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The wave speed of an FKPP equation with jumps via coordinated branching*

Tommaso Rosati† András Tóbiás‡

Abstract

We consider a Fisher–KPP equation with nonlinear selection driven by a Poisson random measure. We prove that the equation admits a unique wave speed $\mathfrak{s} > 0$ given by

$$\frac{\mathfrak{s}^2}{2} = \int_{[0,1]} \frac{\log(1+y)}{y} \mathfrak{R}(\mathrm{d}y) ,$$

where $\mathfrak{R}$ is the intensity of the impacts of the driving noise. Our arguments are based on upper and lower bounds via a quenched duality with a coordinated system of branching Brownian motions.

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1 Introduction

The Fisher-KPP equation is a classical model in spatial population genetics

$$\partial_t u_t = \frac{1}{2}\Delta u_t + \mathbf{r}\, u_t(1-u_t), \quad u(0,x) = u_0(x) \in [0,1], \quad (t,x) \in [0,\infty) \times \mathbb{R} , \tag{1.1}$$

which describes the evolution of the density of one favoured genetic type over another disadvantaged one, where the advantage is given by a selection force of strength $\mathbf{r} > 0$.

Instead of the classical equation, this work is concerned with the analysis of a Fisher-KPP model in which selection acts at discrete jump times. We fix a positive measure $\mathfrak{R}$

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†University of Warwick, UK. E-mail: t.rosati@warwick.ac.uk
‡Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Műegyetem rkp. 3., 1111 Budapest, Hungary, and Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, 1053 Budapest, Hungary. E-mail: tobias@cs.bme.hu

on $[0,1]$ and denote with $\mathcal{R}$ a Poisson random measure on $[0,\infty)\times(0,1]$ with intensity $\mathrm{d}t\otimes\frac{1}{y}\mathfrak{R}(\mathrm{d}y)$. Then we consider the following equation driven by $\mathcal{R}$:

$$\mathrm{d}u_t=\frac{1}{2}\Delta u_t\mathrm{d}t+\mathfrak{r}\,u_t(1-u_t)\mathrm{d}t+\int_{(0,1]}yu_{t-}(1-u_{t-})\mathcal{R}(\mathrm{d}t,\mathrm{d}y),\quad u(0,x)=u_0(x)\ ,\qquad(1.2)$$

for $(t,x)\in[0,\infty)\times\mathbb{R}$, where $\mathfrak{r}=\mathfrak{R}(\{0\})$ corresponds to the continuous component of the equation. The biological motivation behind the choice of such a noise is to model *rare selection*, or better strong temporary selective advantages of fit individuals due to extreme behaviour of a random environment, as opposed to the classical, constantly present but *weak* selection corresponding to models with continuous forcing (represented e.g. by the second term on the right-hand side of (1.2)). Such strong evolutionary events involve a macroscopic portion of the underlying population. They are therefore linked to some form of "coordination" between individuals. We will elaborate on concrete biological examples of rare selection and mathematical models of coordination in Section 1.1.

The purpose of this article is to focus on a prominent dynamical feature of the Fisher-KPP equation – its wave speed – and attempt a first description of how it is affected by extreme selection events. For the classical equation it is well known that there exists a travelling wave solution of speed $\sqrt{2\mathbf{r}}$, that captures the asymptotic evolution of the front of an invading gene, say when the initial distribution is of the form $u_0(x)=\mathbb{1}_{(-\infty,0]}(x)$. Following the convention that $\int_{\{0\}}\log(1+y)\frac{1}{y}\mathfrak{R}(\mathrm{d}y)=\mathfrak{R}(\{0\})=\mathfrak{r}$, our main result states that the wave speed $\mathfrak{s}>0$ of the stochastic equation is given by

$$\frac{\mathfrak{s}^2}{2}=\int_{[0,1]}\log\left(1+y\right)\frac{1}{y}\mathfrak{R}(\mathrm{d}y)\ ,\qquad(1.3)$$

and therefore shows quantitatively how extreme selection events slow down the invading speed, compared to the deterministic equation (1.1) with $\mathbf{r}=\mathfrak{R}([0,1])$ (this is the natural choice because it is consistent with our definitions in the case $\mathfrak{R}=\mathfrak{r}\delta_0$): that the wave speed is strictly smaller than the deterministic one follows for instance by averaging (1.2) and using Jensen's inequality.

The size of the gap between the speed $\mathfrak{s}$ of the stochastic equation and the speed $\sqrt{2\mathbf{r}}$ of the associated deterministic one depends on the nature of the noise. In a so-called *pushed* regime [21] (for example in presence of a genetic drift term) the effect can be surprisingly strong also for small noise, as demonstrated in the seminal work by Mueller, Mytnik and Quastel [27]. In our case, the nonlinearity is smooth and concave: We are in the *pulled* regime, where the effect of noise is weaker and most importantly the speed of the wave is governed by the linearisation of the equation near $u=0$.

Now, the Lyapunov exponent of the linearised equation is in turn described the long-time behaviour of the dual process and the wave speed can be easily rewritten as the speed of the rightmost particle of the dual. This correspondence is very well understood: For the deterministic equation the dual is given by a Branching Brownian Motion (BBM) [16, 25]. In the BBM each particle moves as an independent Brownian motion and branches into two identical offspring at a constant rate $\mathbf{r}$ (see [14, 15, 16]). In our setting, the dual is given by a *coordinated branching Brownian motion* (CBBM), see also the discussion of existing literature below. The main difference with respect to the BBM is that particles tend to reproduce simultaneously rather than independently: If n particles are alive at a given time, for any $1\le k\le n$, any k-tuple of particles decides to simultaneously produce one offspring per particle at rate $\int_{[0,1]}y^k(1-y)^{n-k}\frac{1}{y}\mathfrak{R}(\mathrm{d}y)$.

To study the speed of the rightmost particle of the CBBM, we consider a general approach developed by Kyprianou and Englaender [22, 10] (cf. also the references therein), which uses a martingale argument to study the local survival of branching (or super-) processes. As a rule-of-thumb, it states that the speed of the rightmost particle equals $\sqrt{2\lambda}$, where λ is the Lyapunov exponent of the underlying system. A subtlety when applying their argument in our setting is that the *quenched* (so conditional on the realisation of the jumps) growth rate of the number of particles of the CBBM and *annealed* growth rate (of the expected number of particles of the CBBM, where the expectation is taken also over the random environment) might differ: An environment exhibiting such a behaviour is called *strongly catalytic*. While in the classical BBM the almost sure and the expected growth rate coincide, our coordinated process is strongly catalytic. This imposes a fundamental new challenge to our analysis, as the correct prediction for the wave speed appears by formally using the rule-of-thumb with the *quenched* Lyapunov exponent $\lambda = \frac{\mathfrak{s}^2}{2}$, as in (1.3) (note that this corresponds to the growth rate of the total mass of the CBBM), while an attempt to make the martingale argument rigorous breaks down, because the gap between quenched and annealed implies that now the martingales at hand are not uniformly integrable. We observe that this issue is related to the gap between the stochastic and the deterministic equation we already addressed, as the annealed Lyapunov exponent is given by $\mathbf{r} = \mathfrak{R}([0,1])$ (which is a strict upper bound on our speed, unless $\mathfrak{R} = \mathfrak{r}\delta_0$).

Our approach to overcome this problem is to distinguish between 'large reproduction events' in which individuals participate in an event with probability $y \in (\delta, 1]$, and 'small reproduction events' where individuals participate with probability $y \in (0, \delta]$, for some $\delta \in (0, 1]$. Then we proceed to obtain upper and lower bounds on the speed $\mathfrak{s}$, which depend on δ but converge to the correct speed as $\delta \downarrow 0$. For the upper bound, we use a quenched dual, where we condition on the location and impact of 'large' reproduction events. We then use the martingale argument outlined above to deal with small jumps, which now affect the speed only by a factor $\mathfrak{R}((0, \delta])$. Instead, for large reproduction events we use time changes and a "channeling" argument based on elementary large-deviation estimates to obtain the expected contribution to the speed. For the lower bound we use comparison to remove the mass of $\mathfrak{R}$ in the interval $(0, \delta]$ and then can proceed with similar calculations to those we use the upper bound.

Overall, the novelty of our work consists in quantifying the effect of extreme selective events on the speed of invasion of the favoured of two genetic types. In future, we hope to extend these results to a much broader class of models, potentially including dormancy [5, 6], mutation, genetic drift and spatially localized selective events [3]. Finally, we note that we only consider the highest order (linear) term in the wave speed. For the original Fisher-KPP equation many, more refined results are available [8, 23] and would be interesting to extend to the present setting. Similarly, the existence of a (generalized) travelling wave (and not just the speed of propagation) is left open.

1.1 Related literature

Recently, the study of the effect of extreme selective or reproductive events on evolutionary models has seen a flurry of activity. An archetypal non-spatial model for such an evolution is the Λ–coalescent, in which a measure Λ, corresponding to our $\mathfrak{R}$, determines the proportion of individuals participating in a merger event [30, 31]: see also [13] for one of the first examples of coordinated reproduction in the context of contact processes, and [11] (and the references therein) for a general framework regarding coordination in reproduction, death and migration. In the study of non-spatial models, extreme selection and reproduction events – which are in correspondence via duality – have been recently addressed by [2, 7, 12]. In the study of spatial models such as superprocesses, the effects of strong selection have been analysed for example in [28, 18].

For example, Cordero and Véchambre [7] derive an analogue of our equation, with genetic drift and no spatial component, as the scaling limit of a microscopic particle system and study its long-time behaviour (similar scaling limits and results have been obtained in [2, 12] in related models). Although these works do not consider our spatial setting, they share key aspects of our approach. In particular, the use of duality and the study of the long-time behaviour of the processes through conditioning or averaging over the environment are essential in our arguments.

In more detail, the most well-known population-genetic example of moment duality is the one between the Wright–Fisher diffusion (without selection) and the Kingman coalescent. The Wright–Fisher diffusion appears as the scaling limit of the relative frequency of a neutral allele in the Wright–Fisher model, which runs forwards in time, whereas the Kingman coalescent describes the ancestry of a sample of the haploid and asexually reproducing individuals of the Wright–Fisher model backwards in time. While the introduction of selection into the forwards-in-time process is straightforward, the existence of a moment dual was not known before Krone and Neuhauser [19] introduced this process, called the *ancestral selection graph*. In this graph, while random genetic drift still leads to mergers of ancestral lines in the dual, selection makes ancestral lines branch into multiple potential parents. For example, if the model consists of just one weak and one strong allele, both potential parents of a particle with a weak allele type must be themselves of a weak type. The moment duality between the classical BBM and the solution to the classical FKPP equation can be interpreted similarly; this is a spatial model without random genetic drift, where the forwards-in-time process is deterministic, and the Brownian particles of its dual exhibit branching only. If one introduces a rare selection governed by a Poisson point process just as in Equation (1.2), then the corresponding part of the dual process will be governed by the same Poisson point process. Similarly to the Wright–Fisher diffusion with selection, if (t, y) is a point belonging to the Poisson point process, then the forwards-in-time interpretation of the model is that at time t a fraction y of individuals, chosen uniformly at random, are participating in a large selective resampling event. On the other hand, as in the the ancestral selection graph, backwards-in-time this corresponds to a large scale branching event, in which each particle participates with probability y.

As also mentioned in [7, 12], examples of experimental studies on rare selection can be found in [9, 26]. In [9], lizards with long fingers can hold on stronger and thus avoid being blown away whenever their habitat is hit by a hurricane, which provides them a strong but temporary selective advantage. Further, [26] compares different antibiotic treatment strategies against a bacterial population. Here, the analogue of a continuously present but weak selective pressure is a constant administration of the antibiotic in low concentration dosage, while rare and strong selective events correspond to a less frequent inoculation with higher dosages (possibly of varying concentration and at random times).

We also note that extreme evolutionary events in a spatial setting have received much attention over the past years in relation to the study of spatial Λ–Fleming–Viot (SLFV) models introduced by Barton, Etheridge and Véber [3]. Unlike our equation, in this class of processes reproductive events are localized in space, which is a natural assumption and an interesting direction for future extensions of our result.

After completion of the present paper, we learned that the speed of the rightmost particle of the CBBM can be computed also via the results of [24] on branching random walks in a time-inhomogeneous random environment, using different tools.

Together with our results on well-posedness of (1.2) its duality with respect to the CBBM, this provides an alternative proof of the wave speed of (1.2).

1.2 Structure of the paper

This article is divided as follows. In Section 2.1 we present our model and in Section 2.2 we state our main results, along with the crucial points of their proofs. The technical details of the proofs are carried out in the rest of the paper. In Section 3 we prove the existence and uniqueness of strong solutions to (1.2) as well as (quenched and annealed) duality. These results do not come as a surprise, but require a proof and a precise statement. Section 4 is devoted to upper and lower bounds on the wave speed via quenched duality arguments.

1.3 Notations

We write $\mathbb{N} = \{1, 2, \dots\}$ and denote $[n] = \{1, \dots, n\}$ for any $n \in \mathbb{N}$. Furthermore, let $\mathcal{M}$ be the space of finite positive Borel measures on $[0, 1]$ with the topology of weak convergence. For a set $\mathcal{X}$ and two functions $f, g \colon \mathcal{X} \to \mathbb{R}$ we write $f \lesssim g$ if there exists a constant $c > 0$ such that $f(x) \leqslant cg(x)$ for all $x \in \mathcal{X}$. If the constant c depends on some parameter ϑ we write $f \lesssim_\vartheta g$. We further denote with $C_b^k(\mathbb{R}; \mathcal{O})$ (for $k \in \mathbb{N} \cup \{\infty\}$ and any target set $\mathcal{O} \subseteq \mathbb{R}$) the space of bounded and k times differentiable functions $\varphi \colon \mathbb{R} \to \mathcal{O}$ with continuous and bounded derivatives. Similarly, for $\gamma \in (0, \infty) \setminus \mathbb{N}$ we define C_b^γ to be the space of bounded and $\lfloor \gamma \rfloor$–times differentiable functions with $\gamma - \lfloor \gamma \rfloor$-Hölder continuous and bounded derivatives.

Finally, with $C_{\mathrm{loc}}(\mathbb{R}; \mathcal{O})$ we denote the space of continuous (and not necessarily uniformly bounded) functions with values in $\mathcal{O}$. When $\mathcal{O} = \mathbb{R}$ we may drop the dependence on it in the notation. The spaces C_b^k and C_b^α, for $k \in \mathbb{N}, \alpha \notin \mathbb{N}$ come equipped, respectively, with the norms

$$\|\varphi\|_\infty = \sup_{x \in \mathbb{R}} |\varphi(x)| \;, \quad \|\varphi\|_{C_b^k} = \sum_{i=0}^{k} \|\partial_x^i \varphi\|_\infty \;,$$

and

$$\|\varphi\|_{C_b^\alpha} = \sum_{i=0}^{\lfloor \alpha \rfloor} \|\partial_x^i \varphi\|_\infty + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{\alpha - \lfloor \alpha \rfloor}} \;.$$

Moreover, for any polish space E we indicate with $\mathbb{D}([0, \infty); E)$ the space of càdlàg paths with values in E endowed with the Skorokhod topology (similarly for $[0, \infty)$ replaced by some finite interval $[0, T]$).

2 Setting and main results

2.1 The model

The main object of interest in this work is the wave speed $\mathfrak{s}$ of the solution to (1.2). It will be convenient to consider the following class of initial conditions, for any $\alpha \in (0, 1)$

$$C_{0,1}^\alpha = \Big\{ u \in C_b^\alpha(\mathbb{R}; [0,1]) \text{ such that } \overline{\{x \,:\; u(x) \notin \{0,1\}\}} \text{ is compact} \\ \text{and such that } \lim_{x \to -\infty} u(x) = 1, \quad \lim_{x \to \infty} u(x) = 0 \Big\} \;,$$

for which the wave speed is naturally defined below.

Definition 2.1. *We say that* $\mathfrak{s} \in \mathbb{R}$ *is the wave speed associated to* (1.2) *if for any* $\alpha \in (0, 1)$ *and all* $u_0 \in C_{0,1}^\alpha$ *the following hold.*

1. *For every* $\lambda > \mathfrak{s}$ *and any* $x \in \mathbb{R}$*, we have* $\lim_{t \to \infty} u_t(x + \lambda t) = 0$ *in probability.*
2. *For every* $\lambda < \mathfrak{s}$ *and any* $x \in \mathbb{R}$*, we have* $\lim_{t \to \infty} u_t(x + \lambda t) = 1$ *in probability.*

Remark 2.2. Our initial conditions are chosen to be Hölder continuous, to simplify the statements that will follow. We could consider also discontinuous initial data, e.g. $u_0(x) = \mathbb{1}_{(-\infty,0]}(x)$, at the cost of introducing blow-ups at time $t = 0$ in the solution theory for (1.2).

The study of the wave speed of the solution to (1.2) passes through the analysis of its dual process, which consists of a system of Brownian motions with coordinated branching that run backwards in time and roughly represent the genealogy of types of a sample of particles. In the dual process the parameter y interpolates between no coordination ($y = 0$, so all particles act independently) and full coordination ($y = 1$, so all particles reproduce at once). In this backwards (or dual) picture the measure $\mathfrak{R}$ captures the reproduction rate.

Notation 2.3. *To describe the state space of our particle systems we introduce the set*

$$\mathcal{P} = \bigsqcup_{n\in\mathbb{N}} \mathbb{R}^n .$$

Then to every point $\mathbf{x}$ *of* $\mathcal{P}$ *we can associate a length* $\ell(\mathbf{x}) = n \iff \mathbf{x} \in \mathbb{R}^n$*. In particular,* $\mathcal{P}$ *is a Polish space with the distance* $d(\mathbf{x}, \mathbf{y}) = |\ell(\mathbf{x}) - \ell(\mathbf{y})| + \|\mathbf{x} - \mathbf{y}\|\mathbb{1}_{\{\ell(\mathbf{x})=\ell(\mathbf{y})\}}$*, where* $\|\cdot\|$ *indicates the Euclidean norm. To concisely express our duality formulas, let us introduce the following notation*

$$\varphi^{\mathbf{x}} = \prod_{i=1}^{n} \varphi(x_i), \qquad \forall \varphi \in C_{\text{loc}}(\mathbb{R}), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathcal{P}.$$

In addition, for $\mathbf{x}, \mathbf{y} \in \mathcal{P}$ *we write the concatenation*

$$\mathbf{x} \sqcup \mathbf{y} = (x_1, \dots, x_{\ell(\mathbf{x})}, y_1, \dots, y_{\ell(\mathbf{y})}) \in \mathcal{P}.$$

The way in which we use duality requires the introduction of an additional parameter $\delta \in (0, 1]$. We will then consider a dual *conditional* on jumps with impact $y > \delta$. We start by distinguishing small from large jumps with the following notation.

Definition 2.4. *For any* $\delta \in (0, 1]$ *and* $\mathfrak{R} \in \mathcal{M}([0, 1])$ *define*

$$\mathfrak{R}_\delta^-(A) = \mathfrak{R}(A \cap [0, \delta]) , \qquad \mathfrak{R}_\delta^+(A) = \mathfrak{R}(A \cap (\delta, 1]) , \qquad \mathfrak{R}_\delta = (\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+) .$$

In general, we call **compatible** *with* δ *any ordered pair of measures* $\mu_\delta = (\mu_\delta^-, \mu_\delta^+) \in \mathcal{M}^2$ *with support in* $[0, \delta]$ *and* $[\delta, 1]$ *respectively with* $\mu_\delta^+(\{\delta\}) = 0$*. Finally, for any compatible measures* μ_δ *we introduce the Poisson point process with intensity* $\mathrm{d}t \otimes \frac{1}{y}\mu_\delta^+(\mathrm{d}y)$*:*

$$\mathcal{S}_\delta = \{(t_j, y_j)\}_{j\in\mathbb{N}} \subseteq [0, \infty) \times (\delta, 1] , \tag{2.1}$$

which is characterised by the fact that $0 < t_1 < \cdots < t_j < t_{j+1}$*, and* $t_j \uparrow \infty$*, and is linked to the Poisson random measure*

$$\mathcal{R}_\delta^+(\mathrm{d}t, \mathrm{d}y) = \sum_{j\in\mathbb{N}} \delta_{t_j}(\mathrm{d}t)\delta_{y_j}(\mathrm{d}y) .$$

We observe that formally we can rewrite the noise $\mathcal{R}$ in (1.2) as

$$\mathcal{R} = \mathcal{R}_\delta^- + \mathcal{R}_\delta^+ ,$$

with $\mathcal{R}_\delta^-$ a Poisson random measure with intensity $\mathrm{d}t \otimes \frac{1}{y}\mathfrak{R}_\delta^-(\mathrm{d}y)$. To be precise, $\mathcal{R}$ is in general not a measure, but can only interpreted when integrated against functions that vanish near $y = 0$ sufficiently fast. More precisely, $\mathcal{R}_\delta^-$ is associated with a Poisson point process $\{(s_j, z_j)\colon i \in I\}$ of intensity $\mathrm{d}t \otimes \frac{1}{y}\mathfrak{R}(\mathrm{d}y)$ on $[0,\infty)\times(0,\delta]$, with countable index set $I \subseteq \mathbb{N}$ (so both the points (s_j, z_j) and the index set I are random). Then for measurable functions $f\colon [0,\infty)\times(0,\delta] \to \mathbb{R}$ satisfying

$$\int_{[0,\infty)\times(0,\delta]} \min\{|f(t,y)|, 1\}\frac{1}{y}\mathfrak{R}(\mathrm{d}y)\mathrm{d}t < \infty \ , \tag{2.2}$$

the integral

$$\int_{[0,\infty)\times(0,\delta]} f(t,y)\mathcal{R}(\mathrm{d}t,\mathrm{d}y) := \sum_{j\in I} f(s_j, z_j) \tag{2.3}$$

is almost surely finite, as discussed in the context of Campbell's theorem in [17, Section 3].We can now introduce the dual process to (1.2) *conditional* on the realisation of $\mathcal{S}_\delta$. We highlight that the FKPP equation and its conditional dual share the same jump times $\mathcal{S}_\delta$ (to be precise, the dual process *may* jump at a time contained in $\mathcal{S}_\delta$ but does not necessarily do so), whereas they do not share the jump times associated to smaller impacts.

Definition 2.5 (μ_δ–CBBM)**.** *For any $\delta \in (0,1]$ and any couple $\mu_\delta = (\mu_\delta^-, \mu_\delta^+)$ compatible with δ, let $\mathcal{S}_\delta$ be the Poisson point process defined by* (2.1)*. We say that $(\mathbf{C}_t)_{t\geqslant 0}$ is a μ_δ–coordinated branching Brownian motion (μ_δ–CBBM) with initial condition $\mathbf{C}_0 = \mathbf{x} \in \mathcal{P}$ if, conditional on the realisation of $\mathcal{S}_\delta$, the process $\mathbf{C}_t$ is a $\mathcal{P}$-valued Markov process with the following dynamics:*

1. ***Diffusion.*** *Let $\mathbf{C}_t = (x_1, \ldots, x_n)$ at time $t > 0$. Then each individual x_i moves in $\mathbb{R}$ according to a Brownian motion, independent of all other individuals, until one of the following two jumps occur.*
2. ***Large reproduction events.*** *For every $j \in \mathbb{N}$, assume that at time t_j (of the Poisson point process $\mathcal{S}_\delta$) there are currently n individuals $\mathbf{C}_{t_j} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.*

   Then we observe one of the following transitions, for any subset $\mathcal{I} \subseteq [n]$:

   $$(x_i)_{i\in[n]} \to (x_i)_{i\in[n]} \sqcup (x_i)_{i\in\mathcal{I}} \in \mathbb{R}^{n+|\mathcal{I}|} \quad \textit{with probability} \quad y_j^{|\mathcal{I}|}(1-y_j)^{n-|\mathcal{I}|}.$$

3. ***Small reproduction events.*** *Assume that at time $t \geqslant 0$ there are currently n individuals $\mathbf{C}_t = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for any subset $\varnothing \neq \mathcal{I} \subseteq [n]$ we have the following coordinated transition:*

   $$(x_i)_{i\in[n]} \to (x_i)_{i\in[n]} \sqcup (x_i)_{i\in\mathcal{I}} \in \mathbb{R}^{n+|\mathcal{I}|} \quad \textit{at rate} \quad \int_{[0,\delta]} y^{|\mathcal{I}|}(1-y)^{n-|\mathcal{I}|}\frac{1}{y}\mu_\delta^-(\mathrm{d}y) \ .$$

We observe that for $\delta = 1$ we have $\mathcal{S}_\delta = \varnothing$. Then the dynamics of the CBBM do not have a discrete reproduction component, and in this case the process is the *unconditional* dual of (1.2). The duality between the CBBM and the FKPP equation will be established in Proposition 3.6.

The necessity of dealing with $\delta \in (0,1)$ (and in particular, we will eventually consider the limit $\delta \to 0$) is forced upon us to capture the exact wave speed of (1.2). In fact the martingale problem for the $\mathfrak{R}_1$–CBBM (or alternatively, [11, Lemma 3]) implies the following (in fact $e^{-\mathbf{r}t}I_t$ is a martingale, although it is in general *not* uniformly integrable).

Proposition 2.6 (Invariance of expectation)**.** *Let $\mathfrak{R} \in \mathcal{M}$ be any measure and $\mathbf{C} = (\mathbf{C}_t)_{t\geq 0}$ be an $\mathfrak{R}_1$–CBBM and write $I_t = \ell(\mathbf{C}_t)$ for the total number of particles at time $t \geqslant 0$. Then*

$$\mathbb{E}[I_t] = I_0\mathrm{e}^{\mathbf{r}t} \ , \quad \textit{with} \quad \mathbf{r} = \mathfrak{R}([0,1]) \ . \tag{2.4}$$

In the present case, in which the nonlinearity in (1.2) is concave, the wave speed is determined by the growth of the linearisation near $u = 0$ of the equation: this is referred to as the *pulled* regime [21]. Moreover, the growth of the linearisation is roughly equivalent to that of the dual process. On the other hand, Jensen's inequality guarantees that the speed of the expected value of the solution to (1.2) is strictly slower than in the classical case (with same total mass for the reproduction), since

$$\partial_t \mathbb{E}[u_t] \leqslant \frac{1}{2}\Delta \mathbb{E}[u_t] + \mathbf{r}\mathbb{E}[u_t](1-\mathbb{E}[u_t]) \ ,$$

where we used that $M_t^f = \int_0^t \int_0^1 y f_s \mathcal{R}(\mathrm{d}s, \mathrm{d}y) - \int_0^t \mathfrak{R}((0,1]) f_s \mathrm{d}s$ is a martingale for bounded, adapted f. As an educated guess, one can think that the speed of the expected value of the solution is the same as the wave speed of the solution itself, and Theorem 2.8 below shows that this is indeed true. Hence we are faced with an apparent conundrum, as the speed predicted by (2.4) is exactly the deterministic (annealed) one, which we now know to be incorrect.

The issue is that the coordinated process $\mathbf{C}_t$, unlike the branching Brownian motion, is strongly catalytic (apart from the case $\mathfrak{R} = c\delta_0$, $c > 0$, in which the two processes coincide): Namely its almost sure growth rate is strictly smaller than its annealed growth rate, captured by (2.4). For this reason, classical martingale arguments do not work directly.

Our approach is therefore to use the conditioning as a way to obtain the almost sure growth rate. As usual for Poisson point processes, one has to take particular care of the small jumps: for this reason we consider a fixed parameter $\delta > 0$. Small jumps are then dealt with via the argument we just explained, through Jensen's inequality and martingales. This delivers a wrong estimate, but now with an error of order $\mathcal{O}(\delta)$, in such a way that as $\delta \to 0$ we obtain the correct speed.

2.2 Main results

Now we are ready to present our main results. We start by proving well-posedness of (1.2).

Theorem 2.7. *Fix any $\mathfrak{R} \in \mathcal{M}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a Poisson point process $\mathcal{S}$ on $[0,\infty) \times (0,1]$ with intensity measure $\mathrm{d}t \otimes \frac{1}{y}\mathfrak{R}(\mathrm{d}y)$. Let $\mathcal{F}_t$ be the right-continuous filtration generated by $\mathcal{S}^t = \mathcal{S} \cap ([0,t] \times (0,1])$. For any $\alpha \in (0,1)$ and any initial condition u_0 in $C_b^\alpha(\mathbb{R};[0,1])$ there exists a unique (up to modifications on a nullset) adapted process*

$$u \colon \Omega \to \mathbb{D}([0,\infty); C_b^\alpha([0,1];\mathbb{R}))$$

that solves (1.2) *on $[0,\infty) \times \mathbb{R}$ (with the derivatives interpreted in the sense of distributions and the integral against $\mathcal{R}$ interpreted in the sense of sums over Poisson point processes, cf.* (2.3) *for $\delta = 1$) with $u(0,\cdot) = u_0(\cdot)$.*

This result is a consequence of Proposition 3.4. For the solution we just constructed we can describe the wave speed as follows.

Theorem 2.8. *For every $\mathfrak{R} \in \mathcal{M}$, $\alpha \in (0,1)$ and any initial condition $u_0 \in C_{0,1}^\alpha$, the solution u to the FKPP equation* (1.2) *with initial condition u_0 (as in Theorem 2.7) has wave speed $\mathfrak{s} > 0$ in the sense of Definition 2.1 given by*

$$\frac{\mathfrak{s}^2}{2} = \int_{[0,1]} \log(1+y)\frac{1}{y}\mathfrak{R}(\mathrm{d}y) \ . \tag{2.5}$$

Again, we follow the convention $\int_{\{0\}} \log(1+y)\frac{1}{y}\mathfrak{R}(\mathrm{d}y) = \mathfrak{R}(\{0\}) = \mathfrak{r}$.

Proof. We follow two different arguments for the lower and upper bounds to the wave speed (the two conditions in Definition 2.1).

Step 1. Let us start with the upper bound, so fix any $\lambda > \mathfrak{s}$. Our aim will be to prove that for any $x \in \mathbb{R}$

$$\lim_{t\to\infty} \mathbb{E}u_t(x+\lambda t) = 0 \ ,$$

which implies the required convergence in probability. For this purpose consider $\delta \in (0,1)$ and define $\mathfrak{R}_\delta = (\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+)$ as in Definition 2.4, associated to the decomposition $\mathcal{R} = \mathcal{R}_\delta^- + \mathcal{R}_\delta^+$, where $\mathcal{R}_\delta^+$ is the random measure associated to the Poisson point process $\mathcal{S}_\delta$ with intensity $\mathrm{d}t \otimes \frac{1}{y}\mathfrak{R}_\delta^+(\mathrm{d}y)$. Then let $\mathbb{E}^\delta$ indicate the expectation conditional on $\mathcal{S}_\delta$, namely

$$\mathbb{E}^\delta[f] = \mathbb{E}[f|\mathcal{S}_\delta] \ .$$

Since $u_t(x)$ takes values in $[0,1]$ by dominated convergence it thus suffices to prove that if $\delta = \delta(\lambda) \in (0,1)$ is sufficiently small, then $\mathbb{P}$–almost surely

$$\lim_{t\to\infty} \mathbb{E}^\delta u_t(x+\lambda t) = 0 \ .$$

Here we use the conditional duality of Proposition 3.6 to bound

$$\mathbb{E}^\delta[1-u_t(x+\lambda t)] = \mathbb{E}^\delta\left[(1-u_0)^{\mathbf{C}_t^{\mathbf{x}(\lambda,t)}}\right] \ ,$$

where $\mathbf{C}_t^{\mathbf{x}(\lambda,t)}$ is an $\mathfrak{R}_\delta$–CBBM as in Definition 2.5, started in $\mathbf{x}(\lambda,t) = x+\lambda t \in \mathbb{R}^1$. Now since $u_0 \in C_{0,1}^\alpha$ there exists an $a \in \mathbb{R}$ such that $u_0(x) = 0$ for all $x \geqslant a$. In particular

$$\mathbb{E}^\delta(1-u_0)^{\mathbf{C}_t^{\mathbf{x}(\lambda,t)}} \geqslant \mathbb{E}^\delta(\mathbb{1}_{[a,\infty)})^{\mathbf{C}_t^{\mathbf{x}(\lambda,t)}} = \mathbb{P}^\delta(\mathbf{S}_t \leqslant -a+\lambda t + x) \ ,$$

where $\mathbf{S}_t = \max \mathbf{C}_t^0$ is the rightmost particle of an $\mathfrak{R}_\delta$–CBBM $\mathbf{C}_t^0$ started in $\mathbf{x} = 0 \in \mathbb{R}^1$ (note that by symmetry $\mathbb{P}^\delta(\max \mathbf{C}_t^0 \leq c) = \mathbb{P}^\delta(\min \mathbf{C}_t^0 \geq -c) = \mathbb{E}^\delta(\mathbb{1}_{[-c,\infty)})^{\mathbf{C}_t^0}$). Hence it suffices to show that for any $x_0 \in \mathbb{R}$

$$\lim_{t\to\infty} \mathbb{P}^\delta(\mathbf{S}_t > \lambda t + x_0) = 0 \ .$$

This claim follows from Proposition 4.4, up to choosing δ sufficiently small so that for $\mathfrak{c}_\delta$ as in (4.1)

$$\mathfrak{s} \leqslant \sqrt{2\mathfrak{c}_\delta} < \lambda \ .$$

Step 2. Let us now pass to the lower bound. That is, choose $\lambda < \mathfrak{s}$ and, similarly to above, let us prove that $\lim_{t\to\infty} \mathbb{E}u_t(x+\lambda t) = 1$. As before we can fix $\delta \in (0,1)$, so that it suffices to prove that $\mathbb{P}$–almost surely

$$\lim_{t\to\infty} \mathbb{E}^\delta u_t(x+\lambda t) = 1 \ .$$

Then by the duality of Proposition 3.6 we have

$$\mathbb{E}^\delta(1-u_t)(x+\lambda t) = \mathbb{E}^\delta(1-u_0)^{\mathbf{C}_t^{\mathbf{x}(\lambda,t)}} \leqslant \mathbb{E}^\delta(1-u_0)^{\underline{\mathbf{C}}_t^{\mathbf{x}(\lambda,t)}} \ .$$

Here $\mathbf{C}_t^{\mathbf{x}(\lambda,t)}$ is an $\mathfrak{R}_\delta$–CBBM, with $\mathfrak{R}_\delta = (\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+)$ started in $\mathbf{x}(\lambda,t) = x+\lambda t \in \mathbb{R}^1$, and $\underline{\mathbf{C}}_t^{\mathbf{x}(\lambda,t)}$ is an $\underline{\mathfrak{R}}_\delta$–CBBM associated to compatible measures $\underline{\mathfrak{R}}_\delta = (\mathfrak{r}\delta_0, \mathfrak{R}_\delta^+)$, started in $\mathbf{x}(\lambda,t)$. Then we can use that by definition

$$\mathfrak{r}\delta_0 \leqslant \mathfrak{R}_\delta^- \ ,$$

in the sense of measures, so that we can couple $\mathbf{C}_t$ and $\underline{\mathbf{C}}_t$ in such a way that $\underline{\mathbf{C}}_t \subseteq \mathbf{C}_t$, which implies the desired estimate. In particular it now suffices to prove that

$$\lim_{t\to\infty} \mathbb{E}^\delta (1-u_0)^{\underline{\mathbf{C}}_t^{\mathbf{x}(\lambda,t)}} = 0 \ .$$

Again, we find $b \in \mathbb{R}$ such that $u_0(x) = 1$ for all $x \leqslant b$, so that

$$\mathbb{E}^\delta (1-u_0)^{\underline{\mathbf{C}}_t^{\mathbf{x}(\lambda,t)}} \leqslant \mathbb{P}^\delta(\underline{\mathbf{S}}_t \leqslant -b + \lambda t + x) \ ,$$

where $\underline{\mathbf{S}}_t = \max \underline{\mathbf{C}}_t^0$, the latter being a $(\mathfrak{r}\delta_0, \mathfrak{R}_\delta^+)$–CBBM started in $\underline{\mathbf{C}}_0^0 = 0 \in \mathbb{R}^1$. Now by Proposition 4.5 we know that if $\delta \in (0,1)$ is chosen to be sufficiently small such that for $\underline{\mathfrak{c}}_\delta$ as in (4.4)

$$\lambda < \sqrt{2\underline{\mathfrak{c}}_\delta} \leqslant \mathfrak{s} \ ,$$

then $\lim_{t\to\infty} \mathbb{P}^\delta(\underline{\mathbf{S}}_t \leqslant -b + \lambda t + x) = 0$, $\mathbb{P}$–almost surely. The proof is concluded. □

3 Existence and duality

This section is devoted to proving existence and uniqueness of solution to (1.2), as well as duality. We will first construct unique solutions and observe that they satisfy a certain martingale problem. Then we use the martingale problem to establish duality.

3.1 Existence and uniqueness

Let us start by defining the generator associated to the nonlinearity of (1.2). To be precise, the first definition will be associated to the space-independent equation. The extension to the spatial case passes through cylinder functions, as explained in the subsequent definition of martingale solutions. Here the set C_c^∞ indicates the space of smooth functions with compact support on $\mathbb{R}$. Throughout these construction, we recall that in Definition 2.4 we have divided

$$\mathfrak{R} = \mathfrak{R}_\delta^- + \mathfrak{R}_\delta^+ \ .$$

Definition 3.1. *Fix any $\mathfrak{P} \in \mathcal{M}$. For any n-tuple of smooth functions $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n) \in (C_c^\infty)^n$ and an $F \in C_b^1(\mathbb{R}^n)$ define the cylinder function*

$$C_{\mathrm{loc}}(\mathbb{R}) \ni u \mapsto F_{\boldsymbol{\varphi}}(u) = F(\langle u, \varphi_1\rangle, \ldots, \langle u, \varphi_n\rangle) \ ,$$

where $\langle u, \varphi_i\rangle = \int_\mathbb{R} u\varphi_i \,\mathrm{d}x$, and for any such $F_{\boldsymbol{\varphi}}$ we define the generator

$$\mathcal{L}^\mathfrak{P} F_{\boldsymbol{\varphi}} \colon C_{\mathrm{loc}}(\mathbb{R}) \to \mathbb{R}$$

as follows

$$\mathcal{L}^\mathfrak{P}(F_{\boldsymbol{\varphi}})(u) = \int_{(0,1]} \{F_{\boldsymbol{\varphi}}(u + yu(1-u)) - F_{\boldsymbol{\varphi}}(u)\} \frac{1}{y} \mathfrak{P}(\mathrm{d}y) \ ,$$

for all $u \in C_{\mathrm{loc}}(\mathbb{R})$.

We observe that the integral is well defined at $y = 0$ for every u, since

$$\begin{aligned} &F_{\boldsymbol{\varphi}}(u + yu(1-u)) - F_{\boldsymbol{\varphi}}(u) \\ &\quad = \sum_{i=1}^n \partial_i F(\langle u, \varphi_1\rangle, \ldots, \langle u, \varphi_n\rangle) y \langle u(1-u), \varphi_i\rangle + o(y) \ , \end{aligned}$$

for $y \to 0$ as $F \in C_b^1$.

Remark 3.2. In particular, we can bound for any $F \in C_b^1$ and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$:

$$|\mathcal{L}^{\mathfrak{P}}(F_{\boldsymbol{\varphi}})(u)| \lesssim_{\|F\|_{C_b^1}, \sum_{i=1}^n \|\varphi_i\|_{L^1}} \mathfrak{P}((0,1]) ,$$

where $\|\varphi_i\|_{L^1} = \int_{\mathbb{R}} |\varphi_i(x)| \mathrm{d}x$.

Next we give a precise definition of martingale solutions to the stochastic FKPP equation. We use the following convention. For any $n \in \mathbb{N}$ and $F \in C_b^1(\mathbb{R}^n; \mathbb{R})$ and for any smooth functions $\varphi_i \in C_c^\infty(\mathbb{R}), i = 1, \dots, n$ write, for $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ and any $u \in C_{\mathrm{loc}}(\mathbb{R})$:

$$F_{\boldsymbol{\varphi}}(u) = F\Big(\{\langle u, \varphi_i \rangle\}_{i=1}^n\Big) , \qquad \partial_i F_{\boldsymbol{\varphi}}(u) = \partial_i F\Big(\{\langle u, \varphi_i \rangle\}_{i=1}^n\Big) .$$

We also recall that for $\mathcal{S}_\delta$ as in Definition 2.4 we let $\mathbb{P}^\delta$ be the (random) probability distribution

$$\mathbb{P}^\delta(\mathcal{A}) = \mathbb{P}(\mathcal{A} | \mathcal{S}_\delta) ,$$

and we let $\mathcal{F}_t^\delta$ be the filtration generated by

$$\mathcal{F}_t^\delta = \sigma(\mathcal{S}_\delta^t) , \qquad \mathcal{S}_\delta^t \stackrel{\text{def}}{=} \mathcal{S}^t \cup \mathcal{S}_\delta = (\mathcal{S} \cap ([0,t] \times (0,1])) \cup \mathcal{S}_\delta .$$

Definition 3.3 (Conditional martingale solution)**.** *Fix any $\mathfrak{R} \in \mathcal{M}$ as well as $\delta \in (0,1], \alpha \in (0,1)$ and $u_0 \in C_b^\alpha$. Let u be a stochastic process over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{D}\big([0,\infty); C_b^\alpha(\mathbb{R}; [0,1])\big)$. Let $\mathcal{S}_\delta = \{(t_j, y_j)\}_{j \in \mathbb{N}}$ be a Poisson point process as in Definition 2.4, defined on the same probability space.*

We say that u is a martingale solution to Equation (1.2) *on $[0,\infty)$ with initial condition u_0,* conditional on $\mathcal{S}_\delta$, *if $u(0,\cdot) = u_0(\cdot)$ and the following conditions are satisfied for any $n \in \mathbb{N}$, $F \in C_b^1(\mathbb{R}^n; \mathbb{R})$ and $\varphi_i \in C_c^\infty(\mathbb{R})$ for $i = 1, \dots, n$:*

1. *For all $j \in \mathbb{N}$ the process*

$$M_t^F := F_{\boldsymbol{\varphi}}(u_t) - F_{\boldsymbol{\varphi}}(u_{t_j}) - \int_{t_j}^t \mathcal{L}^\delta(F_{\boldsymbol{\varphi}})(u_s) \mathrm{d}s$$

is an $\mathcal{F}_t^\delta$–càdlàg centered martingale for t in $[t_j, t_{j+1})$, with $\mathcal{L}^\delta(F_{\boldsymbol{\varphi}})$ defined as:

$$\mathcal{L}^\delta(F_{\boldsymbol{\varphi}})(u) = \left(\sum_{i=1}^n \partial_i F_{\boldsymbol{\varphi}}(u) \cdot \Big(\langle u, \frac{1}{2} \Delta \varphi_i \rangle + \langle \mathfrak{r} u(1-u), \varphi_i \rangle \Big) \right) + \mathcal{L}^{\mathfrak{R}_\delta^-}(F_{\boldsymbol{\varphi}})(u) . \quad (3.1)$$

2. *For all $j \in \mathbb{N}$ the martingale M_t^F has predictable quadratic variation, for $t \in [t_j, t_{j+1})$*

$$\langle M^F \rangle_t = \int_{t_j}^t \mathcal{L}^\delta((F_{\boldsymbol{\varphi}})^2)(u_s) - 2F_{\boldsymbol{\varphi}}(u_s) \mathcal{L}^\delta(F_{\boldsymbol{\varphi}})(u_s) \mathrm{d}s . \quad (3.2)$$

3. *And finally for all $j \in \mathbb{N}$ we have $u_{t_j} = u_{t_j-} + y_j u_{t_j-}(1 - u_{t_j-})$.*

In this setting we find the following result.

Proposition 3.4. *Fix any $\mathfrak{R} \in \mathcal{M}$, as well as $\alpha \in (0,1)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a Poisson point process $\mathcal{S} = \mathcal{S}_0$ as in Definition 2.4. For every $u_0 \in C_b^\alpha(\mathbb{R}; [0,1])$ there exists a unique solution*

$$u \colon \Omega \to \mathbb{D}([0,\infty); C_b^\alpha(\mathbb{R}; [0,1])) \cap \mathbb{D}((0,\infty); C_b^\beta(\mathbb{R}; [0,1]))$$

to (1.2) *in the sense of Theorem 2.7, for arbitrary $\beta \geqslant 0$. Moreover such u satisfies for any $\delta \in (0,1]$ the conditional martingale problem of Definition 3.3.*

Proof. To construct solutions our approach is to build an approximating sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ associated to Poisson point processes that has only finitely many jumps. For this reason, for $\varepsilon \in (0,1)$ we recall that the measure $\mathcal{R}^+_\varepsilon$ is the restriction of the Poisson random measure $\mathcal{R}$ to $[0,\infty)\times(\varepsilon,1]$, as in from Definition 2.4 (we call $\mathcal{R}$ a Poisson random measure, but it can only be integrated against functions that vanish sufficiently quickly near $y=0$, cf. (2.3) for $\delta=1$).

In this way the Poisson random measure $\mathcal{R}^+_\varepsilon(\mathrm{d}t,\mathrm{d}y)$ has intensity $\frac{1}{y}\mathfrak{R}^+_\varepsilon(\mathrm{d}y)\otimes\mathrm{d}t$ with finite total mass and we recall the representation through the locally finite Poisson point process $\mathcal{S}_\varepsilon = \{(t_i,y_i)\}_{i\in\mathbb{N}}$

$$\mathcal{R}^+_\varepsilon = \sum_{i\in\mathbb{N}} \delta_{(t_i,y_i)} \ .$$

Let now $u^\varepsilon \in \mathbb{D}([0,\infty);C^\alpha_b)$ be the solution to:

$$\mathrm{d}u^\varepsilon_t = \frac{1}{2}\Delta u^\varepsilon_t\mathrm{d}t + \mathfrak{r}u^\varepsilon_t(1-u^\varepsilon_t)\mathrm{d}t + \int_{(\varepsilon,1]} y u^\varepsilon_{t-}(1-u^\varepsilon_{t-})\mathcal{R}^+_\varepsilon(\mathrm{d}t,\mathrm{d}y), \tag{3.3}$$

with initial condition u_0. Here solutions are defined by the following constraints for any $j\in\mathbb{N}$

$$\begin{aligned} u^\varepsilon_t &= P_{t-t_j}u^\varepsilon_{t_j} + \int_{t_j}^t \mathfrak{r}P_{t-s}[u^\varepsilon_s(1-u^\varepsilon_s)]\mathrm{d}s\ , \qquad \forall t\in[t_j,t_{j+1})\ ,\\ u^\varepsilon_{t_j} &= u^\varepsilon_{t_j-} + y_j u^\varepsilon_{t_j-}(1-u^\varepsilon_{t_j-})\ , \end{aligned}$$

where P_t is the heat semigroup

$$P_t\varphi(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}}\varphi(y)\mathrm{e}^{-\frac{|x-y|^2}{2t}}\mathrm{d}y\ .$$

Step 1: Existence and uniqueness. We start by proving that the sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ is Cauchy in $\mathbb{D}([0,T];C_b(\mathbb{R};[0,1]))$ for $\varepsilon\to 0$ and any $T>0$. Fix any two $0<\overline{\varepsilon}<\varepsilon<1$ and let $v^{\overline{\varepsilon},\varepsilon}$ be the difference $v^{\overline{\varepsilon},\varepsilon} = u^{\overline{\varepsilon}} - u^\varepsilon$. Now we observe that $v^{\overline{\varepsilon},\varepsilon}$ is positive: $v^{\overline{\varepsilon},\varepsilon}_t(x)\geqslant 0$ for all $t\geqslant 0, x\in\mathbb{R}$. This follows because the solution to the ε-discretised FKPP equation is order preserving, meaning that if $g^1_0\geqslant g^2_0$, then the solution g^i_t for $i\in\{1,2\}$ to $\mathrm{d}g^i = \Delta g^i\mathrm{d}t + \mathfrak{r}g^i(1-g^i)\mathrm{d}t + \int_{(0,1]} yg^i(1-g^i)\mathcal{R}_\varepsilon(\mathrm{d}t,\mathrm{d}y)$ satisfies $g^1_t\geqslant g^2_t$. Then observe that $u^{\overline{\varepsilon}}$ solves the same equation as u^ε apart from the jump times t_j such that $y_j\in(\overline{\varepsilon},\varepsilon]$. In particular, if one assumes that at such times t_j one has $v^{\overline{\varepsilon},\varepsilon}_{t_j-}\geqslant 0$, then it follows that $v^{\overline{\varepsilon},\varepsilon}_{t_j}\geqslant 0$, because $u^{\overline{\varepsilon}}$ is increasing at time t_j: we can then conclude that $v^{\overline{\varepsilon},\varepsilon}\geqslant 0$ until the next time $t_{j'}$ such that $y_{j'}\in(\overline{\varepsilon},\varepsilon]$. By induction, since $v^{\overline{\varepsilon},\varepsilon}_0 = 0$ we obtain as desired that $v^{\overline{\varepsilon},\varepsilon}_t\geqslant 0$ for all $t\geqslant 0$. In addition, $v^{\overline{\varepsilon},\varepsilon}$ solves

$$\begin{aligned} \mathrm{d}v^{\overline{\varepsilon},\varepsilon} =& \frac{1}{2}\Delta v^{\overline{\varepsilon},\varepsilon}\mathrm{d}t + \mathfrak{r}v^{\overline{\varepsilon},\varepsilon}(1-u^\varepsilon-u^{\overline{\varepsilon}})\mathrm{d}t + \int_{(0,1]} yv^{\overline{\varepsilon},\varepsilon}(1-u^\varepsilon-u^{\overline{\varepsilon}})\mathcal{R}^+_\varepsilon(\mathrm{d}t,\mathrm{d}y)\\ &+\int_{(0,1]} yu^{\overline{\varepsilon}}(1-u^{\overline{\varepsilon}})\mathbb{1}_{(\overline{\varepsilon},\varepsilon]}(y)\mathcal{R}^+_{\overline{\varepsilon}}(\mathrm{d}t,\mathrm{d}y)\\ \leqslant& \left(\frac{1}{2}\Delta v^{\overline{\varepsilon},\varepsilon} + \mathfrak{r}v^{\overline{\varepsilon},\varepsilon}\right)\mathrm{d}t + \int_{(0,1]} yv^{\overline{\varepsilon},\varepsilon}\mathcal{R}^+_\varepsilon(\mathrm{d}t,\mathrm{d}y) + \int_{(0,1]} y\mathbb{1}_{(\overline{\varepsilon},\varepsilon]}(y)\mathcal{R}^+_{\overline{\varepsilon}}(\mathrm{d}t,\mathrm{d}y)\ , \end{aligned}$$

where we used that the solution takes values in $[0,1]$. Using that $v^{\overline{\varepsilon},\varepsilon}(0,\cdot)=0$, we find via a maximum principle the upper bound:

$$\begin{aligned}\|v_t^{\overline{\varepsilon},\varepsilon}\|_\infty &\leqslant \int_0^t \int_{(0,1]} y \mathbb{1}_{(\overline{\varepsilon},\varepsilon]}(y) \exp\left\{ \mathfrak{r}(t-s) + \int_s^t \int_{(0,1]} \log\left(1+\overline{y}\right) \mathcal{R}_\varepsilon^+(\mathrm{d}r, \mathrm{d}\overline{y}) \right\} \mathcal{R}_{\overline{\varepsilon}}^+(\mathrm{d}s, \mathrm{d}y) \\ &\leqslant \int_0^t \int_{(0,1]} y \mathbb{1}_{(0,\varepsilon]}(y) \exp\left\{ \mathfrak{r}(t-s) + \int_s^t \int_{(0,1]} \log\left(1+\overline{y}\right) \mathcal{R}(\mathrm{d}r, \mathrm{d}\overline{y}) \right\} \mathcal{R}(\mathrm{d}s, \mathrm{d}y) ,\end{aligned}$$

where the integrals are defined in the sense of Campbell's theorem (see (2.3), or [17, Section 3] for further details). Now, the right-hand side decreases to zero as $\varepsilon \to 0$, uniformly over $t \in [0,T]$:

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|v_t^{\overline{\varepsilon},\varepsilon}\|_\infty = 0 .$$

Since the supremum norm (in time) dominates the Skorohod distance, the claimed convergence in $\mathbb{D}([0,T]; C_b(\mathbb{R}; [0,1]))$ follows. We observe that the same argument delivers also uniqueness of solutions, if we replace $u^{\overline{\varepsilon}}$ with any solution u to (1.2).

Step 2: Regularity. Now we focus on the C_b^α regularity of the solutions u^ε. Since the only issue for the regularity comes from the Poisson jumps let us assume without loss of generality that $\mathfrak{r} = 0$. The argument we present works verbatim to obtain the required regularity for the limiting solution u: We provide instead a bound that is uniform in ε instead, which will be useful to obtain the martingale problem in our last step. The bound reads as follows:

$$\sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \|u_t^\varepsilon\|_{C_b^\alpha} < C < \infty ,$$

for some random constant $C > 0$. To prove this statement we can write u^ε in its mild formulation, as a convolution with the heat semigroup:

$$u_t^\varepsilon = P_t u_0 + \int_0^t \int_{(0,1]} y P_{t-s} u_{s-}^\varepsilon (1 - u_{s-}^\varepsilon) \mathcal{R}_\varepsilon^+(\mathrm{d}s, \mathrm{d}y) .$$

Then we can use classical Schauder estimates, which captures the regularisation of the Laplacian

$$\|P_t \varphi\|_{C_b^\alpha} \leqslant C(\alpha, \gamma) t^{-\frac{\gamma}{2}} \|\varphi\|_\infty ,$$

for some $C(\alpha,\gamma) > 0$ and all α, γ such that $\alpha, \gamma \in (0,\infty) \setminus \mathbb{N}$, $\gamma > \alpha$ and $\alpha + \gamma \in (0,\infty) \setminus \mathbb{N}$ (this follows for example from [1, Theorem 2.24] by embedding L^∞ in a Besov space of negative regularity).

Then we find for any $2 > \gamma > \alpha$

$$\begin{aligned}\|u_t^\varepsilon\|_{C_b^\alpha} &\lesssim \|u_0\|_{C_b^\alpha} + \int_0^t \int_{(0,1]} y \|u_{s-}^\varepsilon (1 - u_{s-}^\varepsilon)\|_\infty (t-s)^{-\frac{\gamma}{2}} \mathcal{R}_\varepsilon^+(\mathrm{d}s, \mathrm{d}y) \\ &\lesssim \|u_0\|_{C_b^\alpha} + \int_0^t \int_{(0,1]} (t-s)^{-\frac{\gamma}{2}} y \mathcal{R}(\mathrm{d}s, \mathrm{d}y) ,\end{aligned} \tag{3.4}$$

where the latter integral is again defined by (2.3), since for $\gamma \in (0,2)$

$$\int_0^t \int_{(0,1]} (t-s)^{-\frac{\gamma}{2}} y \, \frac{1}{y} \mathfrak{R}(\mathrm{d}y) \mathrm{d}s < \infty .$$

Hence our upper bound is proven. We observe that we can additionally deduce the following moment bound:

$$\mathbb{E}\Big[\sup_{\varepsilon \in (0,1)} \sup_{0 \leqslant t \leqslant T} \|u_t^\varepsilon\|_{C_b^\alpha} \Big] < \infty .$$

The bound for arbitrary $\beta \geqslant 0$ follows similarly, allowing a blow-up at $t = 0$.

Step 3: Martingale problem. Since the proof does not vary, we restrict to establishing this property for $\delta = 1$. Hence, consider F as in Definition 3.3. It is straightforward to establish the martingale problem for u^ε (where we are in presence of locally finite jumps). Namely, we have that

$$M_t^{F,\varepsilon} := F_{\boldsymbol{\varphi}}(u_t^\varepsilon) - F_{\boldsymbol{\varphi}}(u_0^\varepsilon) - \int_0^t \mathcal{L}^{1,\varepsilon}(F_{\boldsymbol{\varphi}})(u_s^\varepsilon)\mathrm{d}s$$

is a càdlàg martingale, with

$$(\mathcal{L}^{1,\varepsilon} F_{\boldsymbol{\varphi}})(u) = \left(\sum_{i=1}^n \partial_i F_{\boldsymbol{\varphi}}(u) \cdot \Big(\langle u, \frac{1}{2}\Delta \varphi_i \rangle + \langle \mathfrak{r} u(1-u), \varphi_i \rangle \Big) \right) + \mathcal{L}^{\mathfrak{R}_\varepsilon^+}(F_{\boldsymbol{\varphi}})(u)\mathrm{d}s \ ,$$

in analogy to (3.1) (the notation $\mathcal{L}^{1,\varepsilon}$ is used to avoid confusion with $\mathcal{L}^\delta$, since we are in the case $\delta = 1$). Moreover, $M_t^{F,\varepsilon}$ has predictable quadratic variation

$$\langle M^{F,\varepsilon} \rangle_t = \sum_{i=1}^n \int_{t_j}^t \mathcal{L}^{1,\varepsilon}((F_{\boldsymbol{\varphi}})^2)(u_s^\varepsilon) - 2F_{\boldsymbol{\varphi}}(u_s^\varepsilon)\mathcal{L}^{1,\varepsilon}(F_{\boldsymbol{\varphi}})(u_s^\varepsilon)\mathrm{d}s \ .$$

At this point we would like to establish the continuity

$$\mathcal{L}^{\mathfrak{R}_\varepsilon^+}(F_{\boldsymbol{\varphi}})(u_s^\varepsilon) \to \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(u_s) \ , \qquad \mathcal{L}^{\mathfrak{R}_\varepsilon^+}((F_{\boldsymbol{\varphi}})^2)(u_s^\varepsilon) \to \mathcal{L}^{\mathfrak{R}}((F_{\boldsymbol{\varphi}})^2)(u_s) \ . \tag{3.5}$$

Since from the established convergence of u^ε we know that $F_{\boldsymbol{\varphi}}(u_s^\varepsilon) \to F_{\boldsymbol{\varphi}}(u_s)$ almost surely, (3.5) would guarantee that $M_\cdot^{F,\varepsilon}$ converges to $M_\cdot^F$ almost surely, and similarly the quadratic variation at level ε would converge to the desired limiting quadratic variation almost surely. Now, to establish (3.5) we can compute for any pair $u, v \in C_b(\mathbb{R}; [0,1])$:

$$\left| \mathcal{L}^{\mathfrak{R}_\varepsilon^+}(F_{\boldsymbol{\varphi}})(u) - \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(v) \right| \leqslant \left| \mathcal{L}^{\mathfrak{R}_\varepsilon^+}(F_{\boldsymbol{\varphi}})(u) - \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(u) \right| + \left| \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(u) - \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(v) \right| \ .$$

Now, for the first term we have via Remark 3.2

$$\begin{aligned} \left| \mathcal{L}^{\mathfrak{R}_\varepsilon^+}(F_{\boldsymbol{\varphi}})(u) - \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(u) \right| &= \left| \int_{(0,\varepsilon]} \{ F_{\boldsymbol{\varphi}}(u + yu(1-u)) - F_{\boldsymbol{\varphi}}(u) \} \frac{1}{y} \mathfrak{R}(\mathrm{d}y) \right| \\ &\lesssim_{\|F\|_{C_b^1}, \|\varphi_i\|_{L^1}} \mathfrak{R}((0,\varepsilon]) \to 0 \ . \end{aligned}$$

As for the second one we have

$$\left| \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(u) - \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(v) \right| \leqslant \int_{(0,1]} f(u,v,y) \frac{1}{y} \mathfrak{R}(\mathrm{d}y) \ , \tag{3.6}$$

where for $u, v \in C_b(\mathbb{R}; [0,1])$ we have

$$\begin{aligned} f(u,v,y) &= | \{ F_{\boldsymbol{\varphi}}(u + yu(1-u)) - F_{\boldsymbol{\varphi}}(u) \} - \{ F_{\boldsymbol{\varphi}}(v + yv(1-v)) - F_{\boldsymbol{\varphi}}(v) \} | \\ &\leqslant C\big(\|F\|_{C_b^1}, \sum_{i=1}^n \|\varphi_i\|_{L^1} \big) \{ y \wedge \|u - v\|_\infty \} \ . \end{aligned}$$

Then, since $\|u - v\|_\infty \leqslant 1$ we can split up the integral in (3.6) in the two intervals $[0, \sqrt{\|u - v\|_\infty}]$ and $[\sqrt{\|u - v\|_\infty}, 1]$. We then obtain a bound of the form

$$\left| \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(u) - \mathcal{L}^{\mathfrak{R}}(F_{\boldsymbol{\varphi}})(v) \right| \lesssim_{\|F\|_{C_b^1}, \|\varphi_i\|_{L^1}} \mathfrak{R}((0, \sqrt{\|u - v\|_\infty}]) + \sqrt{\|u - v\|_\infty} \mathfrak{R}((0,1]) \ ,$$

which converges to zero if $u \to v$ in $C_b(\mathbb{R};[0,1])$. Here for the first term we have used Remark 3.2, and for the second term we have estimated

$$\int_{\sqrt{\|u-v\|_\infty}}^1 \|u-v\|_\infty \frac{1}{y}\mathfrak{R}(\mathrm{d}y) \leqslant \sqrt{\|u-v\|_\infty}\mathfrak{R}((0,1]) \ .$$

The second convergence in (3.5) follows similarly. Finally, from (3.5) we then find that $M^{F,\varepsilon}_\cdot$ converges almost surely to $M^F_\cdot$ in $\mathbb{D}([0,T];\mathbb{R})$, for M^F as in Definition 3.3. To conclude that M^F is a martingale, we can bound following the same arguments we just outlined:

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}|M^{F,\varepsilon}_T|^2 < \infty \ .$$

Hence by uniform integrability the limiting point M^F is also a martingale, and a similar argument as above shows that it has the required predictable quadratic variation. □

We conclude with an extension of the previous results to a case in which F is not of the prescribed form $F = F_{\boldsymbol{\varphi}}$. To state this assertion, for $\mathbf{x} \in \mathcal{P}$ of the form $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write

$$\mathbf{x} \dagger x_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots x_n) \in \mathbb{R}^{n-1}.$$

In addition for $\mathbf{z} \in \mathcal{P}$, we write $\mathbf{z} \subseteq \mathbf{x}$ if there exists $i_1 < \ldots < i_m$, with $m \leqslant n$, such that

$$\mathbf{z} = (x_{i_1}, \dots, x_{i_m}) \qquad \text{and} \qquad \mathbf{x} \dagger \mathbf{z} = (x_i : \ i \neq i_j \ , \forall j \in \{1, \dots, m\}) \ .$$

Lemma 3.5. *Assume that u is a martingale solution to Equation (1.2) with initial condition u_0, in the sense of Definition 3.3, and $u(0,\cdot) = u_0(\cdot)$ where $u_0 \in C^\alpha_b$ for $\alpha \in (0,1)$. Then, for any fixed $\delta \in (0,1], \mathbf{x} \in \mathcal{P}$ and any jump time $t_j > 0$ of $\mathcal{S}_\delta$ we have that*

$$\begin{aligned}(1-u_t)^{\mathbf{x}} - &\int_{t_j}^t \sum_{x\in\mathbf{x}} (1-u_s)^{\mathbf{x}\dagger x} \left\{ \frac{1}{2}\Delta(1-u_s)(x) + \mathfrak{r}\, u_s(x)(u_s(x)-1) \right\} \mathrm{d}s \\ &- \int_{t_j}^t \sum_{\mathbf{z}\subseteq\mathbf{x}} \int_{(0,\delta]} y^{\ell(\mathbf{z})}(1-y)^{\ell(\mathbf{x})-\ell(\mathbf{z})} \left\{ (1-u_s)^{\mathbf{x}\sqcup\mathbf{z}} - (1-u_s)^{\mathbf{x}} \frac{1}{y}\mathfrak{R}(\mathrm{d}y) \right\} \mathrm{d}s \end{aligned} \tag{3.7}$$

is a square integrable martingale for $t \in [t_j, t_{j+1})$, with respect to the filtration $\mathcal{F}^\delta$.

Note that the quantity in (3.7) is well-defined, since u is smooth for strictly positive times: $u_s \in C^\infty_b(\mathbb{R})$ for any $s > 0$ by Proposition 3.4.

Proof. Fix any non-negative smooth function $\varphi \in C^\infty_c$ such that $\int_\mathbb{R} \varphi(x)\mathrm{d}x = 1$. Then for all $\zeta > 0$ and $y \in \mathbb{R}$ define

$$\varphi_{x,\zeta}(y) = \frac{1}{\zeta}\varphi\Big(\frac{1}{\zeta}(y-x)\Big).$$

Now, consider

$$F^{(\zeta)}(v) = \prod_{x\in\mathbf{x}} \langle 1-v, \varphi_{x,\zeta}\rangle, \qquad v \in C_{\mathrm{loc}}(\mathbb{R}).$$

Since u is a martingale solution to Equation (1.2) with initial condition u_0, conditional on S_δ, and since $v \mapsto F^{(\zeta)}(v)$ is smooth and bounded over $v \in [0,1]$, we can apply Definition 3.3 to obtain that

$$\begin{aligned} F^{(\zeta)}(u_t) - &\int_{t_j}^t \sum_{x\in\mathbf{x}} \Big(\prod_{y\in\mathbf{x}\dagger x} \langle 1-u_s, \varphi_{y,\zeta}\rangle \Big) \cdot \Big\{ \langle 1-u_s, \frac{1}{2}\Delta\varphi_{x,\zeta}\rangle + \langle \mathfrak{r}\, u_s(u_s-1), \varphi_{x,\zeta}\rangle \mathrm{d}s \Big\} \\ &- \int_{t_j}^t \int_{(0,\delta]} \Big(F^{(\zeta)}(u_s + yu_s(1-u_s)) - F^{(\zeta)}(u_s) \Big) \frac{1}{y}\mathfrak{R}(\mathrm{d}y)\mathrm{d}s \end{aligned} \tag{3.8}$$

is a martingale on $[t_j, t_{j+1})$.

In addition, for $F^{(0)}(u) = (1-u)^{\mathbf{x}}$, there exists a constant $C > 0$ such that

$$\sup_{\|u\|_\infty, \|w\|_\infty \leqslant 1} |F^{(0)}(u + yw) - F^{(0)}(u)| \leqslant Cy\ .$$

Therefore, since $F^{(\zeta)}(u + yu(1-u)) = F^{(0)}(w + yr)$ with

$$w(x) = \langle u, \varphi_{x,\zeta} \rangle\ , \qquad r(x) = \langle u(1-u), \varphi_{x,\zeta} \rangle\ ,$$

so that $\|w\|_\infty, \|r\|_\infty \leqslant 1$, we also have the following bound which is uniform over ζ (for the same constant $C > 0$ as above):

$$\sup_{\zeta \in (0,1), \|u\|_\infty, \|w\|_\infty \leqslant 1} |F^{(\zeta)}(u + yw) - F^{(\zeta)}(u)| \leqslant Cy\ . \tag{3.9}$$

Now, for $\zeta \to 0$ we have that $F^{(\zeta)}(u) \to F^{(0)}(u)$ point-wise. The uniform bound (3.9) guarantees that we can pass to the limit $\zeta \to 0$ under the integral over y in (3.8) and moreover via Definition 3.3 we see that limit is still a martingale as the quadratic variation stays uniformly bounded, again by (3.9). We have therefore concluded that

$$\begin{aligned} F^{(0)}(u_t) - \int_{t_j}^t \sum_{x \in \mathbf{x}} (1-u_s)^{\mathbf{x} \dagger x} \cdot \left\{ \frac{1}{2} \Delta (1-u_s)(x) + \mathfrak{r}\, u_s(x)(u_s(x) - 1) \right\} \mathrm{d}s \\ - \int_{t_j}^t \int_{(0,\delta]} \Big(F^{(0)}(u_s + yu_s(1-u_s)) - F^{(0)}(u_s) \Big) \frac{1}{y} \mathfrak{R}(\mathrm{d}y) \mathrm{d}s \end{aligned}$$

is a martingale. Finally, we must obtain that the last term coincides with the term in the statement of the lemma. To see this, we compute

$$(1 - u - yu(1-u))^{\mathbf{x}} = (1-u)^{\mathbf{x}} (1-yu)^{\mathbf{x}} = (1-u)^{\mathbf{x}} (1 - y + y(1-u))^{\mathbf{x}}\ .$$

Then we use the binomial formula:

$$(u + v)^{\mathbf{x}} = \sum_{\mathbf{z} \subseteq \mathbf{x}} u^{\mathbf{z}} v^{\mathbf{x} \dagger \mathbf{z}}\ .$$

In particular, we obtain that

$$(1 - y + y(1-u))^{\mathbf{x}} = \sum_{\mathbf{z} \subseteq \mathbf{x}} y^{\ell(\mathbf{z})} (1-y)^{\ell(\mathbf{x} \dagger \mathbf{z})} (1-u)^{\mathbf{z}}\ ,$$

so we can finally rewrite:

$$\begin{aligned} \int_{(0,\delta]} (1 - u - y(u(1-u)))^{\mathbf{x}} - (1-u)^{\mathbf{x}} \frac{1}{y} \mathfrak{R}(\mathrm{d}y) \\ = \sum_{\mathbf{z} \subseteq \mathbf{x}} \int_{(0,\delta]} y^{\ell(\mathbf{z})} (1-y)^{\ell(\mathbf{x} \dagger \mathbf{z})} \left\{ (1-u)^{\mathbf{x} \sqcup \mathbf{z}} - (1-u)^{\mathbf{x}} \right\} \frac{1}{y} \mathfrak{R}(\mathrm{d}y)\ , \end{aligned}$$

from which our claim follows. □

3.2 Duality

As we have already discussed, we will consider the solution to (1.2) conditional on the large jumps $\mathcal{R}_\delta^+$. In particular, the solution u to (1.2) can be formally rewritten as

$$\begin{aligned} \mathrm{d}u_t =& \frac{1}{2} \Delta u_t \mathrm{d}t + \mathfrak{r}\, u_t(1-u_t) \mathrm{d}t \\ &+ \int_{(0,\delta]} y u_{t-}(1-u_{t-}) \mathcal{R}_\delta^-(\mathrm{d}t, \mathrm{d}y) + \int_{(\delta,1]} y u_{t-}(1-u_{t-}) \mathcal{R}_\delta^+(\mathrm{d}t, \mathrm{d}y)\ . \end{aligned} \tag{3.10}$$

Here the integral against $\mathcal{R}_\delta^-$ should be interpreted in the sense of (2.3). Then, let $\mathbb{E}^\delta$ indicate expectation conditional on $\mathcal{S}_\delta$ as in (2.1), or equivalently conditional on $\mathcal{R}_\delta^+$ as in Definition 2.4:

$$\mathbb{E}^\delta[f] = \mathbb{E}[f|\mathcal{S}_\delta] \ .$$

For $u \in C_{\text{loc}}(\mathbb{R})$ let us recall the notation

$$u^{\mathbf{C}_t} = \prod_{i=1}^{\ell(\mathbf{C}_t)} u(C_t^{(i)}) \ .$$

We find the following duality relation.

Proposition 3.6. *Fix $\mathfrak{R} \in \mathcal{M}$ and, for any $\delta \in (0,1]$, let $\mathfrak{R}_\delta = (\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+)$ be as in Definition 2.4. Then, let u be a martingale solution to* (3.10) *conditioned on $\mathcal{S}^\delta$, associated to $\mathfrak{R}$ in the sense of Definition 3.3. Furthermore, for any $\mathbf{x} \in \mathcal{P}$, let $(\mathbf{C}_t)_{t\geqslant 0}$ be an $\mathfrak{R}_\delta$–CBBM started in $\mathbf{x}$ as in Definition 2.5. Then for any $t > 0$*

$$\mathbb{E}^\delta\left[(1-u_t)^{\mathbf{x}}\right] = \mathbb{E}^\delta\left[(1-u_0)^{\mathbf{C}_t}\right] \ . \tag{3.11}$$

Proof. Since whether Equation (3.11) holds only depends on the marginal laws of the couple $((\mathbf{C}_t)_{t\geqslant 0}, (u_t)_{t\geqslant 0})$ under the (random) probability measure $\mathbb{P}^\delta$, we can without loss of generality assume that the two processes are independent conditional on $\mathcal{S}_\delta$.

Our aim is then to prove an even stronger statement, namely that for any $t > 0$ the process

$$[0,t] \ni s \mapsto \mathbb{E}^\delta(1-u_{t-s})^{\mathbf{C}_s} \ \text{ is constant.} \tag{3.12}$$

From the definition of $\mathfrak{R}_\delta$-CBBM we find that

$$\begin{aligned}(1-u)^{\mathbf{C}_t} - &\int_{t_j}^t \sum_{x\in\mathbf{C}_s} (1-u)^{\mathbf{C}_s \dagger x}\left\{\frac{1}{2}\Delta(1-u)(x) + \mathfrak{r}\, u(x)(u(x)-1)\right\}\mathrm{d}s \\ &- \int_{t_j}^t \sum_{\mathbf{z}\subseteq\mathbf{C}_{s-}} \int_{(0,\delta]} y^{\ell(\mathbf{z})}(1-y)^{\ell(\mathbf{C}_{s-})-\ell(\mathbf{z})}\left\{(1-u)^{\mathbf{C}_{s-}\sqcup\mathbf{z}} - (1-u)^{\mathbf{C}_{s-}}\frac{1}{y}\mathfrak{R}(\mathrm{d}y)\right\}\mathrm{d}s \ ,\end{aligned}$$

is a square integrable martingale on $[t_j, t_{j+1})$, for any fixed $u \in C_b^2$. In addition, by Lemma 3.5 we have that also

$$\begin{aligned}(1-u_t)^{\mathbf{x}} - &\int_{t_j}^t \sum_{x\in\mathbf{x}} (1-u_s)^{\mathbf{x}\dagger x}\left\{\frac{1}{2}\Delta(1-u_s)(x) + \mathfrak{r}\, u_s(x)(u_s(x)-1)\right\}\mathrm{d}s \\ &- \int_{t_j}^t \sum_{\mathbf{z}\subseteq\mathbf{x}} \int_{(0,\delta]} y^{\ell(\mathbf{z})}(1-y)^{|\mathbf{x}|-\ell(\mathbf{z})}\left\{(1-u_s)^{\mathbf{x}\sqcup\mathbf{z}} - (1-u_s)^{\mathbf{x}}\frac{1}{y}\mathfrak{R}(\mathrm{d}y)\right\}\mathrm{d}s\end{aligned}$$

is a square integrable martingale on $[t_j, t_{j+1})$.

In particular, since the two drifts match each other and the processes are assumed to be independent, upon taking expectations we find that for $t \in [t_j, t_{j+1})$

$$s \mapsto \mathbb{E}^\delta(1-u_{t-s})^{\mathbf{C}_s} \text{ is constant on } [t_j, t] \ .$$

Now, at time t_j, we find for $z_t = 1 - u_t$

$$\begin{aligned}(1-u_{t_j})^{\mathbf{x}} &= (1 - u_{t_{j-}} - y_j u_{t_j-}(1-u_{t_j-}))^{\mathbf{x}} \\ &= (z_{t_j-} - y_j(1-z_{t_j-})z_{t_j-})^{\mathbf{x}} \\ &= ((1-y_j)z_{t_{j-}} + y_j z_{t_j-}^2)^{\mathbf{x}} \ .\end{aligned}$$

Hence in particular

$$(1-u_{t_j})^{\mathbf{x}} - (1-u_{t_j-})^{\mathbf{x}} = \left\{ \sum_{\mathbf{z}\subseteq\mathbf{x}} y_j^{\ell(\mathbf{z})}(1-y_j)^{\ell(\mathbf{x}\dagger\mathbf{z})} z_{t_j-}^{\mathbf{x}\dagger\mathbf{z}} (z_{t_j-}^2)^{\mathbf{z}} \right\} - z_{t_j-}^{\mathbf{x}}$$
$$= \sum_{\mathbf{z}\subseteq\mathbf{x}} y_j^{\ell(\mathbf{z})}(1-y_j)^{\ell(\mathbf{x}\dagger\mathbf{z})} \left\{ z_{t_j-}^{\mathbf{x}\dagger\mathbf{z}} (z_{t_j-}^2)^{\mathbf{z}} - z_{t_j-}^{\mathbf{x}} \right\},$$

where we used that

$$\sum_{\mathbf{z}\subseteq\mathbf{x}} y_j^{\ell(\mathbf{z})}(1-y_j)^{\ell(\mathbf{x}\dagger\mathbf{z})} = 1 .$$

This corresponds again to the branching mechanism of $\mathbf{C}_t$, so that we can deduce that $\mathbb{E}^{\delta}(1-u_{t-s})^{\mathbf{C}_s}$ is constant for $s \in (0,t)$. Taking the limit $s \downarrow 0$ and $s \uparrow t$ delivers the result on the closed interval $[0,t]$. □

4 Wave speed

4.1 Conditional dual

Apart from the results of Section 3, the last ingredient we will need in the study of the wave speed is the following almost sure asymptotic property for $\mathcal{S}_\delta$.

Lemma 4.1. *For any $\delta \in (0,1)$ and $\varepsilon \in [0,\infty)$ let $\mathcal{S}_\delta$ be defined as in (2.1) and associated to $\mathfrak{R}_\delta = (\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+)$ as in Definition 2.4. Then for*

$$\mathfrak{d}_{\delta,\varepsilon}(t) = \sum_{j\,:\,t_j \leqslant t} \log(1+y_j+\varepsilon), \quad \mathfrak{d}_{\delta,\varepsilon} = \int_{(\delta,1]} \log(1+y+\varepsilon)\frac{1}{y}\mathfrak{R}(\mathrm{d}y),$$

it holds that almost surely that

$$\lim_{t\to\infty} \frac{1}{t}\mathfrak{d}_{\delta,\varepsilon}(t) = \mathfrak{d}_{\delta,\varepsilon} .$$

For simplicity we will write $\mathfrak{d}_\delta(t)$ for $\mathfrak{d}_{\delta,0}(t)$ in the case $\varepsilon = 0$ (and similarly for $\mathfrak{d}_\delta$).

Proof. We have

$$\frac{1}{t}\mathfrak{d}_{\delta,\varepsilon}(t) = \Big(\frac{1}{\#\{j\,:\,t_j \leqslant t\}} \sum_{j\,:\,t_j\leqslant t} \log(1+y_j+\varepsilon)\Big)\frac{\#\{j\,:\,t_j\leqslant t\}}{t} .$$

Now, the first factor on the right-hand side converges a.s. to $\frac{\int_{(\delta,1]} \frac{\log(1+y+\varepsilon)}{y}\mathfrak{R}(\mathrm{d}y)}{\int_{(\delta,1]}\frac{1}{y}\mathfrak{R}(\mathrm{d}y)}$ due to the strong law of large numbers, while the second factor converges a.s. to $\int_{(\delta,1]}\frac{1}{y}\mathfrak{R}(\mathrm{d}y)$ due to the Poisson law of large numbers [17, Section 4.2]. □

4.2 Upper bound on the wave speed

We start by establishing an upper bound on the quenched (w.r.t. jumps larger than δ) growth rate of the dual process.

Proposition 4.2. *Fix $\mathfrak{R} \in \mathcal{M}$ and, for any $\delta \in (0,1]$, let $\mathfrak{R}_\delta = (\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+)$ be as in Definition 2.4. Let $\mathbf{C}_t$ be an $\mathfrak{R}_\delta$–CBBM started in $\mathbf{x} \in \mathbb{R}^1$. Then for $I_t^\delta = \ell(\mathbf{C}_t)$ we have almost surely*

$$\limsup_{t\to\infty} \frac{1}{t}\log I_t^\delta \leqslant \mathfrak{R}_\delta^-([0,\delta]) + \int_{(\delta,1]} \log(1+y)\frac{1}{y}\mathfrak{R}_\delta^+(\mathrm{d}y) =: \mathfrak{c}_\delta . \tag{4.1}$$

Note that since $\log(1+y)\frac{1}{y} \leqslant 1$ we always have

$$\mathfrak{R}(\{0\}) + \int_{(0,1]} \log(1+y)\frac{1}{y}\mathfrak{R}(\mathrm{d}y) \leqslant \mathfrak{c}_\delta \ ,$$

which reflects the fact that in the following we obtain an *upper* bound on the wave speed. On the other hand, as $\delta \to 0$ we obtain

$$\mathfrak{c}_\delta \to \mathfrak{R}(\{0\}) + \int_{(0,1]} \log(1+y)y^{-1}\mathfrak{R}(\mathrm{d}y) = \mathfrak{c} = \frac{\mathfrak{s}^2}{2} \ , \tag{4.2}$$

where $\mathfrak{s}$ is the wave speed defined by (2.5).

Remark 4.3. Although we only prove an upper bound on the growth rate of I_t^δ, the arguments we present for the lower bound of the wave speed allow to also prove a lower bound to the growth of I^δ by comparing I^δ to the a process where we do not have any jumps with impact $y \in (0,\delta]$ (and everything else being left unvaried). Alternatively, one can also use the same comparison, but in combination with the "channelling" argument that we will use in Section 4.4. Overall, one would thus obtain that almost surely (independently of δ!)

$$\lim_{t\to\infty} \frac{1}{t}\log I_t^\delta = \mathfrak{c} \ ,$$

with $\mathfrak{c}$ as in (4.2).

The proof of Proposition 4.2 will be carried out in Section 4.4. Using the previous result, we obtain an upper bound on the wave speed via a quenched version of the so-called many-to-one lemma, cf. [4, Section 3.6].

Proposition 4.4. *Fix $\mathfrak{R} \in \mathcal{M}$ and, for any $\delta \in (0,1]$, let $\mathfrak{R}_\delta = (\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+)$ be as in Definition 2.4. Let $\mathbf{C}_t$ be an $\mathfrak{R}_\delta$–CBBM started in $\mathbf{x} = 0 \in \mathbb{R}^1$. Then for any $x_0 \in \mathbb{R}$ and $\mathbf{S}_t = \max \mathbf{C}_t$ we have, for any $\lambda > \sqrt{2\mathfrak{c}_\delta}$, $\mathbb{P}$–almost surely*

$$\lim_{t\to\infty} \mathbb{P}^\delta(\mathbf{S}_t > \lambda t + x_0) = 0 \ .$$

Proof. Without loss of generality we restrict to the case $x_0 = 0$. Let us also write $\mathbf{C}_t = (C_t^{(i)})_{i\in[I_t^\delta]}$ for $t > 0$ and $I_t^\delta = \ell(\mathbf{C}_t)$, where we assume that the ordering of particles is exchangeable (which can e.g. be achieved via reshuffling the indices at the time of any reproduction event uniformly in a right-continuous manner). Then, at any given time $t \geq 0$, conditional on I_t^δ, the particles are identically (but not independently) distributed and their marginal law is that of a Brownian motion at time t, so that

$$\mathbb{P}^\delta(C_t^{(i)} > x | I_t^\delta) = \Phi(x/\sqrt{t}) \ ,$$

with $\Phi(z) = \mathbf{P}(\mathcal{N} \geqslant x)$ for a standard Gaussian $\mathcal{N}$. Hence, conditional on the jump times, we obtain that for any $t \geq 0$ and $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}^\delta(\mathbf{S}_t > x | I_t^\delta) &= \mathbb{P}^\delta\big(\exists i \in [I_t^\delta] \colon C_t^{(i)} > x \big| I_t^\delta\big) \\
&\leq \mathbb{E}^\delta\Big[\sum_{i\in[I_t^\delta]} \mathbb{1}_{\{C_t^{(i)}>x\}} \Big| I_t^\delta\Big] \mathbb{1}_{\{I_t^\delta \leqslant \mathrm{e}^{(\mathfrak{c}_\delta+\varepsilon)t}\}} + \mathbb{1}_{\{I_t^\delta > \mathrm{e}^{(\mathfrak{c}_\delta+\varepsilon)t}\}} \\
&\leqslant I_t^\delta\, \Phi(x/\sqrt{t}) \mathbb{1}_{\{I_t^\delta \leqslant \mathrm{e}^{(\mathfrak{c}_\delta+\varepsilon)t}\}} + \mathbb{1}_{\{I_t^\delta > \mathrm{e}^{(\mathfrak{c}_\delta+\varepsilon)t}\}} \ .
\end{aligned}$$

Thus, the Gaussian tail bound

$$\int_t^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2}\mathrm{d}y \leqslant \frac{1}{\sqrt{2\pi}t}\mathrm{e}^{-\frac{t^2}{2}}, \qquad t \geq 0$$

implies that for $\varepsilon, \lambda > 0$

$$\mathbb{P}^\delta(\mathbf{S}_t > \lambda t | I_t^\delta)\mathbb{1}_{\{I_t^\delta \leqslant \mathrm{e}^{(\mathfrak{c}_\delta+\varepsilon)t}\}} \leq \frac{1}{\sqrt{2\pi t}\lambda}\mathrm{e}^{(\mathfrak{c}_\delta+\varepsilon)t-\frac{\lambda^2 t}{2}}$$

holds almost surely, whence for $\lambda > \sqrt{2(\mathfrak{c}_\delta+\varepsilon)}$ we obtain by Proposition 4.2 that almost surely

$$\lim_{t\to\infty} \mathbb{P}^\delta(\mathbf{S}_t > \lambda t) \leqslant \lim_{t\to\infty}\left\{\frac{1}{\sqrt{2\pi t}\lambda}\mathrm{e}^{(\mathfrak{c}_\delta+\varepsilon)t-\frac{\lambda^2 t}{2}} + \mathbb{P}^\delta(I_t^\delta > \mathrm{e}^{(\mathfrak{c}_\delta+\varepsilon)t})\right\} = 0 \,. \tag{4.3}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the result follows. □

4.3 Lower bound on the wave speed

To obtain the lower bound let us introduce the sequence of constants

$$\mathfrak{r} + \int_{(\delta,1]} \log(1+y)\frac{1}{y}\mathfrak{R}_\delta^+(\mathrm{d}y) =: \underline{\mathfrak{c}}_\delta \,, \tag{4.4}$$

which immediately satisfy $\underline{\mathfrak{c}}_\delta \leqslant \frac{\mathfrak{s}^2}{2}$. Then the main result of this section is the next proposition.

Proposition 4.5. *Fix $\mathfrak{R} \in \mathcal{M}$ and, for any $\delta \in (0,1]$, let $\mathfrak{R}_\delta = (\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+)$ be as in Definition 2.4. Let $\mathbf{C}_t$ be an $(\mathfrak{r}\delta_0, \mathfrak{R}_\delta^+)$–CBBM started in $\mathbf{x} = 0 \in \mathbb{R}^1$. Then for any $0 < \lambda < \sqrt{2\underline{\mathfrak{c}}_\delta}$ and $x_0 \in \mathbb{R}$, we have $\lim_{t\to\infty}\mathbb{P}^\delta(\mathbf{S}_t > \lambda t + x_0) = 1$, $\mathbb{P}$–almost surely.*

We chose as approximating sequence the ordered pair of measures $\{(\mathfrak{r}\delta_0, \mathfrak{R}_\delta^+)\}_{\delta\in(0,1]}$, so that the difference to the original measure is just given by $\mathfrak{R}|_{(0,\delta]}$, which vanishes as $\delta \to 0$: In particular, this is why we included the mass at zero in our approximation. To prove this result, let us associate to the $(\mathfrak{r}\delta_0, \mathfrak{R}_\delta^+)$–CBBM $\mathbf{C}_t$ started in $\mathbf{x} = 0 \in \mathbb{R}^1$ a measure-valued process

$$X_t = \sum_{i=1}^{n(t)} \delta_{x_i(t)} \,, \tag{4.5}$$

where we assume that at time $t \geqslant 0$, $\mathbf{C}_t = (x_1(t), \ldots, x_{n(t)}(t))$. Then, in the spirit of [22, 10] we will link the wave speed to the local survival of the $X_t(\cdot + \lambda t)$.

Proof. Consider the measure-valued process X of (4.5). Now, let $I \subseteq \mathbb{R}$ be any compact interval (that is a set of the form $I = [a,b]$ for $a < b$) and observe that the following implication holds, as long as $\lambda' > \lambda$, for any $x_0 \in \mathbb{R}$:

$$\left\{\liminf_{t\to\infty} X_t(I + \lambda' t) > 0\right\} \Longrightarrow \left\{\liminf_{t\to\infty}(\mathbf{S}_t - \lambda t + x_0) > 0\right\} \,.$$

In particular, our result follows if there exists a family

$$\{R_{\lambda,\zeta} : \quad \lambda \in (0, \sqrt{2\underline{\mathfrak{c}}_\delta}) \,, \zeta \in (0,1)\}$$

of positive, $\mathcal{S}_\delta$ adapted random variables such that for the intervals

$$I_{\lambda,\zeta} = [-R_{\lambda,\zeta}, R_{\lambda,\zeta}]$$

the following is satisfied:

$$\mathbb{P}^\delta\left(\liminf_{t\to\infty} X_t(I_{\lambda,\zeta} + \lambda t) > 0\right) \geqslant 1 - \zeta, \quad \mathbb{P}\text{–almost surely,} \quad \text{for all } 0 < \lambda < \sqrt{2\underline{\mathfrak{c}}_\delta} \,.$$

This is exactly the content of Lemma 4.6 below. □

Lemma 4.6. *Let X be the measure-valued process of* (4.5)*. Then for any $0<\lambda<\sqrt{2\mathfrak{c}_\delta}$ and $\zeta\in(0,1)$ there exists an $\mathcal{S}_\delta$–adapted positive random variable $R_{\lambda,\zeta}$ such that $\mathbb{P}$–almost surely*

$$\mathbb{P}^\delta\big(\liminf_{t\to\infty} X_t(I_{\lambda,\zeta}+\lambda t)>0\big)\geqslant 1-\zeta\ ,$$

with $I_{\lambda,\zeta}=[-R_{\lambda,\zeta},R_{\lambda,\zeta}]$.

Proof. First of all we observe that instead of considering $X_t(A+\lambda t)$ for all measurable $A\subseteq\mathbb{R}$ (that is shifting the sets we are measuring) we can and will consider Brownian motions with a drift λ to the left in the definition of X_t and measure $X_t(A)$ instead. Consider intervals of the form $I(R)=[-R,R]$ for $R>0$: below we will choose a sufficiently large value $R(\lambda,\zeta)$ and prove the desired result for $I_{\lambda,\zeta}=I(R(\lambda,\zeta))$.

Step 1: Continuity of the principal eigenvalue. For any $R>0$ let $(P^{\lambda,R}_t)_{t\geqslant 0}$ be the heat semigroup with drift λ and Dirichlet boundary conditions on $\partial I(R)$, acting on $L^2(I(R))$. Namely, for any $\varphi\in L^2(I(R))$, $P^{\lambda,R}_t\varphi$ satisfies $P^{\lambda,R}_0\varphi=\varphi$ and solves

$$\partial_t P^{\lambda,R}_t\varphi(x)=\Big(\frac{1}{2}\Delta+\lambda\partial_x\Big)P^{\lambda,R}_t\varphi(x),\quad x\in(-R,R),\ \ P^{\lambda,R}_t\varphi(\pm R)=0,\ \ \forall t>0.$$

Let us denote with $\mu(\lambda,R)=\sup\frac{1}{t}\log\sigma(P^{\lambda,R}_t)$ the principal eigenvalue of $\frac{1}{2}\Delta+\lambda\partial_x$ on $I(R)$ with Dirichlet boundary conditions (here σ indicates the spectrum, and the definition of μ does not depend on $t>0$). Then, we observe that by [29, Section 4, Theorem 4.1]

$$\lim_{R\to\infty}\mu(\lambda,R)=-\frac{\lambda^2}{2}\ , \tag{4.6}$$

the latter being the principal (generalised) eigenvalue of $\frac{1}{2}\Delta+\lambda\partial_x$ on $\mathbb{R}$.

In particular, for any $\lambda<\lambda'<\sqrt{2\mathfrak{c}_\delta}$ we can find a $R_0(\lambda')$ such that $\mu(\lambda,R)\geqslant-\frac{(\lambda')^2}{2}$ for all $R\geqslant R_0(\lambda')$.

Step 2: Local survival. We now consider λ' and $R_0(\lambda')$ as above and define $I_0=I_0(\lambda,\lambda')=[-R_0(\lambda'),R_0(\lambda')]$. Then we introduce a new process $\overline{X}_t$ in which particles evolve as in X_t but are killed (i.e. the disappear from the measure) upon reaching the boundary of I_0. By comparison we obtain that $\overline{X}_t\leqslant X_t$ in the sense of positive measures. We will then start by considering

$$\eta\overset{\mathrm{def}}{=}\mathbb{P}^\delta\big(\liminf_{t\to\infty}\overline{X}_t(I_0)>0\big)\ ,$$

and proving that η is an $\mathcal{S}_\delta$-adapted random variable satisfying

$$\mathbb{P}(\eta>0)=1\ . \tag{4.7}$$

To prove this result let us fix φ the eigenfunction on $L^2(I_0)$ associated to $\mu=\mu(\lambda,R_0(\lambda'))$ (note that $\varphi(x)>0$ for $x\in(-R_0(\lambda'),R_0(\lambda'))$ by the Krein–Rutman theorem, and $\varphi\in C^\infty_b((-R_0(\lambda'),R_0(\lambda')))$ via classical regularity estimates), so that we can write the martingale problem for $\overline{X}_t(\varphi)$ as follows. Next consider the jump times $\{t_j\}_{j\in\mathbb{N}}$ associated to $\mathcal{S}_\delta$. If we fix some $j\in\mathbb{N}$, then we have that for $t_j\leqslant t<t_{j+1}$ the process

$$[t_j,t_{j+1})\ni t\mapsto \mathrm{e}^{-(\mathfrak{r}+\mu)t}\overline{X}_t(\varphi)=M^j_t$$

is a càdlàg martingale under $\mathbb{P}^\delta$ on $[t_j,t_{j+1})$, with predictable quadratic variation

$$\langle M^j\rangle_t=\int_{t_j}^t \mathrm{e}^{-2(\mathfrak{r}+\mu)r}\Big[\mathfrak{r}\,\overline{X}_r(\varphi^2)+\overline{X}_r((\partial_x\varphi)^2)\Big]\mathrm{d}r\ ,$$

where the first term comes from independent reproduction and the second one from the spatial motion of the particles. Next we consider the jumps at times t_j. We have

$$e^{-(\mathfrak{r}+\mu)t_j}\left\{\overline{X}_{t_j}(\varphi)-(1+y_j)\overline{X}_{t_j-}(\varphi)\right\}=\Delta N_j\ ,$$

where ΔN_j is a martingale increment. Since every particle alive a time t_j reproduces with probability y_j independently of all other particles, N_j has the variance of a Bernoulli random variable with parameter y_j:

$$\langle \Delta N_j\rangle = e^{-2(\mathfrak{r}+\mu)t_j}y_j(1-y_j)\overline{X}_{t_j-}(\varphi^2)\ .$$

Overall we can now conclude that the following is a càdlàg martingale on $[0,\infty)$ under $\mathbb{P}_\delta$:

$$L_t=\mathrm{e}^{-(\mathfrak{r}+\mu)t}\bigg(\prod_{j\,:\,t_j\leqslant t}(1+y_j)^{-1}\bigg)\overline{X}_t(\varphi)\ .$$

In fact, for any $0\leqslant s<t<\infty$ with $j(t)\in\mathbb{N}$ uniquely defined by $t\in[t_{j(t)},t_{j(t)+1})$, and assuming that $j(s)<j(t)$ (otherwise the martingale property is inherited immediately from $M^{j(t)}$) and with the notation $\Delta M_j=M^j_{t_{j+1}-}-M^j_{t_j}$, we find that

$$\begin{aligned}L_t-L_s=&\bigg(\prod_{j\leqslant j(t)}(1+y_j)^{-1}\bigg)\Big[(M_t^{j(t)}-M^{j(t)}_{t_{j(t)}})+\Delta N_j\Big]\\&+\sum_{\ell=j(s)+1}^{j(t)-1}\bigg(\prod_{j\leqslant \ell}(1+y_j)^{-1}\bigg)\Big[\Delta M_\ell+\Delta N_\ell\Big]\\&+\bigg(\prod_{j\leqslant j(s)}(1+y_j)^{-1}\bigg)(M^{j(s)}_{t_{j(s)+1}-}-M^{j(s)}_s)\ ,\end{aligned}$$

which is a sum of martingale increments. Hence we have found a positive martingale L_t, which implies that there exists an almost sure limit $\lim_{t\to\infty}L_t=L_\infty\in[0,\infty)$. We want to prove that $\mathbb{E}^\delta L_\infty=L_0>0$, which amounts to proving that the martingale is uniformly integrable. Hence we will show that

$$\sup_{t\geqslant 0}\mathbb{E}^\delta L_t^2=L_0+\sup_{t\geqslant 0}\mathbb{E}^\delta\langle L\rangle_t<\infty\ . \tag{4.8}$$

We are thus left with computing the expected quadratic variation.

Let us now follow the notation of Lemma 4.1 and write $e^{\mathfrak{d}_\delta(t)}=\prod_{j\leqslant j(t)}(1+y_j)$. Then for $t\in[t_j,t_{j+1})$ we find that

$$\mathrm{d}\langle L\rangle_t=\mathrm{e}^{-2(\mathfrak{r}+\mu)t-2\mathfrak{d}_\delta(t)}\bigg(\Big[\overline{X}_t(\mathfrak{r}\varphi^2+(\partial_x\varphi)^2)\Big]+y_j(1-y_j)\overline{X}_{t_j-}(\varphi^2)\delta_{t_j}(t)\bigg)\mathrm{d}t\ .$$

The last ingredient to bound the expected quadratic variation $\mathbb{E}^\delta\langle L_t\rangle$ is to bound the expected value of $\overline{X}_t$. From the definition of $\overline{X}$ we find for $s\in[0,t]$ and any $\psi\in L^2(I_0)$

$$\mathrm{d}\mathbb{E}^\delta\overline{X}_s(P^{\lambda,R_0(\lambda')}_{t-s}\psi)=\mathfrak{r}\mathbb{E}^\delta\overline{X}_s(P^{\lambda,R_0(\lambda')}_{t-s}\psi)\mathrm{d}s+\sum_{j\in\mathbb{N}}y_j\mathbb{E}^\delta\overline{X}_s(P^{\lambda,R_0(\lambda')}_{t-s}\psi)\delta_{t_j}(s)\mathrm{d}s\ . \tag{4.9}$$

Hence for $\psi=\varphi$ we find

$$\mathbb{E}^\delta\overline{X}_t(\varphi)=\mathrm{e}^{(\mathfrak{r}+\mu)t+\sum_{j\leqslant j(t)}\log(1+y_j)}\varphi(0)\ .$$

In particular, we can rewrite $\mathbb{E}^{\delta}\langle L\rangle_t$ as the sum of two terms:

$$\mathbb{E}^{\delta}\langle L\rangle_t = \int_0^t \mathrm{e}^{-2(\mathfrak{r}+\mu)s-2\mathfrak{d}_\delta(s)}\Big(\mathfrak{r}\mathbb{E}^{\delta}\overline{X}_s(\varphi^2) + y_j(1-y_j)\mathbb{E}^{\delta}\overline{X}_{t_j-}(\varphi^2)\delta_{t_j}(s)\Big)\mathrm{d}s \\ + \int_0^t \mathrm{e}^{-2(\mathfrak{r}+\mu)s-2\mathfrak{d}_\delta(s)}\mathbb{E}^{\delta}\overline{X}_s((\partial_x\varphi)^2)\mathrm{d}s\ .$$

The difference between the first and second line is that in the first line we can estimate $\overline{X}_s(\varphi^2) \leqslant \|\varphi\|_\infty \overline{X}_s(\varphi)$, which is not possible for the term in the second line since $\varphi(\pm R_0(\lambda')) = 0$, which does not hold for $(\partial_x\varphi)^2$ (so here we will need some additional arguments). To fix the key point of the proof let us start with the first term. Using the previous computations we find

$$\int_0^t \mathrm{e}^{-2(\mathfrak{r}+\mu)s-2\mathfrak{d}_\delta(s)}\Big(\mathfrak{r}\mathbb{E}^{\delta}\overline{X}_s(\varphi^2) + y_j(1-y_j)\mathbb{E}^{\delta}\overline{X}_{t_j-}(\varphi^2)\delta_{t_j}(s)\Big)\mathrm{d}s \\ \lesssim_{\mathfrak{r},\|\varphi\|_\infty} \int_0^t \exp\Big\{\Big(-(\mathfrak{r}+\mu)-\mathfrak{d}_\delta+o(1)\Big)s\Big\}\,\mathrm{d}s\ .$$

Here the $o(1)$ term is intended as $s\to\infty$, and is a consequence of Lemma 4.1. Now since by assumption $\lambda < \lambda' < \sqrt{2\underline{\mathfrak{c}}_\delta}$ we have

$$-\mathfrak{r}-\mathfrak{d}_\delta-\mu = -\underline{\mathfrak{c}}_\delta-\mu \leqslant -\underline{\mathfrak{c}}_\delta+\frac{(\lambda')^2}{2} \stackrel{\mathrm{def}}{=} -\varepsilon < 0\ .$$

In particular, the integral under consideration is converging for $t\to\infty$. Now, if we pass to the term involving $\mathbb{E}^{\delta}\overline{X}_s((\partial_x\varphi)^2)$, we find by (4.9)

$$\mathbb{E}^{\delta}\overline{X}_s((\partial_x\varphi)^2) = e^{\mathfrak{r}s+\mathfrak{d}_\delta(s)}P_s^{\lambda,R_0(\lambda')}(\partial_x\varphi)^2(0)\ .$$

Now at time $s=1$ we can control the semigroup $P_1^{\lambda,R_0(\lambda')}(\partial_x\varphi)^2 \leqslant C_{\lambda,R_0}\varphi$, for some $C_{\lambda,R_0} > 0$: indeed both $P_1^{\lambda,R_0(\lambda')}(\partial_x\varphi)^2$ and φ are strictly positive in the interior of $I(R_0(\lambda'))$, vanish at the boundary and are *differentiable* at the boundary (differentiability follows for example from [20, Theorem 1.1]), so that the named constant must exist. Hence the term can be controlled following the same arguments as above, observing that

$$\mathbb{E}^{\delta}\overline{X}_s((\partial_x\varphi)^2) \leqslant e^{\mathfrak{r}s+\mathfrak{d}_0 s+\mu(s-1)}C_{\lambda,R_0}\varphi(0)\ ,$$

which is of the same order as the bound used in the previous discussion. Hence (4.8) is proven, and since $L_0 > 0$ we deduce (4.7) from the fact that $\mathbb{P}$-almost surely

$$\mathbb{P}^{\delta}(\liminf_{t\to\infty} X_t(I_0) > 0) \geqslant \mathbb{P}^{\delta}(\liminf_{t\to\infty} L_t > 0) > 0\ .$$

Here we observe that the inclusion

$$\Big\{\liminf_{t\to\infty} L_t > 0\Big\} \subseteq \Big\{\liminf_{t\to\infty} X_t(I_0) > 0\Big\}$$

holds because in the definition of L_t we find, by our assumptions on μ, that

$$e^{-(\mathfrak{r}+\mu)t}\Bigg(\prod_{j\,:\,t_j\leqslant t}(1+y_j)^{-1}\Bigg) \leqslant \mathrm{e}^{-\varepsilon t+o(1)}\ ,$$

as $t\to\infty$.

Step 3: Almost sure survival. Now we want to use (4.7) to prove that if we choose a suitable larger random interval $I_{\lambda,\zeta}$, depending on λ and $\zeta \in (0,1)$, then

$$\mathbb{P}^\delta\big(\liminf_{t\to\infty} X_t(I_{\lambda,\zeta}) > 0\big) \geqslant 1-\zeta\ , \qquad \mathbb{P}\text{–almost surely}\ .$$

For this purpose let us write, for any $n \in \mathbb{N}$ the interval

$$I_0^n = [-nR_0(\lambda), nR_0(\lambda)]\ ,$$

and consider $\overline{X}_t^n$ the process in which particles evolve as in X_t but are killed upon reaching the boundary of I_0^n. Next, note that we have

$$\mathbb{P}^\delta\big(\liminf_{t\to\infty} \overline{X}_t^n(I_0^n) > 0\big) = \mathbb{P}^\delta(\tau^n = \infty)\ ,$$

where τ^n is the extinction time $\tau^n = \inf\{t \geq 0\colon\ \overline{X}_t^n(I_0^n) = 0\}$ and the equality holds because for $t < \tau^n$ we have $\overline{X}_t^n(I_0^n) \geqslant 1$. Now let us prove that

$$\mathbb{P}^\delta(\tau^n < \infty) \leqslant \mathbb{P}^\delta(\tau^1 < \infty)^n = (1-\eta)^n\ , \tag{4.10}$$

with η as in (4.7). Indeed, let us consider $\overline{X}_t^{n,1} \leqslant \overline{X}_t^n$ the process in which particles evolve as in $\overline{X}_t^n$ but are killed upon reaching the boundary of I_0, coupled so that particles in $\overline{X}_t^{n,1}$ are exactly particles of $\overline{X}_t^n$ that have never left the interval I_0. By construction

$$\tau^{n,1} < \tau^n\ ,$$

where $\tau^{n,1}$ is the extinction time of $\overline{X}^{n,1}$. This means that on the event $\tau^n < \infty$ we have at time $\tau^{n,1}$ at least one particle of $\overline{X}_{\tau^{n,1}}^n$ either in $-R_0(\lambda')$ or in $R_0(\lambda')$. Say the latter is the case and suppose that $n \geqslant 2$, then we can consider the process $\overline{X}_t^{n,2} \leqslant \overline{X}_t^n$ for $t \geqslant \tau^{n,1}$, started with exactly that particle in $R_0(\lambda')$ and in which particles are killed upon reaching the boundary of $[0, 2R_0(\lambda')]$. Observe that $\overline{X}_{\tau^{n,1}+t}^{n,2}(\cdot + R_0(\lambda'))$ has the same law as $\overline{X}_t^{n,1}$ as in the previous step. If we let $\tau^{n,2}$ be the extinction time of $\overline{X}^{n,2}$, then we obtain

$$\begin{aligned}\mathbb{P}^\delta(\tau^n < \infty) &\leqslant \mathbb{P}^\delta(\tau^{n,1} < \infty)\cdot \mathbb{E}^\delta[\mathbb{P}^\delta(\tau^{n,2} < \infty | \mathcal{F}_{\tau^{n,1}})] \\ &= \big[\mathbb{P}^\delta(\tau^1 < \infty)\big]^2 = (1-\eta)^2\ ,\end{aligned}$$

with $\mathcal{F}_t$ the filtration generated by $\overline{X}^n$ and η the random variable from (4.7) (the last line follows from the strong Markov property). We can iterate this procedure at least n times, so that (4.10) is proven. If we choose $n = n(\eta,\zeta)$ (hence n is random) so that

$$(1-\eta)^n \leqslant \zeta\ ,$$

then the claimed result follows. □

4.4 Quenched growth rate

Our goal in this section is to verify Proposition 4.2. Recall that, as in Definition 2.4, $I_t^\delta = \ell(\mathbf{C}_t)$ for $t \geq 0$, where $(\mathbf{C}_t)_{t\geq 0}$ is an $(\mathfrak{R}_\delta^-, \mathfrak{R}_\delta^+)$–CBBM, and recall that we use the notation

$$\mathfrak{c}_\delta = \mathfrak{R}_\delta^-([0,\delta]) + \int_{(\delta,1]} \log(1+y)\frac{1}{y}\mathfrak{R}_\delta^+(\mathrm{d}y)\ . \tag{4.11}$$

Now, Proposition 4.2 is equivalent to the next lemma.

Lemma 4.7. *In the setting of Proposition 4.2, for any $\varepsilon > 0$ we have that $\mathbb{P}$–almost surely*

$$\mathbb{P}^{\delta}(\limsup_{t\to\infty} I_t^{\delta} e^{-(\mathfrak{c}_\delta+\varepsilon)t} > 0) = 0 \ . \tag{4.12}$$

The proof will follow two different arguments for small and large jumps. For small jumps we use a martingale approach: this leads to the term $\mathfrak{R}_\delta^-([0,\delta])$ in our wave speed upper bound. In particular, the martingale argument is not exact and delivers only a rough upper bound (but as $\delta \downarrow 0$ this error will be negligible). Instead for large jumps our argument is exact and builds on a time change argument, which is possible since jump times are now discrete.

Proof. For brevity, let us write I_t for I_t^δ and denote with $j(t) = \max\{j \ : \ t_j \leqslant t\}$. We have

$$I_t = \frac{I_t}{I_{t_{j(t)}}} \cdot \left(\prod_{j \leqslant j(t)} \frac{I_{t_j -}}{I_{t_{j-1}}} \right) \cdot \left(\prod_{j \leqslant j(t)} \frac{I_{t_j}}{I_{t_j -}} \right) \cdot I_0 \ ,$$

with the convention that $t_0 = 0$. Hence our result will follow if we prove the following three inequalities

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{j=1}^{j(t)} \log \frac{I_{t_j-}}{I_{t_{j-1}}} \leqslant \mathfrak{R}_\delta([0,\delta]) \ ,$$

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{j=1}^{j(t)} \log \frac{I_{t_j}}{I_{t_j-}} \leqslant \int_{(\delta,1]} \log(1+y) \frac{1}{y} \mathfrak{R}_\delta^+(\mathrm{d}y) \ ,$$

$$\limsup_{t\to\infty} \frac{1}{t} \log \frac{I_t}{I_{t_{j(t)}}} = 0 \ ,$$

with the convention that $t_0 = 0$. We observe that the last equality follows analogously to the first inequality, so we restrict to proving the first two points.

Step 1: Martingale term. Let us start by proving the first bound. We can define the following discrete time process:

$$M_n = \mathrm{e}^{-t_n \mathfrak{R}_\delta([0,\delta])} \prod_{j=1}^{n} \frac{I_{t_j-}}{I_{t_{j-1}}} \ ,$$

and we observe that $(M_n)_{n\in\mathbb{N}_0}$ is a discrete-time $\mathbb{P}^\delta$-martingale with respect to the filtration $(\mathcal{F}_{t_n})_{n\in\mathbb{N}_0}$, where $\mathcal{F}_t$ is the filtration generated by $(\mathbf{C}_s)_{s\leqslant t}$. To see that the martingale property holds, we observe that $\mathbf{C}_t$ has the law of an $(\mathfrak{R}_\delta^-, 0)$–CBBM on every time interval $[t_j, t_{j+1})$. In particular, by Proposition 2.6, we see that

$$\mathbb{E}^\delta\big[M_n|\mathcal{F}_{t_{n-1}}\big] = \mathrm{e}^{\mathfrak{R}_\delta([0,\delta])(t_n - t_{n-1})} \mathrm{e}^{-t_n \mathfrak{R}_\delta([0,\delta])} \prod_{j=1}^{n-1} \frac{I_{t_j-}}{I_{t_{j-1}}} = M_{n-1}.$$

Since M_n is a positive martingale, it follows from the martingale convergence theorem and Fatou's lemma that

$$\mathbb{E}^\delta\big[\lim_{n\to\infty} M_n\big] \leqslant \liminf_{n\to\infty} \mathbb{E}^\delta M_n \leqslant \mathbb{E}^\delta M_0 = 1 \ .$$

We conclude that almost surely, for any $\varepsilon > 0$

$$\limsup_{n\to\infty} \frac{1}{t_n} \sum_{j=1}^{n} \log \frac{I_{t_j-}}{I_{t_{j-1}}} \leqslant \mathfrak{R}_\delta([0,\delta]) + \varepsilon \ ,$$

which proves the first bound (note that $\frac{t_{j(t)}}{t} \to 1$ as $t \to \infty$).

Step 2: Large jumps. Here we use a different argument, based on large deviation principles. In a nutshell, we will prove that as long as the solution is growing exponentially fast at the correct order, such exponential growth becomes ever more likely (and precise) in future. We observe that the process I_t does not depend on the spatial dynamics of the particles. In particular, for every $j \geqslant 1$, the increment $\frac{I_{t_j}}{I_{t_j-}}$ depends only on the number R_j of particles that participate in the j-th reproduction event:

$$\frac{I_{t_j}}{I_{t_j-}} = 1 + \frac{R_j}{I_{t_j-}} \ .$$

Recall also that every particle reproduces independently of any other particle at time t_j, with probability $y_j \in (\delta, 1]$. Our aim is then to prove that as I_{t_j-} increases, the approximation

$$\frac{R_j}{I_{t_j-}} \simeq y_j$$

becomes ever more likely and precise. Following this description, let us consider, for some $\varepsilon \in (0,1]$ and $M \in \mathbb{N}$

$$\mathcal{G}_{0,M} := \left\{ \Big| \frac{R_j}{I_{t_j-}} - y_j \Big| \leqslant \varepsilon, \ \forall j \in \{1, \ldots, M\} \right\} ,$$

where the letter $\mathcal{G}$ stands for being a "good" set. We can then find a $c(\varepsilon) > 0$ such that for $I_0 \geqslant 1$

$$\begin{aligned} \mathbb{P}^\delta(\mathcal{G}_{0,M}) &\geqslant \mathbb{P}^\delta(\mathcal{G}_{0,M} | \mathcal{G}_{0,M-1}) \mathbb{P}^\delta(\mathcal{G}_{0,M-1}) \\ &\geqslant \left(1 - \exp\left\{ -c(\varepsilon) I_0 \prod_{j=1}^{M-1} (1 + y_j - \varepsilon) \right\} \right) \mathbb{P}^\delta(\mathcal{G}_{0,M-1}) \ , \end{aligned}$$

where we used that on the set $\mathcal{G}_{0,M-1}$ we have $I_{t_M-} \geqslant I_0 \prod_{j=1}^{M-1}(1+y_j-\varepsilon)$ (note that all other reproduction events, not due to large jumps, only increase the value of I_t), together with the following large deviations bound (4.13) for an i.i.d. sequence $\{X_i\}_{i\in\mathbb{N}}$ of Bernoulli random variables of parameter p:

$$\forall n \geqslant 1 \ , \varepsilon \in (0,1] \ , p \in [0,1] \ , \quad \exists c(\varepsilon) > 0 \quad \text{s.t.} \quad \mathbb{P}\Big(\Big| \frac{1}{n} \sum_{i=1}^n X_i - p \Big| \geqslant \varepsilon \Big) \leqslant \mathrm{e}^{-c(\varepsilon) n} \ . \tag{4.13}$$

Indeed, for $p \in [0,1]$ and $\varepsilon \in (0,1]$ we have that for all $n \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned} \mathbb{P}\Big(\Big| \frac{1}{n} \sum_{i=1}^n X_i - p \Big| \geq \varepsilon \Big) &= \mathbb{P}\Big(\frac{1}{n} \sum_{i=1}^n X_i \leq p - \varepsilon \Big) + \mathbb{P}\Big(\frac{1}{n} \sum_{i=1}^n X_i \geq p + \varepsilon \Big) \\ &\leqslant \mathrm{e}^{-n\bar{c}(p-\varepsilon,p)} + \mathrm{e}^{-n\bar{c}(p+\varepsilon,p)}, \end{aligned} \tag{4.14}$$

where the rate function is given by the relative entropy

$$\bar{c}(a,b) = \begin{cases} a \log \frac{a}{b} - (1-a) \log \frac{1-a}{1-b}, & \text{if } a \in [0,1], \\ \infty, & \text{otherwise}, \end{cases}$$

for $b \in [0,1]$, using the convention that $0 \log 0 = 0 \log \frac{0}{0} = 0$. Here the second line of (4.14) follows from Markov's inequality. Thus, in order to show (4.13) it suffices to verify that

$$c(\varepsilon) \stackrel{\text{def}}{=} \inf_{p \in [0,1]} \min \left\{ \bar{c}(p-\varepsilon, p), \bar{c}(p+\varepsilon, p) \right\} > 0 \ ,$$

which follows from the observation that $\bar{c}(a,b) = 0$ if and only if $a = b \in [0,1]$, since $a \mapsto \bar{c}(a,b)$ is convex for fixed $b \in [0,1]$, and since $(a,b) \mapsto \bar{c}(a,b)$ is continuous on $(0,1)^2$. Iterating the previous bound down to $M = 1$ we obtain, always assuming $I_0 \geqslant 1$

$$\begin{aligned}\mathbb{P}^\delta(\mathcal{G}_{0,M}) &\geqslant \prod_{n=1}^{M}\left(1 - \exp\left\{-c(\varepsilon)I_0\prod_{j=1}^{n-1}(1+y_j-\varepsilon)\right\}\right)\\ &\geqslant 1 - \sum_{n=1}^{M}\exp\left\{-c(\varepsilon)I_0\prod_{j=1}^{n-1}(1+y_j-\varepsilon)\right\} .\end{aligned}$$

Here we made use of the inequality $\prod_{n=1}^{M}(1-\alpha_n) \geqslant 1 - \sum_{n=1}^{M}\alpha_n$, which holds for any sequence $\alpha_n \in (0,1)$ by iterating the following bound:

$$\prod_{n=1}^{M}(1-\alpha_n) = \prod_{n=2}^{M}(1-\alpha_n) - \alpha_1\prod_{n=2}^{M}(1-\alpha_n) \geqslant \prod_{n=2}^{M}(1-\alpha_n) - \alpha_1 .$$

Finally, by monotonicity, with $\mathcal{G}_{0,\infty} = \bigcap_{M\in\mathbb{N}}\mathcal{G}_{0,M}$, we find that for $\varepsilon \in (0,\delta/2)$

$$I_0 \geqslant n_0 \quad\Longrightarrow\quad \mathbb{P}^\delta(\mathcal{G}_{0,\infty}) \geqslant 1 - \sum_{n=1}^{\infty}\exp\left\{-c(\varepsilon)n_0\prod_{j=1}^{n-1}(1+y_j-\varepsilon)\right\} \overset{\text{def}}{=} \mathfrak{p}(n_0,\varepsilon) \in \mathbb{R} .$$

The fact that the sum is converging follows for example from the condition $\varepsilon \in (0,\delta/2)$, so that it can be bounded from below by $1 - \sum_n \exp\{-c(\varepsilon)n_0(1+\delta/2)^n\} > -\infty$. In particular, since for any $n_0 \in \mathbb{N}$ and $I_0 \geqslant 1$ we have

$$\mathbb{P}(\exists j \in \mathbb{N} \text{ such that } I_{t_j} \geqslant n_0) = 1 ,$$

we obtain by the strong Markov property that for any $\varepsilon \in (0,\delta/2)$ and $\mathfrak{p}(n_0,\varepsilon)$ as above

$$\mathbb{P}\left(\left|\frac{R_j}{I_{t_j-}} - y_j\right| \leqslant \varepsilon \,,\ \text{for all but finitely many } j \in \mathbb{N}\right) \geqslant \mathfrak{p}(n_0,\varepsilon) \,, \qquad \forall n_0 \in \mathbb{N} .$$

Now, since $\lim_{n_0\to\infty}\mathfrak{p}(n_0,\varepsilon) = 1$ for all $\varepsilon \in (0,\delta/2)$, we conclude

$$\mathbb{P}\left(\left|\frac{R_j}{I_{t_j-}} - y_j\right| \leqslant \varepsilon \,,\ \text{for all but finitely many } j \in \mathbb{N}\right) = 1 .$$

We therefore deduce that on a set of full probability

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\sum_{j=1}^{j(t)}\log\frac{I_{t_j}}{I_{t_j-}} &\leqslant \int_{(\delta,1]}\log(1+y+\varepsilon)\frac{1}{y}\mathfrak{R}^+_\delta(\mathrm{d}y)\\ &\leqslant \int_{(\delta,1]}\log(1+y)\frac{1}{y}\mathfrak{R}^+_\delta(\mathrm{d}y) + \varepsilon\int_{(\delta,1]}\frac{1}{1+y}\frac{1}{y}\mathfrak{R}^+_\delta(\mathrm{d}y) .\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this concludes the proof of the second inequality, and thereby of the lemma. □

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