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# Branching Brownian motion in a periodic environment and existence of pulsating traveling waves*

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**Abstract**

In this paper, we first study the limits of the additive and derivative martingales of one-dimensional branching Brownian motion in a periodic environment. Then we prove the existence of pulsating traveling wave solutions of the corresponding F-KPP equation in the supercritical and critical cases by representing the solutions probabilistically in terms of the limits of the additive and derivative martingales. We also prove that there is no pulsating traveling wave solution in the subcritical case. Our main tools are the spine decomposition and martingale change of measures.

**Keywords:** branching Brownian motion; periodic environment; F-KPP equation; pulsating traveling waves; Bessel-3 process; Brownian motion; spine decomposition.
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## 1 Introduction

### 1.1 Background

A classical branching Brownian motion (BBM) in $\mathbb{R}$ can be constructed as follows. Initially there is a single particle at the origin of the real line. This particle moves as a standard Brownian motion $B = \{B(t), t \geq 0\}$ and produces a random number, $1 + L$, of offspring after an exponential time $\eta$. We assume that $L$ has distribution $\{p_k, k \geq 0\}$ with $m := \sum_{k\geq 0} kp_k < \infty$ and $\eta$ is exponentially distributed with parameter $\beta > 0$. Starting

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from their points of creation, each of these children evolves independently and according to the same law as their parent.

McKean [37] established the connection between BBM and the Fisher-Kolmogorov-Petrovskii-Piskounov (F-KPP) reaction-diffusion equation

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \beta(\mathbf{f}(\mathbf{u}) - \mathbf{u}), \tag{1.1}$$

where $\mathbf{f}(s) = \mathbf{E}(s^{L+1})$ and $\mathbf{u} : \mathbb{R}^+ \times \mathbb{R} \to [0,1]$. More precisely, let $N_t$ be the set of particles alive at time $t$ and $X_v(t)$ be the position at time $t$ of a particle $v \in N_t$. It is shown in [37] that, for any $[0,1]$-valued function $g$ on $\mathbb{R}$, $\mathbf{u}(t,x) = \mathbb{E}\left[\prod_{v\in N_t} g(x + X_v(t))\right]$ is a solution of (1.1) with initial condition $g$. The F-KPP equation has been studied intensively by both analytic techniques (see, for example, Kolmogorov et al. [30] and Fisher [17]) and probabilistic methods (see, for instance, McKean [37], Bramson [6, 7], Harris [24] and Kyprianou [31]).

Particular attention has been paid to solutions of the form $\mathbf{u}(t,x) = \Phi_c(x - ct)$. Substituting this into (1.1) shows that $\Phi_c$ satisfies

$$\frac{1}{2}\Phi_c'' + c\Phi_c' + \beta(\mathbf{f}(\Phi_c) - \Phi_c) = 0, \tag{1.2}$$

and such a solution $\Phi_c$ is known as a traveling wave solution of speed $c$. Kyprianou [31], using the additive and derivative martingales of BBM, gave a probabilistic representation of traveling wave solutions and also gave probabilistic proofs, different from Harris [24], for the existence, asymptotics and uniqueness of traveling wave solutions. Inspired by [31], we consider similar problems for BBMs in a periodic environment.

BBM in a periodic environment (BBMPE) is constructed in the same way as BBM, except that the constant branching rate is replaced by a space-dependent rate function $\mathbf{g}$, where we assume $\mathbf{g} \in C^1(\mathbb{R})$ is strictly positive and 1-periodic, that is, $\mathbf{g}(x) = \mathbf{g}(x+1)$ for any $x \in \mathbb{R}$. More precisely, initially there is a single particle $v$ at $x \in \mathbb{R}$, performing standard Brownian motion and producing a random number, $1 + L$, of offspring at its death time. Let $b_v$ and $d_v$ be the birth time and death time of the particle $v$, respectively, and $X_v(s)$ be the location of the particle $v$ at time $s$, then

$$\mathbb{P}_x\left(d_v - b_v > t \mid b_v, \{X_v(s) : s \ge b_v\}\right) = \exp\left\{-\int_{b_v}^{b_v+t} \mathbf{g}(X_v(s))\mathrm{d}s\right\}.$$

Here, we fictitiously extend $X_v(s)$ beyond its death time $d_v$. We still assume $L$ has distribution $\{p_k : k \ge 0\}$ with $m = \sum_{k\ge 0} kp_k < \infty$. Starting from their points of creation, each of these children evolves independently and according to the same rule as their parent. We always assume that $m > 0$.

As defined before, let $N_t$ be the set of particles alive at time $t$ and $X_u(s)$ be the position of the particle $u \in N_t$ or its ancestor at time $s$ for any $u \in N_t$, $s \le t$. Define

$$Z_t = \sum_{u\in N_t} \delta_{X_u(t)},$$

and $\mathcal{F}_t = \sigma(Z_s : s \le t)$. Then $\{Z_t, t \ge 0\}$ is a point process describing the number and positions of individuals alive at time $t$, and is called a branching Brownian motion in a periodic environment (BBMPE). Let $\mathbb{P}_x$ be the law of BBMPE with one initial particle at $x \in \mathbb{R}$, that is $\mathbb{P}_x(Z_0 = \delta_x) = 1$. We use $\mathbb{E}_x$ to denote the expectation with respect to $\mathbb{P}_x$. For simplicity, $\mathbb{P}_0$ and $\mathbb{E}_0$ will be written as $\mathbb{P}$ and $\mathbb{E}$, respectively. Notice that the distribution of $L$ does not depend on the spatial location. In the remainder of this paper, expectations with respect to $L$ will be written as $\mathbf{E}$.

The F-KPP equation related to BBMPE has the following form:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{g}\cdot(\mathbf{f}(\mathbf{u}) - \mathbf{u}), \tag{1.3}$$

or equivalently, $\mathbf{v} = 1 - \mathbf{u}$ satisfies

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{v}}{\partial x^2} + \mathbf{g}\cdot(1 - \mathbf{v} - \mathbf{f}(1 - \mathbf{v})).$$

In this case, traveling wave solutions, that is solutions satisfying (1.2), do not exist. However, we can consider pulsating traveling waves, that is, solutions $\mathbf{u} : \mathbb{R}^+ \times \mathbb{R} \to [0,1]$ to (1.3) satisfying

$$\mathbf{u}\left(t + \frac{1}{\nu}, x\right) = \mathbf{u}(t, x-1), \tag{1.4}$$

as well as the boundary condition

$$\lim_{x\to-\infty} \mathbf{u}(t,x) = 0, \quad \lim_{x\to+\infty} \mathbf{u}(t,x) = 1,$$

when $\nu > 0$, and

$$\lim_{x\to-\infty} \mathbf{u}(t,x) = 1, \quad \lim_{x\to+\infty} \mathbf{u}(t,x) = 0,$$

when $\nu < 0$. $\nu$ is called the wave speed. It is known that, under the assumption $\mathbf{f} \in C^{1,\alpha}$ for some $\alpha > 0$, there is a constant $\nu^* > 0$ such that when $|\nu| < \nu^*$ no such solution exists, whereas for each $|\nu| \geq \nu^*$ there exists a unique, up to time-shift, pulsating traveling wave. See [1, Theorem 1.14] for existence/non-existence and [21, Theorem 1.1] for uniqueness.

Recently, Lubetzky, Thornett and Zeitouni [34] established the connection between F-KPP equation (1.3) and BBMPE with $1 + L = 2$ and studied the the maximum of BBMPE. In this paper we first study limits of additive and derivative martingales of BBMPE. Then we use these limits to give probabilistic representations of pulsating traveling wave solutions of (1.3) with speed $\nu$ satisfying $|\nu| \geq \nu^*$, where $\nu^*$ is a constant defined below. In the rest of this paper, $|\nu| > \nu^*$ is called the supercritical case, and $|\nu| = \nu^*$ the critical case. We also prove that there is no pulsating traveling wave solution of (1.3) with speed $\nu$ satisfying $|\nu| < \nu^*$ (called the subcritical case). The asymptotic behavior and uniqueness of the pulsating traveling wave solution of (1.3) are studied in the companion paper [40]. Therefore, we extend the results of Kyprianou [31] for classical BBM to BBMPE. It turns out that most of the general ideas in Kyprianou [31] still work for BBMPE. However, as we will see, carrying out the actual argument is much more difficult.

Before we state our main results, we first introduce the minimal speed $\nu^*$. As in [22], for any $\lambda \in \mathbb{R}$, let $\gamma(\lambda)$ and $\psi(\cdot,\lambda)$ be the principal eigenvalue and the corresponding positive eigenfunction of the periodic problem: for all $x \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{2}\psi_{xx}(x,\lambda) - \lambda\psi_x(x,\lambda) + \left(\frac{1}{2}\lambda^2 + m\mathbf{g}(x)\right)\psi(x,\lambda) &= \gamma(\lambda)\psi(x,\lambda), \\ \psi(x+1,\lambda) &= \psi(x,\lambda). \end{aligned} \tag{1.5}$$

Then $\gamma(\lambda)$ is simple, that is, the corresponding eigenspace is 1 dimensional. We normalize $\psi(\cdot,\lambda)$ such that $\int_0^1 \psi(x,\lambda)dx = 1$. Define

$$\nu^* := \min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda}, \quad \lambda^* := \arg\min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda}. \tag{1.6}$$

Then $\nu^*$ defined by (1.6) is the minimal wave speed (see [2, 22]) and the existence of $\lambda^*$ is proved in [34], also see Lemma 2.3 below.

### 1.2 Main results

For any $\lambda \in \mathbb{R}$, define

$$W_t(\lambda) = e^{-\gamma(\lambda)t} \sum_{u \in N_t} e^{-\lambda X_u(t)} \psi(X_u(t), \lambda). \tag{1.7}$$

**Theorem 1.1.** *For any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{(W_t(\lambda))_{t\geq 0}, \mathbb{P}_x\}$ is a martingale. The limit $W(\lambda, x) := \lim_{t\uparrow\infty} W_t(\lambda)$ exists $\mathbb{P}_x$-almost surely.*

(i) *If $|\lambda| > \lambda^*$ then $W(\lambda, x) = 0$ $\mathbb{P}_x$-almost surely.*

(ii) *If $|\lambda| = \lambda^*$ then $W(\lambda, x) = 0$ $\mathbb{P}_x$-almost surely.*

(iii) *If $|\lambda| < \lambda^*$ then $W(\lambda, x) = 0$ $\mathbb{P}_x$-almost surely when $\mathbf{E}(L \log^+ L) = \infty$, and $W(\lambda, x)$ is an $L^1(\mathbb{P}_x)$-limit when $\mathbf{E}(L \log^+ L) < \infty$. Moreover, if $|\lambda| < \lambda^*$ and $\mathbf{E}(L \log^+ L) < \infty$, we have*

$$\mathbb{P}_x(W(\lambda, x) = 0) = 0. \tag{1.8}$$

$\{(W_t(\lambda))_{t\geq 0}, \mathbb{P}_x\}$ is called the additive martingale of the BBMPE starting from $x$.

**Remark 1.2.** For BBM, there is no essential difference between the case when the initial ancestor starting from $x$ and the case starting from the origin, whereas things are different for BBMPE because the branching rate depends on the position. This is why we write the almost sure limit of $W_t(\lambda)$ as $W(\lambda, x)$, instead of $W(\lambda)$. Nevertheless, it can be proved that

$$(W(\lambda, y), \mathbb{P}_y) \overset{d}{=} (e^{-\lambda(y-x)} W(\lambda, x), \mathbb{P}_x), \quad \text{if } y - x \in \mathbb{Z}, \tag{1.9}$$

due to the 1-periodicity of $\mathbf{g}(\cdot)$ and $\psi(\cdot, \lambda)$.

We will show that both $\gamma(\lambda)$ and $\psi(x, \lambda)$ are differentiable with respect to $\lambda$, so we can define

$$\partial W_t(\lambda) := -\frac{\partial}{\partial \lambda} W_t(\lambda) = e^{-\gamma(\lambda)t} \sum_{u \in N_t} e^{-\lambda X_u(t)} \Big( \psi(X_u(t), \lambda)(\gamma'(\lambda)t + X_u(t)) - \psi_\lambda(X_u(t), \lambda) \Big). \tag{1.10}$$

**Theorem 1.3.** *For any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{(\partial W_t(\lambda))_{t\geq 0}, \mathbb{P}_x\}$ is a martingale. For all $|\lambda| \geq \lambda^*$, the limit $\partial W(\lambda, x) := \lim_{t\uparrow\infty} \partial W_t(\lambda)$ exists $\mathbb{P}_x$-almost surely.*

(i) *If $|\lambda| > \lambda^*$ then $\partial W(\lambda, x) = 0$ $\mathbb{P}_x$-almost surely.*

(ii) *If $|\lambda| = \lambda^*$ then $\partial W(\lambda, x) = 0$ $\mathbb{P}_x$-almost surely when $\mathbf{E}(L(\log^+ L)^2) = \infty$, and $\partial W(\lambda, x) \in (0, \infty)$ (respectively $\partial W(\lambda, x) \in (-\infty, 0)$) $\mathbb{P}_x$-almost surely when $\lambda > 0$ (respectively $\lambda < 0$) and $\mathbf{E}(L(\log^+ L)^2) < \infty$.*

$\{(\partial W_t(\lambda))_{t\geq 0}, \mathbb{P}_x\}$ is called the derivative martingale of the BBMPE starting from $x$.

In [31, Theorem 1.3], there is a small gap between the necessary condition and the sufficient condition for $\partial W(\lambda^*) = 0$ $\mathbb{P}_x$-almost surely. Later Yang and Ren [42] fill this gap and give the sharp $\mathbf{E}(L(\log^+ L)^2)$ condition. We adapt some ideas from [42] and get the necessary and sufficient $\mathbf{E}(L(\log^+ L)^2)$ condition for BBMPE.

Pulsating traveling waves have been studied analytically in many papers, see, for example, [1, 2, 20, 21, 22]. The nonlinear reaction-diffusion equations in the analysis literature are more general. The linear parts have general periodic diffusion and drift coefficients, and general periodic domains are considered. The papers [1, 2, 20, 21] all assumed a regularity condition on the non-linear term which, in the setting of this paper, can be stated as $\mathbf{f} \in C^{1,\alpha}$ for some $\alpha > 0$. [2, Theorem 1] proved that the minimal wave speed is given by $\nu^*$ defined (1.6); [20, Theorem 1.3] gave the asymptotic behaviors of pulsating traveling waves; and [21, Theorems 1.1 and 1.3] proved the uniqueness and global stability of pulsating traveling waves. In contrast, in this paper, we only consider nonlinear terms of the form $\mathbf{g}(x)(1 - s - \mathbf{f}(1 - s))$ with $\mathbf{g} \in C^1(\mathbb{R})$ being 1-periodic and with $\mathbf{f}$ being the probability generating function of a non-negative integer-valued random variable. Nonlinear reaction-diffusion equations with nonlinear terms of this form are related to branching Brownian motions. Other more general nonlinear terms covered by analytic method are not considered in this paper since they are not related to branching Brownian motions. On the other hand, we do not assume the condition $\mathbf{f} \in C^{1,\alpha}$ for some $\alpha > 0$, which is stronger than our condition $\mathbf{E}(L(\log^+ L)^2) < \infty$ (see [40, Remark 4.1]).

In this paper, we use the nontrivial limits of the additive and derivative martingales to give probabilistic representations of pulsating traveling wave in the supercritical case $|\nu| > \nu^*$ when $\mathbf{E}(L\log^+ L) < \infty$, and the critical case $|\nu| = \nu^*$ when $\mathbf{E}(L(\log^+ L)^2) < \infty$. Thus we give probabilistic proofs of the existence of pulsating traveling waves in these two cases. We also give a probabilistic proof of the non-existence of pulsating traveling waves in the subcritical case $|\nu| < \nu^*$. In [40], we will give a probabilistic proof of the asymptotic behavior and uniqueness of the pulsating traveling waves.

**Theorem 1.4.** (i) ***Supercriticality case.*** *If $|\nu| > \nu^*$ and $\mathbf{E}(L\log^+ L) < \infty$,*

$$\mathbf{u}(t,x) = \mathbb{E}_x\left(\exp\left\{-e^{\gamma(\lambda)t}W(\lambda,x)\right\}\right) \tag{1.11}$$

*is a pulsating traveling wave with speed $\nu$, where $|\lambda| \in (0,\lambda^*)$ is such that $\nu = \frac{\gamma(\lambda)}{\lambda}$.*

(ii) ***Criticality case.*** *If $|\nu| = \nu^*$ and $\mathbf{E}(L(\log^+ L)^2) < \infty$,*

$$\mathbf{u}(t,x) = \mathbb{E}_x\left(\exp\left\{-e^{\gamma(\lambda^*)t}\partial W(\lambda^*,x)\right\}\right) \tag{1.12}$$

*is a pulsating traveling wave with speed $\nu^*$, and*

$$\mathbf{u}(t,x) = \mathbb{E}_x\left(\exp\left\{-e^{\gamma(\lambda^*)t}\partial W(-\lambda^*,x)\right\}\right)$$

*is a pulsating traveling wave with speed $-\nu^*$.*

(iii) ***Subcriticality case.*** *There is no pulsating traveling wave when $|\nu| < \nu^*$.*

**Remark 1.5.** The theorem above gives a probabilistic representation of the pulsating traveling wave in the supercritical case when $\mathbf{E}(L\log^+ L) < \infty$, and in the critical case when $\mathbf{E}(L(\log^+ L)^2) < \infty$. It is natural to ask whether there are probabilistic representations of pulsating traveling waves when these $L\log L$-type conditions are not satisfied. In the case of BBM, Kyprianou [31, Remark 6] mentioned that Biggins and Kyprianou [4, Theorem 1.2 and the last paragraph of Section 1] (see also [3]) proved that when $\mathbf{E}L < \infty$, there exist Seneta-Heyde norming constants $\{c_t, t \geq 0\}$ such that $c_t W_t(\lambda)$ converges in probability to a limit, whose Laplace transform gives a non-trivial traveling wave. In [31, Remark 6], it was also mentioned that, even for BBM, things are somewhat less clear in the critical case when $\mathbf{E}(L(\log^+ L)^2) = \infty$. These are important questions for BBMPEs also. Tackling them will require techniques very different from those used in this paper. We plan to explore these in a future project.

Now we give a description of the strategy of the proofs of the main results. Our main tools are the spine decomposition and martingale change of measures. We follow the general ideas of [31], but we have to work much harder. A key to the proof of Theorem 1.1 is a strong law of large numbers for the motion of the spine.

The proof of Theorem 1.3 is much more delicate. First, we use the differentiability of $\psi(x,\lambda)$ with respect to $\lambda$ to define the derivative martingale. Next, similar to [31], we define a a non-negative martingale $\{V_t^x(\lambda), \mathbb{P}_x\}$, which is a variant of the derivative martingale. When $\{V_t^x(\lambda), \mathbb{P}_x\}$ is used to define a change of measures, the motion of the spine is no longer a Bessel-3 process. Lemmas 2.12 and 2.14 are crucial for dealing with the motion of the spine. Finally, to get the necessary and sufficient condition $\mathbf{E}(L(\log^+ L)^2) < \infty$, we adapt some ideas from [42].

Using the Markov property and branching property of branching Brownian motion, it is not hard to check that that $\mathbf{u}$ given by (1.11) and (1.12) are pulsating traveling waves. The non-existence follows from a contradiction argument. As a byproduct, we get the linear speed of the maximum of BBMPE.

Since $\gamma(\lambda)$ is an even function (see Lemma 2.3 below), in the remainder of the paper we will deal only with the case that $\lambda \geq 0$ (i.e., $\nu \geq 0$) unless otherwise stated. The case $\lambda < 0$ (i.e., $\nu < 0$) follows from symmetry and the results are the same.

## 2 Preliminaries

Throughout this paper we use $\{B_t, t \geq 0; \Pi_x\}$ to denote a standard Brownian motion starting from $x$. Expectation with respect to $\Pi_x$ will also be denoted by $\Pi_x$. The following many-to-one lemma (see [26], [34] and [36, §2.3]) is fundamental.

**Lemma 2.1** (Many-to-one Lemma)**.** *Let $t > 0$ and $F : C[0,t] \to \mathbb{R}$ be a non-negative measurable function. Then*

$$\mathbb{E}_x\left[\sum_{u\in N_t} F\left(X_u(s), s\in[0,t]\right)\right] = \Pi_x\left[e^{m\int_0^t \mathbf{g}(B_s)\mathrm{d}s} F\left(B_s, s\in[0,t]\right)\right],$$

*where $C[0,t]$ denotes the space of continuous functions from $[0,t]$ to $\mathbb{R}$.*

In particular, for any non-negative function $f$ on $\mathbb{R}$, we have

$$\mathbb{E}_x\left[\sum_{u\in N_t} f\left(X_u(t)\right)\right] = \Pi_x\left[e^{m\int_0^t \mathbf{g}(B_s)\mathrm{d}s} f\left(B_t\right)\right].$$

This says that the mean semigroup of the BBMPE is a Feynman-Kac semigroup of the Brownian motion $B$, and is closely related to the behavior of the BBMPE $\{Z_t, t \geq 0\}$. Now, we define

$$\phi(x,\lambda) := e^{-\lambda x}\psi(x,\lambda), \quad \lambda\in\mathbb{R},\, x\in\mathbb{R}. \tag{2.1}$$

A direct calculation shows that $\phi$ satisfies

$$\frac{1}{2}\phi_{xx}(x,\lambda) + m\mathbf{g}(x)\phi(x,\lambda) = \gamma(\lambda)\phi(x,\lambda). \tag{2.2}$$

The function $\phi$ will play an important role in our proofs.

The following result will be used several times in this paper.

**Lemma 2.2.** *Let $b_1 < b_2$ be two real numbers. It holds that for any $\lambda$,*

$$\phi(x,\lambda) = \Pi_x\left[\phi(B_\tau,\lambda)e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(\lambda))dt}\right], \quad x\in[b_1,b_2],$$

*where $\tau := \inf\{t > 0 : B_t \notin (b_1, b_2)\}$.*

*Proof.* Combining the positivity and continuity of $\psi(\cdot,\lambda)$ with (2.1), we see that, for any $\lambda$, $[b_1,b_2] \ni x \mapsto \psi(x,\lambda)$ is bounded between two positive constants. It follows from [9, Corollary 2] that the gauge function

$$g(x,\lambda) = \Pi_x\left[e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right], \quad x\in[b_1,b_2],$$

is bounded. Now the desired conclusion follows immediately from [9, Theorem 2.3]. □

In the remainder of this section, we first introduce some properties of the principal eigenvalue $\gamma(\lambda)$ and show the eigenfunction $\psi(x,\lambda)$ is differentiable with respect to $\lambda$. Next, we introduce three martingales, $\{\Xi_t(\lambda) : t \geq 0\}$, $\{\Upsilon_t(\lambda) : t \geq 0\}$ and $\{\Lambda_t^{(x,\lambda)} : t \geq 0\}$

with respect to the Brownian filtration. These martingales are related to the additive martingale, the derivative martingale and the non-negative martingale $\{V_t^x(\lambda), \mathbb{P}_x\}$ (defined by (2.34) below) of the BBMPE, respectively. Using the two non-negative martingales $\{\Xi_t(\lambda) : t \geq 0\}$ and $\{\Lambda_t^{(x,\lambda)} : t \geq 0\}$, we get two kinds of measure changes, that are related to the motion of the spine in Sections 3 and 4.

## 2.1 Properties of principal eigenvalue and eigenfunction

We first discuss some properties of $\gamma(\lambda)$,

**Lemma 2.3.** *(1) The function $\gamma$ is analytic and strictly convex on $\mathbb{R}$. There exists a unique $\lambda^* > 0$ such that*

$$\nu^* = \frac{\gamma(\lambda^*)}{\lambda^*} = \min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda} > 0. \tag{2.3}$$

*Furthermore*

$$\lim_{\lambda\to-\infty} \gamma'(\lambda) = -\infty, \quad \lim_{\lambda\to+\infty} \gamma'(\lambda) = +\infty. \tag{2.4}$$

*(2) The function $\gamma$ is even on $\mathbb{R}$.*

*Proof.* The analyticity, convexity of $\gamma$, and the existence and uniqueness of $\lambda^*$ are contained in [34, Lemma 2.5]. For the analyticity and convexity of $\gamma$, one can also see [20, Lemma 2.1]. Now we show (2.4). Recall that $\phi(x, \lambda) = e^{-\lambda x}\psi(x, \lambda)$ satisfies (2.2). If one regards $\phi(\cdot, \lambda)$ as a function of $(t, x)$, then $\frac{\partial \phi}{\partial t} = 0$ and $\phi$ satisfies $\frac{\partial \phi}{\partial t} = \frac{1}{2}\phi_{xx}(x, \lambda) + (m\mathbf{g}(x) - \gamma(\lambda))\phi(x, \lambda)$.

By the Feynman-Kac formula,

$$\phi(x, \lambda) = \Pi_x \left[\phi(B_t, \lambda)e^{-\gamma(\lambda)t+m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\right], \quad x \in \mathbb{R}.$$

Hence,

$$\psi(x, \lambda) = \Pi_x \left[\psi(B_t, \lambda)e^{-\gamma(\lambda)t-\lambda(B_t-x)+m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\right], \quad x \in \mathbb{R}. \tag{2.5}$$

(One can also use Itô's formula to easily get the display above.) Since g is 1-periodic and continuous, we can assume that $0 < \alpha \leq \mathbf{g}(x) \leq \beta < \infty$ for all $x \in \mathbb{R}$. Notice that $\Pi_x[e^{-\lambda B_t}] = e^{-\lambda x+\frac{\lambda^2 t}{2}}$, and by (2.5), we have

$$e^{\left(m\alpha+\frac{\lambda^2}{2}-\gamma(\lambda)\right)t}\Pi_x [\psi(B_t, \lambda)] \leq \psi(x, \lambda) \leq e^{\left(m\beta+\frac{\lambda^2}{2}-\gamma(\lambda)\right)t}\Pi_x [\psi(B_t, \lambda)].$$

Letting $t \to \infty$, the boundedness of $\psi$ implies that

$$\gamma(\lambda) \in \left[\frac{\lambda^2}{2} + m\alpha, \frac{\lambda^2}{2} + m\beta\right]. \tag{2.6}$$

Combining this with the analyticity and convexity of $\gamma$, we get (2.4).

If there exist $\lambda_1 < \lambda_2$ such that $\gamma'(\lambda_1) = \gamma'(\lambda_2)$. The convexity of $\gamma$ would imply that $\gamma'$ is constant on $[\lambda_1, \lambda_2]$, and then analyticity would imply that $\gamma'$ is a constant, which contradicts (2.4). Therefore, $\gamma'(\cdot)$ is strictly increasing on $\mathbb{R}$, that is, $\gamma(\cdot)$ is strictly convex.

(2) Let $\psi(x) := \psi(x, \lambda)$ satisfy (1.5), i.e., $\psi$ is the positive eigenfunction corresponding to the eigenvalue $\gamma(\lambda)$. Let $\bar{\psi} = \psi(x, -\lambda)$ be the positive eigenfunction corresponding to the eigenvalue $\gamma(-\lambda)$, then $\bar{\psi}(x)$ satisfies, for all $x \in \mathbb{R}$,

$$\frac{1}{2}\bar{\psi}_{xx}(x) + \lambda\bar{\psi}_x(x) + \left(\frac{1}{2}\lambda^2 + m\mathbf{g}(x)\right)\bar{\psi}(x) = \gamma(-\lambda)\bar{\psi}(x), \tag{2.7}$$

$$\bar{\psi}(x + 1) = \bar{\psi}(x).$$

Multiplying (2.7) by $\psi$ and integrating over $(0,1)$, we get that

$$
\begin{aligned}
\gamma(-\lambda)\int_0^1 \bar{\psi}\psi \mathrm{d}x =& \int_0^1 \left(\frac{1}{2}\bar{\psi}_{xx}\psi + \lambda\bar{\psi}_x\psi + (\frac{1}{2}\lambda^2 + m\mathbf{g}(x))\bar{\psi}\psi\right)\mathrm{d}x \\
=& \frac{1}{2}\bar{\psi}_x(1)\psi(1) - \frac{1}{2}\bar{\psi}_x(0)\psi(0) + \lambda\bar{\psi}(1)\psi(1) - \lambda\bar{\psi}(0)\psi(0) \\
&+ \int_0^1 \left(-\frac{1}{2}\bar{\psi}_x\psi_x - \lambda\bar{\psi}\psi_x + (\frac{1}{2}\lambda^2 + m\mathbf{g}(x))\bar{\psi}\psi\right)\mathrm{d}x \\
=& \int_0^1 \left(-\frac{1}{2}\bar{\psi}_x\psi_x - \lambda\bar{\psi}\psi_x + (\frac{1}{2}\lambda^2 + m\mathbf{g}(x))\bar{\psi}\psi\right)\mathrm{d}x \\
=& \int_0^1 \left(\frac{1}{2}\bar{\psi}\psi_{xx} - \lambda\bar{\psi}\psi_x + (\frac{1}{2}\lambda^2 + m\mathbf{g}(x))\bar{\psi}\psi\right)\mathrm{d}x \\
=& \gamma(\lambda)\int_0^1 \bar{\psi}\psi \mathrm{d}x.
\end{aligned}
$$

Since $\psi, \bar{\psi} > 0$, we obtain that $\gamma(\lambda) = \gamma(-\lambda)$. □

We compare the values of $\gamma'(\lambda)$ and $\frac{\gamma(\lambda)}{\lambda}$ in the following lemma.

**Lemma 2.4.** *(1)* $\gamma'(\lambda^*) = \dfrac{\gamma(\lambda^*)}{\lambda^*}$. *(2) If* $0 < \lambda < \lambda^*$, $\gamma'(\lambda) < \dfrac{\gamma(\lambda)}{\lambda}$. *(3) If* $\lambda > \lambda^*$, $\gamma'(\lambda) > \dfrac{\gamma(\lambda)}{\lambda}$.

*Proof.* Put $f(\lambda) = \frac{\gamma(\lambda)}{\lambda}$. (1) Note that

$$f'(\lambda) = \frac{\gamma'(\lambda) - \frac{\gamma(\lambda)}{\lambda}}{\lambda}.$$

Since $f(\lambda^*) = \min_{\lambda>0} f(\lambda)$, we have $f'(\lambda^*) = 0$, that is, $\gamma'(\lambda^*) = \frac{\gamma(\lambda^*)}{\lambda^*}$.

(2) If there were $0 < \lambda_1 < \lambda^*$ satisfying $f'(\lambda_1) \geq 0$, then, by the uniqueness of $\lambda^*$,

$$\gamma'(\lambda_1) \geq \frac{\gamma(\lambda_1)}{\lambda_1} > \frac{\gamma(\lambda^*)}{\lambda^*} = \gamma'(\lambda^*).$$

This contradicts the convexity of $\gamma$.

(3) If there were $\lambda_2 > \lambda^*$ satisfying $f'(\lambda_2) < 0$, let $\lambda_3 = \sup\{\lambda : \lambda < \lambda_2 \text{ and } f'(\lambda) \geq 0\}$. Then by the continuity of $f'$, $f'(\lambda_3) = 0$ and $\lambda_3 < \lambda_2$. By the definition of $\lambda_3$,

$$\gamma'(\lambda_3) = \frac{\gamma(\lambda_3)}{\lambda_3} = f(\lambda_3) > \frac{\gamma(\lambda_2)}{\lambda_2} > \gamma'(\lambda_2),$$

which contradicts the convexity of $\gamma$ again.

Suppose there were $\lambda_4 > \lambda^*$ satisfying $f'(\lambda_4) = 0$. For any $\delta \in (0, \lambda_4 - \lambda^*)$, we have

$$\int_{\lambda_4-\delta}^{\lambda_4} \frac{\gamma'(\lambda) - \frac{\gamma(\lambda)}{\lambda}}{\lambda}\mathrm{d}\lambda = \int_{\lambda_4-\delta}^{\lambda_4} f'(\lambda)\mathrm{d}\lambda = f(\lambda_4) - f(\lambda_4 - \delta).$$

We claim that $f(\lambda_4) - f(\lambda_4 - \delta) > 0$. In fact, since $f'(\lambda) \geq 0$ for $\lambda \in [\lambda_4 - \delta, \lambda_4]$, we have $f(\lambda_4) - f(\lambda_4 - \delta) \geq 0$. If $f(\lambda_4) - f(\lambda_4 - \delta) = 0$, then $f'(\lambda) = 0$ for $\lambda \in [\lambda_4 - \delta, \lambda_4]$. Thus we have

$$\gamma'(\lambda_4) = \frac{\gamma(\lambda_4)}{\lambda_4} = f(\lambda_4) = f(\lambda_4 - \delta) = \frac{\gamma(\lambda_4 - \delta)}{\lambda_4 - \delta} = \gamma'(\lambda_4 - \delta),$$

which contradicts the strict convexity of $\gamma$. Thus the claim is true.

For $\lambda \in [\lambda_4-\delta,\lambda_4]$, it holds that $\gamma'(\lambda)\le\gamma'(\lambda_4)=f(\lambda_4)$, and that $f(\lambda)\ge f(\lambda_4-\delta)$ because we have proved $f'(\lambda)\ge 0$ when $\lambda>\lambda^*$. Thus, we obtain

$$f(\lambda_4)-f(\lambda_4-\delta)=\int_{\lambda_4-\delta}^{\lambda_4}\frac{\gamma'(\lambda)-\frac{\gamma(\lambda)}{\lambda}}{\lambda}\mathrm{d}\lambda\le\delta\frac{f(\lambda_4)-f(\lambda_4-\delta)}{\lambda^*}>0,$$

which implies that $\lambda^*\le\delta$ for any $\delta>0$. This contradicts the fact that $\lambda^*>0$. The proof of (3) is complete. □

In the following three lemmas, we will show that $\psi(x,\cdot)\in C(\mathbb{R})\cap C^1(\mathbb{R}\setminus\{0\})$ and that $\psi_\lambda(x,\lambda)$ satisfies

$$\begin{aligned}&\frac{1}{2}\psi_{\lambda xx}(x,\lambda)-\psi_x(x,\lambda)-\lambda\psi_{\lambda x}(x,\lambda)+\left(\frac{1}{2}\lambda^2+m\mathbf{g}(x)\right)\psi_\lambda(x,\lambda)+\lambda\psi(x,\lambda)\\&=\gamma(\lambda)\psi_\lambda(x,\lambda)+\gamma'(\lambda)\psi(x,\lambda).\end{aligned}\tag{2.8}$$

**Lemma 2.5.** *Suppose $\widetilde{\psi}(\cdot,\lambda)$ is a positive eigenfunction of* (1.5) *with $\widetilde{\psi}(0,\lambda)=1$. Then for any $x\in\mathbb{R}$, $\widetilde{\psi}(x,\cdot)\in C(\mathbb{R})\cap C^1(\mathbb{R}\setminus\{0\})$. Moreover, $\widetilde{\psi}_\lambda(x,\lambda)$, the derivative of $\widetilde{\psi}$ with respective to $\lambda$, satisfies, for all $x\in\mathbb{R}$,*

$$\begin{aligned}&\frac{1}{2}\widetilde{\psi}_{\lambda xx}(x,\lambda)-\widetilde{\psi}_x(x,\lambda)-\lambda\widetilde{\psi}_{\lambda x}(x,\lambda)+\left(\frac{1}{2}\lambda^2+m\mathbf{g}(x)\right)\widetilde{\psi}_\lambda(x,\lambda)+\lambda\widetilde{\psi}(x,\lambda)\\&=\gamma(\lambda)\widetilde{\psi}_\lambda(x,\lambda)+\gamma'(\lambda)\widetilde{\psi}(x,\lambda).\end{aligned}\tag{2.9}$$

*Proof.* Since $\widetilde{\psi}(\cdot,\lambda)$ is 1-periodic, to prove $\widetilde{\psi}(x,\cdot)\in C(\mathbb{R})\cap C^1(\mathbb{R}\setminus\{0\})$ for all $x\in\mathbb{R}$, it suffices to verify that, for any $x\in(0,1]$, $\widetilde{\psi}(x,\cdot)$ is continuous and continuously differentiable on $\mathbb{R}\setminus\{0\}$. Define

$$\widetilde{\phi}(x,\lambda):=e^{-\lambda x}\widetilde{\psi}(x,\lambda).\tag{2.10}$$

Since $\widetilde{\psi}(\cdot,\lambda)$ is a positive multiple of $\psi(\cdot,\lambda)$, we have that $\widetilde{\phi}(\cdot,\lambda)$ satisfies (2.2). Since $\widetilde{\phi}(2,\lambda)=e^{-2\lambda}$ and $\widetilde{\phi}(0,\lambda)=1$, we have by Lemma 2.2 that

$$\widetilde{\phi}(x,\lambda)=\Pi_x\left[e^{-\lambda B_\tau}e^{\int_0^\tau(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right],\quad x\in[0,2],\tag{2.11}$$

where $\tau=\inf\{t>0:B_t\notin(0,2)\}$. By (2.10), we only need to prove that for any $x\in\mathbb{R}$, $\widetilde{\phi}(x,\cdot)\in C(\mathbb{R})\cap C^1(\mathbb{R}\setminus\{0\})$.

Note that for any $x\in[0,2]$,

$$\widetilde{\phi}(x,\lambda)=\Pi_x\left[e^{\int_0^\tau(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t},B_\tau=0\right]+e^{-2\lambda}\Pi_x\left[e^{\int_0^\tau(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t},B_\tau=2\right]<\infty.$$

Thus for any $\lambda\in\mathbb{R}$, the gauge function $g(\cdot,\lambda)$ defined by

$$g(x,\lambda):=\Pi_x\left[e^{\int_0^\tau(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]$$

is bounded on $[0,2]$. The function $\gamma(\lambda)$ is analytic, convex and even, so $\gamma(0)$ is the minimum of value of $\gamma(\lambda)$ and for any $\lambda\neq 0$, $\gamma(\lambda)>\gamma(0)$. Hence,

$$\widetilde{\phi}(x,\lambda)\le g(x,\lambda)\le g(x,0)=\Pi_x\left[e^{\int_0^\tau(m\mathbf{g}(B_t)-\gamma(0))\mathrm{d}t}\right]\le\sup_{x\in[0,2]}g(x,0)<\infty,\quad x\in[0,2].$$

Using the dominated convergence theorem with the dominant function $e^{\int_0^\tau(m\mathbf{g}(B_t)-\gamma(0))\mathrm{d}t}$ and the continuity of $\gamma$, we obtain that, for each fixed $x$, $\widetilde{\phi}(x,\lambda)$ is continuous as a function of $\lambda$.

Now we prove $\widetilde{\phi}(x,\cdot) \in C^1(\mathbb{R}\setminus\{0\})$. We only need to deal with the case $\lambda > 0$. Fix a $\lambda_0 > 0$. Note that

$$B_\tau e^{-\lambda B_\tau} e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t} \le 2\, e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(0))\mathrm{d}t}, \tag{2.12}$$

and that for $\lambda \in [\lambda_0/2, 3\lambda_0/2]$,

$$\tau e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t} \le M e^{(\gamma(\lambda)-\gamma(0))\tau} e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t} = M e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(0))\mathrm{d}t}, \tag{2.13}$$

for some large constant $M$. Since $\Pi_x\left[e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(0))\mathrm{d}t}\right] = \widetilde{\phi}(x,0)$, we can differentiate (2.11) with respect to $\lambda$ and change the order of differentiation and expectation to get that when $\lambda = \lambda_0$,

$$\widetilde{\phi}_\lambda(x,\lambda) = -\Pi_x\left[B_\tau e^{-\lambda B_\tau} e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t} + \gamma'(\lambda)\tau e^{-\lambda B_\tau} e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right], \tag{2.14}$$

and, by (2.12) and (2.13), the absolute value of the quantity in the expectation is bounded by

$$\left(2 + M \max_{\lambda\in[\lambda_0/2, 3\lambda_0/2]} \gamma'(\lambda)\right) e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(0))\mathrm{d}t}.$$

Since $\lambda_0 > 0$ is arbitrary, $\widetilde{\phi}(x,\cdot)$ is differentiable on $(0,\infty)$ and (2.14) holds for any $\lambda > 0$. Using the dominated convergence theorem, we obtain $\widetilde{\phi}_\lambda(x,\cdot)$ is continuous.

Using the Markov property of Brownian motion for the second term of the right hand side of (2.14), we have

$$\widetilde{\phi}_\lambda(x,\lambda) = -\Pi_x\left[B_\tau e^{-\lambda B_\tau} e^{\int_0^\tau (m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t} + \gamma'(\lambda)\int_0^\tau e^{\int_0^s (m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\widetilde{\phi}(B_s,\lambda)\mathrm{d}s\right].$$

Combining [10, Theorem 4.7 and 6.4] with regularity theory of second order elliptic equations (see, for instance, [19, Section 8.3]), we can get that, for any $x \in (0,2)$, $\widetilde{\phi}_\lambda(x,\lambda) \in C^2(\mathbb{R})$, and $\widetilde{\phi}_\lambda(x,\lambda)$ satisfies

$$\frac{1}{2}\widetilde{\phi}_{\lambda xx} + (m\mathbf{g} - \gamma(\lambda))\widetilde{\phi}_\lambda = \gamma'(\lambda)\widetilde{\phi}.$$

One can easily check that $\widetilde{\psi} = e^{\lambda x}\widetilde{\phi}(\lambda, x)$ satisfies (2.9). □

**Corollary 2.6.** *Suppose $\widehat{\psi}(\cdot,\lambda)$ is a positive eigenfunction of* (1.5) *with $\widehat{\psi}(0,\cdot) \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$. Then for any $x \in \mathbb{R}$, $\widehat{\psi}(x,\cdot) \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$. Moreover, $\widehat{\psi}_\lambda(x,\lambda)$ satisfies*

$$\begin{aligned}&\frac{1}{2}\widehat{\psi}_{\lambda xx}(x,\lambda) - \widehat{\psi}_x(x,\lambda) - \lambda\widehat{\psi}_{\lambda x}(x,\lambda) + \left(\frac{1}{2}\lambda^2 + m\mathbf{g}(x)\right)\widehat{\psi}_\lambda(x,\lambda) + \lambda\widehat{\psi}(x,\lambda) \\ &= \gamma(\lambda)\widehat{\psi}_\lambda(x,\lambda) + \gamma'(\lambda)\widehat{\psi}(x,\lambda).\end{aligned} \tag{2.15}$$

*Proof.* Since $\gamma(\lambda)$ is simple, we have $\widehat{\psi}(x,\lambda) = \widehat{\psi}(0,\lambda)\widetilde{\psi}(x,\lambda)$ for all $x$ and $\lambda$, where $\widetilde{\psi}$ is the eigenfunction in Lemma 2.5. The results follow immediately from Lemma 2.5. □

**Lemma 2.7.** *It holds that $\psi(x,\cdot) \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$ for $x \in \mathbb{R}$. Moreover, $\psi_\lambda(x,\lambda)$ satisfies* (2.8).

*Proof.* For any $\lambda \in \mathbb{R}$, let $\widetilde{\psi}(\cdot,\lambda)$ be the positive eigenfunction of the periodic problem (1.5) with $\widetilde{\psi}(0,\lambda) = 1$. By Lemma 2.5, for any fixed $x \in [0,1]$, $\widetilde{\psi}(x,\cdot) \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$. Put

$$\widetilde{c}(\lambda) = \int_0^1 \widetilde{\psi}(x,\lambda)\mathrm{d}x.$$

Since $\widetilde{\psi}(\cdot,\lambda)$ is continuous and positive, we know that $\widetilde{c}(\lambda) > 0$. Then $\widetilde{c}(\lambda)^{-1}\widetilde{\psi}(x,\lambda)$ is a positive eigenfunction of the (1.5) with $\int_0^1 \widetilde{c}(\lambda)^{-1}\widetilde{\psi}(x,\lambda)dx = 1$. Thus $\psi(x,\lambda) = \widetilde{c}(\lambda)^{-1}\widetilde{\psi}(x,\lambda)$.

By Corollary 2.6, we only need to prove that $\widetilde{c}(\cdot) \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$. For any $\lambda_0 \in \mathbb{R}$ and $\epsilon > 0$, define $D_\epsilon(\lambda_0) := \{(x,\lambda) : 0 \le x \le 1,\ \lambda_0 - \epsilon \le \lambda \le \lambda_0 + \epsilon\}$. By the proof of Lemma 2.5, $\widetilde{\psi}(x,\lambda)$ is bounded in $D_\epsilon(\lambda_0)$. By the bounded convergence theorem, $\widetilde{c}(\cdot)$ is continuous at $\lambda_0$. Since $\lambda_0 \in \mathbb{R}$ is arbitrary, we get that $\widetilde{c}(\cdot)$ is continuous in $\mathbb{R}$. We now show $\widetilde{c}(\cdot) \in C^1(\mathbb{R}\setminus\{0\})$. By symmetry, it suffices to show $\widetilde{c}(\cdot) \in C^1((0,\infty))$. Let $\lambda_0 > 0$ and $\epsilon > 0$ with $\lambda_0 - \epsilon > 0$. Note that

$$\frac{\partial \widetilde{\psi}}{\partial \lambda}(x,\lambda) = x e^{\lambda x}\widetilde{\phi}(x,\lambda) + e^{\lambda x}\frac{\partial}{\partial \lambda}\widetilde{\phi}(x,\lambda).$$

Combining (2.13) and (2.14), we know that $\widetilde{\psi}_\lambda(x,\lambda)$ is bounded in $D_\epsilon(\lambda_0)$. Therefore, by the bounded convergence theorem, we have for $|h| < \epsilon$,

$$\lim_{h\to 0}\frac{\widetilde{c}(\lambda_0+h)-\widetilde{c}(\lambda_0)}{h} = \int_0^1 \lim_{h\to 0}\frac{\widetilde{\psi}(x,\lambda_0+h)-\widetilde{\psi}(x,\lambda_0)}{h}\mathrm{d}x = \int_0^1 \widetilde{\psi}_\lambda(x,\lambda_0)\mathrm{d}x.$$

Combining this with the arbitrariness of $\lambda_0$, $\widetilde{c}(\cdot)$ is differentiable and $\widetilde{c}'(\lambda) = \int_0^1 \widetilde{\psi}_\lambda(x,\lambda)dx$. By the bounded convergence theorem again, we get that $\widetilde{c}'(\cdot)$ is continuous. □

## 2.2 Measure change for Brownian motion

The martingale in the lemma below is related to the additive martingale $\{W_t(\lambda)\}$ of the BBMPE.

**Lemma 2.8.** *Suppose $\{B_t, t \ge 0; \Pi_x\}$ is a Brownian motion starting from $x \in \mathbb{R}$. Define*

$$\Xi_t(\lambda) := e^{-\gamma(\lambda)t - \lambda B_t + m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\psi(B_t,\lambda).$$

*Then $\{\Xi_t(\lambda), t \ge 0\}$ is a $\Pi_x$-martingale.*

*Proof.* $\psi(x,\lambda)$ is strictly positive, so by Itô's formula, we have

$$\frac{\psi(B_t,\lambda)}{\psi(B_0,\lambda)} = \exp\left\{\log\psi(B_t,\lambda) - \log\psi(B_0,\lambda)\right\} = \exp\left\{\int_0^t \frac{\psi_x}{\psi}\mathrm{d}B_s + \frac{1}{2}\int_0^t \frac{\psi_{xx}\psi - \psi_x^2}{\psi^2}\mathrm{d}x\right\},$$

where we have written $\psi = \psi(B_t,\lambda)$, $\psi_x = \psi_x(B_t,\lambda)$ and $\psi_{xx} = \psi_{xx}(B_t,\lambda)$ for short. Thus,

$$\begin{aligned}\frac{\Xi_t(\lambda)}{\Xi_0(\lambda)} &= \exp\left\{\int_0^t \left(\frac{\psi_x}{\psi} - \lambda\right)\mathrm{d}B_s + \int_0^t \left(\frac{\psi_{xx}\psi - \psi_x^2}{2\psi^2} + m\mathbf{g}(B_s) - \gamma(\lambda)\right)\mathrm{d}s\right\} \qquad (2.16)\\ &= \exp\left\{\int_0^t \left(\frac{\psi_x}{\psi} - \lambda\right)\mathrm{d}B_s - \frac{1}{2}\int_0^t \left(\frac{\psi_x}{\psi} - \lambda\right)^2\mathrm{d}s\right\},\end{aligned}$$

where the second equality follows from (1.5). For fixed $\lambda$, by periodicity, $\frac{\psi_x}{\psi} - \lambda$ is bounded, and so Novikov's condition is satisfied (see, for example, Novikov [38]). Therefore, $\{\Xi_t(\lambda), t \ge 0\}$ is a $\Pi_x$-martingale. □

Since $\frac{\Xi_t(\lambda)}{\Xi_0(\lambda)}$ is a non-negative martingale of mean 1, we can define a probability measure $\Pi_x^\lambda$ by

$$\left.\frac{\mathrm{d}\Pi_x^\lambda}{\mathrm{d}\Pi_x}\right|_{\mathcal{F}_t^B} = \frac{\Xi_t(\lambda)}{\Xi_0(\lambda)}, \qquad (2.17)$$

where $\{\mathcal{F}_t^B : t \geq 0\}$ is the natural filtration of Brownian motion. A direct calculation shows that $\phi(x,\lambda)$ defined by (2.1) satisfies

$$\frac{\phi_x(x,\lambda)}{\phi(x,\lambda)} = \frac{\psi_x(x,\lambda)}{\psi(x,\lambda)} - \lambda.$$

By (2.16) and Girsanov's theorem, $B_t - \int_0^t \frac{\phi_x(B_s,\lambda)}{\phi(B_s,\lambda)} \mathrm{d}s$ is a $\Pi_x^\lambda$-Brownian motion. In other words, under $\Pi_x^\lambda$, $\{B_t, t \geq 0\}$ satisfies

$$\mathrm{d}B_t = \frac{\phi_x(B_t,\lambda)}{\phi(B_t,\lambda)} \mathrm{d}t + \mathrm{d}\widehat{B}_t, \quad B_0 = x, \tag{2.18}$$

where $\{\widehat{B}_t,\, t \geq 0;\, \Pi_x^\lambda\}$ is a Brownian motion. Hence, $\{B_t, \Pi_x^\lambda\}$ is a diffusion with infinitesimal generator

$$(\mathcal{A}f)(x) = \frac{1}{2}\frac{\partial^2 f(x)}{\partial x^2} + \left(\frac{\psi_x(x,\lambda)}{\psi(x,\lambda)} - \lambda\right)\frac{\partial f(x)}{\partial x}. \tag{2.19}$$

Since $\frac{\phi_x(\cdot,\lambda)}{\phi(\cdot,\lambda)}$ is 1-periodic, the law of $\{B_t - B_0 : t \geq 0\}$ under $\Pi_x^\lambda$ is the same as under $\Pi_{x+1}^\lambda$.

*In the remainder of this section and Sections 3–5, we always assume that* $\{Y_t, t \geq 0; \Pi_x^\lambda\}$ *is a diffusion with infinitesimal generator* (2.19)*.* To prove Theorem 1.1, we need some properties of $Y_t$ under $\Pi_x^\lambda$. [34, Lemma 2.6 and Corollary 2.7] gave a strong law of large numbers of $\{Y_t\}$ under $\Pi_x^{\lambda^*}$ for binary branching in the critical case $\lambda = \lambda^*$ and their proofs also work for any $\lambda \in \mathbb{R}$, and thus $Y_t/t \to -\gamma'(\lambda)$, $\Pi_x^\lambda$-almost surely. For the convenience of our readers, we give a proof below. Our proof is slightly different in that we get that the rate function $I(z)$ attains its minimum at the point $z = -\gamma'(\lambda)$ by a simple analysis, see the beginning of the proof of Lemma 2.10. Now we prove the analog for our general case.

**Lemma 2.9.** *Let* $\mu_t$ *be the law of* $\{\frac{Y_t}{t}\}$ *under* $\Pi_x^\lambda$*. Then* $\{\mu_t\}$ *satisfies a large deviation principle with good rate function*

$$I(z) := \gamma^*(z) + \{\lambda z + \gamma(\lambda)\},$$

*where* $\gamma^*(z) = \sup_{\eta\in\mathbb{R}}\{\eta z - \gamma(\eta)\}$ *denotes the Fenchel-Legendre transform of* $\gamma$*.*

*Proof.* For any $x \in \mathbb{R}$, $\lambda > 0$, $t > 0$ and any measurable function $F : C[0,t] \to \mathbb{R}$, by the change of measure (2.17), we have

$$\Pi_x^\lambda\left[F(\{Y_s\}_{s\leq t})\right] = \Pi_x\left[\frac{\psi(B_t,\lambda)}{\psi(x,\lambda)} e^{-\gamma(\lambda)t - \lambda(B_t - x) + m\int_0^t \mathbf{g}(B_s)\mathrm{d}s} F(\{B_s\}_{s\leq t})\right]. \tag{2.20}$$

Since $\psi$ is strictly positive and bounded, taking $F \equiv 1$ in the display above, we get that

$$\Pi_x\left[e^{-\lambda B_t + m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\right] = e^{-\lambda x + \gamma(\lambda)t + O(1)}, \tag{2.21}$$

where for fixed $\lambda$, $O(1)$ is bounded by a constant independent of $(t,x)$.

By (2.20), for any $\eta \in \mathbb{R}$,

$$\begin{aligned}\frac{1}{t}\log \Pi_x^\lambda[e^{\eta Y_t}] &= \frac{1}{t}\log \Pi_x\left[\frac{\psi(B_t,\lambda)}{\psi(x,\lambda)} e^{-\gamma(\lambda)t - \lambda(B_t - x) + m\int_0^t \mathbf{g}(B_s)\mathrm{d}s} e^{\eta B_t}\right] \\ &= \frac{1}{t}\log \Pi_x\left[\frac{\psi(B_t,\lambda)}{\psi(x,\lambda)} e^{-(\lambda-\eta)B_t + m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\right] + \frac{\lambda x}{t} - \gamma(\lambda).\end{aligned}$$

It follows from (2.21) that as $t \to \infty$,

$$\lim_{t\to\infty} \frac{1}{t}\log \Pi_x^\lambda[e^{\eta Y_t}] = \gamma(\lambda - \eta) - \gamma(\lambda).$$

The Fenchel-Legendre transform of $\eta \mapsto \gamma(\lambda-\eta)-\gamma(\lambda)$ is given by

$$\begin{aligned} I(z) :&= \sup_{\eta\in\mathbb{R}} \left\{\eta z - [\gamma(\lambda-\eta)-\gamma(\lambda)]\right\} \\ &= \sup_{\eta\in\mathbb{R}} \left\{(\eta-\lambda)z - \gamma(\lambda-\eta)]\right\} + \lambda z + \gamma(\lambda) \\ &= \gamma^*(z) + \{\lambda z + \gamma(\lambda)\}, \end{aligned}$$

where the last equality follows from the fact that $\gamma$ is an even function. Note that $\gamma(\lambda-\eta)-\gamma(\lambda)$ is differentiable with respect to $\eta$ by Lemma 2.3. By the Gärtner-Ellis theorem (for example, see [11, §2.3]), $\{\mu_t\}$ satisfies a large deviation principle with rate function $I(z)$. By the proof of [34, Lemma 2.6], $I(z)$ is a good rate function. This completes the proof. □

**Lemma 2.10.** *For any $x \in \mathbb{R}$, it holds that $\frac{Y_t}{t} \to -\gamma'(\lambda)$ $\Pi_x^\lambda$-almost surely.*

*Proof.* By Lemma 2.3 (1), $\gamma'(\lambda)$ is strictly increasing. Next, note that $I(z) = \sup_{\eta\in\mathbb{R}}\big\{\eta z - [\gamma(\lambda-\eta)-\gamma(\lambda)]\big\}$ and that the derivative of $\eta z - \gamma(\lambda-\eta)$ with respect to $\eta$ is $z+\gamma'(\lambda-\eta)$. For fixed $z$, define $\eta_z$ such that $\gamma'(\lambda-\eta_z) = -z$. Here the existence of $\eta_z$ is guaranteed by (2.4). By the convexity of $\gamma$, $\eta_z$ is an increasing function of $z$. Moreover, because $\gamma'$ is strictly increasing, we have

$$I(z) = -\eta_z\gamma'(\lambda-\eta_z) - [\gamma(\lambda-\eta_z)-\gamma(\lambda)], \tag{2.22}$$

where $I(z)$ is equal to $0$ when $\eta_z = 0$ or equivalently $z = -\gamma'(\lambda)$. Thus $\frac{\mathrm{d}I(z)}{\mathrm{d}\eta_z} = \eta_z\gamma''(\lambda-\eta_z) \geq 0$ (respectively $\frac{\mathrm{d}I(z)}{\mathrm{d}\eta_z} \leq 0$) when $\eta_z > 0$ (respectively $\eta_z < 0$). Using (2.22) and the fact that $\gamma'$ is strictly increasing, we have for any $\epsilon > 0$,

$$\delta := \inf\{I(z) : |z+\gamma'(\lambda)| \geq \epsilon\} = I(-\gamma'(\lambda)-\epsilon) \wedge I(-\gamma'(\lambda)+\epsilon) > 0. \tag{2.23}$$

Applying the large deviation principle of $\{\mu_t\}$, we get

$$\Pi_x^\lambda\left(\left|\frac{Y_t}{t}+\gamma'(\lambda)\right| > \epsilon\right) \leq Ce^{-\delta t/2}.$$

By (2.20), there is a constant $C_1(\lambda) > 0$ such that

$$\begin{aligned} P(x,T) :&= \Pi_x^\lambda\left(\max_{t\in[0,1]} |Y_t - Y_0 + \gamma'(\lambda)t| > T\epsilon\right) \\ &= \Pi_x\left[\frac{\psi(B_1,\lambda)}{\psi(x,\lambda)} e^{-\gamma(\lambda)-\lambda(B_1-x)+m\int_0^1 \mathbf{g}(B_s)\mathrm{d}s} \mathbb{1}_{\{\max_{t\in[0,1]}|B_t-x+\gamma'(\lambda)t|>T\epsilon\}}\right] \\ &\leq C_1(\lambda)\Pi_x\left[e^{-\lambda(B_1-x)}\mathbb{1}_{\{\max_{t\in[0,1]}|B_t-x+\gamma'(\lambda)t|>T\epsilon\}}\right] \\ &\leq C_1(\lambda)\Pi_0\left[e^{\lambda B_1^*}\mathbb{1}_{\{B_1^*>T\epsilon-|\gamma'(\lambda)|\}}\right], \quad x\in[0,1], \end{aligned}$$

where $B_1^* = \max_{t\in[0,1]}\{|B_t|\}$. Since, under $\Pi_0$, $B_1^*$ has the same distribution as $|B_1|$ and $\int_x^\infty e^{-\frac{y^2}{2}}\mathrm{d}y \leq \frac{1}{x}e^{-\frac{x^2}{2}}$ for $x > 0$, there is a constant $C_2(\lambda) > 0$ such that

$$P(x,T) \leq C_2(\lambda)e^{-\delta T/2}, \quad x\in[0,1].$$

Thus, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, one has

$$\begin{aligned} &\Pi_x^\lambda\bigg(\bigcup_{t\in[n,n+1]}\{|Y_t+\gamma'(\lambda)t| > 2t\epsilon\}\bigg) \\ \leq& \Pi_x^\lambda\left(|Y_n+\gamma'(\lambda)n| > n\epsilon\right) + \Pi_x^\lambda\left(\max_{t\in[0,1]}|Y_{n+t}-Y_n+\gamma'(\lambda)t| > n\epsilon\right) \\ \leq& Ce^{-n\delta/2} + \Pi_x^\lambda\left[P(Y_n,n)\right] \leq Ce^{-n\delta/2}, \end{aligned}$$

where the last inequality follows from the fact that $P(x,T)$ is 1-periodic in $x$. Since $\epsilon > 0$ is arbitrary, by the Borel-Cantelli lemma, we obtain $\frac{Y_t}{t} \to -\gamma'(\lambda)$ $\Pi_x^\lambda$-almost surely. □

The martingale in the following lemma is related to the derivative martingale $\{\partial W_t(\lambda)\}$ of the BBMPE.

**Lemma 2.11.** *Suppose $\{B_t, t \geq 0; \Pi_x\}$ is a Brownian motion starting from $x \in \mathbb{R}$. Define*

$$\Upsilon_t(\lambda) := e^{-\gamma(\lambda)t - \lambda B_t + m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\left(\psi(B_t,\lambda)(\gamma'(\lambda)t + B_t) - \psi_\lambda(B_t,\lambda)\right).$$

*Then $\{\Upsilon_t(\lambda), t \geq 0\}$ is a $\Pi_x$-martingale.*

*Proof.* For convenience, put $J_t := e^{-\gamma(\lambda)t - \lambda B_t + m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}$. A straightforward computation using Itô's formula yields

$$\mathrm{d}J_t = -\lambda J_t \mathrm{d}B_t + \left(\frac{1}{2}\lambda^2 - \gamma(\lambda) + m\mathbf{g}(B_t)\right) J_t \mathrm{d}t,$$

$$\begin{aligned}
&\mathrm{d}\left(\psi(B_t,\lambda)(\gamma'(\lambda)t + B_t) - \psi_\lambda(B_t,\lambda)\right) \\
=&\left[\psi_x(B_t,\lambda)(\gamma'(\lambda)t + B_t) + \psi(B_t,\lambda) - \psi_{\lambda x}(B_t,\lambda)\right]\mathrm{d}B_t \\
&+ \left[\frac{1}{2}\psi_{xx}(B_t,\lambda)(\gamma'(\lambda)t + B_t) + \psi(B_t,\lambda)\gamma'(\lambda) + \psi_x(B_t,\lambda) - \frac{1}{2}\psi_{\lambda xx}(B_t,\lambda)\right]\mathrm{d}t,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{d}\Upsilon_t(\lambda) = \mathrm{d}\left[J_t(\psi(B_t,\lambda)(\gamma'(\lambda)t + B_t) - \psi_\lambda(B_t,\lambda)\right] \\
=&J_t\left[(\psi_x - \lambda\psi)(B_t,\lambda)(\gamma'(\lambda)t + B_t) + \lambda\psi_\lambda(B_t,\lambda) + \psi(B_t,\lambda) - \psi_{x\lambda}(B_t,\lambda)\right]\mathrm{d}B_t \\
&+ J_t(\gamma'(\lambda)t + B_t)\left(\frac{1}{2}\psi_{xx} - \lambda\psi_x + \left(\frac{1}{2}\lambda^2 - \gamma(\lambda) + m\mathbf{g}(B_t)\right)\psi\right)(B_t,\lambda)\mathrm{d}t \\
&- J_t\left(\frac{1}{2}\psi_{\lambda xx} - \psi_x - \lambda\psi_{\lambda x} + \left(\frac{1}{2}\lambda^2 - \gamma(\lambda) + m\mathbf{g}(B_t)\right)\psi_\lambda + \lambda\psi - \gamma'(\lambda)\psi\right)(B_t,\lambda)\mathrm{d}t \\
=&J_t\left[(\gamma'(\lambda)t + B_t)(\psi_x - \lambda\psi) + \lambda\psi_\lambda + \psi - \psi_{\lambda x}\right](B_t,\lambda)\mathrm{d}B_t,
\end{aligned}$$

where in the last equality we used (1.5) and (2.15). Note that $\mathbf{g}$, $\psi$, $\psi_\lambda$ and $\psi_{\lambda x}$ are 1-periodic in $x$, so they are bounded for fixed $\lambda \in \mathbb{R}$. Using this, one can easily check that

$$\Pi_x \int_0^T J_t^2\left[(\gamma'(\lambda)t + B_t)(\psi_x - \lambda\psi) + \lambda\psi_\lambda + \psi - \psi_{\lambda x}\right]^2(B_t,\lambda)\mathrm{d}t < \infty, \quad \text{for all } T > 0.$$

Thus $\{\Upsilon_t(\lambda), t \geq 0\}$ is a $\Pi_x$-martingale. □

The martingale $\{(\Upsilon_t(\lambda))_{t\geq 0}, \Pi_x\}$ may take negative values. Now we introduce a related non-negative martingale. Before giving its definition, we first give some properties of the function $h$ defined by

$$h(x) := x - \frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}.$$

Clearly $h(x)$ is continuous and satisfies $h(x+1) = h(x)+1$. Recall that we always suppose $\lambda > 0$ unless explicitly stated otherwise. Since $\phi(x,\lambda) = e^{-\lambda x}\psi(x,\lambda)$, we have

$$h(x) = -\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}. \tag{2.24}$$

It is easy to see that $h'$ is 1-periodic and continuous. Thus, $h'$ is bounded.

**Lemma 2.12.** *$h'$ is strictly positive.*

*Proof.* Recall that $\phi(x,\lambda)=e^{-\lambda x}\psi(x,\lambda)$ satisfies (2.2). For any $-\infty\le y<z\le\infty$, define

$$\tau_{(y,z)}:=\inf\{t>0:B_t\notin(y,z)\},$$

and $\tau_y:=\inf\{t>0:B_t=y\}$. It follows from Lemma 2.2 that for any $x\in(y,z)$,

$$\begin{aligned}\phi(x,\lambda)&=\Pi_x\left[\phi(B_{\tau_{(y,z)}},\lambda)e^{\int_0^{\tau_{(y,z)}}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]\\&=\Pi_x\left[\phi(y,\lambda)\mathbb{1}_{\{\tau_y<\tau_z\}}e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]+\Pi_x\left[\phi(z,\lambda)\mathbb{1}_{\{\tau_z<\tau_y\}}e^{\int_0^{\tau_z}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right].\end{aligned}\tag{2.25}$$

By the monotone convergence theorem, the first term on the right-hand side of (2.25) converges to

$$\Pi_x\left[\phi(y,\lambda)e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\right]$$

as $z\to\infty$. The second term of (2.25) is equal to

$$e^{-\lambda z}\Pi_x\left[\psi(z,\lambda)\mathbb{1}_{\{\tau_z<\tau_y\}}e^{\int_0^{\tau_z}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right],\tag{2.26}$$

thus bounded by $Ce^{-\lambda z}$, if we can show that

$$\Pi_x\left[\psi(z,\lambda)\mathbb{1}_{\{\tau_z<\tau_y\}}e^{\int_0^{\tau_z}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]\le C$$

for some constant $C>0$. Let $\psi(x,-\lambda)$ be the positive eigenfunction corresponding to the eigenvalue $\gamma(-\lambda)$ and let $\phi(x,-\lambda)=e^{\lambda x}\psi(x,-\lambda)$. Since $\gamma$ is even, we have $\phi(x,-\lambda)$ satisfies (2.2). Therefore,

$$\begin{aligned}&\phi(x,-\lambda)\\=&\Pi_x\left[\phi(y,-\lambda)\mathbb{1}_{\{\tau_y<\tau_z\}}e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]+\Pi_x\left[\phi(z,-\lambda)\mathbb{1}_{\{\tau_z<\tau_y\}}e^{\int_0^{\tau_z}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]\\\ge&e^{\lambda z}\Pi_x\left[\psi(z,-\lambda)\mathbb{1}_{\{\tau_z<\tau_y\}}e^{\int_0^{\tau_z}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right],\end{aligned}$$

that is,

$$\Pi_x\left[\mathbb{1}_{\{\tau_z<\tau_y\}}e^{\int_0^{\tau_z}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]\le e^{\lambda(x-z)}\frac{\psi(x,-\lambda)}{\psi(z,-\lambda)}.$$

Since $x<z$, $\lambda>0$ and $\psi$ is bounded between two positive constants, (2.26) is bounded by $Ce^{-\lambda z}$ and so converges to zero when $z\to\infty$. Therefore we have for any $x>y$ and $\lambda>0$,

$$\phi(x,\lambda)=\Pi_x\left[\phi(y,\lambda)e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\right].\tag{2.27}$$

Since $\Pi_x(\tau_y<\infty)=1$, we have

$$\phi(x,\lambda)=\Pi_x\left[\phi(y,\lambda)e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right].\tag{2.28}$$

Hence,

$$\ln\phi(x,\lambda)=\ln\phi(y,\lambda)+\ln\Pi_x\left[e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right].$$

Differentiating both sides of the previous equation with respect to $\lambda$ gives

$$\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}=\frac{\phi_\lambda(y,\lambda)}{\phi(y,\lambda)}-\frac{\Pi_x\left[\gamma'(\lambda)\tau_y e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]}{\Pi_x\left[e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]}<\frac{\phi_\lambda(y,\lambda)}{\phi(y,\lambda)}.$$

In other words, $\phi_\lambda(\cdot,\lambda)/\phi(\cdot,\lambda)$ is strictly decreasing. Thus by (2.24), $h$ is strictly increasing.

By (2.1) and (2.28), we have

$$C = \max_{y\in[x-1,x]} \Pi_x \left[e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right] = \max_{y\in[x-1,x]} \frac{\phi(x,\lambda)}{\phi(y,\lambda)} \le \frac{\max_{z\in[0,1]}\psi(z,\lambda)}{\min_{z\in[0,1]}\psi(z,\lambda)} < +\infty.$$

It is well known, see [14, Theorem 8.5.7] for example, that for any $b>0$,

$$\Pi_x[e^{-b\tau_y}] = e^{-(x-y)\sqrt{2b}}.$$

Differentiating both sides of the previous equation with respect to $b$, we get

$$\Pi_x\left[\tau_y e^{-b\tau_y}\right] = \frac{x-y}{\sqrt{2b}} e^{-(x-y)\sqrt{2b}}.$$

Recall that $0<\alpha\le \min_{x\in[0,1]}\mathbf{g}(x)$ and that (2.6) implies $\gamma(\lambda)>m\alpha$. Thus we have

$$\begin{aligned} h(x)-h(y) &\ge \Pi_x\left[\gamma'(\lambda)\tau_y e^{\int_0^{\tau_y}(m\mathbf{g}(B_t)-\gamma(\lambda))\mathrm{d}t}\right]/C \\ &\ge \Pi_x\left[\gamma'(\lambda)\tau_y e^{(m\alpha-\gamma(\lambda))\tau_y}\right]/C \\ &= \frac{x-y}{C\sqrt{2(\gamma(\lambda)-m\alpha)}}\gamma'(\lambda)e^{-(x-y)\sqrt{2(\gamma(\lambda)-m\alpha)}}. \end{aligned}$$

Therefore

$$h'(x) = \lim_{y\to x}\frac{h(x)-h(y)}{x-y} \ge \frac{\gamma'(\lambda)}{C\sqrt{2(\gamma(\lambda)-m\alpha)}} > 0.$$

This completes the proof. □

For any $x\in\mathbb{R}$, define

$$\tau_\lambda^x := \inf\left\{t\ge 0:\ h(B_t)\le -x-\gamma'(\lambda)t\right\}.$$

Since $h(x)$ is strictly increasing, for any $x\in\mathbb{R}$, we may rewrite the definition of $\tau_\lambda^x$ as

$$\tau_\lambda^x = \inf\left\{t\ge 0: B_t\le h^{-1}(-x-\gamma'(\lambda)t)\right\}.$$

Hence, $\tau_\lambda^x$ is an $\{\mathcal{F}_t^B\}$-stopping time. Consider the barrier $\mathbf{\Gamma}^{(-x,\lambda)}$ described by $z = h^{-1}(-x-\gamma'(\lambda)t)$ on the space-time half plane $\{(z,t): z\in\mathbb{R}, t\in\mathbb{R}^+\}$, then $\tau_\lambda^x$ is the first time when the Brownian motion hits this barrier.

Define

$$\Lambda_t^{(x,\lambda)} := e^{-\gamma(\lambda)t-\lambda B_t+m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\psi(B_t,\lambda)\left(x+\gamma'(\lambda)t+h(B_t)\right)\mathbf{1}_{\{\tau_\lambda^x>t\}}.$$

It is easy to see that $\Lambda_t^{(x,\lambda)}$ is non-negative. We now prove that $\{\Lambda_t^{(x,\lambda)}, t\ge 0\}$ is a martingale. Notice that $\Lambda_t^{(x,\lambda)} = \left(\Upsilon_t(\lambda)+x\Xi_t(\lambda)\right)\mathbf{1}_{\{\tau_\lambda^x>t\}}$. The key to proving the martingale property is that $\left(\Upsilon_{\tau_\lambda^x}(\lambda)+x\Xi_{\tau_\lambda^x}(\lambda)\right)=0$. This is also the reason why we consider the barrier $z=h^{-1}(-x-\gamma'(\lambda)t)$ instead of the line considered in [31]. As stated at the beginning of this section, this martingale is related to the martingale $\{V_t^x(\lambda)\}$.

**Lemma 2.13.** *For any $x,y\in\mathbb{R}$ with $y>h^{-1}(-x)$, $\{\Lambda_t^{(x,\lambda)}, t\ge 0\}$ is a $\Pi_y$-martingale.*

*Proof.* By Lemmas 2.8 and 2.11, $\{\Xi_t(\lambda): t\ge 0\}$ and $\{\Upsilon_t(\lambda): t\ge 0\}$ are $\Pi_y$-martingales. Put $F(B_t) := \Upsilon_t(\lambda) + x\Xi_t(\lambda)$. Then $\{F(B_t): t\ge 0\}$ is a $\Pi_y$-martingale. The fact $\tau_\lambda^x$ is an $\{\mathcal{F}_t^B\}$-stopping time yields that $\{F(B_{\tau_\lambda^x\wedge t}): t\ge 0\}$ is a $\Pi_y$-martingale. Note that

$$\Lambda_t^{(x,\lambda)} = (\Upsilon_t(\lambda) + x\Xi_t(\lambda))\,\mathbf{1}_{\{\tau_\lambda^x>t\}} = F(B_t)\mathbf{1}_{\{\tau_\lambda^x>t\}}$$

and

$$F(B_{\tau_\lambda^x}) = 0.$$

Thus

$$F(B_{\tau_\lambda^x\wedge t}) = F(B_t)\mathbf{1}_{\{\tau_\lambda^x>t\}} + F(B_{\tau_\lambda^x})\mathbf{1}_{\{\tau_\lambda^x\le t\}} = \Lambda_t^{(x,\lambda)},$$

and hence $\{\Lambda_t^{(x,\lambda)}, t\ge 0\}$ is a $\Pi_y$-martingale. □

For $x, y\in\mathbb{R}$ with $y > h^{-1}(-x)$, consider a new probability measure $\Pi_y^{(x,\lambda)}$ defined by

$$\left.\frac{\mathrm{d}\Pi_y^{(x,\lambda)}}{\mathrm{d}\Pi_y}\right|_{\mathcal{F}_t^B} = \frac{\Lambda_t^{(x,\lambda)}}{\Lambda_0^{(x,\lambda)}}.$$

Recall that under $\Pi_x^\lambda$, $\{Y_t\}$ satisfies $\mathrm{d}Y_t = \frac{\phi_x(Y_t,\lambda)}{\phi(Y_t,\lambda)}\mathrm{d}t + \mathrm{d}\widehat{B}_t$, where $\{\widehat{B}_t, t\ge 0; \Pi_x^\lambda\}$ is a Brownian motion starting from $x$. The following lemma, which says that $h(Y_t) + \gamma'(\lambda)t$ is a martingale under $\Pi_x^\lambda$, is crucial to study the behavior of the spatial motion under $\Pi_y^{(x,\lambda)}$.

**Lemma 2.14.** *Define*

$$M_t := \gamma'(\lambda)t + h(Y_t) - h(Y_0), \quad t\ge 0. \tag{2.29}$$

*Then* $\{M_t, t\ge 0; \Pi_x^\lambda\}$ *is a martingale. Moreover, there exist two constants* $c_2 > c_1 > 0$ *such that the quadratic variation* $\langle M\rangle_t \in [c_1 t, c_2 t]$.

*Proof.* By (2.24), $h(Y_t) = -\frac{\phi_\lambda(Y_t,\lambda)}{\phi(Y_t,\lambda)}$. Using (2.18) and Itô's formula, we have

$$\begin{aligned}
\mathrm{d}\left(\frac{\phi_\lambda(Y_t,\lambda)}{\phi(Y_t,\lambda)}\right) &= \frac{\partial}{\partial x}\left(\frac{\phi_\lambda(Y_t,\lambda)}{\phi(Y_t,\lambda)}\right)\mathrm{d}Y_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\phi_\lambda(Y_t,\lambda)}{\phi(Y_t,\lambda)}\right)\mathrm{d}t\\
&= \frac{\partial}{\partial x}\left(\frac{\phi_\lambda(Y_t,\lambda)}{\phi(Y_t,\lambda)}\right)\mathrm{d}\widehat{B}_t + \left[\frac{\phi_x(Y_t,\lambda)}{\phi(Y_t,\lambda)}\frac{\partial}{\partial x}\left(\frac{\phi_\lambda(Y_t,\lambda)}{\phi(Y_t,\lambda)}\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\phi_\lambda(Y_t,\lambda)}{\phi(Y_t,\lambda)}\right)\right]\mathrm{d}t,
\end{aligned}$$

where $\{\widehat{B}_t, t\ge 0; \Pi_x^\lambda\}$ is a Brownian motion. Notice that

$$\begin{aligned}
\frac{1}{2}\phi_{xx}(x,\lambda) + m\mathbf{g}(x)\phi(x,\lambda) &= \gamma(\lambda)\phi(x,\lambda),\\
\frac{1}{2}\phi_{\lambda xx}(x,\lambda) + m\mathbf{g}(x)\phi_\lambda(x,\lambda) &= \gamma'(\lambda)\phi(x,\lambda) + \gamma(\lambda)\phi_\lambda(x,\lambda).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\phi_x}{\phi}\frac{\partial}{\partial x}\frac{\phi_\lambda}{\phi} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\frac{\phi_\lambda}{\phi} &= \frac{\phi_x}{\phi}\frac{\phi_{\lambda x}\phi - \phi_\lambda\phi_x}{\phi^2} + \frac{1}{2}\frac{\phi_{\lambda xx}}{\phi} - \frac{\phi_{\lambda x}\phi_x}{\phi^2} + \frac{\phi_\lambda\phi_x^2}{\phi^3} - \frac{1}{2}\frac{\phi_\lambda\phi_{xx}}{\phi^2}\\
&= \frac{1}{2}\frac{\phi_{\lambda xx}}{\phi} - \frac{1}{2}\frac{\phi_\lambda\phi_{xx}}{\phi^2}\\
&= \frac{\gamma'(\lambda)\phi + \gamma(\lambda)\phi_\lambda - m\mathbf{g}\phi_\lambda}{\phi} - \frac{1}{2}\frac{\phi_\lambda\phi_{xx}}{\phi^2}\\
&= \gamma'(\lambda) - \frac{\phi_\lambda}{\phi^2}\left[\frac{1}{2}\phi_{xx} + (m\mathbf{g} - \gamma(\lambda))\phi\right] = \gamma'(\lambda).
\end{aligned}$$

This yields

$$\mathrm{d}\left(h(Y_t)\right) = h'(Y_t)\mathrm{d}\widehat{B}_t - \gamma'(\lambda)\mathrm{d}t. \tag{2.30}$$

It follows from the boundedness of $h'$ and Lemma 2.12 that $h'(x) \in [\sqrt{c_1}, \sqrt{c_2}]$ for two constants $c_1 \le c_2$. Integrating both sides of (2.30) gives

$$h(Y_t) - h(Y_0) = \int_0^t h'(Y_s)\mathrm{d}\widehat{B}_s - \gamma'(\lambda)t.$$

Hence,

$$M_t = \int_0^t h'(Y_s)\mathrm{d}\widehat{B}_s$$

is a martingale with quadratic variation

$$\langle M\rangle_t = \int_0^t \left(h'(Y_s)\right)^2 \mathrm{d}s \in [c_1 t, c_2 t]. \tag{2.31}$$

□

Define

$$T(s) = \inf\{t > 0 : \langle M\rangle_t > s\} \tag{2.32}$$

Thanks to the Dambis-Dubins-Schwarz theorem, $\widetilde{B}_t := M_{T(t)}$ is a standard Brownian motion. Note that for $y > h^{-1}(-x)$,

$$\begin{aligned}\left.\frac{\mathrm{d}\Pi_y^{(x,\lambda)}}{\mathrm{d}\Pi_y^{\lambda}}\right|_{\mathcal{F}_t^B} &= \left.\frac{\mathrm{d}\Pi_y^{(x,\lambda)}}{\mathrm{d}\Pi_y}\right|_{\mathcal{F}_t^B} \times \left.\frac{\mathrm{d}\Pi_y}{\mathrm{d}\Pi_y^{\lambda}}\right|_{\mathcal{F}_t^B} \\ &= \frac{x + \gamma'(\lambda)t + h(Y_t)}{x + h(y)} \mathbf{1}_{\{\forall s \le t:\, x + h(y) + M_s > 0\}} \\ &= \frac{x + h(y) + M_t}{x + h(y)} \mathbb{1}_{\{\forall s \le t:\, x + h(y) + M_s > 0\}}.\end{aligned}$$

Put

$$\begin{aligned}\widetilde{\Lambda}_t = \Lambda_{T(t)}^{(x,\lambda)} &= \left(x + h(y) + M_{T(t)}\right) \mathbb{1}_{\{\forall s \le t:\, x + h(y) + M_{T(s)} > 0\}} \\ &= \left(x + h(y) + \widetilde{B}_t\right) \mathbb{1}_{\{\forall s \le t:\, x + h(y) + \widetilde{B}_s > 0\}}\end{aligned}$$

and $\mathcal{G}_t = \mathcal{F}_{T(t)}^{\widetilde{B}}$. We have

$$\left.\frac{\mathrm{d}\Pi_y^{(x,\lambda)}}{\mathrm{d}\Pi_y^{\lambda}}\right|_{\mathcal{G}_t} = \frac{x + h(y) + M_{T(t)}}{x + h(y)} \mathbb{1}_{\{\forall s \le t:\, x + h(y) + M_{T(s)} > 0\}},$$

that is,

$$\left.\frac{\mathrm{d}\Pi_y^{(x,\lambda)}}{\mathrm{d}\Pi_y^{\lambda}}\right|_{\mathcal{G}_t} = \frac{\widetilde{\Lambda}_t}{\widetilde{\Lambda}_0}.$$

By [27], $\left\{x + h(y) + \widetilde{B}_t, t \ge 0; \Pi_y^{(x,\lambda)}\right\}$ is a standard Bessel-3 process starting at $x + h(y)$, i.e.,

$$\left\{x + h(y) + M_{T(t)}, t \ge 0; \Pi_y^{(x,\lambda)}\right\} \text{ is a standard Bessel-3 process starting at } x + h(y). \tag{2.33}$$

### 2.3 Martingales for branching Brownian motion

In this subsection we give three martingales that will play important roles for BBMPE. First we prove that for any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{W_t(\lambda), t \geq 0\}$ is a martingale.

**Lemma 2.15.** *For any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{W_t(\lambda), t \geq 0; \mathbb{P}_x\}$ is a non-negative martingale and the limit $W(\lambda, x) := \lim_{t\uparrow\infty} W_t(\lambda)$ exists $\mathbb{P}_x$-almost surely.*

*Proof.* Recall that $\{\Xi_t(\lambda) : t \geq 0\}$ is a martingale by Lemma 2.8. Combining this with Lemma 2.1, we get that

$$\mathbb{E}_x W_t(\lambda) = \Pi_x \left[ e^{m\int_0^t \mathbf{g}(B_s)\mathrm{d}s - \gamma(\lambda)t - \lambda B_t} \psi(B_t, \lambda) \right] = \Pi_x \Xi_t(\lambda) = \Pi_x \Xi_0(\lambda) = e^{-\lambda x}\psi(x, \lambda).$$

Thus by the branching property, for $s < t$ we have

$$\begin{aligned}
\mathbb{E}_x \left[ W_t(\lambda) \mid \mathcal{F}_s \right] &= \mathbb{E}_x \left[ e^{-\gamma(\lambda)t} \sum_{u\in N_t} e^{-\lambda X_u(t)} \psi(X_u(t), \lambda) \,\Big|\, \mathcal{F}_s \right] \\
&= e^{-\gamma(\lambda)s} \sum_{v\in N_s} \mathbb{E}_x \left[ e^{-\gamma(\lambda)(t-s)} \sum_{u\in N_t, u>v} e^{-\lambda X_u(t)} \psi(X_u(t), \lambda) \,\Big|\, \mathcal{F}_s \right] \\
&= e^{-\gamma(\lambda)s} \sum_{v\in N_s} \mathbb{E}_{X_v(s)} W_{t-s}^{(v)}(\lambda) \\
&= e^{-\gamma(\lambda)s} \sum_{v\in N_s} e^{-\lambda X_v(s)} \psi(X_v(s), \lambda) = W_s(\lambda),
\end{aligned}$$

where $u > v$ denotes that $u$ is a descendant of $v$ and $W_{t-s}^{(v)}(\lambda)$ is the additive martingale for the BBMPE starting from $X_v(s)$. Therefore, $\{W_t(\lambda) : t \geq 0\}$ is a non-negative martingale and the limit $W(\lambda, x) = \lim_{t\uparrow\infty} W_t(\lambda)$ exists $\mathbb{P}_x$-almost surely. □

The second martingale is given by the following lemma.

**Lemma 2.16.** *For any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{\partial W_t(\lambda), t \geq 0; \mathbb{P}_x\}$ is a martingale.*

*Proof.* It follows from Lemma 2.1 and the martingale property in Lemma 2.11 that

$$\mathbb{E}_x \partial W_t(\lambda) = \Pi_x \Upsilon_t(\lambda) = \Pi_x \Upsilon_0(\lambda) = e^{-\lambda x} \left( x\psi(x, \lambda) - \psi_\lambda(x, \lambda) \right).$$

Using the branching property and the Markov property, it is easy to show that for any $t > s > 0$,

$$\begin{aligned}
&\mathbb{E}_x \left[ \partial W_t(\lambda) \mid \mathcal{F}_s \right] \\
=& e^{-\gamma(\lambda)s} \sum_{v\in N_s} \mathbb{E}_{X_v(s)} \left( \partial W_{t-s}^{(v)}(\lambda) + \gamma'(\lambda) s W_{t-s}^{(v)}(\lambda) \right) \\
=& e^{-\gamma(\lambda)s} \sum_{v\in N_s} \left( e^{-\lambda X_v(s)} \left( X_v(s)\psi(X_v(s), \lambda) - \psi_\lambda(X_v(s), \lambda) \right) + \gamma'(\lambda) s e^{-\lambda X_v(s)} \psi(X_v(s), \lambda) \right) \\
=& \partial W_s(\lambda),
\end{aligned}$$

where, for each $v \in N_s$, $W_{t-s}^{(v)}(\lambda)$ and $\partial W_{t-s}^{(v)}(\lambda)$ are respectively the additive and derivative martingales for the BBMPE starting from $X_v(s)$. Therefore, $\{\partial W_t(\lambda), t \geq 0; \mathbb{P}_x\}$ is a martingale. □

The martingale $\{\partial W_t(\lambda)\}$ may take negative values. To study its limit we need to consider a related non-negative martingale. Note that

$$\partial W_t(\lambda) + xW_t(\lambda) = e^{-\gamma(\lambda)t} \sum_{u\in N_t} e^{-\lambda X_u(t)}\psi(X_u(t),\lambda)\left(x+\gamma'(\lambda)t + X_u(t) - \frac{\psi_\lambda(X_u(t),\lambda)}{\psi(X_u(t),\lambda)}\right)$$
$$= e^{-\gamma(\lambda)t} \sum_{u\in N_t} e^{-\lambda X_u(t)}\psi(X_u(t),\lambda)\left(x+\gamma'(\lambda)t + h(X_u(t))\right).$$

Put

$$\widetilde{N}_t^x = \{u \in N_t : \forall s \le t,\ x + \gamma'(\lambda)s + h(X_u(s)) > 0\}.$$

In the spirit of [31], we define for each $x \in \mathbb{R}$,

$$V_t^x(\lambda) = \sum_{u\in \widetilde{N}_t^x} e^{-\gamma(\lambda)t-\lambda X_u(t)}\psi(X_u(t),\lambda)\left(x+\gamma'(\lambda)t + h(X_u(t))\right). \tag{2.34}$$

Recall that the barrier $\mathbf{\Gamma}^{(-x,\lambda)}$ is described by $z = h^{-1}(-x-\gamma'(\lambda)t)$ on the space-time half plane $\{(z,t) : z \in \mathbb{R}, t \in \mathbb{R}^+\}$. Then $\widetilde{N}_t^x$ is the set of the particles alive at time $t$ that never hit this barrier before time $t$. Now we show that $V_t^x(\lambda)$ is a martingale.

**Lemma 2.17.** *For any $x, y \in \mathbb{R}$ with $y > h^{-1}(-x)$, $\{V_t^x(\lambda), t \ge 0; \mathbb{P}_y\}$ is a martingale with respect to $\{\mathcal{F}_t : t \ge 0\}$.*

*Proof.* By stopping the lines of descent the first time they hit the barrier $\mathbf{\Gamma}^{(-x,\lambda)}$ we produce a random collection $\mathbf{C}(-x,\lambda)$ of individuals. Let $\sigma_u$ denote the first hitting time for each $u \in \mathbf{C}(-x,\lambda)$. Consider the stopping "line"

$$\mathcal{L}(t) = \{u \in \mathbf{C}(-x,\lambda) : \sigma_u \le t\} \cup \widetilde{N}_t^x.$$

Let $\mathcal{F}_{\mathcal{L}(t)}$ be the natural filtration generated by the spatial paths and the number of offspring of the individuals before hitting the stopping line $\mathcal{L}(t)$. By the strong Markov branching property of $\{Z_t, t \ge 0\}$ (see Jagers [28, Theorem 4.14], also see Dynkin [15, Theorem 1.5] for the corresponding property for superprocesses, where this property is called the special Markov property),

$$\mathbb{E}_y\left[\partial W_t(\lambda) + xW_t(\lambda) \mid \mathcal{F}_{\mathcal{L}(t)}\right] = V_t^x(\lambda).$$

Thus

$$\mathbb{E}_y V_t^x(\lambda) = \mathbb{E}_y\left[\partial W_t(\lambda) + xW_t(\lambda)\right] = e^{-\lambda y}\psi(y,\lambda)\left(x + h(y)\right).$$

Hence we have for $0 \le s \le t$,

$$\begin{aligned}
&\mathbb{E}_y\left[V_t^x(\lambda) \mid \mathcal{F}_s\right] \\
=& \sum_{v\in\widetilde{N}_s^x} e^{-\gamma(\lambda)s+\lambda(\gamma'(\lambda)s+\delta)}\mathbb{E}_y\Bigg[\sum_{u\in\widetilde{N}_t^x, u>v} e^{-\gamma(\lambda)(t-s)}e^{-\lambda(X_u(t)+\gamma'(\lambda)s+\delta)}\psi(X_u(t),\lambda) \\
&\qquad \times\left((x-\delta)+\gamma'(\lambda)(t-s)+(\gamma'(\lambda)s+\delta)+h(X_u(t))\right)\ \Bigg|\ \mathcal{F}_s\Bigg] \\
=& \sum_{v\in\widetilde{N}_s^x} e^{-\gamma(\lambda)s+\lambda(\gamma'(\lambda)s+\delta)}\mathbb{E}_{X_v(s)+\gamma'(\lambda)s+\delta}V_{t-s}^{x-\delta}(\lambda,v) \\
=& \sum_{v\in\widetilde{N}_s^x} e^{-\gamma(\lambda)s+\lambda(\gamma'(\lambda)s+\delta)}e^{-\lambda(X_v(s)+\gamma'(\lambda)s+\delta)}\psi(X_v(s),\lambda)\times(x-\delta+(\gamma'(\lambda)s+\delta)+h(X_v(s))) \\
=& \sum_{v\in\widetilde{N}_s^x} e^{-\gamma(\lambda)s-\lambda X_v(s)}\psi(X_v(s),\lambda)\left(x+\gamma'(\lambda)s+h(X_v(s))\right) \\
=& V_s^x(\lambda),
\end{aligned}$$

where $\delta$ is such that $(\delta+\gamma'(\lambda)s)\in\mathbb{Z}$, and in second equality, for each $v$, $V^{x-\delta}_{t-s}(\lambda,v)$ is the counterpart of $V^{x-\delta}_{t-s}$ for the BBMPE starting from $X_v(s)+\gamma'(\lambda)s+\delta$ and we used the periodicity of $h$ and $\psi$. So $\{V^x_t(\lambda),t\ge 0;\mathbb{P}_y\}$ is a martingale. □

## 3 Proof of Theorem 1.1

### 3.1 Measure change by the additive martingale

The spine decomposition theorem has been studied in many papers (for example, see [5], [8], [26], [35], and [39]). For any $u\in N_t$, define

$$\Xi_t(u,\lambda):=e^{-\gamma(\lambda)t-\lambda X_u(t)+m\int_0^t \mathbf{g}(X_u(s))\mathrm{d}s}\psi(X_u(t),\lambda),$$

which is similar to the martingale $\{\Xi_t(\lambda),\Pi_x\}$ for Brownian motion. Then we may rewrite $W_t(\lambda)$ as

$$W_t(\lambda)=\sum_{u\in N_t}\Xi_t(u,\lambda)e^{-m\int_0^t \mathbf{g}(X_u(s))\mathrm{d}s}.$$

Define a new probability measure $\mathbb{P}^\lambda_x$ by

$$\left.\frac{\mathrm{d}\mathbb{P}^\lambda_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t}=\frac{W_t(\lambda)}{W_0(\lambda)}. \tag{3.1}$$

Now we construct the space of Galton-Watson trees with a spine. Here we use the same notation as those in [31]. Let $(\mathcal{T},\mathcal{F},\mathcal{F}_t,\mathbb{P}_x)$ be the filtered probability space in which the BBMPE $\{Z_t:t\geqslant 0\}$ is defined. Let $\mathbb{T}$ be the space of Galton-Watson trees. A Galton-Watson tree $\tau\in\mathbb{T}$ is a point in the space of possible Ulam-Harris labels

$$\Omega=\emptyset\cup\bigcup_{n\in\mathbb{N}}(\mathbb{N})^n,$$

where $\mathbb{N}=\{1,2,3,\ldots\}$ such that

(i) $\emptyset\in\tau$ (the ancestor);

(ii) if $u,v\in\Omega$, $uv\in\tau$ implies $u\in\tau$;

(iii) for all $u\in\tau$, there exists $A_u\in\{0,1,2,\ldots\}$ such that for $j\in\mathbb{N}$, $uj\in\tau$ if and only if $1\le j\le 1+A_u$.

(Here $1+A_u$ is the number of offspring of $u$, and $A_u$ has the same distribution as $L$.)

Each particle $u\in\tau$ has a mark $(\eta_u,B_u)\in\mathbb{R}^+\times C(\mathbb{R}^+,\mathbb{R})$, where $\eta_u$ is the lifetime of $u$ and $B_u$ is the motion of $u$ relative to its birth position. Then the birth time of $u$ can be written as $b_u=\sum_{v<u}\eta_v$, the death time of $u$ is $d_u=\sum_{v\le u}\eta_v$ and the position of $u$ at time $t$ is given by $X_u(t)=\sum_{v<u}B_v(\eta_v)+B_u(t-b_u)$. We write $(\tau,B,\eta)$ as a short hand for the marked Galton-Watson tree $\{(u,\eta_u,B_u):u\in\tau\}$, and $\mathcal{T}=\{(\tau,B,\eta):\tau\in\mathbb{T}\}$. The $\sigma$-field $\mathcal{F}_t$ is generated by

$$\left\{\begin{array}{l}(u,A_u,\eta_u,\{B_u(s):s\in[0,\eta_u]\}:u\in\tau\text{ with }d_u\le t)\text{ and}\\(u,\{B_u(s):s\in[0,t-b_u]\}:u\in\tau\text{ with }t\in[b_u,d_u)):\tau\in\mathbb{T}\end{array}\right\}.$$

A spine is a distinguished genealogical line of descent from the ancestor. A spine will be written as $\xi=\{\xi_0=\emptyset,\xi_1,\xi_2,\ldots\}$, where $\xi_n\in\tau$ is the label of $\xi$'s node in the $n$th generation. We write $u\in\xi$ if $u=\xi_i$ for some $i\ge 0$. Now let

$$\widetilde{\mathcal{T}}=\{(\tau,B,\eta,\xi):\xi\subseteq\tau\in\mathbb{T}\}$$

be the space of marked trees in $\mathcal{T}$ with distinguished spine, $\xi$, let

$$\widetilde{\mathcal{F}}_t = \sigma\left(\mathcal{F}_t, \{(\xi : u \in \xi) : u \in N_t\}\right)$$

and $\widetilde{\mathcal{F}} = \cup_{t\geq 0}\widetilde{\mathcal{F}}_t$. The $\sigma$-field $\widetilde{\mathcal{F}}$ contains all the information about the marked tree and the nodes on the spine, and $\mathcal{F} = \cup_{t\geq 0}\mathcal{F}_t$ contains all the information about the marked tree but does not contain information about what nodes are on the spine. We denote by $\{X_\xi(t) : t \geq 0\}$ the spatial path followed by the spine $\xi$ and write $n = \{n_t : t \geq 0\}$ for the counting process of points of fission along the spine. Let $\mathcal{G} = \sigma(X_\xi(t) : t \geq 0)$ and $\widetilde{\mathcal{G}} = \sigma\left((X_\xi(t) : t \geq 0), (A_{\xi_k}, d_{\xi_k} : k \in \mathbb{N})\right)$. The $\sigma$-field $\mathcal{G}$ contains all the information about the motion of the spine, and the $\sigma$-field $\widetilde{\mathcal{G}}$ contains all the information about the spatial path of the spine, the fission times along the spine and the number of offspring born at these fission times.

Hardy and Harris [23] noticed that it is convenient to consider $\{\mathbb{P}_x\}$ as measures on the enlarged space $(\widetilde{\mathcal{T}}, \mathcal{F})$ rather than on $(\mathcal{T}, \mathcal{F})$. We extend the probability measures $\{\mathbb{P}_x\}$ to probability measures $\{\widetilde{\mathbb{P}}_x\}$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$. Under $\widetilde{\mathbb{P}}_x$, if $v$ is the particle in the $n$th generation on the spine, then for the next generation, the spine is chosen uniformly from the $1 + A_v$ offspring of $v$. Therefore, we have

$$\widetilde{\mathbb{P}}_x(u \in \xi) = \prod_{v<u} \frac{1}{1 + A_v}.$$

Define

$$\tilde{\zeta}_t = \sum_{u\in N_t} \prod_{v<u} (1 + A_v) \frac{\Xi_t(u, \lambda)}{\Xi_0(u, \lambda)} e^{-m\int_0^t \mathbf{g}(X_u(s))\mathrm{d}s} \mathbb{1}_{\{\xi_t=u\}}.$$

According to [23] or [39], $\{\tilde{\zeta}_t, \widetilde{\mathcal{F}}_t\}$ is a martingale and

$$\frac{W_t(\lambda)}{W_0(\lambda)} = \widetilde{\mathbb{P}}_x(\tilde{\zeta}_t \mid \mathcal{F}_t),$$

in other words, $W_t(\lambda)$ is the projection of $\tilde{\zeta}_t$ onto $\mathcal{F}_t$.

Now we define a probability measure $\widetilde{\mathbb{P}}_x^\lambda$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ by

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}_x^\lambda}{\mathrm{d}\widetilde{\mathbb{P}}_x}\right|_{\widetilde{\mathcal{F}}_t} = \tilde{\zeta}_t. \tag{3.2}$$

According to [23] or [39], under $\widetilde{\mathbb{P}}_x^\lambda$:

(i) the ancestor starts from $x$ and the spine process $\xi$ moves according to $\Pi_x^\lambda$, that is, the spine moves as a diffusion with infinitesimal generator given by (2.19);

(ii) given the trajectory $X_\xi$ of the spine, the branching rate is given by $(m + 1)\mathbf{g}(X_\xi(t))$;

(iii) at the fission time of node $v$ on the spine, the single spine particle is replaced by $1 + A_v$ offspring, with $A_v$ being independent identically distributed with common distribution $\{\tilde{p}_k : k \geq 0\}$, where $\tilde{p}_k = \frac{(k+1)p_k}{m+1}$;

(iv) the spine is chosen uniformly from the $1 + A_v$ offspring at the fission time of $v$;

(v) the remaining $A_v$ particles $O_v$ give rise to the independent subtrees $\{(\tau, B, \eta)_j^v\}$, $j \in O_v$, evolving as independent processes determined by the measure $\mathbb{P}_{X_v(d_v)}$ shifted to their point and time of creation, where $O_v$ is the set of particle labels with $A_v = |O_v|$.

Moreover, the measure $\mathbb{P}_x^\lambda$ defined by (3.1) satisfies

$$\mathbb{P}_x^\lambda = \widetilde{\mathbb{P}}_x^\lambda|_{\mathcal{F}}. \tag{3.3}$$

### 3.2 Proof of Theorem 1.1

*Proof of Theorem 1.1.* By Lemma 2.15, we know $\{W_t(\lambda), \mathbb{P}_x\}$ is a non-negative martingale. Let $\overline{W}(\lambda, x) = \limsup_{t\uparrow\infty} W_t(\lambda)$ so that $\overline{W}(\lambda, x) = W(\lambda, x)$ $\mathbb{P}_x$-a.s. By (3.3) and [14, Theorem 5.3.3],

$$\overline{W}(\lambda, x) = \infty,\ \widetilde{\mathbb{P}}_x^{\lambda}\text{-a.s.} \quad \Longleftrightarrow \quad \overline{W}(\lambda, x) = 0,\ \mathbb{P}_x\text{-a.s.}$$
$$\overline{W}(\lambda, x) < \infty,\ \widetilde{\mathbb{P}}_x^{\lambda}\text{-a.s.} \quad \Longleftrightarrow \quad \int \overline{W}(\lambda, x) \mathrm{d}\mathbb{P}_x = 1.$$

(i) When $\lambda > \lambda^*$, we have

$$W_t(\lambda) \geq C \exp\left\{-\lambda X_\xi(t) - \gamma(\lambda)t\right\} = C \exp\left\{-\lambda t \left(\frac{X_\xi(t)}{t} + \frac{\gamma(\lambda)}{\lambda}\right)\right\}. \tag{3.4}$$

By Lemmas 2.10 and 2.4, we have

$$\lim_{t\uparrow\infty} \frac{X_\xi(t)}{t} + \frac{\gamma(\lambda)}{\lambda} = -\gamma'(\lambda) + \frac{\gamma(\lambda)}{\lambda} < 0.$$

Thus $\overline{W}(\lambda, x) = \infty$, $\widetilde{\mathbb{P}}_x^{\lambda}$-a.s. and hence $W(\lambda, x) = 0$, $\mathbb{P}_x$-a.s.

(ii) According to the paragraph before Lemma 2.9, under $\widetilde{\mathbb{P}}_x^{\lambda^*}$, we have $\frac{X_\xi(t)}{t} \to -\gamma'(\lambda^*)$ as $t \to \infty$. Define hitting times:

$$T_k := \inf\left\{t \geq 0 : X_\xi(t) \leq x - k\right\}, \quad k \in \mathbb{N}.$$

Then $T_k$ is a $\widetilde{\mathbb{P}}_x^{\lambda^*}$-almost surely finite stopping time. Thanks to the strong Markov property and 1-periodicity, $\{T_k - T_{k-1}\}_{k\geq 1}$ are independent and identically distributed. Moreover, $T_k \to \infty$ as $k \to \infty$, $\widetilde{\mathbb{P}}_x^{\lambda^*}$-almost surely, and

$$\widetilde{\mathbb{P}}_x^{\lambda^*} T_1 = \lim_{k\to\infty} \frac{T_k}{k} = \lim_{k\to\infty} \frac{T_k}{x - X_\xi(T_k)} = \frac{1}{\gamma'(\lambda^*)}.$$

It follows that $T_k - \frac{k}{\gamma'(\lambda^*)}$ is a mean zero (non-trivial) random walk. Thus

$$\begin{aligned}\liminf_{t\to\infty} \left(X_\xi(t) + \frac{\gamma(\lambda^*)}{\lambda^*} t\right) &\leq \liminf_{k\to\infty} \left(X_\xi(T_k) + \gamma'(\lambda^*) T_k\right) \\ &= \liminf_{k\to\infty} \left(\frac{x-k}{\gamma'(\lambda^*)} + T_k\right) \gamma'(\lambda^*) = -\infty, \quad \widetilde{\mathbb{P}}_x^{\lambda^*}\text{-a.s.}\end{aligned}$$

where the last equality follows from the property of mean zero (non-trivial) random walk. The lower bound (3.4) also holds when $\lambda = \lambda^*$. Hence, $\overline{W}(\lambda^*, x) = \infty$, $\widetilde{\mathbb{P}}_x^{\lambda^*}$-a.s. and consequently $W(\lambda^*, x) = 0$, $\mathbb{P}_x$-a.s.

(iii) The proof of this part is similar to that of [31, Theorem 1 (iii)]. Suppose that $\lambda \in [0, \lambda^*)$. Then we have

$$\lim_{t\to\infty} \frac{X_\xi(t)}{t} + \frac{\gamma(\lambda)}{\lambda} = -\gamma'(\lambda) + \frac{\gamma(\lambda)}{\lambda} > 0, \tag{3.5}$$

in other words, $\{X_\xi(t) + \frac{\gamma(\lambda)}{\lambda} t\}$ is a diffusion with strictly positive drift. (When $\lambda = 0$, $\{\lambda X_\xi(t) + \gamma(\lambda)t\} = \{\gamma(0)t\}$ is a deterministic drift to the right and can be regarded as a degenerate diffusion. In this case, the proof below still works.)

Suppose $\mathbf{E}(L \log^+ L) = \infty$. Let $\{d_{\xi_i} : i \geq 0\}$ be the fission times along the spine. Note that

$$W_{d_{\xi_k}}(\lambda) \geq C A_{\xi_k} \exp\left\{-\lambda \left(X_\xi(d_{\xi_k}) + \frac{\gamma(\lambda)}{\lambda} d_{\xi_k}\right)\right\}, \tag{3.6}$$

where $\{A_{\xi_k} : k \geq 0\}$ are iid with distribution $\{\tilde{p}_k, k \geq 0\}$. The assumption $\mathbf{E}(L \log^+ L) = \infty$ implies that $\widetilde{\mathbb{P}}_x^\lambda \log^+ A_{\xi_k} = \infty$, and thus $\limsup_{k\to\infty} k^{-1} \log A_{\xi_k} = \infty$, $\widetilde{\mathbb{P}}_x^\lambda$-a.s. By (3.5) and (3.6), $\overline{W}(\lambda, x) = \infty$, $\widetilde{\mathbb{P}}_x^\lambda$-a.s. and hence $W(\lambda, x) = 0$, $\mathbb{P}_x$-a.s.

Suppose $\mathbf{E}(L \log^+ L) < \infty$. Recall that $\widetilde{\mathcal{G}}$ is the $\sigma$-field generated by the motion of the spine and the genealogy along the spine. By the spine decomposition and the martingale property of $W_t(\lambda)$, we have

$$\widetilde{\mathbb{P}}_x^\lambda \left( W_t(\lambda) \mid \widetilde{\mathcal{G}} \right) = \sum_{i=1}^{n_t} A_{\xi_{i-1}} e^{-\lambda X_\xi(d_{\xi_{i-1}}) - \gamma(\lambda) d_{\xi_{i-1}}} \psi(X_\xi(d_{\xi_{i-1}}), \lambda) + e^{-\lambda X_\xi(t) - \gamma(\lambda) t} \psi(X_\xi(t), \lambda).$$

The assumption $\mathbf{E}(L \log^+ L) < \infty$ implies that $\widetilde{\mathbb{P}}_x^\lambda \log^+ A_{\xi_k} < \infty$, and thus

$$\limsup_{k\to\infty} k^{-1} \log A_{\xi_k} = 0.$$

Since $\psi$ is bounded, by (3.5), we have

$$\limsup_{t\uparrow\infty} \widetilde{\mathbb{P}}_x^\lambda \left( W_t(\lambda) \mid \widetilde{\mathcal{G}} \right) < \infty \quad \widetilde{\mathbb{P}}_x^\lambda\text{-a.s.} \tag{3.7}$$

Hence $\liminf_{t\uparrow\infty} W_t(\lambda) < \infty$ $\widetilde{\mathbb{P}}_x^\lambda$-a.s. By [25] and (3.3), $W_t(\lambda)^{-1}$ is a non-negative $\widetilde{\mathbb{P}}_x^\lambda$-supermartingale, which implies that the limit of $W_t(\lambda)^{-1}$ exists as $t \to \infty$ $\widetilde{\mathbb{P}}_x^\lambda$-a.s. Hence $\lim_{t\uparrow\infty} W_t(\lambda) < \infty$ $\widetilde{\mathbb{P}}_x^\lambda$-a.s. Therefore for $\lambda \in [0, \lambda^*)$ and $\mathbf{E}(L \log^+ L) < \infty$, $W(\lambda, x)$ is a $L^1(\mathbb{P}_x)$-limit.

Now we prove (1.8). Noticing that

$$W_t(\lambda) = e^{-\gamma(\lambda)s} \sum_{u\in N_s} e^{-\gamma(\lambda)(t-s)} \sum_{v\in N_t, v>u} e^{-\lambda X_v(t)} \psi(X_u(t), \lambda),$$

we get that under $\mathbb{P}_x$,

$$W_t(\lambda) \overset{d}{=} \sum_{u\in N_s} e^{-\gamma(\lambda)s} W_{t-s}^{(u)}(\lambda, X_u(s)),$$

where $W_{t-s}^{(u)}(\lambda, X_u(s))$ is the additive martingale of the BBMPE starting from $X_u(s)$, and given $\mathcal{F}_s$, $\{W_{t-s}^{(u)}(\lambda, X_u(s)), u \in N_s\}$ are independent. Hence, letting $t \to \infty$, we have under $\mathbb{P}_x$

$$W(\lambda, x) \overset{d}{=} e^{-\gamma(\lambda)s} \sum_{u\in N_s} W^{(u)}(\lambda, X_u(s)), \tag{3.8}$$

where $W^{(u)}(\lambda, X_u(s))$ is the limit of the additive martingale for the BBMPE starting from $X_u(s)$, and given $\mathcal{F}_s$, $\{W^{(u)}(\lambda, X_u(s)) : u \in N_s\}$ are independent.

Define

$$p(x) := \mathbb{P}_x(W(\lambda, x) = 0), \quad x \in \mathbb{R}.$$

For any $y \in \mathbb{R}$, we define the following two stopping times $\tau_y^B$ and $\tau_y^Z$ with respect to Brownian motion and BBMPE, respectively:

$$\tau_y^B := \inf\{t \geq 0 : B_t = y\},$$
$$\tau_y^Z := \inf\{t \geq 0 : \exists u \in N_t \ s.t. \ X_u(t) = y\}.$$

At each fission time, there is at least one offspring. As a result, for any $M \geq 0$ and $x \in \mathbb{R}$,

$$\mathbb{P}_x(\tau_y^Z \leq M) \geq \Pi_x(\tau_y^B \leq M).$$

By the decomposition (3.8) and the strong Markov property, we have

$$(1 - p(x)) \geq (1 - p(y))\mathbb{P}_x(\tau_y^Z \leq M) \geq (1 - p(y))\Pi_x(\tau_y^B \leq M). \tag{3.9}$$

Since $\Pi_x(\tau_y^B < \infty) = 1$, letting $M \to \infty$, we obtain $p(x) \le p(y)$. By symmetry, we get

$$\mathbb{P}_x(W(\lambda, x) = 0) = \mathbb{P}_y(W(\lambda, y) = 0), \quad \forall x, y \in \mathbb{R}.$$

Let $p$ denote this common value. It follows from (3.8) that $p$ satisfies $p = \mathbb{E}_x p^{|N_s|}$ for any $s > 0$. Since $|N_s| \ge 1$ almost surely and $\mathbb{P}_y(|N_s| > 1) > 0$, $p = \mathbb{E}_y p^{|N_s|}$ implies $p = 0$ or 1. Since $W(\lambda, x)$ is an $L^1(\mathbb{P}_x)$-limit in this case, we have $p < 1$ and hence $p = 0$. This completes the proof. □

## 4 Proof of Theorem 1.3

The martingale $\{V_t^x(\lambda)\}$ will play an important role in the proof of the following result.

**Proposition 4.1.** *Suppose that $\lambda \ge \lambda^*$. Then $\partial W(\lambda, y) = \lim_{t\uparrow\infty} \partial W_t(\lambda)$ exists $\mathbb{P}_y$-almost surely in $[0, \infty)$. Furthermore, $\mathbb{P}_y(\partial W(\lambda, y) = 0) = 0$ or 1.*

*Proof.* Let $x \in \mathbb{R}$ be such that $y > h^{-1}(-x)$. Since $V_t^x(\lambda)$ is a non-negative martingale, it has an almost sure limit. Let $\gamma^{(-x,\lambda)}$ denote the event that the BBMPE remains entirely to the right of $\mathbf{\Gamma}^{(-x,\lambda)}$, where the barrier $\mathbf{\Gamma}^{(-x,\lambda)}$ is described by $z = h^{-1}(-x - \gamma'(\lambda)t)$ on the half plane $\{(z, t) : z \in \mathbb{R}, t \in \mathbb{R}^+\}$. On this event we have $V_t^x(\lambda) = \partial W_t(\lambda) + xW_t(\lambda)$. Hence, on $\gamma^{(-x,\lambda)}$, $\lim_{t\uparrow\infty}(\partial W_t(\lambda)+xW_t(\lambda))$ exists and equals $\lim_{t\uparrow\infty} V_t^x(\lambda) \ge 0$. Note that when $\lambda \ge \lambda^*$ we have $W(\lambda, y) = 0$ $\mathbb{P}_y$-almost surely. Therefore, we have $\lim_{t\uparrow\infty} V_t^x(\lambda) = \lim_{t\uparrow\infty} \partial W_t(\lambda)$ on $\gamma^{(-x,\lambda)}$.

Let $m_t := \min\{X_u(t) : u \in N_t\}$. Then

$$W_t(\lambda) = e^{-\gamma(\lambda)t} \sum_{u\in N_t} e^{-\lambda X_u(t)} \psi(X_u(t), \lambda) \ge c e^{-\gamma(\lambda)t - \lambda m_t},$$

where $c = \min_{z\in[0,1]} \psi(z, \lambda)$ is a positive constant. Since $W(\lambda^*, y) = 0$, we have $\lim_{t\uparrow\infty} e^{-\gamma(\lambda^*)t - \lambda^* m_t} = 0$, that is, $\lim_{t\uparrow\infty}(m_t + \gamma'(\lambda^*)t) = \infty$. Hence $\inf_{t\ge0}\{m_t + \gamma'(\lambda)t\} > -\infty$ $\mathbb{P}_y$-almost surely for all $\lambda \ge \lambda^*$. Therefore

$$\mathbb{P}_y(\gamma^{(-x,\lambda)}) \ge \mathbb{P}_y\left(\inf_{t\ge0}\{m_t + \gamma'(\lambda)t\} > -x + \max_{z\in[0,1]} \frac{\psi_\lambda(z,\lambda)}{\psi(z,\lambda)}\right) \uparrow 1 \quad \text{as } x \uparrow \infty. \tag{4.1}$$

Thus $\partial W(\lambda, y) = \lim_{t\uparrow\infty} \partial W_t(\lambda)$ exists $\mathbb{P}_y$-almost surely in $[0, \infty)$.

It remains to prove that $\partial W(\lambda, y)$ is either strictly positive or zero with probability one. Noticing that

$$\begin{aligned}\partial W_t(\lambda) =& e^{-\gamma(\lambda)s} \sum_{u\in N_s} e^{-\gamma(\lambda)(t-s)} \sum_{v\in N_t, v>u} e^{-\lambda X_v(t)} \\ & \times (\psi(X_u(t), \lambda)(\gamma'(\lambda)(t-s) + X_u(t)) - \psi_\lambda(X_u(t), \lambda) + \gamma'(\lambda)s\psi(X_u(t), \lambda)),\end{aligned}$$

we have under $\mathbb{P}_y$,

$$\partial W_t(\lambda) \overset{d}{=} e^{-\gamma(\lambda)s} \sum_{u\in N_s} \left(\partial W_{t-s}^{(u)}(\lambda, X_u(s)) + \gamma'(\lambda)sW_{t-s}^{(u)}(\lambda, X_u(s))\right),$$

where $W_{t-s}^{(u)}(\lambda, X_u(s))$ and $\partial W_{t-s}^{(u)}(\lambda, X_u(s))$ are the additive and derivative martingales for the BBMPE starting from $X_u(s)$, and given $\mathcal{F}_s$, $\{(W_{t-s}^{(u)}(\lambda, X_u(s)), \partial W_{t-s}^{(u)}(\lambda, X_u(s))) : u \in N_s\}$ are independent. Letting $t \to \infty$ and noticing that $W(\lambda, y) = 0$ $\mathbb{P}_y$-almost surely for $\lambda \ge \lambda^*$, we have under $\mathbb{P}_y$

$$\partial W(\lambda, y) \overset{d}{=} e^{-\gamma(\lambda)s} \sum_{u\in N_s} \partial W^{(u)}(\lambda, X_u(s)), \tag{4.2}$$

where $\partial W^{(u)}(\lambda, X_u(s))$ is the limit of the derivative martingale for the BBMPE starting from $X_u(s)$. Define

$$p(y) := \mathbb{P}_y(\partial W(\lambda, y) = 0), \quad y \in \mathbb{R}. \tag{4.3}$$

Thanks to (4.2), an argument similar to the one used in the proof of (1.8) shows that (3.9) still holds for $p(\cdot)$ defined by (4.3), and $\mathbb{P}_y(\partial W(\lambda, y) = 0) = \mathbb{P}_z(\partial W(\lambda, z) = 0)$ holds for any $y, z \in \mathbb{R}$. Let $p$ denote this common value. It follows from (4.2) that $p$ satisfies $p = \mathbb{E}_y p^{|N_s|}$ for any $s > 0$. Since $|N_s| \geq 1$ almost surely and $\mathbb{P}_y(|N_s| > 1) > 0$, $p = \mathbb{E}_y p^{|N_s|}$ implies $p = 0$ or $1$. This completes the proof. □

### 4.1 Measure change by $V$

We have shown in Lemma 2.17 that, for any $x, y \in \mathbb{R}$ with $y > h^{-1}(-x)$, $\{V_t^x(\lambda), t \geq 0; \mathbb{P}_y\}$ is a martingale. We now assume $x, y \in \mathbb{R}$ with $y > h^{-1}(-x)$ and use $V_t^x(\lambda)$ to define a probability measure $\mathbb{P}_y^{(x,\lambda)}$ on $(\mathcal{T}, \mathcal{F})$:

$$\left.\frac{\mathrm{d}\mathbb{P}_y^{(x,\lambda)}}{\mathrm{d}\mathbb{P}_y}\right|_{\mathcal{F}_t} = \frac{V_t^x(\lambda)}{V_0^x(\lambda)}. \tag{4.4}$$

According to [23] or [39], there exists a probability measure $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ such that

$$\mathbb{P}_y^{(x,\lambda)} = \widetilde{\mathbb{P}}_y^{(x,\lambda)}|_{\mathcal{F}}, \tag{4.5}$$

and under $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$:

(i) the ancestor starts from $y$ and the spine $\xi$ moves according to $\Pi_y^{(x,\lambda)}$.

(ii) given the trajectory $X_\xi$ of the spine, the branching rate is given by $(m+1)\mathbf{g}(X_\xi(t))$;

(iii) at the fission time of node $v$ on the spine, the single spine particle is replaced by $1 + A_v$ offspring, with $A_v$ being independent identically distributed as $\{\tilde{p}_k : k \geq 0\}$, where $\tilde{p}_k = \frac{(k+1)p_k}{m+1}$;

(iv) the spine is chosen uniformly from the $1 + A_v$ offspring at the fission time of $v$;

(v) the remaining $A_v$ particles $O_v$ give rise to the independent subtrees $\{(\tau, B, \eta)_j^v\}$, $j \in O_v$, and they evolve as independent processes determined by the measure $\mathbb{P}_{X_v(d_v)}$ shifted to their point and time of creation, where $O_v$ is the set of particle labels with $A_v = |O_v|$.

Define

$$R_t := -h(y) + \gamma'(\lambda)T(t) + h(X_\xi(T(t))), \quad t \geq 0.$$

By (2.29) and (2.33), $\{x + h(y) + R_t : t \geq 0\}$ is a standard Bessel-3 process started at $x + h(y)$, where

$$T(t) = \inf\{s \geq 0 : \langle M \rangle_s > t\} \text{ and } \langle M \rangle_t = \int_0^t [h'(X_\xi(s))]^2 \mathrm{d}s, \tag{4.6}$$

(see (2.32) and (2.31)).

### 4.2 Proof of Theorem 1.3

Assume $x, y \in \mathbb{R}$ satisfy $y > h^{-1}(-x)$. Let $V^x(\lambda) = \limsup_{t\uparrow\infty} V_t^x(\lambda)$ and using the fundamental measure theoretic result again, we have

$$V^x(\lambda) = \infty,\ \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.} \iff V^x(\lambda) = 0,\ \mathbb{P}_y\text{-a.s.} \tag{4.7}$$

$$V^x(\lambda) < \infty,\ \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.} \iff \int V^x(\lambda)\mathrm{d}\mathbb{P}_y = 1. \tag{4.8}$$

**Theorem 4.2.** *For $x, y \in \mathbb{R}$ satisfying $y > h^{-1}(-x)$, the almost sure limit $V^x(\lambda)$ has the following properties:*

*(i) If $\lambda > \lambda^*$ then $V^x(\lambda) = 0$ $\mathbb{P}_y$-almost surely.*

*(ii) If $\lambda = \lambda^*$ then $V^x(\lambda) = 0$ $\mathbb{P}_y$-almost surely or is an $L^1(\mathbb{P}_y)$-limit according to $\mathbf{E}(L(\log^+ L)^2) = \infty$ or $\mathbf{E}(L(\log^+ L)^2) < \infty$.*

*(iii) If $\lambda \in [0, \lambda^*)$ then $V^x(\lambda) = 0$, $\mathbb{P}_y$-almost surely or is an $L^1(\mathbb{P}_y)$-limit according to $\mathbf{E}(L \log^+ L) = \infty$ or $\mathbf{E}(L \log^+ L) < \infty$.*

*Proof.* Recall that $\tilde{p}_k = (k+1)p_k/(m+1)$. Suppose $q > 0$. A simple calculation shows that, for any fixed $c > 0$, $\mathbf{E}(L(\log^+ L)^q) < \infty$ if and only if

$$\sum_{n \geq 1} \widetilde{\mathbf{P}}(\log L > cn^{1/q}) < \infty,$$

where under $\widetilde{\mathbf{P}}$, $L$ has distribution $\{\tilde{p}_k : k \geq 0\}$. Therefore, if $\{A_n : n \geq 0\}$ is a sequence of independent copies of $L$ under $\widetilde{\mathbf{P}}$, then by the Borel-Cantelli lemma,

$$\limsup_{n \to \infty} n^{-1/q} \log A_n = \begin{cases} 0 & \text{if } \mathbf{E}\left(L(\log^+ L)^q\right) < \infty, \\ \infty & \text{if } \mathbf{E}\left(L(\log^+ L)^q\right) = \infty \end{cases}$$

$\widetilde{\mathbf{P}}$-almost surely.

(i) Suppose that $\lambda > \lambda^*$. By Lemma 2.4, $\gamma'(\lambda) > \gamma(\lambda)/\lambda$. Then by the definition of $V_t^x(\lambda)$ in (2.34),

$$\begin{aligned} V^x_{T(t)}(\lambda) \geq & e^{-\lambda\left(X_\xi(T(t)) + \frac{\gamma(\lambda)}{\lambda}T(t)\right)} \psi(X_\xi(T(t)), \lambda)\left(x + \gamma'(\lambda)T(t) + h(X_\xi(T(t)))\right) \\ = & e^{-\lambda\left(X_\xi(T(t)) + \gamma'(\lambda)T(t)\right) + \lambda\left(\gamma'(\lambda) - \frac{\gamma(\lambda)}{\lambda}\right)T(t)} \psi(X_\xi(T(t)), \lambda)\left(x + h(y) + R_t\right) \\ \geq & c(\lambda) e^{-\lambda R_t + \lambda\left(\gamma'(\lambda) - \frac{\gamma(\lambda)}{\lambda}\right)T(t)} \left(x + h(y) + R_t\right), \end{aligned}$$

where the constant $c(\lambda) := e^{-\lambda h(y)} \inf_{x \in \mathbb{R}} \{e^{-\lambda(x - h(x))}\psi(x, \lambda)\}$. Under $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$, $\{x + h(y) + R_t\}$ is a Bessel-3 process, and so for any $\epsilon > 0$, this process eventually grows no faster than $t^{1/2+\epsilon}$ and no slower than $t^{1/2-\epsilon}$. By Lemma 2.14, there exist two positive constants $c_1 \leq c_2$ such that $\langle M \rangle_t \in [c_1 t, c_2 t]$ and hence by (4.6), $\frac{t}{c_2} \leq T(t) \leq \frac{t}{c_1}$. Combining these with $\gamma'(\lambda) > \gamma(\lambda)/\lambda$, we get that

$$V^x(\lambda) = \limsup_{t \uparrow \infty} V^x_{T(t)}(\lambda) \geq c(\lambda) \limsup_{t \uparrow \infty} e^{-c\lambda t^{1/2+\epsilon} + \lambda(\gamma'(\lambda) - \frac{\gamma(\lambda)}{\lambda})\frac{t}{c_2}} t^{1/2-\epsilon} = \infty, \quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.}$$

Hence, by (4.7), $V^x(\lambda) = 0$, $\mathbb{P}_y$-almost surely.

(ii) Suppose that $\lambda = \lambda^*$ which, by Lemma 2.4, implies that $\gamma'(\lambda^*) = \gamma(\lambda^*)/\lambda^*$. We first consider the case that $\mathbf{E}(L(\log^+ L)^2) = \infty$. Recall that $d_{\xi_k}$ is the death time of the particle $\xi_k$ on the spine and $1 + A_{\xi_k}$ is the number of its offspring. We have

$$V^x_{d_{\xi_n}}(\lambda^*) \geq A_{\xi_n}\left(x + \gamma'(\lambda^*)d_{\xi_n} + h(X_\xi(d_{\xi_n}))\right) e^{-\gamma(\lambda^*)d_{\xi_n} - \lambda^* X_\xi(d_{\xi_n})} \psi(X_\xi(d_{\xi_n}), \lambda^*).$$

We only need to prove that, $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-almost surely,

$$\limsup_{n \to \infty} A_{\xi_n}\left(x + \gamma'(\lambda^*)d_{\xi_n} + h(X_\xi(d_{\xi_n}))\right) e^{-\gamma(\lambda^*)d_{\xi_n} - \lambda^* X_\xi(d_{\xi_n})} \psi(X_\xi(d_{\xi_n}), \lambda^*) = +\infty.$$

Define $v_n$ such that $T(v_n) = d_{\xi_n}$, that is, $\langle M \rangle_{d_{\xi_n}} = v_n$. Then

$$x + h(y) + R_{v_n} = x + \gamma'(\lambda^*)d_{\xi_n} + h(X_\xi(d_{\xi_n})).$$

It suffices to show that for any $M>0$, $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-almost surely,

$$\sum_{n=0}^{+\infty}\mathbb{1}_{\left\{A_{\xi_n}(x+h(y)+R_{v_n})e^{-\lambda^*\left(X_\xi(T(v_n))+\gamma'(\lambda^*)T(v_n)\right)}\psi(X_\xi(T(v_n)),\lambda^*)\geq M\right\}}=+\infty.$$

Since $\inf_{z\in\mathbb{R}}\psi(z,\lambda^*)>0$, it suffices to show that for any $M>0$,

$$\sum_{n=0}^{+\infty}\mathbb{1}_{\left\{A_{\xi_n}(x+h(y)+R_{v_n})e^{-\lambda^* R_{v_n}}\geq M\right\}}=+\infty\quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.} \tag{4.9}$$

Recall that $\mathcal{G}$ denotes the $\sigma$-field generated by $X_\xi$ (the spatial path of the spine). For any set $B\in\mathcal{B}[0,+\infty)\times\mathcal{B}(\mathbb{Z}_+)$, define

$$\varphi(B):=\#\{n:(v_n,A_{\xi_n})\in B\}. \tag{4.10}$$

We first show that, conditioned on $\mathcal{G}$, $\varphi$ is a Poisson random measure on $[0,+\infty)\times\mathbb{Z}_+$ with intensity $(m+1)\mathbf{g}(X_\xi(T(t)))\mathrm{d}T(t)\sum_{k\in\mathbb{Z}_+}\tilde{p}_k\delta_k(\mathrm{d}y)$. For simplicity, given $\mathcal{G}$, put $f(t)=\langle M\rangle_t=\int_0^t\left[h'(X_\xi(s))\right]^2\mathrm{d}s$. Then, it is known that $f(t)$ is a strictly increasing $C^1$-function and $f'(t)\in[c_1,c_2]$. Hence $T(t)=f^{-1}(t)$ and $T'(t)\in[1/c_2,1/c_1]$. Define

$$\widetilde{\varphi}(B):=\#\{n:(d_{\xi_n},A_{\xi_n})\in B\}.$$

Using the spine decomposition, it is easy to show that, conditioned on $\mathcal{G}$, $\widetilde{\varphi}$ is a Poisson random measure on $[0,+\infty)\times\mathbb{Z}_+$ with intensity $(m+1)\mathbf{g}(X_\xi(t))\mathrm{d}t\sum_{k\in\mathbb{Z}_+}\tilde{p}_k\delta_k(\mathrm{d}y)$. Note that, given the spatial path of the spine, $f(t)$ is a deterministic increasing function and $v_n=f(d_{\xi_n})$. It is not difficult to verify $\varphi$ satisfies the definition of Poisson random measure. Moreover, for any $D\subset\mathbb{Z}_+$, $\widetilde{\varphi}([0,t]\times D)=\varphi([0,f(t)]\times D)$. By making the change of variables $s=T(u)$, we have

$$\int_0^t(m+1)\mathbf{g}(X_\xi(s))\mathrm{d}s=\int_0^{f(t)}(m+1)\mathbf{g}(X_\xi(T(u)))\mathrm{d}T(u).$$

Hence, conditioned on $\mathcal{G}$, the intensity of $\varphi$ is

$$(m+1)\mathbf{g}(X_\xi(T(t)))\mathrm{d}T(t)\sum_{k\in\mathbb{Z}_+}\tilde{p}_k\delta_k(\mathrm{d}y).$$

Thus for any $t\in(0,+\infty)$, given $\mathcal{G}$,

$$N_t:=\#\left\{n:v_n\leq t,A_{\xi_n}(x+h(y)+R_{v_n})e^{-\lambda^* R_{v_n}}\geq M\right\}$$

is a Poisson random variable with parameter

$$\int_0^t(m+1)\mathbf{g}(X_\xi(T(s)))\sum_k\tilde{p}_k\mathbb{1}_{\left\{k(x+h(y)+R_s)e^{-\lambda^* R_s}\geq M\right\}}\mathrm{d}T(s).$$

Since $\min_{z\in\mathbb{R}}\mathbf{g}(z)>0$ and $T'(t)\in[1/c_2,1/c_1]$, to prove (4.9), it suffices to show that

$$\int_0^{+\infty}(m+1)\sum_k\tilde{p}_k\mathbb{1}_{\left\{k(x+h(y)+R_t)e^{-\lambda^* R_t}\geq M\right\}}\mathrm{d}t=+\infty,\quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.} \tag{4.11}$$

For any $c\in(0,+\infty)$, put

$$A_c:=\left\{\omega:\int_0^{+\infty}\sum_k\tilde{p}_k\mathbb{1}_{\left\{k(x+h(y)+R_t)e^{-\lambda^* R_t}\geq M\right\}}\mathrm{d}t\leq c\right\}. \tag{4.12}$$

Using arguments similar to those in the proof of [42, Theorem 1], we get that $\widetilde{\mathbb{P}}_y^{(x,\lambda)}(A_c) = 0$ (see Lemma 7.1 in the Appendix for a proof), which implies (4.11) holds. Therefore, we have $V^x(\lambda) = \infty$, $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-a.s. Hence $V^x(\lambda) = 0$, $\mathbb{P}_y$-a.s.

Now we consider the case that $\mathbf{E}(L(\log^+ L)^2) < \infty$. Recall that $\widetilde{\mathcal{G}}$ is the $\sigma$-field generated by the motion of the spine and the genealogy along the spine and $\{n_t : t \geq 0\}$ is the counting process of fission points along the spine. Using the spine decomposition and the martingale property of $V_t^x(\lambda^*)$, we have

$$\widetilde{\mathbb{P}}_y^{(x,\lambda)}(V_t^x(\lambda^*)|\widetilde{\mathcal{G}}) = \left(x + \gamma'(\lambda^*)t + X_\xi(t) - \frac{\psi_\lambda(X_\xi(t), \lambda^*)}{\psi(X_\xi(t), \lambda^*)}\right) e^{-\lambda^* X_\xi(t) - \gamma(\lambda^*)t} \psi(X_\xi(t), \lambda^*)$$
$$+ \sum_{k=0}^{n_t-1} A_{\xi_k} \left(x + \gamma'(\lambda^*)d_{\xi_k} + X_\xi(d_{\xi_k}) - \frac{\psi_\lambda(X_\xi(d_{\xi_k}), \lambda^*)}{\psi(X_\xi(d_{\xi_k}), \lambda^*)}\right) e^{-\lambda^* X_\xi(d_{\xi_k}) - \gamma(\lambda^*)d_{\xi_k}} \psi(X_\xi(d_{\xi_k}), \lambda^*).$$

Next we show that $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-almost surely,

$$\sum_{n=0}^{+\infty} A_{\xi_n} \left(x + \gamma'(\lambda^*)d_{\xi_n} + X_\xi(d_{\xi_n}) - \frac{\psi_\lambda(X_\xi(d_{\xi_n}), \lambda^*)}{\psi(X_\xi(d_{\xi_n}), \lambda^*)}\right) e^{-\lambda^* X_\xi(d_{\xi_n}) - \gamma(\lambda^*)d_{\xi_n}} \psi(X_\xi(d_{\xi_n}), \lambda^*) \tag{4.13}$$
$$< +\infty.$$

Using an argument similar to the one above, it is equivalent to prove

$$\sum_{n=0}^{+\infty} A_{\xi_n} \left(x - \frac{\phi_\lambda(y, \lambda^*)}{\phi(y, \lambda^*)} + R_{v_n}\right) e^{-\lambda^* R_{v_n}} < +\infty, \quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.}$$

For simplicity, we use $\tilde{x}$ to denote $x - \frac{\phi_\lambda(y,\lambda^*)}{\phi(y,\lambda^*)}$ and we will show

$$\sum_{n=0}^{+\infty} A_{\xi_n} \left(\tilde{x} + R_{v_n}\right) e^{-\lambda^* R_{v_n}} < +\infty \quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.}$$

Choose any $\epsilon \in (0, \lambda^*)$,

$$\begin{aligned}\sum_{n=0}^{+\infty} A_{\xi_n} \left(\tilde{x} + R_{v_n}\right) e^{-\lambda^* R_{v_n}} &= \sum_{n=0}^{+\infty} A_{\xi_n} \left(\tilde{x} + R_{v_n}\right) e^{-\lambda^* R_{v_n}} \mathbf{1}_{\{A_{\xi_n} \leq e^{\epsilon R_{v_n}}\}} \\ &\quad + \sum_{n=0}^{+\infty} A_{\xi_n} \left(\tilde{x} + R_{v_n}\right) e^{-\lambda^* R_{v_n}} \mathbf{1}_{\{A_{\xi_n} > e^{\epsilon R_{v_n}}\}} \\ &=: \mathrm{I} + \mathrm{II}.\end{aligned}$$

We will prove that both I and II are finite $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-almost surely.

Recall that $\varphi$ is defined by (4.10). We rewrite I as

$$\mathrm{I} = \int_{[0,+\infty)\times\mathbb{Z}_+} (\tilde{x} + R_s) y e^{-\lambda^* R_s} \mathbb{1}_{\{y \leq e^{\epsilon R_s}\}} \varphi(\mathrm{d}s \times \mathrm{d}y).$$

Since $\widetilde{\mathbb{P}}_y^{(x,\lambda)}(\mathrm{I}) = \widetilde{\mathbb{P}}_y^{(x,\lambda)}(\widetilde{\mathbb{P}}_y^{(x,\lambda)}(I|\mathcal{G}))$, by the compensation formula of Poisson random measures,

$$\widetilde{\mathbb{P}}_y^{(x,\lambda)}(\mathrm{I}) = \widetilde{\mathbb{P}}_y^{(x,\lambda)}\left(\int_0^{+\infty}(m+1)\mathbf{g}(X_\xi(T(s)))(\tilde{x}+R_s)e^{-\lambda^* R_s}\sum_k \tilde{p}_k k \mathbb{1}_{\{k\le e^{\epsilon R_s}\}}\mathrm{d}T(s)\right) \tag{4.14}$$

$$\lesssim \widetilde{\mathbb{P}}_y^{(x,\lambda)}\left(\int_0^{+\infty}(\tilde{x}+R_s)e^{-\lambda^* R_s}\sum_k \tilde{p}_k k \mathbb{1}_{\{k\le e^{\epsilon R_s}\}}\mathrm{d}s\right)$$

$$\le \sum_k \tilde{p}_k \int_0^{+\infty}\widetilde{\mathbb{P}}_y^{(x,\lambda)}\left((\tilde{x}+R_s)e^{-(\lambda^*-\epsilon)R_s}\mathbb{1}_{\{R_s\ge \epsilon^{-1}\log^+ k\}}\right)\mathrm{d}s.$$

In the display above and also in the sequel, we write $A \lesssim B$ when there exists a constant $c>0$, such that $A \le cB$. Under $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$, $\tilde{x}+R_s$ is a Bessel-3 process, which has the same distribution as $|W_t+\hat{x}|$ under $\mathbf{P}_w$, where $(W_t, \mathbf{P}_w)$ is a 3-dimensional standard Brownian motion starting from 0 and $\hat{x}$ is a point in $\mathbb{R}^3$ with norm $\tilde{x}$. Thus

$$\begin{aligned}
\widetilde{\mathbb{P}}_y^{(x,\lambda)}(\mathrm{I}) &\lesssim \sum_k \tilde{p}_k \int_0^{+\infty}\mathbf{P}_w\left(|W_s+\hat{x}|\,e^{-(\lambda^*-\epsilon)|W_s+\hat{x}|}\mathbb{1}_{\left\{|W_s+\hat{x}|\ge \epsilon^{-1}\log^+ k+\tilde{x}\right\}}\right)\mathrm{d}s\\
&\lesssim \sum_k \tilde{p}_k \int_{\left\{|y+\hat{x}|\ge \epsilon^{-1}\log^+ k+\tilde{x}\right\}}|y+\hat{x}|e^{-(\lambda^*-\epsilon)|y+\hat{x}|}\mathrm{d}y\int_0^{+\infty}s^{-3/2}e^{-|y|^2/2\pi s}\,\mathrm{d}s\\
&= \sum_k \tilde{p}_k \int_{\left\{|y+\hat{x}|\ge \epsilon^{-1}\log^+ k+\tilde{x}\right\}}\frac{|y+\hat{x}|}{|y|}e^{-(\lambda^*-\epsilon)|y+\hat{x}|}\mathrm{d}y\int_0^{+\infty}t^{-1/2}e^{-t/2\pi}\mathrm{d}t\\
&\lesssim \sum_k \tilde{p}_k \int_{\left\{|y+\hat{x}|\ge \epsilon^{-1}\log^+ k+\tilde{x}\right\}}\frac{|y+\hat{x}|}{|y|}e^{-(\lambda^*-\epsilon)|y+\hat{x}|}\mathrm{d}y\\
&\le \sum_k \tilde{p}_k \int_{\left\{|y|\ge \epsilon^{-1}\log^+ k\right\}}\frac{|y|+\tilde{x}}{|y|}e^{-(\lambda^*-\epsilon)(|y|-\tilde{x})}\mathrm{d}y.
\end{aligned}$$

Using spherical coordinates in the last integral, we get

$$\widetilde{\mathbb{P}}_y^{(x,\lambda)}(\mathrm{I}) \lesssim \sum_k \tilde{p}_k \int_{\epsilon^{-1}\log^+ k}^{+\infty}(r^2+\tilde{x}r)e^{-(\lambda^*-\epsilon)r}\mathrm{d}r < +\infty,$$

and therefore, $\widetilde{\mathbb{P}}_y^{(x,\lambda)}(\mathrm{I}<+\infty)=1$.

On the other hand, similar calculation yields

$$\begin{aligned}
&\widetilde{\mathbb{P}}_y^{(x,\lambda)}\left(\sum_{n=0}^{+\infty}\mathbb{1}_{\left\{A_{\xi_n}>e^{\epsilon R_{v_n}}\right\}}\right)\\
=&(1+m)\sum_k \tilde{p}_k \widetilde{\mathbb{P}}_y^{(x,\lambda)}\left(\int_0^{+\infty}\mathbf{g}(X_\xi(T(s)))\mathbb{1}_{\{\tilde{x}+R_s<\epsilon^{-1}\log^+ k+\tilde{x}\}}\mathrm{d}T(s)\right)\\
\lesssim&\sum_k \tilde{p}_k \int_0^{+\infty}\mathbf{P}_w\left(|W_s+\hat{x}|<\epsilon^{-1}\log^+ k+\tilde{x}\right)\mathrm{d}s\\
\lesssim&\sum_k \tilde{p}_k \int_{\left(|y+\hat{x}|<\epsilon^{-1}\log^+ k+\tilde{x}\right\}}\mathrm{d}y\int_0^{+\infty}s^{-3/2}\mathrm{e}^{-|y|^2/2\pi s}\mathrm{d}s\\
\lesssim&\sum_k \tilde{p}_k \int_{\left\{|y|\le \epsilon^{-1}\log^+ k+2\tilde{x}\right\}}|y|^{-1}\,\mathrm{d}y\\
\lesssim&\sum_k \tilde{p}_k\left(\epsilon^{-1}\log^+ k+2\tilde{x}\right)^2.
\end{aligned}$$

The assumption that $\mathbf{E}(L(\log^+ L)^2) < +\infty$ implies that $\sum_{k\in\mathbb{Z}_+} \tilde{p}_k(\log^+ k)^2 < +\infty$, which implies that the right side of the last inequality is finite. Hence, $\sum_{n=0}^{+\infty} \mathbb{1}_{\left\{A_{\xi_n} > e^{\epsilon R_{v_n}}\right\}} < +\infty$, $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-almost surely, that is, II is a sum of finitely many terms. It follows that $\widetilde{\mathbb{P}}_y^{(x,\lambda)}(\mathrm{II} < +\infty) = 1$. Hence (4.13) is valid, which implies that

$$\limsup_{t\uparrow\infty} \widetilde{\mathbb{P}}_y^{(x,\lambda)}(V_t^x(\lambda^*) \,|\, \widetilde{\mathcal{G}}) < \infty \quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.}$$

By Fatou's lemma, $\liminf_{t\uparrow\infty} V_t^x(\lambda^*) < \infty$, $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-a.s. It follows from (4.4) and (4.5) that $V_t^x(\lambda^*)^{-1}$ is a $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-supermartingale and therefore has a limit $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-almost surely. It follows that

$$\limsup_{t\uparrow\infty} V_t^x(\lambda^*) = \liminf_{t\uparrow\infty} V_t^x(\lambda^*) < \infty, \quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.}$$

Hence, by (4.8), $V^x(\lambda^*)$ is an $L^1(\mathbb{P}_y)$ limit when $\mathbf{E}(L(\log^+ L)^2) < \infty$.

(iii) Now suppose $\lambda \in [0, \lambda^*)$ and $\mathbf{E}(L\log^+ L) = \infty$. By Lemma 2.4, $\gamma'(\lambda) < \gamma(\lambda)/\lambda$. We have

$$\begin{aligned} V^x_{d_{\xi_n}}(\lambda) &\geq A_{\xi_n}\,(x + \gamma'(\lambda)d_{\xi_n} + h(X_\xi(d_{\xi_n})))\, e^{-\gamma(\lambda)d_{\xi_n} - \lambda X_\xi(d_{\xi_n})}\psi(X_\xi(d_{\xi_k}), \lambda) \\ &\gtrsim A_{\xi_n}\left(x - \frac{\phi_\lambda(y,\lambda)}{\phi(y,\lambda)} + R_{v_n}\right) e^{-\lambda R_{v_n} - \lambda(\frac{\gamma(\lambda)}{\lambda} - \gamma'(\lambda))T(v_n)}. \end{aligned}$$

Since a Bessel-3 process eventually grows no faster than $t^{1/2+\epsilon}$, the leading order in the exponent is $-\lambda(\frac{\gamma(\lambda)}{\lambda} - \gamma'(\lambda))T(v_n)$. The assumption $\mathbf{E}(L\log^+ L) = \infty$ implies that $\limsup_{n\to\infty} n^{-1}\log A_{\xi_n} = \infty$, $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$-a.s. Thus we have $V^x(\lambda) = \infty$, $\mathbb{P}_y^{(x,\lambda)}$-a.s. and hence $V^x(\lambda) = 0$ $\mathbb{P}_y$-a.s.

When $\lambda \in [0, \lambda^*)$ and $\mathbf{E}(L\log^+ L) < \infty$, using the spine decomposition, we have

$$\begin{aligned} &\widetilde{\mathbb{P}}_y^{(x,\lambda)}(V_t^x(\lambda)|\widetilde{\mathcal{G}}) \\ &= (x + \gamma'(\lambda)t + h(X_\xi(t)))\, e^{-\lambda(X_\xi(t) + \gamma'(\lambda)t)} e^{-\lambda(\frac{\gamma(\lambda)}{\lambda} - \gamma'(\lambda))t}\psi(X_\xi(t), \lambda) \\ &\quad + \sum_{k=0}^{n_t - 1} A_{\xi_k}\,(x + \gamma'(\lambda)d_{\xi_k} + h(X_\xi(d_{\xi_k})))\, e^{-\lambda(X_\xi(d_{\xi_k}) + \gamma'(\lambda)d_{\xi_k}) - \lambda(\frac{\gamma(\lambda)}{\lambda} - \gamma'(\lambda))d_{\xi_k}}\psi(X_\xi(d_{\xi_k}), \lambda). \end{aligned}$$

Similar to the case of part (ii), to prove that $V^x(\lambda)$ is an $L^1(\mathbb{P}_y)$-limit, it suffices to show

$$\limsup_{t\uparrow\infty} \widetilde{\mathbb{P}}_y^{(x,\lambda)}(V_t^x(\lambda^*) \,|\, \widetilde{\mathcal{G}}) < \infty, \quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.}$$

which is equivalent to prove

$$\sum_{n=0}^{\infty} A_{\xi_n}\left(x - \frac{\phi_\lambda(y,\lambda)}{\phi(y,\lambda)} + R_{v_n}\right) e^{-\lambda R_{v_n} - \lambda(\frac{\gamma(\lambda)}{\lambda} - \gamma'(\lambda))T(v_n)} < \infty, \quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.} \tag{4.15}$$

The assumption $\mathbf{E}(L\log^+ L) < \infty$ implies that $\widetilde{\mathbb{P}}_y^{(x,\lambda)} \log^+ A_{\xi_n} < \infty$, and hence

$$\limsup_{n\to\infty} n^{-1}\log A_{\xi_n} = 0.$$

Therefore, using $T(v_n) = d_{\xi_n}$ and the proof of (3.7), we have

$$\sum_{n=0}^{\infty} A_{\xi_n} e^{-\lambda(\frac{\gamma(\lambda)}{\lambda} - \gamma'(\lambda))T(v_n)} < \infty, \quad \widetilde{\mathbb{P}}_y^{(x,\lambda)}\text{-a.s.}$$

which implies (4.15). Hence, $V^x(\lambda)$ is an $L^1(\mathbb{P}_y)$-limit. □

*Proof of Theorem 1.3.* By Lemma 2.16, $\{\partial W_t(\lambda), \mathbb{P}_y\}$ is a martingale. Suppose $\lambda \geq \lambda^*$. The case $\lambda \leq -\lambda^*$ follows by symmetry. For a given $x \in \mathbb{R}$, let $y \in \mathbb{R}$ be such that $y > h^{-1}(-x)$. By Proposition 4.1, we know that on the event $\gamma^{(-x,\lambda)}$, $V^x(\lambda) = \partial W(\lambda, y)$ $\mathbb{P}_y$-almost surely. Combining this with Theorem 4.2, we get that $\partial W(\lambda, y) = 0$ almost surely on the event $\gamma^{(-x,\lambda)}$ when $\lambda > \lambda^*$ or when $\lambda = \lambda^*$ and $\mathbf{E}(L(\log^+ L)^2) = \infty$. By (4.1), we know that

$$\mathbb{P}_y(\gamma^{(-x,\lambda)}) \to 1 \quad \text{as } x \to \infty.$$

Hence $\partial W(\lambda, y) = 0$ $\mathbb{P}_y$-almost surely when $\lambda > \lambda^*$ or when $\lambda = \lambda^*$ and $\mathbf{E}(L(\log^+ L)^2) = \infty$.

Now we deal with the case of $\lambda = \lambda^*$ and $\mathbf{E}(L(\log^+ L)^2) < \infty$. Let $z > x$, then the monotonicity of $h$ implies that $y > h^{-1}(-x) > h^{-1}(-z)$. By (2.34), we get that $V_t^x(\lambda) \leq V_t^z(\lambda)$ and hence $V^x(\lambda) \leq V^z(\lambda)$ $\mathbb{P}_y$-almost surely. So on the event $\gamma^{(-z,\lambda)}$, $V^x(\lambda) \leq \partial W(\lambda, y)$ $\mathbb{P}_y$-almost surely. Letting $z \to \infty$, we have $V^x(\lambda) \leq \partial W(\lambda, y)$ $\mathbb{P}_y$-almost surely. It follows from Proposition 4.1 that $\mathbb{P}_y(\partial W(\lambda, y) = 0) = 0$ or $1$. Therefore, when $V^x(\lambda)$ is an $L^1(\mathbb{P}_y)$-limit, we have $\partial W(\lambda, y) \in (0, \infty)$. So Theorem 4.2 implies Theorem 1.3. ☐

## 5 Proof of Theorem 1.4

It was proved analytically in [1, Theorem 1.2] and [20, Proposition 1.2] that pulsating traveling waves exist if and only if $|\nu| \geq \nu^*$. In this section, we will use probabilistic methods to prove the existence in the supercritical case $|\nu| > \nu^*$ and critical case $|\nu| = \nu^*$, and the non-existence in the subcritical case $|\nu| < \nu^*$.

### 5.1 Existence in the supercritical case ($\nu > \nu^*$)

In this subsection, we consider the case $\nu > \nu^*$ and $\mathbf{E}(L \log^+ L) < \infty$. By (2.6) and Lemma 2.4, $\frac{\gamma(\lambda)}{\lambda}$ strictly decreases from $+\infty$ to $\nu^*$ on $[0, \lambda^*]$. Therefore, for any $\nu > \nu^*$ there exists a unique $\lambda \in (0, \lambda^*)$ such that $\nu = \frac{\gamma(\lambda)}{\lambda}$. Recall that the additive martingale $W_t(\lambda)$ is defined in (1.7). As stated in Remark 1.2, we have that for $y - x \in \mathbb{Z}$,

$$(W(\lambda, y), \mathbb{P}_y) \overset{d}{=} (e^{-\lambda(y-x)} W(\lambda, x), \mathbb{P}_x).$$

We know that (3.8) holds, that is, under $\mathbb{P}_x$,

$$W(\lambda, x) \overset{d}{=} e^{-\gamma(\lambda)s} \sum_{u \in N_s} W^{(u)}(\lambda, X_u(s)),$$

where $W^{(u)}(\lambda, X_u(s))$ is the limit of the additive martingale for the BBMPE starting from $X_u(s)$, and given $\mathcal{F}_s$, $\{W^{(u)}(\lambda, X_u(s)) : u \in N_s\}$ are independent.

**Theorem 5.1.** *Suppose $|\nu| > \nu^*$ and $\mathbf{E}(L \log^+ L) < \infty$. Define*

$$\mathbf{u}(t, x) := \mathbb{E}_x \exp\left\{-e^{\gamma(\lambda)t} W(\lambda, x)\right\}, \tag{5.1}$$

*where $|\lambda| \in (0, \lambda^*)$ is such that $\nu = \frac{\gamma(\lambda)}{\lambda}$. Then $\mathbf{u}$ is a pulsating traveling wave with speed $\nu$.*

*Proof.* We assume that $\lambda \geq 0$. The case $\lambda < 0$ can be analyzed by symmetry. By (3.8) and the Markov property, we have, for any $t \geq s \geq 0$,

$$\begin{aligned}\mathbf{u}(t,x) &= \mathbb{E}_x \exp\left\{-e^{\gamma(\lambda)t}W(\lambda,x)\right\} = \mathbb{E}_x \exp\left\{-e^{\gamma(\lambda)t}e^{-\gamma(\lambda)s}\sum_{u\in N_s}W^{(u)}(\lambda,X_u(s))\right\} \\ &= \mathbb{E}_x\left(\mathbb{E}_x\left[\exp\big\{-e^{\gamma(\lambda)(t-s)}\sum_{u\in N_s}W^{(u)}(\lambda,X_u(s))\big\}\,\Big|\,\mathcal{F}_s\right]\right) \\ &= \mathbb{E}_x\prod_{u\in N_s}\mathbb{E}_{X_u(s)}\exp\left\{-e^{\gamma(\lambda)(t-s)}W^{(u)}(\lambda,X_u(s))\right\} \\ &= \mathbb{E}_x\prod_{u\in N_s}\mathbf{u}(t-s,X_u(s)).\end{aligned}$$

In particular, setting $s=t$, we have

$$\mathbf{u}(t,x) = \mathbb{E}_x\prod_{u\in N_t}\mathbf{u}(0,X_u(t)).$$

For convenience, let $f(x) = \mathbf{u}(0,x)$. The following argument is similar to the one used in [37]. Let $T$ be the first fission time of the initial particle. Then the expectation $\mathbb{E}_x\prod_{u\in N_t}f(X_u(t))$ in the display above can be split into two pieces, according to whether $T\leq t$ or not. Conditioned on the first fission time of the initial particle and on the number of its offspring, we have

$$\begin{aligned}\mathbf{u}(t,x) &= \Pi_x\left[f(B_t),\,T>t\right] + \int_0^t\int_R\sum_k p_k\mathbf{u}(t-s,y)^{k+1}\Pi_x\left[T\in \mathrm{d}s,\,B_s\in\mathrm{d}y\right] \\ &= \Pi_x\left[e^{-\int_0^t\mathbf{g}(B_s)\mathrm{d}s}f(B_t)\right] + \int_0^t\Pi_x\left[\mathbf{g}(B_s)e^{-\int_0^s\mathbf{g}(B_r)\mathrm{d}r}\mathbf{f}(\mathbf{u}(t-s,B_s))\right]\mathrm{d}s.\end{aligned}$$

By [16, Lemma 1.5 on p. 1211], the above equation can be written as

$$\mathbf{u}(t,x) = \Pi_x\left[f(B_t)\right] + \Pi_x\int_0^t\mathbf{g}(B_s)\left[\mathbf{f}(\mathbf{u}(t-s,B_s)) - \mathbf{u}(t-s,B_s)\right]\mathrm{d}s. \tag{5.2}$$

By (5.1), $\mathbf{u}$ is bounded. Using the display above, a routine argument (for details, see Appendix B) shows that $\mathbf{u}(t,x)$ satisfies

$$\frac{\partial\mathbf{u}}{\partial t} = \frac{1}{2}\frac{\partial^2\mathbf{u}}{\partial x^2} + \mathbf{g}\cdot(\mathbf{f}(\mathbf{u}) - \mathbf{u}).$$

Moreover, since $\frac{\gamma(\lambda)}{\lambda}=\nu$, by (1.9), we have

$$\begin{aligned}\mathbf{u}\left(t+\frac{1}{\nu},x\right) &= \mathbb{E}_x\exp\left\{-e^{\gamma(\lambda)(t+\frac{1}{\nu})}W(\lambda,x)\right\} = \mathbb{E}_x\exp\left\{-e^{\gamma(\lambda)t+\lambda}W(\lambda,x)\right\} \\ &= \mathbb{E}_{x-1}\exp\left\{-e^{\gamma(\lambda)t}W(\lambda,x-1)\right\} = \mathbf{u}(t,x-1).\end{aligned}$$

In order to prove that $\mathbf{u}(t,x)$ is a pulsating traveling wave, it remains to show that

$$\lim_{x\to-\infty}\mathbf{u}(t,x)=0,\quad \lim_{x\to+\infty}\mathbf{u}(t,x)=1.$$

Let $\lfloor x\rfloor$ denote the integer part of $x$. By (1.9),

$$\lim_{x\to+\infty}\mathbf{u}(t,x) = \lim_{x\to\infty}\mathbb{E}_{x-\lfloor x\rfloor}\exp\left\{-e^{\gamma(\lambda)t}e^{-\lambda\lfloor x\rfloor}W(\lambda,x-\lfloor x\rfloor)\right\}.$$

Since $\lim_{x\to+\infty}e^{-\lambda\lfloor x\rfloor}=0$ and $y=x-\lfloor x\rfloor\in[0,1)$, we have

$$e^{-\lambda n}W(\lambda,y)\to 0,\quad \mathbb{P}_y\text{-almost surely as } n\to\infty.$$

It follows from the bounded dominated convergence theorem that for fixed $t \geq 0$,

$$\lim_{n\to\infty} \mathbf{u}(t, y+n) = \mathbb{E}_y \lim_{n\to\infty} \exp\left\{-e^{\gamma(\lambda)t}e^{-\lambda n}W(\lambda, y)\right\} = 1.$$

For any $y \in [0,1]$, let $f_n(y) = \mathbf{u}(t, y+n)$ and thus $f_n(y) = \mathbb{E}_y \exp\left\{-e^{\gamma(\lambda)t}e^{-\lambda n}W(\lambda, y)\right\}$. Then $f_n(y) \leq f_{n+1}(y)$ and $f_n(y) \to 1$ as $n \to \infty$ for any $y \in [0,1]$. By Dini's theorem, we have $\lim_{n\to\infty} f_n(y) = 1$ uniformly for $y \in [0,1]$, that is,

$$\lim_{x\to+\infty} \mathbf{u}(t, x) = 1.$$

By (1.8) and (1.9), we have $\lim_{x\to-\infty} \mathbf{u}(t, x) = 0$. Therefore $\mathbf{u}$ defined by (5.1) is a pulsating traveling wave. □

## 5.2 Existence in the critical case ($\nu = \nu^*$)

Recall that the derivative martingale is defined by (1.10). Using the 1-periodicity of $\mathbf{g}(\cdot)$ and $\psi(\cdot, \lambda)$, and the fact $W(\lambda, y) = 0$ $\mathbb{P}_y$-almost surely for $\lambda \geq \lambda^*$, we get that for $y - z \in \mathbb{Z}$,

$$(\partial W(\lambda, y), \mathbb{P}_y) \stackrel{d}{=} (e^{-\lambda(y-z)}\partial W(\lambda, z), \mathbb{P}_z).$$

Recall that we have under $\mathbb{P}_y$,

$$\partial W(\lambda, y) \stackrel{d}{=} e^{-\gamma(\lambda)s} \sum_{u\in N_s} \partial W^{(u)}(\lambda, X_u(s)).$$

An argument similar to the one used in Section 5.1 leads to the following result.

**Theorem 5.2.** *Suppose* $|\nu| = \nu^*$ *and* $\mathbf{E}(L(\log^+ L)^2) < \infty$. *Define*

$$\mathbf{u}(t, x) := \mathbb{E}_x\left(\exp\left\{-e^{\gamma(\lambda^*)t}\partial W(\lambda^*, x)\right\}\right).$$

*Then* $\mathbf{u}$ *is a pulsating traveling wave with speed* $\nu^*$, *and*

$$\mathbf{u}(t, x) = \mathbb{E}_x\left(\exp\left\{-e^{\gamma(\lambda^*)t}\partial W(-\lambda^*, x)\right\}\right)$$

*is a pulsating traveling wave with speed* $-\nu^*$.

*Proof.* We assume that $\lambda \geq 0$. The case $\lambda < 0$ can be analyzed by symmetry. The proof of $\mathbf{u}(t, x)$ being a pulsating traveling wave is similar to the proof of Theorem 5.1. The decomposition (4.2) implies $\mathbf{u}(t, x)$ satisfies the F-KPP equation (1.3). For the derivative martingale, we also have for $y - x \in \mathbb{Z}$,

$$(\partial W(\lambda^*, y), \mathbb{P}_y) \stackrel{d}{=} (e^{-\lambda^*(y-x)}\partial W(\lambda^*, x), \mathbb{P}_x).$$

It follows that

$$\mathbf{u}\left(t + \frac{1}{\nu^*}, x\right) = \mathbf{u}(t, x-1).$$

By Theorem 1.3, we obtain $\mathbb{P}_y(\partial W(\lambda^*, y) = 0) = 0$ when $\mathbf{E}(L(\log^+ L)^2) < \infty$. Therefore,

$$\lim_{x\to-\infty} \mathbf{u}(t, x) = 0, \quad \lim_{x\to+\infty} \mathbf{u}(t, x) = 1.$$

So $\mathbf{u}(t, x)$ is a pulsating traveling wave with speed $\nu^*$. □

### 5.3 Proof of Theorem 1.4

The following result about the extremes of BBMPE is a consequence of Theorem 1.1 and (1.8).

**Lemma 5.3.** *Let* $\widetilde{m}_t := \max\{X_u(t) : u \in N_t\}$ *and* $m_t := \min\{X_u(t) : u \in N_t\}$*. If* $\mathbf{E}(L\log^+ L) < \infty$*, then for any* $x \in \mathbb{R}$*,*

$$\lim_{t\uparrow\infty} \frac{\widetilde{m}_t}{t} = \nu^* \quad \textit{and} \quad \lim_{t\uparrow\infty} \frac{m_t}{t} = -\nu^*, \quad \mathbb{P}_x\textit{-a.s.}$$

*Proof.* We first show that

$$\limsup_{t\uparrow\infty} \frac{-m_t}{t} \le \nu^*.$$

If this were not true, there would exist $\lambda > \lambda^*$ such that

$$\limsup_{t\uparrow\infty} \frac{-m_t}{t} > \frac{\gamma(\lambda)}{\lambda} > \nu^*.$$

Hence,

$$W_t(\lambda) \ge e^{-\lambda m_t - \gamma(\lambda)t}\psi(m_t,\lambda) = e^{\lambda t\left(\frac{-m_t}{t} - \frac{\gamma(\lambda)}{\lambda}\right)}\psi(m_t,\lambda).$$

Then we have

$$\limsup_{t\uparrow\infty} W_t(\lambda) \ge \limsup_{t\uparrow\infty} e^{\lambda t\left(\frac{-m_t}{t} - \frac{\gamma(\lambda)}{\lambda}\right)}\psi(m_t,\lambda) = +\infty,$$

which contradicts Theorem 1.1.

Next we show that

$$\liminf_{t\uparrow\infty} \frac{-m_t}{t} \ge \nu^*.$$

For any small $\delta, \epsilon > 0$, let $\lambda = \lambda^* - \delta$. By the mean value theorem, there exists $\tilde{\lambda} \in (\lambda - \epsilon, \lambda)$ with

$$\gamma(\lambda) - \gamma(\lambda - \epsilon) = \gamma'(\tilde{\lambda})\epsilon. \tag{5.3}$$

For any fixed $\lambda$ and $\lambda - \epsilon$, there exist $C_1, C_2 > 0$ such that $C_1 \le \psi(x,\lambda), \psi(x,\lambda-\epsilon) \le C_2$ for any $x \in \mathbb{R}$. Using an argument similar to that of [32, Corollary 3.2], we get that

$$\begin{aligned}
&\limsup_{t\uparrow\infty} e^{-\gamma(\lambda)t} \sum_{u\in N_t} e^{-\lambda X_u(t)}\psi(X_u(t),\lambda)\mathbf{1}_{\{X_u(t)\ge(-\gamma'(\tilde{\lambda})+\epsilon)t\}} \\
&\le \limsup_{t\uparrow\infty} C_2 e^{-\gamma(\lambda)t} \sum_{u\in N_t} e^{-(\lambda-\epsilon)X_u(t)} e^{-\epsilon X_u(t)} \mathbf{1}_{\{X_u(t)\ge(-\gamma'(\tilde{\lambda})+\epsilon)t\}} \\
&\le \limsup_{t\uparrow\infty} C_2 e^{-\gamma(\lambda)t} \sum_{u\in N_t} e^{-(\lambda-\epsilon)X_u(t)} e^{-\epsilon(-\gamma'(\tilde{\lambda})+\epsilon)t} \\
&= \limsup_{t\uparrow\infty} C_2 e^{-(\gamma(\lambda)-\gamma'(\tilde{\lambda})\epsilon)t} \sum_{u\in N_t} e^{-(\lambda-\epsilon)X_u(t)} e^{-\epsilon^2 t} \\
&\le \limsup_{t\uparrow\infty} \frac{C_2}{C_1} e^{-\gamma(\lambda-\epsilon)t} \sum_{u\in N_t} e^{-(\lambda-\epsilon)X_u(t)}\psi(X_u(t),\lambda-\epsilon) e^{-\epsilon^2 t} \\
&= \limsup_{t\uparrow\infty} \frac{C_2}{C_1} e^{-\epsilon^2 t} W_t(\lambda-\epsilon) = 0,
\end{aligned}$$

where in the last inequality we used (5.3). Therefore,

$$\lim_{t\uparrow\infty} e^{-\gamma(\lambda)t} \sum_{u\in N_t} e^{-\lambda X_u(t)}\psi(X_u(t),\lambda)\mathbf{1}_{\{X_u(t)<(-\gamma'(\tilde{\lambda})+\epsilon)t\}} = W(\lambda,x), \quad \mathbb{P}_x\text{-a.s.}$$

By (1.8), $\mathbb{P}_x(W(\lambda,x) = 0) = 0$. Thus the previous limit implies that

$$\liminf_{t\uparrow\infty} \mathbb{1}_{\{\exists\, u\in N_t : X_u(t)<(-\gamma'(\tilde{\lambda})+\epsilon)t\}} > 0.$$

This yields

$$\liminf_{t\uparrow\infty}\frac{-m_t}{t}\geq \gamma'(\tilde{\lambda})-\epsilon.$$

Since $\epsilon,\delta$ are arbitrary and $\gamma'$ is continuous, we obtain $\liminf_{t\uparrow\infty}\frac{-m_t}{t}\geq\gamma'(\lambda^*)=\nu^*$. Thus

$$\lim_{t\uparrow\infty}\frac{m_t}{t}=-\nu^*.$$

Since $\gamma(\cdot)$ is an even function, using an argument similar as above, we can easily get that $\lim_{t\uparrow\infty}\frac{\widetilde{m}_t}{t}=\nu^*$. □

*Proof of Theorem 1.4.* (i) follows from Theorems 5.1. (ii) follows from Theorems 5.2. Now we prove (iii). If the conclusion were false, let $\mathbf{u}(t,x)$ denote the pulsating traveling wave with speed $\nu<\nu^*$. By the uniqueness of solutions of initial value problem

$$\frac{\partial\mathbf{u}}{\partial t}=\frac{1}{2}\frac{\partial^2\mathbf{u}}{\partial x^2}+\mathbf{g}\cdot(\mathbf{f}(\mathbf{u})-\mathbf{u}),$$

with initial value $\mathbf{u}(0,x)$, we have

$$\mathbf{u}(t,x)=\mathbb{E}_x\prod_{u\in N_t}\mathbf{u}(0,X_u(t)).$$

Noting that $\mathbf{u}(t+\frac{1}{\nu},x)=\mathbf{u}(t,x-1)$, we get that for $\nu t\in\mathbb{N}$,

$$\mathbf{u}(0,x)=\mathbf{u}(t,x+\nu t)=\mathbb{E}_{x+\nu t}\prod_{u\in N_t}\mathbf{u}(0,X_u(t))=\mathbb{E}_x\prod_{u\in N_t}\mathbf{u}(0,X_u(t)+\nu t),$$

where the last equality follows from the periodicity. Since $0\leq\mathbf{u}(t,x)\leq 1$, by the dominated convergence theorem, we have

$$\begin{aligned}\mathbf{u}(0,x)&=\lim_{t\to\infty,\nu t\in\mathbb{N}}\mathbb{E}_x\prod_{u\in N_t}\mathbf{u}(0,X_u(t)+\nu t)=\mathbb{E}_x\lim_{t\to\infty,\nu t\in\mathbb{N}}\prod_{u\in N_t}\mathbf{u}(0,X_u(t)+\nu t)\\&\leq\mathbb{E}_x\lim_{t\to\infty,\nu t\in\mathbb{N}}\mathbf{u}(0,m_t+\nu t)=0,\end{aligned}$$

here we used $\lim_{t\uparrow\infty}(m_t+\nu t)=-\infty$ $\mathbb{P}_x$-almost surely, which follows from Lemma 5.3. This leads to a contradiction. □

## 6 Extensions

The problem studied in this paper can be generalized to a more general setup. We can consider the case where the inhomogeneity is not just in the branching rate, but also in the spatial motion, just as in Lubetzky et al. [34, Section 5]. We still assume that $L$ does not depend on the spatial location.

In this section, we consider a branching diffusion in a periodic environment, where both the branching rate and underlying dynamics are spatially dependent. Each particle moves as a symmetric diffusion $X=\{X_t,t\geq 0\}$ and, it produces a random number, $1+L$, of offspring with a branching rate $\mathbf{g}\in C^1(\mathbb{R})$.

More precisely, let $\{X_t,t\geq 0;\Pi_x\}$ be a diffusion process with infinitesimal generator

$$\mathcal{L}=\rho(x)^{-1}\frac{\partial}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial}{\partial x}\right),\tag{6.1}$$

where $\rho$ and $\sigma$ are positive and 1-periodic $C^1$ functions. In other words, $\{X_t,t\geq 0\}$ satisfies

$$\mathrm{d}X_t=\mu(X_t)\mathrm{d}t+\sigma(X_t)\mathrm{d}W_t,\quad X_0=x,\tag{6.2}$$

where

$$\mu(x) = \sigma'(x)\sigma(x) + \frac{\rho'(x)}{\rho(x)}\frac{\sigma^2(x)}{2},$$

and $\{W_t\}$ is a standard Brownian motion. Recall that $L$ has distribution $\{p_k, k \geq 0\}$ and $m := \sum_{k\geq 0} kp_k < \infty$.

In this case, we need to reformulate the eigen-problem. For every $\lambda \in \mathbb{R}$, let $\gamma(\lambda)$ and $\psi(\cdot, \lambda)$ be the principal eigenvalue and the corresponding positive eigenfunction of the periodic problem: for all $x \in \mathbb{R}$,

$$\begin{aligned} \frac{\sigma^2(x)}{2}\psi_{xx} + (\mu(x) - \lambda\sigma^2(x))\psi_x + \left(\frac{\lambda^2\sigma^2(x)}{2} - \lambda\mu(x) + m\mathbf{g}(x)\right)\psi = \gamma(\lambda)\psi, \\ \psi(x+1, \lambda) = \psi(x, \lambda). \end{aligned} \tag{6.3}$$

We normalize $\psi(\cdot, \lambda)$ such that $\int_0^1 \psi(x, \lambda)\rho(x)\mathrm{d}x = 1$.

## 6.1 Properties of principal eigenvalue and martingales

First, we will show that Lemma 2.3 still holds for branching diffusion in a periodic environment.

*Proof of Lemma 2.3.* The analyticity and convexity of $\gamma$ are proved in [20, Lemma 2.1]. Since g is 1-periodic and continuous, we can assume that $0 < \alpha \leq \mathbf{g}(x) \leq \beta < \infty$ for all $x \in \mathbb{R}$.

We first prove (2.3). Assume $\lambda > 0$. Since $\psi(\cdot, \lambda)$ is 1-periodic and continuous, there exists $x_0 \in [0, 1]$ such that $\psi(\cdot, \lambda)$ attains its minimum at $x_0$. Using $\psi_x(x_0, \lambda) = 0$ and $\psi_{xx}(x_0, \lambda) \geq 0$, we get that

$$\gamma(\lambda)\psi(x_0, \lambda) \geq \left(\frac{\lambda^2\sigma^2(x_0)}{2} - \lambda\mu(x_0) + m\mathbf{g}(x_0)\right)\psi(x_0, \lambda).$$

Hence

$$\gamma(\lambda) \geq \frac{\lambda^2}{2}\min_{x\in[0,1]}\sigma^2(x) - \lambda\|\mu\|_\infty + m\alpha.$$

Similarly, by considering the maximum value, we get that

$$\gamma(\lambda) \leq \frac{\lambda^2}{2}\max_{x\in[0,1]}\sigma^2(x) + \lambda\|\mu\|_\infty + m\beta.$$

Thus

$$\gamma(\lambda) \in \left[\frac{\lambda^2}{2}\min_{x\in[0,1]}\sigma^2(x) - \lambda\|\mu\|_\infty + m\alpha, \frac{\lambda^2}{2}\max_{x\in[0,1]}\sigma^2(x) + \lambda\|\mu\|_\infty + m\beta\right].$$

It follows that $\frac{\gamma(\lambda)}{\lambda} \to \infty$ both as $\lambda \to 0^+$ and as $\lambda \to \infty$, and hence

$$\nu^* = \frac{\gamma(\lambda^*)}{\lambda^*} = \min_{\lambda>0}\frac{\gamma(\lambda)}{\lambda}$$

exists. The proofs of (2.4) and the uniqueness of $\lambda^*$ are the same as those in Lemma 2.3.

Now we will show that $\gamma(\cdot)$ is an even function. Let $\psi(x, \lambda)$ be the positive eigenfunction corresponding to the eigenvalue $\gamma(\lambda)$ and define $\phi(x) := \phi(x, \lambda) = e^{-\lambda x}\psi(x, \lambda)$. A direct calculation shows that $\phi$ satisfies

$$\rho(x)^{-1}\frac{\partial}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial}{\partial x}\right)\phi + m\mathbf{g}(x)\phi = \gamma(\lambda)\phi, \tag{6.4}$$

that is,

$$\frac{\sigma^2(x)}{2}\phi_{xx} + \mu(x)\phi_x + m\mathbf{g}(x)\phi = \gamma(\lambda)\phi. \tag{6.5}$$

Let $\bar{\psi}(x) = \psi(x, -\lambda)$ be the positive eigenfunction corresponding to the eigenvalue $\gamma(-\lambda)$ and define $\bar{\phi}(x) = e^{\lambda x}\bar{\psi}(x)$, then $\bar{\phi}(x)$ satisfies

$$\begin{aligned} \rho(x)^{-1}\frac{\partial}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial}{\partial x}\right)\bar{\phi} + m\mathbf{g}(x)\bar{\phi} = \gamma(-\lambda)\bar{\phi}, \\ \bar{\phi}(x+1) = e^{\lambda}\bar{\phi}(x). \end{aligned} \tag{6.6}$$

Multiplying (6.6) by $\phi$ and integrating over (0,1) with respect to the measure $\rho(x)\mathrm{d}x$, we get that

$$\begin{aligned} \gamma(-\lambda)\int_0^1 \bar{\phi}\phi\rho\mathrm{d}x &= \int_0^1 \phi\frac{\partial}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial}{\partial x}\right)\bar{\phi} + m\mathbf{g}(x)\phi\bar{\phi}\mathrm{d}x \\ &= -\int_0^1 \frac{\partial\phi}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial\bar{\phi}}{\partial x}\right)\mathrm{d}x + \int_0^1 m\mathbf{g}(x)\phi\bar{\phi}\mathrm{d}x \\ &= -\int_0^1 \frac{\partial\bar{\phi}}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial\phi}{\partial x}\right)\mathrm{d}x + \int_0^1 m\mathbf{g}(x)\phi\bar{\phi}\mathrm{d}x \\ &= \int_0^1 \bar{\phi}\frac{\partial}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial}{\partial x}\right)\phi + m\mathbf{g}(x)\phi\bar{\phi}\mathrm{d}x \\ &= \gamma(\lambda)\int_0^1 \bar{\phi}\phi\rho\mathrm{d}x, \end{aligned}$$

where in the second equality we used integration by parts, the fact

$$\phi(x)\frac{\partial\bar{\phi}(x)}{\partial x} = e^{-\lambda x}\psi(x)\frac{\partial}{\partial x}(e^{\lambda x}\bar{\psi}(x)) = e^{-\lambda x}\psi(x)\left(\lambda e^{\lambda x}\bar{\psi}(x) + e^{\lambda x}\bar{\psi}_x(x)\right) = (\lambda\bar{\psi} + \bar{\psi}_x)\psi,$$

and the periodicity of $\rho, \sigma, \psi, \bar{\psi}$. Since $\phi, \bar{\phi}, \rho > 0$, we obtain that $\gamma(\lambda) = \gamma(-\lambda)$.

Since $\gamma$ is convex and even, we have that $\gamma(0)$ is the minimum of $\gamma$. Hence, $\gamma(\lambda) \geq \gamma(0) > 0$ and $\nu^* > 0$. □

With Lemma 2.3 in hand, one can immediately get that Lemma 2.4 holds in the present general case. In order to prove Lemma 2.5 and Lemma 2.7, we need to use $\{X_t\}$ to substitute $\{B_t\}$. Let $\widetilde{\psi}(\cdot, \lambda)$ be the positive eigenfunction of (6.3) with $\widetilde{\psi}(0, \lambda) = 1$. Define $\widetilde{\phi}(x, \lambda) = e^{-\lambda x}\widetilde{\psi}(x, \lambda)$, then $\widetilde{\phi}$ satisfies (6.4). Note that the argument of [9] for Brownian motion and the Laplacian still works for any symmetric diffusion and its generator. Hence Lemma 2.2 holds for our branching diffusion in periodic environments. Thus $\widetilde{\phi}$ has the probabilistic representation

$$\widetilde{\phi}(x, \lambda) = \Pi_x\left[e^{-\lambda X_\tau}e^{\int_0^\tau (m\mathbf{g}(X_t) - \gamma(\lambda))\mathrm{d}t}\right], \quad x \in [0, 2],$$

where $\{X_t\}$ has infinitesimal generator (6.1) and $\tau = \inf\{t > 0 : X_t \notin (0, 2)\}$. Using arguments similar to those in the proof of Lemma 2.5, we have

$$\widetilde{\phi}_\lambda(x, \lambda) = -\Pi_x\left[X_\tau e^{-\lambda X_\tau}e^{\int_0^\tau (m\mathbf{g}(X_t) - \gamma(\lambda))\mathrm{d}t} + \gamma'(\lambda)\int_0^\tau e^{\int_0^s (m\mathbf{g}(X_t) - \gamma(\lambda))\mathrm{d}t}\widetilde{\phi}(X_s, \lambda)\mathrm{d}s\right].$$

Repeating the argument of Lemma 2.5, we get that $\widetilde{\phi}_\lambda(\cdot, \lambda) \in C^2(\mathbb{R})$, and $\widetilde{\phi}_\lambda$ satisfies

$$\rho(x)^{-1}\frac{\partial}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial}{\partial x}\right)\widetilde{\phi}_{\lambda xx} + (m\mathbf{g}(x) - \gamma(\lambda))\widetilde{\phi}_\lambda = \gamma'(\lambda)\widetilde{\phi}.$$

By defining $\widetilde{c}(\lambda) = \int_0^1 \widetilde{\psi}(x,\lambda)\rho(x)\mathrm{d}x$ and $\psi(x,\lambda) = \widetilde{c}^{-1}(\lambda)\widetilde{\psi}(x,\lambda)$, we get $\int_0^1 \psi(x,\lambda)\rho(x)dx = 1$.

Define

$$\Xi_t(\lambda) := e^{-\gamma(\lambda)t - \lambda X_t + m\int_0^t \mathbf{g}(X_s)\mathrm{d}s}\psi(X_t,\lambda).$$

Using Itô's formula, one can easily check that $\{\Xi_t(\lambda), t \geq 0\}$ is a $\Pi_x$-martingale. Since $\frac{\Xi_t(\lambda)}{\Xi_0(\lambda)}$ is a non-negative martingale of mean 1, we can define a probability measure $\Pi_x^\lambda$ by

$$\left.\frac{\mathrm{d}\Pi_x^\lambda}{\mathrm{d}\Pi_x}\right|_{\mathcal{F}_t^X} = \frac{\Xi_t(\lambda)}{\Xi_0(\lambda)}, \tag{6.7}$$

where $\{\mathcal{F}_t^X : t \geq 0\}$ is the natural filtration of $\{X_t\}$. Similar to the proof of Lemma 2.8, we have

$$\begin{aligned}\frac{\Xi_t(\lambda)}{\Xi_0(\lambda)} &= \exp\left\{\int_0^t \left(\frac{\psi_x}{\psi} - \lambda\right)\mathrm{d}X_s + \int_0^t \left(\frac{\psi_{xx}\psi - \psi_x^2}{2\psi^2}\sigma^2 + m\mathbf{g} - \gamma(\lambda)\right)\mathrm{d}s\right\} \\ &= \exp\left\{\int_0^t \left(\frac{\psi_x}{\psi} - \lambda\right)\sigma(X_s)\mathrm{d}W_s - \frac{1}{2}\int_0^t \left(\frac{\psi_x}{\psi} - \lambda\right)^2 \sigma^2(X_s)\mathrm{d}s\right\},\end{aligned}$$

where in the second equality we used (6.2) and (6.3). Then by Girsanov's theorem, under $\Pi_x^\lambda$, $\{X_t\}$ satisfies

$$\mathrm{d}X_t = \left(\mu(X_t) + \left(\frac{\psi_x(X_t,\lambda)}{\psi(X_t,\lambda)} - \lambda\right)\sigma^2(X_t)\right)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x. \tag{6.8}$$

So under $\Pi_x^\lambda$, $\{X_t\}$ has infinitesimal generator

$$(\mathcal{A}f)(x) = \rho(x)^{-1}\frac{\partial}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial f(x)}{\partial x}\right) + \left(\sigma^2(x)\frac{\psi_x(x,\lambda)}{\psi(x,\lambda)} - \lambda\right)\frac{\partial f(x)}{\partial x}. \tag{6.9}$$

For convenience, in this section, we always assume that $\{Y_t, t \geq 0; \Pi_x^\lambda\}$ is a diffusion with infinitesimal generator (6.9). One can show that, under $\Pi_x^\lambda$, $\{Y_t/t\}$ satisfies a large deviation principle with good rate function

$$I(z) = \gamma^*(z) + \{\lambda z + \gamma(\lambda)\} = \sup_{\eta\in\mathbb{R}}\{\eta z - \gamma(\eta)\} + \{\lambda z + \gamma(\lambda)\}.$$

In order to prove Lemma 2.10, it suffices to show that

$$P(x,T) := \Pi_x^\lambda\left(\max_{t\in[0,1]}|Y_t - Y_0 + \gamma'(\lambda)t| > T\epsilon\right) \leq C_2(\lambda)e^{-\delta T/2}, \quad x \in [0,1],$$

where $\delta$ is defined by (2.23). By (6.8), we have

$$P(x,T) \leq \Pi_x^\lambda\left(\max_{t\in[0,1]}|\int_0^t \sigma(Y_s)\mathrm{d}W_s| > T\epsilon - C\right),$$

where $C$ is a constant depending only on $\lambda$, since the functions $\mu$, $\sigma$ and $\psi$ are bounded. Thanks to the Dambis-Dubins-Schwarz theorem, there exists a standard Brownian motion $\widetilde{B}_t$ such that $\widetilde{B}_{\langle A\rangle_t} = \int_0^t \sigma(Y_s)\mathrm{d}W_s$ with $\langle A\rangle_t = \int_0^t \sigma^2(Y_s)\mathrm{d}s$. Let $\sigma_{\max} = \max_{x\in[0,1]}\sigma(x)$, then we have

$$P(x,T) \leq \Pi_x^\lambda\left(\max_{t\in[0,1]}|\widetilde{B}_{\langle A\rangle_t}| > T\epsilon - C\right) \leq \Pi_x^\lambda\left(\max_{t\in[0,\sigma_{\max}^2]}|\widetilde{B}_t| > T\epsilon - C\right) \leq C_2(\lambda)e^{-\delta T/2},$$

where the last inequality follows from the tail probability of normal distribution. This gives Lemma 2.10 in the present situation.

Define

$$\Upsilon_t(\lambda) := e^{-\gamma(\lambda)t-\lambda X_t+m\int_0^t \mathbf{g}(X_s)\mathrm{d}s}\left(\psi(X_t,\lambda)(\gamma'(\lambda)t+X_t)-\psi_\lambda(X_t,\lambda)\right).$$

Notice that

$$\Upsilon_t(\lambda) = e^{-\gamma(\lambda)t+m\int_0^t \mathbf{g}(X_s)\mathrm{d}s}\left(\gamma'(\lambda)t\phi(X_t,\lambda)-\phi_\lambda(X_t,\lambda)\right). \tag{6.10}$$

Using (6.2), (6.5) and (6.10), repeating the argument of Lemma 2.11, we show that $\{\Upsilon_t(\lambda), t\geq 0\}$ is a $\Pi_x$-martingale. As in Section 2,

$$\Lambda_t^{(x,\lambda)} := e^{-\gamma(\lambda)t-\lambda X_t+\int_0^t m\mathbf{g}(X_s)\mathrm{d}s}\psi(X_t,\lambda)\left(x+\gamma'(\lambda)t+h(X_t)\right)\mathbf{1}_{\{\tau_\lambda^x>t\}}$$

is a non-negative martingale, where $h(x)=x-\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}$. For $x,y\in\mathbb{R}$ with $y>h^{-1}(-x)$, we have a new probability measure $\Pi_y^{(x,\lambda)}$ defined by

$$\left.\frac{\mathrm{d}\Pi_y^{(x,\lambda)}}{\mathrm{d}\Pi_y}\right|_{\mathcal{F}_t^X} = \frac{\Lambda_t^{(x,\lambda)}}{\Lambda_0^{(x,\lambda)}}.$$

Next, we prove Lemma 2.12 in the general case.

*Proof of Lemma 2.12.* Since $\phi(x,\lambda)=e^{-\lambda x}\psi(x,\lambda)$ satisfies (6.5) and $\gamma(\cdot)$ is even on $\mathbb{R}$, similar to (2.27), we have for all $x>y$ and $\lambda>0$,

$$\phi(x,\lambda)=\Pi_x\left[\phi(y,\lambda)e^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\right], \tag{6.11}$$

where $\tau_y=\inf\{t>0: X_t=y\}$ under $\Pi_x$. Therefore

$$\ln\phi(x,\lambda)=\ln\phi(y,\lambda)+\ln\Pi_x\left[e^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\right]. \tag{6.12}$$

Fix a $\lambda_0>0$. Since $\gamma(\lambda)>\gamma(\lambda_0)$ for all $\lambda>\lambda_0$, we have for $\lambda>\lambda_0$,

$$\begin{aligned}\tau_y e^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}} &\leq Me^{(\gamma(\lambda)-\gamma(\lambda_0))\tau_y}e^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\\ &= Me^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda_0))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\end{aligned}$$

for some large $M$. Since $\Pi_x\left[e^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda_0))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\right]=\frac{\phi(x,\lambda_0)}{\phi(y,\lambda_0)}$, we can differentiate (6.12) with respect to $\lambda$ and change the order of differentiation and expectation to get that for $\lambda>\lambda_0$,

$$\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}=\frac{\phi_\lambda(y,\lambda)}{\phi(y,\lambda)}-\frac{\Pi_x\left[\gamma'(\lambda)\tau_y e^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\right]}{\Pi_x\left[e^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t}\mathbb{1}_{\{\tau_y<\infty\}}\right]}. \tag{6.13}$$

Since $\lambda_0>0$ is arbitrary, (6.13) holds for all $\lambda>0$.

Now we show that $\Pi_x^\lambda(\tau_y)=\frac{1}{\gamma'(\lambda)}(h(x)-h(y))$, where $\tau_y=\inf\{t>0: Y_t=y\}$ and $\{Y_t, t\geq 0; \Pi_x^\lambda\}$ is a diffusion with infinitesimal generator (6.9). By (6.7) and the optional stopping theorem, we have

$$\begin{aligned}\Pi_x^\lambda((\tau_y\wedge t)\mathbb{1}_{\{\tau_y\leq t\}}) &= \Pi_x\left((\tau_y\wedge t)\mathbb{1}_{\{\tau_y\leq t\}}\frac{\Xi_t(\lambda)}{\Xi_0(\lambda)}\right)=\Pi_x\left((\tau_y\wedge t)\mathbb{1}_{\{\tau_y\leq t\}}\Pi_x\left[\frac{\Xi_t(\lambda)}{\Xi_0(\lambda)}\Big|\mathcal{F}_{\tau_y\wedge t}^X\right]\right)\\ &= \Pi_x\left((\tau_y\wedge t)\mathbb{1}_{\{\tau_y\leq t\}}\frac{\Xi_{\tau_y\wedge t}(\lambda)}{\Xi_0(\lambda)}\right)=\Pi_x\left(\tau_y\mathbb{1}_{\{\tau_y\leq t\}}\frac{\Xi_{\tau_y}(\lambda)}{\Xi_0(\lambda)}\right).\end{aligned}$$

Letting $t \to \infty$, the monotone convergence theorem implies that

$$\Pi_x^\lambda(\tau_y \mathbb{1}_{\{\tau_y<\infty\}}) = \Pi_x\left(\tau_y \mathbb{1}_{\{\tau_y<\infty\}} \frac{\Xi_{\tau_y}(\lambda)}{\Xi_0(\lambda)}\right).$$

Since $Y_t/t \to -\gamma'(\lambda)$ $\Pi_x^\lambda$-almost surely, we have $\Pi_x^\lambda(\tau_y = \infty) = 0$. Therefore,

$$\begin{aligned}\Pi_x^\lambda(\tau_y) &= \Pi_x\left(\tau_y \mathbb{1}_{\{\tau_y<\infty\}} e^{-\gamma(\lambda)\tau_y - \lambda(y-x) + \int_0^{\tau_y} m\mathbf{g}(X_t)\mathrm{d}t} \frac{\psi(y,\lambda)}{\psi(x,\lambda)}\right) \\ &= \Pi_x\left(\tau_y \mathbb{1}_{\{\tau_y<\infty\}} e^{\int_0^{\tau_y}(m\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t} \frac{\phi(y,\lambda)}{\phi(x,\lambda)}\right).\end{aligned}$$

Using (6.11), (6.13) and that $h(x) = -\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}$, we obtain

$$\begin{aligned}\Pi_x^\lambda(\tau_y) &= \frac{\Pi_x\left[\tau_y e^{\int_0^{\tau_y}(m(X_t)\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t} \mathbb{1}_{\{\tau_y<\infty\}}\right]}{\Pi_x\left[e^{\int_0^{\tau_y}(m(X_t)\mathbf{g}(X_t)-\gamma(\lambda))\mathrm{d}t} \mathbb{1}_{\{\tau_y<\infty\}}\right]} \\ &= \frac{1}{\gamma'(\lambda)}\left(\frac{\phi_\lambda(y,\lambda)}{\phi(y,\lambda)} - \frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}\right) = \frac{1}{\gamma'(\lambda)}\left(h(x) - h(y)\right).\end{aligned}$$

Notice that $Y_t$ satisfies (6.8), that is,

$$\mathrm{d}Y_t = \left(\mu(Y_t) + \left(\frac{\psi_x(Y_t,\lambda)}{\psi(Y_t,\lambda)} - \lambda\right)\sigma^2(Y_t)\right)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t, \quad Y_0 = x.$$

Let $Z_t^{(1)}$ satisfy

$$\mathrm{d}Z_t^{(1)} = -\mu_1 \mathrm{d}t + \sigma(Z_t^{(1)})\mathrm{d}W_t, \quad Z_0^{(1)} = x, \tag{6.14}$$

where $-\mu_1 \le \min_{z\in[0,1]}\left(\mu(z) + \left(\frac{\psi_x(z,\lambda)}{\psi(z,\lambda)} - \lambda\right)\sigma^2(z)\right)$ with $\mu_1 > 0$. By the comparison theorem for solutions of stochastic differential equations (see, for instance, [29, Proposition 5.2.18]), we have

$$\Pi_x^\lambda(Y_t \ge Z_t^{(1)}, \forall\, 0 \le t < \infty) = 1. \tag{6.15}$$

Let $\tau_y^{(1)} = \inf\{t > 0 : Z_t^{(1)} = y\}$, then $\Pi_x^\lambda(\tau_y \ge \tau_y^{(1)}) = 1$ and $\Pi_x^\lambda(\tau_y) \ge \Pi_x^\lambda(\tau_y^{(1)})$ with $x > y$. Define

$$Z_t^{(2)} = \int_0^t \sigma(Z_s^{(1)})\mathrm{d}W_s, \tag{6.16}$$

then $\{Z_t^{(2)}, \Pi_x^\lambda\}$ is a martingale with quadratic variation

$$\langle Z^{(2)}\rangle_t = \int_0^t \sigma^2(Z_s^{(1)})\mathrm{d}s.$$

Let

$$T^{(2)}(t) = \inf\{s \ge 0 : \langle Z^{(2)}\rangle_s > t\}. \tag{6.17}$$

Due to the Dambis-Dubins-Schwarz theorem, $\widehat{W}_t = Z^{(2)}_{T^{(2)}(t)}$ is a standard Brownian motion. It follows easily from the assumption on $\sigma$ that $\sigma_1 := \min_{z\in[0,1]}\sigma^2(z)$ and $\sigma_2 := \max_{z\in[0,1]}\sigma^2(z)$ are positive numbers. Hence

$$\langle Z^{(2)}\rangle_t \in [\sigma_1 t, \sigma_2 t] \text{ and } T^{(2)}(t) \in \left[\frac{t}{\sigma_2}, \frac{t}{\sigma_1}\right]. \tag{6.18}$$

Moreover, we have

$$Z^{(1)}_{T^{(2)}(t)} = -\mu_1 T^{(2)}(t) + Z^{(2)}_{T^{(2)}(t)} \ge -\frac{\mu_1}{\sigma_1}t + \widehat{W}_t \quad \Pi_x^\lambda\text{-a.s.} \tag{6.19}$$

If we define

$$\tau_y^{(2)} = \inf\{t > 0 : Z^{(1)}_{T^{(2)}(t)} = y\} \text{ and } \tau_y^{(3)} = \inf\{t > 0 : -\frac{\mu_1}{\sigma_1}t + \widehat{W}_t = y\}, \tag{6.20}$$

then $\tau_y^{(2)} \geq \tau_y^{(3)}$ and $\tau_y^{(1)} = T^{(2)}(\tau_y^{(2)}) \geq \frac{\tau_y^{(2)}}{\sigma_2}$, $\Pi_x^\lambda$-almost surely. Hence, we obtain

$$\Pi_x^\lambda(\tau_y) \geq \Pi_x^\lambda(\tau_y^{(1)}) \geq \frac{1}{\sigma_2}\Pi_x^\lambda(\tau_y^{(2)}) \geq \frac{1}{\sigma_2}\Pi_x^\lambda(\tau_y^{(3)}).$$

Since $\tau_y^{(3)}$ is the hitting time of a Brownian motion with drift $-\mu_1/\sigma_1$, it follows from [29, Exercise 3.5.10] that

$$\Pi_x^\lambda(e^{-\alpha\tau_y^{(3)}}) = \exp\left\{\frac{\mu_1}{\sigma_1}(x-y) - (x-y)\sqrt{\frac{\mu_1^2}{\sigma_1^2} + 2\alpha}\right\}.$$

Fix an $\alpha_0 > 0$. For any $\alpha > 2\alpha_0$, $\tau_y^{(3)} e^{-\alpha\tau_y^{(3)}} \leq M e^{-\alpha_0\tau_y^{(3)}}$ for some large constant $M$. By the dominated convergence theorem, we can differentiate both sides of the previous equation with respect to $\alpha$ and get that for $\alpha > 2\alpha_0$,

$$\Pi_x^\lambda(\tau_y^{(3)} e^{-\alpha\tau_y^{(3)}}) = \frac{x-y}{\sqrt{\frac{\mu_1^2}{\sigma_1^2} + 2\alpha}} \exp\left\{\frac{\mu_1}{\sigma_1}(x-y) - (x-y)\sqrt{\frac{\mu_1^2}{\sigma_1^2} + 2\alpha}\right\}. \tag{6.21}$$

Since $\alpha_0 > 0$ is arbitrary, (6.21) holds for any $\alpha > 0$. Notice that $\Pi_x^\lambda(\tau_y^{(3)} < \infty) = 1$ and $\tau_y^{(3)} e^{-\alpha\tau_y^{(3)}} \to \tau_y^{(3)} \mathbb{1}_{\{\tau_y^{(3)} < \infty\}}$ as $\alpha \to 0^+$. By the monotone convergence theorem, we get that

$$\Pi_x^\lambda(\tau_y^{(3)}) = \frac{\sigma_1}{\mu_1}(x-y).$$

Therefore,

$$h'(x) = \lim_{y\to x}\frac{h(x)-h(y)}{x-y} = \lim_{y\to x}\frac{\gamma'(\lambda)\Pi_x^\lambda(\tau_y)}{x-y} \geq \lim_{y\to x}\frac{\gamma'(\lambda)\Pi_x^\lambda(\tau_y^{(3)})}{(x-y)\sigma_2} = \frac{\gamma'(\lambda)\sigma_1}{\mu_1\sigma_2} > 0.$$

This completes the proof. □

Recall that $\phi(x,\lambda) = e^{-\lambda x}\psi(x,\lambda)$ and $\phi(x,\lambda)$ satisfies (6.5). Differentiating both sides of (6.5) with respect to $\lambda$, we get that

$$\frac{\sigma^2(x)}{2}\phi_{\lambda xx} + \mu(x)\phi_{\lambda x} + m\mathbf{g}(x)\phi_\lambda = \gamma(\lambda)\phi_\lambda + \gamma'(\lambda)\phi.$$

Recall that $h(x) = -\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}$. A straightforward computation similar to that in the proof of Lemma 2.14 yields that

$$\mathrm{d}h(Y_t) = \sigma(Y_t)h'(Y_t)\mathrm{d}\widehat{B}_t - \gamma'(\lambda)\mathrm{d}t,$$

where $\{\widehat{B}_t, t\geq 0; \Pi_x^\lambda\}$ is a Brownian motion. Define $M_t = \gamma'(\lambda)t + h(Y_t) - h(Y_0)$. Then $\{M_t, t \geq 0\}$ is a $\Pi_x^\lambda$-martingale with the quadratic variation

$$\langle M\rangle_t = \int_0^t \left(\sigma(Y_s)h'(Y_s)\right)^2 \mathrm{d}s \in [c_1 t, c_2 t].$$

Thus, we can use the Dambis-Dubins-Schwarz theorem to make a time change of the martingale $M_t$ and get that

$$\left\{x + h(y) + M_{T(t)}, t\geq 0; \Pi_y^{(x,\lambda)}\right\} \text{ is a standard Bessel-3 process starting at } x + h(y),$$

where $T(t) = \inf\{s > 0 : \langle M \rangle_s > t\}$.

Now, we consider martingales for branching diffusion in a periodic environment. In this case, the many-to-one formula is

$$\mathbb{E}_x\left[\sum_{u\in N_t} F\left(X_u(s), s\in[0,t]\right)\right] = \Pi_x\left[e^{m\int_0^t \mathbf{g}(X_s)\mathrm{d}s} F\left(X_s, s\in[0,t]\right)\right],$$

where $F : C[0,t] \to \mathbb{R}$ is a non-negative measurable function. So $W_t(\lambda)$, $\partial W_t(\lambda)$ and $V_t^x(\lambda)$, defined by (1.7), (1.10) and (2.34) respectively, are still martingales for our branching diffusion in a periodic environment.

### 6.2 Extensions of Theorems 1.1, 1.3 and 1.4

For branching diffusion in a periodic environment, we can define the probability measure $\mathbb{P}_x^\lambda$ similarly by (3.1) and $\widetilde{\mathbb{P}}_x^\lambda$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ by (3.2). The spine decomposition described at the beginning of Subsection 3.1 is still valid and the only difference is the spatial motion of the particles: now the infinitesimal generator of the motion of the spine is given by (6.9) and the infinitesimal generator of the motion of a non-spine particle is given by (6.1). Since Lemma 2.10 still works for the motion of the spine and Lemma 2.4 yields the relationship between $\gamma'(\lambda)$ and $\gamma(\lambda)/\lambda$, the proof of Theorem 1.1 also works for branching diffusions in periodic environments. Therefore, Theorem 1.1 holds for branching diffusions in periodic environments.

For the derivative martingale, recall that for each $x \in \mathbb{R}$, $V_t^x(\lambda)$ is defined by

$$V_t^x(\lambda) = \sum_{u\in\widetilde{N}_t^x} e^{-\gamma(\lambda)t-\lambda X_u(t)}\psi(X_u(t),\lambda)\left(x+\gamma'(\lambda)t+h(X_u(t))\right),$$

where $\widetilde{N}_t^x = \{u \in N_t : \forall s \le t,\ x + \gamma'(\lambda)s + h(X_u(s)) > 0\}$. Define a probability measure $\mathbb{P}_y^{(x,\lambda)}$ on $(\mathcal{T}, \mathcal{F})$:

$$\left.\frac{\mathrm{d}\mathbb{P}_y^{(x,\lambda)}}{\mathrm{d}\mathbb{P}_y}\right|_{\mathcal{F}_t} = \frac{V_t^x(\lambda)}{V_0^x(\lambda)}.$$

According to [23] or [39], there exists a probability measure $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ such that

$$\mathbb{P}_y^{(x,\lambda)} = \widetilde{\mathbb{P}}_y^{(x,\lambda)}|_{\mathcal{F}},$$

and under $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$ the spine decomposition is same as that in Section 4. The spine $\xi$ moves according to $\Pi_y^{(x,\lambda)}$, that is, $\{x + \gamma'(\lambda)T(t) + h(X_\xi(T(t))) :\ t \ge 0\}$ is a standard Bessel-3 process started at $x + h(y)$, where

$$T(t) = \inf\{s \ge 0 : \langle M\rangle_s > t\} \text{ and } \langle M\rangle_t = \int_0^t [\sigma(X_\xi(s))h'(X_\xi(s))]^2 \mathrm{d}s,$$

Therefore, Theorem 1.3 holds in this case.

Finally, the proof of Theorem 1.4 is still valid, since (1.9) and (3.8) still hold. In this case, the F-KPP equation will be

$$\frac{\partial \mathbf{u}}{\partial t} = \rho(x)^{-1}\frac{\partial}{\partial x}\left(\rho(x)\frac{\sigma^2(x)}{2}\frac{\partial}{\partial x}\right)\mathbf{u} + \mathbf{g}\cdot(\mathbf{f}(\mathbf{u}) - \mathbf{u}), \tag{6.22}$$

where $\mathbf{f}(s) = \mathbf{E}(s^{L+1})$.

**Remark 6.1.** Now we briefly discuss the case when the offspring distribution is allowed to change periodically. More precisely, if a particle dies at position $x$, it produces a random number, $1+L(x)$, of offspring. We assume $L(x)$ has distribution $\{p_k(x), k\geq 0\}$ with $m(x) := \sum_{k\geq 0} kp_k(x)$, and for each $k$, $p_k(\cdot)$ and $m(\cdot)$ are 1-periodic $C^1$ function with $m(\cdot)>0$. In this case, we can show that Theorem 1.1 still holds, but we could not prove Theorem 1.3.

Note that $\{Y_t, t\geq 0; \Pi_x^\lambda\}$ has the infinitesimal generator (6.9) and all the coefficients are 1-periodic. If we regard $\{\{Y_t\}, t\geq 0; \Pi_x^\lambda\}$ as a process on $[0,1)$, where $\{Y_t\}$ is the fractional part of $Y_t$, then $\{\{Y_t\}, t\geq 0; \Pi_x^\lambda\}$ is a Markov process. Thus $\{\{Y_t\}, t\geq 0; \Pi_x^\lambda\}$ is a diffusion on a compact set and has an invariant distribution on $[0,1)$, say $n(\mathrm{d}x)$. Using this and adapting some ideas from [33], we can show that Theorem 1.1 holds with $\mathbf{E}(L\log^+ L)$ replaced by $\int_0^1 \mathbf{E}(L(x)\log^+ L(x))n(\mathrm{d}x)$ when $L$ is allowed to be 1-periodic.

However, we were not able to prove Theorem 1.3 when $L$ depends on $x$ periodically. First, using the change of measure by the martingale $V_t^x(\lambda)$, the spatial motion of the spine is a time-changed Bessel-3 process. The fractional part of the Bessel-3 process $\{\{R_t\}, t\geq 0\}$ is not a Markov process, and does not have an invariant distribution. Moreover, since $p_k$ is a function of the spatial location $x$, for estimates like (4.14) and (7.1), one can not take $p_k$ out of the expectation as we did before.

### 6.3 The case when extinction is allowed

In this subsection, we relax the assumption on the offspring distribution to allow extinction and generalize our results further. More precisely, we assume that each particle produces $L$ offspring with $\mathbf{E}L>1$. We also suppose that the spatial motion of each particle is a symmetric periodic diffusion with infinitesimal generator (6.1).

First we consider the additive martingale and the pulsating traveling waves in the supercritical case. Under $\widetilde{\mathbb{P}}_x^\lambda$, the spine decomposition still holds, see [39]. Therefore, Theorem 1.1 (iii) still holds in this case except that (1.8) has to be modified slightly: when $|\lambda|<\lambda^*$ and $\mathbf{E}(L\log^+ L)<\infty$, conditioned on non-extinction, $W(\lambda,x)>0$, $\mathbb{P}_x$-almost surely.

Now we prove the assertion above. Let $p$ be the unique fixed point of $\mathbf{f}(x)=x$ in $(0,1)$. Define

$$p_1(x)=\mathbb{P}_x(W(\lambda,x)=0) \text{ and } p_2(x)=\mathbb{P}_x(\exists\, t \text{ such that } \langle Z_t,1\rangle=0),$$

then $p_2(x)$ is the extinction probability and $p_2(x)\leq p_1(x)$ for any $x\in\mathbb{R}$. Both $p_1(x)$ and $p_2(x)$ are 1-periodic. Since $W(\lambda,x)$ is still an $L^1(\mathbb{P}_x)$-limit in this case, we get that $p_1(x)<1$ for any $x\in\mathbb{R}$. We first claim that

$$p_1(x)=p_2(x)=p, \quad x\in\mathbb{R}. \tag{6.23}$$

To prove (6.23), we first show that $p_1(\cdot)$ is continuous. Define stopping times $\tau_y^X$ and $\tau_y^Z$ with respect to the symmetric diffusion $\{X_t, t\geq 0\}$ and the branching diffusion in a periodic environment respectively by

$$\tau_y^X := \inf\{t\geq 0: X_t=y\},$$
$$\tau_y^Z := \inf\{t\geq 0: \exists u\in N_t \ s.t.\ X_u(t)=y\}.$$

Using an argument similar to that at the end of the proof of Theorem 1.1, we get

$$(1-p_1(x))\geq (1-p_1(y))\mathbb{P}_x(\tau_y^Z\leq t). \tag{6.24}$$

The probability $\mathbb{P}_x(\tau_y^Z\leq t)$ is no less than the probability that the initial particle has reached the position $y$ and has not split before time $t$. Hence, we have

$$\begin{aligned}(1-p_1(x)) &\geq (1-p_1(y))\mathbb{P}_x(\tau_y^Z\leq t)\geq (1-p_1(y))\Pi_y\left(e^{-\int_0^t \mathbf{g}(X_s)\mathrm{d}s}\mathbb{1}_{\{\tau_y^X\leq t\}}\right)\\ &\geq e^{-\beta t}(1-p_1(y))\Pi_x(\tau_y^X\leq t),\end{aligned}$$

where $\beta = \sup_{z\in[0,1]} \mathbf{g}(z)$. Using a symmetry argument, we get that

$$e^{-\beta t}(1-p_1(y))\Pi_x(\tau_y^X \le t) \le (1-p_1(x)) \le e^{\beta t}(1-p_1(y))/\Pi_y(\tau_x^X \le t). \tag{6.25}$$

Now we use an argument similar to the proof of Lemma 2.12 to prove that

$$\lim_{y\to x} \Pi_x(\tau_y^X \le t) = 1. \tag{6.26}$$

Without loss of generality, we suppose $y > x$. Recall that $X = \{X_t, t \ge 0\}$ satisfies (6.2). Let $Z_t^{(1)}$ satisfy (6.14) with $-\mu_1 \le \min_{z\in[0,1]} \mu(z)$. Similar to (6.15), we have

$$\Pi_x(X_t \ge Z_t^{(1)}, \forall\, 0 \le t < \infty) = 1.$$

Recall that $\tau_y^{(1)} = \inf\{t > 0 : Z_t^{(1)} = y\}$, and $Z_t^{(2)}$, $T^{(2)}(t)$, $\tau_y^{(2)}$ and $\tau_y^{(3)}$ are defined by (6.16), (6.17), (6.20) and (6.20), respectively. Since $y > x$, we have $\Pi_x(\tau_y^X \le \tau_y^{(1)}) = 1$. Combining this with (6.18), (6.19) and (6.20), we get that

$$\tau_y^X \le \tau_y^{(1)} = T^{(2)}(\tau_y^{(2)}) \le \frac{\tau_y^{(2)}}{\sigma_1} \le \frac{\tau_y^{(3)}}{\sigma_1},$$

and thus $\Pi_x(\tau_y^X \le t) \ge \Pi_x(\tau_y^{(3)} \le \sigma_1 t)$. It follows from [29, Section 3.5.C] that the passage time $\tau_y^{(3)}$ for Brownian motion with drift $-\mu_1/\sigma_1$ has density

$$\Pi_x(\tau_y^{(3)} \in \mathrm{d}s) = \frac{y-x}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(y-x+\frac{\mu_1}{\sigma_1}s)^2}{2s}\right\} \mathrm{d}s, \quad s > 0.$$

Letting $b = y - x$, $\mu = \mu_1/\sigma_1$ and $r = b^2/s$, using the bounded convergence theorem, we have

$$\begin{aligned}
\lim_{y\to x^+} \Pi_x(\tau_y^{(3)} \le \sigma_1 t) &= \lim_{b\to 0^+} \int_0^{\sigma_1 t} \frac{b}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(b+\mu s)^2}{2s}\right\} \mathrm{d}s \\
&= \lim_{b\to 0^+} \int_{\frac{b^2}{\sigma_1 t}}^{\infty} \frac{1}{\sqrt{2\pi r}} \exp\left\{-\frac{r}{2} - b\mu - \frac{\mu^2 b^2}{2r}\right\} \mathrm{d}r \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi r}} \exp\left\{-\frac{r}{2}\right\} \mathrm{d}r = 1.
\end{aligned}$$

This yields (6.26).

Combining (6.26) and (6.25), we get

$$\limsup_{y\to x} e^{-\beta t}(1-p_1(y)) \le (1-p_1(x)) \le \liminf_{y\to x} e^{\beta t}(1-p_1(y)). \tag{6.27}$$

Letting $t \to 0$, we get $1 - p_1(x) = \lim_{y\to x}(1-p_1(y))$. This implies the continuity of $p_1(\cdot)$. Since $p_1(x) < 1$ for any $x \in \mathbb{R}$, we have that $\sup_{x\in[0,1]} p_1(x) = \max_{x\in[0,1]} p_1(x) < 1$. It follows from the periodicity of $p_1(\cdot)$ that $\sup_{x\in\mathbb{R}} p_1(x) < 1$. Hence $\sup_{x\in\mathbb{R}} p_2(x) \le \sup_{x\in\mathbb{R}} p_1(x) < 1$. Using similar arguments, we can show that $p_2(\cdot)$ is continuous in $\mathbb{R}$. More precisely, (6.24) also holds when $p_1(\cdot)$ is replaced by $p_2(\cdot)$. Hence (6.25) holds for $p_2(\cdot)$ instead of $p_1(\cdot)$. Combining this with (6.26), we get that (6.27) holds for $p_2(\cdot)$. This implies the continuity of $p_2(\cdot)$.

Recall that $X_\emptyset(d_\emptyset)$ is the position of the ancestor at its death time. By the branching property and the monotonicity of $\mathbf{f}$, we have

$$p_2(x) = \mathbb{E}_x(\mathbf{f}(p_2(X_\emptyset(d_\emptyset)))) \le \mathbb{E}_x(\mathbf{f}(\sup_{y\in\mathbb{R}} p_2(y))) = \mathbf{f}(\sup_{y\in\mathbb{R}} p_2(y)).$$

Thus $\sup_{x\in\mathbb{R}} p_2(x) \le \mathbf{f}(\sup_{x\in\mathbb{R}} p_2(x))$. Since the probability generating function $\mathbf{f}(s) = \mathbf{E}(s^L)$ satisfies $\mathbf{f}(s) > s$ for $s \in [0,p)$ and $\mathbf{f}(s) < s$ for $s \in (p,1)$, we have $\sup_{x\in\mathbb{R}} p_2(x) \in [0,p]$. Similarly, $\inf_{x\in\mathbb{R}} p_2(x) \ge \mathbf{f}(\inf_{x\in\mathbb{R}} p_2(x))$ and hence $\inf_{x\in\mathbb{R}} p_2(x) \in [p,1)$. Therefore $\inf_{x\in\mathbb{R}} p_2(x) = \sup_{x\in\mathbb{R}} p_2(x) = p$, that is, $p_2(x) \equiv p$.

By the branching property, the equation $p_1(x) = \mathbb{E}_x(\mathbf{f}(p_1(X_\emptyset(d_\emptyset))))$ holds and thus the proofs also work for $p_1(x) \equiv p$. Thus (6.23) is valid. Hence, conditioned on non-extinction, $W(\lambda, x) > 0$, $\mathbb{P}_x$-almost surely.

Define $\mathbf{u}$ as in (5.1). It follows from (1.9) that

$$\lim_{x\to-\infty} \mathbf{u}(t,x) = p.$$

Repeating the argument of Subsection 5.1, we get that $\mathbf{u}(t,\cdot)$ defined by (1.11) is a solution of (6.22) and (1.4), and satisfies the boundary condition:

$$\lim_{x\to-\infty} \mathbf{u}(t,x) = p, \quad \lim_{x\to+\infty} \mathbf{u}(t,x) = 1. \tag{6.28}$$

Thus we may say $\mathbf{u}$ is a pulsating traveling wave of equation (6.22) connecting $p$ at $-\infty$ and $1$ at $\infty$.

If extinction is allowed, we can treat the critical case ($\nu = \nu^*$) similar to what just did in the supercritical case. First, under $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$ the spine decomposition still holds, see [39]. Next, define $p_3(x) = \mathbb{P}_x(\partial W(\lambda^*, x) = 0)$, then $p_3(x) \equiv p$ and the proof is the same as above. The analogue of Theorem 1.3 holds and is given below.

**Theorem 6.2.** *For any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{(\partial W_t(\lambda))_{t\ge0}, \mathbb{P}_x\}$ is a martingale. For all $|\lambda| \ge \lambda^*$, the limit $\partial W(\lambda,x) := \lim_{t\uparrow\infty} \partial W_t(\lambda)$ exists $\mathbb{P}_x$-almost surely.*

(i) *If $|\lambda| > \lambda^*$ then $\partial W(\lambda,x) = 0$ $\mathbb{P}_x$-almost surely.*

(ii) *If $|\lambda| = \lambda^*$ then $\partial W(\lambda,x) = 0$ $\mathbb{P}_x$-almost surely when $\mathbf{E}(L(\log^+ L)^2) = \infty$, and $\partial W(\lambda,x) \in (0,\infty)$ (respectively $\partial W(\lambda,x) \in (-\infty,0)$) conditioned on non-extinction $\mathbb{P}_x$-almost surely when $\lambda > 0$ (respectively $\lambda < 0$) and $\mathbf{E}(L(\log^+ L)^2) < \infty$.*

Similarly, $\mathbf{u}(t,x)$ defined by (1.12) is a solution of (6.22) and (1.4) with boundary condition (6.28).

# 7 Appendix

## 7.1 Appendix A

Recall that, for any $c > 0$, $A_c$ is defined in (4.12).

**Lemma 7.1.** *For any $c > 0$, $\widetilde{\mathbb{P}}_y^{(x,\lambda)}(A_c) = 0$.*

*Proof.* The proof is almost the same as the proof [42, (8)]. The only changes are some notation and fixing of a few typos. Note that under $\widetilde{\mathbb{P}}_y^{(x,\lambda)}$, $x - \frac{\phi_\lambda(y,\lambda^*)}{\phi(y,\lambda^*)} + R_t$ is a Bessel-3 process starting from $x - \frac{\phi_\lambda(y,\lambda^*)}{\phi(y,\lambda^*)}$. For simplicity, we still use $\tilde{x}$ to denote $x - \frac{\phi_\lambda(y,\lambda^*)}{\phi(y,\lambda^*)}$. Then $\tilde{x} + R_t$ has the same law as the modulus process of $W_t + \hat{x}$, where $\{W_t, t \ge 0; \mathbf{P}_w\}$ is a three-dimensional standard Brownian motion and $\hat{x}$ is a point in $\mathbb{R}^3$ with norm $\tilde{x}$. We still use $A_c$ to denote the same set corresponding to $\{W_t, t \ge 0; \mathbf{P}_w\}$.

$$\begin{aligned} c &\ge \widetilde{\mathbb{P}}_y^{(x,\lambda)}\left(\mathbb{1}_{A_c}\int_0^{+\infty}\sum_{k\in\mathbb{Z}_+}\tilde{p}_k\mathbb{1}_{\left\{k(\tilde{x}+R_t)e^{-\lambda^* R_t}\ge M\right\}}\mathrm{d}t\right) \qquad (7.1)\\ &= \int_0^{+\infty}\sum_{k\in\mathbb{Z}_+}\tilde{p}_k\widetilde{\mathbb{P}}_y^{(x,\lambda)}(\mathbb{1}_{A_c}\mathbb{1}_{\left\{(\tilde{x}+R_t)e^{-\lambda^*(\tilde{x}+R_t)}\ge Mk^{-1}e^{-\lambda^*\tilde{x}}\right\}})\mathrm{d}t\\ &= \sum_{k\in\mathbb{Z}_+}\tilde{p}_k\int_0^{+\infty}\mathbf{P}_w\left(\mathbb{1}_{A_c}\mathbb{1}_{\left\{|W_t+\hat{x}|e^{-\lambda^*|W(t)+\hat{x}|}\ge Mk^{-1}e^{-\lambda^*\tilde{x}}\right\}}\right)\mathrm{d}t. \qquad (7.2)\end{aligned}$$

We claim that there exists $K_1 > 1$ such that when $k \geq K_1$

$$\left\{ y \in \mathbb{R}^3 : 1 + \tilde{x} \leq |y| \leq \frac{\log k}{\lambda^*} \right\} \subset \left\{ y \in \mathbb{R}^3 : |y + \hat{x}| e^{-\lambda^* |y + \hat{x}|} \geq M k^{-1} e^{-\lambda^* \tilde{x}} \right\}. \tag{7.3}$$

In fact, $1 + \tilde{x} \leq |y| \leq \frac{\log k}{\lambda^*}$ implies $1 \leq |y + \hat{x}| \leq \frac{\log k}{\lambda^*} + \tilde{x}$. Consider the function $f(x) = \tilde{x} e^{-\lambda^* \tilde{x}}$. On the positive half line, it increases to a supremum and then decreases to 0 as $x$ goes to infinity. Thus we can find $K_1 > 1$ large enough such that when $k \geq K_1$,

$$\begin{aligned} 1 + \tilde{x} \leq |y| \leq \frac{\log k}{\lambda^*} &\Rightarrow f(|y + \hat{x}|) \geq f\left( \frac{\log k}{\lambda^*} + \tilde{x} \right) \\ &\Rightarrow |y + \hat{x}| e^{-\lambda^* |y + \hat{x}|} \geq \left( \frac{\log k}{\lambda^*} + \tilde{x} \right) k^{-1} e^{-\lambda^* \tilde{x}}. \end{aligned}$$

Thus (7.3) is valid.

We continue the estimate (7.2) when $k \geq K_1$,

$$\begin{aligned} c &\geq \sum_{k: k \geq K_1} \tilde{p}_k \int_0^{+\infty} \mathbf{P}_w \left( \mathbb{1}_{A_c} \mathbb{1}_{\left\{1 + \tilde{x} \leq |W_t| \leq \frac{\log k}{\lambda^*}\right\}} \right) \mathrm{d}t \\ &= \sum_{k: k \geq K_1} \tilde{p}_k \mathbf{P}_w \left( \mathbb{1}_{A_c} \int_0^{+\infty} \mathbb{1}_{\left\{1 + \tilde{x} \leq |W_t| \leq \frac{\log k}{\lambda^*}\right\}} \mathrm{d}t \right). \end{aligned} \tag{7.4}$$

$(|W_t|, t \geq 0; \mathbf{P}_w)$ is a Bessel-3 process starting from 0. Let $\{l^a : a \geq 0\}$ be the family of its local times, then the process $\{l^a_\infty, a \geq 0\}$ is a $\mathrm{BESQ}^2(0)$ process which implies $l^a_\infty \overset{d}{=} a l^1_\infty$ and $\mathbf{P}_w \left( l^1_\infty = 0 \right) = 0$ (see Revuz and Yor [41], p. 425, Ex. 2.5). Thus

$$\begin{aligned} \mathbf{P}_w \left( \mathbb{1}_{A_c} \int_0^{+\infty} \mathbb{1}_{\left\{1 + \tilde{x} \leq |W_t| \leq \frac{\log k}{\lambda^*}\right\}} \mathrm{d}t \right) &= \mathbf{P}_w \left( \mathbb{1}_{A_c} \int_{1 + \tilde{x}}^{\frac{\log k}{\lambda^*}} l^a_\infty \mathrm{d}a \right) \\ &= \mathbf{P}_w \left( \mathbb{1}_{A_c} \int_{1 + \tilde{x}}^{\frac{\log k}{\lambda^*}} a \mathrm{d}a \int_0^{a^{-1} l^a_\infty} \mathrm{d}u \right) \\ &= \int_{1 + \tilde{x}}^{\frac{\log k}{\lambda^*}} a \mathrm{d}a \int_0^{+\infty} \mathbf{P}_w \left( \mathbb{1}_{A_c} \mathbb{1}_{\{u \leq a^{-1} l^a_\infty\}} \right) \mathrm{d}u. \end{aligned} \tag{7.5}$$

Note that

$$\mathbf{P}_w \left( \mathbb{1}_{A_c} \mathbb{1}_{\{u \leq a^{-1} l^a_\infty\}} \right) \geq \left( \mathbf{P}_w (A_c) - \mathbf{P}_w \left( a^{-1} l^a_\infty < u \right) \right)^+ = \left( \mathbf{P}_w (A_c) - \mathbf{P}_w \left( l^1_\infty < u \right) \right)^+,$$

and there exist $C > 0$ and $K_2 > 1$ such that for $k \geq K_2$

$$\int_{1 + \tilde{x}}^{\frac{\log k}{\lambda^*}} a \mathrm{d}a = \frac{1}{2} \left( \left( \frac{\log k}{\lambda^*} \right)^2 - (1 + \tilde{x})^2 \right) \geq C (\log k)^2.$$

Then (7.5) implies

$$\mathbf{P}_w \left( \mathbb{1}_{A_c} \int_0^{+\infty} \mathbb{1}_{\left\{1 + \tilde{x} \leq |W_t| \leq \frac{\log k}{\lambda^*}\right\}} \mathrm{d}t \right) \geq C (\log k)^2 \int_0^{\infty} \left( \mathbf{P}_w (A_c) - \mathbf{P}_w \left( l^1_\infty < u \right) \right)^+ \mathrm{d}u. \tag{7.6}$$

Set $K = K_1 \vee K_2$. Using (7.4) and (7.6) we get

$$\sum_{k: k \geq K} \tilde{p}_k (\log k)^2 \int_0^{+\infty} \left( \mathbf{P}_w (A_c) - \mathbf{P}_w \left( l^1_\infty < u \right) \right)^+ \mathrm{d}u < +\infty. \tag{7.7}$$

The assumption $\mathbf{E}(L\left(\log^+ L\right)^2) = +\infty$ is equivalent to $\sum_{k\in\mathbb{Z}_+}\tilde{p}_k\left(\log^+ k\right)^2 = +\infty$. Then by (7.7),

$$\int_0^{+\infty}\left(\mathbf{P}_w\left(A_c\right) - \mathbf{P}_w\left(l^1_\infty < u\right)\right)^+ \mathrm{d}u = 0.$$

Thus $\mathbf{P}_w(A_c) = 0$ since $\mathbf{P}_w\left(l^1_\infty = 0\right) = 0$, consequently $\widetilde{\mathbb{P}}^{(x,\lambda)}_y(A_c) = 0$. □

### 7.2 Appendix B

In this subsection, we show that how to get (1.3) from (5.2).

Let $\{X_t, t \geq 0; \Pi_x\}$ be a diffusion process with infinitesimal generator $\mathcal{L}$ given by (6.1). In particular, when $\sigma \equiv 1$ and $\rho \equiv 1$, $\{X_t, t \geq 0; \Pi_x\}$ is a Brownian motion and $\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$. By [12, Theorem 1.XVII.6 on p. 303] (for Brownian motion) and [18, Theorem 1.10] (for diffusion), we have the following lemma.

**Lemma 7.2.** *Let $\{X_t, t \geq 0; \Pi_x\}$ be a diffusion process in $\mathbb{R}$ with generator $\mathcal{L}$. Suppose $w$ is a Borel function on $[0,\infty)\times\mathbb{R}$ and bounded on $[0,T]\times\mathbb{R}$ for any $T > 0$. Define*

$$F(t,x) = \Pi_x\int_0^t w(s, X_{t-s})\mathrm{d}s, \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

*Then $F$ and $\frac{\partial F}{\partial x}$ are continuous in $(0,\infty)\times\mathbb{R}$.*

*If, in addition, $w$ is continuous and if, for every $(r,x) \in [0,\infty)\times\mathbb{R}$, there exist a neighbourhood $U$ and constants $\alpha \in (0,1]$ and $C > 0$ such that*

$$|w(s,y) - w(s,z)| \leq C|y-z|^\alpha \quad \textit{for all } (s,y),(s,z) \in U, \tag{7.8}$$

*then $F$, $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$ are continuous in $(0,\infty)\times\mathbb{R}$ and $F$ satisfies*

$$\frac{\partial F}{\partial t} = \mathcal{L}F + w \quad \textit{in } (0,\infty)\times\mathbb{R}.$$

Now we derive (1.3) from the integral equation (5.2). Actually we prove the result for general $\mathcal{L}$ given by (6.1) which include the special case of $\frac{1}{2}\frac{\partial^2}{\partial x^2}$. Suppose $\mathbf{u}$ is bounded and satisfies

$$\mathbf{u}(t,x) = \Pi_x\left[f(X_t)\right] + \Pi_x\int_0^t \mathbf{g}(X_s)\left[\mathbf{f}(\mathbf{u}(t-s,X_s)) - \mathbf{u}(t-s,X_s)\right]\mathrm{d}s. \tag{7.9}$$

For simplicity, we write $F \in C^1((0,\infty)\times\mathbb{R})$ if $F, \frac{\partial F}{\partial x}$ are continuous in $(0,\infty)\times\mathbb{R}$, and write $F \in C^2((0,\infty)\times\mathbb{R})$ if $F, \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$ are continuous in $(0,\infty)\times\mathbb{R}$. The equation (7.9) can be written as

$$\mathbf{u} = l + F, \text{ in } (0,\infty)\times\mathbb{R}, \tag{7.10}$$

where $F(t,x) = \Pi_x\int_0^t \mathbf{g}(X_{t-s})\left[\mathbf{f}(\mathbf{u}(s,X_{t-s})) - \mathbf{u}(s,X_{t-s})\right]\mathrm{d}s$ and $l(t,x) = \Pi_x\left[f(X_t)\right]$. We know that $l \in C^2((0,\infty)\times\mathbb{R})$. Since $\mathbf{u}$ is bounded, $\mathbf{g}(x)\left[\mathbf{f}(\mathbf{u}(s,x)) - \mathbf{u}(s,x)\right]$, as a function of $(s,x)$, is bounded in $(0,\infty)\times\mathbb{R}$. By Lemma 7.2, we have $F \in C^1((0,\infty)\times\mathbb{R})$. Therefore, by (7.10), $\mathbf{u} \in C^1((0,\infty)\times\mathbb{R})$ holds. Since $\mathbf{g}$ and $\mathbf{f}$ are continuously differentiable, $\mathbf{g}(x)\left[\mathbf{f}(\mathbf{u}(s,x)) - \mathbf{u}(s,x)\right]$ satisfies (7.8). Using Lemma (7.2) again, we have $F \in C^2((0,\infty)\times\mathbb{R})$. Hence we have $\mathbf{u} \in C^2((0,\infty)\times\mathbb{R})$ and $\frac{\partial F}{\partial t} - \mathcal{L}F = \mathbf{g}\cdot(\mathbf{f}(\mathbf{u}) - \mathbf{u})$. Since $\frac{\partial l}{\partial t} - \mathcal{L}l = 0$, we have

$$\frac{\partial \mathbf{u}}{\partial t} - \mathcal{L}\mathbf{u} = \mathbf{g}\cdot(\mathbf{f}(\mathbf{u}) - \mathbf{u}) \quad \text{in } (0,\infty)\times\mathbb{R}.$$

In particular, if $\{X_t, t \geq 0; \Pi_x\}$ is a Brownian motion, we get (1.3) from (5.2).

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