respect to the works of J. Bertoin and coauthors where in some sense a development of the fragmentation is given by studying partitions and associated fragmentation chains [11, 12, 13, 48]. In particular important results on the asymptotic behavior, shattering and formation of dusts are obtained.

Many studies are devoted also to the coagulation-fragmentation equation. In some sense the enthusiasm and growing interest of the probabilistic approach for the Smoluchowski's coagulation equation goes back to the seminal paper of D. Aldous [3] which gave a rich review of the domain and its open problems. In particular D. Aldous mentioned already that the study of general kernels keeps difficult and gives rise to interesting mathematical questions.

The analysis of the coagulation-fragmentation model involving discrete masses with a probabilistic approach has been considered in many works as for example in I. Jeon [56] who considered the coagulation-fragmentation discrete model interpreted by a sequence of finite Markov chains and formulated a criterion on the finite chain which detect the gelation phenomena, B. Jourdain [57] who constructed a non-linear stochastic process and gave a stochastic algorithm with a constant number of particles in the discrete coagulation-fragmentation case.

Results on the probabilistic approach for coagulation-fragmentation processes are obtained in [44], where the interpretation is given by the solution of a stochastic differential equation with jumps.

Another interesting direction is given by the growth-fragmentation. In this setting two opposite dynamics, growth and fragmentation are balanced in the equation. This equation is a linear, partial integro-differential equation and models population dynamics in biology, physics, neurociences, etc. One of the main challenges is here the long-time behavior. We refer to the paper of J.A. Cañizo, P. Gabriel and H. Yoldaş [22] and the references therein for an interesting overview of the topic. We list here some probabilistic approaches which match our problem [15, 20, 23].

This formulation is also important as it shows that the binary fragmentation can be seen in our context in a more general framework. As we are following

the evolution of a typical particle the fragmentation can be interpreted as the splitting in the size of the particle that we follow and a "second" system that can be formed by one or more particles with smaller masses.

An important topic is also given by the homogeneous fragmentation developed for example by W. Wagner [78], that we will detail later on in the paper.

Organization of the paper After this introductory part, we derive the Fokker-Planck equation from a microscopic description in Section 2. In Section 3, we study the existence of the solutions of fragmentation equation using fixed point theorems. In Section 4, we study the Kolmogorov forward and backward equations, which are the keys to construct the transition probabilities of Markov processes. In Section 5, we construct a Markov chain from the characteristics of the fragmentation equations using two constructions. In Section 6, we pass from the Markov chain to a continuous time stochastic process. For this, we propose several approaches: embedding the Markov chain, thinning a Poisson process, using martingale problem, using Marked Point Processes. In Section 7, we construct a stochastic differential equation. In Section 8, we give some results on the shattering effects. In Section 9, we study from elementary means the binary fragmentation. Finally, in Section 10, we give a simulation algorithm. The articles ends with three appendices recalling elementary results.

2. From a microscopic description to a Fokker-Planck equation

2.1. Standard notations

We start with notations we use through all the document.

Notation 2.1 (Borel sets)**.** We denote by $\mathrm{Bor}(\mathbb{Y})$ the σ-algebra of Borel sets of a topological space $\mathbb{Y}$.

Notation 2.2 (Borel measures)**.** We denote by $\mathbb{M}(\mathbb{Y})$ (resp. $\mathbb{M}^+(\mathbb{Y})$) the set of $\mathbb{R}$-valued (resp. $\mathbb{R}_+$-valued) Borel measures on a topological space $\mathbb{Y}$, that is the set of σ-additive measures defined on $\mathrm{Bor}(\mathbb{Y})$.

There are several ways to equip $\mathbb{M}(\mathbb{Y})$ with a metric so that it becomes a complete, separable metric space with respect to the vague (resp. weak) topology, *i.e.* a family $\{\mu_n\}_n$ converges vaguely to μ if and only if $\int f \,\mathrm{d}\mu_n$ converges to $\int f \,\mathrm{d}\mu$ for any non-negative function f with compact support (resp. any non-negative function f uniformly bounded) [45, 46, 76].

Notation 2.3 (Space of continuous functions)**.** Given two locally compact metric spaces $\mathbb{Y}$ and $\mathbb{Y}'$, we denote by $\mathcal{C}(\mathbb{Y},\mathbb{Y}')$ (resp. $\mathcal{C}_\mathrm{b}(\mathbb{Y},\mathbb{R})$; $\mathcal{C}_\mathrm{c}(\mathbb{Y},\mathbb{Y}')$; $\mathcal{C}_0(\mathbb{Y},\mathbb{Y}')$) the space of continuous functions (resp. and bounded; with compact support; vanishing at infinity) from $\mathbb{Y}$ to $\mathbb{Y}'$.

Notation 2.4 (Space of bounded, measurable functions)**.** We denote by $\mathcal{B}(\mathbb{Y},\mathbb{Y}')$ the space of bounded, $\mathrm{Bor}(\mathbb{Y})$-measurable function from $\mathbb{Y}$ to $\mathbb{Y}'$.

Notation 2.5. For $\mu \in \mathbb{M}(\mathbb{Y})$ and $f : \mathbb{Y} \to \mathbb{R}$ a measurable function, we set

$$\langle \mu, f \rangle := \int_{\mathbb{Y}} f(x)\mu(\mathrm{d}x)$$

whenever the integral is well defined.

2.2. A microscopic description of the fragmentation process

A *fragmentation* is a physical phenomenon in which a particle of a given characteristics x, say the mass, breaks into particles with characteristics $x_1, \ldots, x_k$. Here, the number k of fragments may be fixed or random, as well as the x_i's. A particle of characteristics x is called a *x-mer*. A simple model of fragmentation is when a x-mer breaks into two $x/2$-mers.

During all this article, we consider that the state-space of the characteristics is

$$\mathbb{X} := \mathbb{R}_+ \text{ and } \mathbb{X}^* := (0, +\infty).$$

We refer to x as the mass, but it could correspond to other parameters, such as the diameter of a bubble in a cavitation phenomenon (see *e.g.* [68]).

In this section, we give a microscopic description of the fragmentation as a random phenomena. We follow [78] and we refer to this article for more details.

Definition 2.1 ((Probability) kernel)**.** A *kernel* K on $\mathbb{Y}$ is a function on $\mathbb{Y} \times \mathrm{Bor}(\mathbb{Y})$ such that $K(x, \cdot) \in \mathbb{M}^+(\mathbb{Y})$ for any $x \in \mathbb{Y}$ and $x \mapsto K(x, \Lambda)$ is measurable for any $\Lambda \in \mathrm{Bor}(\mathbb{Y})$. If $K(x, \mathbb{Y}) = 1$ for any $x \in \mathbb{Y}$, then we refer to K as a *probability kernel*.

We describe the *breakage mechanism* of a x-mer as follows:

- First, the number $k^\dagger$ of fragments is defined by a discrete probability $\mathsf{p}(x) := \{\mathsf{p}_\ell(x)\}_{\ell \geq 2}$ with $\mathsf{p}_\ell \geq 0$, $\sum_{\ell \geq 2} \mathsf{p}_\ell(x) = 1$, where $x \mapsto \mathsf{p}_\ell(x)$ is measurable for any $\ell \geq 2$. Therefore, $\mathbb{P}[k^\dagger = \ell] = \mathsf{p}_\ell(x)$. Note that any x-mer breaks necessarily into two or more fragments.
- Second, the distribution of the resulting particles is ruled by a family of probability kernels f_k on $\mathbb{X}^{*k}$, $k \geq 2$, that is $\mathsf{f}_k(x, \cdot)$ is the distribution of $(x_1, \ldots, x_k)$ given the number of fragments is exactly k.

As we do not order the x_i-mer after a breakage, we do not distinguished between $(x_1, \ldots, x_k)$ and $(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ whatever the permutation σ of $\{1, \ldots, k\}$. Therefore, we introduce the space $\underline{\mathbb{X}^{*k}}$ by quotienting $\mathbb{X}^{*k}$ with the equivalence relation $(x_1, \ldots, x_k) \sim (y_1, \ldots, y_k)$ whenever there exists a permutation σ of $\{1, \ldots, k\}$ such that $(y_1, \ldots, y_k) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$. The kernel f_k is then transformed into

$$\underline{\mathsf{f}}_k(x, \mathrm{d}x_1, \ldots, \mathrm{d}x_k) := \frac{1}{k!} \sum_\sigma \mathsf{f}_k(x, \mathrm{d}x_{\sigma(1)}, \ldots, \mathrm{d}x_{\sigma(k)})$$

which is a kernel on $\underline{\mathbb{X}^*}^k$. With this kernel, the marginal distribution of one fragment picked randomly is then

$$\mathsf{f}_k^{\mathrm{marg}}(x, \mathrm{d}y) := \underline{\mathsf{f}}_k(x, \mathrm{d}y, \mathbb{X}^*, \dots, \mathbb{X}^*).$$

Such $\mathsf{f}_k^{\mathrm{marg}}(x, \cdot)$ is also a probability distribution.

Therefore, the law of a particle picked randomly after the breakage of a x-mer is

$$\mathsf{f}^{\mathrm{marg}}(x, \mathrm{d}y) = \sum_{k \geq 2} \mathsf{p}_k(x) \cdot \mathsf{f}_k^{\mathrm{marg}}(x, \mathrm{d}y).$$

In particular, for any $\Lambda \in \mathrm{Bor}(\mathbb{X}^*)$,

$$M(x, \Lambda) := \sum_{k \geq 2} k \cdot \mathsf{p}_k(x) \cdot \mathsf{f}_k^{\mathrm{marg}}(x, \Lambda) \tag{2.1}$$

is the mean number of particles falling into Λ, since $\mathsf{f}_k^{\mathrm{marg}}(x, \Lambda)$ is the probability that a randomly picked particle falls in Λ. The mean number of fragments after the breakage of a x-mer is $M(x, \mathbb{X}^*)$.

We will show in Section 2.3 just below that M summarizes the macroscopic behavior of the fragmentation process.

Definition 2.2 (Daughter distribution)**.** The kernel $M : \mathbb{X}^* \times \mathrm{Bor}(\mathbb{X}^*) \to \mathbb{R}_+$ is called the *daughter distribution*.

In spite of its name, $M(x, \cdot)$ is not necessarily a distribution as its total mass is not necessarily 1.

Up to now, we have only described the way a x-mer breaks. To turn into an evolution phenomena, we have to describe also at what time a breakage occurs. We assume that each of the particles breaks independently according to an exponential random time at a given rate $F(x)$, called the *fragmentation rate*. The probability that an x-mer breaks during $[t, t + \Delta t]$ is $1 - \exp(-F(x)\Delta t) \approx F(x) \cdot \Delta t$, for small Δt.

2.3. Derivation of the Fokker-Planck equation

We are now able to write the equation for the fluxes of a cloud of particles: we consider an infinite number of particles which are distributed at time $t \geq 0$ according to the distribution $\mu(t, \cdot)$ on $\mathbb{X}^*$. Then

$$\begin{aligned} \mu(t + \Delta t, \mathrm{d}x) \ &\approx \mu(t, \mathrm{d}x) \\ &+ \int (\text{inflow in } \mathrm{d}x \text{ of particles from } y \text{ during } \Delta t\ ) \times \mu(t, \mathrm{d}y) \\ &\qquad - (\text{outflow of } x\text{-mers during } \Delta t) \times \mu(t, \mathrm{d}x). \end{aligned} \tag{2.2}$$

The inflow itself is decomposed as

$$\begin{aligned}&(\text{inflow in } \mathrm{d}x \text{ of particles from } y \text{ during } \Delta t)\\ &\quad = (\text{prob. breakage}) \times (\text{mean \# of created particles in } \mathrm{d}x)\\ &\quad \approx F(y)\cdot\Delta t \times M(y,\mathrm{d}x)\end{aligned}$$

while

$$(\text{outflow of } x\text{-mers during } \Delta t) = (\text{prob. breakage}) \approx F(x)\cdot\Delta t.$$

We thus rewrite (2.2) as

$$\mu(t+\Delta t,\mathrm{d}x) \approx \mu(t,\mathrm{d}x) + \int_{\mathbb{X}^*} F(y)M(y,\mathrm{d}x)\mu(t,\mathrm{d}y) - F(x)\mu(t,\mathrm{d}x)\cdot\Delta t.$$

Passing to the limit, we obtain

$$\frac{\mathrm{d}\mu(t,\mathrm{d}x)}{\mathrm{d}t} = \int_{\mathbb{X}^*} \mu(t,\mathrm{d}y)F(y)M(y,\mathrm{d}x) - F(x)\mu(t,\mathrm{d}x). \tag{2.3}$$

Such equation is a *Fokker-Planck equation*, which relates the evolution of the distribution of the x-mers from a simple balance mechanism. Although we have given a microscopic description of the mechanism involving the rate F, the kernels f_k's and the p_k's only the kernel M and the rate F are used in this equation.

2.4. Mass-biased evolution and mass conservation

It could be convenient to introduce the *mass-biased* measure $\nu(t,\mathrm{d}x)$, which is the measure with Radon-Nikodym derivative x with respect to the measure $\mu(t,\mathrm{d}x)$, that is

$$\nu(t,\mathrm{d}x) = x\cdot\mu(t,\mathrm{d}x). \tag{2.4}$$

Integrating (2.3) against a continuous test function ϕ with compact support on $\mathbb{X}^* := (0,+\infty)$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{X}^*}\frac{\phi(x)}{x}x\cdot\mu(t,\mathrm{d}x)\\ &\quad = \int_{\mathbb{X}^*}\int_{\mathbb{X}^*} y\cdot\mu(t,\mathrm{d}y)F(y)M(y,\mathrm{d}x)\frac{x}{y}\frac{\phi(x)}{x}\,\mathrm{d}x - \int_{\mathbb{X}^*}\frac{\phi(x)}{x}F(x)x\cdot\mu(t,\mathrm{d}x).\end{aligned}$$

With $\psi(x) := \phi(x)/x$ for $x\in\mathbb{X}^*$, this writes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{X}^*}\psi(x)\nu(t,\mathrm{d}x) = \int_{\mathbb{X}^*}\int_{\mathbb{X}^*}\nu(t,\mathrm{d}y)F(y)M(y,\mathrm{d}x)\frac{x}{y}\psi(x) - \int_{\mathbb{X}^*}\psi(x)F(x)\nu(t,\mathrm{d}x).$$

Hence, ν solves the *Fokker-Planck* equation

$$\frac{\mathrm{d}\nu(t,\mathrm{d}x)}{\mathrm{d}t} = \int_{\mathbb{X}^*}\nu(t,\mathrm{d}y)F(y)\mathsf{G}(y,\mathrm{d}x) - F(x)\nu(t,\mathrm{d}x) \tag{2.5}$$

with

$$\mathsf{G}(y,\mathrm{d}x) := \frac{x}{y} M(y,\mathrm{d}x).$$

Such approach could work with any other non-vanishing Radon-Nikodym density, yet the choice of this $\nu(t,\mathrm{d}x)$ is of particular interest under the additional property that the mass is conserved.

Let us consider a x-mer after its breakage into $k^\dagger$ fragments $(x_1,\dots,x_{k^\dagger})$ (here $k^\dagger$ is random). For a measurable function $\phi : \mathbb{X}^* \to \mathbb{R}_+$, since we do not distinguish between the particles,

$$\mathbb{E}[\phi(x_1)+\cdots+\phi(x_{k^\dagger}) \mid k^\dagger = k] = k \int_{\mathbb{X}^*} \mathsf{f}^{\mathrm{marg}}(x,\mathrm{d}y)\phi(y)$$

and then, using the definition of M in (2.1),

$$\mathbb{E}[\phi(x_1)+\cdots+\phi(x_{k^\dagger})] = \int_{\mathbb{X}^*} M(x,\mathrm{d}y)\phi(y). \tag{2.6}$$

In particular, taking $\phi(x) = x^k$, we obtain the following moments

$$M_k(x) := \int_{\mathbb{X}^*} y^k M(x,\mathrm{d}y)$$

and in particular

$$M_0(x) := \mathbb{E}[k^\dagger] \text{ and } M_1(x) := \mathbb{E}[x_1+\cdots+x_{k^\dagger}].$$

An important property is that of the *mass conservation*.

Hypothesis 2.1 (Mass conservation). We say that the *mass conservation* holds when $x_1+\cdots+x_k = x$ almost surely for any x-mer breaks into $(x_1,\dots,x_k)$.

We note that the mass conservation implies that all the fragments have necessarily a smaller mass than the initial particle. Mathematically, the support of $M(x,\cdot)$ is contained in $[0,x]$.

Hypothesis 2.2 (Mass conservation on average). We say that the *mass conservation on average* holds when

$$x = \int_{\mathbb{X}^*} M(x,\mathrm{d}y)\cdot y = M_1(x) \text{ for any } x \in \mathbb{X}^*. \tag{2.7}$$

The next results are then immediate.

Lemma 2.1. *If the mass conservation holds (Hypothesis 2.1), then the mass conservation on average holds (Hypothesis 2.2).*

Lemma 2.2. *If the mass conservation on average holds, then* $\mathsf{G}(x,\mathbb{X}^*) = \mathsf{G}(x,[0,x]) = 1$ *for any* $x \in \mathbb{X}^*$.

Let us now consider computing the moments

$$\mu_k(t) := \int_{\mathbb{X}^*} \mu(t, \mathrm{d}x) \cdot x^k$$

of $\mu(t, \mathrm{d}x)$, the solution to (2.3). Multiplying (2.3) by x^k and integrating with respect to the x variable, we obtain that

$$\frac{\mathrm{d}\mu_k(t)}{\mathrm{d}t} = \int_{\mathbb{X}^*} \int_{\mathbb{X}^*} \mu(t, \mathrm{d}y) F(y) M(y, \mathrm{d}x) x^k - \int_{\mathbb{X}^*} F(x) x^k \mu(t, \mathrm{d}x).$$

In particular, if mass conservation holds on average,

$$\frac{\mathrm{d}\mu_1(t)}{\mathrm{d}t} = \int_{\mathbb{X}^*} \mu(t, \mathrm{d}y) F(y) y - \int_{\mathbb{X}^*} F(x) x \mu(t, \mathrm{d}x).$$

However, one may not conclude that $\mu_1(t) = \nu(t, \mathbb{X}^*)$ is constant unless one ensures that $\int_{\mathbb{X}^*} \mu(t, \mathrm{d}y) F(y) y$ is finite.

There are actually situations, called *shattering fragmentation*, where particles of mass zero are produced (See Section 8).

Under (2.7), if $\nu(0, \mathbb{X}^*) \leq 1$, then one may expect to construct a $\mathbb{X}^*$-valued stochastic process ξ such that $\nu(t, \cdot)$ is the distribution of ξ. If shattering occurs, then ξ has a finite lifetime.

The goal of this paper is to summarize the various ways one may construct such a process and to study its properties.

An alternative direction, that we do not consider here is to consider the families $\{(\mathsf{p}_k, \mathsf{f}_k)\}_{k \geq 2}$ and the rates F to describe the evolution of a cloud of particles evolving according to the fragmentation process. This gives rise to a class of *branching processes*.

2.5. A condition on the daughter distribution

We give a condition on the daughter distribution. Basically, it comes from the fact that at most one fragment has a mass greater than half of the mass of its parent.

Proposition 2.1 ([69] or [9, Section 8.2, p. 202])**.** *The daughter distribution shall satisfy*

$$\int_0^z y M(x, y) \, \mathrm{d}y \geq \int_{x-z}^x (x - y) M(x, y) \, \mathrm{d}y \text{ for } 0 \leq z \leq \frac{x}{2}. \tag{2.8}$$

We will see that on the case of binary fragmentation, that is when $M_0(x) = 2$ for any x, then M is symmetric $M(x + y, x) = M(x + y, y)$.

The condition (2.8) holds whenever $y \mapsto M(x, y)$ is non-increasing.

2.6. Classical examples of daughter distribution

Let us give some classical examples of the daughter distribution.

Example 2.1 (Binary fragmentation, [78, Sect. 2.3]). Let $p_2 = 1$ and

$$\mathsf{f}_2(x, \mathrm{d}x_1, \mathrm{d}x_2) := \rho(x, \mathrm{d}x_1)\delta_{x-x_1}(\mathrm{d}x_2)$$

for a probability kernel ρ. Define $\rho^\sharp$ by

$$\int_{\mathbb{X}^*} \phi(y)\rho^\sharp(x, \mathrm{d}y) = \frac{1}{2}\int_{\mathbb{X}^*} (\phi(y) + \phi(x-y))\rho(x, \mathrm{d}y).$$

We note that $\rho^\sharp(x, \mathscr{S}_x(\cdot)) = \rho^\sharp(x, \cdot)$ with $\mathscr{S}_x(\Lambda) = \{y \mid x - y \in \Lambda\}$ for $\Lambda \in \mathrm{Bor}(\mathbb{X}^*)$. Then

$$\underline{\mathsf{f}}_2(x, \mathrm{d}x_1, \mathrm{d}x_2) = \rho^\sharp(x, \mathrm{d}x_1)\delta_{x-x_1}(\mathrm{d}x_2) \text{ and } M(x, \mathrm{d}y) = 2\rho^\sharp(x, \mathrm{d}y).$$

With $\mathscr{S}_x$ above,

$$M(x, \mathrm{d}y) = M(x, \mathscr{S}_x(\mathrm{d}y)),\ \forall x \in \mathbb{X}^*. \tag{2.9}$$

If $M(x, \mathrm{d}y) = m(x, y)\mathrm{d}y$ has a density, (2.9) becomes

$$m(x + y, x) = m(x + y, y) \text{ for all } x, y \in \mathbb{X}^*. \tag{2.10}$$

We develop the specific case of the binary fragmentation in Section 9. In particular, we show it could be summarized by a single function from $\mathbb{X}^{*2}$ to $\mathbb{R}_+$ in one-to-one correspondence with the daughter distribution M and the rate F.

Example 2.2 (Binary fragmentation: equi-partition). With the Example 2.1, set $\rho(x, \cdot) := \delta_{x/2}$, in which case

$$\rho^\sharp(x, \cdot) = \delta_{x/2},\ M(x, \cdot) = 2\delta_{x/2} \text{ and } \mathsf{G}(x, \cdot) = \delta_{x/2}.$$

Example 2.3 (Binary fragmentation: fixed size). With the Example 2.1, set $\rho(x, \cdot) := \delta_{\lambda x}$ for some $\lambda \in (0, 1)$. Hence

$$\rho^\sharp(x, \cdot) = \frac{1}{2}\delta_{\lambda x} + \frac{1}{2}\delta_{(1-\lambda)x},\ M(x, \cdot) = \delta_{\lambda x} + \delta_{(1-\lambda)x}$$
$$\text{and } \mathsf{G}(x, \mathrm{d}y) = \lambda\delta_{\lambda x} + (1 - \lambda)\delta_{(1-\lambda)x}.$$

Example 2.4 (Binary fragmentation: proportional splitting). With the Example 2.1, set $\rho(x, \mathrm{d}y) := x^{-1}\mathbb{1}_{[0,x]}(y)\,\mathrm{d}x$, in which case

$$\rho^\sharp(x, \mathrm{d}y) = \rho(x, \mathrm{d}y),\ M(x, \mathrm{d}y) = 2\rho(x, \mathrm{d}y) \text{ and } \mathsf{G}(x, \mathrm{d}y) = \frac{2y}{x^2}\mathbb{1}_{[0,x]}(y)\,\mathrm{d}y.$$

Example 2.5 (Homogeneous fragmentation, [78, Sect. 2.4]). Let $\rho(\mathrm{d}s)$ be a probability measures on the set of partitions

$$\mathbb{S} := \{s_1, \ldots, s_k > 0 \mid s_1 + \cdots + s_k = 1,\ k \geq 2\}.$$

A fragmentation kernel is *homogeneous* if

$$\mathsf{f}(x,\mathrm{d}z)=\sum_{k\geq 2}\mathsf{p}_k(x)\mathsf{f}_k(x,\mathrm{d}x_1,\ldots,\mathrm{d}x_k)=\int_{\mathbb{S}}\delta_{s_1x,\ldots,s_kx}(\mathrm{d}z)\rho(\mathrm{d}s).$$

In this case (see the details in [78]),

$$M(x,\mathrm{d}y)=\int_0^1\delta_{ux}(\mathrm{d}y)M(1,\mathrm{d}u)$$

and the mass conservation condition is

$$\int_0^1 uM(1,\mathrm{d}u)=1.$$

The average number of fragments is $\int_0^1 M(1,\mathrm{d}u)$ and does not depend on x. Besides,

$$\mathsf{G}(x,\mathrm{d}y)=\int_0^1\delta_{ux}(\mathrm{d}y)uM(1,\mathrm{d}u).$$

If $M(x,\cdot)$ has a density $m(x,\cdot)$, then $\mathsf{G}(x,\cdot)$ has a density $\mathsf{g}(x,\cdot)$. These densities satisfy

$$m(x,y)=\frac{1}{x}\vartheta\Big(\frac{y}{x}\Big)\text{ and }\mathsf{g}(x,y)=\frac{y}{x^2}\vartheta\Big(\frac{y}{x}\Big)\text{ with }\vartheta:=m(1,\cdot).$$

Example 2.6. We easily see see that Examples 2.1-2.4 also describe homogeneous fragmentation.

Example 2.7 (Homogeneous fragmentation: Power law). In the homogeneous fragmentation, the daughter distribution is characterized by a function $\vartheta:\mathbb{X}^*\to\mathbb{R}_+$. A classical case is that of a *power law*, that is $\vartheta(x)=x^\nu(\nu+2)$ for some $\nu>-2$. For $\nu=0$, the fragmentation corresponds to the one of Example 2.2.

The $(\nu+2)$-factor ensures the mass conservation, that is $\int_0^1 u\vartheta(u)\,\mathrm{d}u=1$. The condition of Proposition 2.1 implies that $\nu\leq 0$ [9, Lemma 8.2, p. 202].

The mean number $M_0(x)$ of particles $M(x,\mathbb{X}^*)$ after a breakage of an x-mer does not depend on x then is equal to

$$M_0(x)=\int_0^1\vartheta(z)\,\mathrm{d}z=\begin{cases}\dfrac{\nu+2}{\nu+1} & \text{if } -1<\nu\leq 0,\\ +\infty & \text{if } -2<\nu\leq -1.\end{cases}$$

We give more insight on this example below in Example 2.10.

Examples of such functions ϑ, their characteristics, inference and applications may be found in [31].

Example 2.8 (Separable fragmentation). We follow [9, Section 8.2.3, p. 204]. The *separable fragmentation* is when the daughter distribution has a density of the form $m(x,y)=\beta(y)\gamma(x)$. In this case, the mean number $M_0(x)$ of particles $M(x,\mathbb{X}^*)$ after a breakage of an x-mer is

$$M_0(x)=\frac{x\int_0^x\beta(z)\,\mathrm{d}z}{\int_0^x z\beta(z)\,\mathrm{d}z}$$

and therefore may vary with the mass x.

2.7. Some pathologies of the Fokker-Planck equation

We consider that the daughter distribution has a density m.

Assuming that the solution $\mu(t,\mathrm{d}x)$ has a density $c(t,x)$, we rewrite the Fokker-Planck equation (2.3) as

$$\frac{\mathrm{d}c(t,x)}{\mathrm{d}t} = \int_{\mathbb{X}^*} c(t,y)F(y)m(y,x) - F(x)c(t,x)$$

and (2.5) as

$$\frac{\mathrm{d}p(t,x)}{\mathrm{d}t} = \int_{\mathbb{X}^*} p(t,y)F(y)m(y,x) - F(x)c(t,x)$$

with $p(t,x) = x\cdot c(t,x)$, which is the density of $\nu(t,\mathrm{d}x)$, where ν is given by (2.4).

Although these equations seem to be a simple first-order equation, we recall classical examples which are based on analytical computations.

Example 2.9 (Non uniqueness, [9, Section 8.2, p 201]). With $F(x) = x$ and $m(x,y) = 2/x$ (See Example 2.3), then

$$c_1(t,x) = \frac{e^t}{(1+x)^3},$$
$$c_2(t,x) = e^{-xt}\left(\frac{1}{(1+x)^3} + \int_x^{+\infty} \frac{2t+t^2(y-x)}{(1+x)^3}\,\mathrm{d}y\right)$$

are both solutions to (2.3) with the initial condition $c(0,x) = 1/(1+x)^3$. Note that c_1 does not conserve the mass, while c_2 does.

Example 2.10 (Shattering fragmentation, [69]). With

$$m(x,y) = (\nu+2)\frac{y^\nu}{x^{\nu+1}} \text{ and } F(x) = x^{\beta+1},$$

explicit expressions for the solution may be found in [69]. Such expressions involve special functions. This example shows a phase transition: when $\beta \geq -1$, the mass is conserved that is $\mu_1(t) := \int_{\mathbb{X}^*} p(t,x)\,\mathrm{d}x$. If $\beta < -1$, then $\mu_1(t)$ is time dependent. More precisely, if the mass is initially concentrated at $x = \ell$,

$$\mu_1(t) = \ell \exp\left(-\frac{t}{\ell^{1+\nu/2}}\right)\left(1 + \frac{t}{\ell^{1+\nu/2}} + \frac{t^2}{2\ell^{\nu+2}}\right).$$

The loss of mass is due to infinitesimally small particles (*dust*).

3. Existence of solutions to the fragmentation equation

3.1. A perturbation principle

We will make use of the following lemma.

Lemma 3.1 (A perturbation lemma)**.** *Let us consider $\mathscr{B}$ and $\mathscr{D}$ be bounded operators on a Banach space $\mathbb{U}$. For some $T > 0$, we also consider $\mathscr{A}$ from $\mathbb{Y} := \mathcal{C}^1([0,T],\mathbb{U})$ to itself*[1]*. We assume that for any $f \in \mathbb{Y}$,*

$$\partial_t \mathscr{A} f(t) = \mathscr{B} f(t) + \mathscr{D}\mathscr{A} f(t),\ t \in [0,T] \tag{3.1}$$

and $\mathscr{A} f(0) = 0$. We assume that there exists a solution $g \in \mathbb{Y}$ to

$$\partial_t g(t) := \mathscr{D} g(t),\ t \in [0,T] \tag{3.2}$$

as well as a solution $f \in \mathbb{Y}$ to

$$(\mathrm{Id} - \mathscr{A}) f = g. \tag{3.3}$$

Then f solves

$$\partial_t f(t) = (\mathscr{B} - \mathscr{D}) f(t),\ t \in [0,T]. \tag{3.4}$$

In addition, $f(0) = g(0)$.

Proof. First, using (3.1) and (3.3),

$$\begin{aligned}&\partial_t(\mathrm{Id} - \mathscr{A}) f(t) = \partial_t f(t) - \mathscr{B} f(t) + \mathscr{D}\mathscr{A} f(t)\\ &= \partial_t f(t) - \mathscr{B} f(t) - \mathscr{D}(\mathrm{Id} - \mathscr{A}) f(t) + \mathscr{D} f(t) = \partial_t f(t) - \mathscr{B} f(t) - \mathscr{D} g f(t) + \mathscr{D} f(t).\end{aligned}$$

On the other hand, with (3.2),

$$\partial_t(\mathrm{Id} - \mathscr{A}) f(t) - \mathscr{D} f(t) = \partial_t g(t) = \mathscr{D} g(t).$$

Substracting these two equations leads to (3.4). The equality $f(0) = g(0)$ follows immediately from (3.3). □

Remark 3.1 (Duhamel principle). If $\{\mathscr{T}^{\mathscr{D}}(t)\}_{t\geq 0}$ is the semi-group generated by $\mathscr{D}$, then the solution of (3.2) is given by $g(t) = \mathscr{T}^{\mathscr{D}}(t) g(0)$. Lemma 3.1 corresponds to the *Duhamel formula* (or a perturbation principle [9, Chapter 4] or [36]), by setting

$$\mathscr{A} f(t) := \int_0^t \mathscr{T}^{\mathscr{D}}(t-s)\mathscr{B} f(s)\,\mathrm{d}s$$

which satisfies (3.1). Therefore,

$$f(t) = \mathscr{T}^{\mathscr{D}}(t) g(0) + \int_0^t \mathscr{T}^{\mathscr{D}}(t-s)\mathscr{B} f(s)\,\mathrm{d}s.$$

Remark 3.2 (Solving (3.3)). Assume $\|\mathscr{A}\|_{\mathbb{Y}\to\mathbb{Y}} < 1$. Eq. (3.3) is easy to solve through a Picard principle by setting

$$f^{(0)} = g,\ f^{(n+1)} = \mathscr{A} f^{(n)} \text{ and } f := \sum_{n\geq 0} f^{(n)}.$$

The bound $\|\mathscr{A}\|_{\mathbb{Y}\to\mathbb{Y}} < 1$ ensures the convergence of the series defining f in $\mathbb{Y}$.

[1] We may replace $\mathcal{C}^1$ by the space of absolutely continuous functions.

Remark 3.3 (Preservation of some properties). Let us assume that $\mathscr{D}$ is such that if $g(0)$ satisfies a given property (P), then for any $t \in [0,T]$, $g(t)$ given by (3.2) also satisfies (P). We also assume that for any $t \in [0,T]$, $\mathscr{A} f(t)$ also satisfies (P) whenever $f(t)$ satisfies (P). For a function k, such a property (P) could be "$k(x) \geq 0$ for any x", "$k(x) \leq M$ for any x" or "$k(x) = 0$ for any $x \geq K$". From the construction of Remark 3.2, if $g(0)$ satisfies (P), then $f(t)$ satisfies (P) for any $t \geq 0$.

3.2. The fragmentation equation with absolutely continuous fragmentation kernel

In a first time, we solve the Fokker-Planck equation (2.5) when the fragmentation kernel is absolutely continuous with respect to the Lebesgue measure. We follow the approach of [71].

Hypothesis 3.1 (Boundedness of the rate). We consider $F : \mathbb{X} \to \mathbb{R}_+$, measurable with $F(x) \leq B$, $x \in \mathbb{X}$, for some $B > 0$. This function is the *rate*.

Hypothesis 3.2 (Kernel). We consider a kernel G with a density g with respect to the Lebesgue measure that satisfies

$$0 \leq \mathsf{g}(x,y) \leq C,\ \mathsf{g}(x,y) = 0 \text{ if } y \geq x,$$
$$\int_0^x \frac{y}{x}\mathsf{g}(x,y)\,\mathrm{d}y \leq 1 \text{ and } \int_0^x F(x)\mathsf{g}(x,y)\,\mathrm{d}y \leq E,$$

for some constants C and E.

Notation 3.1. We define

$$g(x,y) := F(x)\mathsf{g}(x,y) \text{ for } x,y \in \mathbb{X}$$

so that $\int_0^x g(x,y)\,\mathrm{d}y \leq E$.

We also define for any measurable function $f : \mathbb{X} \to \mathbb{R}$ the following operators:

$$\mathscr{G}^\flat f(x) := \int_x^{+\infty} g(y,x)f(y)\,\mathrm{d}y,\ \mathscr{M}^\flat f(x) := F(x)f(x),$$
$$\mathscr{F}^\flat f(x) := \int_x^{+\infty} g(y,x)f(y)\,\mathrm{d}y - F(x)f(x) = (\mathscr{G}^\flat - \mathscr{M}^\flat)f(x)$$

for $x \in \mathbb{X}$.

Notation 3.2 (Space of integrable functions)**.** We denote by $\mathrm{L}^1(\mathbb{X})$ the space of measurable, integrable functions from $\mathbb{X}$ to $\mathbb{R}$. The corresponding norm is $\|f\|_{\mathrm{L}^1} := \int_0^{+\infty} |f(x)|\,\mathrm{d}x$. With this norm, $\mathrm{L}^1(\mathbb{X})$ is a Banach space.

Lemma 3.2. *The operators $\mathscr{G}^\flat$, $\mathscr{M}^\flat$ and $\mathscr{F}^\flat$ are bounded from $\mathrm{L}^1(\mathbb{X})$ to $\mathrm{L}^1(\mathbb{X})$ respectively with*

$$\|\mathscr{G}^\flat\|_{\mathrm{L}^1\to\mathrm{L}^1} \leq E,\ \|\mathscr{M}^\flat\|_{\mathrm{L}^1\to\mathrm{L}^1} \leq B \text{ and } \|\mathscr{F}^\flat\|_{\mathrm{L}^1\to\mathrm{L}^1} \leq E + B.$$

Proof. Interverting integrals thanks to the Fubini's theorem leads to,

$$\|\mathscr{G}^\flat f\|_{\mathrm{L}^1} \le \int_0^{+\infty} |\mathscr{G}^\flat f(x)|\,\mathrm{d}x \le \int_0^{+\infty}\int_x^{\infty} g(y,x)|f(y)|\,\mathrm{d}y\,\mathrm{d}x \le \int_0^{+\infty} |f(x)| \int_0^x g(y,x)\,\mathrm{d}x\,\mathrm{d}y \le E\|f\|_{\mathrm{L}^1}.$$

The bounds on the norm of $\mathscr{M}^\flat$ and $\mathscr{F}^\flat$ are immediate. □

In a first time, we consider the equation

$$\frac{\partial f(t,x)}{\partial t} = \mathscr{F}^\flat f(t,x) \text{ with } f(0,x) = a_0(x) \tag{3.5}$$

where $a_0 \in \mathrm{L}^1(\mathbb{X})$ is given.

Using Lemma 3.2, we then look at solutions in the Banach space $\mathbb{Y} := \mathcal{C}([0,T],\mathrm{L}^1(\mathbb{X}))$ with the norm $\|f\|_{\mathbb{Y}} := \sup_{t\in[0,T]}\|f(t,\cdot)\|_{\mathrm{L}^1}$.

Proposition 3.1. *For $T > 0$, there exists a unique solution f of (3.5) in $\mathbb{Y}$ which could be written as*

$$f(t,x) = \sum_{k\ge 0} a_k(x)t^k \text{ with } a_k \in \mathrm{L}^1(\mathbb{X}).$$

Besides, if a_0 is non-negative, then f is non-negative. If a_0 has a support contained in $[0,x_0]$, then $f(t,\cdot)$ has also a support contained in $[0,x_0]$.

Proof. If $f \in \mathbb{Y}$ solves (3.5), then

$$\|f(t,\cdot)\|_{\mathrm{L}^1} \le (B+E)\int_0^t \|f(s,\cdot)\|_{\mathrm{L}^1}\,\mathrm{d}s + \|a_0\|_{\mathrm{L}^1}.$$

Using the Grönwall lemma, for $t \le T$,

$$\|f\|_{\mathbb{Y}} \le \|a_0\|_{\mathrm{L}^1}\exp(T(B+E)).$$

Since (3.5) is linear, we see that the solution of (3.5), if any, is necessarily unique.

Define from $a_0 \in \mathrm{L}^1(\mathbb{X})$ the sequence

$$a_{k+1}(x) := \frac{1}{k+1}\mathscr{F}^\flat a_k(x), \text{ for } k \ge 0. \tag{3.6}$$

Using Lemma 3.2, $\|a_{k+1}\|_{\mathrm{L}^1} \le (B+E)^{k+1}/(k+1)!$. Therefore, $\{\sum_{i=0}^k a_i\}_{k\ge 0}$ is a Cauchy sequence in $\mathrm{L}^1(\mathbb{X})$. From this, $\{\sum_{i=0}^k a_i(x)t^i\}_{k\ge 0}$ is a Cauchy sequence in $\mathbb{Y}$. We define for any $t > 0$,

$$f(t,x) := \sum_{k=0}^{+\infty} a_k(x)t^k = \sum_{k=0}^{+\infty}\frac{1}{k!}(\mathscr{F}^\flat)^k a_0(x)t^k. \tag{3.7}$$

This function is such that $t \mapsto f(t,\cdot)$ is analytic around 0 in $\mathrm{L}^1(\mathbb{X})$. Moreover, using (3.6),

$$\frac{\partial^\ell f(t,x)}{\partial t^\ell} = (\mathscr{F}^\flat)^\ell f(t,x) \text{ } x\text{-a.e., } t \in [0,T] \text{ for any } \ell \geq 1 \tag{3.8}$$

so that in particular, f solves (3.5) in $\mathbb{Y}$.

The solution to $\partial g(t,x) = -\mathscr{M} g(t,x) = -F(x)g(t,x)$ is given by $g(t,x) = \exp(-tF(x))g_0(x)$. The results of Section 3.1, in particular Remark 3.1 suggests to introduce the operator

$$\mathscr{A}^\flat f(t,x) := \int_0^t e^{-(t-s)F(x)} \mathscr{G}^\flat f(s,x)\,\mathrm{d}s, \tag{3.9}$$

which is continuous from $\mathbb{Y}$ to itself with a norm smaller than TE. Hence, for $T < E$, $\mathscr{A}^\flat$ is contractive. With Lemma 3.1, the unique fixed point f in $\mathbb{Y}$ to $f = \exp(-tF)a_0 + \mathscr{A}^\flat f$ solves (3.5).

If $a_0 \geq 0$, since $\mathscr{G}^\flat$ and $\mathscr{M}^\flat$ preserve positivity, then $\mathscr{A}^\flat$ preserves positivity and then $f(t,\cdot) \geq 0$ for any $t \in [0,T]$ by Remark 3.3.

Finally, assume that for $f \in \mathrm{L}^1(\mathbb{X})$, $f(x) = 0$ when $x \geq x_0$ for some x_0. Then $\mathscr{G}^\flat f(x) = \int_x^{+\infty} f(y)F(y)\mathsf{g}(y,x)\,\mathrm{d}y = 0$ and $\mathscr{M}^\flat f(x) = 0$ for any $x \geq x_0$. Hence, if $a_0(x) = 0$ for $x \geq x_0$, a_k shares this property for any $k \geq 0$ by construction and so is $f(t,\cdot)$ for any $t \geq 0$. □

Clearly, variants of Proposition 3.1 hold whenever $\mathrm{L}^1(\mathbb{X})$ is replaced by a Banach space $\mathbb{U}$ such that $\mathscr{G}^\flat$ and $\mathscr{M}^\flat$ are bounded operators from $\mathbb{U}$ to $\mathbb{U}$.

Hypothesis 3.3 (Continuous kernel and rate). Assume in addition to Hypothesis 3.2 that $(x,y) \mapsto g(x,y)$ and $x \mapsto F(x)$ are continuous.

To prove the proposition below, one has just to replace $\mathbb{U} := \mathrm{L}^1(\mathbb{X})$ by $\mathbb{U} := \mathrm{L}^1(\mathbb{X}) \cap \mathcal{C}(\mathbb{X})$ with the norm $\|f\|_{\mathbb{U}} := \|f\|_{\mathrm{L}^1} + \|f\|_\infty$.

Proposition 3.2 ([71]). *Under Hypotheses 3.1 and 3.3, the solution given by Proposition 3.1 is such that $x \mapsto f(t,x)$ is continuous for any $t \in [0,T]$.*

We could also see Proposition 3.1 from the point of view of functional analysis. The key point behind the proof of this proposition is the construction of the family of operators

$$\mathscr{T}^\flat(t) := \sum_{k=0}^{+\infty} \frac{t^k}{k!} (\mathscr{F}^\flat)^k, \ t \in \mathbb{R}_+.$$

This family $\{\mathscr{T}^\flat(t)\}_{t\in\mathbb{R}_+}$ is a semi-group, that is $\mathscr{T}^\flat(t)\mathscr{T}^\flat(t') = \mathscr{T}^\flat(t+t')$ for any $t,t' \in \mathbb{R}_+$, $\mathscr{T}^\flat(0) = \mathrm{Id}$. Hence, $f(t,x) = \mathscr{T}^\flat(t)a_0(x)$. The construction (3.9) is obtained by a perturbation argument from the two semi-groups generated respectively by $\mathscr{G}^\flat$ and $\mathscr{M}^\flat$. This functional analysis point of view is extensively developed in the work of J. Banasiak and his co-authors (see *e.g.* [9]). Other conditions using functional analysis may be found for example in [70, 75] and many other papers.

As pointed out in Section 2.7, despite its apparent simplicity, the fragmentation equation is not that simple.

The condition of boundedness of the coefficients may be relaxed (see *e.g.*, [9, 64, 74]) to grant uniqueness. Note that however, when the rate is not bounded, the situation may become intricate: the solution may not be unique or the mass conservation may be lost [9, Chapter 8], due either to the rate, but also to the kernel [9, Remark 8.14].

3.3. The fragmentation equation in the space of measures

We now come back to the Fokker-Planck equation (2.5), which we consider solving in some space of measures. For considerations on measures and their norms, we refer to [1, 19, 34].

Notation 3.3 (Space of Radon measures)**.** Let us denote by $\mathbb{M}_{\mathrm{fr}}(\mathbb{X}) \subset \mathbb{M}(\mathbb{X})$ the space of finite Radon measures over $\mathbb{X}$, and $\mathbb{M}^+_{\mathrm{fr}}(\mathbb{X})$ the cone of positive measures. Radon measures acts linearly on the space $\mathcal{C}_{\mathrm{c}}(\mathbb{X}, \mathbb{R})$ of continuous functions with compact support.

For a probability kernel $\mathsf{G} : \mathbb{X} \to \mathbb{M}^+_{\mathrm{fr}}(\mathbb{X})$ and a rate $F : \mathbb{X} \to \mathbb{R}_+$, we define

$$G(x, \mathrm{d}y) = F(x)\mathsf{G}(x, \mathrm{d}y), \text{ for } x \in \mathbb{X}$$

as well as the operators acting on the left by

$$\mu\mathscr{G}(\mathrm{d}y) := \int_{\mathbb{R}_+} \mu(\mathrm{d}x)G(x, \mathrm{d}y) \text{ and } \mu\mathscr{M}(\mathrm{d}y) := F(y)\mu(\mathrm{d}y).$$

We use the action on the left as we also define $\mathscr{G}$ and $\mathscr{M}$ as acting on $\mathcal{C}_{\mathrm{c}}(\mathbb{X}, \mathbb{R})$ by

$$\mathscr{G}\phi(x) := \int_{\mathbb{X}} G(x, \mathrm{d}y)\phi(y) \text{ and } \mathscr{M}\phi(x) := F(x)\phi(x).$$

Notation 3.4. We define $\mathscr{F} := \mathscr{G} - \mathscr{M}$, so that

$$\mathscr{F}\phi(x) = \int_{\mathbb{X}} (\phi(y) - \phi(x))G(x, \mathrm{d}y) = F(x) \int_{\mathbb{X}} (\phi(y) - \phi(x))\mathsf{G}(x, \mathrm{d}y)$$

for $\phi \in \mathcal{C}_{\mathrm{b}}(\mathbb{X}, \mathbb{R})$, whenever the integral is well defined.

We rewrite (2.5) as

$$\begin{aligned} \frac{\partial \mu(t, \mathrm{d}x)}{\partial t} &= \mu(t, \cdot)\mathscr{F}(\mathrm{d}x) \\ \text{with } \mu(0, \cdot) &= \mu_0 \in \mathbb{M}^+_{\mathrm{fr}}(\mathbb{R}_+) \end{aligned} \tag{3.10}$$

for a given μ_0.

Remark 3.4. If μ is a measure with a Radon-Nikodym density m with respect to the Lebesgue measure, then $\mu\mathscr{G}(\mathrm{d}y) = \mathscr{G}^\flat m(y)\mathrm{d}y$ and $\mu\mathscr{M}(\mathrm{d}y) = \mathscr{M}^\flat m(y)\mathrm{d}y$. Thus, (3.10) is a natural expansion of (3.5) in the context of measures.

In a first time, we apply a strategy similar to the one of Section 3.2 to solve (3.10). For this, we follow the approach from *e.g.* [1, 2]. It is however more cumbersome as the space of measures has to be properly identified.

Below, in Section 4.1, we consider again solving simultaneously this problem for $\mu_0 = \delta_x$ for any x. This way, we construct a transition probability by following W. Feller [40]. The latter construction requires less regularity and is naturally related to a probabilistic interpretation. Yet, the approach proposed here easily extends to a wider class of equations including coagulation, growth, and so on [1].

Definition 3.1 (Bounded-Lipschitz and total variation norms)**.** On $\mathbb{M}_{\mathrm{fr}}(\mathbb{R}_+)$, we define the *Bounded-Lipschitz norm*[2] by

$$\|\mu\|_{\mathrm{BL}} := \sup_{\substack{\phi\in \mathrm{W}^{1,\infty}(\mathbb{X},\mathbb{R}) \\ \|\phi\|_{\mathrm{W}^{1,\infty}}\leq 1}} \int_{\mathbb{X}} \phi(x)\mu(\mathrm{d}x),$$

where $\mathrm{W}^{1,\infty}(\mathbb{X},\mathbb{R})$ is the Sobolev space of measurable, bounded functions with generalized bounded first order derivative[3] equipped with the norm $\|\phi\|_{\mathrm{W}^{1,\infty}} := \|\phi\|_\infty + \|\phi'\|_\infty$. We also define the *total variation norm* by

$$\|\mu\|_{\mathrm{TV}} := \sup_{\substack{f\in \mathcal{C}_{\mathrm{c}}(\mathbb{X},\mathbb{R}) \\ \|f\|_\infty\leq 1}} \int_{\mathbb{R}_+} f(x)\mu(\mathrm{d}x).$$

These norms are obviously related by $\|\cdot\|_{\mathrm{BL}} \leq \|\cdot\|_{\mathrm{TV}}$.

Proposition 3.3. *The space* $(\mathbb{M}_{\mathrm{fr}}(\mathbb{X}), \|\cdot\|_{\mathrm{TV}})$ *is complete.*

With the total variation norm, the continuity properties are difficult to deal with[4], while the Bounded-Lipschitz norm is more adapted to this task. Unfortunately, $(\mathbb{M}_{\mathrm{fr}}(\mathbb{X}), \|\cdot\|_{\mathrm{BL}})$ is *not* complete.

We fix a radius $R > 0$ and we define

$$B_{\mathrm{BL}}(R) := \{\mu \in \mathbb{M}_{\mathrm{fr}}(\mathbb{X}) \mid \|\mu\|_{\mathrm{BL}} \leq R\}$$
$$\text{and } B_{\mathrm{TV}}(R) := \{\mu \in \mathbb{M}_{\mathrm{fr}}(\mathbb{X}) \mid \|\mu\|_{\mathrm{TV}} \leq R\}.$$

Proposition 3.4. *The set* $\mathbb{Y}_{T,R} := \mathcal{C}_{\mathrm{b}}([0,T), B_{\mathrm{TV}}(R))$ *is a complete metric space when equipped with the norm*

$$\|\mu\|_{\mathbb{Y}_{T,R}} := \sup_{t\in[0,T)} \|\mu(t)\|_{\mathrm{BL}}.$$

Under Hypothesis 3.1, $\mathscr{M}$ is a bounded endomorphism on $(\mathbb{M}_{\mathrm{fr}}(\mathbb{X}), \|\cdot\|_{\mathrm{TV}})$.

Hypothesis 3.4. The rate F belongs to $\mathrm{W}^{1,\infty}(\mathbb{X})$.

[2] or flat norm, Dudley norm, Fortet-Mourier norm [1, p. 2475].

[3] Since $\mathbb{X} \subset \mathbb{R}$, every function in $\mathrm{W}^{1,\infty}(\mathbb{X},\mathbb{R})$ has a continuous version, which we always consider.

[4] A classical example is that if $\{x_n\}_n$ converges to x, $x_n \neq x$, then $\|\delta_{x_n} - \delta_x\|_{\mathrm{TV}} = 2$ whatever n.

Under Hypothesis 3.4, $\mathscr{M}$ is a bounded endomorphism on $(\mathbb{M}_{\rm fr}(\mathbb{X}), \|\cdot\|_{\rm BL})$.

Hypothesis 3.5. The kernel G is such that $\mathsf{G}(x,\cdot)$ has support in $[0,x]$, $\mathsf{G}(x,\mathbb{X}) \leq E$ for any $x \in \mathbb{R}_+$, $\mathscr{G}$ is a bounded endomorphism on $(\mathbb{M}_{\rm fr}(\mathbb{X}), \|\cdot\|_{\rm TV})$ and on $(\mathbb{M}_{\rm fr}(\mathbb{X}), \|\cdot\|_{\rm BL})$.

Proposition 3.5. *Assume Hypothesis 3.1 and 3.5. Assume that $\mu_0 \in B_{\rm TV}(r)$ for some $r \geq 0$. Then for any $T > 0$, there exists R such that (3.10) has a unique solution in $\mathbb{Y}_{T,R}$.*

The Fokker-Planck equation may be written

$$\frac{\partial \langle \mu(t), f\rangle}{\partial t} = \langle \mu(t)\mathcal{L}, f\rangle$$

for any $f \in \mathcal{C}_{\rm c}(\mathbb{X},\mathbb{R})$, as $\mathbb{M}_{\rm fr}(\mathbb{X})$ is the dual space of $\mathcal{C}_{\rm c}(\mathbb{X},\mathbb{R})$.

We are then led to define $\mathscr{G}$ and $\mathscr{M}$ as acting on continuous functions as

$$\mathscr{G}f(x) = \int_{\mathbb{X}} G(x,\,{\rm d}y) f(y) \text{ and } \mathscr{M}f(x) = F(x)f(x) \text{ for } f \in \mathcal{C}_{\rm c}(\mathbb{X},\mathbb{R}_+).$$

Again with Notation 2.5,

$$\langle \mu\mathscr{G}, f\rangle = \langle \mu, \mathscr{G}f\rangle \text{ and } \langle \mu\mathscr{M}, f\rangle = \langle \mu, \mathscr{M}f\rangle.$$

The conditions on $\mathscr{G}$ and $\mathscr{M}$ as acting on measures may then be transformed as conditions on $\mathscr{G}$ and $\mathscr{M}$ as acting on $\mathrm{W}^{1,\infty}(\mathbb{X},\mathbb{R})$. In particular, if these operators should be bounded endomorphisms on $\mathrm{W}^{1,\infty}(\mathbb{X},\mathbb{R})$, then they are bounded endomorphisms on $\mathbb{M}_{\rm fr}(\mathbb{X})$.

4. The Kolmogorov forward and backward equations

We intend in this part to come closer to a stochastic description as we construct a transition probability. For this, we follow the work of W. Feller [40]. This also allows one to consider the well-posedness and uniqueness of the fragmentation equation under mild hypotheses on the kernel F.

4.1. The Kolmogorov forward equations

We now consider solving (3.10) with the family of initial conditions δ_x. We are then led to consider

$$\frac{\partial P(t,x,\Lambda)}{\partial t} = \int_{\mathbb{X}} P(t,x,{\rm d}y) G(y,\Lambda) - \int_{\mathbb{X}} P(t,x,{\rm d}y) F(y) \mathbb{1}_\Lambda(y) \tag{4.1}$$

$$\text{with } P(0,x,\Lambda) = \mathbb{1}_\Lambda(x) \tag{4.2}$$

for any $t \in \mathbb{R}_+$, $x \in \mathbb{X}$ and $\Lambda \in {\rm Bor}(\mathbb{X})$. The system (4.1)-(4.2) is stronger than (9.10) as we are concerned with solving a family of equations, one per starting point.

Hypothesis 4.1. The rate is a measurable function $F : \mathbb{X} \to \mathbb{R}_+$.

Hypothesis 4.2. The kernel $\mathsf{G} : \mathbb{X} \to \mathbb{M}^+(\mathbb{X})$ is a probability kernel on $\mathbb{X}$ and for any $x \in \mathbb{X}$, $\mathsf{G}(x, \{x\}) = 0$.

Notation 4.1. We denote by $\mathbb{M}_{\mathrm{tr}}(\mathbb{R}_+ \times \mathbb{X})$ the space of parametric measures $(t, x) \mapsto \mu(t, x, \cdot)$ such that

(i) $\mu(t, x, \cdot) \in \mathbb{M}^+(\mathbb{X})$ for any $t \in \mathbb{R}_+$, $x \in \mathbb{X}$.
(ii) $t \mapsto \mu(t, x, \Lambda)$ is continuous for any $x \in \mathbb{X}$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$.
(iii) $x \mapsto \mu(t, x, \Lambda)$ is $\mathrm{Bor}(\mathbb{X})$-measurable for any $t \geq 0$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$.

Definition 4.1 (Transition probability)**.** A *transition probability* is a family $P = \{P(t, x, \Lambda)\}_{t \geq 0, x \in \mathbb{X}, \Lambda \in \mathrm{Bor}(\mathbb{X})}$ such that

(i) P belongs to $\mathbb{M}_{\mathrm{tr}}(\mathbb{R}_+ \times \mathbb{X})$.
(ii) $0 \leq P(t, x, \Lambda) \leq 1$ for any $t \geq 0$, $x \in \mathbb{X}$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$. If $P(t, x, \mathbb{X}) = 1$ for any $t \geq 0$, $x \in \mathbb{X}$, then P is *stochastic*. Otherwise, it is *substochastic*.
(iii) For any $x \in \mathbb{X}$, $\Lambda \in \mathrm{Bor}(\mathbb{X})$,

$$\lim_{t \to 0+} P(t, x, \Lambda) = \mathbb{1}_\Lambda(x).$$

Roughly speaking, a transition probability is such that $P(t, \cdot, \cdot)$ is a sub-probability kernel on $\mathbb{X}$ for any $t \geq 0$, with $P(0, x, \cdot) = \delta_x$.

Definition 4.2 (Kolmogorov forward equation)**.** We say that the transition probability P *solves the Kolmogorov forward equation* whenever for any $x \in \mathbb{X}$, any bounded set $\Lambda \in \mathrm{Bor}(\mathbb{X})$, $\partial_t P(t, x, \mathrm{d}y)$ exists for almost every time $t > 0$ and (4.1)-(4.2) are satisfied.

Remark 4.1. The Kolmogorov forward equation is also called the *Fokker-Planck equation* or the *master equation*.

Following the method of Section 3.1, we define the operator $\mathscr{A}^+$ which acts on $\mathbb{M}_{\mathrm{tr}}(\mathbb{R}_+ \times \mathbb{X})$ by

$$\mathscr{A}^+ P(t, x, \mathrm{d}z) := \int_0^t P(s, x, \mathrm{d}y) e^{-(t-s)F(z)} G(y, \mathrm{d}z)\, \mathrm{d}s. \tag{4.3}$$

A simple computation shows that for any $\Lambda \in \mathrm{Bor}(\mathbb{X})$,

$$\begin{aligned} &\frac{\partial \mathscr{A}^+ P(t, x, \Lambda)}{\partial t} \\ &\quad = (P\mathscr{G})(t, x, \Lambda) - \int_0^t \mathrm{d}s \int_{\mathbb{X}} P(s, x, \mathrm{d}y) e^{-(t-s)F(z)} F(z) G(y, \mathrm{d}z) \mathbb{1}_\Lambda(z) \\ &\quad = (P\mathscr{G})(t, x, \Lambda) - \int_{\mathbb{X}} (\mathscr{A}^+ P)(t, x, \mathrm{d}z) F(z) \mathbb{1}_\Lambda(z) \\ &\quad = P\mathscr{G}(t, x, \Lambda) - (\mathscr{A}^+ P)\mathcal{F}(t, x, \Lambda). \end{aligned} \tag{4.4}$$

Defining

$$P^{(0)}(t, x, \mathrm{d}y) := \delta_x(\mathrm{d}y) \exp(-tF(y)),$$

it holds that $P^{(0)}(t,x,\Lambda) := \exp(-tF(x))\mathbb{1}_\Lambda(x)$ and

$$\partial_t P^{(0)}(t,x,\,\mathrm{d}y) = -\int_{\mathbb{X}} P^{(0)}(t,x,\,\mathrm{d}y)F(y)\mathbb{1}_\Lambda(y) = -P^{(0)}\mathscr{M}(t,x,\mathrm{d}y).$$

The solution to

$$\partial_t P^{(0)}(t,x,\Lambda) = -\int_{\mathbb{X}} P^{(0)}(t,x,\,\mathrm{d}y)F(y)\mathbb{1}_\Lambda(y) \text{ with } P(t,x,\Lambda) = \mathbb{1}_\Lambda(x)$$

is given by

$$P^{(0)}(t,x,\Lambda) := \mathbb{1}_\Lambda(x)\exp(-tF(x)).$$

To solve $(\mathrm{Id} - \mathscr{A}^+)P(t,x,\Lambda) = P^{(0)}(t,x,\Lambda)$, we define recursively

$$P^{(n)}(t,x,\Lambda) := \mathscr{A}^+ P^{(n-1)}(t,x,\Lambda) \text{ for } n \geq 1 \tag{4.5}$$

and we set

$$P(t,x,\Lambda) := \sum_{n=0}^{+\infty} P^{(n)}(t,x,\Lambda) = \sum_{n=0}^{+\infty} (\mathscr{A}^+)^n \mathbb{1}_\Lambda(x)\exp(-tF(x)). \tag{4.6}$$

The arguments of Section 3.1 apply here, up to details that are covered in [40].

Proposition 4.1 ([39, Theorem 4.1],[40, Theorem 1])**.** *Under Hypotheses 4.1 and 4.2, the family P given by (4.6) is a transition probability and solves the Kolmogorov forward equation (4.1)-(4.2).*

Remark 4.2. It follows from [40, Eq. (23)] that $\partial_t P(t,x,\Lambda)$ exists for any time for all bounded sets $\Lambda \in \mathrm{Bor}(\mathbb{X})$.

4.2. The Kolmogorov backward equation

We now derive another equation solved by the transition probability of Proposition 4.1.

Similarly to $\mathscr{A}^+$ given by (4.3), we define for $t \geq 0$, $x \in \mathbb{X}$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$,

$$\mathscr{A}^- P(t,x,\Lambda) := \int_0^t \mathrm{d}s \int_{\mathbb{X}} e^{-sF(x)} G(x,\mathrm{d}z) P(t-s,z,\Lambda). \tag{4.7}$$

Again, assuming that $t \mapsto P(t,x,\Lambda)$ is differentiable,

$$\partial_t \mathscr{A}^- P(t,x,\Lambda) = \int_{\mathbb{X}} G(x,\mathrm{d}z)P(t,z,\Lambda) + \int_0^t \mathrm{d}s \int_{\mathbb{X}} e^{-sF(x)} G(x,\mathrm{d}z)\partial_t P(t-s,z,\lambda).$$

Using an integration by parts, assuming that $P(0,x,\Lambda) = \mathbb{1}_\Lambda(x)$,

$$\begin{aligned}
&\int_0^t \mathrm{d}s \int_{\mathbb{X}} e^{-sF(x)} G(x,\mathrm{d}z)\partial_t P(t-s,z,\lambda) \\
&\quad = \left[\int_{\mathbb{X}} G(x,\mathrm{d}z)e^{-sF(x)}P(t-s,z,\Lambda)\right]_0^t + F(x)\mathscr{A}^- P(t,x,\Lambda) \\
&\quad = \int_{\mathbb{X}} G(x,\mathrm{d}z)e^{-tF(x)}\mathbb{1}_\Lambda(x) - \mathscr{G}P(t,x,\Lambda) + \mathscr{M}(\mathscr{A}^- P)(t,x,\Lambda).
\end{aligned}$$

Therefore, whenever $P(0,x,\Lambda)=\mathbb{1}_\Lambda(x)$,

$$\partial_t \mathscr{A}^- P(t,x,\Lambda) = \mathscr{G} P(t,x,\Lambda) - \mathscr{M}(\mathscr{A}^- P)(t,x,\Lambda).$$

We define recursively

$$\begin{aligned} Q^{(0)}(t,x,\Lambda) &:= \exp(-tF(x))\mathbb{1}_\Lambda(x) = P^{(0)}(t,x,\Lambda) \\ \text{and } Q^{(n+1)}(t,x,\Lambda) &:= \mathscr{A}^- Q^{(n)}(t,x,\Lambda) \end{aligned}$$

as well as

$$Q(t,x,\Lambda) := \sum_{n=0}^{+\infty} Q^{(n)}(t,x,\Lambda). \tag{4.8}$$

We see that $Q^{(0)}(t,x,\Lambda)$ solves $\partial_t Q^{(0)}(t,x,\Lambda) = \mathscr{M} Q^{(0)}(t,x,\Lambda)$. The same arguments as in Section 3.1 show that $Q(t,x,\Lambda)$ solves $\partial_t Q(t,x,\Lambda) = \mathscr{F} Q(t,x,\Lambda)$. We refer to [40] for the details. Proposition 4.2 below summarizes the results.

Lemma 4.1. *For any $t \geq 0$, $x \in \mathbb{X}$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$, $Q(t,x,\Lambda) = P(t,x,\Lambda)$ where P is given by (4.6) and Q by (4.8).*

Proof. First, $P^{(0)}(t,x,\mathrm{d}y) = \delta_x(y)\exp(-tF(y)) = Q^{(0)}(t,x,\mathrm{d}y)$. In addition, using the Fubini-Tonelli theorem, the operators $\mathscr{A}^+$ and $\mathscr{A}^-$ commute so that if $P^{(m)} = Q^{(m)}$ for any $1 \leq m \leq n$,

$$\begin{aligned} P^{(n+1)} := \mathscr{A}^+ P^{(n)} &= \mathscr{A}^+ \mathscr{A}^- Q^{(n-1)} \\ &= \mathscr{A}^- \mathscr{A}^+ P^{(n-1)} = \mathscr{A}^- P^{(n)} = \mathscr{A}^- Q^{(n)} = Q^{(n+1)}. \end{aligned}$$

This proves the result. □

Definition 4.3 (Kolmogorov backward equation)**.** A transition function P *solves the Kolmogorov backward equation*[5] if $\partial_t P(t,x,\Lambda)$ exists for any $t \geq 0$, $x \in \mathbb{X}$, $\Lambda \in \mathrm{Bor}(\mathbb{X})$ and

$$\frac{\partial P(t,x,\Lambda)}{\partial t} = \int_{\mathbb{X}} G(x,\mathrm{d}y) P(t,y,\Lambda) - F(x) P(t,x,\Lambda), \tag{4.9}$$

$$P(0,x,\Lambda) = \mathbb{1}_\Lambda(x). \tag{4.10}$$

We introduce a new property of the transition probability.

Definition 4.4 (Chapman-Kolmogorov equation)**.** For a transition probability, the *Chapman-Kolmogorov* equation is that

$$P(t+s,x,\Lambda) = \int_{\mathbb{X}} P(t,x,\mathrm{d}y) P(s,y,\Lambda) = \langle P(t,x,\cdot), P(s,\cdot,\Lambda)\rangle \tag{4.11}$$

for any $s,t \geq 0$, $x \in \mathbb{X}$, $\Lambda \in \mathrm{Bor}(\mathbb{X})$.

[5] Here, backward refers to the fact that the operator acts on the value x which represents the initial point.

Proposition 4.2 ([40, Theorems 2-5], [41], [39, Theorem 3.1])**.** *Under Hypotheses 4.1 and 4.2, the function P given by (4.6) solves the Kolmogorov backward equation (4.9)-(4.10). Finally, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{X}$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$,*

$$P(t,x,\Lambda) = (1 - F(x)t)\mathbb{1}_\Lambda(x) + tF(x)\mathsf{G}(x,\Lambda) + \mathrm{o}(t). \tag{4.12}$$

When $P(t,x,\mathbb{X}) = 1$ for any $t > 0$, $x \in \mathbb{X}$, then it is the unique transition probability to do so among all the $P(t,x,\Lambda)$ that solves (4.9)-(4.10) for any t. Besides, it satisfies the Kolmogorov-Chapman equation (4.11).

As a by-product of the use of Lemma 3.1, we obtain that

$$P(t,x,\Lambda) = \exp(-tF(x))\mathbb{1}_\Lambda(x) + \mathscr{A}^- P(t,x,\Lambda) \tag{4.13}$$

for any $(t,x,\Lambda) \in \mathbb{R}_+ \times \mathbb{X} \times \mathrm{Bor}(\mathbb{X})$, from which we deduce (4.12).

Proposition 4.3 ([42, Sect. X.3, p. 330])**.** *Assume that F is bounded (Hypothesis 3.1) and Hypothesis 4.2 holds. Then $P(t,x,\mathbb{X}) = 1$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{X}$, and the Kolmogorov backward equation has a unique solution.*

Remark 4.3. The question of non-uniqueness of the solution of the Kolmogorov forward and backward equations is a tricky one [42, X.3, p. 330]. A solution P to the Kolmogorov forward or backward equation is called *minimal* if for any other solution Q, it holds that $P(t,x,\Lambda) \leq Q(t,x,\Lambda)$ for any $t \in \mathbb{R}_+$, $x \in \mathbb{X}$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$. Actually, the solution constructed by the iterated schemes (4.6) and (4.8) is minimal.

4.3. Comments on the Kolmogorov equations

It has to be noted that Proposition 4.2 ensures uniqueness of the solution to the Kolmogorov backward equation, which is not the case of Proposition 4.1.

Nothing ensures that $P(t,x,\mathbb{X}) = 1$ for any $x > 0$, $t > 0$, meaning that the mass is conserved. It is always possible to extend the state-space $\mathbb{X}$ to get such a condition by adding an external cemetery point. The article [40] contains conditions ensuring mass conservations. More details are given in Section 8.

We now detail the link between the Kolmogorov forward and backward equations using operators.

With the operators $\mathscr{G}$ and $\mathscr{M}$ acting on the left on measures and on the right of functions (see Section 3.3), the Kolmogorov forward equation (4.1) is written using Notation 3.4 as

$$\partial_t P(t,x,\mathrm{d}y) = P(t,x,\cdot)\mathscr{F}(\mathrm{d}y), \tag{4.14}$$

while the Kolmogorov backward equation (4.9) is written as

$$\partial_t P(t,x,\mathrm{d}y) = \mathscr{F} P(t,\cdot,\mathrm{d}y)(x). \tag{4.15}$$

The relationship between (4.14) and (4.15) which follows from Lemma 4.1 may also be understood from the Chapman-Kolmogorov equation (4.11): Since

$$\frac{P(t+\varepsilon,x,\Lambda) - P(t,x,\Lambda)}{\varepsilon} = \frac{1}{\varepsilon}\int_{\mathbb{X}} P(t,x,\mathrm{d}z)(P(\varepsilon,z,\Lambda) - \mathbb{1}_\Lambda(z)),$$

it follows formally that letting $\varepsilon \to 0$,

$$\partial_t P(t,x,\Lambda) = \int_{\mathbb{X}} P(t,x,\mathrm{d}z)\partial_t P(0,z,\Lambda)$$
$$= \int_{\mathbb{X}} P(t,x,\mathrm{d}z)(\mathscr{F}P(0,z,\lambda)) = \int_{\mathbb{X}} P(t,x,\mathrm{d}z)(\mathscr{F}\mathbb{1}_\Lambda(z)).$$

On the other hand, since formally $P(0,x,\cdot)=\delta_x$, we get

$$\frac{P(t+\varepsilon,x,\Lambda)-P(t,x,\Lambda)}{\varepsilon} = \frac{1}{\varepsilon}\int_{\mathbb{X}}(P(\varepsilon,x,\mathrm{d}z)-\delta_x(\mathrm{d}z))P(t,z,\Lambda)$$
$$= \frac{1}{\varepsilon}\int_{\mathbb{X}}\int_0^\varepsilon P(\eta,x,\mathrm{d}z)(\mathscr{G}-\mathscr{M})P(t,z,\Lambda)\,\mathrm{d}\eta\,\mathrm{d}z.$$

Letting ε converges to 0 leads to the Kolmogorov backward equation. The difficulty to apply rigorously this approach is explained in [42, Sect. X.3, p. 328]. Basically, it is due to the possible lack of uniform control in the expansion of the Kolmogorov forward equation in short time, while such a problem does not appear with (4.12) as the starting point is left fixed.

4.4. Construction of a semi-group

For convenience, we now use $\mathbb{X}^* = \mathbb{X}\setminus\{0\}$ as we will avoid 0 as a starting point.

Notation 4.2 (Semi-group)**.** We set for any $\phi \in \mathcal{B}(\mathbb{X}^*,\mathbb{R})$,

$$P_t\phi(x) := \int_{\mathbb{X}^*} P(t,x,\mathrm{d}z)\phi(z) = \langle P(t,x,\cdot),\phi\rangle \text{ for } t\geq 0,\ x\in\mathbb{X}^*.$$

The family $\{P_t\}_{t\geq 0}$ is called a *semi-group* on $(\mathcal{B}(\mathbb{X}^*,\mathbb{R}),\|\cdot\|_\infty)$ as $P_t(\mathcal{B}(\mathbb{X}^*,\mathbb{R}))\subset\mathcal{B}(\mathbb{X}^*,\mathbb{R})$, $P_{t+s}=P_sP_s$ for any $s,t\geq 0$ (from the Chapman-Kolmogorov equation (4.11)), and $P_0=\mathrm{Id}$.

Since $0\leq P(t,x,\mathbb{X}^*)\leq 1$, it is *contractive* on $\mathcal{B}(\mathbb{X}^*,\mathbb{R})$, meaning that $\|P_t\phi\|_\infty \leq \|\phi\|_\infty$ for any $\phi\in\mathcal{B}(\mathbb{X}^*,\mathbb{R})$. It is also *preserves the positivity*, meaning that if $\phi\geq 0$ on $\mathbb{X}^*$, then $P_t\phi\geq 0$ on $\mathbb{X}^*$.

With $\phi\in\mathcal{B}(\mathbb{X}^*,\mathbb{R})$, we then rewrite (4.14) as

$$\partial_t P_t\phi(x) = \int_{\mathbb{X}^*} P(t,x,\mathrm{d}z)G(z,\mathrm{d}y)\phi(y) - \int_{\mathbb{X}^*} P(t,x,\,\mathrm{d}y)F(y)\phi(y)$$

and (4.15) as

$$\partial_t P_t\phi(x) = \int_{\mathbb{X}^*} G(x,\mathrm{d}z)P(t,z,\mathrm{d}y)\phi(y) - \int_{\mathbb{X}^*} F(x)P(t,x,\,\mathrm{d}y)\phi(y).$$

Integrating against a measure μ, whenever the integrals are well defined, we obtain for the Kolmogorov forward equation

$$\partial_t\langle\mu,P_t\phi\rangle = \langle\mu,P_t\mathscr{F}\phi\rangle.$$

On the other hand, with the Kolmogorov backward equation (4.15),

$$\partial_t \langle \mu, P_t \phi \rangle = \langle \mu, \mathscr{F} P_t \phi \rangle = \langle \mu \mathscr{F}, P_t \phi \rangle.$$

In Notation 4.2, we consider semi-groups as acting on $\mathcal{B}(\mathbb{X}^*, \mathbb{R})$. There are sub-classes of semi-groups of particular interest.

Definition 4.5 (Strongly continuous semi-group)**.** A semi-group $\{T_t\}_{t \geq 0}$ on a Banach space $\mathbb{Y}$ is *strongly continuous* on $\mathbb{Y}$ whenever $T_t f$ converges to f for any $f \in \mathbb{Y}$ as $t \to 0$.

Definition 4.6 (Generator)**.** Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semi-group on $\mathbb{Y}$. The operator $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$ defined by

$$\mathrm{Dom}(\mathscr{L}) := \left\{ \phi \in \mathbb{Y} \;\middle|\; \lim_{\substack{t \to 0 \\ t > 0}} \frac{T_t \phi - \phi}{t} \text{ exists} \right\} \subset \mathbb{Y},$$
$$\mathscr{L} \phi := \lim_{\substack{t \to 0 \\ t > 0}} \frac{T_t \phi - \phi}{t}$$

is the *generator* of $\{T_t\}_{t \geq 0}$. The space $\mathrm{Dom}(\mathscr{L})$ is the domain of $\mathscr{L}$.

When a semi-group with generator $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$ is strongly continuous on $\mathbb{Y}$, then

$$P_t \phi = \phi + \int_0^t P_s \mathscr{L} \phi \, \mathrm{d}s, \text{ for all } \phi \in \mathrm{Dom}(\mathscr{L}), \tag{4.16}$$

$$P_t \phi = \phi + \int_0^t \mathscr{L} P_s \phi \, \mathrm{d}s, \text{ for all } \phi \in \mathbb{Y}. \tag{4.17}$$

A classical result is the following one: not only $\mathrm{Dom}(\mathscr{L})$ is not empty, but it is dense in $\mathbb{Y}$.

Lemma 4.2. *Assume that the semi-group $\{T_t\}_{t \geq 0}$ is strongly continuous on $\mathbb{Y}$ with generator $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$. The domain $\mathrm{Dom}(\mathscr{L})$ is dense in $\mathbb{Y}$, and $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$ is a closed operator*[6]*.*

Conversely, the infinitesimal generator $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$ of a strongly continuous semi-group determines the semi-group uniquely provided that $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$ is closed and densely defined [36, Theorem II.1.4, p. 53].

Unfortunately, there is no reason for the semi-group $\{P_t\}_{t \geq 0}$ associated to the couple (F, G) to be strongly continuous without further controls.

Lemma 4.3. *Assume that the rate is bounded (Hypothesis 3.1). Then the semi-group $\{P_t\}_{t \geq 0}$ generated by $(\mathscr{F}, \mathcal{B}(\mathbb{X}^*, \mathbb{R}))$ is strongly continuous on $\mathcal{B}(\mathbb{X}^*, \mathbb{R})$.*

Proof. Since the rate is bounded, $\|\mathscr{F}\phi\|_\infty \leq 2\|F\|_\infty \|\phi\|_\infty$ for any $\phi \in \mathcal{B}(\mathbb{X}^*, \mathbb{R})$. Therefore, $\mathscr{F}(\mathcal{B}(\mathbb{X}^*, \mathbb{R})) \subset \mathcal{B}(\mathbb{X}^*, \mathbb{R})$.

[6] This means that if ϕ_n converges to ϕ in $\mathbb{Y}$ for some sequence $\{\phi_n\}$ and $\mathscr{L}\phi_n$ converges to some $\psi \in \mathbb{Y}$, then $\psi \in \mathrm{Dom}(\mathscr{L})$ and $\psi = \mathscr{L}\phi$.

Let $\phi(x) := \sum_{i=1}^n a_i \mathbb{1}_{\Lambda_i}(x)$ be a step function. Then from the Kolmogorov backward equation,

$$P_t\phi(x) - \phi(x) = \int_0^t F(x) \int G(x, \mathrm{d}y) P(s, y, \Lambda_i) a_i \,\mathrm{d}s - \int_0^t F(x) P(s, x, \Lambda_i) a_i \,\mathrm{d}s$$

from which

$$\|P_t\phi - \phi\|_\infty \leq 2t\|F\|_\infty \sup_{i=1,\dots,n} |a_i|.$$

By approximating any bounded, measurable function by a step function, such inequality is sufficient to conclude that $\{P_t\}_{t\geq 0}$ is strongly continuous on $\mathcal{B}(\mathbb{X}^*, \mathbb{R})$. In addition,

$$\left\| \frac{P_t\phi - \phi}{t} - \mathscr{F}\phi \right\|_\infty \leq \frac{1}{t} \int_0^t 2s\|F\|_\infty \|P_s\phi - \phi\|_\infty \,\mathrm{d}s \leq t\|F\|_\infty^2 \|\phi\|_\infty.$$

Hence, $(\mathscr{F}, \mathcal{B}(\mathbb{X}^*, \mathbb{R}))$ is the infinitesimal generator of $\{P_t\}_{t>0}$. □

Actually, from the proof of Lemma 4.3, the semi-group $\{P_t\}_{t\geq 0}$ is *uniformly continuous*, that is

$$\|P_t - \mathrm{Id}\|_{\infty\to\infty} \leq 2\|F\|_\infty.$$

Such inequality implies that the infinitesimal generator $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$ of $\{P_t\}_{t\geq 0}$ is actually bounded, that is $\mathscr{L}$ is a bounded linear operator and $\mathrm{Dom}(\mathscr{L}) = \mathbb{Y}$ [36, Corollary II.1.5, p. 52].

When using semi-groups, we could replace $\mathcal{B}(\mathbb{X}^*, \mathbb{R})$ by particular subspaces. Two kinds of semi-groups are of particular importance in stochastic analysis.

Definition 4.7 (Feller-Dynkin and Feller semi-group)**.** Let $\{P_t\}_{t\geq 0}$ be a strongly continuous, positivity-preserving and contractive semi-group on $\mathbb{Y}$. Then

1. If $\mathbb{Y} = (\mathcal{C}_0(\mathbb{X}^*, \mathbb{R}), \|\cdot\|_\infty)$, then $\{P_t\}_{t\geq 0}$ is a *Feller-Dynkin semi-group* (meaning that $P_t(\mathcal{C}_0(\mathbb{X}^*, \mathbb{R})) \subset \mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$).
2. If $\mathbb{Y} = (\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R}), \|\cdot\|_\infty)$, then $\{P_t\}_{t\geq 0}$ is a *Feller semi-group* (meaning that $P_t(\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})) \subset \mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$).

Let us give an example of the Feller semi-group.

Hypothesis 4.3. The rate F is bounded (Hypothesis 3.1) and continuous. Besides, and $x \in \mathbb{X} \mapsto G(x, \cdot) \in \mathbb{M}^+(\mathbb{X})$ (space of positive measures) is continuous with respect to the weak topology of $\mathbb{M}^+(\mathbb{X})$.

The proof of the next lemma is immediate.

Lemma 4.4. *Under Hypothesis 4.3, $(\mathscr{F}, \mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R}))$ is a bounded operator on $\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$ and the associated semi-group is Feller (and is continuous).*

Lemma 4.5. *Assume that $\{P_t\}_{t\geq 0}$ is contractive and strongly continuous on a Banach space $\mathbb{Y} \subset \mathcal{B}(\mathbb{X}^*, \mathbb{R})$ — either $\mathcal{B}(\mathbb{X}^*, \mathbb{R})$, $\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$, or $\mathcal{C}_\mathrm{c}(\mathbb{X}^*, \mathbb{R})$ — then its generator is $(\mathscr{L}, \mathrm{Dom}(\mathscr{L})) = (\mathscr{F}, \mathrm{Dom}(\mathscr{F}))$ with*

$$\mathrm{Dom}(\mathscr{F}) := \{\phi \in \mathbb{Y} \mid \mathscr{F}\phi \in \mathbb{Y}\}. \tag{4.18}$$

Proof. Let $\phi \in \mathrm{Dom}(\mathscr{F}) \subset \mathcal{B}(\mathbb{X}^*, \mathbb{R})$. Then there exists a family of step functions $\{\phi_m\}_m$ that approximate uniformly ϕ, with $\phi_m \leq \phi$.

Since $\int_0^t \int_{\mathbb{X}^*} P(s,x,\mathrm{d}y)G(y,\Lambda)\,\mathrm{d}s$ and $\int_0^t \int_{\mathbb{X}^*} P(s,x,\mathrm{d}y)F(y)\mathbb{1}_\Lambda(y)\,\mathrm{d}s$ are well defined for any $\Lambda \in \mathrm{Bor}(\mathbb{R}_+)$, they are well defined in particular for $\Lambda \in \mathbb{X}^*$. Thefore

$$\int_{\mathbb{X}^*} G(y,\mathrm{d}z)\phi_m(z) \leq \|\phi\|_\infty F(y)G(y,\mathbb{X}^*) \text{ and } F(y)\phi_m(y) \leq \|\phi\|_\infty F(y)\mathbb{1}_{\mathbb{X}^*}(y)$$

and $F(y)$ and $G(y,\mathbb{X}^*)$ are integrable on $[0,t]\times\mathbb{X}^*$ with respect to $P(s,x,\mathrm{d}y)\,\mathrm{d}s$. From the Lebesgue dominated convergence theorem, we then obtain that

$$P_t\phi(x) = \phi(x) + \int_0^t P_s\mathscr{F}\phi(x)\,\mathrm{d}s, \quad \text{for all } t \geq 0,\ x \in \mathbb{X}^*.$$

Since $\{P_t\}_{t\geq 0}$ is contractive on $\mathbb{Y}$, $\|P_s\mathscr{F}\phi\|_\infty \leq \|\mathscr{F}\phi\|_\infty$ and then

$$\left\|\frac{P_t\phi - \phi}{t} - \mathscr{F}\phi\right\|_\infty \leq \left\|\frac{1}{t}\int_0^t (P_s\mathscr{F}\phi - \mathscr{F}\phi)\,\mathrm{d}s\right\|_\infty.$$

The strong continuity of $\{P_t\}_{t\geq 0}$ and the property of the Riemann integral implies that

$$\frac{P_t\phi - \phi}{t} = \mathscr{F}\phi \text{ for any } \phi \in \mathrm{Dom}(\mathscr{F}).$$

Hence, the generator $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$ of $\{P_t\}_{t\geq 0}$ extends $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}))$ in the sense that $\mathrm{Dom}(\mathscr{F}) \subset \mathrm{Dom}(\mathscr{L})$ and $\mathscr{L}\phi = \mathscr{F}\phi$ for any $\phi \in \mathrm{Dom}(\mathscr{F})$.

On the other hand, let $\phi \in \mathrm{Dom}(\mathscr{L})$ be such that $\mathscr{L}\phi = \phi$. Then

$$P_t\phi = \phi + \int_0^t P_s\mathscr{L}\phi\,\mathrm{d}s = \phi + \int_0^t P_s\phi\,\mathrm{d}s.$$

Multiplying both sides by $\exp(-t)$ and performing an integration by parts,

$$\int_0^{+\infty} \exp(-t)P_t\phi\,\mathrm{d}t = \phi + \int_0^{+\infty} \exp(-t)P_t\phi\,\mathrm{d}t$$

so that $\phi = 0$. By [73, Lemma III.4.17, p. 237], this proves that $(\mathscr{L}, \mathrm{Dom}(\mathscr{L})) = (\mathscr{F}, \mathrm{Dom}(\mathscr{F}))$. □

4.5. Probabilistic representation

Our main point of interest is to use the transition P as the transition probability for a stochastic process ξ defined on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ with a filtration $(\mathcal{H}_t)_{t\geq 0}$. The filtration is a family of σ-algebras which contains the information up to time t. Here, for any $\omega \in \Omega$, $t \mapsto \xi_t(\omega)$ is a path which describes the evolution of a typical particle. Averaging over many such paths gives the average behavior of the evolution of the mass and allows one to recover the concentration.

More precisely, the relation between ξ and P that solves (4.1)–(4.2) or (4.9)–(4.10) is the following

$$\mathbb{P}[\xi(t+s) \in \Lambda \mid \mathcal{H}_t] = P(s, \xi(t), \Lambda) \text{ for any } s,t \geq 0 \text{ and } \Lambda \in \mathrm{Bor}(\mathbb{X}).$$

Knowing the stochastic process contains much more information than just knowing its marginal distributions $\xi(t)$ for any time t, as it encodes a dynamical behavior.

We now focus on constructing several possible probabilistic representations of the fragmentation equation. Each such representation underlines a different aspect of the probability theory yet showing its unity. Besides, they have some practical impact in term of Monte Carlo simulations as well as inference from observed data.

5. Evolution of the mass as a stochastic process from a Markov chain

We now present several structures that specify the dynamical evolution of the mass of a typical fragment as a stochastic process, that is, as a random evolution in time. The mass may only decrease, which we specify with the next hypothesis.

Hypothesis 5.1. In addition to Hypotheses 4.1 and 4.2 (mainly $\mathsf{G}(x,\{x\}) = 0$ for any $x \in \mathbb{X}$), for any $x \in \mathbb{X}$, the support of $\mathsf{G}(x,\cdot)$ is contained in $[0,x]$ that is $\mathsf{G}(x,\Lambda) = 0$ for any $\Lambda \in \mathrm{Bor}(\mathbb{X})$ with $\Lambda \cap [0,x] = \emptyset$.

We first describe the evolution of the mass as a discrete event indexed by the number of breakages, with

- Y_k, the mass after k-th breakage,
- σ_k, the time span between two breakages,
- τ_k the time at which the k-th breakage occurs.

5.1. A description by a Markov chain

Let us consider first a Markov chain $\{(\sigma_k, Y_k)\}_{k\geq 0}$ living in $\mathbb{R}_+ \times \mathbb{X}$ defined by $Y_0 = x$, $\sigma_0 = 0$ and given Y_k, Y_{k+1} and σ_k are drawn independently with the distributions

$$\sigma_{k+1} \sim \mathcal{E}(F(Y_k)),\ Y_{k+1} \sim \mathsf{G}(Y_k,\cdot), \tag{5.1}$$

where $\mathcal{E}(\lambda)$ is the exponential distribution of parameter λ. The distribution of (σ_{k+1}, Y_{k+1}) given $Y_k = x$ has distribution

$$\mathsf{R}_x(\mathrm{d}s, \mathrm{d}y) = F(x)\exp(-sF(x))\mathsf{G}(x,\mathrm{d}y)\,\mathrm{d}s. \tag{5.2}$$

We note that $\sigma_k > 0$ a.s. for any $k \geq 1$. The time of the events are defined as

$$\tau_k = \sigma_0 + \sigma_1 + \cdots + \sigma_k \text{ for } k \geq 0. \tag{5.3}$$

The sequence $\{\tau_k\}_{k\geq 0}$ is increasing. Since $G(x,\Lambda) = G(x,\Lambda \cap [0,x))$ for any $x \in \mathbb{X}$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$, the sequence $\{Y_k\}_{k\geq 0}$ is decreasing almost surely.

The sequence $\{(\tau_k, Y_k)\}_{k\geq 0}$ is also a Markov chain: it represents the typical evolution of a fragment. At time τ_k, the Y_{k-1}-mer breaks up into two new fragments, one being a Y_k-mer. The σ_k's represent the time span between two breakages.

The algorithmic construction of realizations of such a chain is easy. We detail this in Algorithm 1 in Section 10.2 below.

5.2. An alternative description using a Poisson point process

We introduce now another construction of the above chain. With this new construction, the chain depends on a single stochastic process, called a Poisson point process, which serves as a "random generator" for all the other random variables. Later, this construction will be helpful to understand the formulation through stochastic differential equations.

We consider a PPP N (see Appendix B) on $\mathbb{R}_+^3$ with intensity $n(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v) = \mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v$ on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

Hypothesis 5.2. In addition to Hypothesis 5.1, for any $x \in \mathbb{X}$, $\mathsf{G}(x, \cdot)$ has a density $\mathsf{g}(x, \cdot)$ with respect to the Lebesgue measure on $\mathbb{X}$.

Notation 5.1 (Density of G)**.** We set $g(x, y) := F(x)\mathsf{g}(x, \cdot)$, the density of $G(x, \cdot)$.

Notation 5.2. For a point $x \in \mathbb{R}_+$, we set

$$\Gamma(x) := \big\{(u, v) \in \mathbb{R}_+^2 \;\big|\; v \leq g(x, u)\big\}.$$

Proposition 5.1. *Assume Hypotheses 3.1 and 4.1. Let us construct iteratively a family $\{(\zeta_k, U_k, V_k)\}_{k\geq 0}$ with values in $\mathbb{R}_+^3$ by setting $\zeta = 0$, $U_0 := 0$ and defining $(\zeta_{k+1}, U_{k+1}, V_{k+1})$ as the leftmost point of the PPP N in the subset $I_k \times \Gamma(U_k)$ for an interval $I_k := [\theta_0 + \cdots + \theta_k, +\infty)$, for any $k \geq 0$. Then $\{(\zeta_k, U_k)\}_{k\geq 0}$ is a Markov chain with the same distribution as $\{(\sigma_k, Y_k)\}_{k\geq 0}$ defined in Section 5.1.*

Proof. We set $\nu(\mathrm{d}u, \mathrm{d}v) := \mathrm{d}u\,\mathrm{d}v$ so that N has intensity measure $\mathrm{d}s\,\nu(\mathrm{d}u, \mathrm{d}v)$. Hence,

$$\nu(\Gamma(x)) = \int_0^{+\infty}\int_0^{+\infty} \mathbb{1}_{v\leq g(x,u)}\,\mathrm{d}u\,\mathrm{d}v = \int_0^{+\infty} g(x, u)\,\mathrm{d}u = G(x, \mathbb{X}) = F(x).$$

With Lemma B.1 and Remark B.1 in Appendix B, the leftmost point (ζ, U, V) of N in $[t, +\infty) \times \Gamma(x)$ satisfies:

(i) (U, V) is independent from ζ and uniform in $\Gamma(x)$,

(ii) $\zeta \sim \mathcal{E}(F(x))$.

With Lemma C.1, U has for density $F(x)^{-1}g(x, \cdot) = \mathsf{g}(x, \cdot)$. With the rejection principle (Lemma C.2), this proves the result. □

With this construction, the probability space $(\Omega, \mathcal{H}, \mathbb{P})$ of the chain is the one supporting the PPP. There exists a measurable function Φ from Ω to $\mathbb{R}_+^\infty \times \mathbb{X}^\infty$ such that $\{(\sigma_k(\omega), Y_k(\omega))\}_{k\geq 0} = \Phi(\omega)$. In Section 5.1, the probability space has to be constructed inductively as a limit of the finite chains $\{(\sigma_k, Y_k)\}_{0\leq k\leq n}$ thanks to the Ionescu Tulcea theorem [52].

6. From Markov chain to jump process

We now give three ways to embed the Markov chain to a continuous time stochastic process:

- Sect. 6.1 The embedding of a Markov chain into a jump process and the construction of the associated transition probability, which is the probabilistic counterpart of the construction of the solution of the Kolmogorov forward et backward equations.
- Sect. 6.2 The thinning of a Poisson process, where the associated semi-group is constructed through the infinitesimal generator. However, this construction is valid only for bounded rates.
- Sect. 6.6 The construction of a marked point process. The distribution follows from the martingale problem.

6.1. Embedding the Markov chain into a jump process

We have given two constructions of Markov chains in Sections 5.1 and 5.2.

We now denote by $\Xi := \{(\tau_k, Y_k)\}_{k\geq 0}$, encoding the times $\{\tau_k\}_{k\geq 0}$ at which the breakages occurs, and $\{Y_k\}_{k\geq 0}$, the mass of the fragments.

Notation 6.1 (Cemetery point and point at infinity)**.** We append to $\mathbb{X}$ a cemetery point †. We write $\overline{\mathbb{X}} := \mathbb{X} \cup \{\dagger\}$ and $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. Actually, as we consider only a decreasing mass, the cemetery point could also be identified with 0. Any function ϕ on $\mathbb{X}$ are extended to functions on $\overline{\mathbb{X}}$ by setting $\phi(\dagger) = 0$.

Our goal is now to construct a $\overline{\mathbb{X}}$-valued continuous process $\xi = \{\xi(t)\}_{t\geq 0}$ indexed by the time and to identify its distribution.

Notation 6.2 (Embedding the Markov chain)**.** From a $(\mathbb{R}_+ \times \overline{\mathbb{X}})$-valued Markov chain $\Xi := \{(\tau_k, Y_k)\}_{k\geq 0}$, we define

$$\xi(t) := \begin{cases} Y_k & \text{if } \tau_k \leq t < \tau_{k+1} \text{ and } Y_0 \in \mathbb{X}, \\ \dagger & \text{if } t \geq \tau_\infty := \lim_n \tau_n \text{ or } Y_0 = \dagger. \end{cases} \tag{6.1}$$

Definition 6.1 (Conservative/regular process; explosive process)**.** Let $\tau_\infty := \lim_n \tau_n$. If almost surely $\tau_\infty = +\infty$, then the process is said to be *conservative* or *regular*. This is equivalent to $P(t, x, \mathbb{X}) = 1$ for any $(t, x) \in \mathbb{R}_+ \times \mathbb{X}^*$. Otherwise, the process is *explosive*.

As we saw in Section 4.2 (See also Section 8), the process ξ is not always conservative. This justifies the introduction of the cemetery point, which is useless for conservative processes (for example when the rate is bounded, see Proposition 4.3). At the end of this section, we give conditions for the process to be conservative.

Definition 6.2 (Characteristics of a process)**.** The process ξ defined by (6.1) is called the *(minimal[7]) process with characteristics* (F, G).

[7] As we only consider minimal processes, see Remarks 4.3 and 6.1, we drop this adjective.

Notation 6.3 (Space of càdlàg process)**.** We set $\mathcal{D}(\mathbb{R}_+, \mathbb{X})$ the space of paths from $\mathbb{R}_+$ to $\mathbb{X}$ which are left-continuous with right-limits at each time[8].

The process ξ defined above takes its values in $\mathcal{D}(\mathbb{R}_+, \mathbb{X})$ and is piecewise constant on $[\tau_k, \tau_{k+1})$. It has jumps only when breakages occur.

Notation 6.4 (Counting process)**.** The *counting process* $[\![\xi]\!]$ is

$$[\![\xi]\!]J := \#\{\tau_k \mid \tau_k \in J\} \text{ for } J \in \mathrm{Bor}(\mathbb{R}_+),$$

i.e., $[\![\xi]\!]J$ is the number of events occurring in J. We also set $[\![\xi]\!](\tau) := [\![\xi]\!](0, \tau]$.

We now define properly the underlying probabilistic structure as in [39, Sect. 2]:

- **Probability space.** We denote by Ω the set of all sequences $\{(t_k, x_k)\}_{k\geq 0}$ in $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$ such that (i) $x_0 \in \mathbb{X}$ and $t_0 = 0$; (ii) if $t_k < +\infty$, then $t_k < t_{k+1}$ and $x_k \in \mathbb{X}$; (iii) if $t_k = +\infty$, then $t_k = t_{k+1}$ and $x_k = \dagger$. We also set $t_\infty := \lim_{k\to\infty} t_k$ and $x_\infty = \dagger$. This space Ω is a measurable subspace of $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$.
- **σ-algebra.** We then denote by $\mathcal{H}$ the σ-field of measurable subsets of Ω, that is the set of elements of type $\Lambda \cap \Omega$ when Λ is measurable in $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$, where the σ-algebra on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$ is the product σ-algebra.
- **Filtration.** For any $t \geq 0$, we define $\mathcal{H}_t := \sigma(\mathrm{Bor}(\mathbb{X}), \mathcal{H}'_t)$ with

$$\mathcal{H}'_t := \sigma(\mathbb{1}_{x_k \in \Lambda} \mathbb{1}_{t_k \leq s} \mid 0 \leq s \leq t,\ \Lambda \in \mathrm{Bor}(\mathbb{X}),\ k \geq 0).$$

This filtration $\{\mathcal{H}_t\}_{t\geq 0}$ is an *a.s. jumping filtration* on $\mathbb{R}_+$ [55, Definition 1, p. 15], *i.e.* $\mathcal{H}_t$ and $\mathcal{H}_{t_k}$ coincide on $\{t_k \leq t < t_{k+1}\} \in \mathcal{H}_t$.

- **Stopped σ-algebra.** A *stopping time* is a random time τ such that $\{\tau \leq t\} \in \mathcal{H}_t$. For a stopping time τ, we define the *stopped σ-algebra* as

$$\mathcal{H}_\tau := \left\{A \in \mathcal{G} \;\middle|\; A \cap \{\tau \leq t\} \in \mathcal{H}_t \text{ for all } t \in \mathbb{R}_+\right\}.$$

With the above filtration, for any $\Lambda \in \mathcal{H}$, $\Lambda \cap \{t_k \leq t\} \in \mathcal{H}_t$ so that each t_k is a stopping time.

- **Process.** This is the process ξ defined by (6.1). Its distribution is denoted by $\mathbb{P}$. Besides, we write $\mathbb{P}_x$ for the distribution of ξ given $\xi(0) = x$, $x \in \mathbb{X}$.

Notation 6.5. For $u \geq 0$, we denote respectively by $\mathsf{K}_u(\mathrm{d}s, \mathrm{d}y)$ and $\mathsf{H}_u(\mathrm{d}s)$ the regular conditional distributions of $(\tau_{[\![\xi]\!](u)+1}, \xi([\![\xi]\!](u)+1))$ given $\mathcal{H}_u$ and of $\tau_{[\![\xi]\!](u)+1}$ given $\mathcal{H}_u$. This means that we consider by K_u (resp. H_u) the distribution of the mass and time (resp. time only) of the first breakage occurring after the time u.

We skip the technical details, especially regarding the measurability, of the next lemma. However, thanks to the memoryless property of the exponential distribution (see Lemma A.1), its meaning is clear.

[8] or *continu à droite avec une limite à gauche* (*càdlàg*).

Lemma 6.1 ([39, Lemma 2.1]). *For all $t \geq u \geq 0$,*

$$\mathsf{H}_u(\mathrm{d}t) = F(\xi(u))\exp(-(t-u)F(\xi(u)))\,\mathrm{d}t \text{ on } \{[\![\xi]\!](u) < +\infty\},$$
$$\mathsf{K}_u(\mathrm{d}t,\mathrm{d}y) = \exp(-(t-u)F(\xi(u)))G(\xi(u),\mathrm{d}y)\,\mathrm{d}t \text{ on } \{[\![\xi]\!](u) < +\infty\}.$$

Now, let us consider computing $\mathbb{P}[\xi(t) \in \Lambda | \mathcal{H}_u]$ for $0 \leq u \leq t$ and $\Lambda \in \mathrm{Bor}(\mathbb{X})$.

Lemma 6.2. *For any $u < t$, $\Lambda \in \mathrm{Bor}(\mathbb{X})$ and $n \geq 0$,*

$$P^{(n)}(t-u,\xi(u),\Lambda) = \mathbb{P}[\xi(t) \in \Lambda, [\![\xi]\!](u,t] = n \mid \mathcal{H}_u], \tag{6.2}$$

where $P^{(n)}$ is defined in (4.5).

Sketch of the proof. We give only the main argument of the proof. We left the details on measurability and cite the steps in [39].

First, with the notations of Section 4,

$$\mathbb{P}[\xi(t) \in \Lambda, [\![\xi]\!](u,t] = 0 \mid \mathcal{H}_u] = \mathbb{P}\big[\xi(t) \in \Lambda, \tau_{[\![\xi]\!](u)+1} > t \,\big|\, \mathcal{H}_u\big]$$
$$= \mathbb{P}\big[\xi(u) \in \Lambda, \tau_{[\![\xi]\!](u)+1} > t \,\big|\, \mathcal{H}_u\big] = \mathbb{1}_\Lambda(\xi(u))\mathsf{H}_u([t,+\infty)) = P^{(0)}(t-u,\xi(u),\Lambda).$$

Now, for some $n \geq 0$, our aim is to show by induction that (6.2) holds at level n. Then

$$\mathbb{P}[\xi(t) \in \Lambda, [\![\xi]\!](u,t] = n+1 \mid \mathcal{H}_u]$$
$$= \int_u^t \int_{\mathbb{X}} \mathbb{P}\big[\xi(t) \in \Lambda, [\![\xi]\!](\tau_{[\![\xi]\!](u)+1},t] = n \,\big|\, \mathcal{H}_u, \tau_{[\![\xi]\!](u)+1} = s, Y_{[\![\xi]\!](u)+1} = y\big] \times \mathsf{K}_u(\mathrm{d}s,\mathrm{d}y)$$
$$= \int_u^t \int_{\mathbb{X}} \mathbb{P}\big[\xi(t) \in \Lambda, [\![\xi]\!](s,t] = n \,\big|\, \mathcal{H}_{\tau_{[\![\xi]\!](u)+1}}, \tau_{[\![\xi]\!](u)+1} = s, Y_{[\![\xi]\!](u)+1} = y\big] \times \mathsf{K}_u(\mathrm{d}s,\mathrm{d}y).$$

Using the induction hypothesis and the definition of $\mathscr{A}^+$ in (4.3) as well as the expression of K_u in Lemma 6.1, we obtain

$$\mathbb{P}[\xi(t) \in \Lambda, [\![\xi]\!](u,t] = n+1 \mid \mathcal{H}_u] = \int_u^t \int_{\mathbb{X}} P^{(n)}(t-s,y,\Lambda)\mathsf{K}_u(\mathrm{d}s,\mathrm{d}y)$$
$$= \mathscr{A}^+ P^{(n)}(t-u,\xi(u),\Lambda) = P^{(n+1)}(t-u,\xi(u),\Lambda). \tag{6.3}$$

Hence we get the result. □

Proposition 6.1. *The process ξ with characteristics (F,G) (Definition 6.2) is a Markov process whose probability transition P satisfies the Kolmogorov forward and backward equations, and is given by (4.6) or (4.8). In particular, P is minimal.*

Proof. From Lemma 6.2, by decomposition over the possible values in $[\![\xi]\!](u,t]$,

$$\mathbb{P}[\xi(t)\in\Lambda \mid \mathcal{H}_u] = \sum_{n\geq 0}\mathbb{P}[\xi(t)\in\Lambda,\ [\![\xi]\!](u,t]=n \mid \mathcal{H}_u] = \sum_{n\geq 0} P^{(n)}(t-u,\xi(u),\Lambda) = P(t-u,\xi(u),\Lambda).$$

The result follows from Propositions 4.1 and 4.2. □

Proposition 6.1 shows that the process ξ with characteristics (F,G) is easy to simulate thanks to Algorithm 1. See Section 4.5 for some comments on practical applications.

This process is not only Markov but also strongly Markov, *i.e.* the independence with respect to the σ-algebra $\mathcal{H}_\tau$ for any suitable stopping time τ can be assessed also when a fixed time is replaced by a stopping time.

Proposition 6.2 ([29, 37, 55])**.** *The process ξ with characteristics (F,G) is a strong Markov process.*

Such a process ξ is a *Pure Jump Markov Process*. This class of stochastic processes was introduced by W. Feller in [40] under the name *Purely Discontinuous Markov Process*. This class of processes is included in the class of *Piecewise Deterministic Markov Process* (PDMP) [29]. itself included in the class of *Jump Markov Process* [55] (see also the work [37]). Heuristically, this process is Markov as a consequence of the memoryless property of the exponential distribution.

Lemma 6.3 ([55, Theorem 9])**.** *Let ξ be a strong Markov process on a filtered probability space $(\Omega,\mathcal{H},(\mathbb{P}_x)_{x\in\overline{\mathbb{X}}},(\mathcal{H})_{t\geq 0})$ which is piecewise constant. Assume that the distribution of $(\tau,\Delta_\tau\xi)$, where τ is the time of its first jump and $\Delta_t\xi = \xi(t)-\xi(t-)$, is $\mathcal{E}(F(x))\otimes\mathsf{G}(x,\cdot)$. Then its probability transition function P satisfies the Kolmogorov backward equation. Besides, if ξ is such that $\xi(t)=\dagger$ whenever $t>\lim_{k\geq 0}\tau_k$, where τ_k are the successive jumps of t, then P is minimal and thus corresponds to the one constructed in Section 4.2.*

Proof. Fix $t>0$, $\Lambda\in\mathrm{Bor}(\overline{\mathbb{X}})$. By the strong Markov property,

$$\begin{aligned}
P(t,x,\Lambda) &= \mathbb{P}_x[\xi(t)\in\Lambda] = \mathbb{P}_x[\xi(t)\in\Lambda \mid \tau>t] \\
&\quad + \int_0^t\int_{\overline{\mathbb{X}}} \mathbb{P}_x[\xi(t)\in\Lambda \mid (\tau,\Delta_\tau\xi)=(s,y)]F(x)\exp(-F(x)s)\mathsf{G}(x,\mathrm{d}y)\,\mathrm{d}s \\
&= \mathbb{1}_\Lambda(x)\exp(-F(x)t) + \int_0^t\int_{\overline{\mathbb{X}}} \mathbb{P}_y[\xi(t-s)\in\Lambda]F(x)\exp(-F(x)s)\mathsf{G}(x,\mathrm{d}y)\,\mathrm{d}s \\
&= \int_0^t\int_{\overline{\mathbb{X}}} P(t-s,y,\Lambda)\exp(-F(x)s)G(x,\mathrm{d}y) \\
&= \mathbb{1}_\Lambda(x)\exp(-F(x)t) + \int_0^t\int_{\overline{\mathbb{X}}} P(t-s,y,\Lambda)\exp(-F(x)s)G(x,\mathrm{d}y).
\end{aligned}$$

This is (4.13), so that P also satisfies (4.9)-(4.10). □

Remark 6.1. Using (6.2), we see that since $P(t,x,\mathbb{X}^*) = \sum_{n\geq 0} P^{(n)}(t,x,\mathbb{X}^*)$,

$$P(t,x,\mathbb{X}^*) = 1 - \mathbb{P}_x[[\![\xi]\!](0,t] = +\infty].$$

Therefore, the probability transition function is substochastic if and only if (as we consider here only processes which are minimal in the sense of Lemma 6.3) their probability transition functions are also the minimal ones solving the Kolmogorov forward and backward equations (See Remark 4.3).

We give a necessary and sufficient condition for the process to be conservative/regular. A thoughtful discussion may be found in [77] (More references are given in Section 8).

Proposition 6.3 (See *e.g.,* [58, Proposition 13.5, p. 281])**.** *Under* $\mathbb{P}_x$, $\tau_\infty = +\infty$ *almost surely if and only if* $\sum_{k=1}^{+\infty} 1/F(Y_k) = +\infty$ *almost surely.*

From this proposition, it is easily seen that if F is bounded (Hypothesis 3.1), then the process is conservative (See Proposition 4.3).

6.2. Thinning Poisson processes

Following [38, Section 4.2, p. 163], we now give another way to embed a Markov chain into a jump process. This leads naturally to the martingale problem, yet in a restrictive setting. We assume Hypothesis 3.1 (boundedness of the rate) and we set

$$\widehat{\mathsf{G}}(x,\mathrm{d}y) := \left(1 - \frac{F(x)}{\varkappa}\right)\delta_x + \frac{F(x)}{\varkappa}\mathsf{G}(x,\mathrm{d}y),\ \widehat{F}(x) := \varkappa := \|F\|_\infty.$$

We define $\widehat{\xi}$ as the process with characteristics $(\widehat{F},\widehat{\mathsf{G}})$. The difference with ξ is that $\widehat{\xi}$ may stay on the same position. Actually, if a "jump" occurs at time $\widehat{\tau}$, there is a probability $F(\xi(\tau-))/\varkappa$ that $\widehat{\xi}(\widehat{\tau}) \neq \widehat{\xi}(\widehat{\tau}-)$. Note that this was excluded in Hypothesis 4.2. The jump times $\widehat{\tau}_k$ are all distributed as $\mathcal{E}(\varkappa)$.

Lemma 6.4. *[38, Section 4.2, p. 163] Let* $\{Z_k\}_{k\geq 0}$ *be a Markov chain with transition kernel* $\widehat{\mathsf{G}}$ *and* N *be an independent Poisson process with rate* $\varkappa$*. Then* $\xi' := \{Z_{N(t)}\}_{t\geq 0}$*,* $\widehat{\xi}$ *and* ξ *have the same finite dimensional distributions.*

On the topic, see also Corollary 6.3 and Theorem 6.1 in Section 6.4 on the martingale problem. In particular, ξ', ξ and $\widehat{\xi}$ are actually equal in distribution.

We define

$$\mathscr{P}f(x) := \int_{\mathbb{X}} f(y)\widehat{\mathsf{G}}(x,\mathrm{d}y).$$

Clearly, $\mathscr{P}$ is a bounded linear operator on $\mathcal{B}(\mathbb{X}^*,\mathbb{R})$ with norm 1. In addition,

$$\begin{aligned}\mathscr{F}f(x) &= F(x)\int_{\mathbb{X}}(f(y)-f(x))\mathsf{G}(x,\mathrm{d}y) = \varkappa\int_{\mathbb{X}}(f(y)-f(x))\widehat{\mathsf{G}}(x,\mathrm{d}y)\\ &= \varkappa(\mathscr{P}-\mathrm{Id})f(x).\end{aligned}$$

The (unique) strongly continuous semi-group generated by $(\mathscr{F}, \mathcal{B}(\mathbb{X}^*, \mathbb{R}))$ of Lemma 4.3 may be written

$$P_t := \sum_{k\geq 0} e^{-\varkappa t} \frac{(\varkappa t)^k}{k!} \mathscr{P}^k, \ t \geq 0.$$

This follows from the fact that formally,

$$P_t = \exp(t\varkappa\mathscr{P} - t\varkappa \mathrm{Id}) = \exp(t\varkappa\mathscr{P}) \exp(-t\varkappa \mathrm{Id})$$

since the operators $\varkappa\mathscr{P}$ and $\varkappa\mathrm{Id}$ commute.

Lemma 6.5 ([38, Section 4.2, p. 163])**.** *Let $\{\mathcal{H}'_t\}_{t\geq 0}$ be the filtration generated by the Poisson process N and the Markov chain $\{Z_k\}_{k\geq 0}$ in Lemma 6.4. Then for any $\phi \in \mathcal{C}_c(\mathbb{X}^*, \mathbb{R})$,*

$$\mathbb{E}[\phi(\xi'(s+t)) \mid \mathcal{H}'_s] = P_t\phi(\xi'(s)) \text{ for any } s, t \geq 0, \tag{6.4}$$

where $\xi' := \{Z_{N(t)}\}_{t\geq 0}$. Therefore, ξ' satisfies the Markov property and $\{P_t\}_{t\geq 0}$ is its associated semi-group.

Thinning algorithm The above construction also gives rise to the *thinning algorithm*, which is a simplified version of the Ogata algorithm [72, 66]. To simulate ξ, one has to simulate $\widehat{\xi}$ or ξ'. For this, we simulate first a Poisson process with rate $\varkappa$, with successive times $\{\tau_k\}$. At each time τ_k, $\widehat{\xi}$ jumps according to the distribution $\mathsf{G}(\widehat{\xi}(\tau_k-), \cdot)$, with probability $F(\widehat{\xi}(\tau_k-))$, otherwise does not change. Algorithm 1 is simpler here. Yet this thinning algorithm is useful in presence of growth or abrasion.

6.3. The extended generator

We have defined the notion of generator associated to the semi-group. However, this notion requires the semi-group to be strongly continuous and thus refers to an ambient Banach space. In a probabilistic context, it is natural to consider it as the space of continuous or bounded functions. On the other hand, in functional analysis approaches, the ambient space is that of integrable functions.

We now present another notion of generator, called the *extended generator* [26, 29].

Through all this chapter, we work under the following hypothesis, which is stronger than Hypothesis 4.1, yet weaker than Hypothesis 3.1.

Hypothesis 6.1. The rate F is bounded on any compact subset of $\mathbb{X}^* = (0, +\infty)$.

Notation 6.6. Let ξ be the process with characteristics (F, G). Let ϕ be a measurable function. We denote by

$$M_t^\phi := \phi(\xi(t)) - \phi(\xi(0)) - \int_0^t \mathscr{F}^\sharp\phi(\xi(s))\,\mathrm{d}s, \ t \geq 0. \tag{6.5}$$

Definition 6.3 (Extended generator)**.** Let ξ be the process with characteristics (F, G). Let $(\mathscr{F}^\sharp, \mathrm{Dom}(\mathscr{F}^\sharp))$ be an operator that satisfies

(i) $\mathrm{Dom}(\mathscr{F}^\sharp)$ contains measurable functions.

(ii) For any $\phi \in \mathrm{Dom}(\mathscr{F}^\sharp)$, $s \mapsto \mathscr{F}^\sharp\phi(\xi(s))$ is Lebesgue integrable almost surely on $[0,t]$ for any $t > 0$, and $\{M_t^\phi\}_{t\geq 0}$ defined by (6.5) is a local martingale.

Lemma 6.6. *Let $\Lambda \in \mathrm{Bor}(\mathbb{X}^*)$. Then $\mathbb{1}_\Lambda \in \mathrm{Dom}(\mathscr{F}^\sharp)$ where $\mathscr{F}^\sharp$ is the extended generator of ξ, $\mathscr{F}^\sharp\mathbb{1}_\Lambda(x) = \mathscr{F}\mathbb{1}_\Lambda(x) = G(x,\Lambda) - F(x)\mathbb{1}_\Lambda(x)$, and $\{M_t^{\mathbb{1}_\Lambda}\}_{t\geq 0}$ is a martingale.*

Proof. First, for any $x \in \mathbb{X}^*$ and $t \geq 0$,

$$\mathbb{E}\left[\int_0^t F(\xi(s))\mathbb{1}_\Lambda(\xi(s))\,\mathrm{d}s\right] = \int_0^t P(s,x,\mathrm{d}y)F(y)\mathbb{1}_\Lambda(y) < +\infty$$

as well as

$$\mathbb{E}\left[\int_0^t G(\xi(s),\Lambda)\,\mathrm{d}s\right] = \int_0^t P(s,x,\mathrm{d}y)G(y,\Lambda) < +\infty$$

from the existence of the solution to the Kolmogorov forward equation. Hence, $(\omega,t) \mapsto \mathscr{F}\mathbb{1}_\Lambda(\xi(t,\omega))$ is Lebesgue integrable for $\mathbb{P}\otimes \mathrm{d}t$.

For any $s \leq t$,

$$\begin{aligned}\mathbb{E}[M_t^{\mathbb{1}_\Lambda} \mid \mathcal{H}_s] &= \mathbb{E}\left[\mathbb{1}_\Lambda(\xi(t)) - \mathbb{1}_\Lambda(\xi(0)) - \int_0^t \mathscr{F}\mathbb{1}_\Lambda(\xi(r)) \,\middle|\, \mathcal{H}_s\right] \\ &= \mathbb{E}[\mathbb{1}_\Lambda(\xi(t)) \mid \mathcal{H}_s] - \mathbb{1}_\Lambda(\xi(s)) - \int_s^t \mathbb{E}[\mathscr{F}\mathbb{1}_\Lambda(\xi(r)) \mid \mathcal{H}_s]\,\mathrm{d}s + M_s^{\mathbb{1}_\Lambda}.\end{aligned}$$

Owing to the Markov property,

$$\begin{aligned}\mathbb{E}[M_t^{\mathbb{1}_\Lambda} \mid \mathcal{H}_s] &= M_s^{\mathbb{1}_\Lambda} + P(t-s,\xi(s),\Lambda) - \mathbb{1}_\Lambda(\xi(s)) \\ &\quad - \int_s^t \int_{\mathbb{X}^*} P(r-s,\xi(s),\mathrm{d}y)G(y,\Lambda) - \int_s^t P(r-s,\xi(s),\mathrm{d}y)F(y)\mathbb{1}_\Lambda, \text{ a.s.}.\end{aligned}$$

Thanks to the Kolmogorov forward equation (4.1)-(4.2), $\{M_t^{\mathbb{1}_\Lambda}\}_{t\geq 0}$ is a martingale and $\mathbb{1}_\Lambda \in \mathrm{Dom}(\mathscr{F}^\sharp)$. □

We now focus on subspaces of the domain of the extended generator.

Notation 6.7. We the notations of Definition 6.3, we set

$$\mathrm{Mart}(\mathscr{F}^\sharp) := \left\{ \phi \in \mathcal{B}(\mathbb{X}^*,\mathbb{R}) \;\middle|\; \begin{array}{l} \mathbb{E}_x\left[\int_0^t |\mathscr{F}^\sharp\phi|(\xi(s))\,\mathrm{d}s\right] < +\infty \\ \text{for any } (t,x) \in \mathbb{R}_+ \times \mathbb{X}^* \end{array} \right\} \subset \mathrm{Dom}(\mathscr{F}^\sharp).$$

A consequence of Lemma 6.6 is the so-called *Dynkin formula*.

Proposition 6.4 (Dynkin formula)**.** *For any $\phi \in \mathrm{Mart}(\mathscr{F}^\sharp)$, M^ϕ is a martingale. Besides, $\mathscr{F}^\sharp\phi = \mathscr{F}$ and the* Dynkin formula *holds:*

$$P_t\phi(x) = \phi(x) + \int_0^t P_s\mathscr{F}\phi(x)\,\mathrm{d}s. \tag{6.6}$$

Moreover, for any $t \geq 0$, $P_t\phi \in \mathrm{Mart}(\mathscr{F}^\sharp)$, $P_t\mathscr{F}\phi = \mathscr{F}P_t\phi$ for any $t \geq 0$, and

$$P_t\phi(x) = \phi(x) + \int_0^t \mathscr{F}P_s\phi(x)\,\mathrm{d}s.$$

At last,

$$\lim_{t\to 0} P_t\phi(x) = \phi(x) \text{ for any } x \in \mathbb{X}^*. \tag{6.7}$$

Remark 6.2. Eq (6.7) means that the semi-group with *weakly continuous*. If the semi-group maps $\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$ to itself, then it it is strongly continuous on $\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$.

Proof. The proof of (6.6) is the same as the one of Lemma 4.5. That M^ϕ is a martingale follows from the same arguments as the one of Lemma 6.6.

Fix $v \geq 0$ and $\phi \in \mathrm{Mart}(\mathscr{F}^\sharp)$. We define

$$N_t := P_v\phi(\xi(t)) - P_v\phi(\xi(0)) - \int_0^t P_v\mathscr{F}\phi(\xi(r))\,\mathrm{d}r,\ t \geq 0.$$

As in the poof of Lemma 6.6 and using the semi-group property of $\{P_t\}_{t\geq 0}$,

$$\mathbb{E}[N_t - N_s \mid \mathcal{H}_s] = P_{t-s+v}\phi(\xi(s)) - P_v\phi(\xi(s)) - \int_0^t P_{r-s+v}\mathscr{F}\phi(\xi(s))\,\mathrm{d}r.$$

This quantity is equal to 0 thanks to the Dynkin formula (6.6) so that $\{N_t\}_{t\geq 0}$ is a martingale. Therefore, $P_v\phi \in \mathrm{Mart}(\mathscr{F}^\sharp)$ and $\mathscr{F}^\sharp P_v\phi = \mathscr{F}P_v\phi = P_v\mathscr{F}\phi$.

The weak continuity of P_t follows from the integrability condition of $(s,\omega) \mapsto \phi(\xi(s,\omega))$ on $[0,t] \times \Omega$. □

Proposition 6.5. (i) *Let $\{P_t\}_{t\geq 0}$ be a contractive, strongly continuous semi-group with generator $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}))$ on $\mathbb{Y} \subset \mathcal{B}(\mathbb{X}^*, \mathbb{R})$ (We refer to Lemma 4.5). Then* $\mathrm{Dom}(\mathscr{F}) \in \mathrm{Mart}(\mathscr{F}^\sharp)$.

(ii) *Under Hypothesis 3.1 (boundedness of the rate), then* $\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R}) \subset \mathrm{Mart}(\mathscr{F}^\sharp)$.

(iii) *Under Hypothesis 6.1 (local boundedness of the rate), then* $\mathcal{C}_\mathrm{c}(\mathbb{X}^*, \mathbb{R}) \subset \mathrm{Mart}(\mathscr{F}^\sharp)$.

Proof. These properties holds because for ϕ in either $\mathrm{Dom}(\mathscr{F})$, $\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$ or $\mathcal{C}_\mathrm{c}(\mathbb{X}^*, \mathbb{R})$, $\mathscr{F}\phi$ is bounded. More specifically, for (iii), recall that $G(x, [0,x]) = 1$ so that $\int_{\mathbb{X}^*} G(x, \mathrm{d}y)\phi(y) = 0$ whenever x close to 0 does not belong to the support of ϕ. □

With Proposition 6.4, we get a solution of the fragmentation equation.

Corollary 6.1. *Let $\phi \in \mathrm{Mart}(\mathscr{F}^\sharp)$, then*

$$u : (t,x) \mapsto P_t\phi(x) \in \mathrm{L}^1([0,T], \mathcal{B}(\mathbb{X}^*, \mathbb{R}))$$

solves the fragmentation equation $\partial_t u(t,x) = \mathscr{F}u(t,x)$ with initial condition $u(0,x) = \phi(x)$.

6.4. The martingale problem

We now discuss an approach through the martingale problem. We just give some hints, a complete study is found in [38].

Hypothesis 6.2. The operator $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}))$ is a closed operator on $\mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$ with

$$\mathrm{Dom}(\mathscr{F}) := \{\phi \in \mathcal{C}_0(\mathbb{X}^*, \mathbb{R}) \mid \mathscr{F}\phi \in \mathcal{C}_0(\mathbb{X}^*, \mathbb{R})\}.$$

Besides, the rate F is bounded (in particular, the associated process generated by (F, G) is conservative, see Proposition 4.3).

When the rate F is bounded, the operator $\mathscr{F}$ is bounded on $\mathcal{B}(\mathbb{X}^*, \mathbb{R})$, and thus $\lambda - \mathscr{F}$ is invertible for $\lambda > 0$ large enough. However, it does not mean that $\mathrm{Dom}(\mathscr{F})$ is not empty unless additional hypotheses are added on the pair (F, G).

Example 6.1. Assume that $F \in \mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$ and $x \mapsto G(x, \cdot)$ is weakly continuous. Then Hypothesis 6.2 holds and $\mathrm{Dom}(\mathscr{F}) = \mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$.

Example 6.2. Assume that the rate F is continuous, bounded and that $G(x, \mathrm{d}y) = \delta_{\lambda x}$ for some $\lambda > 0$. Then Hypothesis 6.2 holds and $\mathrm{Dom}(\mathscr{F}) = \mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$.

Lemma 6.7. *Under Hypothesis 6.2,* $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}))$ *is* conservative *(that is that there exists* ϕ_n *such that* $\lim_{n\to\infty} \mathscr{F}\phi_n(x) = 0$ *and* $\lim_{n\to\infty} \phi_n(x) = 1$*), and satisfies the* positive maximum principle *(that is for any* $\phi \in \mathrm{Dom}(\mathscr{F})$ *and* y *such that* $\sup_{x\in\mathbb{X}^*} \phi(x) = \phi(y) \geq 0$*, then* $\mathscr{F}\phi(y) \leq 0$*). The last condition implies that the operator is* dissipative *(that is* $\|(\lambda - \mathscr{F})\phi\|_\infty \geq \|\phi\|_\infty$ *for any* $\phi \in \mathrm{Dom}(\mathscr{F})$ *for any* $\lambda > 0$*, [38, Lemma 4.2.1, p. 165]).*

The first result is the existence of a strongly continuous semi-group[9] as well as the existence of a strong Markov process.

Theorem 6.1 ([38, Theorems 1.2.6 (p. 13), 4.2.2 (p. 165), and 4.2.7, (p. 169)])**.** *Assume Hypothesis 6.2 and moreover that* $\mathrm{Dom}(\mathscr{F})$ *is dense in* $\mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$ *and that* $\mathrm{Dom}(\lambda - \mathscr{F}) = \mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$ *for some* $\lambda > 0$*. Then:*

(i) *The operator* $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}))$ *generates (See Definition 4.6) a strongly continuous, positive, contractive semi-group* $\{Q_t\}_{t\geq 0}$ *on* $\mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$*, that is a Feller semi-group.*

(ii) *For any probability measure* ν*, there exists a Markov process* $\zeta \in \mathcal{D}(\mathbb{R}_+, \mathbb{X}^*)$ *(see Notation 6.3) with initial distribution* ν *and whose corresponding semi-group is* $\{Q_t\}_{t\geq 0}$*. Besides,* ζ *is strong Markov with respect to the filtration* $\{\mathcal{H}_t\}_{t\geq 0}$*,* $\mathcal{H}_t := \cap_{\epsilon>0}\sigma(\zeta(s), s \leq t + \epsilon)$*.*

Remark 6.3. The process ζ constructed above is the one constructed in Section 6.2.

Definition 6.4 (Martingale problem, [38, Section 4.3, p. 173])**.** A probability measure $\mathbb{P}$ solves the *martingale problem* for $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}), \nu)$ if there exists a stochastic process ζ on a filtered probability space $(\Omega, \mathcal{H}, \mathbb{P}, \{\mathcal{H}_t\}_{t\geq 0})$ such that

$$M_t^\phi := \phi(\zeta(t)) - \phi(\zeta(0)) - \int_0^t \mathscr{F}\phi(\zeta(s))\,\mathrm{d}s$$

[9] This is actually a consequence of the Hille-Yosida theorem.

is a martingale for any $\phi \in \mathrm{Dom}(\mathscr{F})$ and $\zeta(0)$ has a given distribution ν.

The following result is a consequence of Proposition (i) for the process ξ and of [38, Proposition 4.1.7, p. 162] for ζ.

Corollary 6.2. *Under the conditions of Theorem 6.1, both the processes ξ with characteristics (F, G) and ζ of Theorem 6.1 solve the martingale problem for $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}), \nu)$ for any initial distribution ν.*

We have constructed possibly two different processes ξ and ζ, one associated with a semi-group $\{P_t\}_{t\geq 0}$ given by the Kolmogorov equations, and the other associated with a semi-group $\{Q_t\}_{t\geq 0}$.

Corollary 6.3. *The processes ξ and ζ have the same distribution.*

Proof. As the processes ξ and ζ are Markov processes, it is sufficient to show that $\xi(t)$ and $\zeta(t)$ have the same marginal distribution for any $t \geq 0$. From Proposition 6.4 and Remark 6.2, the semi-group $\{P_t\}_{t\geq 0}$ of ξ is strongly continuous on $\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$ and then on $\mathcal{C}_0(\mathbb{X}^*, \mathbb{R})$, and with Lemma 4.5, its infinitesimal generator is $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}))$. It is then uniquely defined [36, Theorem II.1.4, p. 53] and thus $Q_t = P_t$ for any $t \geq 0$. This implies the equality of the marginal distributions of ξ and ζ. Finally, since $\mathbb{X}^*$ is separable, this is sufficient to determine their distributions in $\mathcal{D}(\mathbb{R}_+, \mathbb{X}^*)$ [38, Proposition 3.7.1, p. 127]. □

From now, we have constructed two solutions ξ and ζ of the martingale problem, which are furthermore Markov processes. More may arise.

Definition 6.5 (Uniqueness, [38, Section 4.4, p. 182])**.** Let ν be a distribution on $\mathbb{X}^*$. Then *uniqueness holds* for the martingale problem $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}), \nu)$ whenever any two solutions have the same finite-dimensional distribution.

Definition 6.6 (Well-posedness, [38, Section 4.4, p. 182])**.** The martingale problem (Definition 6.4) is *well-posed* whenever for any $y \in \mathbb{X}^*$, there exists one and only one solution to the martingale problem with the initial distribution δ_y.

We now state two uniqueness results. Recall that Theorem 6.1 gives one construction of the solution of the martingale problem. Yet other may exists.

Theorem 6.2 ([38, Theorem 4.4.2, p. 184])**.** *Assume that for any starting distribution μ, any two solutions of the martingale problem with respect to $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}), \mu)$ have the same finite-dimensional distributions. Then the solution is Markov, has paths in $\mathcal{D}(\mathbb{R}_+, \mathbb{X}^*)$, and is unique in the sense of Definition 6.5.*

Theorem 6.3 (Direct consequence of [38, Theorem 4.4.1, p. 182])**.** *Under the conditions of Theorem 6.1, the martingale problem is well-posed and any solution is Markov.*

Dealing with functions in $\mathcal{C}_\mathrm{c}(\mathbb{X}^*, \mathbb{R})$ may be to stringent. Using some localization argument, one may relax $\mathcal{C}_\mathrm{c}(\mathbb{X}^*, \mathbb{R})$ to $\mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$ and even assume that the rate F is not bounded. We now give some related results from V.N. Kolokoltsov on the subject. We refer to [61] for the proofs.

Proposition 6.6 (Proposition 2.2(i), [61]). *Under Hypothesis 4.3, the operator* $(\mathscr{F}, \mathcal{C}_{\mathrm{b}}(\mathbb{X}^*, \mathbb{R}))$ *is bounded and the associated martingale problem is well-posed.*

Hypothesis 6.3. There exists a function $\phi \in \mathcal{C}(\mathbb{X}^*, \mathbb{R}_+)$ such that F is uniformly bounded on $U_\alpha := \{x \in \mathbb{X} \mid \psi(x) \leq \alpha\}$, $\alpha > 0$ and $\mathscr{F}\psi(x) \leq b + c\psi(x)$ for any $x \in \mathbb{X}^*$ and some constants $b, c \geq 0$. The rate $x \in \mathbb{X}^* \mapsto G(x, \cdot) \in \mathbb{M}^+(\mathbb{X}^*)$ is weakly continuous.

Proposition 6.7 (Proposition 2.2(ii), [61]). *Under Hypothesis 6.3, for any* $x \in \mathbb{X}$, *there exists a unique Markov process* ξ *whose infinitesimal generator is* $\mathscr{F}$ *with* $\mathcal{C}_{\mathrm{c}}(\mathbb{X}^*, \mathbb{R}) \subset \mathrm{Dom}(\mathscr{F})$, *and* ξ *is the unique solution to the martingale problem with initial measure* δ_x.

6.5. A description using a marked point process

Using the rate F and the kernel G, we have constructed a Markov chain $\{(\tau_k, Y_k)\}_{k\geq 0}$ which we have embedded into a continuous time stochastic process $\{\xi(t)\}_{t\geq 0}$ with jumps at times τ_k such that $\xi(\tau_k) = Y_k$. Here, Y_k represents the mass after a breakage that occurs at time τ_k.

Notation 6.8 (Loss of mass at breakage). The loss of mass at breakage is the quantity $\Delta_k\xi := -(\xi(\tau_{k+1}) - \xi(\tau_k)) \geq 0$.

Notation 6.9 (Kernel for the loss of mass). We define the kernel G^{LM} as

$$\mathsf{G}^{\mathrm{LM}}(x, [0, u]) := \begin{cases} \mathsf{G}(x, [x-u, x]) & \text{for } 0 \leq u \leq x, \\ \mathsf{G}(x, [0, u]) = 1 & \text{for } u \geq x. \end{cases} \tag{6.8}$$

This means that if X is distributed as $\mathsf{G}(x, \cdot)$, then $x - X$ is distributed as $\mathsf{G}^{\mathrm{LM}}(x, \cdot)$. The indice LM refers to the *loss of mass* as it corresponds to the remaining mass after a breakage. We also set

$$G^{\mathrm{LM}}(x, \cdot) := F(x)\mathsf{G}^{\mathrm{LM}}(x, \cdot). \tag{6.9}$$

The sequence $\{(\tau_k, \Delta_k\xi)\}_{k\geq 0}$ is not Markov as it depends on the remaining mass.

We consider now another representation using random measures on $[0, +\infty] \times \mathbb{X}$. For this aim, we define the random measure

$$\mu(\omega; \mathrm{d}t, \mathrm{d}x) := \sum_{k\geq 1} \delta_{\tau_k(\omega), \Delta_k\xi(\omega)}(\mathrm{d}t, \mathrm{d}x)\mathbb{1}_{\tau_k(\omega) < +\infty}, \tag{6.10}$$

where $\delta_{(s,x)}$ is the Dirac mass at (s, x). This measure is identified with points on $\mathbb{R}_+ \times \mathbb{X}$, where the coordinates of the points represent the time and the loss of mass when a breakage occurs.

Such a measure can be integrated against any measurable, bounded function f on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$ with $f(+\infty, x) = f(t, \dagger) = 0$ for any $(t, x) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$ by

(omitting the ω),

$$\int_0^{+\infty} \int_{\mathbb{X}} f(s,x)\mu(\mathrm{d}t,\mathrm{d}x) = \sum_{k\geq 1} f(\tau_k, \Delta_k\xi). \tag{6.11}$$

The set $\{(\tau_k, \Delta_k\xi)\}_{\geq 1}$ is called a *multivariate point process*, or a *marked point process*.

From the knowledge of μ and (6.11), we recover many information on ξ. For example, with $f : (t; s, x) \mapsto \mathbb{1}_{(0,t]}(s)$,

$$\int_0^{+\infty} \int_{\mathbb{X}} f(t;s,x)\mu(\mathrm{d}t,\mathrm{d}x) = \sum_{\substack{k\geq 1\\ \tau_k\leq t}} 1 = [\![\xi]\!](0,t].$$

Using the function $f : (t; s, x) \mapsto -\mathbb{1}_{(0,t]}(s)x$,

$$\int_0^{+\infty} \int_{\mathbb{X}} f(t;s,x)\mu(\mathrm{d}t,\mathrm{d}x) = -\sum_{\substack{k\geq 1\\ \tau_k\leq t}} \Delta_k\xi = \xi(t) - \xi(0). \tag{6.12}$$

The random measure is defined on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$. As it contains temporal evolution, we consider also a filtration $(\mathcal{H}_t)_{t\geq 0}$ such that $\mathcal{H}_t \subset \mathcal{H}$ and $\mu((0,t],\cdot)$ is $\mathcal{H}_t$-measurable for any $t > 0$. Given $\Lambda \in \mathrm{Bor}(\mathbb{X})$ and $t \geq 0$, $h > 0$, one may ask what is the mean number of points of μ falling in $[t, t+h)\times\Lambda$ given $\mathcal{H}_{t-}$, where $\mathcal{H}_{t-}$ represents the information seen up to time $t-$. For h small,

$$\mathbb{P}[[\![\xi]\!][t,t+h) = \ell \mid \mathcal{H}_{t-}] = \begin{cases} \exp(-hF(\xi(\tau_{[\![\xi]\!](t-)}))) & \text{if } \ell = 0,\\ hF(\xi(\tau_{[\![\xi]\!](t-)}))) & \text{if } \ell = 1,\\ \mathrm{o}(h) & \text{if } \ell \geq 2. \end{cases} \tag{6.13}$$

Most likely, there will be no points or one point. Given there is one point, the probability it falls in Λ is $\mathsf{G}^{\mathrm{LM}}(\xi(\tau_{[\![\xi]\!](t-)}), \Lambda)$.

Note that since ξ is piecewise constant, $\xi(\tau_{[\![\xi]\!](t-)}) = \xi(t-)$ for any $t \geq 0$.

Thus, the mean number of points in the small volume $[t, t+\Delta t)\times\Delta y$ is approximately

$$F(\xi(\tau_{[\![\xi]\!](t-)}))\mathsf{G}^{\mathrm{LM}}(\xi(\tau_{[\![\xi]\!](t-)}), \Delta y)\Delta t = G^{\mathrm{LM}}(\xi(\tau_{[\![\xi]\!](t-)}), \Delta y)\Delta t.$$

This suggests to define a new random measure

$$\begin{aligned} \nu(\mathrm{d}t,\mathrm{d}x) &= \sum_{k\geq 0} \mathbb{1}_{\tau_k < t \leq \tau_{k+1}} F(\xi(\tau_k))\mathsf{G}^{\mathrm{LM}}(\xi(\tau_k),\mathrm{d}x)\,\mathrm{d}t\\ &= \sum_{k\geq 0} \mathbb{1}_{\tau_k < t \leq \tau_{k+1}} G^{\mathrm{LM}}(\xi(\tau_k),\mathrm{d}x)\,\mathrm{d}t\\ &= G^{\mathrm{LM}}(\xi(\tau_{[\![\xi]\!](t-)}),\mathrm{d}x)\,\mathrm{d}t = G^{\mathrm{LM}}(\xi(t-),\mathrm{d}x)\,\mathrm{d}t. \end{aligned} \tag{6.14}$$

This random measure ν has two main features: it is $(\mathcal{H}_{t-})_{t\geq 0}$-measurable, hence depends only on the past. Besides, $\mathbb{E}[\mu([t,t+h),\Lambda)] = \mathbb{E}[\nu([t,t+h),\Lambda)]$.

We now formalize this construction.

Hypothesis 6.4. On a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ supporting $\{(\tau_k, \Delta_k\xi)\}_{k\geq 0}$, we assume that there exists a filtration $\{\mathcal{H}_t\}_{t\geq 0}$ which is right-continuous, such that $\mathcal{H}_t \subset \mathcal{H}$, τ_k is a stopping time with respect to $(\mathcal{H}_t)_{t\geq 0}$ and $\Delta_k\xi$ is $\mathcal{H}_{\tau_k}$-measurable. Furthermore, we assume that

$$\mathcal{H}_t = \mathcal{H}_0 \vee \sigma(\mu((0,s)\times\Lambda);\ s\leq t,\ \Lambda\in \mathrm{Bor}(\mathbb{X})) \text{ for any } t\geq 0. \tag{6.15}$$

Such a filtration may be called the *intrinsic history*[10] [28, p. 358].

In (6.15), $\mu((0,s)\times B)$ is the number of points of $\{(\tau_k, \Delta_k\xi)\}_{k\geq 0}$ that falls in $(0,s)\times B$.

Definition 6.7 (Predictable process)**.** A process $X : \Omega\times\mathbb{R}_+$ is *predictable* if $(\omega, t)\mapsto X_t(\omega)$ is measurable with respect to the σ-algebra $\mathcal{P}$ generated by left-continuous process which are $\{\mathcal{H}_t\}_{t\geq 0}$-adapted.

Definition 6.8. A random measure ν is called *predictable* if for each process $X : \Omega\times\mathbb{R}_+\times\mathbb{X}\to\mathbb{R}_+$ which is measurable with respect to $\mathcal{P}\otimes\mathrm{Bor}(\mathbb{X})$, the process $(\nu X) : \Omega\times\mathbb{R}_+\to\mathbb{R}_+$ defined by

$$(\nu X)_t(\omega) := \int_{\mathbb{X}}\int_0^t X(\omega, s, x)\nu(\omega; \mathrm{d}s, \mathrm{d}x)$$

is predictable.

Our measure ν given by (6.14) is clearly predictable.

Theorem 6.4 ([53, Theorem (2.1), (2.4)])**.** *There exists a unique predictable random measure ν (up to a $\mathbb{P}$-null set) such that*

$$\mathbb{E}\left[\int_{\mathbb{X}} X(t,x)\mu(\mathrm{d}t, \mathrm{d}x)\right] = \mathbb{E}\left[\int_{\mathbb{X}} X(t,x)\nu(\mathrm{d}t, \mathrm{d}x)\right] \tag{6.16}$$

for any process $X : \Omega\times\mathbb{R}_+\times\mathbb{X}\to\mathbb{R}_+$ which is measurable with respect to $\mathcal{P}\otimes\mathrm{Bor}(\mathbb{X})$. Moreover, it may be chosen so that $\nu(\{t\}\times\mathbb{X})\leq 1$ for any $t\geq 0$ and $\nu([\tau_\infty, +\infty[\times\mathbb{X}) = 0$. This measure ν is called the dual predictable projection *of μ. This measure ν is also characterized by the fact that for any $\Lambda\in\mathrm{Bor}(\mathbb{X})$, $\{\nu((0,t]\times\Lambda)\}_{t\geq 0}$ is predictable and $\{\mu((0,t\wedge\tau_k]\times\Lambda) - \nu((0, t\wedge\tau_k\times\Lambda))\}_{t\geq 0}$ is a uniformly integrable martingale.*

Proposition 6.8 ([53, Proposition (3.1)])**.** *The dual predictable projection ν of μ is given by* (6.14).

Remark 6.4. In [53, Theorem (3.6)], it is possible to construct μ from a suitable predictable random measure. We note that however, to apply such a construction, we need to construct ξ as in our setting, ν depends on ξ. The issue of simultaneously constructing μ and ξ is solved when considering the approach *via* stochastic differential equations.

The next result is a consequence of (6.13).

Corollary 6.4. *The counting process $t\mapsto [\![\xi]\!](t)$ is a non-homogeneous Poisson process with random intensity $t\mapsto F(\xi(t))$.*

[10] Using more general filtrations hinders the explicit computation of the compensator, see [28, p. 367].

6.6. Marked point processes and the martingale problem

The construction by the marked point process also allows one to easily recover that the process so constructed is the solution to the martingale problem.

Since ξ is piecewise constant and of finite variation, the following change of variable formula is obtained through a telescopic sum:

$$\begin{aligned}\phi(\xi(t)) &= \phi(\xi(r)) + \sum_{\substack{k\geq 0\\ r<\tau_k\leq t}} (\phi(\xi(\tau_k)) - \phi(\xi(\tau_k-)))\\ &= \phi(\xi(r)) + \sum_{\substack{k\geq 0\\ r<\tau_k\leq t}} (\phi(\xi(\tau_k-) - \Delta_k\xi) - \phi(\xi(\tau_k-))).\end{aligned} \tag{6.17}$$

Let $\phi \in \mathcal{C}_{\mathrm{b}}(\mathbb{X},\mathbb{R})$ and $t \geq r \geq 0$. Eq. (6.17) suggests to introduce the process $X_{r,t,\phi}$ from $\Omega \times \mathbb{R}_+ \times \mathbb{X}$ to $\mathbb{R}$ by

$$X_{r,t,\phi}(\omega; s, z) := \mathbb{1}_{(r,t]}(s) X_\phi(\omega, s, z)$$

with

$$X_\phi(\omega; s, z) := -(\phi(\xi(\omega; s-) - z) - \phi(\xi(\omega; s-))).$$

Since $X_{r,t,\phi}(\omega; s, z)$ involves $s \mapsto \xi(s-)$ on $(r, t]$ and is left-continuous in s, $X_{r,t,\phi}$ is $\{\mathcal{H}_t\}_{t\geq 0}$-adapted and is $\mathcal{P} \otimes \mathrm{Bor}(\mathbb{X})$-measurable.

With (6.17) and the definition of μ in (6.10), we have

$$\begin{aligned}\phi(\xi(\omega; t)) &= \phi(\xi(\omega; r)) + \int_0^{+\infty} \int_{\mathbb{X}} X_{r,t,\phi}(\omega; s, z)\mu(\omega;\, \mathrm{d}s, \mathrm{d}z)\\ &= \phi(\xi(\omega; r)) + \int_{(r,t]} \int_{\mathbb{X}} X_\phi(\omega, s, z)\mu(\omega; \mathrm{d}s, \mathrm{d}z).\end{aligned}$$

With (6.14), we obtain

$$\begin{aligned}&\int_0^{+\infty} \int_{\mathbb{X}} X_{r,t,\phi}(\omega; r, z)\nu(\omega;\, \mathrm{d}s, \mathrm{d}z)\\ &\quad= \sum_{\substack{k\geq 0\\ r<\tau_k\leq t}} \int_{\tau_{k-1}}^{\tau_k} (\phi(\xi(s-) - z) - \phi(\xi(s-)))G^{\mathrm{LM}}(\xi(s-), \mathrm{d}z)\,\mathrm{d}s\\ &\quad= \int_s^t (\phi(\xi(s) - z) - \phi(\xi(s)))G^{\mathrm{LM}}(\xi(s), \mathrm{d}z)\,\mathrm{d}s.\end{aligned}$$

With Notation 3.4, we remark that

$$\mathscr{F}\phi(x) := \int_{\mathbb{X}} (\phi(x - y) - \phi(x))G^{\mathrm{LM}}(x, \mathrm{d}y).$$

We then rewrite

$$\int_0^{+\infty} \int_{\mathbb{X}} X_{r,t,\phi}(\omega; r, z)\nu(\omega; \mathrm{d}s, \mathrm{d}z) = \int_r^t \mathscr{F}\phi(\xi(s))\,\mathrm{d}s.$$

Proposition 6.9 ([53, Proposition 5.3]). *Let X be a finite $\mathcal{P}$-measurable function. Let us assume that for any $t \geq 0$,*

$$\int_0^t \int_{\mathbb{X}} |X(\omega; r, z)| \nu(\omega; \mathrm{d}s, \mathrm{d}z) < +\infty \text{ a.s. on } \{t < \tau_\infty\}. \tag{6.18}$$

There exists a sequence $\{\theta_k\}_k$ of stopping times, increasing almost surely toward τ_∞ such that $\{Z_{t\wedge\theta_k}\}_{t\geq 0}$ is a uniformly integrable martingale for each $k \geq 0$, where Z is the right-continuous $\{\mathcal{H}_t\}_{t\geq 0}$-adapted process defined by

$$Z_t := Z_0 + \int_0^t \int_{\mathbb{X}} X(s,x)(\mu(\mathrm{d}s, \mathrm{d}z) - \nu(\mathrm{d}s, \mathrm{d}z)) \text{ a.s. on } \{T < \tau_\infty\}.$$

With our choice of X_ϕ as above, we define Z as

$$Z_t := \int_{(0,t]} X_\phi(s,z)(\mu(\mathrm{d}s, \mathrm{d}z) - \nu(\mathrm{d}s, \mathrm{d}z)),\ t \geq 0,$$

with $Z_0 = 0$ so that

$$\phi(\xi(t)) - \phi(\xi(0)) = Z_t + \int_0^t \mathscr{F}\phi(\xi(s))\,\mathrm{d}s.$$

The conditions of the next corollary are met when the rate is bounded.

Corollary 6.5 ([53]). *If (6.18) holds with X_ϕ and $\tau_\infty = +\infty$ almost surely, the distribution of the process ξ solves the local martingale problem.*

7. Stochastic differential equations

7.1. Motivation

The martingale problem (Definition 6.4) was defined in terms of the distribution of the process, through the notion of test functions. The process ξ constructed by embedding a Markov chain solves the local martingale problem:

$$\phi(\xi(t)) = \phi(\xi(0)) + M_t^\phi + \int_0^t \mathscr{F}\phi(\xi(s))\,\mathrm{d}s,\ t \geq 0$$

for any suitable ϕ and a local martingale M^ϕ. Can we describe M_t^ϕ?

7.2. Stochastic differential equations driven by a Poisson point process

We assume through all this section that Hypothesis 5.2, that is $\mathsf{G}(x,\cdot)$ has a density $\mathsf{g}(x.\cdot)$ for any $x \in \mathbb{X}^*$. We also write $g^{\mathrm{LM}}(x,\cdot)$ for the density of $G^{\mathrm{LM}}(x,\cdot)$.

Definition 7.1 (Compensated Poisson process)**.** Let N be a Poisson Point Process[11] on $\mathbb{R}_+^3$ with intensity $\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v$. The *compensated Poisson process* of N is

$$\widetilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v) := N(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v) - \mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v. \tag{7.1}$$

We set

$$H(y, u, v) := -u \cdot \mathbb{1}_{u \le y} \mathbb{1}_{v \le g^{\mathrm{LM}}(y,u)}, \tag{7.2}$$

$$\text{and } \ell(y) := \int_{\mathbb{X}} \int_{\mathbb{X}} H(y, u, v)\,\mathrm{d}u\,\mathrm{d}v = \int_0^y z \cdot g^{\mathrm{LM}}(y, z)\,\mathrm{d}z. \tag{7.3}$$

We consider finding a process $X \in \mathcal{D}(\mathbb{R}_+, \mathbb{R})$ and a PPP N such that

$$X_t = X_0 + \int_0^t \int_{\mathbb{X}} \int_{\mathbb{X}} H(X_{s-}, u, v) N(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v) \text{ for } t \ge 0 \tag{7.4}$$

or equivalently using the compensated Poisson process,

$$X_t = X_0 + \int_0^t \int_{\mathbb{X}} \int_{\mathbb{X}} H(X_{s-}, u, v) \widetilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v) + \int_0^t \ell(X_s)\,\mathrm{d}s \text{ for } t \ge 0. \tag{7.5}$$

Eq. (7.4) is called a *stochastic differential equation (SDE) driven by a random measure*. For general accounts on this theory, see for example [10, 51].

Remark 7.1. As X has a countable number of discontinuities, we could write either $\int_0^t \ell(X_{s-})\,\mathrm{d}s$ or $\int_0^t \ell(X_s)\,\mathrm{d}s$, as these two quantities are almost surely equal.

The meaning of the solution to (7.5) is clear from the results of Section 5.2: the function H filters out the points of a Poisson Point Process and return a sample of a jump.

7.3. Weak solutions

We consider only existence of weak solutions. The problem of existence of a strong solution remains open, although this seems plausible thanks to the results of Section 5.2 and the approach of [62]. However, strong existence does not follow from the usual results (see *e.g.* [51]) which require some Lipschitz continuity not satisfied by the function H.

Definition 7.2 (Weak solution)**.** Given a distribution ν on $\mathbb{X}$, a *weak solution* to (7.4) is any 3-uple (X, N, ζ) defined on a probability space $(\Omega, \mathcal{H}, (\mathcal{H}_t)_{t\ge0}, \mathbb{P})$ such that the distribution of ζ is ν, the distribution of N is that of a PPP with intensity $\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}u$ and X solves (7.4), while ζ is $\mathcal{H}_0$-measurable, and X and N are both $(\mathcal{H}_t)_{t\ge0}$-adapted.

Weak solutions are also solutions to the martingale problem.

[11] See Appendix B.

Proposition 7.1 (Itô's formula)**.** *Let us consider X to be a weak solution to the SDE (7.4). Let $\phi \in \mathrm{Mart}(\mathscr{F}^\sharp) \cap \mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$. Then for any $t \geq 0$,*

$$\phi(X_t) = \phi(\zeta) + \int_0^t \int_{\mathbb{R}_+^2} (\phi(X_{s-} + H(X_{s-}, u, v)) - \phi(X_{s-}))\widetilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v) + \int_0^t \mathscr{F}\phi(X_s)\,\mathrm{d}s, \quad (7.6)$$

for the compensated Poisson Process $\widetilde{N}$ of N. Moreover, X solves the martingale problem with respect to $(\mathscr{F}, \mathrm{Dom}(\mathscr{F}), \mu)$ when ζ has distribution μ.

Proof. Let $\phi \in \mathcal{C}_\mathrm{b}(\mathbb{X}^*, \mathbb{R})$. Since X is piecewise constant,

$$\phi(X_t) - \phi(\zeta) = \sum_{s \text{ s.t. } \Delta_s X \neq 0} (\phi(X_{s-} + \Delta_s X) - \phi(X_{s-})) \text{ with } \Delta_t X = X_s - X_{s-}. \quad (7.7)$$

Using (7.5), we obtain

$$\phi(X_t) = \phi(\zeta) + \int_0^t \int_{\mathbb{R}_+^2} (\phi(X_{s-} + H(X_{s-}, u, v)) - \phi(Y_{s-}))N(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v). \quad (7.8)$$

For any event A and any $0 \leq u \leq y$,

$$\phi(y - u\mathbb{1}_A) - \phi(y) = (\phi(y - u) - \phi(y))\mathbb{1}_A.$$

Hence,

$$\begin{aligned}\int_0^t \int_{\mathbb{R}_+^2} &(\phi(X_{s-} + H(X_{s-}, u, v)) - \phi(X_{s-}))\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s \\ &= \int_0^t \int_0^{X_{s-}} (\phi(X_{s-} - u) - \phi(X_{s-}))g^{\mathrm{LM}}(X_{s-}, u)\,\mathrm{d}u\,\mathrm{d}s \\ &= \int_0^t \mathscr{F}\phi(X_{s-})\,\mathrm{d}s = \int_0^t \mathscr{F}\phi(X_s)\,\mathrm{d}s.\end{aligned}$$

Using the compensated Poisson process, we rewrite (7.7) and (7.8) as (7.6). As ϕ is bounded,

$$M_t^\phi := \int_0^t \int_{\mathbb{R}_+^2} (\phi(X_{s-} + H(X_{s-}, u, v)) - \phi(Y_{s-}))\widetilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v),\ t \geq 0$$

is a martingale [51, Lemma 3.1]. □

Corollary 7.1. *If the fragmentation kernel is such that the martingale problem is well posed, then any weak solution to (7.4), which is necessarily a Markov process, has a transition kernel which satisfies the Kolmogorov forward equation.*

Proof. Let X be a weak solution to (7.4), which is also a Markov process by Theorem 6.2. Let $\phi \in \mathcal{C}_{\mathrm{b}}(\mathbb{X}, \mathbb{R}) \cap \mathrm{Mart}(\mathscr{F}^\sharp)$. Using (7.6),

$$\mathbb{E}[\phi(X_t)] = \mathbb{E}[\phi(\zeta)] + \int_0^t \mathbb{E}[\mathscr{F}\phi(X_s)]\,\mathrm{d}s, \ \forall t \geq 0.$$

As X is a Markov process, it is characterized by its probability transition function $P(t, x, \Lambda)$, $t \geq 0$, $x \in \mathbb{X}$, $\Lambda \in \mathrm{Bor}(\mathbb{X})$. Therefore, if μ is the distribution of the initial condition ζ,

$$\begin{aligned}\int_{\mathbb{X}}\int_{\mathbb{X}} \mu(\mathrm{d}x)P(t,x,\mathrm{d}y)\phi(y) &= \mathbb{E}[\phi(X_t)] \\ &= \int_{\mathbb{X}} \mu(\mathrm{d}x)\phi(x) + \int_0^t \int_{\mathbb{X}}\int_{\mathbb{X}} \mu(\mathrm{d}x)P(t,x,\mathrm{d}y)\mathscr{F}\phi(y). \quad (7.9)\end{aligned}$$

Using Notation 4.2,

$$\langle \mu, \partial_t P_t \phi\rangle = \langle \mu, P_t \mathscr{F}\phi\rangle,$$

which is nothing more than the Kolmogorov forward equation. □

7.4. A representation theorem

The process ξ with characteristics (F, G) solves the SDE (7.4) provided that F is bounded. The proof relies on a Poisson representation theorem. As the subject is technical, we give only the main points of the proofs. We refer to [26, 25, 35, 54] for the details and proofs.

Proposition 7.2. *Assume that the rate is bounded (Hypothesis 3.1). Let ξ be the process with characteristics (F, G). Then for any $x \in \mathbb{X}^*$, there exists an extension of the filtered probability space carrying a Poisson Point Process N with intensity* $\mathrm{d}t\mathrm{d}u\mathrm{d}v$ *such that on this extension, (ξ, N, x) is a weak solution to the SDE (7.4).*

Sketch of the proof. The process ξ has paths of finite variation and is non-decreasing. Therefore, $\mathrm{Id} : x \mapsto x$ belongs to $\mathrm{Mart}(\mathscr{F})$ as the rate F is bounded under $\mathbb{P}_x$ for any $x \in \mathbb{X}^*$.

Thus, ξ is a semi-martingale, which we may write

$$\xi(t) = \xi(0) + M_t^{\mathrm{Id}} + \int_0^t \mathscr{F}\mathrm{Id}(\xi(s))\,\mathrm{d}s.$$

We associate ξ with the measure μ given by (6.10). According to Proposition 6.8, the dual predictable projection of μ is $\nu(\mathrm{d}t, \mathrm{d}y) = G^{\mathrm{LM}}(\xi(t-), \mathrm{d}y)\,\mathrm{d}t$. (With Notation 6.8, we use the convention that $\Delta_t\xi \geq 0$, which explains the change of signs with respect to the cited results).

With H defined in (7.2), it holds for any $\Lambda \in \mathrm{Bor}(\mathbb{X}^*)$ (hence $0 \notin \Lambda$),

$$G^{\mathrm{LM}}(y, \Lambda) = \int_{\mathbb{R}_+} \mathbb{1}_\Lambda(u) g^{\mathrm{LM}}(y, u)\,\mathrm{d}u$$

$$= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbb{1}_\Lambda(u) \mathbb{1}_{v \le g^{\mathrm{LM}}(y,u)} \, \mathrm{d}v \, \mathrm{d}u = \int \mathbb{1}_\Lambda(-H(y,u,v)) \, \mathrm{d}v \, \mathrm{d}u.$$

According to [25, Proposition (3.19), p. 202] (note that the jumps are bounded), there exists an extension of $(\Omega, \mathcal{H}, \mathbb{P}_x, \{\mathcal{H}_t\}_{t\ge 0})$ supporting a Poisson random measure N on $\mathbb{R}_+ \times \mathbb{R}_+^2$ such that

$$\xi(t) = \xi(0) + \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} H(\xi(s-), u, v) N(\mathrm{d}s, \mathrm{d}u, \mathrm{d}v)$$

with mean measure $\mathrm{d}t \, \mathrm{d}u \, \mathrm{d}v$.

Additional comments may be found in [25, Section (3.57), p. 214]. □

7.5. A weak existence condition with unbounded rate

We now give a condition for existence of weak solution of SDE for an unbounded rate. We refer to [44] for the proofs and additional results.

Hypothesis 7.1. Hypothesis 5.2 (existence of a density for the kernel) holds, and $(x, y) \mapsto g(x, y)$ is continuous. The map ℓ defined by (7.3) is continuous with

$$\ell(x) \le C(1 + x^p) \tag{7.10}$$

for some constants $C \ge 0$ and $p \ge 1$. Moreover,

$$\lim_{n\to\infty} \sup_{\varepsilon \le x \le \varepsilon^{-1}} \ell_n(x) = 0 \text{ for any } \varepsilon \in (0, 1)$$

with

$$\ell_n(x) := \int_0^x y \cdot g^{\mathrm{LM}}(x, y) \mathbb{1}_{xg(x,y) \ge ny} \, \mathrm{d}y.$$

Proposition 7.3 ([44, Proposition 3.1])**.** *Assume that Hypothesis 7.1 holds and that ζ is finite moment of order $p + 1$ (where p appears in (7.10)). Then*

(i) *There exists a weak solution to the SDE (7.4) with $X_0 = \zeta$ and X_t has finite moments of order $p + 1$ for any $t \ge 0$.*

(ii) *If moreover, $\ell(x) \ge \rho x^\gamma$ for $\gamma \in (0, 1)$ and $\rho > 0$. Then $\mathbb{E}[\tau_0] < +\infty$ where $\tau_0 := \inf\{t > 0 \,|\, X_t = 0\}$, and $X_{\tau_0+t} = 0$ almost surely. Identifying 0 to the cemetery point †, this means that the process is almost surely explosive. In addition, $\mathbb{P}[X_t = 0] > 0$ for any $t > 0$ and $t \mapsto \mathbb{P}[X_t = 0]$ is increasing with limit 1. If $\mathbb{P}[\zeta > 0] > 0$ then $\mathbb{P}[\tau_0 > t] > 0$ for any $t \ge 0$.*

8. Mass conservation

The *mass conservation* means that the density satisfies $P(t, x, \mathbb{X}) = 1$ for any time $t > 0$ and any starting point $x > 0$. In the framework of Markov chains, it means that Y_k never reaches the cemetery point † in finite time, *i.e.* $\lim_{k\to\infty} \tau_k = +\infty$. In our case, as the mass is non-increasing with respect to the time, one can use 0 as the cemetery point †.

If $P(t, x, \mathbb{X}) = 1$ for any $t > 0$, then the associated semi-group is said to be *stochastic*. Otherwise, it is *substochastic*.

Several sufficient conditions on the kernel can be given to ensure that the semi-group is stochastic [70, 75, 77, 80]. We just give one such condition. Discussions about cases where the mass is not conserved due to the creation an infinite number of particles of zero mass (dust) may be found in [8, 47, 49, 69, 77, 78, 79] among others. Such a phenomena is called *shattering* and was pointed out first by A.F. Filippov [43] and McGrady and R.M. Ziff [69].

We give a criteria (see also [75, Corollary 6.1]). A similar criteria is that $\limsup_{x\to 0+} F(x) < +\infty$ [9, Theorem 8.5].

Proposition 8.1 ([75, Corollary 6.1])**.** *If F is bounded on bounded subsets of $(0, +\infty)$, then the mass is conserved.*

9. Binary fragmentation

We now consider the specific case of the *binary fragmentation*, where each fragment is split in exactly two parts. In this case, the stochastic process represents exactly the typical path of a particle. This is no longer the case of simultaneous multiple breakages. However, the binary fragmentation equation, though the average of the kernel, summarizes the density of the mass of the particles.

Our aim is to examplify the derivations of the various forms of the fragmentation equation by elementary arguments.

Definition 9.1 (Binary fragmentation kernel)**.** The *binary fragmentation kernel* is a measurable function $F : \mathbb{X}^2 \to \mathbb{R}_+$ specifies how a $(x + y)$-mer breaks up into a x-mer and a y-mer. More precisely, $(x+y)$-mers break into y-mers at rate $F(x, y)$ and into x-mers at are $F(y, x)$.

We add a supplementary hypothesis.

Hypothesis 9.1. The binary fragmentation kernel F is symmetric: $F(x, y) = F(y, x)$, for any $x, y \geq 0$.

Several examples of classical binary fragmentation kernels may be found in Table 1.

9.1. The fragmentation equation from the balance condition

The quantity of interest is $c(t, x)$, the concentration of x-mers at time t. Using Definition 9.1, the balance condition is the following: In a short time Δt,

$$c(t + \Delta t, x) \approx c(t, x) - \frac{\Delta t}{2} c(t, x) \int_0^x F(x - y, y)\, \mathrm{d}y + \Delta t \int_0^{+\infty} c(t, x + y) F(y, x)\, \mathrm{d}y. \quad (9.1)$$

The 1/2 term is to avoid to count twice the fragmentation of a x-mer as $(y, x-y)$ and as $(x-y, y)$.

Passing to the limit as $\Delta t \to 0$, we recover the fragmentation equation (1.1):

$$\frac{\partial c(t,x)}{\partial t} = -\frac{1}{2}\int_0^x c(t,x)F(x-y,y)\,\mathrm{d}y + \int_0^{+\infty} c(t,x+y)F(y,x)\,\mathrm{d}y, \tag{9.2}$$

which we rewrite as

$$\frac{\partial c(t,x)}{\partial t} = \int_0^{+\infty} c(t,x+y)F(y,x)\,\mathrm{d}y - c(t,x)\times\frac{1}{2}\int_0^x F(x-y,y)\,\mathrm{d}y.$$

Using the mass-biased mechanism, with $p(t,x) := c(t,x)\cdot x$,

$$\frac{\partial p(t,x)}{\partial t} = \int_x^{+\infty} p(t,y)F(y-x,x)\frac{x}{y}\,\mathrm{d}y - p(t,x)\times\frac{1}{2}\int_0^x F(x-y,y)\,\mathrm{d}y. \tag{9.3}$$

9.2. Recovering the rate and the daughter distribution

We now relate the kernel F and the daughter distribution m.

Notation 9.1. We define

$$F(x) := \int_0^x F(x-y,y)\frac{x-y}{x}\,\mathrm{d}y \text{ for } x \geq 0. \tag{9.4}$$

Lemma 9.1. *Under Hypothesis 9.1,*

$$F(x) = \frac{1}{2}\int_0^x F(x-y,y)\,\mathrm{d}y \text{ for any } x \geq 0. \tag{9.5}$$

Proof. Let $x \geq 0$ be fixed. We evaluate

$$\int_0^x F(y,x-y)\frac{x-y}{x}\,\mathrm{d}y = \int_0^x F(y,x-y)\,\mathrm{d}y - \int_0^x F(y,x-y)\frac{y}{x}\,\mathrm{d}y.$$

With the change of variable $z \leftarrow x-y$, since F is symmetric in its arguments,

$$\begin{aligned}\int_0^x F(y,x-y)\frac{x-y}{x}\,\mathrm{d}y &= \int_0^x F(y,x-y)\,\mathrm{d}y - \int_0^x F(y,x-y)\frac{y}{x}\,\mathrm{d}y \\ &= \int_0^x F(y,x-y)\,\mathrm{d}y - \int_0^x F(z,x-z)\frac{x-z}{x}\,\mathrm{d}z.\end{aligned}$$

This leads to (9.5). □

With Lemma 9.1, (9.2) is rewritten

$$\frac{\partial c(t,x)}{\partial t} = -c(t,x)F(x) + \int_0^{+\infty} c(t,x+y)F(y,x)\,\mathrm{d}y. \tag{9.6}$$

Therefore, $F(x)$ is a *rate* at which a x-mer breaks.

Proposition 9.1. *Under Hypothesis 9.1, the fragmentation kernel* $F : \mathbb{X}^* \times \mathbb{X}^* \to \mathbb{R}_+$ *is in one-to-one correspondence with the daughter distribution* M *and the rate* F*.*

Proof. Using a change of variable $y \leftarrow y-x$, we rewrite (9.6) (which is equivalent to (9.2)) as

$$\frac{\partial c(t,x)}{\partial t} = -F(x)c(t,x) + \int_x^{+\infty} c(t,y)F(y)m(y,x)\,\mathrm{d}y \tag{9.7}$$

with

$$m(y,x) := \frac{F(y-x,x)}{F(y)}\mathbb{1}_{(0,y)}(x) \text{ and } M(y,\mathrm{d}x) = m(y,x)\mathrm{d}x. \tag{9.8}$$

This is the Fokker-Planck equation (2.3).

By combining (9.4) and (9.5),

$$\int_0^x m(x,y)\frac{y}{x}\,\mathrm{d}y = 1. \tag{9.9}$$

Therefore, the mass conservation in average is satisfied (Hypothesis 2.2).

Hence, from the knowledge of a symmetric fragmentation kernel $F : \mathbb{X}^{*2} \to \mathbb{R}_+$, we recover the rate $F : \mathbb{X}^* \to \mathbb{R}_+$ and the mean number of fragments M.

Conversely, given $y \mapsto m(y,x)$ for any $x \in \mathbb{X}^*$ and a rate function $x \mapsto F(x)$, we may recover the kernel $(x,y) \mapsto F(x,y)$ by setting

$$F(x,y) := F(x+y)m(x+y,y).$$

Hypothesis 9.1 is equivalent to (2.10). □

9.3. The Kolmogorov forward and backward equations

We presented previously a formal derivation of the fragmentation equation (9.2). We now rewrite the Kolmogorov forward and backward equations using the fragmentation kernel F.

We introduce the following operators.

Notation 9.2. We set for any $\phi \in \mathcal{C}_{\mathrm{c}}(\mathbb{X}^*)$

$$\mathscr{F}\phi(x) := \int_0^x F(x-y,y)\frac{x-y}{x}(\phi(x-y)-\phi(x))\,\mathrm{d}y$$

and for any $\mu \in \mathbb{M}(\mathbb{X}^*)$

$$\mathscr{F}^\star\mu(\mathrm{d}x) := \left(\int_x^{+\infty} \mu(\mathrm{d}y)\frac{x}{y}F(y-x,x)\right)\mathrm{d}x - F(x)\mu(\mathrm{d}x)$$

whenever the integral is well defined.

Proposition 9.2. *With Notation 2.5, for any $\mu \in \mathbb{M}(\mathbb{X}^*)$ and $\phi \in \mathcal{C}_c(\mathbb{X}^*, \mathbb{R})$,*

$$\langle \mathscr{F}^\star \mu, \phi \rangle = \langle \mu, \mathscr{F} \phi \rangle.$$

In other words, the operators $\mathscr{F}$ and $\mathscr{F}^\star$ are in duality.

Proof of Proposition 9.2. Let $\phi \in \mathcal{C}_c(\mathbb{X}^*)$ be a continuous function with compact support on $\mathbb{R}_+$. After two changes of variables, by inverting first x and y and then changing y to $x - y$,

$$\begin{aligned}\int_0^{+\infty} \phi(x)\,\mathrm{d}x \int_x^{+\infty} \mu(\mathrm{d}y)\frac{x}{y}F(y-x,x) &= \int_0^{+\infty}\int_0^{+\infty} \mu(\mathrm{d}y)\,\mathrm{d}x\,\phi(x)\frac{x}{y}F(y-x,x)\mathbb{1}_{x\le y} \\ &= \int_0^{+\infty} \mu(\mathrm{d}x)\int_0^{x} \mathrm{d}y\,\phi(y)\frac{y}{x}F(x-y,y) \\ &= \int_0^{+\infty} \mu(\mathrm{d}x)\int_0^{x} \mathrm{d}y\,\phi(x-y)\frac{x-y}{x}F(x-y,y).\end{aligned}$$

On the other hand, from the very definition of F in (9.4), we have

$$\int_0^{+\infty} \mu(\mathrm{d}x)F(x)\phi(x) = \int_0^{+\infty}\int_0^{x} \mu(\mathrm{d}x)F(y-x,x)\frac{y-x}{x}\phi(x)\,\mathrm{d}y.$$

This proves the result. □

Let $t \mapsto p(t, \mathrm{d}x)$ be a $\mathbb{M}(\mathbb{X}^*)$-valued solution to

$$\frac{\partial}{\partial t}\int_0^{+\infty} p(t,\mathrm{d}x)\phi(x) = \int_0^{+\infty}\int_0^{x} p(t,\mathrm{d}x)F(x-y,y)\frac{x-y}{x}(\phi(x-y)-\phi(x))\,\mathrm{d}y \tag{9.10}$$

for any $\phi \in \mathcal{C}_c(\mathbb{X}^*, \mathbb{R})$. This is a rewriting of (9.3). Using Notation 2.5, (9.10) is

$$\frac{\partial}{\partial t}\langle p(t,\cdot), \phi\rangle = \langle p(t,\cdot), \mathscr{F}\phi\rangle. \tag{9.11}$$

Using the duality of Proposition 9.2, we obtain that

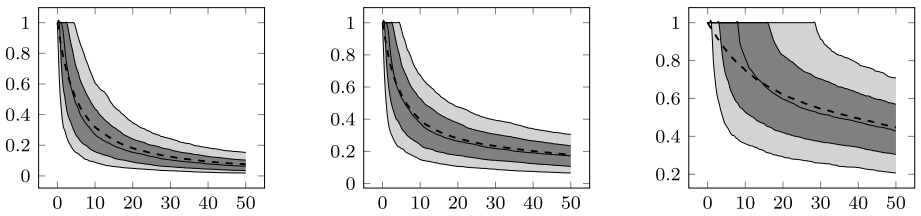
$$\frac{\partial}{\partial t}\langle p(t,\cdot), \phi\rangle = \langle \mathscr{F}^\star p(t,\cdot), \phi\rangle. \tag{9.12}$$

Hence, (9.11) is the Kolmogorov forward equation while (9.12) is the Kolmogorov backward equation. The existence of such equations have been studied in Sections 4.1 and 4.2.

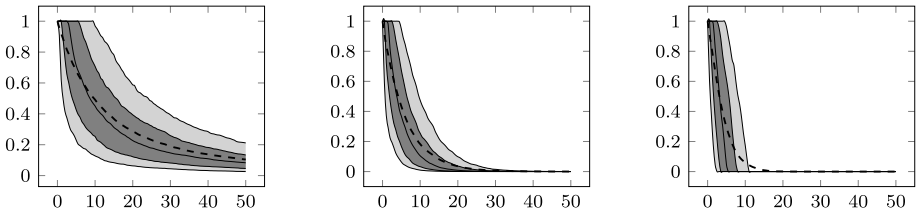
10. A simulation algorithm

10.1. Computation for some kernels

In Table 1, we consider a variety of standard kernels and compute the quantities associated to some usual fragmentation kernels.



(a) $F(x,y)=1$ (b) $F(x,y)=x+y$ (c) $F(x,y)=xy$



(d) $F(x,y)=\dfrac{1}{1+x+y}$ (e) $F(x,y)=\dfrac{1}{x+y}$ (f) $F(x,y)=\dfrac{1}{(x+y)^2}$

FIG 1. *Statistics of the evolution of a cloud of particles with the time. Each line represents a quantile (10 %, 20 %, 50 %, 75 %, 90 %, the lines are ordered from bottom to top), and the dashed line represents the mean. The initial mass is* 1. *The cloud has* 2000 *particles.*

In Figures 1 and Figures 2, we present the quantiles (10 %, 20 %, 50 %, 75 %, 90 %) and the means at different masses for the evolution of a set of particles over the time, with a constant initial mass, and with an initial mass being log-normally distributed. Although almost all the kernels in Table 1 have the same breaking dynamics $\mathsf{G}(x,\cdot)$, the rate strongly influence the behavior of the dynamics of the cloud of particles.

We summarize the relations between the different kernels and functions for a symmetric kernel $F(x,y)$:

$$
\begin{aligned}
F(x) &= \int_0^x F(x-y,y)\frac{x-y}{x}\,\mathrm{d}y = \int_0^x F(x-y,y)\frac{y}{x}\,\mathrm{d}y \qquad \text{at (9.4)},\\
G(x,\mathrm{d}y) &= \mathbb{1}_{[0,x]}(y)F(x-y,y)\frac{y}{x}\,\mathrm{d}y,\\
\mathsf{G}(x,\mathrm{d}y) &= \frac{G(x,\mathrm{d}y)}{F(x)},\\
G^{\mathrm{LM}}(x,\mathrm{d}y) &= F(x-y,y)\frac{x-y}{x}\mathbb{1}_{0\le y\le x}\,\mathrm{d}y \qquad \text{at (6.9)},\\
\mathsf{G}^{\mathrm{LM}}(x,\mathrm{d}y) &= \frac{G^{\mathrm{LM}}(x\,\mathrm{d}y)}{F(x)} \qquad \text{at (6.9)},\\
M(x,y)\,\mathrm{d}y &= \frac{F(x-y,y)}{F(x)}\,\mathrm{d}y = \mathsf{G}(x,\mathrm{d}y)\frac{x}{y} \qquad \text{at (9.8)}.
\end{aligned}
$$

The distribution function of the mass of a fragment after a breakage of a

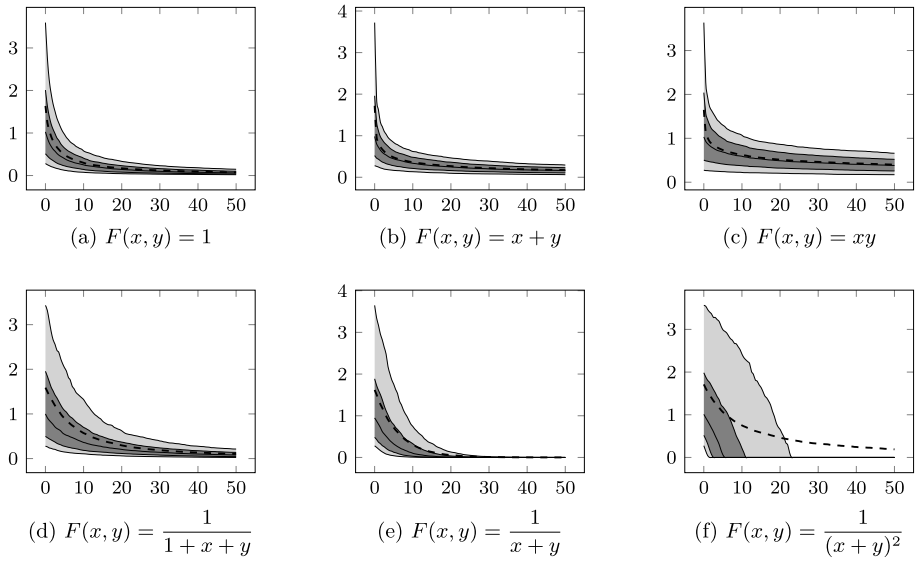


FIG 2. *Statistics of the evolution of a cloud of particles with the time. Each line represents a quantile (10 %, 20 %, 50 %, 75 %, 90 %, the lines are ordered from bottom to top), and the dashed line represents the mean. The initial mass follows the log-normal distribution of log-mean* 0 *and log-variance* 1.0. *The cloud has* 2000 *particles.*

x-mer is then $\mathsf{G}(x,[0,y]) = \int_0^y \mathsf{G}(x,\mathrm{d}z)$.

Self-similar fragmentation In many situations, there exists a function H such that $\mathsf{G}(x,[0,y]) = H(y/x)$ for $y \leq x$. This is the case for homogeneous fragmentation (Example 2.5) with $H(z) = z \cdot \vartheta(z)$.

For $x = 1$, $y \in [0,1] \mapsto H(y)$ is the distribution function of a random variable we denote by Y. Then $Y(x) = xY$ has for distribution function $H : y \in [0,x] \mapsto H(y/x) = \mathsf{G}(x,[0,y])$. In such situations, this means that the breakage of the particle x to new particles of mass y and $x-y$ depends only on the relative mass y/x. Regarding simulation, one has only to know how to simulate the random variable Y.

Time-change by scaling the kernel Let $\lambda > 0$ be a scalar factor. We consider replacing a kernel $F(x,y)$ by $F^{[\lambda]}(x,y) := \lambda F(x,y)$. The rate then become $F^{[\lambda]}(x) = \lambda F(x)$. This means that the mean of the exponential distribution drawn at each jump is divided by λ.

Let $p^{[\lambda]}(t,\mathrm{d}y)$ be the solution to the corresponding fragmentation equation. Then, a simple computation shows that $p^{[\lambda]}(t,\cdot) = p^{[1]}(\lambda t,\cdot)$ for any $t > 0$. Thus, scaling the kernel corresponds to scale the time dynamic, and to scale accordingly the parameter of the exponential distributions giving the breakages' times.

TABLE 1
Computations for different fragmentation kernels for $y \in [0, x]$.

$F(x, y)$	$G(x, y)$	$F(x)$	$\mathsf{G}(x, [0, y])$
1	$\frac{y}{x}$	$\frac{x}{2}$	$\frac{y^2}{x^2}$
$x + y$	y	$\frac{x^2}{2}$	$\frac{y^2}{x^2}$
xy	$\frac{y^2(x-y)}{x}$	$\frac{x^3}{12}$	$\frac{4y^3}{x^3} - \frac{3y^4}{x^4}$
$\frac{1}{1+x+y}$	$\frac{y}{x(x+1)}$	$\frac{x}{2(1+x)}$	$\frac{y^2}{x^2}$
$\frac{1}{(x+y)^n}$, $n \geq 1$	$\frac{y}{x^{n+1}}$	$\frac{1}{2x^{n-1}}$	$\frac{y^2}{x^2}$

Data: A fragmentation kernel F, a time T and a mass m
Result: The mass of a particle initially of mass m at time T
$t \leftarrow 0$;
$\xi \leftarrow m$;
while $t < T$ **do**
  Draw a random variate ζ with law $\mathcal{E}(F(\xi))$;
  Draw a random variate ξ' with law $\mathsf{G}(\xi, \cdot)$;
  $\xi \leftarrow \xi'$;
  $t \leftarrow \min\{T, t + \zeta\}$;
end
return ξ;

Algorithm 1: Monte Carlo algorithm for the simulation of the mass of a typical particle at time T after a fragmentation process.

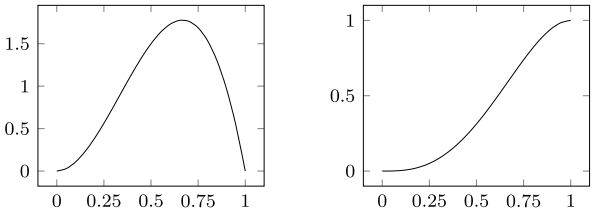
10.2. A general simulation algorithm

The algorithm follows the considerations of Section 5.1. Unless the kernel has a simple form, we implement a rejection principle as in Section 5.2. The algorithm is really simple to set up. The memoryless property of the exponential (Lemma A.1) means that by restarting the algorithm using the position of the particle at time T up to time T', the particles' distribution is the same as the one at horizon $T + T'$.

10.3. Practical implementation

Algorithm 1 is easy to implement.

The ways to draw a random variable Y with density $\mathsf{G}(x, \cdot)$ depends on the expression of its distribution function $y \mapsto \mathsf{G}(x, [0, y])$. If $H := y \in [0, x] \mapsto$



(a) pdf (b) df

FIG 3. *Probability density function (a) and distribution function (b) of* $\mathsf{G}(x,\cdot)$ *for the multiplicative kernel* $F(x,y)=xy$.

$\mathsf{G}(x,[0,y]) \in [0,1]$ is easy to invert, then $H^{-1}(U)$ for a uniform random variable U on $[0,1]$.

For example, when $H(y)=y^2/x^2$, then

$$H^{-1}(u|x) = x\sqrt{u} \text{ for } 0 \leq u \leq 1.$$

This is the case for several kernels in Table 1. In Figure 3, we present the density and distribution functions associated to the multiplicative kernel.

If H^{-1} has no tractable form or is not easily computed numerically, we may rely on the rejection algorithm 2 in Appendix C. Other *ad hoc* simulation algorithms may be found in [30].

11. Conclusion

We have presented a survey on the probabilistic representations of the solution to the fragmentation equation. These representations have the important advantage to allow to construct easy to implement numerical methods and to develop Monte Carlo simulation. Besides, they can be extended in several directions, among them

- The breakage mechanism may occurs from different phenomena: it is easy to deal with several competing kernels.
- The mass can be subject to a continuous abrasion [21] or mass accumulation (growth-fragmentation) [14, 15, 24]. In such case, the times of the next breakage is the first time of a time-inhomogeneous Poisson process [65].
- The evolution of the mass may be coupled with the position and the speed.
- The evolution of mass may be of "mean-field type", meaning that the fragmentation kernel actually depends on the distribution of the mass at a given time.

The question of the inference of the kernel remains largely opened. Some methods and algorithms may be found in [6, 7, 67] for Monte Carlo approach, [32, 50] for an approach based on PDE, or [4, 5] for inference from the observations of a single path.

Appendix A: Properties of the exponential distribution

We recall here some fundamental properties of the exponential distribution.

Lemma A.1 (Memoryless property)**.** *Let $\theta \sim \mathcal{E}(\lambda)$. Then*

$$\mathbb{P}[\theta > t + u \,|\, \theta > t] = \mathbb{P}[\theta > u] \text{ for any } u, t \geq 0.$$

Lemma A.2. *Let $\xi_1 \sim \mathcal{E}(\lambda_1)$, ..., $\xi_n \sim \mathcal{E}(\lambda_n)$ be independent for some positive parameters $\lambda_1, \ldots, \lambda_n$. Then $\zeta := \min\{\xi_1, \ldots, \xi_n\}$ has the distribution $\mathcal{E}(\lambda)$ with $\lambda := \lambda_1 + \cdots + \lambda_n$. Besides, $J := \arg\min_{i=1,\ldots,n} \xi_i$ is independent from ζ and*

$$\mathbb{P}[J = i] = \frac{\lambda_i}{\lambda}.$$

Appendix B: Random measures and Poisson point processes

We present here some facts regarding Poisson Point Processes (PPP) and random measures. More may be found in [45, 46, 59].

Notation B.1 (Poisson distribution)**.** We denote by $\mathcal{P}(\lambda)$ the Poisson distribution of parameter λ, that is $X \sim \mathcal{P}(\lambda)$ has its distribution characterized by $\mathbb{P}[X = k] = \lambda^k \exp(-\lambda)/k!$ for $k = 0, 1, 2, \ldots$.

We start by recalling standard definitions on Poisson Point process (PPP). Some properties of PPP are recalled in Section B. More information is given in [27, 28, 45, 46, 59].

Definition B.1 (Counting measure)**.** Fix $d \geq 1$. A *counting measure* μ on $\mathbb{R}^d$ is a locally finite measure such that $\mu(\Lambda) \in \mathbb{N}$ for any relatively compact set $\Lambda \in \mathrm{Bor}(\mathbb{R}^d)$. The set of counting measures is denoted by $\mathbb{M}_c$. We write $\mathrm{Bor}(\mathbb{M}_c)$ the smallest σ-algebra such that $\mu \mapsto \mu(\Lambda)$ is measurable for any relatively compact $\Lambda \in \mathrm{Bor}(\mathbb{R}^d)$.

Definition B.2 (Point process and its intensity measure)**.** Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space. A *point process* N is a measurable mapping from $(\Omega, \mathcal{H}, \mathbb{P})$ to $(\mathbb{M}_c, \mathrm{Bor}(\mathbb{M}_c))$, meaning that N is a random counting measure. Its *intensity measure* n is $n(\Lambda) := \mathbb{E}[N(\Lambda)]$ for any $\Lambda \in \mathrm{Bor}(\mathbb{R}^d)$ (note that possibly, $n(B) = +\infty$).

Definition B.3 (Poisson Point process (PPP))**.** A point process N on $\mathbb{R}^d$ whose intensity n is locally finite is called a *Poisson Point Process* (PPP) if $N(B_1), \ldots, N(B_k)$ are independent with $N(B_i) \sim \mathcal{P}(n(B_1))$ for any family of pairwise disjoint Borel sets $(B_1, \ldots, B_k)$ of $\mathbb{R}^d$ and any $k \geq 1$.

Lemma B.1. *Let N be a PPP on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure n of the form $n(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s \cdot \nu(\mathrm{d}x)$ for a measure ν on $\mathbb{R}^d$. Let $\Lambda \in \mathrm{Bor}(\mathbb{R}^d)$ such that $\nu(\Lambda) < +\infty$. Denote by (ζ, Y) the coordinate of the leftmost point of $N \cap \mathbb{R}_+ \times \mathbb{R}^d$. Then $\zeta \sim \mathcal{E}(\nu(\Lambda))$ while $\mathbb{P}[Y \in \Lambda'] = \nu(\Lambda')/\nu(\Lambda)$ for any $\Lambda' \in \mathrm{Bor}(\mathbb{R}^d)$, $\Lambda' \subset \Lambda$. At last, ζ and Y are independent.*

Remark B.1. In the above statement, we may replace the leftmost point of $N \cap \mathbb{R}_+ \times \mathbb{R}^d$ by the leftmost point of $N \cap [t, +\infty) \times \mathbb{R}^d$ for any $t \geq 0$.

Proof. For any $t, h \geq 0$ and $\Lambda' \in \mathrm{Bor}(\mathbb{R}^d)$, set $\Lambda'(t, t+h) = [t, t+h) \times \Lambda'$. Remark that $n(\Lambda(t, t+h)) = h\nu(\Lambda)$.

The key point is that when h is small,

$$\begin{aligned}\mathbb{P}[(\xi, Y) \in \Lambda'(t, t+h)] &= \mathbb{P}[N(\Lambda'(t, t+h)) = 1,\ N(\Lambda(0,t)) = 0] + \mathrm{o}(h^2) \\ &= \mathbb{P}[N(\Lambda'(t, t+h)) = 1]\mathbb{P}[N(\Lambda(0,t)) = 0] + \mathrm{o}(h^2) \\ &= h\nu(\Lambda') \exp(-h\nu(\Lambda')) \exp(-t\nu(\Lambda)) + \mathrm{o}(h^2).\end{aligned}$$

Thus, set $h = 1/n$ for some integer n and $t_k = a + (b-a)kh$ for $0 \leq a \leq b$. Hence,

$$\begin{aligned}\mathbb{P}[(\xi, Y) \in \Lambda'(a, b)] &= \sum_{k=0}^{n-1} \mathbb{P}[(\xi, Y) \in \Lambda'(t_k, t_{k+1})] \\ &= \sum_{k=0}^{n-1} \frac{b-a}{n} \nu(\Lambda') \exp\left(-\frac{1}{n}\nu(\Lambda')\right) \exp(-t_k \nu(\Lambda)) + \mathrm{o}(n^{-1}) \\ &\xrightarrow[n\to\infty]{} \frac{\nu(\Lambda')}{\nu(\Lambda)} (\exp(-b\nu(\Lambda)) - \exp(-a\nu(\Lambda))).\end{aligned}$$

This proves the result. □

Appendix C: The rejection principle

Introduced by J. Von Neumann, the *rejection principle* is a convenient way to simulate random variables from the knowledge of their densities [30].

Lemma C.1. *Let p be proportional to the density of a random variable Y with $p(x) = 0$ when $x \notin [0, K]$. Let (U, V) be two random variables that are uniformly distributed under $\{(u, v) \mid 0 \leq v \leq p(u)\}$. Then U is distributed as Y.*

Proof. Let (U, V) be as in the statement of the lemma. Then

$$\begin{aligned}\mathbb{P}[U \leq x] &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \mathbb{P}\left[(U, V) \in \left[\frac{kx}{n}, \frac{(k+1)x}{n}\right] \times \left[0, p\left(\frac{(k+1/2)x}{n}\right)\right]\right] \\ &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{\sum \frac{x}{n} p\left(\frac{(k+1/2)x}{n}\right)}{\int_0^K p(y)\,\mathrm{d}y} = \frac{\int_0^x p(y)\,\mathrm{d}y}{\int_0^K p(y)\,\mathrm{d}y}. \quad \text{(C.1)}\end{aligned}$$

Hence the result. □

From this, we derive the following result, as the distribution of a random variable $Z \in \mathbb{R}^2$ given $Z \in \Lambda$ for a Borel set Λ is uniformly distributed in Λ when Z is uniformly distributed in $\mathbb{R}^2$, or any subset of $\mathbb{R}^2$ containing Λ. Using a rectangular set Λ in Lemma C.2, this leads to the simple Monte Carlo algorithm 2.

Lemma C.2 (Monte Carlo simulation using the rejection principle)**.** *Let p be proportional to the density of a random variable Y with $0 \leq p(x) \leq M$ for $x \in [0, K]$ and $p(x) = 0$ otherwise. Let (U, V) be two uniform random variables on $[0, K] \times [0, M]$. Then the distribution of U given $V \leq p(U)$ is the one of Y.*

Data: A function $p : [0, K] \to [0, M]$.
Result: A random variate whose density is proportional to p
repeat
  Draw a random variate $U \sim \mathcal{U}(0, 1)$;
  Draw a random variate $V \sim \mathcal{U}(0, 1)$;
until $V \cdot M \leq p(K \cdot U)$;
return U;

Algorithm 2: Monte Carlo rejection algorithm.

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