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Dynamic programming approach to reflected backward stochastic differential equations

Hun O* Mun-Chol Kim*† Kon-Gun Kim*

Abstract

By introducing a new type of minimality condition, this paper gives a novel approach to the reflected backward stochastic differential equations (RBSDEs) with càdlàg obstacles. Our first step is to prove the dynamic programming principles for nonlinear optimal stopping problems with g-expectations. We then use the nonlinear Doob-Meyer decomposition theorem for g-supermartingales to get the existence of the solution. With a new type of minimality condition, we prove a representation formula of solutions to RBSDEs, in an efficient way. Finally, we derive some a priori estimates and stability results.

Keywords: reflected backward stochastic differential equation; nonlinear optimal stopping; dynamic programming principle; g-supermartingale decomposition; non-Skorohod-type minimality condition, second order reflected backward stochastic differential equation.
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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space with a natural filtration $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ generated by a d-dimensional Brownian motion B.

In 1997, El Karoui et al. [15] first introduced the following reflected backward stochastic differential equation (RBSDE) with a Skorohod-type minimality condition:

$$\begin{cases} Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\,ds - \int_t^T Z_s\,dB_s + K_t - K_t, \\ Y_t \geq L_t,\ \int_0^T (Y_t - L_t) dK_t = 0, \end{cases} \tag{1.1}$$

where ξ is an $\mathcal{F}_T$-measurable random variable, $g(\cdot, y, z)$ is $\mathbb{F}$-progressively measurable for any $(y, z) \in \mathbb{R} \times \mathbb{R}^{1\times d}$, and L is an $\mathbb{F}$-adapted, continuous obstacle process. Hamadéne [19], Lepeltier and Xu [22] studied RBSDEs in the case where the obstacle process is no

*Faculty of mathematics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea.
E-mail: cioc13@ryongnamsan.edu.kp, mc.kim0523@ryongnamsan.edu.kp, kg.kim0425@ryongnamsan.edu.kp
†Corresponding author.

longer continuous and assumed only càdlàg. In that case, the minimality condition is replaced by

$$Y_t \geq L_t, \ \int_0^T (Y_{t-} - L_{t-}) dK_t = 0. \tag{1.2}$$

The Skorohod-type minimality condition (1.2) is convenient to prove the existence and uniqueness of solutions to RBSDEs by a standard Picard iteration method. With this minimality condition, the investigation of RBSDEs is rich enough. There are two prototype methods for dealing with the reflected dynamics–Snell envelope method and Penalization method. Most of the papers which are concerned with RBSDEs, are based on these two methods. Recently, Qian and Xu [34] studied RBSDEs with a nonlinear resistance by using the Skorohod representation for the increasing process K, without the help of the Snell envelope theory. More recently, O and Kim [27] proposed a fixed-point problem approach which provides benefits for a wide class of RBSDEs with path-dependent coefficients.

In this paper, we introduce a new type of minimality condition (see (1.8) below), which is not Skorohod-type, and then propose a completely different approach from existing methods, by using the stochastic control technique. In what follows, we heuristically describe the motivation of a new non-Skorohod-type minimality condition.

Once it is proved that the RBSDE (1.1) has a unique solution, we can represent the first component of solution as the value process of the nonlinear optimal stopping problem with g-expectation in terms of Peng [31]:

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} \mathcal{E}^g_{t,\tau}(\widetilde{L}_\tau) := \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} y_t(\tau, \widetilde{L}_\tau), \tag{1.3}$$

where $\mathbb{T}^{t,T}$ is the set of all $\mathbb{F}$-stopping times taking values in $[t,T]$, $\widetilde{L}_\tau := L_\tau \mathbb{1}_{\tau<T} + \xi \mathbb{1}_{\tau=T}$ and $(y_\cdot, z_\cdot) := (y_\cdot(\tau, \widetilde{L}_\tau), z_\cdot(\tau, \widetilde{L}_\tau))$ is a solution of the standard BSDE

$$y_t = \widetilde{L}_\tau + \int_t^\tau g(s, y_s, z_s)\, ds - \int_t^T z_s\, dB_s, \tag{1.4}$$

$\mathcal{E}^g$ is the g-expectation operator introduced in [31], i.e. $\mathcal{E}^g_{t,\tau}(\cdot) := y_t(\tau, \cdot)$ (see El Karoui et al. [16] or Quenez and Sulem [35]).

Unlike the traditional approach to RBSDEs, our starting point is the dynamic nonlinear optimal stopping problem (1.3). We aim to look for a dynamics by which the value process of (1.3) is characterized. This idea leads to a new formulation of the minimality condition. To motivate a definition of RBSDEs, let us take a closer look at the classical optimal stopping problem

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} \mathbb{E}_t[\widetilde{L}_\tau], \tag{1.5}$$

where $\mathbb{E}_t[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_t$. It is well known that Y_t (which is Snell envelope of càdlàg process $\widetilde{L}$) is a supermartingale (see e.g. El Karoui [14]). Then we can use the Doob-Meyer decomposition theorem and the martingale representation theorem for Brownian martingales to get the existence of $\mathbb{F}$-progressively measurable process Z and a non-decreasing process K such that

$$Y_t = Y_\tau - \int_t^\tau Z_s\, dB_s + K_\tau - K_t, \ t \in [0,T], \ \tau \in \mathbb{T}^{t,T}. \tag{1.6}$$

Fix $t \in [0,T], \ \tau \in \mathbb{T}^{t,T}$. If we set $y^\tau_u := \mathbb{E}_{u\wedge\tau}[\widetilde{L}_\tau], \ u \in [0,T]$, again by the martingale representation theorem, one has

$$y^\tau_u = \widetilde{L}_\tau - \int_{u\wedge\tau}^\tau z_s\, dB_s, \ u \in [0,T], \tag{1.7}$$

for some $\mathbb{F}$-progressively measurable process z. Combining (1.6) and (1.7), we have

$$Y_t - y_t^\tau = (Y_\tau - \widetilde{L}_\tau) - \int_t^\tau (Z_s - z_s)\, dB_s + K_\tau - K_t.$$

By taking conditional expectations $\mathbb{E}_t[\cdot]$ on both sides of the above equality, we obtain

$$Y_t - y_t^\tau = \mathbb{E}_t[(Y_\tau - \widetilde{L}_\tau) + K_\tau] - K_t.$$

Observe that

$$\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t,T}}(Y_t - y_t^\tau) = Y_t - \operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}} \mathbb{E}_t[\widetilde{L}_\tau] = 0.$$

Finally we derive

$$K_t = \operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t,T}} \mathbb{E}_t[(Y_\tau - \widetilde{L}_\tau) + K_\tau], \tag{1.8}$$

which is the minimality condition of this paper.

The main advantage of our minimality condition is that it can be very suitable for the complete formulation of the second order reflected backward stochastic differential equations (2RBSDEs) or RBSDEs driven by G-Brownian motion (G-RBSDEs), which are RBSDEs under nonlinear expectation which carries a family of possibly non-dominated probability measures. It is proved in Section 2 that our minimality condition is equivalent to the Skorohod-type minimality condition in the framework of classical linear expectation, which carries only one probability measure. However, they are not equivalent under nonlinear expectation, which leads to the 2RBSDEs or G-RBSDEs. Indeed, our minimality condition is weaker than the classical Skorohod condition in the uncertainty setting and has nice properties in the sense that it gives the definition of 2RBSDEs as a natural extension of standard RBSDEs and that it is adapted to any kind of generators which are not necessarily Lipschitz or monotonic. This will be described in the future publications.

Our first step is to prove, without using the theory of RBSDEs, the dynamic programming principle for nonlinear optimal stopping problem with g-expectation, which immidiately gives us the (strong) g-supermartingale property of the value process. This is the main purpose of Section 3. Section 4 is devoted to prove the representation formula. With the help of a new minimality condition, we prove that the solution must satisfy the representation formula (1.3) if it does exist. This formula gives us the natural candidate of solutions to RBSDEs as the essential supremum of solutions to ordinary BSDEs over a family of stopping times. We then use the nonlinear Doob-Meyer decomposition theorem for g-supermartingales, proposed by Peng [32] and further developed by Bouchard et al. [4], in order to get the existence of the solution to RBSDEs, as explored in Section 5. Finally, we derive some a priori estimates and stability results. To focus on the novelty of our approach, we assume that the generator satisfies the standard Lipschitz condition, which is the prototypical case. However it can be generalized to the case where the generator satisfies the monotonicity condition. Our results in this paper, as usual, are equipped with $\mathbb{L}^2$-data (i.e., square integrable parameters). But this is only for the ease of presentation and all the results remain true for the dynamics with $\mathbb{L}^p$-data ($p > 1$) with obvious changes, in view of the work of Briand et al. [8], where the authors studied the L^p-solution of BSDEs.

Spaces and norms We conclude this introduction with a list of most frequently used spaces and norms. For $p \geq 1$,

- $\mathbb{L}^p$ denotes the space of all $\mathcal{F}_T$-measurable scalar random variables ξ with

$$\|\xi\|_{\mathbb{L}^p}^p := \mathbb{E}\left[|\xi|^p\right] < +\infty.$$

- $\mathbb{S}^p$ denotes the space of $\mathbb{R}$-valued, $\mathbb{F}$-adapted processes Y, with $\mathbb{P}-a.s.$ càdlàg paths, such that
$$\|Y\|_{\mathbb{S}^p}^p := \mathbb{E}\left[\sup_{0\le t\le T}|Y_t|^p\right] < +\infty.$$
- $\mathbb{H}^p$ (resp $\mathbb{H}_1^p$) denotes the space of all $\mathbb{F}$-progressively measurable $\mathbb{R}^{1\times d}$-valued (resp. $\mathbb{R}$-valued) processes Z with
$$\|Z\|_{\mathbb{H}^p}^p := \mathbb{E}\left[\left(\int_0^T |Z_t|^2\,dt\right)^{p/2}\right] < +\infty.$$
- $\mathbb{I}^p$ denotes the space of $\mathbb{R}$-valued, $\mathbb{F}$-adapted processes K, with $\mathbb{P}-a.s.$ càdlàg non-decreasing paths K, such that
$$\|K\|_{\mathbb{I}^p}^p := \mathbb{E}\left[|K_T|^p\right] < +\infty,$$
and $K_0 = 0$.

2 A new formulation of RBSDE

We consider a terminal condition $\xi \in \mathbb{L}^2$, a generator $g : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^{1\times d} \to \mathbb{R}$ and a lower obstacle process $L \in \mathbb{S}^2$. We will always assume that L is a càdlàg, adapted process with $L_T \le \xi$. Throughout this paper, we work under the following standing assumptions on the generator.

Assumption 2.1. *(i) The process* $(t,\omega) \mapsto g(t,\omega,y,z)$ *is* $\mathbb{F}$*-progressively measurable for all* $(y,z) \in \mathbb{R}\times\mathbb{R}^{1\times d}$.

(ii) $\mathbb{E}\left[\int_0^T |g(t,0,0)|^2\,dt\right] < +\infty$.

(iii) There exists a constant $C > 0$ *such that*
$$|g(t,y,z) - g(t,y',z')| \le C(|y-y'| + |z-z'|),$$
for all $(t,\omega,y,z,y',z') \in [0,T]\times\Omega\times(\mathbb{R}\times\mathbb{R}^{1\times d})^2$.

By the pioneer work of Pardoux and Peng [30], we know that, under Assumption 2.1, the standard BSDE with terminal condition $\xi \in \mathbb{L}^2$ and generator g has a unique solution.

Then, we shall consider the following reflected backward stochastic differential equation (RBSDE for short) with lower obstacle L:
$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)\,ds - \int_t^T Z_s\,dB_s + K_T - K_t,\ 0\le t\le T,\ \mathbb{P}-a.s. \tag{2.1}$$
We recall that for each $t \in [0,T]$, $\mathbb{T}^{t,T}$ is the set of $\mathbb{F}$-stopping times taking values in $[t,T]$, $\widetilde{L}_\tau := L_\tau \mathbb{1}_{\tau<T} + \xi\mathbb{1}_{\tau=T}$ and $\mathbb{E}_t[\cdot]$ means the conditional expectation with respect to $\mathcal{F}_t$.

Definition 2.2. *We say* $(Y,Z,K) \in \mathbb{S}^2\times\mathbb{H}^2\times\mathbb{I}^2$ *is a solution to RBSDE* (2.1) *if*

- $Y_T = \xi,\ \mathbb{P}-a.s.$
- *The process* K *has non-decreasing paths,* $\mathbb{P}-a.s.$ *and satisfies:*
$$K_t = Y_0 - Y_t - \int_0^t g(s,Y_s,Z_s)\,ds + \int_0^t Z_s\,dB_s,\ 0\le t\le T,\ \mathbb{P}-a.s. \tag{2.2}$$
- *The process* K *satisfies the following minimality condition:*
$$K_t = \operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t,T}} \mathbb{E}_t[Y_\tau - \widetilde{L}_\tau + K_\tau],\ 0\le t\le T,\ \mathbb{P}-a.s. \tag{2.3}$$

At this stage, we recall the classical Skorohod-type minimality condition

$$Y_t \geq L_t,\ \int_0^T (Y_{t-} - L_{t-})dK_t = 0,\ 0 \leq t \leq T,\ \mathbb{P}-a.s. \tag{2.4}$$

Proposition 2.3. *If we assume that* $(Y, L, K) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}^1$,[1] *then the minimality condition* (2.3) *is equivalent to the Skorohod-type minimality condition* (2.4).

Proof. **(1)** Suppose that (2.3) holds true. Define $\overline{L}_t := Y_t - \widetilde{L}_t + K_t$. Then it follows that

$$-K_t = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} \mathbb{E}_t[-\overline{L}_\tau].$$

Hence $-K$ is a Snell envelope of $-\overline{L}$. Moreover, since K is non-decreasing, K is just the non-decreasing component in the decomposition of supermartingale $-K$. Therefore, according to El Karoui [14], one obtains

$$-K_t \geq -\overline{L}_t,\ \int_0^T (-K_{t-} + \overline{L}_{t-})dK_t = 0,\ 0 \leq t \leq T,\ \mathbb{P}-a.s.,$$

which leads to

$$Y_t \geq L_t,\ \int_0^T (Y_{t-} - L_{t-})dK_t = 0.$$

(2) Next, let us assume that (2.4) holds true. One has for any $\tau \in \mathbb{T}^{t,T}$,

$$\mathbb{E}_t[Y_\tau - \widetilde{L}_\tau + K_\tau] \geq \mathbb{E}_t[K_\tau] \geq K_t,\ \mathbb{P}-a.s.$$

Taking infimum implies

$$K_t \leq \operatorname*{ess\,inf}_{\tau \in \mathbb{T}^{t,T}} \mathbb{E}_t[Y_\tau - \widetilde{L}_\tau + K_\tau]. \tag{2.5}$$

We now prove the reverse inequality. For this purpose, we define a stopping time

$$\tau_t^\varepsilon := \inf\{s \geq t:\ Y_s \leq \widetilde{L}_s + \varepsilon\},\ t \in [0, T],$$

where $\varepsilon > 0$ is an arbitrary constant. Since $\widetilde{L}$ is right-continuous, it follows that

$$Y_{\tau_t^\varepsilon} \leq \widetilde{L}_{\tau_t^\varepsilon} + \varepsilon,\ \mathbb{P}-a.s. \tag{2.6}$$

Now, we prove that for $a.e.\ \omega \in \Omega$, the map $t \mapsto K_t(\omega)$ is constant on $[t, \tau_t^\varepsilon]$. For $a.e.\ \omega \in \Omega$, if $s \in [t, \tau_t^\varepsilon)$ then $Y_s > \widetilde{L}_s + \varepsilon > \widetilde{L}_s$. Therefore, for $a.e.\ \omega \in \Omega$, $t \mapsto K_t^c(\omega)$ is constant on $[t, \tau_t^\varepsilon]$ and $t \mapsto K_t^d(\omega)$ is constant on $[t, \tau_t^\varepsilon)$, where K^c (resp. K^d) is the continuous (resp. discontinuous) part of K. On the other hand, since $Y_{\tau_t^\varepsilon -} \geq \widetilde{L}_{\tau_t^\varepsilon -} + \varepsilon > \widetilde{L}_{\tau_t^\varepsilon -}$, one has $\Delta K^d_{\tau_t^\varepsilon} = 0,\ \mathbb{P}-a.s.$ Hence we have

$$K_t = K_{\tau_t^\varepsilon}, \mathbb{P}-a.s. \tag{2.7}$$

Using (2.6) and (2.7), we deduce that

$$\operatorname*{ess\,inf}_{\tau \in \mathbb{T}^{t,T}} \mathbb{E}_t[Y_\tau - \widetilde{L}_\tau + K_\tau] \leq \mathbb{E}_t[Y_{\tau_t^\varepsilon} - \widetilde{L}_{\tau_t^\varepsilon} + K_{\tau_t^\varepsilon}] \leq \mathbb{E}_t[\varepsilon + K_t] = \varepsilon + K_t.$$

By the arbitrariness of $\varepsilon > 0$, we obtain

$$\operatorname*{ess\,inf}_{\tau \in \mathbb{T}^{t,T}} \mathbb{E}_t[Y_\tau - \widetilde{L}_\tau + K_\tau] \leq K_t. \tag{2.8}$$

Combining (2.7) with (2.8), it is proved that

$$K_t = \operatorname*{ess\,inf}_{\tau \in \mathbb{T}^{t,T}} \mathbb{E}_t[Y_\tau - \widetilde{L}_\tau + K_\tau].$$ □

[1] Here, we remark that the integrability assumption is only used to define the Snell envelope. It is not optimal and may be further weakened (see e.g. [14]).

Remark 2.4. If $L=-\infty$, then the minimality condition (2.3) implies

$$K_t=\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t,T}}\mathbb{E}_t[Y_\tau-\widetilde{L}_\tau+K_\tau]=\mathbb{E}_t[Y_T-\xi+K_T]=\mathbb{E}_t[K_T],\ t\in[0,T],\ \mathbb{P}-a.s.$$

In particular, $0=K_0=\mathbb{E}[K_T]$ and thus $K=0,\ \mathbb{P}-a.s.$ Hence the RBSDE (2.1) is equivalent to the following standard BSDE:

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s)\,ds-\int_t^T Z_s\,dB_s,\ 0\le t\le T,\ \mathbb{P}-a.s.$$

Remark 2.5. We emphasize that Proposition 2.3 is no longer true under nonlinear expectation which encompasses a family of probability measures. In this uncertainty setting, our minimality condition (2.3) is actually weaker than the Skorohod-type minimality condition (2.4), so that it can be very effective for 2RBSDEs (or G-RBSDEs), which are the RBSDEs under nonlinear expectation. The details are postponed to future publications.

3 Dynamic programming principle for nonlinear optimal stopping problems under g-expectations

Consider the following optimal stopping problem under g-expectations:

$$V=\operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{0,T}}\mathcal{E}^g_{0,\tau}(\widetilde{L}_\tau)=\operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{0,T}}y_0(\tau,\widetilde{L}_\tau),$$

where $\mathcal{E}^g(\cdot)$ denotes the g-expectation operator, that is, $\mathcal{E}^g_{\cdot,\tau}(\widetilde{L}_\tau):=y_\cdot(\tau,\widetilde{L}_\tau)$ is the first component of solution to standard BSDE with terminal pair $(\tau,\widetilde{L}_\tau)$ and generator g (see (1.4)). We then define, as usual, the following value process to make the problem dynamic:

$$Y_t=\operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}}\mathcal{E}^g_{t,\tau}(\widetilde{L}_\tau)=\operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}}y_t(\tau,\widetilde{L}_\tau),\ t\in[0,T]. \tag{3.1}$$

In the rest of this paper, we shall denote by C a generic constant which may vary from line to line. Our purpose is to study the above optimal stopping problem without the help of the RBSDE theory.

Lemma 3.1 (DPP for deterministic times)**.** *For any $0\le t<t'\le T$, we have*

$$Y_t=\operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}}\mathcal{E}^g_{t,\tau\wedge t'}(\widetilde{L}_\tau\mathbb{1}_{\tau<t'}+Y_{t'}\mathbb{1}_{\tau\ge t'}).$$

Proof. We first prove the forward inequality " $\le$ ". Using the DPP (or flow property) for BSDEs, one has for any $\tau\in\mathbb{T}^{t,T}$

$$\begin{aligned}\mathcal{E}^g_{t,\tau}(\widetilde{L}_\tau)&=\mathcal{E}^g_{t,t'\wedge\tau}\left(\mathcal{E}^g_{t'\wedge\tau,\tau}(\widetilde{L}_\tau)\right)\\&=\mathcal{E}^g_{t,t'\wedge\tau}\left(\mathbb{1}_{\tau<t'}\widetilde{L}_\tau+\mathbb{1}_{\tau\ge t'}\mathcal{E}_{t',\tau}(\widetilde{L}_\tau)\right)\\&\le\mathcal{E}^g_{t,t'\wedge\tau}\left(\mathbb{1}_{\tau<t'}\widetilde{L}_\tau+Y_{t'}\mathbb{1}_{\tau\ge t'}\right),\end{aligned}$$

where we used the comparison principle for standard BSDEs. Therefore,

$$Y_t=\operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}}\mathcal{E}^g_{t,\tau}(\widetilde{L}_\tau)\le\operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}}\mathcal{E}^g_{t,t'\wedge\tau}\left(\mathbb{1}_{\tau<t'}\widetilde{L}_\tau+Y_{t'}\mathbb{1}_{\tau\ge t'}\right).$$

We now prove the reverse inequality " $\ge$ ". First, we observe that

$$Y_{t'}=\sup_{n\ge1}\mathcal{E}^g_{t',\tau_n}(\widetilde{L}_{\tau_n}),$$

for some $\tau_n \in \mathbb{T}^{t',T}$ (see e.g. [26] or Theorem 1.1.4 of [39]). Define

$$E_n := \{\mathcal{E}^g_{t',\tau_n}(\widetilde{L}_{\tau_n}) \geq Y_{t'} - \varepsilon\} \text{ and } \widetilde{E}_n := E_n \setminus \cup_{i=1}^n E_i.$$

We see that $\{\widetilde{E}_n\}$ is a partition of Ω and

$$Y_{t'} \leq \mathcal{E}^g_{t',\tau_n}(\widetilde{L}_{\tau_n}) + \varepsilon \qquad \text{on} \quad \widetilde{E}_n. \tag{3.2}$$

Define a stopping time

$$\widetilde{\tau} := \sum_n \tau_n \mathbb{1}_{\widetilde{E}_n}.$$

We then define

$$y_{t'} := \mathcal{E}^g_{t',\widetilde{\tau}}(\widetilde{L}_{\widetilde{\tau}}) \text{ and } y^n_{t'} := \mathcal{E}^g_{t',\tau_n}(\widetilde{L}_{\tau_n}).$$

Note that

$$\begin{aligned} y_{t'}\mathbb{1}_{\widetilde{E}_n} &= \widetilde{L}_{\widetilde{\tau}}\mathbb{1}_{\widetilde{E}_n} + \int_{t'}^{\widetilde{\tau}} \mathbb{1}_{\widetilde{E}_n} g(s, \mathbb{1}_{\widetilde{E}_n} y_s, \mathbb{1}_{\widetilde{E}_n} z_s)\, ds - \int_{t'}^{\widetilde{\tau}} \mathbb{1}_{\widetilde{E}_n} z_s\, dB_s \\ &= \widetilde{L}_{\tau_n}\mathbb{1}_{\widetilde{E}_n} + \int_{t'}^{\tau_n} \mathbb{1}_{\widetilde{E}_n} g(s, \mathbb{1}_{\widetilde{E}_n} y_s, \mathbb{1}_{\widetilde{E}_n} z_s)\, ds - \int_{t'}^{\tau_n} \mathbb{1}_{\widetilde{E}_n} z_s\, dB_s, \end{aligned}$$

and

$$y^n_{t'}\mathbb{1}_{\widetilde{E}_n} = \widetilde{L}_{\tau_n}\mathbb{1}_{\widetilde{E}_n} + \int_{t'}^{\tau_n} \mathbb{1}_{\widetilde{E}_n} g(s, \mathbb{1}_{\widetilde{E}_n} y^n_s, \mathbb{1}_{\widetilde{E}_n} z^n_s)\, ds - \int_{t'}^{\tau_n} \mathbb{1}_{\widetilde{E}_n} z^n_s\, dB_s.$$

By the uniqueness of solution to standard BSDEs, we deduce that

$$y_{t'}\mathbb{1}_{\widetilde{E}_n} = y^n_{t'}\mathbb{1}_{\widetilde{E}_n},\ \mathbb{P}-a.s.,$$

hence,

$$\mathcal{E}^g_{t',\widetilde{\tau}}(\widetilde{L}_{\widetilde{\tau}})\mathbb{1}_{\widetilde{E}_n} = \mathcal{E}^g_{t',\tau_n}(\widetilde{L}_{\tau_n})\mathbb{1}_{\widetilde{E}_n},\ \mathbb{P}-a.s. \tag{3.3}$$

Using (3.2) and (3.3), we obtain

$$\begin{aligned} Y_{t'} &= \sum_n Y_{t'}\mathbb{1}_{\widetilde{E}_n} \leq \sum_n \mathbb{1}_{\widetilde{E}_n}\mathcal{E}^g_{t',\tau_n}(\widetilde{L}_{\tau_n}) + \varepsilon \\ &= \sum_n \mathbb{1}_{\widetilde{E}_n}\mathcal{E}^g_{t',\widetilde{\tau}}(\widetilde{L}_{\widetilde{\tau}}) + \varepsilon = \mathcal{E}^g_{t',\widetilde{\tau}}(\widetilde{L}_{\widetilde{\tau}}) + \varepsilon. \end{aligned}$$

Then, by the comparison principle for BSDEs, it follows that

$$\mathcal{E}^g_{t,\tau\wedge t'}\left[\widetilde{L}_\tau\mathbb{1}_{\tau<t'} + Y_{t'}\mathbb{1}_{\tau\geq t'}\right] \leq \mathcal{E}^g_{t,\tau\wedge t'}\left[\widetilde{L}_\tau\mathbb{1}_{\tau<t'} + \mathcal{E}^g_{t',\widetilde{\tau}}(\widetilde{L}_{\widetilde{\tau}})\mathbb{1}_{\tau\geq t'} + \varepsilon\right]. \tag{3.4}$$

We now define a stopping time

$$\widehat{\tau} := \tau\mathbb{1}_{\tau<t'} + \widetilde{\tau}\mathbb{1}_{\tau\geq t'}.$$

Then, it follows that

$$\begin{aligned} &\mathcal{E}^g_{t,\tau\wedge t'}\left[\widetilde{L}_\tau\mathbb{1}_{\tau<t'} + \mathcal{E}^g_{t',\widetilde{\tau}}(\widetilde{L}_{\widetilde{\tau}})\mathbb{1}_{\tau\geq t'} + \varepsilon\right] \\ &= \mathcal{E}^g_{t,\widehat{\tau}\wedge t'}\left[\widetilde{L}_{\widehat{\tau}}\mathbb{1}_{\widehat{\tau}<t'} + \mathcal{E}^g_{t'\wedge\widehat{\tau}.\widehat{\tau}}(\widetilde{L}_{\widehat{\tau}})\mathbb{1}_{\widehat{\tau}\geq t'} + \varepsilon\right] \\ &= \mathcal{E}^g_{t,\widehat{\tau}\wedge t'}\left[\mathcal{E}^g_{t'\wedge\widehat{\tau},\widehat{\tau}}(\widetilde{L}_{\widehat{\tau}})\mathbb{1}_{\widehat{\tau}<t'} + \mathcal{E}^g_{t'\wedge\widehat{\tau}.\widehat{\tau}}(\widetilde{L}_{\widehat{\tau}})\mathbb{1}_{\widehat{\tau}\geq t'} + \varepsilon\right] \\ &= \mathcal{E}^g_{t,\widehat{\tau}\wedge t'}\left[\mathcal{E}^g_{t'\wedge\widehat{\tau},\widehat{\tau}}(\widetilde{L}_{\widehat{\tau}}) + \varepsilon\right] \\ &\leq \mathcal{E}^g_{t,\widehat{\tau}\wedge t'}\left[\mathcal{E}^g_{t'\wedge\widehat{\tau},\widehat{\tau}}(\widetilde{L}_{\widehat{\tau}})\right] + C\varepsilon \\ &= \mathcal{E}^g_{t,\widehat{\tau}}\left(\widetilde{L}_{\widehat{\tau}}\right) + C\varepsilon \\ &\leq Y_t + C\varepsilon, \end{aligned} \tag{3.5}$$

where we used the stability of BSDEs (see Lemma 5.2), and the DPP for BSDEs. Combining (3.4) with (3.5), leads to

$$\mathcal{E}^g_{t,\tau\wedge t'}\left[\widetilde{L}_\tau \mathbb{1}_{\tau<t'} + Y_{t'}\mathbb{1}_{\tau\geq t'}\right] \leq Y_t + C\varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we have

$$\mathcal{E}^g_{t,\tau\wedge t'}\left[\widetilde{L}_\tau \mathbb{1}_{\tau<t'} + Y_{t'}\mathbb{1}_{\tau\geq t'}\right] \leq Y_t,$$

from which we get the reverse inequality. □

From Lemma 3.1, by taking $\tau := t$, we see that Y is a (weak) g-supermartingale[2] in terms of Peng [32]. Our next goal is to show that there is a càdlàg modification of Y. When the generator g does not depend on y with $g(s,y,0) = 0$, we know from Lemma 5.2 of Coquet et al. [11] that Y has a càdlàg modification. The following result generalizes that of [11] to the arbitrary generators.

Lemma 3.2. *The value process Y has a càdlàg modification.*

Proof. Since Y is a g-supermartingale, we can use the downcrossing inequality for g-supermartingales (see e.g. Lemma A.1 of [4]; see also Theorem 6 of [9] and Proposition 2.6 of [11] for first attempts) in order to show that Y has left- and right- limits outside an evanescent. Define $Y^+_t := \lim_{s\downarrow t} Y_s$. Since the filtration is right-continuous, we see that Y^+ is $\mathbb{F}$-adapted process. Our aim is to show that $Y_t = Y^+_t$, for all $t \in [0,T],\ \mathbb{P}-a.s.$ Fix $t \in [0,T]$. For any $\tau \in \mathbb{T}^{t,T}$,

$$Y_t \geq \mathcal{E}^g_{t,\tau}(\widetilde{L}_\tau).$$

Since the g-solution is right-continuous, it follows that

$$Y^+_t \geq \lim_{s\downarrow t} \mathcal{E}^g_{s,\tau}(\widetilde{L}_\tau) = \mathcal{E}^g_{t,\tau}(\widetilde{L}_\tau),$$

which leads to $Y^+_t \geq Y_t$. It remains to show that $Y_t \geq Y^+_t$. Since Y is a g-supermartingale, one has $s \geq t$,

$$\mathcal{E}_{t,s}[Y_s] \leq Y_t. \tag{3.6}$$

For any $\varepsilon > 0$, we denote by $(\bar{Y}^\varepsilon, \bar{Z}^\varepsilon)$ the solution of the BSDE:

$$\bar{Y}^\varepsilon_r = Y_{t+\varepsilon} + \int_r^T g(s,\bar{Y}^\varepsilon_s,\bar{Z}^\varepsilon_s)\mathbb{1}_{s\leq t+\varepsilon}\,ds - \int_r^T \bar{Z}^\varepsilon_s\,ds,\ r\in[0,T].$$

Notice that $\bar{Z}^\varepsilon_r = 0$ for all $r \in (t+\varepsilon, T]$ and therefore $\bar{Y}^\epsilon_r = \mathcal{E}^g_{r,t+\varepsilon}[Y_{t+\varepsilon}]$ for all $r \in [0, t+\varepsilon]$. In particular, $\bar{Y}^\varepsilon_t = \mathcal{E}^g_{t,t+\varepsilon}[Y_{t+\varepsilon}]$. By the stability of solutions to standard BSDEs (see Lemma 5.2), we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{r\in[0,T]}|\bar{Y}^\varepsilon_r - \bar{Y}^0_r|^2\right] = 0,$$

where $\bar{Y}^0$ is a solution of the BSDE:

$$\bar{Y}^0_r = Y^+_t + \int_r^T g(s,\bar{Y}^0_s,\bar{Z}^0_s)\mathbb{1}_{s\leq t}\,ds - \int_r^T \bar{Z}^0_s\,ds,\ r\in[0,T].$$

[2] We note that, in some special cases, the weak g-supermartingale property of the value process may be obtained without using the DPP for deterministic times. For instance, when $g(s,y,0) = 0$, one can easily prove that the family $\{\mathcal{E}^g_t(\widetilde{L}_\tau):\ \tau\in\mathbb{T}^{t,T}\}$ is upward directed, which in turn derives the g-supermartingale property of Y due to the monotonic continuity of BSDEs (see e.g. Lemma 3.1 and Proposition 3.1 of [37]).

In particular, we have $\mathbb{P}-a.s.$,

$$\mathcal{E}^g_{t,t+\varepsilon}[Y_{t+\varepsilon}] = \bar{Y}^\varepsilon_t \to \bar{Y}^0_t = Y^+_t,\ \varepsilon \to 0.$$

This, combined with (3.6), leads to

$$Y^+_t \le Y_t.$$

□

Now, we are in a position to state the main result of this section.

Theorem 3.3 (DPP for stopping times)**.** *For any $t \in [0,T]$ and $\widetilde{\tau} \in \mathbb{T}^{t,T}$, we have*

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} \mathcal{E}^g_{t,\tau\wedge\widetilde{\tau}}(\widetilde{L}_\tau \mathbb{1}_{\tau<\widetilde{\tau}} + Y_{\widetilde{\tau}} \mathbb{1}_{\tau \ge \widetilde{\tau}}).$$

Proof. **(i)**. In this step, we prove the forward inequality

$$Y_t \le \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} \mathcal{E}^g_{t,\tau\wedge\widetilde{\tau}}(\widetilde{L}_\tau \mathbb{1}_{\tau<\widetilde{\tau}} + Y_{\widetilde{\tau}} \mathbb{1}_{\tau \ge \widetilde{\tau}}). \tag{3.7}$$

First, we assume that $\widetilde{\tau}$ takes only finitely many values $t_1, \cdots, t_n$. In view of Lemma 3.1, one has for any $\tau \in \mathbb{T}^{t,T}$,

$$\begin{aligned} Y_t &= \sum_{i=1}^n Y_t \mathbb{1}_{\widetilde{\tau}=t_i} \\ &\le \sum_{i=1}^n \mathcal{E}^g_{t,t_i\wedge\tau}\left(\mathbb{1}_{\tau<t_i}\widetilde{L}_\tau + Y_{t_i}\mathbb{1}_{\tau\ge t_i}\right)\mathbb{1}_{\widetilde{\tau}=t_i} \\ &= \sum_{i=1}^n \mathcal{E}^g_{t,\widetilde{\tau}\wedge\tau}\left(\mathbb{1}_{\tau<\widetilde{\tau}}\widetilde{L}_\tau + Y_{\widetilde{\tau}}\mathbb{1}_{\tau\ge \widetilde{\tau}}\right)\mathbb{1}_{\widetilde{\tau}=t_i} \\ &= \mathcal{E}^g_{t,\widetilde{\tau}\wedge\tau}\left(\mathbb{1}_{\tau<\widetilde{\tau}}\widetilde{L}_\tau + Y_{\widetilde{\tau}}\mathbb{1}_{\tau\ge \widetilde{\tau}}\right), \end{aligned}$$

which gives (3.7). For general $\widetilde{\tau}$, there exist $\widetilde{\tau}_n \in \mathbb{T}^{t,T}$ such that $\widetilde{\tau}_n \downarrow \widetilde{\tau}$ and each $\widetilde{\tau}_n$ takes only finitely many values. Indeed, we can set $\widetilde{\tau}_n$ as

$$\widetilde{\tau}_n := \sum_{i=1}^{2^n} t_i \mathbb{1}_{(t_{i-1},t_i]}(\tau),\ t_i = \frac{Ti}{2^n}.$$

Then, we have for any $n \ge 1$,

$$Y_t \le \mathcal{E}^g_{t,\widetilde{\tau}_n\wedge\tau}\left(\mathbb{1}_{\tau<\widetilde{\tau}_n}\widetilde{L}_\tau + Y_{\widetilde{\tau}_n}\mathbb{1}_{\tau\ge \widetilde{\tau}_n}\right).$$

Sending $n \to \infty$, by Lemma 3.2, the stability and the comparison theorems for BSDEs,

$$\begin{aligned} Y_t &\le \mathcal{E}^g_{t,\widetilde{\tau}\wedge\tau}\left(\mathbb{1}_{\tau\le\widetilde{\tau}}\widetilde{L}_\tau + Y_{\widetilde{\tau}}\mathbb{1}_{\tau> \widetilde{\tau}}\right) \\ &\le \mathcal{E}^g_{t,\widetilde{\tau}\wedge\tau}\left(\mathbb{1}_{\tau<\widetilde{\tau}}\widetilde{L}_\tau + Y_{\widetilde{\tau}}\mathbb{1}_{\tau\ge \widetilde{\tau}}\right). \end{aligned}$$

Since $\tau \in \mathbb{T}^{t,T}$ is arbitrary, we obtain the forward inequality (3.7).

(ii) We now prove the reverse inequality. As in step (i), one can easily show that this result holds for stopping times $\widetilde{\tau}$ taking only finitely many values. For general $\widetilde{\tau}$, we again choose a sequence of stopping times $\widetilde{\tau}_n$ such that $\widetilde{\tau}_n \downarrow \widetilde{\tau}$ and each $\widetilde{\tau}_n$ takes only finitely many values. Then for any $\tau \in \mathbb{T}^{t,T}$, by denoting $\tau_m := (\tau + \frac{1}{m}) \wedge T$, we have

$$Y_t \ge \mathcal{E}^g_{t,\tau_m\wedge\widetilde{\tau}_n}\left[\widetilde{L}_{\tau_m}\mathbb{1}_{\tau_m<\widetilde{\tau}_n} + Y_{\widetilde{\tau}_n}\mathbb{1}_{\tau_m\ge\widetilde{\tau}_n}\right].$$

Sending $n \to \infty$, by Lemma 3.2 and the stability result for BSDEs,

$$Y_t \geq \mathcal{E}^g_{t,\tau_m \wedge \widetilde{\tau}} \left[\widetilde{L}_{\tau_m} \mathbb{1}_{\tau_m \leq \widetilde{\tau}} + Y_{\widetilde{\tau}} \mathbb{1}_{\tau_m > \widetilde{\tau}} \right].$$

Since L is right continuous, we obtain by sending $m \to \infty$,

$$Y_t \geq \mathcal{E}^g_{t,\tau \wedge \widetilde{\tau}} \left[\widetilde{L}_{\tau} \mathbb{1}_{\tau < \widetilde{\tau}} + Y_{\widetilde{\tau}} \mathbb{1}_{\tau \geq \widetilde{\tau}} \right],$$

where we again used the stability of BSDEs. □

Remark 3.4. We would like to mention that, when the generator takes a special form $g(s,y,z) = C|z|$ or more generally $g(s,y,z) = \tilde{g}(s,z)$ with $\tilde{g}(s,0) = 0$, the DPP for stopping times or strong g-supermartingale property of Y can be easily deduced as a direct consequence of existing results on optimal stopping problems under nonlinear expectations (see e.g. Theorem 4.3 of [13], or Proposition 2.2 of [2]). In the above, we provided a self-contained proof of the generalized DPP under g-expectations.

Remark 3.5. Once it is proved that the RBSDE has a unique solution (by using the Skorohod condition and the classical argument), then the DPP can be easily obtained thanks to the flow property of the RBSDE and the characterization theorem (see e.g. El Karoui et al. [16] or Quenez and Sulem [35]). On the other hand, the author in [37] used the relation between doubly reflected BSDE and Dynkin's game, in order to prove the DPP. Hence these two methods are heavily based on the theory of RBSDEs. Our framework consists of applying the DPP to the study of RBSDEs.

4 Representation formula

Theorem 4.1 (Representation formula)**.** *Let Assumption 2.1 holds. Assume that* $(Y,Z,K) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$ *is a solution to RBSDE* (2.1)*. Then for any* $t \in [0,T]$*,*

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} y_t(\tau, \widetilde{L}_\tau), \quad \mathbb{P} - a.s. \tag{4.1}$$

Proof. **(i)** For any fixed $\tau \in \mathbb{T}^{t,T}$, we note that

$$Y_t = Y_\tau + \int_t^\tau g(s, Y_s, Z_s)\, ds - \int_t^\tau Z_s\, dB_s + K_\tau - K_t.$$

Since $Y_\tau \geq \widetilde{L}_\tau$, by the general comparison theorem for BSDEs studied in [21] (see Theorem A.1 therein), we have $Y_t \geq y_t(\tau, \widetilde{L}_\tau)$, and thus

$$Y_t \geq \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} y_t(\tau, \widetilde{L}_\tau), \ \mathbb{P} - a.s. \tag{4.2}$$

(ii) We now prove the reverse inequality. Define for any $\tau \in \mathbb{T}^{t,T}$,

$$\Delta Y := Y - y(\tau, \widetilde{L}_\tau) \text{ and } \Delta Z := Z - z(\tau, \widetilde{L}_\tau).$$

We then use the classical linearization procedure. By the Lipschitz condition, there exist two bounded processes λ and μ such that

$$\Delta Y_t = (Y_\tau - \widetilde{L}_\tau) + \int_t^\tau (\Delta Y_s \lambda_s + \Delta Z_s \mu_s)\, ds - \int_t^\tau \Delta Z_s\, dB_s + K_\tau - K_t.$$

Using Itô's formula, we get

$$\Delta Y_t = \mathbb{E}_t \left[\Gamma^t_\tau (Y_\tau - \widetilde{L}_\tau) + \int_t^\tau \Gamma^t_s\, dK_s \right], \tag{4.3}$$

where the adjoint[3] process Γ is defined on $[t,\tau]$:

$$\Gamma_s^t = 1 + \int_t^s \lambda_r \Gamma_r^t\, dr + \int_t^s \mu_r \Gamma_r^t\, dB_r,\ s \in [t,\tau].$$

By the boundedness of λ and μ, for every $p \geq 1$, we have

$$\mathbb{E}_t\left[\sup_{t\leq s\leq \tau} |\Gamma_s^t|^p + \sup_{t\leq s\leq \tau} |\Gamma_s^t|^{-p}\right] \leq C_p,\ \mathbb{P}-a.s.$$

Using this estimate together with Hölder's inequality, we obtain

$$\begin{aligned}
\Delta Y_t &\leq \mathbb{E}_t\left[\Gamma_\tau^t \cdot (Y_\tau - \widetilde{L}_\tau) + \left(\sup_{t\leq s\leq \tau}|\Gamma_s^t|\right)\cdot (K_\tau - K_t)\right] \\
&\leq \mathbb{E}_t\left[\left(\sup_{t\leq s\leq \tau}|\Gamma_s^t|\right)\cdot (Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)\right] \\
&\leq \left(\mathbb{E}_t\left[\sup_{t\leq s\leq \tau}|\Gamma_s^t|^3\right]\right)^{1/3}\cdot\left(\mathbb{E}_t\left[\left(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t\right)^{3/2}\right]\right)^{2/3} \\
&\leq C\cdot\left(\mathbb{E}_t\left[Y_\tau - \widetilde{L}_\tau + K_\tau - K_t\right]\cdot \mathbb{E}_t\left[(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)^2\right]\right)^{1/3}.
\end{aligned}$$

We shall prove in step (iii) below that

$$C_t := \operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}} \mathbb{E}_t\left[\left(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t\right)^2\right] < \infty,\ \mathbb{P}-a.s. \tag{4.4}$$

Then it follows from the last inequality that

$$Y_t - \operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}} y_t(\tau,\widetilde{L}_\tau) \leq C\cdot (C_t)^{1/3}\cdot \operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t,T}}\left(\mathbb{E}_t\left[Y_\tau - \widetilde{L}_\tau + K_\tau - K_t\right]\right)^{1/3} = 0,\ \mathbb{P}-a.s.,$$

by the minimality condition (2.3).

(iii) It remains to show that the estimate (4.4) holds. For any $\tau \in \mathbb{T}^{t,T}$, we observe that

$$\mathbb{E}_t\left[\left(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t\right)^2\right] \leq 2\mathbb{E}_t\left[\left(Y_\tau - \widetilde{L}_\tau\right)^2 + (K_\tau - K_t)^2\right].$$

But

$$\mathbb{E}_t\left[\left(Y_\tau - \widetilde{L}_\tau\right)^2\right] \leq 2\mathbb{E}_t\left[|Y_\tau|^2 + |\widetilde{L}_\tau|^2\right] \leq 2\mathbb{E}_t\left[\sup_{t\leq s\leq T}|Y_s|^2 + \sup_{t\leq s\leq T}|\widetilde{L}_s|^2\right] < \infty,$$

and

$$\begin{aligned}
\mathbb{E}_t\left[(K_\tau - K_t)^2\right] &\leq \mathbb{E}_t\left[(K_T - K_t)^2\right] \\
&\leq C\cdot \mathbb{E}_t\left[\sup_{t\leq s\leq T}|Y_s|^2 + \int_t^T |Z_s|^2\, ds + \int_t^T |g_0(s)|^2\, ds\right] < \infty.
\end{aligned}$$

Thus, the estimate (4.4) follows. □

Remark 4.2. In the above proof, we used the linearization argument, which is efficient for dynamics with Lipschitz or monotonic generators. However, if one aims to consider non-monotonic generators like in [17, 18, 28], then the linearization argument is no longer efficient. In this respect, we give another proof method of Theorem 4.1, which does not use any linearization argument, in Appendix.

[3]Note that $\Gamma_s^t = \Gamma_t^{-1}\Gamma_s$, $\Gamma_s := \exp\left(\int_0^s (\lambda_r - \frac{1}{2}|\mu_r|^2)\, dr + \int_0^s \mu_r\, dB_r\right)$.

Theorem 4.3. *Assume that*

$$^{o}L \geq L_{-}, \tag{4.5}$$

where ^{o}L *is the optional projection ([10, Theorem 7.6.2]) of* L*. Let* $(Y, Z, K) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$ *be a solution to RBSDE* (2.1)*. Then, for any* $t \in [0, T]$*, we have*

$$Y_t = y_t(\tau_t^*, \widetilde{L}_{\tau_t^*}),\ \mathbb{P} - a.s., \tag{4.6}$$

where $\tau_t^* := \inf\{s \geq t : Y_s = L_s\} \wedge T$*.*

Proof. It suffices to show that

$$\int_t^{\tau_t^*} dK_s = 0. \tag{4.7}$$

Let $A := \{(s, \omega) \in]\!]t, \tau_t^*]\!] : Y_{s-}(\omega) = L_{s-}(\omega), \Delta K_s > 0\}$.

We will show that $\mathbb{P}(\pi(A)) := \mathbb{P}(\{\omega : (\omega, t) \in A \text{ for some } t\}) = 0$. Assume that $\mathbb{P}(\pi(A)) > 0$. By the measurable section theorem, for every $\varepsilon > 0$, there exists a stopping time τ such that

$$[\![\tau]\!] \subset A,\ \mathbb{P}(\pi(A)) \leq \mathbb{P}(\tau < \infty) + \varepsilon.$$

On the set $\{\tau < \infty\}$, we have

$$Y_\tau - L_{\tau-} = -\Delta K_\tau.$$

Since $Y_\tau \geq L_\tau$, it follows from (4.5) and the projection theorem ([10, Theorem 7.6.2]) that $\mathbb{E}\mathbb{1}_{\{\tau<\infty\}}(Y_\tau - L_{\tau-}) \geq 0$. Hence, since K is non-decreasing, $\mathbb{E}\mathbb{1}_{\{\tau<\infty\}}\Delta K_\tau = 0$. Thus, we have $\mathbb{P}(\pi(A)) = 0$.

Finally, from Proposition 2.3 and the minimality condition, the result follows. □

As a direct consequence of Theorem 4.1, we get the comparison theorem for RBSDEs.

Corollary 4.4 (Comparison theorem)**.** *Let* (Y^i, Z^i, K^i) *be the solution of the RBSDE* (2.1) *with Lipschitz generator* g^i*, the terminal condition* $\xi^i \in \mathbb{L}^2$*, and the obstacle* $L^i \in \mathbb{S}^2$*. Suppose that*

- $\xi^1 \leq \xi^2,\ L_t^1 \leq L_t^2$ *for all* $0 \leq t \leq T,\ \mathbb{P} - a.s.$
- $g^1(t, y, z) \leq g^2(t, y, z)$ *for all* (y, z)*;* $d\mathbb{P} \times dt - a.s.$

Then we have for all $t \in [0, T]$*,*

$$Y_t^1 \leq Y_t^2,\ \mathbb{P} - a.s.$$

Proof. From Theorem 4.1, we have

$$Y_t^i = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^{t,T}} y_t^i(\tau, \widetilde{L}_\tau^i), \quad \mathbb{P} - a.s.,\ i = 1, 2.$$

From the comparison theorem for BSDEs, it holds for any $\tau \in \mathbb{T}^{t,T}$ that $y_t^1(\tau, \widetilde{L}_\tau^1) \leq y_t^2(\tau, \widetilde{L}_\tau^2)$. Thus we have $Y_t^1 \leq Y_t^2,\ \mathbb{P} - a.s.$ □

5 Well-posedness, a priori estimate and stability

We first state our main existence and uniqueness result.

Theorem 5.1. *Let Assumption 2.1 holds true. Then the RBSDE* (2.1) *has a unique solution* $(Y, Z, K) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$*.*

Proof. The uniqueness is proved in Theorem 5.3.

Let us prove the existence result. Thanks to Theorem 3.3, we notice that the nonlinear value process Y defined by (3.1) is a strong g-supermartingale. Moreover, we have $Y \in \mathbb{S}^2$ (see the proof of Theorem 5.3 below). Then we can use the nonlinear Doob-Meyer decomposition theorem introduced by Peng [32, Theorem 3.3] (and further developed by Bouchard et al. [4]), to obtain the semi-martingale decomposition[4] of Y:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s + K_T - K_t,$$

for some $(Z, K) \in \mathbb{H}^2 \times \mathbb{I}^2$.

We claim that (Y, Z, K) is the solution to the RBSDE (2.1). It only remains to check that the minimality condition holds. For any $\tau \in \mathbb{T}^{t,T}$, we define

$$\Delta Y := Y - y(\tau, \widetilde{L}_\tau) \text{ and } \Delta Z := Z - z(\tau, \widetilde{L}_\tau).$$

By (4.3), we have

$$\Delta Y_t = \mathbb{E}_t\left[\Gamma^t_\tau (Y_\tau - \widetilde{L}_\tau) + \int_t^\tau \Gamma^t_s\, dK_s\right] \geq \mathbb{E}_t\left[\inf_{t\leq s\leq \tau} \Gamma^t_s \cdot (Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)\right].$$

Denote $\mathcal{K}_t := Y_\tau - \widetilde{L}_\tau + K_\tau - K_t$. We then have

$$\begin{aligned}
\mathbb{E}_t[\mathcal{K}_t] &= \mathbb{E}_t\left[\left(\inf_{t\leq s\leq\tau} \Gamma^t_s\right)^{1/3} \cdot \left(\inf_{t\leq s\leq\tau} \Gamma^t_s\right)^{-1/3} \cdot (\mathcal{K}_t)^{1/3} \cdot (\mathcal{K}_t)^{2/3}\right] \\
&\leq C \cdot \left(\mathbb{E}_t\left[\inf_{t\leq s\leq\tau} \Gamma^t_s \cdot \mathcal{K}_t\right]\right)^{1/3} \cdot \left(\mathbb{E}_t\left[\sup_{t\leq s\leq\tau} (\Gamma^t_s)^{-1/2} \mathcal{K}_t\right]\right)^{2/3} \\
&\leq C \cdot \left(\mathbb{E}_t\left[\inf_{t\leq s\leq\tau} \Gamma^t_s \cdot \mathcal{K}_t\right]\right)^{1/3} \cdot \left(\mathbb{E}_t\left[\sup_{t\leq s\leq\tau} (\Gamma^t_s)^{-1}\right]\right)^{1/3} \cdot \left(\mathbb{E}_t\left[(\mathcal{K}_t)^2\right]\right)^{1/3} \\
&\leq C \cdot (\Delta Y_t)^{1/3} \cdot (C_t)^{1/3}.
\end{aligned}$$

Therefore, we obtain

$$\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t,T}} \mathbb{E}_t[Y_\tau - \widetilde{L}_\tau + K_\tau - K_t] \leq C(C_t)^{1/3} \operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t,T}} (\Delta Y_t)^{1/3} = 0,$$

which is the desired result. □

To study the estimates and stability for RBSDEs, we first recall the classical estimates for the solutions of standard BSDEs (see e.g. Lemma 4.2 of [36] or Section 4 of Bouchard et al. [5]).

Lemma 5.2. *Let us consider the terminal conditions ξ^i and the generator functions g^i satisfying Assumption 2.1 ($i = 1, 2$). We denote by (y^i, z^i) the solution of the BSDE with g^i and ξ^i. Then, for any $p \in (1, 2]$, there exists some constant $C > 0$ such that for all $t \in [0, T]$,*

$$|y^i_t| \leq C\left(\mathbb{E}_t\left[|\xi^i|^p + \int_t^T |g^i(s, 0, 0)|^p\, ds\right]\right)^{1/p},$$

$$\left\|y^i\right\|^2_{\mathbb{S}^2} + \left\|z^i\right\|^2_{\mathbb{H}^2} \leq C\left(\left\|\xi^i\right\|^2_{\mathbb{L}^2} + \|g(\cdot, 0, 0)\|^2_{\mathbb{H}^2_1}\right).$$

[4] We remark that the original paper of Peng [32] actuaully uses only the monotonic limit theorem for BSDE (without the theory of RBSDEs) to obtain the nonlinear decomposition theorem of Doob-Meyer type.

Denoting $\Delta\xi := \xi^1 - \xi^2$*,* $\Delta y := y^1 - y^2$*,* $\Delta z := z^1 - z^2$*,* $\Delta g := (g^1 - g^2)(\cdot, y^1, z^1)$*, we also have*

$$|\Delta y_t| \leq C\left(\mathbb{E}_t\left[|\Delta\xi|^p + \int_t^T |\Delta g_s|^p\,ds\right]\right)^{1/p},$$
$$\|\Delta y\|_{\mathbb{S}^2}^2 + \|\Delta z\|_{\mathbb{H}^2}^2 \leq C\left(\|\Delta\xi\|_{\mathbb{L}^2}^2 + \|\Delta g\|_{\mathbb{H}_1^2}^2\right).$$

Theorem 5.3 (A priori estimate and stability)**.** *Let us consider the terminal conditions* ξ^i *and the generator functions* g^i *satisfying Assumption 2.1 (*$i = 1, 2$*). We denote by* (Y^i, Z^i, K^i) *the solution of the RBSDE with* g^i *and* ξ^i*. Then, for any* $p \in (1, 2]$*, there exists some constant* $C > 0$ *such that for all* $t \in [0, T]$*,*

$$|Y_t^i| \leq C\left(\mathbb{E}_t\left[\sup_{t\leq s\leq T}|L_s^{i,+}|^p + |\xi^i|^p + \int_t^T |g^i(s,0,0)|^p\,ds\right]\right)^{1/p},$$
$$\left\|Y^i\right\|_{\mathbb{S}^2}^2 + \left\|Z^i\right\|_{\mathbb{H}^2}^2 + \left\|K^i\right\|_{\mathbb{I}^2}^2 \leq C\left(\left\|\xi^i\right\|_{\mathbb{L}^2}^2 + \left\|L^{i,+}\right\|_{\mathbb{S}^2}^2 + \left\|g^i(\cdot,0,0)\right\|_{\mathbb{H}_1^2}^2\right),$$

where $L_s^{i,+} := \max\{L_s^i, 0\}$*. Denoting* $\Delta\xi := \xi^1 - \xi^2$*,* $\Delta Y := Y^1 - Y^2$*,* $\Delta Z := Z^1 - Z^2$*,* $\Delta K := K^1 - K^2$ *and assuming that* $g^1 = g^2 = g$*, we also have for any* $p \in (1, 2]$*,*

$$\begin{aligned}
&|\Delta Y_t| \leq C\left(\mathbb{E}_t\left[|\Delta\xi|^p + \sup_{t\leq s\leq T}|\Delta L_s|^p\right]\right)^{1/p},\\
&\|\Delta Y\|_{\mathbb{S}^2}^2 \leq C\left(\|\Delta L\|_{\mathbb{S}^2}^2 + \|\Delta\xi\|_{\mathbb{L}^2}^2\right),\\
&\|\Delta Z\|_{\mathbb{H}^2}^2 + \|\Delta K\|_{\mathbb{S}^2}^2 \leq C\left(\|\Delta\xi\|_{\mathbb{L}^2}^2 + \|\Delta L\|_{\mathbb{S}^2}^2\right)\\
&\qquad\cdot\left(1 + \left\|\xi^1\right\|_{\mathbb{L}^2} + \left\|\xi^2\right\|_{\mathbb{L}^2} + \|g(\cdot,0,0)\|_{\mathbb{H}_1^2} + \left\|L^{1,+}\right\|_{\mathbb{S}^2} + \left\|L^{2,+}\right\|_{\mathbb{S}^2}\right).
\end{aligned}$$

Consequently, the RBSDE (2.1) *has at most one solution in* $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$*.*

Proof. By Theorem 4.1 and the comparison theorem for BSDEs,

$$y_t(T,\xi^i) \leq Y_t^i = \operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}} y_t(\tau, \widetilde{L}_\tau^i) \leq \operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}} y_t(\tau, L_\tau^{i,+}\mathbb{1}_{\tau<T} + \xi^i\mathbb{1}_{\tau=T}).$$

By Lemma 5.2, it follows that

$$\begin{aligned}
&\left|\operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}} y_t(\tau, L_\tau^{i,+}\mathbb{1}_{\tau<T} + \xi^i\mathbb{1}_{\tau=T})\right|\\
&\leq C\left(\mathbb{E}_t\left[\sup_{t\leq s\leq T}|L_s^{i,+}|^p + |\xi^i|^p + \int_t^T |g^i(s,0,0)|^p\,ds\right]\right)^{1/p},
\end{aligned}$$

$$|y_t(T,\xi^i)| \leq C\left(\mathbb{E}_t\left[|\xi^i|^p + \int_t^T |g^i(s,0,0)|^p ds\right]\right)^{1/p}.$$

Therefore, we have

$$|Y_t^i| \leq C\left(\mathbb{E}_t\left[\sup_{t\leq s\leq T}|L_s^{i,+}|^p + |\xi^i|^p + \int_t^T |g^i(s,0,0)|^p\,ds\right]\right)^{1/p}.$$

Using Doob's maximal inequality together with Hölder's inequality, we also have

$$\begin{aligned}\left\|Y^i\right\|_{\mathbb{S}^2}^2 &\le C\mathbb{E}\left[\sup_{0\le t\le T}\left(\mathbb{E}_t\left[\sup_{0\le s\le T}|L_s^{i,+}|^p+|\xi^i|^p+\int_0^T|g^i(s,0,0)|^p\,ds\right]\right)^{2/p}\right]\\&\le C\mathbb{E}\left[\left(\sup_{0\le s\le T}|L_s^{i,+}|^p+|\xi^i|^p+\int_0^T|g^i(s,0,0)|^p\,ds\right)^{2/p}\right]\\&\le C\mathbb{E}\left[\sup_{0\le s\le T}|L_s^{i,+}|^2+|\xi^i|^2+\int_0^T|g^i(s,0,0)|^2\,ds\right]\\&= C\left(\left\|L^{i,+}\right\|_{\mathbb{S}^2}^2+\left\|\xi^i\right\|_{\mathbb{L}^2}^2+\left\|g^i(\cdot,0,0)\right\|_{\mathbb{H}_1^2}^2\right).\end{aligned}$$

Next, assuming that $g^1=g^2=g$, we get from Theorem 4.1 and Lemma 5.2 that

$$|\Delta Y_t|\le \operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t,T}}\left|y_t(\tau,\widetilde{L}_\tau^1)-y_t(\tau,\widetilde{L}_\tau^2)\right|\le C\left(\mathbb{E}_t\left[|\Delta\xi|^p+\sup_{t\le s\le T}|\Delta L_s|^p\right]\right)^{1/p},$$

and then

$$\|\Delta Y\|_{\mathbb{S}^2}^2\le C\mathbb{E}\left[\sup_{0\le t\le T}\left(\mathbb{E}_t\left[\sup_{0\le s\le T}|\Delta L_s|^p+|\Delta\xi|^p\right]\right)^{2/p}\right]\le C\left(\|\Delta L\|_{\mathbb{S}^2}^2+\|\Delta\xi\|_{\mathbb{L}^2}^2\right).$$

It remains to get the estimates for Z,K and $\Delta Z,\Delta K$. But these can be easily obtained via standard approach, to a priori estimates for supersolutions of BSDEs, which does not require the use of minimality condition. Indeed, the philosophy of the estimates of Bouchard et al. [5] is that

"It is sufficient to control the norm of Y to control the norm of (Y,Z,K)."

The estimates for Z,K is immediately proved by using Theorem 2.1 of [5] together with the estimate for Y. Next, applying Itô's formula to $|\Delta Y|^2$, we get as usual

$$\begin{aligned}&\mathbb{E}\left[\int_0^T|\Delta Z_t|^2\,dt\right]\\&\le C\left(\mathbb{E}\left[|\Delta\xi|^2+\int_0^T|\Delta Y_t|d(\Delta K_t)\right]+\int_0^T|\Delta Y_t|(|\Delta Y_t|+|\Delta Z_t|)dt\right)\\&\le C\left(\|\Delta\xi\|_{\mathbb{L}^2}^2+\|\Delta Y\|_{\mathbb{S}^2}^2\right)+\frac{1}{2}\mathbb{E}\left[\int_0^T|\Delta Z_t|^2\,dt\right]+C\,\|\Delta Y\|_{\mathbb{S}^2}\left(\mathbb{E}\left[\sum_{i=1}^2(K_T^i)^2\right]\right)^{1/2}.\end{aligned}$$

This, combined with the estimates for K_T^i and ΔY, gives the estimate for ΔZ.

Finally, we derive the estimate for ΔK. From (2.2), we have

$$\begin{aligned}\Delta K_t=\Delta Y_0-\Delta Y_t&-\int_0^t[g(s,Y_s^1,Z_s^1)-g(s,Y_s^2,Z_s^2)]\,ds\\&+\int_0^t\Delta Z_s\,dB_s,\ 0\le t\le T,\ \mathbb{P}-a.s.\end{aligned}$$

Again, by using the classical linearization argument as in the proof of Theorem 4.1, we derive

$$\begin{aligned}|\Delta K_t|^2\le|\Delta Y_0|^2+|\Delta Y_t|^2+C&\int_0^T[|\Delta Y_s|^2+|\Delta Z_s|^2]\,ds\\&+\left(\int_0^t\Delta Z_s\,dB_s\right)^2.\end{aligned}$$

Taking supremum and expectation and using Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}\left(\sup_{0\le t\le T}|\Delta K_t|^2\right) \le C\left(\|\Delta Y\|_{\mathbb{S}^2}^2 + \|\Delta Z\|_{\mathbb{H}^2}^2\right).$$

This, combined with the estimates for ΔY and ΔZ, gives the estimate for ΔK. □

A Appendix

The aim of this section is to prove the represention formula without using any linearization argument. We shall work under the following assumption.

Assumption A.1. *The generator g satisfies the followings.*

- *g satisfies Assumption 2.1 (i)–(ii).*
- *g is continuous and has a general growth with respect to y.*
- *(p-weak monotonicity condition) There exist a constant $p \in [1,2)$ and a non-decreasing, concave function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(t) > 0$ for $t > 0$ and $\int_{0+} \frac{dt}{\rho(t)} = +\infty$ such that for any $y, y' \in \mathbb{R}$ and $z \in \mathbb{R}^{1\times d}$,*

$$|y-y'|^{p-1}\operatorname{sgn}(y-y')(g(t,y,z)-g(t,y',z)) \le \rho(|y-y'|^p). \tag{A.1}$$

- *g is Lipschitz with respect to z, that is, there exists a constant C such that for all $(t,y,z,z') \in [0,T]\times\mathbb{R}\times(\mathbb{R}^{1\times d})^2$,*

$$|g(t,y,z)-g(t,y,z')| \le C|z-z'|.$$

We note that the usual monotonicity condition corresponds to $\rho(x) = \mu x$. Under Assumption A.1, the standard BSDE with a generator g and a square integrable terminal condition, has a unique solution (see [18]).

An alternative Proof of Theorem 4.1. The proof of the uniqueness is exactly the same as that of Theorem 4.1. In view of the general comparison theorem proved in [28] (see Proposition 3.1 therein), the forward inequality is immediate.

We now show the reverse inequality. Using the growth assumption and the definition of K, we can easily prove the estimate (4.4). Fix $t_1 \in [0,T]$. For every $\tau \in \mathbb{T}^{t_1,T}$, we set $\Delta Y := Y - y(\tau,\widetilde{L}_\tau)$ and $\Delta Z := Z - z(\tau,\widetilde{L}_\tau)$. Then, by assumptions, there exist two processes λ and μ satisfying $|\Delta Y_s|^p \mathbb{1}_{\Delta Y_s\neq 0}\lambda_s \le \rho(|\Delta Y_s|^p)$ and $|\mu| \le C$ such that

$$\Delta Y_t = (Y_\tau - \widetilde{L}_\tau) + \int_t^\tau (\Delta Y_s\lambda_s + \Delta Z_s\mu_s)\,ds - \int_t^\tau \Delta Z_s\,dB_s + K_\tau - K_t,\ t\le\tau. \tag{A.2}$$

Using Itô-Tanaka's formula to $|\Delta Y_t|^p$, we get the following expression.

$$\begin{aligned}
&|\Delta Y_t|^p + \frac{p(p-1)}{2}\int_t^\tau |\Delta Y_s|^{p-2}\mathbb{1}_{\Delta Y_s\neq 0}|\Delta Z_s|^2\,ds \\
&\le |Y_\tau - \widetilde{L}_\tau|^p + p\int_t^\tau |\Delta Y_s|^{p-1}\operatorname{sgn}(\Delta Y_s)(\lambda_s\Delta Y_s + \mu_s\Delta Z_s)\,ds \\
&\quad + p\int_t^\tau |\Delta Y_{s-}|^{p-1}\operatorname{sgn}(\Delta Y_{s-})dK_s - \int_t^\tau |\Delta Y_s|^{p-1}\operatorname{sgn}(\Delta Y_s)\Delta Z_s\,dB_s,\ t\le\tau.
\end{aligned} \tag{A.3}$$

Using an inequality: $p\mu y^{p-1}z \le \frac{p\mu^2}{p-1}|y|^p + \frac{p(p-1)}{2}\mathbb{1}_{y\neq 0}|y|^{p-2}z^2$, we obtain

$$\begin{aligned}
&p|\Delta Y_s|^{p-1}\operatorname{sgn}(\Delta Y_s)(\lambda_s\Delta Y_s + \mu_s\Delta Z_s) \le p\rho(|\Delta Y_s|^p) + p|\mu_s|\cdot|\Delta Y_s|^{p-1}|\Delta Z_s| \\
&\le p\rho(|\Delta Y_s|^p) + \frac{pC^2}{p-1}|\Delta Y_s|^p + \frac{p(p-1)}{2}|\Delta Y_s|^{p-2}\mathbb{1}_{\Delta Y_s\neq 0}|\Delta Z_s|^2.
\end{aligned} \tag{A.4}$$

Define a new function $\bar{\rho}(u) := \rho(u) + C^2/(p-1) \cdot u$. Then it is again a non-decreasing and concave function with $\bar{\rho}(0) = 0$ and $\bar{\rho}(u) > 0$ for $u > 0$. Also, one can easily check that $\int_{0^+} \frac{du}{\bar{\rho}(u)} = +\infty$ using an inequality: $\rho(u) \geq \rho(1)u$, $u \in [0,1]$, which follows from the concavity of $\rho(\cdot)$. From (A.3) and (A.4), we get for $t_1 \leq t \leq \tau$,

$$\begin{aligned} |\Delta Y_t|^p \leq |Y_\tau - \widetilde{L}_\tau|^p + p\int_t^\tau \bar{\rho}(|\Delta Y_s|^p)\, ds + p\int_t^\tau |\Delta Y_{s-}|^{p-1}\operatorname{sgn}(\Delta Y_{s-})\, dK_s \\ - p\int_t^\tau |\Delta Y_s|^{p-1}\operatorname{sgn}(\Delta Y_s)\Delta Z_s\, dB_s. \end{aligned} \tag{A.5}$$

The term $\int_0^\cdot |\Delta Y_s|^{p-1}\operatorname{sgn}(\Delta Y_s)\Delta Z_s\, dB_s$ is a uniformly integrable martingale thanks to the Burkholder-Davis-Gundy's inequality. Therefore, by taking conditional expectations $\mathbb{E}_{t_1}[\cdot]$ on both sides of (A.5), we obtain

$$\begin{aligned} \mathbb{E}_{t_1}[|\Delta Y_t|^p] \leq \mathbb{E}_{t_1}[|Y_\tau - \widetilde{L}_\tau|^p] + p\int_t^\tau \bar{\rho}(\mathbb{E}_{t_1}[|\Delta Y_s|^p])\, ds \\ + p\mathbb{E}_{t_1}\left[\int_t^\tau |\Delta Y_{s-}|^{p-1}\operatorname{sgn}(\Delta Y_{s-})\, dK_s\right], \end{aligned} \tag{A.6}$$

where we used the concavity of $\bar{\rho}(\cdot)$ and Jensen's inequality. On the other hand, we get by Hölder's inequality

$$\begin{aligned} (*) :&= \mathbb{E}_{t_1}[|Y_\tau - \widetilde{L}_\tau|^p] + p\mathbb{E}_{t_1}\left[\int_t^\tau |\Delta Y_{s-}|^{p-1}\operatorname{sgn}(\Delta Y_{s-})\, dK_s\right] \\ &\leq p\mathbb{E}_{t_1}\left[\sup_{t\leq s\leq\tau}|\Delta Y_s|^{p-1}\cdot(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)\right] \\ &\leq p\mathbb{E}_{t_1}\left[\sup_{t\leq s\leq\tau}|\Delta Y_s|^2\right]^{\frac{p-1}{2}}\cdot \mathbb{E}_{t_1}\left[(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)^q\right]^{1/q}, \end{aligned} \tag{A.7}$$

where $q := \frac{2}{3-p} \in (1,2)$. We observe that $1 < \frac{1}{2-q} < +\infty$. Using Hölder's inequality again, we get

$$\begin{aligned} &\mathbb{E}_{t_1}\left[(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)^q\right] \\ &= \mathbb{E}_{t_1}\left[(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)^{2-q}[(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)^2]^{q-1}\right] \\ &\leq (\mathbb{E}_{t_1}[Y_\tau - \widetilde{L}_\tau + K_\tau - K_t])^{2-q}\cdot\left(\mathbb{E}_{t_1}[(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)^2]\right)^{q-1}. \end{aligned} \tag{A.8}$$

Using (4.4), (A.8) and the minimality condition (2.3), we have

$$\begin{aligned} &\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}} \mathbb{E}_{t_1}\left[(Y_\tau - \widetilde{L}_\tau + K_\tau - K_t)^q\right] \\ &\leq (C_{t_1})^{q-1}\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}}(\mathbb{E}_{t_1}[Y_\tau - \widetilde{L}_\tau + K_\tau - K_{t_1}])^{2-q} = 0. \end{aligned} \tag{A.9}$$

In view of Proposition 3.3 of [28] and the fact that $Y \in \mathbb{S}^2$, we deduce that

$$\mathbb{E}_{t_1}\left[\sup_{t\leq s\leq\tau}|\Delta Y_s|^2\right] \leq \operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t_1,T}}\mathbb{E}_{t_1}\left[\sup_{t\leq s\leq\tau}|\Delta Y_s|^2\right] < +\infty,\ \mathbb{P}-a.s.$$

Thus (A.7), combined with (A.9), implies that

$$\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}}(*) = 0. \tag{A.10}$$

Using the classical results for the essential infimum (see e.g. Neveu [26] or Theorem 1.1.4 of [39]), we see that

$$\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}}(*) = \lim_{n\to\infty}\downarrow(*^n), \tag{A.11}$$
$$\text{with } (*^n) := \mathbb{E}_{t_1}[|Y_{\tau_n} - \widetilde{L}_{\tau_n}|^p] + p\mathbb{E}_{t_1}\left[\int_t^{\tau_n} |\Delta Y_{s-}|^{p-1}\operatorname{sgn}(\Delta Y_{s-})\,dK_s\right],$$

for some sequence $\tau_n \in \mathbb{T}^{t_1,T}$. Using Bihari's inequality (see [1]), the expression (A.6) yields

$$\mathbb{E}_{t_1}[|\Delta Y_t|^p] \le \Xi^{-1}\left(\Xi(*) + \tau - t\right), \tag{A.12}$$

where $\Xi(x) := \int_0^x \frac{1}{p\bar{\rho}}(u)\,du,\ x > 0$ is a strictly increasing real-valued function, and Ξ^{-1} is the reverse function of Ξ. Since $\int_{0^+} \frac{du}{p\bar{\rho}(u)}\,du = +\infty$, we see that $\Xi(x) \to +\infty$, $x \to 0$ and $\Xi^{-1}(x) \to 0$, $x \to +\infty$. Using this and expressions (A.10)–(A.12), we obtain

$$\begin{aligned}\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}} \mathbb{E}_{t_1}[|\Delta Y_t|^p] &\le \operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}} \Xi^{-1}\left(\Xi(*) + \tau - t\right)\\ &\le \lim_{n\to\infty}\downarrow \Xi^{-1}\left(\Xi(*^n) + T - t\right)\\ &= \lim_{\varepsilon\downarrow 0}\Xi^{-1}\left(\Xi(\varepsilon) + \tau - t\right) = 0.\end{aligned}$$

Hence, we have proved that

$$\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}} \mathbb{E}_{t_1}[|\Delta Y_t|^p] = \operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}} \mathbb{E}_{t_1}[|\Delta Y_t|] = 0,\ \mathbb{P}-a.s.$$

In particular, we have at $t = t_1$,

$$\operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}} |Y_{t_1} - y_{t_1}(\tau, \widetilde{L}_\tau)| = 0,\ \mathbb{P}-a.s.$$

This, combining with forward inequality, leads to

$$0 = \operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}} |Y_{t_1} - y_{t_1}(\tau,\widetilde{L}_\tau)| = \operatorname*{ess\,inf}_{\tau\in\mathbb{T}^{t_1,T}}\left(Y_{t_1} - y_{t_1}(\tau,\widetilde{L}_\tau)\right) = Y_{t_1} - \operatorname*{ess\,sup}_{\tau\in\mathbb{T}^{t_1,T}} y_{t_1}(\tau,\widetilde{L}_\tau),\ \mathbb{P}-a.s.,$$

which is the desired result. The proof is then complete. □

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