Limit theorem for sub-ballistic Random Walks in Dirichlet Environment in dimension $d \geq 3$

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Abstract

We look at random walks in Dirichlet environment. It was known that in dimension $d \geq 3$, if the walk is sub-ballistic, the displacement of the walk is polynomial of order κ for some explicit κ. We show that the walk, after renormalization, actually converges to a κ-stable completely asymmetric Lévy Process.

1 Introduction and results

1.1 Introduction

Random walks in random environments (RWRE) have been studied for several decades and are now rather well understood in the one dimensional case (see Solomon [29], Kesten, Kozlov, Spitzer [18] and Sinaï [27]). Important progress has been made in higher dimension, mainly in 3 directions: under a ballisticity condition, for small perturbation of the simple random walk [9, 34, 6, 23, 19] and in Dirichlet environment.

The most studied ballisticity conditions come from the conditions (T) and (T') introduced by Sznitman in [32, 33]. They have been shown to be equivalent in iid uniformly elliptic environments in [17] and also to be equivalent to an effective polynomial condition [3, 10]. By assuming any of these, in the ballistic regime, directional transience, ballisticity, and a CLT have been proved. Quenched CLTs have also been proved in various cases, either by assuming an annealed CLT, uniform ellipticity and a condition introduced by Kalikow [31], or by assuming the existence of high enough moments for the renewal times (see [30] for a definition of the renewal times) and uniform ellipticity of the environment [22] and [4] in dimension $d \geq 4$.

All these results show limit theorems in the ballistic case, that is to say that the walk has a positive speed. In dimension 2 and higher no complete limit theorems are

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known for the RWRE in the sub-ballistic case. However in dimension 1 we know that a sub-ballistic regime exists, where the walk can behave like the inverse of a stable subordinator [18, 13]. This sub-ballistic regime is caused by the existence of traps where the walk spends most of its time. This trapping phenomenon appears in other models closely related to the RWRE for instance the Bouchaud Trap Model (see [1] for a precise definition and an overview of the results). The model of random walks in random conductances also exhibits a similar trapping phenomenon. Indeed an annealed limit theorem (the limit is the inverse of a stable subordinator) and an equivalent to the CLT [16] have been proved for the biased random walk in random conductances. Similar results have been obtained for the biased walk in the percolation cluster and in Galton-Watson trees, but in both cases there is no convergence to a limit law [2, 15]. In the special case of iid, elliptic (but not uniformaly elliptic) RWRE a trapping phenomenon that leads to sub-ballistic behaviour has been identified in [8, 7] and [14] but no limit theorem has been proved.

The random walk in Dirichlet environment (RWDE) is a model where the transition probabilities are iid Dirichlet random variables (see [26] for an overview). It was first introduced because of its link to the linearly directed-edge reinforced random walk [21, 12]. It also has a property of invariance by time reversal that allows explicit calculations (see [24]). In particular, it gives a simple criterion for existence of absolutely continuous invariant distribution from the point of view of the particle, directional transience and ballisticity in dimension $d \geq 3$ [35, 8, 36, 25]). In the non-ballistic case the walk is directionally transient but the limit law was still unknown [8], it was only known that for some explicit $\kappa \in (0,1]$, $\frac{\log(|X_n|)}{\log(n)} \xrightarrow[n \to \infty]{} \kappa$.

In this paper we prove the annealed limit law for the sub-ballistic regime ($\kappa \leq 1$) in dimension $d \geq 3$. In the case $\kappa = 1$ we prove that $\frac{1}{n \log(n)} Y_n$ (where Y is the random walk) converges in law and we give its limit while for $\kappa < 1$ we prove that the process converges and give its limit. To the best of our knowledge, this is the first stable limit theorem for non reversible RWRE in iid environment, in dimension $d \geq 3$.

1.2 Notations

As the article is quite long, it involves a large number of notations. We put them here to help the reader.

- d_α is the drift defined in 1.3.
- $\mathrm{E}_{\mathrm{P}_0^\omega}$ is the expectation with respect to P_0^ω.
- $\mathbb{E}_{\mathbb{P}^{(\alpha)}}$, $\mathbb{E}_{\mathbb{P}}$ be the expectation with respect to $\mathbb{P}^{(\alpha)}$.
- $\gamma_\omega^m(x)$ is the speed up function defined in (1.1).
- $\gamma_{G,\omega}^\Lambda$ is the modification of $\gamma_\omega^m(x)$ for graph different from $\mathbb{Z}^d$ defined in 2.5.
- $\gamma_G^\omega(x)$ is the acceleration function defined in 2.7.
- I_i is the set of configuration of size i defined in 2.16.
- $I^n = \bigcup_{1 \leq i \leq n} I_i$ (2.16).
- κ, κ' and κ_j are parameters describing the strength of traps defined in 1.4.
- $L^\tau(n) = \min\{i, \tau_i \geq n\}$.
- ℓ_i^j is the time Y spends in the ith trap in the direction j.
- $(\overline{\omega_t^m})_{t \in [0,+\infty)}$ is the process $(X_t^m)_{t \in [0,+\infty)}$ from the point of view of the particle.
- ω is the environment (the set of the transition probabilities).
- Ω is the set of all environments.
- $\tilde{\omega}$ is the partially-forgotten environment defined in 2.15.
- $\tilde{\Omega}$ is the set of all equivalence classes for the trap-equivalence relation (2.14).

- P_0^ω is the expectation associated with the discrete time walk in environment ω and started at 0.
- $\mathbb{P}^{(\alpha)}$, $\mathbb{P}$ is the measure on Ω where the transition probabilities are iid Dirichlet random variables of parameter α at each site.
- $\mathbb{P}_0^{(\alpha)}[.] := \mathbb{E}_{\mathbb{P}^{(\alpha)}}[P_0^\omega(.)]$ is the annealed law of the process starting at 0.
- $\mathbb{Q}^{(\alpha)}$, $\mathbb{Q}$ is the invariant measure from the point of view of the particle.
- $\mathbb{Q}_0^{m,\alpha}(\cdot)$ is the invariant measure from the point of view of the particle for $(\overline{\omega_t^m})_{t \in [0,+\infty)}$.
- $\mathcal{S}^\kappa$ is the κ-stable Lévy process defined in 1.8.
- $N_{x \to x}, N_{x \to y}, N_{y \to x}, N_{y \to y}$ are the number of crossings of the trap (see 2.16).
- N_i^j is the number of times the walk enters the ith trap in the direction j.
- s_i^j is the strength of the ith trap in the direction j.
- $\tilde{\mathcal{S}}^\kappa$ is the inverse of $\mathcal{S}^\kappa$.
- $(\tau_i)_{i \in \mathbb{N}}$ are the renewal times defined in 1.1.
- $(T_i^m)_{i \in \mathbb{N}^*}$ are the renewal times associated with $(X_t^m)_{t \in [0,+\infty)}$.
- $\mathcal{T}^\omega$, $\mathcal{T}$ is the set of traps (2.13)
- $\tilde{\mathcal{T}}^\omega$, $\tilde{\mathcal{T}}$ is the set of vertices in traps (2.13).
- $\mathcal{T}_J^\omega$, $\mathcal{T}_J$ is the set of traps with direction in J (2.13).
- $\tilde{\mathcal{T}}_J^\omega$, $\tilde{\mathcal{T}}_J$ is the set of vertices in traps in direction in J (2.13).
- $(X_t^m)_{t \in [0,+\infty)}$ is the accelerated version of Y where the speed up is given by γ_ω^m.
- $(Y_n)_{n \in \mathbb{N}}$ is the discrete time random walk in Dirichlet environment.
- $(\tilde{Y}_n)_{n \in \mathbb{N}}$ is the partially-forgotten walk associated with $(Y_n)_{n \in \mathbb{N}}$ defined in 2.15.
- $\|z\|$ is the L_1-norm of z.

1.3 Definitions and statement of the results

In all the paper we set $d \geq 3$. Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{Z}^d$ and for any $j \in [\![d+1, 2d]\!]$, set $e_j = -e_{j-d}$. For any $z \in \mathbb{Z}^d$, let $\|z\| := \sum_{i=1}^d |z_i|$ be the L_1-norm of z. For any $x, y \in \mathbb{Z}^d$ we will write $x \sim y$ if $\|y - x\| = 1$. Let $E = \{(x,y) \in (\mathbb{Z}^d)^2, x \sim y\}$ be the set of directed edges of $\mathbb{Z}^d$ and let $\tilde{E} = \{\{x,y\}, (x,y) \in (\mathbb{Z}^d)^2, x \sim y\}$ be the set of non-directed edges. Let Ω be the set of environments on $\mathbb{Z}^d$:

$$\Omega = \left\{ \omega = (\omega(x,y))_{x \sim y} \in (0,1]^E \text{ such that } \forall x \in \mathbb{Z}^d, \sum_{i=1}^{2d} \omega(x, x+e_i) = 1 \right\}.$$

For each $\omega \in \Omega$, let $(Y_n)_{n \in \mathbb{N}}$ be the Markov chain on $\mathbb{Z}^d$ defined by $Y_0 = 0$ almost surely and the following transition probabilities:

$$\forall y \in \mathbb{Z}^d, \forall i \in [\![1, 2d]\!], \ P_0^\omega(Y_{n+1} = y + e_i | Y_n = y) = \omega(y, y + e_1).$$

Let $E_{P_0^\omega}$ be the expectation with respect to P_0^ω.

Given a family of positive weights $(\alpha_1, \ldots, \alpha_{2d})$, we consider the case where the transition probabilities at each site are iid Dirichlet random variables of parameter $\alpha := (\alpha_1, \ldots, \alpha_{2d})$, that is with density:

$$\frac{\Gamma\left(\sum_{i=1}^{2d} \alpha_i\right)}{\prod_{i=1}^{2d} \Gamma(\alpha_i)} \left(\prod_{i=1}^{2d} x_i^{\alpha_i - 1}\right) dx_1 \ldots dx_{2d-1}$$

on the simplex

$$\left\{ (x_1, \ldots, x_{2d}) \in (0,1]^{2d}, \sum_{i=1}^{2d} x_i = 1 \right\}.$$

Let $\mathbb{P}^{(\alpha)}$ be the law obtained on Ω this way. Let $\mathbb{E}_{\mathbb{P}^{(\alpha)}}$ be the expectation with respect to $\mathbb{P}^{(\alpha)}$ and let $\mathbb{P}_0^{(\alpha)}[.] := \mathbb{E}_{\mathbb{P}^{(\alpha)}}[\mathrm{P}_0^\omega(.)]$ be the annealed law of the process starting at 0. Let $(\tau_i)_{i \in \mathbb{N}^*}$ be the renewal times in the direction e_1 (see [30]):

Definition 1.1. *We define $(\tau_i)_{i \in \mathbb{N}^*}$, the renewal times in the direction e_1, by:*

$$\tau_1 = \inf\{n \in \mathbb{N}, \forall i < n, Y_i \cdot e_1 < Y_n \cdot e_1 \text{ and } \forall i > n, Y_i \cdot e_1 > Y_n \cdot e_1\}$$

and for all $i > 1$:

$$\tau_{i+1} = \inf\{n > \tau_i, \forall i < n, Y_i \cdot e_1 < Y_n \cdot e_1 \text{ and } \forall i > n, Y_i \cdot e_1 > Y_n \cdot e_1\}.$$

The renewal times are used to create independence thanks to the following theorem (Theorem 1.4 of [30]).

Proposition 1.2. *For all $k \in \mathbb{N}^*$, let $\mathcal{G}_k$ be the σ-field defined by:*

$$\mathcal{G}_k := \sigma(\tau_1, \ldots, \tau_k, (Y_n)_{0 \leq n \leq \tau_k}, (\omega(x, \cdot))_{x \cdot e_1 < Y_{\tau_k} \cdot e_1}).$$

We have, for all $k \geq 1$:

$$\mathbb{P}_0^{(\alpha)} \left((Y_{\tau_k + n})_{n \geq 0} \in \cdot, (\omega(Y_{\tau_k} + x, \cdot))_{x \cdot e_1 \geq 0} \in \cdot | \mathcal{G}_k \right)$$
$$= \mathbb{P}_0^{(\alpha)} \left((Y_n)_{n \geq 0} \in \cdot, (\omega(x, \cdot))_{x \cdot e_1 \geq 0} \in \cdot | \tau_1 = 0 \right).$$

This means that the trajectories and the transition probabilities inside slabs between two consecutive renewal times (after the first one) are i.i.d random variables.

Definition 1.3. *We define the drift d_α by:*

$$d_\alpha := \sum_{i=1}^{2d} \alpha_i e_i.$$

If $d_\alpha \neq 0$, we will assume, without loss of generality, that $\alpha_1 > \alpha_{1+d}$.

Definition 1.4. *We define the two parameters κ and κ' by:*

$$\kappa = 2 \left(\sum_{i=1}^{2d} \alpha_i \right) - \max_{i=1,\ldots,d} (\alpha_i + \alpha_{i+d})$$

and

$$\kappa' = 3 \left(\sum_{i=1}^{2d} \alpha_i \right) - 2 \max_{i=1,\ldots,d} (\alpha_i + \alpha_{i+d}).$$

For any direction $j \in [\![1, d]\!]$ we also define the parameter κ_j by:

$$\kappa_j := 2 \left(\sum_{i=1}^{2d} \alpha_i \right) - (\alpha_j + \alpha_{j+d})$$

In [25], it was proved that, for $d \geq 3$, when $\kappa > 1$, there exists an invariant probability measure $\mathbb{Q}^{(\alpha)}$ for the environment from the point of view of the particle, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$. From that it is possible to show that directional transience and ballisticity are equivalent when $\kappa > 1$. Furthermore, we know for which parameter the walk is directionally transient.

Theorem 1.5 (Corollary 1 of [36]). *If $d \geq 3$ and $d_\alpha \neq 0$, then for $\mathbb{P}^{(\alpha)}$ almost every environment, the walk is directionally transient with asymptotic direction d_α, that is to say:*

$$\frac{Y_n}{\|Y_n\|} \xrightarrow[n \to \infty]{} \frac{d_\alpha}{\|d_\alpha\|}, \ P_0^\omega \text{ almost surely.}$$

However, when $\kappa \leq 1$, such an invariant probability does not exist because of traps. But, in [8], it was proved that, by accelerating the walk, we can get an invariant probability for this accelerated walk, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$.

This leads to the following limit theorem in [8]:

Proposition 1.6. *If $\kappa \leq 1$, $d \geq 3$ and $d_\alpha \neq 0$. Let $l \in \{e_1, \ldots, e_{2d}\}$ be such that $d_\alpha \cdot l > 0$. Then we have the following convergence in probability (for the annealed law):*

$$\frac{\log(Y_n \cdot l)}{\log(n)} \xrightarrow[n \to \infty]{} \kappa.$$

We will now give a precise definition of the accelerated walk. We call directed path a sequence of vertices $\sigma = (x_0, \ldots, x_n)$ such that $(x_i, x_{i+1}) \in E$ for all i. To simplify notations, we will write $\omega_\sigma := \prod_{i=0}^{n-1} \omega(x_i, x_{i+1})$. For any positive integer m, we define the accelerating function $\gamma_\omega^m(x)$ by:

$$\gamma_\omega^m(x) := \frac{1}{\sum \omega_\sigma}, \tag{1.1}$$

where the sum is on all finite simple (each vertex is visited at most once) paths σ in $x + [\![-m, m]\!]^d$, starting from x, going to the border of $x + [\![-m, m]\!]^d$ and stopped the first time they reach this border. We will call X_t^m the continuous-time Markov chain whose jump rate from x to y is $\gamma_\omega^m(x)\omega(x, y)$, with $X_0^m = 0$. This means that $Y_n = X_{t_n^m}^m$ and $X_t^m = \sum_k Y_k 1_{t_k^m \leq t < t_{k+1}^m}$, for $t_n^m = \sum_{k=1}^n \frac{1}{\gamma_\omega^m(Y_k)} \mathcal{E}_k$, where the $\mathcal{E}_i$ are iid exponential random variables of parameter 1. The walk X_t^m can be viewed as an accelerated version of the walk Y_n.

Now, we need to introduce an other object: the walk seen from the point of view of the particle. First, let $(\theta_x)_{x \in \mathbb{Z}^d}$ be the shift on the environment defined by: $\theta_x \omega(y, z) := \omega(x + y, x + z)$. We call process seen from the point of view of the particle the process defined by $\overline{\omega_t^m} = \theta_{X_t^m} \omega$. Unlike the walk Y, under $\mathbb{P}_0^{(\alpha)}$, $\overline{\omega_t^m}$ is a Markov process on Ω. Its generator R is given by:

$$Rf(\omega) = \sum_{i=1}^{2d} \gamma_\omega^m(0)\omega(0, e_i)f(\theta_{e_i}\omega),$$

for all bounded measurable functions f on Ω.

Theorem 1.7 (Theorem 2.1 of [8]). *In dimension $d \geq 3$, if m is large enough then the process $\left(\overline{\omega_t^m}\right)_{t \in \mathbb{R}^+}$ has a stationary distribution $\mathbb{Q}^{m,\alpha}$. For any $\beta > 1$ there exists an m such that $\frac{d\mathbb{Q}^{m,\alpha}}{d\mathbb{P}^{(\alpha)}}$ is in $L^\beta\left(\mathbb{P}^{(\alpha)}\right)$.*

We will write $\mathbb{Q}_0^{m,\alpha}(\cdot)$ for $\mathbb{Q}^{m,\alpha}\left(P_0^\omega(\cdot)\right)$ To simplify the notations, we will drop the (α) from $\mathbb{P}^{(\alpha)}, \mathbb{P}_0^{(\alpha)}, \mathbb{Q}^{m,\alpha}$ and $\mathbb{Q}_0^{m,\alpha}$ when there is no ambiguity. We will also write $X_t, \mathbb{Q}$ and $\mathbb{Q}_0$ instead of $X_t^m, \mathbb{Q}^m$ and $\mathbb{Q}_0^m$ when there is no ambiguity on m.

We need a last definition to be able to state the limit theorems.

Definition 1.8. *For any $\kappa \in (0, 1)$ let $\mathcal{S}^\kappa$ be the Lévy process where the increments are completely asymmetric κ-stable random variables. The increments have the following characterizations:*

$$\forall \lambda \in \mathbb{R}, \forall s \in \mathbb{R}^+, \mathbb{E}\left(\exp\left(i\lambda \mathcal{S}_s^\kappa\right)\right) = \exp\left(-s|\lambda|^\kappa \left(1 - i\,\mathrm{sgn}(\lambda)\tan\left(\frac{\pi\kappa}{2}\right)\right)\right)$$

and for any $s \in \mathbb{R}^+$, $\mathcal{S}_s^\kappa$ and $s^{\frac{1}{\kappa}} \mathcal{S}_1^\kappa$ have the same law.

Since this process is non-decreasing and càdlàg we can define the càdlàg inverse $\tilde{\mathcal{S}}^\kappa$ by:

$$\tilde{\mathcal{S}}_t^\kappa := \inf\{s, \mathcal{S}_s^\kappa \geq t\}.$$

The following two theorems, which are the main results of this paper, give a full annealed limit theorem:

Theorem 1.9. *Set $d \geq 3$ and $\alpha \in (0, \infty)^{2d}$. Let $Y^n(t)$ be defined by:*

$$Y^n(t) = n^{-\kappa} Y_{\lfloor nt \rfloor}.$$

If $\kappa < 1$ and $d_\alpha \neq 0$ (e_1 is such that $d_\alpha \cdot e_1 > 0$), there exist positive constants c_1, c_2, c_3 such that for the J_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$\left(u \mapsto n^{-\frac{1}{\kappa}} \tau_{\lfloor nu \rfloor} \right) \xrightarrow[n \to \infty]{} c_1 \mathcal{S}^\kappa,$$

for the M_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$\left(x \mapsto n^{-\frac{1}{\kappa}} \inf\{t \geq 0, Y(t) \cdot e_1 \geq nx\} \right) \xrightarrow[n \to \infty]{} c_2 \mathcal{S}^\kappa$$

and for the J_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$Y^n \xrightarrow[n \to \infty]{} c_3 \tilde{\mathcal{S}}^\kappa d_\alpha.$$

Remark 1.10. We will give a quick explanation on what the M_1 and J_1 topologies are, for a precise definition see [28, 37]. They were both introduced as a generalization of the infinite norm for càdlàg functions. In the J_1 topology, a sequence of càdlàg functions f_n converges to f if there exists a sequence of increasing homeomorphisms $\lambda_n : [0, 1] \to [0, 1]$ such that

$$\sup_{t \in [0,1]} |\lambda_n(t) - t| \xrightarrow[n \to \infty]{} 0,$$

and

$$\sup_{t \in [0,1]} |f_n(\lambda_n(t)) - f(t)| \xrightarrow[n \to \infty]{} 0.$$

It is essentially the same as the infinite norm except that the function are allowed to "wiggle" time-wise. The M_1 topology is a topology on the graphs of the functions where we add vertical segments every time there is a jump. The main difference between the M_1 and J_1 topology is that there is almost no difference between one jump and small consecutive jumps in the M_1 topology while the difference is significant in the J_1 topology. The reason why we only have a convergence in M_1 for the hitting times $n^{-\frac{1}{\kappa}} \inf\{t \geq 0, Y(t) \cdot e_1 \geq nx\}$ is because there are consecutive jumps (this does not happen for the renewal times τ as the increments are independent). Indeed, if there is a large jump between $\inf\{t \geq 0, Y(t) \cdot e_1 \geq n\}$ and $\inf\{t \geq 0, Y(t) \cdot e_1 \geq n+1\}$ it is likely that there is a trap with high strength close-by which means that it is likely that there also is a large jump between $\inf\{t \geq 0, Y(t) \cdot e_1 \geq n+1\}$ and $\inf\{t \geq 0, Y(t) \cdot e_1 \geq n+2\}$.

Theorem 1.11. *If $d \geq 3, d_\alpha \neq 0$ (e_1 is such that $d_\alpha \cdot e_1 > 0$) and $\kappa = 1$, there exist positive constants c_1, c_2, c_3 such that we have the following convergences in probability (for the annealed law):*

$$\frac{1}{n \log(n)} \tau_n \xrightarrow[n \to \infty]{} c_1,$$

$$\frac{1}{n \log(n)} \inf\{i, Y_i \cdot e_1 \geq n\} \xrightarrow[n \to \infty]{} c_2,$$

$$\frac{\log(n)}{n} (Y_n) \xrightarrow[n \to \infty]{} c_3 d_\alpha.$$

Remark 1.12. We cannot replace the convergence in probability by an almost sure convergence. This is because if we look at a sum of iid random variables Z_i with a heavy tail $\mathbb{P}(Z_i \geq t) \sim ct^{-1}$ then we do not have an almost sure convergence. In fact, there are infinitely many i such that:

$$Z_i \geq i \log(i) \log(\log(i)).$$

However we might expect, as is the case in dimension 1, to have a convergence of the process:

$$t \mapsto \frac{Y_{\lfloor tn \rfloor} - v_n n \log(n)}{n},$$

for some deterministic sequence $(v_n)_{n \in \mathbb{N}}$ as n goes to infinity.

Remark 1.13. As the result we obtain are similar as the ones that exist in dimension 1, it seems that the same results should be true in dimension 2. The reason we do not treat this case is that there is no proof for the existence of an invariant measure for the walk from the point of view of the particle. It is actually possible that no such measure exists in the symmetric case ($d_{\alpha=0} = 0$). However it should exist in the transient case and then the result could easily be extended to dimension 2.

Remark 1.14. The Dirichlet environment is mostly used for two reasons. First, the existence of the invariant measure which is an essential part of the proof. Second, to get information on the probability density of the strength of a trap knowing the partially-forgotten walk 2.15. For this second part, weaker assumptions on the environment would suffice. This means that the proof could theoretically be adapted for other environments once the existence of an invariant measure is proved.

A tool that will be central in the proof is the study of traps. We now give a precise definition of traps.

Definition 1.15. *A trap is any undirected edge $\{x, y\}$ such that $\omega(x, y) + \omega(y, x) > \frac{3}{2}$.*

The strength of a trap is the quantity $\frac{1}{(1-\omega(x,y))+(1-\omega(y,x))}$.

Remark 1.16. The strength of a trap is of the same order as the time the walk spends in the trap every time it enters it. The value $\frac{3}{2}$ has been chosen because it ensures that $\omega(x, y), \omega(y, x) > \frac{1}{2}$ which in turn means that for every point x, there is at most one point y such that (x, y) is a trap.

1.4 Sketch of the proof

The proofs for $\kappa < 1$ and $\kappa = 1$ are mostly the same and therefore we will explain both at the same time. One of the key elements of the proof is the existence of an invariant measure which is a consequence of the invariance by time-reversal property of the random walk in Dirichlet environment. This invariant measure will allow us to bound the expectation of various quantities (the number of vertices visited between two renewal times for instance) which is necessary for our proof.

1.4.1 Only the renewal times matter

We first show that the number of points visited between two renewal times has a finite expectation (Lemma 2.2). This means that the walk does not "wander far" between two renewal times. So we only have to know the renewal times and the position of the walk at the renewal times to prove both theorems (Lemma 2.3). By Proposition 1.2, the random variables $(\tau_{i+1} - \tau_i)$ are iid which simplifies the study of the process $i \mapsto \tau_i$

1.4.2 The time between renewal times only depends on the strength of the traps

Then we use the stationary law of the accelerated walk to get two results: firstly, the time spent outside of traps is negligible (Lemma 2.27); secondly, the number N of times the walk enters a trap has a finite moment of order $\kappa + \varepsilon$ for some $\varepsilon > 0$ if $\kappa < 1$. If $\kappa = 1$, then N has a finite expectation (Lemma 2.19). This means the time spent in a trap mostly depends on its strength.

Now we want to show that the number of times the walk enters a trap and the time it stays in the trap each time are approximately independent.

We get two different results in this direction:

1.4.3 The strengths of the traps are essentially independent

The first result (Lemma 2.17) is that in a way the time spent in traps are independent random variables. These random variables have a tail in $Ct^{-\kappa}$ where the constant C depends on where the walk enters and exits the trap and how many times it does. More precisely, we first set an environment and a path in this environment. Then we forget all the transition probabilities in the traps, this means that if $\{x, y\}$ is a trap, then we only remember the "renormalized" transition probabilities:

$$\left(\frac{\omega(x, z)}{1 - \omega(x, y)} \right)_{z \sim x, z \neq y} \text{ and } \left(\frac{\omega(y, z)}{1 - \omega(y, x)} \right)_{z \sim y, z \neq x}.$$

Then every time the path visits a trap we only remember where it enters the trap and where it exits the trap, we forget the number of back and forths inside the trap. Then, only knowing these information, the strengths of the traps are independent.

1.4.4 The number of times a trap is visited and its strength are essentially independent

The second result (Lemma 2.20) allows us to bound the probability that both the number of times the walk enters a trap and the strength of the trap are high. We use the fact that for an edge (x, y) if we know all the transition probabilities outside of x, y and we know the $\left(\frac{\omega(x,z)}{1-\omega(x,y)} \right)_{z \sim x}$ and the $\left(\frac{\omega(y,z)}{1-\omega(y,x)} \right)_{z \sim y}$ then the number of times the walk enters the trap is essentially independent of the strength of the trap (it depends mostly on $\frac{1-\omega(x,y)}{1-\omega(y,x)}$ and hardly on the strength of the trap). This means that it is unlikely that the traps with a high strength are visited many times.

1.4.5 Conclusion

Thanks to these results we get that if we set an integer A and we only look at traps that are entered less than A times then we have a good approximation of the total time spent in traps (Lemma 2.22). The higher A is, the better the approximation gets. Now if we only look at the traps the walk enters less than A times, we get a finite sum of sums of iid random variables by Lemma 2.17. This means that, after renormalization, the time spent in traps entered less than A times converges to a stable distribution if $\kappa < 1$. It converges to a constant if $\kappa = 1$ (Lemma 2.23). Then the only thing left is to make A go to infinity and we get the first two results of both theorems.

Finally to prove the last part of both theorems we just use basic inversion arguments.

2 The proof

For the remainder of the article we assume that $d_\alpha \neq 0$ and $\alpha_1 > \alpha_{1+d}$, so $d_\alpha \cdot e_1 > 0$.

2.1 Number of points visited between renewal times

In this section we show that the expectation of the number of points visited between two renewal times is finite. This means that only knowing the values of the renewal times will be enough to prove Theorems 1 and 2.

Lemma 2.1. *For m such that $\mathbb{Q}^m$ exists, let $(T_i^m)_{i \in \mathbb{N}^*}$ be the renewal times for the walk X^m i.e $T_i^m := t_{\tau_i}^m$ or to put it another way $X_{T_i^m}^m = Y_{\tau_i}$. There exists a constant C_m such that for all $i \in \mathbb{N}^*$, $\mathbb{E}_{\mathbb{P}_0^{(\alpha)}}(T_{i+1}^m - T_i^m) = C_m$ and $\mathbb{P}_0^{(\alpha)}$ almost surely:*

$$\frac{1}{n} T_n^m \xrightarrow[n \to \infty]{} C_m.$$

Proof. Let D be the random distance defined by $D := (Y_{\tau_2} - Y_{\tau_1}) \cdot e_1$. First we will show that $\mathbb{E}_{\mathbb{P}_0}(D) < \infty$.

Let $(\tau_i)_{i \in \mathbb{N}^*}$ be the different renewal times along the direction e_1. Now let $(d_i)_{i \in \mathbb{N}^*}$ be the sequence defined by:

$$\forall i \in \mathbb{N}^*, d_i = Y_{\tau_i} \cdot e_1.$$

Let $\tilde{L}^\tau(i)$ be the number of renewal times before the walk travels a distance i in the direction e_1 ie:

$$\forall i \in \mathbb{N}^*, \tilde{L}^\tau(i) = \inf\{n, d_n \geq i\}.$$

The sequence of random variables $(d_{i+1} - d_i)_{i \in \mathbb{N}^*}$ is iid by Lemma 1.2. Therefore, if the expectation of $D = d_2 - d_1$ is infinite then $\frac{d_n}{n} \xrightarrow[n \to \infty]{} \infty$, $\mathbb{P}_0$ almost surely. Now, for every $i \in \mathbb{N}^*$, we have $d_{\tilde{L}^\tau(i)} \geq i$ and therefore $\frac{\tilde{L}^\tau(i)}{i} \leq \frac{\tilde{L}^\tau(i)}{d_{\tilde{L}^\tau(i)}}$. If $\mathbb{P}_0$ almost surely $\frac{n}{d_n} \xrightarrow[n \to \infty]{} 0$ we would have $\frac{\tilde{L}^\tau(i)}{i} \xrightarrow[i \to \infty]{} 0$ $\mathbb{P}_0$ almost surely. Since $\frac{\tilde{L}^\tau(i)}{i+1} \leq 1$ we would get that $\mathbb{E}_{\mathbb{P}_0}\left(\frac{\tilde{L}^\tau(i)}{i}\right) \xrightarrow[i \to \infty]{} 0$. However, there is a constant $C > 0$ such that every time the walk reaches a new height along e_1, it is a renewal time with probability C (independent of the walk up to that time) so $\mathbb{E}_{\mathbb{P}_0}\left(\frac{\tilde{L}^\tau(i)}{i}\right) \geq C$. Therefore we get that the expectation of the distance the walk travels in the direction e_1 between two renewal times is finite.

Now we can look at the accelerated walk X^m. We would like the sequence $(T_{i+1}^m - T_i^m)_{i \in \mathbb{N}^*}$ to be a sequence of iid random variables. Unfortunately, the definition of the accelerated random walk uses vertices in a box of size m around the vertex on which the walk currently is, so we need to wait at least $2m + 3$ renewal times to be sure to be at a distance at least $2m + 1$ of all the vertices visited before time $T_{i+1}^m - 1$. So we only have that for any $j \in \mathbb{N}$, the sequence $\left(T_{(2m+3)i+j+1}^m - T_{(2m+3)i+j}^m\right)_{i \in \mathbb{N}^*}$ is a sequence of iid random variables. Furthermore the sequence $(T_{i+1}^m - T_i^m)_{i \geq m+2}$ is identically distributed.

We know that there exists a constant $c > 0$ such that $\mathbb{P}_0$ almost surely $\frac{X_t^m \cdot e_1}{t} \xrightarrow[t \to \infty]{} c > 0$. If the expectation of the time the accelerated walk spends between two renewal times is infinite then $\frac{T_i^m}{i} \xrightarrow[i \to \infty]{} \infty$, $\mathbb{P}_0$ almost surely since the random variables $\left(T_{(2m+3)i+1}^m - T_{(2m+3)i}^m\right)_{i \in \mathbb{N}^*}$ are iid. Therefore we would have $\frac{X_{T_i^m}^m \cdot e_1}{T_i^m} \frac{T_i^m}{i} \xrightarrow[i \to \infty]{} \infty$ so $\frac{Y_{\tau_i} \cdot e_1}{i} \xrightarrow[i \to \infty]{} \infty$ which is absurd because: $\frac{Y_{\tau_i} \cdot e_1}{i} = \frac{d_i}{i}$ and $\frac{d_i}{i}$ satisfies a law of large numbers. Therefore the expectation of time the accelerated walk spends between two renewal times is finite and there exists a constant $C > 0$ such that:

$$\forall i \geq m+2, \ \mathbb{E}_{\mathbb{P}_0}(T_{i+1}^m - T_i^m) = C.$$

And by the law of large numbers, $\mathbb{P}_0$ almost surely:

$$\frac{1}{i} T_i^m \xrightarrow[i \to \infty]{} C. \qquad \square$$

Lemma 2.2. *The number of different points the walk visits between two renewal times has a finite expectation (Note that the number of different points visited between two renewal times is the same for the walk Y and the accelerated walks X^m).*

Proof. We choose m large enough such that $\frac{\mathrm{d}\mathbb{Q}^m}{\mathrm{d}\mathbb{P}}$ is in L^γ for some $\gamma > 1$. In the following we will write T_i instead of T_i^m to simplify the notations. Let β be such that $\frac{1}{\gamma} + \frac{1}{\beta} = 1$. Let c_∞ be the constant such that $\mathbb{P}_0$ almost surely: $\frac{1}{i} T_i \xrightarrow[i \to \infty]{} c_\infty$, it exists by Lemma 2.1. Let $(R_i)_{i \in \mathbb{N}^*}$ be the sequence defined by: $\forall i \in \mathbb{N}^*, R_i = \#\{x, \exists j \leq \tau_i, Y_j = x\}$. The random variables $(R_{i+1} - R_i)_{i \geq 1}$ are iid by Proposition 1.2. Thus if the number of different points the walk visits between two renewal times has an infinite expectation (for $\mathbb{P}_0$) then $\frac{R_i}{i} \xrightarrow[i \to \infty]{} \infty$, $\mathbb{P}_0$ almost surely and therefore $\mathbb{Q}_0^m$ almost surely. However we have for any $C > 0$:

$$
\begin{aligned}
\mathbb{Q}_0^m(R_n \geq Cn) &\leq \mathbb{Q}_0^m(T_n \geq 2c_\infty n) + \mathbb{Q}(R_n \geq Cn \text{ and } T_n < 2c_\infty n) \\
&= o(1) + \mathbb{Q}_0^m(R_n \geq Cn \text{ and } T_n < 2c_\infty n) \\
&\leq o(1) + \mathbb{Q}_0^m\left(\sum_{0 \leq i < 2c_\infty n} \#\{x, \exists t \in [i, i+1), X_t = x\} \geq Cn \right) \\
&\leq o(1) + \frac{1}{Cn} \mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{0 \leq i < 2c_\infty n} \#\{x, \exists t \in [i, i+1), X_t = x\} \right) \\
&\leq o(1) + \frac{2c_\infty}{C} \mathbb{E}_{\mathbb{Q}_0^m}\left(\#\{x, \exists t \in [0, 1), X_t = x\} \right).
\end{aligned}
$$

Now we just have to prove that $\mathbb{E}_{\mathbb{Q}_0^m}\left(\#\{x, \exists t \in [0, 1), X_t = x\} \right)$ is finite. We use the fact that $\frac{\mathrm{d}\mathbb{Q}^m}{\mathrm{d}\mathbb{P}}$ is in L^γ and therefore $\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}$ is also in L^γ.

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}_0^m}\left(\#\{x, \exists t \in [0, 1), X_t = x\} \right) &= \mathbb{E}_{\mathbb{P}_0}\left(\#\{x, \exists t \in [0, 1), X_t = x\} \frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0} \right) \\
&\leq \mathbb{E}_{\mathbb{P}_0}\left(\#\{x, \exists t \in [0, 1), X_t = x\}^\beta \right)^{\frac{1}{\beta}} \left(\mathbb{E}_{\mathbb{P}_0}\left(\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0} \right)^\gamma \right)^{\frac{1}{\gamma}}.
\end{aligned}
$$

So we just need to prove that $\mathbb{E}_{\mathbb{P}}\left(\#\{x, \exists t \in [0, 1), X_t = x\}^\beta \right)$ is finite. This is an immediate consequence of Lemma 4 of [8]. Therefore, for C large enough, we get:

$$\mathbb{Q}_0^m(R_n \geq Cn) \leq o(1) + \frac{1}{2}.$$

This contradicts $R_n / n \xrightarrow[n \to \infty]{} \infty$ $\mathbb{Q}_0^m$ almost surely and thus we have the desired result. $\square$

Now, we show that the trajectory of the walk cannot deviate too much from a straight line.

Lemma 2.3. *Let $L^\tau(n) = \min\{i, \tau_i \geq n\}$. There exists $D \in \mathbb{R}^d$ such that $\mathbb{P}_0$ almost surely:*

$$\frac{Y_n}{L^\tau(n)} \xrightarrow[n \to \infty]{} D.$$

Proof. By Proposition 1.2, $(Y_{\tau_{i+1}} - Y_{\tau_i})_{i \geq 1}$ is a sequence of iid random variables (for $\mathbb{P}_0$). Let $R_i := \# \left\{ x \in \mathbb{Z}^d, \exists j < \tau_i, \ Y_j = x \right\}$ be the number of different points visited before time τ_i. By Lemma 2.2, $R_i - R_{i-1}$ has a finite expectation and since $\|Y_{\tau_{i+1}} - Y_{\tau_i}\| \leq R_{i+1} - R_i$, we get that $\|Y_{\tau_{i+1}} - Y_{\tau_i}\|$ also has a finite expectation. So there exists $D \in \mathbb{R}^d$ such that $\mathbb{P}_0$ almost surely:

$$\frac{Y_{\tau_n}}{n} \xrightarrow[n \to \infty]{} D.$$

Now we want to show that $\frac{\|Y_n - Y_{\tau(L^\tau(n))}\|}{L^\tau(n)} \xrightarrow[n \to \infty]{} 0$, $\mathbb{P}_0$ almost surely. We clearly have:

$$\frac{\|Y_n - Y_{\tau(L^\tau(n))}\|}{L^\tau(n)} \leq \frac{R_{L^\tau(n)} - R_{L^\tau(n)-1}}{L^\tau(n)}$$

but since $\mathbb{E}_{\mathbb{P}_0}(R_i - R_{i-1})$ is finite, $\frac{R_i - R_{i-1}}{i} \xrightarrow[i \to \infty]{} 0$, $\mathbb{P}_0$ almost surely, so:

$$\frac{\|Y_n - Y_{\tau(L^\tau(n))}\|}{L^\tau(n)} \xrightarrow[n \to \infty]{} 0, \ \mathbb{P}_0 \text{ almost surely .}$$

So we get that $\mathbb{P}_0$ almost surely: $\frac{Y_n}{L^\tau(n)} \xrightarrow[n \to \infty]{} D$. $\qquad\square$

2.2 Number of visits of traps

This section is devoted to refining some results of [8] to get an upper bound on the number of visits of traps. First we must get some results on finite graphs and then we will extend these results on $\mathbb{Z}^d$.

Definition 2.4. *Let $G = (V, E)$ be a finite, directed graph. A vertex $\delta \in V$ is a cemetery vertex if*

- *no edge exits δ, ie $\forall x \in V, (\delta, x) \notin E$,*
- *for every vertex $x \in V$ there exists a directed path from x to δ.*

In this section we will only consider graphs with no multiple edges, no elementary loops (one edge starting and ending at the same point), and such that for every $x, y \in V \setminus \{\delta\}$, $(x, y) \in E$ if and only if $(y, x \in E)$.

We will first extend the definition of $\gamma_\omega^m(x)$ for those graphs. Let $G = (V \cup \{\delta\}, E)$ be a finite directed graph, $(\alpha(e))_{e \in E}$ be a family of positive real numbers, and $\mathbb{P}^\alpha$ be the corresponding Dirichlet distribution (independent at each site).

Definition 2.5. *For $x \in G$ and $\Lambda \subset V \cup \{\delta\}$, we define the following generalization of γ_ω^m:*

$$\gamma_{G,\omega}^\Lambda(x) := \frac{1}{\sum_\sigma \omega_\sigma},$$

where we sum on simple paths from x to the border of Λ (i.e $\{y \in \Lambda, \exists z \notin \Lambda, \{x, y\} \in V\}$) that stay in Λ.

Remark 2.6. We notice that, in $\mathbb{Z}^d$, for any $m \in \mathbb{N}^*$:

$$\forall x \in \mathbb{Z}^d, \ \gamma_\omega^m(x) = \gamma_{\mathbb{Z}^d,\omega}^{x + [\![-m,m]\!]^d}(x).$$

We will also use the following acceleration function.

Definition 2.7. *For any graph G and any environment ω on G we define the partial acceleration function γ_G^ω by:*

$$\gamma_G^\omega(x) = \max_{y \sim x} \left(\frac{1}{1 - \omega(x,y) + 1 - \omega(y,x)} \right).$$

When there is no ambiguity we will write $\gamma^\omega(x)$ instead of $\gamma_G^\omega(x)$

Remark 2.8. Let x be a vertex in $\mathbb{Z}^d$. If it is in a trap then $\gamma^\omega(x)$ is equal to the strength of the trap. Otherwise $\gamma^\omega(x) \leq 2$.

We have the following result, in the case of finite graphs:

Lemma 2.9 (Proposition A.2 of [8])**.** *Let $n \in \mathbb{N}^*$. Let $G = (V \cup \{\delta\}, E)$ be a finite directed graph possessing at most n edges and such that every vertex is connected to δ by a directed path. We furthermore suppose that G has no multiple edges, no elementary loop, and that if $(x, y) \in E$ and $y \neq \delta$, then $(y, x) \in E$. Let $(a(e))_{e \in E}$ be positive real numbers. Then, for every vertex $x \in V$, there exist real numbers $C, r > 0$ such that, for small $\varepsilon > 0$,*

$$\mathbb{P}^{(a)} \left(\gamma_{G,\omega}^{V \setminus \{\delta\}}(x) \geq \frac{1}{\varepsilon} \right) \leq C \varepsilon^\beta (-\ln \varepsilon)^r$$

where the value of β is explicit and given in [8] but to simplify the notations we will only use the fact that it is bigger than or equal to κ' in the case we will look at.

Lemma 2.10 (Lemma 9 of [35])**.** *Let $(p_i^{(1)})_{1 \leq i \leq n_1}, \ldots, (p_i^{(r)})_{1 \leq i \leq n_r}$ be independent Dirichlet random variables with respective parameters $(\alpha_i^{(1)})_{1 \leq i \leq n_1}, \ldots, (\alpha_i^{(r)})_{1 \leq i \leq n_r}$. Let $m_1, \ldots, m_r$ be integers such that $\forall i \leq r, 1 \leq m_i < n_i$, and let $\Sigma = \sum_{j=1}^r \sum_{i=1}^{m_j} p_i^{(j)}$ and $\beta = \sum_{j=1}^r \sum_{i=1}^{m_j} \alpha_i^{(j)}$. There exist positive constants C, C' such that, for any positive measurable function $f : \mathbb{R} \times \mathbb{R}^{\sum_j m_j} \to \mathbb{R}$,*

$$\mathbb{E}\left[f\left(\Sigma, \frac{p_1^{(1)}}{\Sigma}, \ldots, \frac{p_{m_1}^{(1)}}{\Sigma}, \ldots, \frac{p_1^{(r)}}{\Sigma}, \ldots, \frac{p_{m_r}^{(r)}}{\Sigma} \right) \right] \leq C \tilde{\mathbb{E}}\left[f\left(\tilde{\Sigma}, \tilde{p}_1^{(1)}, \ldots, \tilde{p}_{m_1}^{(1)}, \ldots, \tilde{p}_1^{(r)}, \ldots, \tilde{p}_{m_r}^{(r)} \right) \right],$$

where, under the probability $\tilde{\mathbb{P}}$, $(\tilde{p}_1^{(1)}, \ldots, \tilde{p}_{m_1}^{(1)}, \ldots, \tilde{p}_1^{(r)}, \ldots, \tilde{p}_{m_r}^{(r)})$ is sampled from a Dirichlet distribution of parameter $(\alpha_1^{(1)}, \ldots, \alpha_{m_1}^{(1)}, \ldots, \alpha_1^{(r)}, \ldots, \alpha_{m_r}^{(r)})$, $\tilde{\Sigma}$ is bounded and satisfies $\tilde{\mathbb{P}}(\tilde{\Sigma} < \varepsilon) \leq C' \varepsilon^\beta$ for every $\varepsilon > 0$, and those two random variables are independent.

The following lemma shows that the value of the acceleration function $\gamma_\omega^m(x)$ depends mostly on the strength of the trap that contains x (if there is one). This means that the number of visits to a vertex depends mostly on the strength of the trap containing this vertex.

Lemma 2.11. *Set $\alpha \in (0, \infty)^{2d}$. In $\mathbb{Z}^d$, for any $\beta \in \left[\kappa, \frac{\kappa + \kappa'}{2} \right)$, for any $m \geq 2$:*

$$\mathbb{E}_{\mathbb{P}_0^{(\alpha)}} \left(\left(\frac{\gamma_\omega^m(0)}{\gamma_{\mathbb{Z}^d}^\omega(0)} \right)^\beta \right) < \infty.$$

Proof. Let $m \geq 2$ be an integer. We will use the results we have on finite graphs for this lemma. First we notice that the value of $\left(\frac{\gamma_\omega^m(0)}{\gamma^\omega(0)} \right)^\beta$ only depends on a finite amount of edges and vertices around 0. This means that we can look at this quantity on a finite graph and have the same law. The finite graph $G^m = (V^m, E^m)$ we want is obtained by contracting all the points $x \in \mathbb{Z}^d$ such that $\|x\| \geq m$ in a single point δ (the cemetery vertex) and deleting all the edges going from this vertex to the rest of the environment. For any environment ω on $\mathbb{Z}^d$ we have an equivalent environment ω^m on G^m: if $(x, y) \in E$ and $(x, y) \in E^m$ then $\omega(x, y) = \tilde{\omega}(x, y)$ and for any $x \in V^m \setminus \{\delta\}$, $\tilde{\omega}(x, \delta) = \sum_{y \in \mathbb{Z}^d, \|y\| = m} \omega(x, y)$. Now we have:

$$\gamma_\omega^m(0) = \gamma_{G^m, \omega^m}^{V^m \setminus \{\delta\}}(0)$$

and

$$\gamma_{\mathbb{Z}^d}^\omega(0) = \gamma_{G^m}^{\omega^m}(0).$$

So we just have to show that

$$\mathbb{E}_{\mathbb{P}^{(\alpha)}}\left(\left(\frac{\gamma_{G^m,\omega^m}^{V^m\setminus\{\delta\}}(0)}{\gamma_{G^m}^{\omega^m}(0)}\right)^\beta\right) < \infty.$$

For any point $y \sim 0$ and any environment ω we define Σ_y^ω by:

$$\Sigma_y^\omega = 2 - \omega(0,y) - \omega(y,0).$$

For any point $x \in G^m$ such that $x \sim 0$, we define $G_x^m = (V_x^m, E_x^m)$ by contracting the vertices 0 and x into a single vertex 0 and deleting the edges $(0,x)$ and $(x,0)$. The edges $(0,y)$ and $(y,0)$ stay the same for any $y \sim 0$ such that $x \neq y$. However, the edges (x,y) and (y,x) become $(0,y)$ and $(y,0)$ respectively, for any $y \sim x$ such that $0 \neq y$. We can also define ω_x^m by:

$$\forall (y,z) \in E^m, y \notin \{0,x\},\ \omega_x^m(y,z) := \omega^m(y,z)$$

$$\forall (y,z) \in E^m, y \in \{0,x\}, (y,z) \in E_x^m,\ \omega_x^m(y,z) := \frac{\omega^m(y,z)}{\Sigma_y^\omega}$$

Let $x \sim 0$ be a vertex of G^m. If we think of $\frac{1}{\gamma_{G^m,\omega^m}^{V^m\setminus\{\delta\}}}$ as a sum on simple paths, we have:

$$\frac{1}{\gamma_{G^m,\omega^m}^{V^m\setminus\{\delta\}}} \geq \Sigma_x^{\omega^m}\omega^m(0,x)\frac{1}{\gamma_{G^m,\omega^m}^{V^m\setminus\{\delta\}}}$$

Indeed, if we look at $\frac{1}{\gamma_{G_x^m,\omega_x^m}^{V\setminus\{\delta\}}}$ as a sum on simple paths σ from 0 to δ ($\sigma_0 = 0$), either the first vertex σ_1 visited by the path is such that $(0,\sigma_1) \in E^m$ or $(x,\sigma_1) \in E^m$. We define $\tilde{\sigma}$ by: if $(0,\sigma_1) \in E^m$ then $\tilde{\sigma} := \sigma$ and we have:

$$\omega^m(\tilde{\sigma}) = \Sigma_x^{\omega^m}\omega_x^m(\sigma) \geq \Sigma_x^{\omega^m}\omega^m(0,x)\omega_x^m(\sigma),$$

and if $(x,\sigma_1) \in E^m$ then $\tilde{\sigma}_i := \sigma_{i-1}$ for $i \geq 2$ and $\tilde{\sigma}_0 := 0$ and $\tilde{\sigma}_1 := x$ and we get:

$$\omega^m(\tilde{\sigma}) = \Sigma_x^{\omega^m}\omega^m(0,x)\omega_x^m(\sigma).$$

For any environment ω, let $x(\omega^m)$ be the point that maximises $(y \mapsto \omega^m(0,y))$. We have $\tilde{\omega}(0,x(\omega^m)) \geq \frac{1}{2d}$ and therefore:

$$\frac{1}{\gamma_{G^m,\omega^m}^{V\setminus\{\delta\}}} \geq \frac{1}{2d}\Sigma_{x(\omega^m)}^{\omega^m}\frac{1}{\gamma_{G_{x(\omega^m)}^m,\omega_{x(\omega^m)}^m}^{V^m\setminus\{\delta\}}}.$$

So we get, for any $\varepsilon > 0$:

$$\mathbb{P}^{(\alpha)}\left(\left(\frac{\gamma_{G^m,\omega^m}^{V^m\setminus\{\delta\}}(0)}{\gamma_{G^m}^{\omega^m}(0)}\right) \geq \frac{1}{\varepsilon}\right) \leq \mathbb{P}^{(\alpha)}\left(\frac{2d\varepsilon}{\gamma_{G^m}^{\omega^m}(0)} \geq \Sigma_{x(\omega^m)}^{\omega^m}\frac{1}{\gamma_{G_{x(\omega^m)}^m,\omega_{x(\omega^m)}^m}^{V^m\setminus\{\delta\}}}\right)$$

$$= \sum_{y\sim 0}\mathbb{P}^{(\alpha)}\left(y = x(\omega^m) \text{ and } \frac{2d\varepsilon}{\gamma_{G^m}^{\omega^m}(0)} \geq \Sigma_y^{\omega^m}\frac{1}{\gamma_{G_y^m,\omega_y^m}^{V^m\setminus\{\delta\}}}\right)$$

$$\leq \sum_{y\sim 0}\mathbb{P}^{(\alpha)}\left(\frac{2d\varepsilon}{\gamma_{G^m}^{\omega^m}(0)} \geq \Sigma_y^{\omega^m}\frac{1}{\gamma_{G_y^m,\omega_y^m}^{V^m\setminus\{\delta\}}}\right).$$

by definition of $\gamma_{G^m}^{\omega^m}(0)$:

$$\forall y \sim 0,\ \gamma_{G^m}^{\omega^m}(0)\Sigma_y^{\omega^m} \geq 1.$$

Therefore:

$$\mathbb{P}^{(\alpha)}\left(\left(\frac{\gamma_{G^m,\omega^m}^{V^m\backslash\{\delta\}}(0)}{\gamma_{G^m}^{\omega^m}(0)}\right)\geq\frac{1}{\varepsilon}\right)\leq\sum_{y\sim 0}\mathbb{P}^{(\alpha)}\left(2d\varepsilon\geq\frac{1}{\gamma_{G_y^m,\omega_y^m}^{V^m\backslash\{\delta\}}}\right).$$

Now we can apply Lemma 2.10 which gives, for any $y\sim 0$:

$$\mathbb{P}^{(\alpha)}\left(2d\varepsilon\geq\frac{1}{\gamma_{G_y^m,\omega_y^m}^{V^m\backslash\{\delta\}}}\right)\leq C\tilde{\mathbb{P}}\left(2d\varepsilon\geq\frac{1}{\gamma_{G_y^m,\omega_y^m}^{V^m\backslash\{\delta\}}}\right),$$

where under $\tilde{\mathbb{P}}$, ω_y^m are independent Dirichlet random variables (on the graph G_y^m and the parameters of the Dirichlet are the same as in $\mathbb{Z}^d$). Now, according to Lemma 2.9 there exist two constants C', r such that:

$$\forall\varepsilon\quad\text{small enough,}\quad\tilde{\mathbb{P}}\left(2d\varepsilon\geq\frac{1}{\gamma_{G_y^m,\omega_y^m}^{V\backslash\{\delta\}}}\right)\leq C'\varepsilon^{\kappa'}\left(-\log(\varepsilon)\right)^r.$$

This means that by changing the constant C', we get:

$$\forall\varepsilon\geq 0,\tilde{\mathbb{P}}\left(2d\varepsilon\geq\frac{1}{\gamma_{G_y^m,\omega_y^m}^{V^m\backslash\{\delta\}}}\right)\leq C'\varepsilon^{\frac{\kappa+\kappa'}{2}}.$$

So there exists a constant D that does not depend on ε such that:

$$\mathbb{P}^{(\alpha)}\left(\left(\frac{\gamma_{G^m,\omega^m}^{V^m\backslash\{\delta\}}(0)}{\gamma_{G^m}^{\omega^m}(0)}\right)\geq\frac{1}{\varepsilon}\right)\leq D\varepsilon^{\frac{\kappa+\kappa'}{2}}.$$

We have the result we want. $\qquad\square$

Unfortunately this statement cannot be efficiently used with the invariant distribution $\mathbb{Q}^m$ because we can visit multiple points between times 0 and 1 since the time is continuous. So we need a version of the previous lemma that takes this continuity into account.

Lemma 2.12. *Set $\alpha\in(0,\infty)^{2d}$. For every $\beta<\frac{\kappa+\kappa'}{2}$, there exists an integer m such that:*

$$\mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{x\in\mathbb{Z}^d}\left(\int_0^1\frac{\gamma_\omega^m(x)}{\gamma_{\mathbb{Z}^d}^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta\right)<\infty.$$

Proof. Let $p\in(1,\infty)$ be a constant such that $\beta p^2<\frac{\kappa+\kappa'}{2}$ and let γ be such that $\frac{1}{p}+\frac{1}{\gamma}=1$. Now let m be an integer such that $\frac{\mathrm{d}\mathbb{Q}^m}{\mathrm{d}\mathbb{P}}$ is in L^γ. This means that $\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}$ is also in L^γ. We will only work in $\mathbb{Z}^d$ so we will write γ^ω instead of $\gamma_{\mathbb{Z}^d}^\omega$.

$$\mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{x\in\mathbb{Z}^d}\left(\int_0^1\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta\right)$$

$$=\sum_{x\in\mathbb{Z}^d}\mathbb{E}_{\mathbb{P}_0}\left(\left(\int_0^1\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}\right)$$

$$\leq\sum_{x\in\mathbb{Z}^d}\mathbb{E}_{\mathbb{P}_0}\left(\left(\int_0^1\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^{p\beta}\right)^{\frac{1}{p}}\mathbb{E}_{\mathbb{P}_0}\left(\left(\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}\right)^\gamma\right)^{\frac{1}{\gamma}}.$$

This means we just need to show that $\sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\mathbb{P}_0} \left(\left(\int_0^1 \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^{p\beta} \right)^{\frac{1}{p}}$ is finite. Let D_1^m be the random variable defined by:

$$D_1^m := \sum_{i=1}^d \max_{t \in [0,1]} |X_t^m . e_i|.$$

We have:

$$\sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\mathbb{P}_0} \left(\left(\int_0^1 \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^{p\beta} \right)^{\frac{1}{p}}$$

$$\leq \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\mathbb{P}_0} \left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} \right)^{p\beta} 1_{\exists t \in [0,1], X_t^m = x} \right)^{\frac{1}{p}}$$

$$\leq \sum_{x \in \mathbb{Z}^d} \left(\mathbb{E}_{\mathbb{P}_0} \left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} \right)^{p\beta} 1_{D_1^m \geq \|x\|_\infty} \right) \right)^{\frac{1}{p}}$$

$$\leq \sum_{x \in \mathbb{Z}^d} \left(\mathbb{E}_{\mathbb{P}_0} \left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} \right)^{p^2\beta} \right) \right)^{\frac{1}{p^2}} \left(\mathbb{E}_{\mathbb{P}_0} \left(1_{D_1^m \geq \|x\|_\infty} \right) \right)^{\frac{1}{\alpha}}$$

$$= \sum_{x \in \mathbb{Z}^d} \left(\mathbb{E}_{\mathbb{P}} \left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} \right)^{p^2\beta} \right) \right)^{\frac{1}{p^2}} \left(\mathbb{E}_{\mathbb{P}_0} \left(1_{D_1^m \geq \|x\|_\infty} \right) \right)^{\frac{1}{\alpha}}$$

Now since the environment for $\mathbb{P}$ is iid, $\mathbb{E}_{\mathbb{P}} \left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} \right)^{p^2\beta} \right)$ does not depend on x and we get:

$$\sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\mathbb{P}_0} \left(\left(\int_0^1 \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^{p\beta} \right)^{\frac{1}{p}}$$

$$\leq \left(\mathbb{E}_{\mathbb{P}} \left(\left(\frac{\gamma_\omega^m(0)}{\gamma^\omega(0)} \right)^{p^2\beta} \right) \right)^{\frac{1}{p^2}} \sum_{x \in \mathbb{Z}^d} \left(\mathbb{E}_{\mathbb{P}_0} \left(1_{D_1^m \geq \|x\|_\infty} \right) \right)^{\frac{1}{\gamma}} .$$

And since there exists a constant C such that for every $i \geq 1$ there are at most Ci^{d-1} points x such that $\|x\|_\infty = i$, we get:

$$\sum_{x \in \mathbb{Z}^d} \left(\mathbb{E}_{\mathbb{P}_0} \left(1_{D_1 \geq \|x\|_\infty} \right) \right)^{\frac{1}{\gamma}} \leq 1 + C \sum_{i \geq 1} i^{d-1} \left(\mathbb{E}_{\mathbb{P}_0} \left(1_{D_1 \geq i} \right) \right)^{\frac{1}{\gamma}}$$

which is finite by lemma 4 of [8]. And by Lemma 2.11 we get:

$$\mathbb{E}_{\mathbb{P}} \left(\left(\frac{\gamma_\omega^m(0)}{\gamma^\omega(0)} \right)^{p^2\beta} \right) < \infty.$$

So we get the result we want. $\qquad \square$

2.3 Independence of the traps

This section will be devoted to the precise study of traps. The notion of trap was defined in the introduction in definition 1.15. In the previous section we have essentially

shown that the total amount of time spent on a trap mostly depends on its strength. Now, we need a way to create independence between the times spent in the different traps. We will do it in two steps. First we will show that the strengths of the traps are essentially independent and then we will show that the strength of a trap and the number of times it is visited are essentially independent. However, we first need to introduce a few objects to characterize this independence precisely.

Definition 2.13. *Let $\mathcal{T}^\omega$ be the set of traps $\{x, y\} \in \tilde{E}$ for the environment ω.*
$\tilde{\mathcal{T}}^\omega$ is the set of vertices $x \in \mathbb{Z}^d$ such that there exists y such that $\{x, y\} \in \mathcal{T}^\omega$.
For any subset J of $[\![1, d]\!]$ we define $\mathcal{T}_J^\omega$, the traps with direction in J by:

$$\mathcal{T}_J^\omega = \{\{x, y\} \in \mathcal{T}, \ \exists j \in J, y = x + e_j \text{ or } y = x - e_j\}.$$

For any subset J of $[\![1, d]\!]$, $\tilde{\mathcal{T}}_J^\omega$ is the set of vertices $x \in \mathbb{Z}^d$ such that there exists y such that $\{x, y\} \in \mathcal{T}_J^\omega$.
In the following we will omit the ω when there is no ambiguity.

Definition 2.14. *We say that two environments ω_1 and ω_2 are trap-equivalent if:*

- *they have the same traps:*
$$\mathcal{T}^{\omega_1} = \mathcal{T}^{\omega_2},$$

- *at each vertex not in a trap, the transition probabilities are the same for both environment:*
$$\forall x \notin \tilde{\mathcal{T}}^{\omega_1}, \ \forall y \sim x, \ \omega_1(x, y) = \omega_2(x, y),$$

- *at each vertex x in a trap $\{x, y\}$, the transition probabilities conditioned on not crossing the trap are the same:*
$$\forall (x, y) \in E, \ \{x, y\} \in \mathcal{T}^{\omega_1}, \ \forall z \sim x, z \neq y, \frac{\omega_1(x, z)}{1 - \omega_1(x, y)} = \frac{\omega_2(x, y)}{1 - \omega_2(x, y)}.$$

We will denote by $\tilde{\Omega}$ the set of all equivalence classes for the trap-equivalence relation.

Definition 2.15. *Set $\tilde{\omega} \in \tilde{\Omega}$. Let $\mathcal{T}$ be its set of traps and σ a path starting at 0 that only stays a finite amount of time every time it enters a trap. We want to define a path, with the same trajectory as σ outside the traps, which does not keep information regarding the time spent in the traps. We essentially want to erase all the back and forths inside traps. To that extent we define the sequences of integer times $(t_i), (s_i)$ by:*

$$t_0 = 0,$$
$$s_i = \inf\{n \geq t_i, (\sigma_n = \sigma_{t_i} \text{ or } \{\sigma_n, \sigma_{t_i}\} \in \mathcal{T}) \text{ and } (\sigma_{n+1} \neq \sigma_{t_i} \text{ and } \{\sigma_{n+1}, \sigma_{t_i}\} \notin \mathcal{T})\},$$
$$t_{i+1} = \begin{cases} s_i + 1 & \text{if } \sigma_{s_i} = \sigma_{t_i} \\ s_i & \text{otherwise.} \end{cases}$$

If σ_{t_i} is in a trap then $[t_i, s_i]$ is the interval of time spent in this trap before leaving it.
The partially forgotten path $\tilde{\sigma}$ associated with σ in the environment $\tilde{\omega}$ is defined by:

$$\tilde{\sigma}_i := \sigma_{t_i}.$$

Similarly we can define the partially-forgotten walk $(\tilde{Y}_n)_{n \in \mathbb{N}}$ associated with $(Y_n)_{n \in \mathbb{N}}$

Definition 2.16. *For all $i \in \mathbb{N}^*$, let I_i be the set defined by:*

$$I_i = [\![1, d]\!] \times \{a, b, c, d \in \mathbb{N}, a \geq 1, a + b + c + d = i\}.$$

And I^n be defined by:

$$I^n = \bigcup_{1 \le i \le n} I_i.$$

Let $\tilde{e} \in \tilde{E}$ be an undirected edge. We define the sequences (t_i^{in}) (the times when the walk $\tilde{Y}$ enters $\tilde{e}$) and (t_i^{out}) (the times when the walk $\tilde{Y}$ exits $\tilde{e}$) by:

$$
\begin{aligned}
t_1^{\text{in}} &= \inf\{n, \tilde{Y}_n \in \tilde{e}\}, \\
t_{i+1}^{\text{in}} &= \inf\{n > t_i^{\text{in}}, \tilde{Y}_n \in \tilde{e} \text{ and } \tilde{Y}_{n-1} \notin \tilde{e}\}, \\
t_i^{\text{out}} &= \begin{cases} t_i^{\text{in}} & \text{if } \tilde{Y}_{t_i^{\text{in}}+1} \notin \tilde{e} \\ t_i^{\text{in}} + 1 & \text{otherwise} \end{cases}
\end{aligned}
$$

Since the walk is almost surely transient by Theorem 1.5, we have that for i large enough $t_i^{\text{in}} = t_i^{\text{out}} = \infty$ almost surely.

Now let $\hat{Y} := \sigma_{t_1^{\text{in}}}$ and y be such that $\{x, y\} = \tilde{e}$. Let $j \in [\![1, d]\!]$ be such that either $x = y + e_j$ or $x = y - e_j$ (j is the direction of the edge) and n be such that $t_n^{\text{in}} < \infty$ and $t_{n+1}^{\text{in}} = \infty$. Now we can define $N_{x \to x}, N_{x \to y}, N_{y \to x}, N_{y \to y}$ by:

$$
\begin{aligned}
N_{x \to x} &= \#\{i \le n, \tilde{Y}_{t_i^{\text{in}}} = x \text{ and } \tilde{Y}_{t_i^{\text{out}}} = x\}, \\
N_{x \to y} &= \#\{i \le n, \tilde{Y}_{t_i^{\text{in}}} = x \text{ and } \tilde{Y}_{t_i^{\text{out}}} = y\}, \\
N_{y \to x} &= \#\{i \le n, \tilde{Y}_{t_i^{\text{in}}} = y \text{ and } \tilde{Y}_{t_i^{\text{out}}} = x\}, \\
N_{y \to y} &= \#\{i \le n, \tilde{Y}_{t_i^{\text{in}}} = y \text{ and } \tilde{Y}_{t_i^{\text{out}}} = y\}.
\end{aligned}
$$

The configuration p of the edge $\tilde{e}$, for the walk $\tilde{Y}$, is the element of I_n defined by:

$$p_{\{x,y\}}^{\tilde{Y}} := (j, N_{x \to x}, N_{x \to y}, N_{y \to x}, N_{y \to y}).$$

Now we can say in what way the strengths of the traps are independent.

Lemma 2.17. *For any environment $\omega \in \Omega$, let $\tilde{\omega} \in \tilde{\Omega}$ be its equivalence class for the trap-equivalent relation. Now let $(\tilde{Y}_i)$ be the partially forgotten walk. We will write $\overline{\alpha} := \sum_{1 \le i \le 2d} \alpha_i$ and for any vertex z and integer i we will use the notation $\alpha(z, z+e_i) := \alpha_i$. Knowing $\tilde{\omega}$ and $(\tilde{Y}_i)$, the strengths of the various traps are independent. Furthermore, let $\{x, y\}$ be a trap and $p = (j, N_{x \to x}, N_{x \to y}, N_{y \to x}, N_{y \to y})$ its configuration. To simplify notations we will write $N_x := N_{x \to x} + N_{y \to x}$, $N_y := N_{x \to y} + N_{y \to y}$ and $N := N_x + N_y$. Let (r, k) be defined by $(1 - \omega(x, y), 1 - \omega(y, x)) = ((1+k)r, (1-k)r)$. The density of law of (r, k) (with respect to the Lebesgue measure) knowing $\tilde{\omega}$ and $\tilde{Y}$ is:*

$$C_p r^{\kappa_j - 1}(1+k)^{N_x + \overline{\alpha} - \alpha(x,y) - 1}(1-k)^{N_y + \overline{\alpha} - \alpha(y,x) - 1} h_p(r(1+k), r(1-k)) 1_{0 \le r \le \frac{1}{4}} 1_{-1 \le k \le 1},$$

where C_p is a constant that only depends on p and α, and h_p is a function that only depends on p and α and that satisfies the following bound:

$$\forall r \le \frac{1}{4}, \ \ |\log(h_p(r(1+k), r(1-k)))| \le 5(N + 2\overline{\alpha})r.$$

And for the law of the strength s of the trap, there exists a constant D_p that only depends on the configuration of the trap such that for any $A \ge 2$:

$$D_p A^{-\kappa_j} \exp\left(-\frac{5(N + 2\overline{\alpha})}{A}\right) \le \mathbb{P}_0\big(s \ge A \big| \tilde{\omega}, \tilde{Y}\big) \le D_p A^{-\kappa_j} \exp\left(\frac{5(N + 2\overline{\alpha})}{A}\right).$$

Proof. In the following, we will write $\overline{\alpha} := \sum_{i=1}^{2d} \alpha_i$ and if $y = x + e_i$ we will write $\alpha(x, y) := \alpha_i$. First we need to show that the strengths of the traps are approximately

independent of the trajectory of the walk. We will take an environment ω and let $\tilde{\omega}$ be the set of all environments that are trap-equivalent to ω. Now for any path σ starting at 0, let $\tilde{\sigma}^{\tilde{\omega}}$ be the set of all paths that start at 0 and that have the same partially-forgotten path as σ. We want to see how the law of the environment is changed knowing the partially-forgotten path and the equivalence class of the environment. We get that the density of the environment (we look at an environment of finite size, large enough to contain the path we look at) (for $\mathbb{P}^{(\alpha)}$) knowing the equivalence class of the environment is equal to:

$$C \prod_{\{x,y\} \in \mathcal{T}} (\varepsilon_x)^{\overline{\alpha}-\alpha(x,y)-1}(1-\varepsilon_x)^{\alpha(x,y)-1}(\varepsilon_y)^{\overline{\alpha}-\alpha(y,x)-1}(1-\varepsilon_y)^{\alpha(y,x)-1}1_{\varepsilon_x+\varepsilon_y<\frac{1}{2}}\mathrm{d}\varepsilon_x\mathrm{d}\varepsilon_y, \quad (2.1)$$

where $\varepsilon_x = 1-\omega(x,y)$ and $\varepsilon_y = 1-\omega(y,x)$. Now, knowing the environment, the probability of having the given partially-forgotten walk is the same in parts of the environment where there is no trap. The only thing that depends on the specific environment is the times when the walk crosses the traps. Let $\{x,y\}$ be a trap, and for any $z_1, z_2 \in \{x,y\}$ let $\tilde{p}(z_1, z_2)$ be the probability to exit the path by z_2, starting at z_1, we get:

$$\tilde{p}(x,x) = \frac{\varepsilon_x}{\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y}, \qquad \tilde{p}(y,y) = \frac{\varepsilon_y}{\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y},$$

$$\tilde{p}(x,y) = \frac{\varepsilon_y(1-\varepsilon_x)}{\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y}, \qquad \tilde{p}(y,x) = \frac{\varepsilon_x(1-\varepsilon_y)}{\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y}.$$

So for any environment ω, we get that the probability of a partially-forgotten path (for $\mathrm{P}_0^{(\alpha)}$), is equal to:

$$C \prod_{\{x,y\} \in \mathcal{T}} \tilde{p}(x,x)^{N_{x \to x}}\tilde{p}(x,y)^{N_{x \to y}}\tilde{p}(y,x)^{N_{y \to x}}\tilde{p}(y,y)^{N_{y \to y}}$$

$$=C \prod_{\{x,y\} \in \mathcal{T}} \frac{\varepsilon_x^{N_{x \to x}}(\varepsilon_y(1-\varepsilon_x))^{N_{x \to y}}(\varepsilon_x(1-\varepsilon_y))^{N_{y \to x}}\varepsilon_y^{N_{y \to y}}}{(\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y)^{N_{x \to x}+N_{x \to y}+N_{y \to x}+N_{y \to y}}}$$

$$=C \prod_{\{x,y\} \in \mathcal{T}} \frac{\varepsilon_x^{N_{x \to x}+N_{y \to x}}\varepsilon_y^{N_{x \to y}+N_{y \to y}}}{(\varepsilon_x + \varepsilon_y)^{N_{x \to x}+N_{x \to y}+N_{y \to x}+N_{y \to y}}} \frac{(1-\varepsilon_x)^{N_{x \to y}}(1-\varepsilon_y)^{N_{y \to x}}}{\left(1-\frac{\varepsilon_x\varepsilon_y}{\varepsilon_x+\varepsilon_y}\right)^{N_{x \to x}+N_{x \to y}+N_{y \to x}+N_{y \to y}}}. \quad (2.2)$$

We define $h_{\{x,y\}}$ by:

$$h_{\{x,y\}}(\varepsilon_x, \varepsilon_y) = \frac{(1-\varepsilon_x)^{N_{x \to y}}(1-\varepsilon_y)^{N_{y \to x}}}{\left(1-\frac{\varepsilon_x\varepsilon_y}{\varepsilon_x+\varepsilon_y}\right)^{N_{x \to x}+N_{x \to y}+N_{y \to x}+N_{y \to y}}}(1-\varepsilon_x)^{\alpha(x,y)-1}(1-\varepsilon_y)^{\alpha(y,x)-1}.$$

Now we get that the probability density of having a given environment knowing the equivalence class of the environment and the partially forgotten path is equal to the product of (2.1) and (2.2) up to a multiplicative constant C that depends on the partially-forgotten path:

$$C \prod_{\{x,y\} \in \mathcal{T}} \frac{\varepsilon_x^{N_{x \to x}+N_{y \to x}+\overline{\alpha}-\alpha(x,y)-1}\varepsilon_y^{N_{x \to y}+N_{y \to y}+\overline{\alpha}-\alpha(y,x)-1}}{(\varepsilon_x + \varepsilon_y)^{N_{x \to x}+N_{x \to y}+N_{y \to x}+N_{y \to y}}}h_{\{x,y\}}(\varepsilon_x, \varepsilon_y)1_{\varepsilon_x+\varepsilon_y<\frac{1}{2}}.$$

This means that for $\mathbb{P}_0^{(\alpha)}$, knowing the equivalence class of the environment and the partially forgotten path, the transition probabilities for each trap are independent, so we will look at each trap independently. Let's fix a trap $\{x,y\}$ and to simplify notations, we will write $N_x = N_{x \to x} + N_{y \to x}$, $N_y = N_{x \to y} + N_{y \to y}$ and $N = N_x + N_y$. We define r and k

by $r = \frac{\varepsilon_x + \varepsilon_y}{2}$ and $k = \frac{\varepsilon_x - \varepsilon_y}{\varepsilon_x + \varepsilon_y}$ which gives $\varepsilon_x = r(1 + k)$ and $\varepsilon_y = r(1 - k)$ the law of the transition probabilities becomes:

$$C \frac{r}{2} \frac{(r(1+k))^{N_x + \overline{\alpha} - \alpha(x,y) - 1} (r(1-k))^{N_y + \overline{\alpha} - \alpha(y,x) - 1}}{(2r)^{N_x + N_y + 2}} h_{\{x,y\}}(r(1+k), r(1-k)) 1_{r < \frac{1}{2}} \mathrm{d}r\mathrm{d}k$$

$$= C' r^{\kappa_j - 1} (1+k)^{N_x + \overline{\alpha} - \alpha(x,y) - 1} (1-k)^{N_y + \overline{\alpha} - \alpha(y,x) - 1} h_{\{x,y\}}(r(1+k), r(1-k)) 1_{r < \frac{1}{2}} \mathrm{d}r\mathrm{d}k.$$

Now we want to give bounds on $h_{\{x,y\}}$. Since for all $r \leq \frac{1}{2}$, $|\log(1 - r)| \leq 2r$, we get:

$$\begin{aligned}
&|\log(h_{\{x,y\}}(r(1+k), r(1-k)))| \\
&\leq |(N(x,y) + \alpha(x,y) - 1) \log(1 - r(1+k))| + |(N(y,x) + \alpha(y,x) - 1) \log(1 - r(1-k))| \\
&\quad + |N \log(1 - \frac{r(1 - k^2)}{2})| \\
&\leq (N(x,y) + \alpha(x,y))4r + (N(y,x) + \alpha(y,x))4r + Nr \\
&\leq 5(N_x + N_y + \alpha_x + \alpha_y)r.
\end{aligned}$$

Let $D := \int_{k=-1}^{1} C'(1+k)^{N_x + \overline{\alpha} - \alpha(x,y) - 1} (1-k)^{N_y + \overline{\alpha} - \alpha(y,x) - 1} \mathrm{d}k$. For any $A \geq 2$, we have:

$$DA^{-\kappa_j} \exp\left(-\frac{5(N + 2\alpha)}{A}\right) \leq \mathbb{P}_0\left(s \geq A | \tilde{\omega}, \tilde{Y}\right) \leq DA^{-\kappa_j} \exp\left(\frac{5(N + 2\alpha)}{A}\right). \qquad \square$$

Now we want to show that there cannot be too many traps that are visited many times.

Lemma 2.18. *Set $\alpha \in (0, \infty)^{2d}$. For any $\beta \in \left[\kappa, \frac{\kappa + \kappa'}{2}\right)$ with $\beta \leq 1$ there exists a finite constant $C > 0$ such that for every $i \in \mathbb{N} \setminus \{0, 1\}$:*

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\} \in \mathcal{T}} \#\{j \in [\![\tau_i, \tau_{i+1} - 1]\!], Y_j \in \{x, y\} \text{ and } Y_{j+1} \notin \{x, y\}\}^{\beta}\right) = C.$$

Proof. We want to show that

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\} \in \mathcal{T}} \#\{j \in [\![\tau_i, \tau_{i+1} - 1]\!], Y_j \in \{x, y\} \text{ and } Y_{j+1} \notin \{x, y\}\}^{\beta}\right)$$

can be bounded away from infinity by using the inequality from Lemma 2.12:

$$\mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{x \in \mathbb{Z}^d} \left(\int_{t=0}^{1} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t = x} \mathrm{d}t\right)^{\beta}\right) < \infty,$$

which is true for any $\beta \in \left[\kappa, \frac{\kappa + \kappa'}{2}\right)$, and for any integer m such that $\mathbb{Q}_0^m$ exists.

To that end we need to introduce the intermediate quantity S_n^m:

$$S_n^m := \sum_{i=0}^{n} \sum_{\{x,y\} \in \mathcal{T}} \left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t\right)^{\beta} + \left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m = y} \mathrm{d}t\right)^{\beta},$$

where (T_i^m) are the renewal times for the walk (X_t^m), with the convention that $T_0^m := 0$. By definition of X^m, the time the walk X^m spends in a vertex x is a sum of ℓ_x iid

exponential random variables of expectation $\frac{1}{\gamma_\omega^m(x)}$, where ℓ_x is the number of times the walk Y visits the point x. Therefore the quantity

$$\int_0^\infty \gamma_\omega^m(x) 1_{X_t=x} \mathrm{d}t$$

should be close to ℓ_x. Then, every time the walk Y enters the trap $\{x,y\}$ it stays a time of order $\gamma^\omega(x)$. This means that $\frac{\ell_x}{\gamma^\omega(x)}$ should be almost equal to the number of times the trap is entered. Finally, we get that for every trap the quantities

$$\sum_{\{x,y\}\in\mathcal{T}} \#\{j\in[\![\tau_i,\tau_{i+1}-1]\!], Y_j\in\{x,y\} \text{ and } Y_{j+1}\notin\{x,y\}\}^\beta$$

and

$$\sum_{\{x,y\}\in\mathcal{T}} \left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t\right)^\beta + \left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t\right)^\beta$$

should be of the same order. Then we just need to bound the second quantity with Lemma 2.12 and a law of large numbers.

For any $k\in[\![0,2m+3]\!]$ the random variables $(S_{(2m+3)i+k+1}^m - S_{(2m+3)i+k}^m)_{i\geq 1}$ are iid (the definition of $\gamma_\omega^m(x)$ depends on a box of size m around x and traps span over 2 vertices that's why we cannot consider the sequence $(S_{i+1}^m - S_i^m)_{i\geq 1}$). This means that there is a positive constant C_0 that can be infinite such that $\mathbb{E}_{\mathbb{P}_0}\left(S_{2m+3}^m - S_{2m+2}^m\right) = C_0$ and

$$\frac{1}{n} S_n^m \xrightarrow[n\to\infty]{} C_0 \; \mathbb{P}_0 \text{ a.s and therefore } \mathbb{Q}_0 \text{ a.s.}$$

For any $x\in\mathbb{Z}^d$ there is at most one integer i such that $\left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t\right)$ is non-zero and therefore:

$$S_n^m = \sum_{\{x,y\}\in\mathcal{T}} \left(\int_0^{T_{n+1}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t\right)^\beta + \left(\int_0^{T_{n+1}^m} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t\right)^\beta.$$

Define $\tilde{S}_u^m$ by:

$$\tilde{S}_u^m := \sum_{\{x,y\}\in\mathcal{T}} \left(\int_0^u \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t\right)^\beta + \left(\int_0^u \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t\right)^\beta.$$

By Lemma 2.1 there is a finite constant D^m such that $\frac{1}{n} T_n^m \xrightarrow[n\to\infty]{} D^m \; \mathbb{P}_0$ and $\mathbb{Q}_0$ almost surely. And therefore (keep in mind that $u\mapsto \tilde{S}_u^m$ is non-decreasing) for any $\varepsilon\in(0,1)$, $\mathbb{P}_0$ and $\mathbb{Q}_0$ almost surely:

$$\liminf_{n\to\infty} \frac{1}{n} \tilde{S}_{D^m n}^m \geq \liminf_{n\to\infty} \frac{1}{n} S_{\lfloor(1-\varepsilon)n\rfloor}^m = (1-\varepsilon)C_0 \text{ and}$$

$$\limsup_{n\to\infty} \frac{1}{n} \tilde{S}_{D^m n}^m \geq \limsup_{n\to\infty} \frac{1}{n} S_{\lfloor(1+\varepsilon)n\rfloor}^m = (1+\varepsilon)C_0.$$

And therefore

$$\frac{1}{n} \tilde{S}_{D^m n}^m \xrightarrow[n\to\infty]{} C_0 \; \mathbb{Q}_0 \text{ a.s.}$$

In turn this means that,

$$\liminf \frac{1}{n} \mathbb{E}_{\mathbb{Q}_0} \left(\sum_{\{x,y\}\in\mathcal{T}} \left(\int_0^{D^m n} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t\right)^\beta + \left(\int_0^{D^m n} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t\right)^\beta\right) \geq C_0.$$

Since $\beta \leq 1$ we have:

$$
\frac{1}{n} \mathbb{E}_{\mathbb{Q}_0} \left(\sum_{\{x,y\} \in \mathcal{T}} \left(\int_0^{D^m n} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta + \left(\int_0^{D^m n} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m = y} \mathrm{d}t \right)^\beta \right)
$$

$$
\leq \frac{1}{n} \sum_{i=0}^{\lfloor D^m n \rfloor} \mathbb{E}_{\mathbb{Q}_0} \left(\sum_{\{x,y\} \in \mathcal{T}} \left(\int_i^{i+1} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta + \left(\int_i^{i+1} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m = y} \mathrm{d}t \right)^\beta \right)
$$

$$
= \frac{\lfloor D^m n \rfloor + 1}{n} \mathbb{E}_{\mathbb{Q}_0} \left(\sum_{\{x,y\} \in \mathcal{T}} \left(\int_0^1 \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta + \left(\int_0^1 \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m = y} \mathrm{d}t \right)^\beta \right)
$$

$$
< \infty \text{ by Lemma 2.12.}
$$

So C_0 is finite.

Now we want to get a bound on Y from a bound on X^m. For any trap $\{x,y\} \in \mathcal{T}$ let $N_{\{x,y\}}$ be the number of times the trap $\{x,y\}$ is entered. Let $\mathcal{T}^{\omega,n}$ be the subset of $\mathcal{T}^\omega$ defined by:

$$
\mathcal{T}^{\omega,n} := \{\{x,y\} \in \mathcal{T}^\omega, \ Y_{\tau_1} \cdot e_1 \leq x \cdot e_1 \leq Y_{\tau_1} \cdot e_1 + n \text{ and } Y_{\tau_1} \cdot e_1 \leq y \cdot e_1 \leq Y_{\tau_1} \cdot e_1 + n \} .
$$

We chose a partially-forgotten path σ and we look at the law of the total time the walk X spends in a trap $\{x,y\} \in \mathcal{T}^\omega$ knowing Y_{τ_1} and $\tilde{Y} = \sigma$, where $\tilde{Y}$ is the partially forgotten walk. We now have two sources of randomness: the number of back and forths the walk does every time it visits a trap and the time the accelerated walk X^m spends for every step.

Knowing the partially-forgotten walk, $N_{\{x,y\}}$ is deterministic. Let $t_{\{x,y\}}^j$ be the jth time the walk Y enters the trap $\{x,y\}$ and $\tilde{t}_{\{x,y\}}^j$ be the jth time the walk Y exits the trap $\{x,y\}$. We define $H_{\{x,y\}}^j$ by $H_{\{x,y\}}^j := \left\lfloor \frac{\tilde{t}_{\{x,y\}}^j - t_{\{x,y\}}^j}{2} \right\rfloor$, the number of back and forths in the trap $\{x,y\}$ during the jth visit to the trap. For any integer n and for any trap $\{x,y\} \in \mathcal{T}^{\omega,n}$ we have that knowing the environment, Y_{τ_1} and the partially forgotten walk, $\left(H_{\{x,y\}}^j \right)_{j \in \mathbb{N}, \{x,y\} \in \mathcal{T}}$ is a sequence of independent geometric random variables of parameter $1 - (1 - \omega(x,y))(1 - \omega(y,x))$. Finally, for every $x \in \tilde{\mathcal{T}}$, let ℓ_x^j be the number of times x is visited between times $t_{\{x,y\}}^j$ and $\tilde{t}_{\{x,y\}}^j$. We define ε_x^j by $\varepsilon_x^j := \ell_x^j - H_{\{x,y\}}^j$. Knowing the partially forgotten walk, ε_x^j is deterministic (it is equal to 0 iff the walk enters and leaves the trap by y during the jth visit) and $\varepsilon_x^j \in \{0,1\}$. We have:

$$
\int_0^\infty \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t = \sum_{j=1}^{N_{\{x,y\}}} \sum_{k=1}^{\varepsilon_x^j + H_{\{x,y\}}^j} \mathcal{E}_{m,x}^{k,j} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)},
$$

where the $(\mathcal{E}_{m,x}^{k,j})_{x \in \mathbb{Z}^d, k,j \in \mathbb{N}}$ are independent exponential random variables of parameter $\gamma_\omega^m(x)$, they correspond to the time the accelerated walk spends on each vertex. By technical Lemma 3.4 (the proof of which is in the annex) we get that there exists a constant $C_1 > 0$ such that for any integer n and any trap $\{x,y\} \in \mathcal{T}^{\omega,n}$:

$$
C_1 (N_{\{x,y\}})^\beta \leq \mathbb{E}_{\mathrm{P}_0^\omega} \left(\left(\int_0^\infty \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta + \left(\int_0^\infty \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m = y} \mathrm{d}t \right)^\beta \Big| \tilde{Y}, Y_{\tau_1} \right). \quad (2.3)
$$

Unfortunately, we cannot directly use this inequality to conclude because it does not behave nicely with the renewal times. Indeed if you know that a trap spans over two renewal blocks, it means that you cannot do any back and forth inside the trap and the previous inequality becomes false. Instead we will have to first consider traps in $\mathcal{T}^{\omega,n}$. First, by definition of the renewal times, no trap in $\mathcal{T}^{\omega,n}$ can be visited before time τ_1 or after time τ_{n+2} since $Y_{\tau_{n+2}} \cdot e_1 \geq Y_{\tau_1} \cdot e_1 + n + 1$. Therefore:

$$\sum_{\{x,y\} \in \mathcal{T}^{\omega,n}} \left(\int_0^\infty \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta + \left(\int_0^\infty \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m = y} \mathrm{d}t \right)^\beta$$

$$\leq \sum_{\{x,y\} \in \mathcal{T}^\omega} \left(\int_{T_1^m}^{T_{n+2}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta + \left(\int_{T_1^m}^{T_{n+2}^m} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m = y} \mathrm{d}t \right)^\beta$$

Therefore we get:

$$\frac{1}{n+1} \mathbb{E}_{\mathbb{P}_0} \left(\sum_{\{x,y\} \in \mathcal{T}^{\omega,n}} \left(\int_0^\infty \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta + \left(\int_0^\infty \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m = y} \mathrm{d}t \right)^\beta \right) \leq C_0 < \infty$$

This in turns gives:

$$\frac{1}{n+1} \mathbb{E}_{\mathbb{P}_0} \left(\sum_{\{x,y\} \in \mathcal{T}^{\omega,n}} \left(N_{\{x,y\}} \right)^\beta \right) \leq \frac{C_0}{C_1} < \infty$$

Now let $C_2 := \mathbb{E}_{\mathbb{P}_0} \left(\sum_{\{x,y\} \in \mathcal{T}} \#\{j \in [\![\tau_i, \tau_{i+1} - 1]\!], Y_j \in \{x,y\} \text{ and } Y_{j+1} \notin \{x,y\}\}^\beta \right)$ be the quantity we want to bound. By the law of large numbers, we have that $\mathbb{P}_0$ a.s and therefore $\mathbb{Q}_0$ a.s:

$$\frac{1}{n} \sum_{i=1}^n \sum_{\{x,y\} \in \mathcal{T}} \#\{j \in [\![\tau_i, \tau_{i+1} - 1]\!], Y_j \in \{x,y\} \text{ and } Y_{j+1} \notin \{x,y\}\}^\beta \xrightarrow[n \to \infty]{} C_2$$

Now, as a consequence of Lemma 2.2 and the law of large numbers, there exists a finite constant $D > 0$ such that $\mathbb{P}_0$ a.s and therefore $\mathbb{Q}_0$ a.s, $\frac{1}{n} Y_{\tau_n} \cdot e_1 \xrightarrow[n \to \infty]{} D$. Furthermore, a trap spans over at most two renewal blocks so for any trap $\{x,y\}$:

$$\sum_{i \geq 1} \#\{j \in [\![\tau_i, \tau_{i+1} - 1]\!], Y_j \in \{x,y\} \text{ and } Y_{j+1} \notin \{x,y\}\}^\beta \leq 2(N_{\{x,y\}})^\beta.$$

As a consequence, $\mathbb{P}_0$ a.s:

$$\liminf \frac{1}{n} \frac{1}{n+1} \mathbb{E}_{\mathbb{P}_0} \left(\sum_{\{x,y\} \in \mathcal{T}^{\omega,Dn}} \left(N_{\{x,y\}} \right)^\beta \right) \geq \frac{C_2}{2}.$$

Finally we get:

$$\frac{C_2}{2} \leq D \frac{C_0}{C_1}$$

so C_2 is finite. $\qquad\qquad\square$

The next lemma is just a variation of the previous one, with the difference that the sum has a deterministic number of terms instead of a random one which makes it simpler to use.

Lemma 2.19. *For any $j \in [\![1,d]\!]$ let (x_i^j, y_i^j) be the ith trap in the direction j the walk encounters after τ_2. Let N_i^j be the number of times the walk enters this trap.*

If $\kappa \leq 1$, for any $\beta \in [\kappa, \frac{\kappa + \kappa'}{2})$ with $\beta \leq 1$ there is a constant C such that for any $j \in [\![1,d]\!]$:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^n (N_i^j)^\beta\right) \leq Cn.$$

If $\kappa = 1$ there exists a positive concave function ϕ defined on $[0, \infty)$ such that $\phi(t)$ goes to infinity when t goes to infinity and such that if $\Phi(t) = \int_{x=0}^t \phi(x)\mathrm{d}x$ then there exists a constant C such that for any $n \in \mathbb{N}$:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^n \Phi(N_i^j)\right) \leq Cn.$$

Those results are also true if $\{x_i^j, y_i^j\}$ is the ith trap in the direction j the walk encounters after τ_2 such that $x_i^j \cdot e_1, y_i^j \cdot e_1 \geq Y_{\tau_2} \cdot e_1$.

Proof. Let $p > 0$ be the probability, for $\mathbb{P}_0$, that there is at least one trap in the direction j between times τ_2 and $\tau_3 - 1$. Let $\mathcal{T}_j$ be the set of traps in the direction j. Now let the sequence (n_i) be defined by:

$$\begin{aligned} n_0 &= 1, \\ n_{i+1} &= \min\{k > n_i, \exists \{x,y\} \in \mathcal{T}_j, \exists n \in [\![\tau_k, \tau_{k+1} - 1]\!], Y_n \in \{x,y\}\}. \end{aligned}$$

Now, if $\kappa \leq 1$, let $Z_i^j = \sum_{\{x,y\} \in \mathcal{T}_j} \#\{m \in [\![\tau_{n_i}, \tau_{n_{i+1}} - 1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta$. The $(Z_i^j)_{i \geq 1}$ are clearly identically distributed and we have:

$$\mathbb{E}_{\mathbb{P}_0}(Z_i^j) = \frac{1}{p}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\} \in \mathcal{T}_j} \#\{m \in [\![\tau_2, \tau_3 - 1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right).$$

So let $C_j = \mathbb{E}_{\mathbb{P}_0}(Z_i^j)$ which is finite by Lemma 2.18. We clearly have:

$$\sum_{i=1}^m (N_i^j)^\beta \leq \sum_{i=1}^{2m} Z_i^j.$$

The sum has to go up to $2m$ because in the second sum some traps can appear twice if they are in between two renewal slabs. Indeed, in this case they can be visited before and after the renewal time (if they are in the direction e_1). We now have:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^m (N_i^j)^\beta\right) \leq 2C_j m.$$

Similarly, if $\{\overline{x}_i, \overline{y}_i\}$ is the ith trap in the direction j the walk encounters after τ_2 such that $x_i \cdot e_1, y_i \cdot e_1 \geq Y_{\tau_2} \cdot e_1$ and $\overline{N}_i^j$ the number of times the walk enters this trap then we have:

$$\sum_{i=1}^m (\overline{N}_i^j)^\beta \leq \sum_{i=1}^{2m+1} Z_i.$$

If $\kappa = 1$, by Lemma 2.18,

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\} \in \mathcal{T}_j} \#\{m \in [\![\tau_2, \tau_3 - 1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right) < \infty.$$

Therefore, by forthcoming technical Lemma 3.1 there exists an increasing, positive, concave function ϕ defined on $[0, \infty)$ such that $\phi(t)$ goes to infinity when t goes to infinity and such that, if $\Phi(t) := \int_{x=0}^{t} \phi(x)\mathrm{d}x$ then:

$$\mathbb{E}_{\mathbb{P}_0}\left(\Phi\left(2 \sum_{\{x,y\}\in\mathcal{T}_j} \#\{m \in [\![\tau_2, \tau_3 - 1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^{\beta}\right)\right) < \infty,$$

where $\Phi(t) := \int_{x=0}^{t} \phi(x)\mathrm{d}x$. We have that $x \mapsto \frac{\Phi(x)}{x}$ is increasing and therefore, by writing $g(x) = \frac{\Phi(x)}{x}$, for any non-negative sequence $(a_i)_{1 \le i \le n}$:

$$\sum_{1 \le i \le n} \Phi(a_i) = \sum_{1 \le i \le n} a_i g(a_i)$$

$$\le \sum_{1 \le i \le n} a_i g\left(\sum_{1 \le j \le n} a_j\right)$$

$$= \left(\sum_{1 \le i \le n} a_i\right) g\left(\sum_{1 \le i \le n} a_i\right)$$

$$= \Phi\left(\sum_{1 \le i \le n} a_i\right).$$

So we get:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathcal{T}_j} \Phi\left(2\#\{m \in [\![\tau_2, \tau_3 - 1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^{\beta}\right)\right)$$

$$\le \mathbb{E}_{\mathbb{P}_0}\left(\Phi\left(2 \sum_{\{x,y\}\in\mathcal{T}_j} \#\{m \in [\![\tau_2, \tau_3 - 1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^{\beta}\right)\right) < \infty.$$

Let $Z_i^j := \sum_{\{x,y\}\in\mathcal{T}_j} \Phi\left(\#\{m \in [\![\tau_{n_i}, \tau_{n_i+1} - 1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^{\beta}\right)$. The $(Z_i^j)_{i \ge 1}$ are clearly identically distributed and we have:

$$\mathbb{E}_{\mathbb{P}_0}(Z_i) = \frac{1}{p}\mathbb{E}_{\mathbb{P}_0}\left(2 \sum_{\{x,y\}\in\mathcal{T}_j} \Phi\left(\#\{m \in [\![\tau_2, \tau_3 - 1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^{\beta}\right)\right) < \infty.$$

So let $C_j = \mathbb{E}_{\mathbb{P}_0}(Z_i^j)$, which is finite. We clearly have:

$$\sum_{i=1}^{m} \Phi(N_i^j) \le \sum_{i=1}^{2m} Z_i.$$

Once again, the sum has to go up to $2m$ because in the second sum some traps can appear twice if they are in between two renewal slabs. Indeed, in this case they can be visited before and after the renewal time (if they are in the direction e_1). So:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{m} \Phi(N_i^j)\right) \le 2C_j m.$$

Similarly, if $\{\overline{x}_i, \overline{y}_i\}$ is the ith trap in the direction j the walk encounters after τ_2 such that $x_i \cdot e_1, y_i \cdot e_1 \ge Y_{\tau_2} \cdot e_1$ and $\overline{N}_i^j$ the number of times the walk enters this trap then we

have:

$$\sum_{i=1}^{m} \Phi(\overline{N}_i^j) \leq \sum_{i=1}^{2m+1} Z_i^j.$$

and we get the result we want. □

The following lemma gives us some independence between the strength of a trap and the number of times the walk enters this trap.

Lemma 2.20. *Let $j \in [\![1, d]\!]$ be an integer that represents the direction of the traps we will consider. Let $\{x_i^j, y_i^j\}$ be the ith trap in the direction j (ie $x_i^j - y_i^j \in \{e_j, -e_j\}$) to be visited after time τ_2 and such that $x_i^j \cdot e_1 \geq Y_{\tau_2} \cdot e_1$ and $y_i^j \cdot e_1 \geq Y_{\tau_2} \cdot e_1$. Now let s_i^j be the strength of the trap. Let N_i^j be the number of times the trap $\{x_i^j, y_i^j\}$ is exited. Let $\kappa_j = 2 \sum_{i=1}^{2d} \alpha_i - \alpha_j - \alpha_{j+d}$. For any $\gamma \in [0, 1]$, there exists a constant C that does not depend on i such that:*

$$\forall A \geq 2, \; \mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^\gamma 1_{s_i^j \geq A}\right) \leq \frac{C}{A^{\kappa_j}} \mathbb{E}_{\mathbb{P}_0}((N_i^j)^\gamma).$$

We also have that for any positive concave function ϕ such that $\phi(0) = 1$ with $\Phi(t) = \int_{x=0}^{t} \phi(x) \mathrm{d}x$ we get:

$$\forall A \geq 2, \; \mathbb{E}_{\mathbb{P}_0}\left(\Phi(N_i^j) 1_{s_i^j \geq A}\right) \leq \frac{C}{A^{\kappa_j}} \mathbb{E}_{\mathbb{P}_0}(\Phi(N_i^j)).$$

Proof. First if H is a geometric random variable of parameter p then for any $\gamma \in [0, 1]$ we have the following three inequalities:

$$\mathbb{E}((1 + H)^\gamma) \geq 1 = p^\gamma \frac{1}{p^\gamma}, \tag{2.4}$$

$$\mathbb{E}((1 + H)^\gamma) \geq \mathbb{P}\left(1 + H \geq \frac{1}{p}\right) \frac{1}{p^\gamma} \geq (1-p)^{\frac{1}{p}-1} \frac{1}{p^\gamma}, \tag{2.5}$$

$$\mathbb{E}((1 + H)^\gamma) \leq \mathbb{E}((1 + H))^\gamma = \frac{1}{p^\gamma}. \tag{2.6}$$

Inequalities (2.4) and (2.5) give us that there is a constant C_γ such that $\mathbb{E}((1 + H)^\gamma) \geq C_\gamma \frac{1}{p^\gamma}$, inequality (2.4) gives us the result for $p \geq \frac{1}{2}$ and since $(1-p)^{\frac{1}{p}-1}$ converges to $\exp(-1)$ when p goes to 0, inequality (2.5) gives us the result for $p \leq \frac{1}{2}$.

By Lemma 3.2 we get that there is a constant C_ϕ such that:

$$\frac{1}{2} \frac{1}{p} \phi\left(\frac{1}{p}\right) \leq \mathbb{E}(\Phi(1 + H)) \leq C_\phi \frac{1}{p} \phi\left(\frac{1}{p}\right). \tag{2.7}$$

Let $t \in \mathbb{N}$ be an integer. In the following we will call renewal hyperplane the set of vertices $\{x, x \cdot e_1 = Y_t \cdot e_1\}$. We look at the nth time, after time t, that the walk encounters a vertex that touches a trap $\{x, y\}$ in the direction j that has never been visited before and such that $x \cdot e_1, y \cdot e_1 \geq Y_t \cdot e_1$. We want to show that the strength of the trap is basically independent from the number of times the walk leaves the trap and from the random variable $1_{\tau_2 = t}$. Let x, y be the corresponding trap with x being the first vertex visited.

Now we look at the trap $\{x, y\}$. Let i be such that $y = x + e_i$, we will write $\alpha_x := \alpha_i$, $\alpha_y := \alpha_{i+d}$ and $\overline{\alpha} := \sum_{k=1}^{2d} \alpha_k$. The probability density (for $\mathbb{P}^{(\alpha)}$) for the transition probabilities $\omega(x, y)$ and $\omega(y, x)$, knowing all the transition probabilities $(\omega(z_1, z_2))_{z_1 \in \mathbb{Z}^d \setminus \{x, y\}}$, the renormalized transition probabilities $(\frac{\omega(x,z)}{1-\omega(x,y)})_{z \neq y}$, $(\frac{\omega(y,z)}{1-\omega(y,x)})_{z \neq x}$ and that $\{x, y\}$ is a trap is:

$$C\omega(x, y)^{\alpha_x - 1}(1 - \omega(x, y))^{\overline{\alpha} - \alpha_x - 1} \omega(y, x)^{\alpha_y - 1}(1 - \omega(y, x))^{\overline{\alpha} - \alpha_y - 1} 1_{\omega(x,y) + \omega(y,x) \geq \frac{3}{2}}.$$

Now we make the change of variables:

$$1 - \omega(y, x) = r(1 - k), \ 1 - \omega(x, y) = r(1 + k),$$

which gives a probability density of:

$$2rCr^{\kappa_j - 2}(1 - k)^{\overline{\alpha} - \alpha_y - 1}(1 + k)^{\overline{\alpha} - \alpha_x - 1}(1 - r(1 + k))^{\alpha_x - 1}(1 - r(1 - k))^{\alpha_y - 1}1_{r \leq \frac{1}{4}}\mathrm{d}r\mathrm{d}k.$$

Let $h(r, k)$ be defined by:

$$h(r, k) = (1 - r(1 + k))^{\alpha_x - 1}(1 - r(1 - k))^{\alpha_y - 1}.$$

For $0 \leq r \leq \frac{1}{4}$ and $-1 \leq k \leq 1$ we have:

$$\log(h(r, k)) \leq |\alpha_x - 1|\left|\log\left(\frac{1}{2}\right)\right| + |\alpha_y - 1|\left|\log\left(\frac{1}{2}\right)\right| \leq (\alpha_x + \alpha_y + 2)\log(2).$$

So for $0 \leq r \leq \frac{1}{4}$ and $-1 \leq k \leq 1$ we have:

$$2^{-(\alpha_x + \alpha_y + 2)} \leq h(r, k) \leq 2^{\alpha_x + \alpha_y + 2}.$$

Now the probability density is:

$$2Ch(r, k)r^{\kappa_j - 1}(1 - k)^{\overline{\alpha} - \alpha_y - 1}(1 + k)^{\overline{\alpha} - \alpha_x - 1}1_{r \leq \frac{1}{4}}\mathrm{d}r\mathrm{d}k.$$

Now we look at a specific environment ω and an edge $\{x', y'\}$ in that environment. To simplify the notation we will write $\varepsilon_{x'} = 1 - \omega(x', y')$ and $\varepsilon_{y'} = 1 - \omega(y', x')$. When the walk leaves the trap there are three possibilities:

- the walk goes to infinity before going back to the trap or the renewal hyperplane
- the walk goes to the renewal hyperplane before it goes back to the trap (this does not necessarily mean that the walk will go back to the trap after going to the renewal hyperplane)
- the walk goes back to the trap before it goes to the renewal hyperplane (this does not necessarily mean that the walk will eventually go to the renewal hyperplane).

If the walk is in x' let $\beta_{x'}^\infty$ be the probability, knowing that the next step isn't crossing the trap, that the walk goes to infinity without going to the renewal hyperplane or the trap. Similarly, let $\beta_{x'}^0$ be the probability, knowing that the next step isn't crossing the trap, that the walk goes to the renewal hyperplane before it goes back to the trap (this does not mean that the walk necessarily goes back to the trap). We will also define $\beta_{x'}$ by $\beta_{x'} := \beta_{x'}^\infty + \beta_{x'}^0$. Similarly we will define $\beta_{y'}, \beta_{y'}^\infty, \beta_{y'}^0$.

Now, if the walk is in x', the probability that when the walk leaves the trap it either never comes back to the trap or goes to the renewal hyperplane before it goes back to the trap is:

$$\frac{\varepsilon_{x'}}{\varepsilon_{x'} + \varepsilon_{y'} - \varepsilon_{x'}\varepsilon_{y'}}\beta_{x'} + \frac{\varepsilon_{y'}(1 - \varepsilon_{x'})}{\varepsilon_{x'} + \varepsilon_{y'} - \varepsilon_{x'}\varepsilon_{y'}}\beta_{y'} = \frac{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'} - \varepsilon_{x'}\varepsilon_{y'}}.$$

Similarly, if the walk is in y', this probability is:

$$\frac{\varepsilon_{x'}(1 - \varepsilon_{y'})\beta_{x'} + \varepsilon_{y'}\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'} - \varepsilon_{x'}\varepsilon_{y'}}.$$

Now we want to show that that both these quantities are almost equal to:

$$\frac{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'}}.$$

We will only show it for the first quantity, the proof is the same for the second one. We recall that $\varepsilon_{x'}, \varepsilon_{y'} \leq \frac{1}{2}$, therefore:

$$0 \leq \varepsilon_{x'}\varepsilon_{y'} \leq \frac{1}{2}(\varepsilon_{x'} + \varepsilon_{y'})$$

and

$$0 \leq \varepsilon_{x'}\varepsilon_{y'}\beta_{y'} \leq \frac{1}{2}(\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}).$$

So we get:

$$\frac{1}{2}\frac{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'}} \leq \frac{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'} - \varepsilon_{x'}\varepsilon_{y'}} \leq 2\frac{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'}}.$$

Similarly, if the walk is in x', the probability that the walk goes to infinity knowing that the walk either goes to infinity or the renewal hyperplane before coming to the trap is:

$$\frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}^\infty}{\varepsilon_{x'} + \varepsilon_{y'} - \varepsilon_{x'}\varepsilon_{y'}} \frac{\varepsilon_{x'} + \varepsilon_{yx'} - \varepsilon_{x'}\varepsilon_{y'}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}} = \frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}^\infty}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}}.$$

And if it is in y' this probability is:

$$\frac{\varepsilon_{x'}(1 - \varepsilon_{y'})\beta_{x'}^\infty + \varepsilon_{y'}\beta_{y'}^\infty}{\varepsilon_{x'}(1 - \varepsilon_{y'})\beta_{x'} + \varepsilon_{y'}\beta_{y'}}.$$

We want to show that both these probabilities are almost equal to $\frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}\beta_{y'}^\infty}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}$. We will only show it for the first one:

$$\frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}^\infty}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}} \leq \frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}\beta_{y'}^\infty}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}}$$

$$\leq \frac{1}{(1 - \varepsilon_x)}\frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}\beta_{y'}^\infty}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}$$

$$\leq 2\frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}\beta_{y'}^\infty}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}.$$

And we also get, the same way:

$$\frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}^\infty}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1 - \varepsilon_{x'})\beta_{y'}} \geq \frac{1}{2}\frac{\varepsilon_{x'}\beta_{x'}^\infty + \varepsilon_{y'}\beta_{y'}^\infty}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}.$$

Now we get back to the trap $\{x, y\}$. Let N be the number of times the walk leaves the trap $\{x, y\}$ before going to the renewal hyperplane (so if the walk never goes to the renewal hyperplane, N is just the number of times the walk leaves the trap $\{x, y\}$). We get that knowing $\varepsilon_x, \varepsilon_y$ and N, the probability (for P_0^ω) that the walk never goes to the renewal hyperplane is between $\frac{1}{2}\frac{\varepsilon_x\beta_x^\infty + \varepsilon_y\beta_y^\infty}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}$ and $2\frac{\varepsilon_x\beta_x^\infty + \varepsilon_y\beta_y^\infty}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}$.

We also have that there exist two geometric random variables N^- and N^+ respectively of parameter $1 - \frac{1}{2}\frac{\varepsilon_x\beta_x + \varepsilon_y\beta_y}{\varepsilon_x + \varepsilon_y}$ and $1 - 2\frac{\varepsilon_x\beta_x + \varepsilon_y\beta_y}{\varepsilon_x + \varepsilon_y}$ such that P_0^ω almost surely:

$$1 + N^- \leq N \leq 1 + N^+.$$

Therefore, by equations (2.4), (2.5), (2.6) and (2.7) there exist two positive constants C_1 and C_2 (that depend on γ and Φ) such that for f equal to either $x \mapsto x^\gamma$ or Φ:

$$C_1 f\left(\frac{\varepsilon_x + \varepsilon_y}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}\right) \leq \mathbb{E}_{\mathrm{P}_0^\omega}(f(N)) \leq C_2 f\left(\frac{\varepsilon_x + \varepsilon_y}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}\right). \qquad (2.8)$$

Now let f be either $x \mapsto x^\gamma$ or Φ. We need to show that N is almost independent from $1_{\tau_2=t}$. Let t_{xy} be the first time the walk is in x or y and let B be the event that "τ_2 can be equal to t" ie there exists $t' < t$ (t' plays the role of τ_1) such that:

- $\forall i < t',\ X_i \cdot e_1 < X_{t'} \cdot e_1$,
- $\forall i \in [\![t', t-1]\!],\ X_{t'} \cdot e_1 \leq X_i \cdot e_1 < X_t \cdot e_1$,
- $\forall i \in [\![t, t_{xy}]\!],\ X_i \cdot e_1 \geq X_t \cdot e_1$,
- $\forall i \in [\![0, t'-1]\!] \cup [\![t'+1, t-1]\!], (\exists j < i,\ X_j \cdot e_1 \geq X_i \cdot e_1)$ or $(\exists j \in [\![i+1, t-1]\!] X_j \cdot e_1 < X_i \cdot e_1)$.

We have that if B isn't true then τ_2 cannot be equal to t. If B is true then $\tau_2 = t$ iff the walk never crosses the renewal hyperplane after time t_{xy}. So, for any environment ω:

$$\frac{1}{2}\frac{\varepsilon_x \beta_x^\infty + \varepsilon_y \beta_y^\infty}{\varepsilon_x \beta_x + \varepsilon_y \beta_y} \mathrm{P}_0^\omega(B) \leq \mathrm{P}_0^\omega(\tau_2 = t | N) \leq 2\frac{\varepsilon_x \beta_x^\infty + \varepsilon_y \beta_y^\infty}{\varepsilon_x \beta_x + \varepsilon_y \beta_y} \mathrm{P}_0^\omega(B) \qquad (2.9)$$

To simplify notations we will write

$$\overline{h}(k) := \frac{(1+k)\beta_x^\infty + (1-k)\beta_y^\infty}{(1+k)\beta_x + (1-k)\beta_y}.$$

We have (in the following, the constant C will depend on the line):

$$\mathbb{E}_{\mathbb{P}_0}\left(f(N)1_{\varepsilon_x + \varepsilon_y \leq \frac{1}{A}} 1_{\tau_2 = t}\right)$$

$$\leq 2\mathbb{E}_{\mathbb{P}_0}\left(f(N)\frac{\varepsilon_x \beta_x^\infty + \varepsilon_y \beta_y^\infty}{\varepsilon_x \beta_x + \varepsilon_y \beta_y} 1_{\varepsilon_x + \varepsilon_y \leq \frac{1}{A}}\right) \text{ by (2.9)}$$

$$\leq C\mathbb{E}_{\mathbb{P}_0}\left(f\left(\frac{\varepsilon_x + \varepsilon_y}{\varepsilon_x \beta_x + \varepsilon_y \beta_y}\right)1_{\varepsilon_x + \varepsilon_y \leq \frac{1}{A}}\frac{\varepsilon_x \beta_x^\infty + \varepsilon_y \beta_y^\infty}{\varepsilon_x \beta_x + \varepsilon_y \beta_y}\right) \text{ by (2.8).}$$

Now we use the fact that the various β only depend on $\{x, y\}$ being a trap, the trajectory of the walk up to the time it encounters the nth trap in the direction j after time t, the transition probabilities $(\omega(z_1, z_2))_{z_1 \in \mathbb{Z}^d \setminus \{x,y\}}$ and the renormalized transition probabilities $(\frac{\omega(x,z)}{1-\omega(x,y)})_{z \neq y}, (\frac{\omega(y,z)}{1-\omega(y,x)})_{z \neq x}$. But the law of $(\omega(x,y), \omega(x,y))$ is independent of this so we get:

$$\mathbb{E}_{\mathbb{P}_0}\left(f(N)1_{\varepsilon_x + \varepsilon_y \leq \frac{1}{A}} 1_{\tau_2 = t}\right)$$

$$\leq C\mathbb{E}_{\mathbb{P}_0}\left(\int_{r=0}^{\frac{1}{2A}}\int_{k=-1}^{1} f\left(\frac{r}{r(1+k)\beta_x + r(1-k)\beta_y}\right) 2Ch(r,k)r^{\kappa_j-1}(1-k)^{\alpha_y}(1+k)^{\alpha_x}\overline{h}(k)\mathrm{d}k\mathrm{d}r\right)$$

$$\leq C\mathbb{E}_{\mathbb{P}_0}\left(\int_{r=0}^{\frac{1}{2A}} r^{\kappa_j-1}\mathrm{d}r \int_{k=-1}^{1} f\left(\frac{1}{(1+k)\beta_x + (1-k)\beta_y}\right)(1-k)^{\alpha_y}(1+k)^{\alpha_x}\overline{h}(k)\mathrm{d}k\right)$$

$$= C\left(\frac{2}{A}\right)^{\kappa_j}\mathbb{E}_{\mathbb{P}_0}\left(\int_{r=0}^{\frac{1}{4}} r^{\kappa_j-1}\mathrm{d}r \int_{k=-1}^{1} f\left(\frac{1}{(1+k)\beta_x + (1-k)\beta_y}\right)(1-k)^{\alpha_y}(1+k)^{\alpha_x}\overline{h}(k)\mathrm{d}k\right)$$

$$\leq \frac{C}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}\left(\int_{r=0}^{\frac{1}{4}}\int_{k=-1}^{1} f\left(\frac{r}{r(1+k)\beta_x + r(1-k)\beta_y}\right) 2Ch(r,k)r^{\kappa_j-1}(1-k)^{\alpha_y}(1+k)^{\alpha_x}\overline{h}(k)\mathrm{d}k\mathrm{d}r\right)$$

$$\leq \frac{C}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}\left(f(N)1_{\varepsilon_x + \varepsilon_y \leq \frac{1}{2}}\frac{\varepsilon_x \beta_x^\infty + \varepsilon_y \beta_y^\infty}{\varepsilon_x \beta_x + \varepsilon_y \beta_y}\right)$$

$$\leq \frac{C}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}\left(f(N)1_{\varepsilon_x + \varepsilon_y < \frac{1}{2}} 1_{\tau_2 = t}\right).$$

Then, by summing on all t we get the result. $\qquad \square$

2.4 The time the walk spends in traps

Now that we have some independence, we can start to look at the precise behaviour of the time spent in the traps. First we want to show that the number of times the walk enters a trap times the strength of said trap is a good approximation of the total time spent in this trap.

Lemma 2.21. *Let* $j \in [\![1, d]\!]$ *be a direction. Now let* $\{x_i^j, y_i^j\}$ *be the* i^{th} *trap in the direction* j *entered after time* τ_2 *and such that* $x_i^j \cdot e_1, y_i^j \cdot e_1 \geq Y_{\tau_2} \cdot e_1$. *Let* s_i^j *be the strength of this trap,* N_i^j *the number of times the walk enters this trap and* $\ell_i^j = \#\{n, Y_n \in \{x_i^j, y_i^j\}\}$ *the time spent in the trap. We have for any environment* ω, *for any* $A, B \geq 0$, *for any integer* m *and for any* $C \in \mathbb{R}^+ \cup \{\infty\}$:

$$P_0^\omega \left(\sum_{i=1}^n \ell_i^j 1_{N_i^j \geq m} 1_{s_i^j \leq C} \geq A \text{ and } \sum_{i=1}^n N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq C} \leq B \right) \leq \frac{5B}{A}.$$

Proof. Let ω be an environment, $(\tilde{Y}_i)_{i \in \mathbb{N}}$ be the partially forgotten walk on this environment. Let $p_i^j = \omega(x_i^j, y_i^j)\omega(y_i^j, x_i^j)$. Now the number of back and forths inside the trap (x_i^j, y_i^j) during its kth visit is equal to $H_{i,k}^j$ where $H_{i,k}^j$ is a geometric random variable of parameter $1 - p_i^j$. Knowing the partially-forgotten walk and p_i^j, the $H_{i,k}^j$ are independent and we get for any j:

$$\mathbb{E}_{P_0^\omega} \left(\sum_{i=1}^n 1_{N_i^j \geq m} 1_{s_i^j \leq C} \sum_{k=1}^{N_i^j} 2 H_{i,k}^j | \tilde{Y} \right) = \sum_{i=1}^n 1_{N_i^j \geq m} 1_{s_i^j \leq C} \sum_{k=1}^{N_i^j} 2 \frac{p_i^j}{1 - p_i^j}$$

$$\leq 2 \sum_{i=1}^n 1_{N_i^j \geq m} 1_{s_i^j \leq C} N_i^j \frac{1}{1 - p_i^j}.$$

Now we use the fact that $\omega(x_i^j, y_i^j) \geq \frac{1}{2}$ to show that $1 - p_i^j \geq \frac{1}{2s_i^j}$:

$$1 - p_i^j = 1 - (1 - (1 - \omega(x_i^j, y_i^j)))(1 - (1 - \omega(y_i^j, x_i^j))$$

$$= (1 - \omega(x_i^j, y_i^j)) + (1 - \omega(y_i^j, x_i^j)) - (1 - \omega(x_i^j, y_i^j))(1 - \omega(y_i^j, x_i^j))$$

$$\geq (1 - \omega(x_i^j, y_i^j)) + (1 - \omega(y_i^j, x_i^j)) - \frac{1}{2}(1 - \omega(y_i^j, x_i^j))$$

$$\geq \frac{(1 - \omega(x_i^j, y_i^j)) + (1 - \omega(y_i^j, x_i^j))}{2}$$

$$= \frac{1}{2s_i^j}.$$

So we get:

$$\mathbb{E}_{P_0^\omega} \left(\sum_{i=1}^n 1_{N_i^j \geq m} 1_{s_i^j \leq C} \sum_{k=1}^{N_i^j} 2 H_{i,k}^j | \tilde{Y} \right) \leq 4 \sum_{i=1}^n N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq C}.$$

The actual value of ℓ_i^j can be slightly larger than $\sum_{k=1}^{N_i^j} 2 H_{i,k}^j$ because this only counts the back and forths, so we miss the correct amount by 1 every time the walk crosses the trap an even number of times and by 2 every time the walk crosses the trap an odd number of times. So we get that the time ℓ_i^j the walk spends in the ith trap is smaller than $2N_i^j + \sum_{j=1}^{N_i^j} 2 H_{i,k}^j$. For any positive constants $A, B > 0$, let $E^n(B)$ be the event

$\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} 1_{s_i^j \leq C} s_i^j \leq B$, we have:

$$\mathrm{P}_0^{\omega}\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m} 1_{s_i^j \leq C} \geq A \text{ and } \sum N_i^j 1_{s_i^j \leq C} 1_{N_i^j \geq m} s_i^j \leq B\right)$$

$$=\mathbb{E}_{\mathrm{P}_0^{\omega}}\left(\mathrm{P}_0^{\omega}\left(\sum_{i=1}^{n} \ell_i^j 1_{s_i^j \leq C} 1_{N_i^j \geq m} \geq A | \tilde{Y}\right) 1_{E^n(B)}\right)$$

$$\leq \mathbb{E}_{\mathrm{P}_0^{\omega}}\left(\frac{\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} 1_{s_i^j \leq C}(4s_i^j + 2)}{A} 1_{E^n(B)}\right)$$

$$\leq \mathbb{E}_{\mathrm{P}_0^{\omega}}\left(\frac{\sum_{i=1}^{n} 5N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq C}}{A} 1_{E^n(B)}\right) \text{ since } s_i^j > 2$$

$$\leq \mathbb{E}_{\mathrm{P}_0^{\omega}}\left(\frac{5B}{A} 1_{E^n(B)}\right) \leq \frac{5B}{A}. \qquad \square$$

Now we want to show that we can neglect the time spent in traps in directions j such that $\kappa_j \neq \kappa$ and in traps that are visited a lot of times. This will allow us to have traps that are rather similar so that the time spent in those traps are almost identically distributed.

Lemma 2.22. *Let $j \in [\![1, d]\!]$ be an integer that represents the direction of the trap we will consider. Let $\{x_i, y_i\}$ be the ith trap in the direction j visited by the walk after time τ_2 and such that $x_i \cdot e_1 \geq Y_{\tau_2} \cdot e_1$ and $y_i \cdot e_1 \geq Y_{\tau_2} \cdot e_1$. Let $\kappa_j = 2\alpha - \alpha_j - \alpha_{j+d} \geq \kappa$.*

If $\kappa < 1$ there are two cases: If $\kappa_j = \kappa$, for any $\varepsilon > 0$ there exists an integer m_ε such that for n large enough:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m_\varepsilon} \geq \varepsilon n^{\frac{1}{\kappa}}\right) \leq \varepsilon.$$

If $\kappa_j > \kappa$, for any $\varepsilon > 0$, for n large enough:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j \geq \varepsilon n^{\frac{1}{\kappa}}\right) \leq \varepsilon.$$

If $\kappa = 1$ there are two cases: If $\kappa_j = \kappa$, for any $\varepsilon > 0$ there exists an integer m_ε such that for n large enough:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m_\varepsilon} \geq \varepsilon n \log(n)\right) \leq \varepsilon.$$

If $\kappa_j > 1$, for any $\varepsilon > 0$, for n large enough:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j \geq \varepsilon n \log(n)\right) \leq \varepsilon.$$

Proof. For all $i \geq 0$ let t_i be the time at which the walk Y enters its ith trap ($\{x_i, y_i\}$) in the direction j after τ_2 and such that $x_i \cdot e_1 \geq Y_{\tau_2} \cdot e_1$ and $y_i \cdot e_1 \geq Y_{\tau_2} \cdot e_1$. We will write x_i the vertex such that $x_i = Y_{t_i}$. Let s_i^j be the strength of the trap $\{x_i^j, y_i^j\}$. For any

$A, B > 0$:

$$\mathbb{P}_0(\exists i \leq n, s_i^j \geq A \text{ and } N_i^j \geq B) \leq \mathbb{P}_0\left(\left(\sum_{i=1}^n N_i^j 1_{s_i^j \geq A}\right)^\kappa \geq B^\kappa\right)$$

$$\leq \mathbb{P}_0\left(\sum_{i=1}^n (N_i^j)^\kappa 1_{s_i^j \geq A} \geq B^\kappa\right)$$

$$\leq \frac{1}{B^\kappa} \mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^n (N_i^j)^\kappa 1_{s_i^j \geq A}\right)$$

$$\leq \frac{c}{B^\kappa}\frac{1}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^n (N_i^j)^\kappa\right) \quad \text{by Lemma 2.20}$$

$$\leq \frac{c}{B^\kappa}\frac{1}{A^{\kappa_j}}Cn \quad \text{by Lemma 2.19.} \tag{2.10}$$

We will first look at the case $\kappa < 1$.

Now, we want to show that we can neglect traps with a high N_i^j or a low s_i^j. We get that for any positive integer M, any real $A \geq 2$ and any $\beta \in [\kappa, 1]$ and $\eta > 0$ such that $\beta + \eta \leq \min\left(\frac{\kappa + \kappa'}{2}, 1\right)$:

$$\mathbb{P}_0\left(\sum_{i=1}^n N_i^j s_i^j 1_{s_i^j < A} 1_{N_i^j \geq M} \geq (an)^{\frac{1}{\kappa}}\right)$$

$$\leq \mathbb{P}_0\left(\sum_{i=1}^n (N_i^j)^\beta (s_i^j)^\beta 1_{s_i^j < A} 1_{N_i^j \geq M} \geq (an)^{\frac{\beta}{\kappa}}\right)$$

$$\leq (an)^{-\frac{\beta}{\kappa}} \mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^n (N_i^j)^\beta (s_i^j)^\beta 1_{s_i^j < A} 1_{N_i^j \geq M}\right)$$

$$\leq (an)^{-\frac{\beta}{\kappa}} M^{-\eta} \mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^n (N_i^j)^{\beta+\eta} (s_i^j)^\beta 1_{s_i^j < A}\right)$$

$$\leq (an)^{-\frac{\beta}{\kappa}} M^{-\eta} \mathbb{E}_{\mathbb{P}_0}\left(\int_{t=0}^{A^\beta} \sum_{i=1}^n (N_i^j)^{\beta+\eta} 1_{(s_i^j)^\beta \geq t} dt\right)$$

$$\leq (an)^{-\frac{\beta}{\kappa}} M^{-\eta} \sum_{i=1}^n \int_{t=0}^{A^\beta} \mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta} 1_{(s_i^j)^\beta \geq t}\right) dt$$

$$\leq (an)^{-\frac{\beta}{\kappa}} M^{-\eta} \sum_{i=1}^n \left(2\mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta}\right) + \int_{t=2^\beta}^{A^\beta} \mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta} 1_{s_i \geq t^{\frac{1}{\beta}}}\right) dt\right).$$

By Lemma 2.20, there exists a constant c such that $\mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta} 1_{s_i^j \geq t^{\frac{1}{\beta}}}\right) \leq \mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta}\right) c t^{-\frac{\kappa}{\beta}}$, for $t \geq 2^\beta$ so:

$$\mathbb{P}_0\left(\sum_{i=1}^n N_i^j s_i^j 1_{s_i^j < A} 1_{N_i^j \geq M} \geq (an)^{\frac{1}{\kappa}}\right)$$

$$\leq (an)^{-\frac{\beta}{\kappa}} M^{-\eta} \sum_{i=1}^n \left(2\mathbb{E}_{\mathbb{P}_0}((N_i^j)^{\beta+\eta}) + \mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta}\right) \int_{t=2^\beta}^{A^\beta} c t^{-\frac{\kappa}{\beta}} dt\right) \tag{2.11}$$

$$\leq (an)^{-\frac{\beta}{\kappa}} M^{-\eta} \sum_{i=1}^{n} \left(2 + c \int_{t=2^{\beta}}^{A^{\beta}} t^{-\frac{\kappa}{\beta}} \, \mathrm{d}t \right) E_{\mathbb{P}_0}((N_i^j)^{\beta+\eta})$$

$$\leq dn(an)^{-\frac{\beta}{\kappa}} M^{-\eta} \left(2 + c \int_{t=2^{\beta}}^{A^{\beta}} t^{-\frac{\kappa}{\beta}} \, \mathrm{d}t \right) \text{ by Lemma 2.19.}$$

Now for $\kappa_j = \kappa$ if we take $\beta \in (\kappa, 1]$ such that $\beta < \frac{\kappa+\kappa'}{2}$, $\eta = 0$ and $A = bn^{\frac{1}{\kappa}}$ we get:

$$\mathbb{P}_0 \left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j < bn^{\frac{1}{\kappa}}} 1_{N_i^j \geq M} \geq (an)^{\frac{1}{\kappa}} \right) \leq \frac{d}{a} n^{1-\frac{\beta}{\kappa}} \left(2 + \frac{\beta c}{\beta - \kappa} \left(bn^{\frac{1}{\kappa}} \right)^{\beta-\kappa} \right)$$

$$\leq \frac{d}{a} n^{1-\frac{\beta}{\kappa}} \left(2 + \frac{\beta c}{\beta - \kappa} b^{\beta-\kappa} n^{\frac{\beta-\kappa}{\kappa}} \right) \qquad (2.12)$$

$$= 2\frac{d}{a} n^{1-\frac{\beta}{\kappa}} + \frac{d}{a} \frac{\beta c}{\beta - \kappa} b^{\beta-\kappa}.$$

Now, we get by Lemma 2.21 that for any positive constants A, B and any positive integer m:

$$\mathbb{P}_0 \left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m} \geq A \right)$$

$$\leq \mathbb{P}_0 \left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m} \geq A \text{ and } \sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j \leq B \right) + \mathbb{P}_0 \left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j \geq B \right) \quad (2.13)$$

$$\leq \frac{5B}{A} + \mathbb{P}_0 \left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j \geq B \right).$$

So for any $\varepsilon > 0$, for any $a > 0$, by taking $B = \varepsilon^2 n^{\frac{1}{\kappa}}$ and $A = \varepsilon n^{\frac{1}{\kappa}}$ in (2.13), we have for any positive integer m:

$$\mathbb{P}_0 \left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m} \geq \varepsilon n^{\frac{1}{\kappa}} \right) \leq 5\varepsilon + \mathbb{P}_0 \left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j \geq \varepsilon^2 n^{\frac{1}{\kappa}} \right).$$

And we have for any $b > 0$:

$$\mathbb{P}_0 \left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j \geq \varepsilon^2 n^{\frac{1}{\kappa}} \right)$$

$$\leq \mathbb{P}_0 \left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq bn^{\frac{1}{\kappa}}} \geq \varepsilon^2 n^{\frac{1}{\kappa}} \right) + \mathbb{P}_0 \left(\exists i \leq n, N_i^j \geq m \text{ and } s_i^j \geq bn^{\frac{1}{\kappa}} \right).$$

We have by (2.10):

$$\mathbb{P}_0 \left(\exists i \leq n, N_i^j \geq m \text{ and } s_i^j \geq bn^{\frac{1}{\kappa}} \right) \leq \frac{cdn}{(mb)^{\kappa} n} = \frac{cd}{(mb)^{\kappa}}.$$

And by (2.12), taking $b = \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}}$:

$$\mathbb{P}_0 \left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}} n^{\frac{1}{\kappa}}} \geq \varepsilon^2 n^{\frac{1}{\kappa}} \right) \leq \frac{d}{\varepsilon^{2\kappa}} \left(2n^{1-\frac{\beta}{\kappa}} + \frac{\beta c}{\beta - \kappa} \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}(\beta-\kappa)} \right)$$

$$= \frac{d}{\varepsilon^{2\kappa}} \left(2n^{1-\frac{\beta}{\kappa}} + \frac{\beta c}{\beta - \kappa} \varepsilon^{2\kappa+1} \right).$$

So for n large enough:

$$\mathbb{P}_0 \left(\sum_{i=1}^n N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}} n^{\frac{1}{\kappa}}} \geq \varepsilon^2 n^{\frac{1}{\kappa}} \right) \leq \frac{2d\beta c}{\beta - \kappa} \varepsilon$$

which means that for n large enough and m_ε such that $m_\varepsilon \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}} \geq \varepsilon^{-\frac{1}{\kappa}}$ we have:

$$\mathbb{P}_0 \left(\sum_{i=1}^n \ell_i^j 1_{N_i^j \geq m_\varepsilon} \geq \varepsilon n^{\frac{1}{\kappa}} \right) \leq 5\varepsilon + cd\varepsilon + \frac{2d\beta c}{\beta - \kappa} \varepsilon.$$

And we have the result we want.

If $\kappa_j > \kappa$ there exists $\beta \in (\kappa, \kappa_j)$ such that $\beta \leq 1$ and $\beta \leq \frac{\kappa + \kappa'}{2}$ we get by taking $M = 1$ and $A = \infty$ in (2.11):

$$
\begin{aligned}
\mathbb{P}_0 \left(\sum_{i=1}^n N_i^j s_i^j \geq (an)^{\frac{1}{\kappa}} \right) &\leq da^{-\frac{\beta}{\kappa}} n^{1-\frac{\beta}{\kappa}} \left(2 + \int_{t=2}^{\infty} t^{-\frac{\kappa_j}{\beta}} \mathrm{d}t \right) \\
&= da^{-\frac{\beta}{\kappa}} n^{1-\frac{\beta}{\kappa}} \left(2 + \frac{\beta}{\kappa_j - \beta} 2^{1-\frac{\kappa_j}{\beta}} \right) \\
&= Ca^{-\frac{\beta}{\kappa}} n^{1-\frac{\beta}{\kappa}} \text{ for some constant } C.
\end{aligned}
$$

And then Lemma 2.21 gives us the result we want.

Now we can look at the case $\kappa = 1$.

Let ϕ be a positive concave function such that $\phi(t)$ goes to infinity when t goes to infinity. We define Φ by $\Phi(x) := \int_{t=0}^x \phi(t) \, \mathrm{d}t$. Let f be defined by $f(0) := \phi(0) > 0$ and $\forall x > 0$, $f(x) := \frac{\Phi(x)}{x}$. As ϕ is concave and goes to infinity, it is increasing. As ϕ is positive and increasing, it follows that f is increasing. We choose ϕ such that $E_{\mathbb{P}_0}(\Phi(N_i^j)) < \infty$ which is possible by Lemma 2.19. We get that for any positive integer M and any real $A \geq 2$:

$$\mathbb{P}_0 \left(\sum_{i=1}^n N_i^j s_i^j 1_{s_i^j < A} 1_{N_i^j \geq M} \geq an \log(n) \right)$$

$$\leq \frac{1}{an \log(n)} \mathbb{E}_{\mathbb{P}_0} \left(\sum_{i=1}^n N_i^j s_i^j 1_{s_i^j < A} 1_{N_i^j \geq M} \right)$$

$$\leq \frac{1}{an \log(n) f(M)} \mathbb{E}_{\mathbb{P}_0} \left(\sum_{i=1}^n N_i^j f(N_i^j) s_i^j 1_{s_i^j < A} \right)$$

$$\leq \frac{1}{an \log(n) f(M)} \sum_{i=1}^n \int_{t=0}^A \mathbb{E}_{\mathbb{P}_0} \left(\Phi(N_i^j) 1_{s_i^j \geq t} \right) \mathrm{d}t$$

$$\leq \frac{1}{an \log(n) f(M)} \sum_{i=1}^n \left(2\mathbb{E}_{\mathbb{P}_0}(\Phi(N_i^j)) + \int_{t=2}^A \mathbb{E}_{\mathbb{P}_0} \left(\Phi(N_i^j) 1_{s_i^j \geq t} \right) \mathrm{d}t \right).$$

Now, by Lemma 2.20 we get:

$$\mathbb{P}_0 \left(\sum_{i=1}^n N_i^j s_i^j 1_{s_i^j < A} 1_{N_i^j \geq M} \geq an \log(n) \right)$$

$$\leq \frac{1}{an \log(n) f(M)} \sum_{i=1}^n \left(2\mathbb{E}_{\mathbb{P}_0}(\Phi(N_i^j)) + \mathbb{E}_{\mathbb{P}_0} \left(\Phi(N_i^j) \right) \int_{t=2}^A ct^{-\kappa_j} \mathrm{d}t \right)$$

$$\leq \frac{1}{an\log(n)f(M)} \sum_{i=1}^{n} \left(2 + c\int_{t=2}^{A} t^{-\kappa_j}\,\mathrm{d}t\right) E_{\mathbb{P}_0}(\Phi(N_i^j))$$

$$\leq \frac{dn}{an\log(n)f(M)} \left(2 + c\int_{t=2}^{A} t^{-\kappa_j}\,\mathrm{d}t\right) \text{ by definition of } \phi. \tag{2.14}$$

If $\kappa_j = 1$, we get, by taking $A = n^2$ (for $n \geq 2$) in (2.14):

$$\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j \leq A} 1_{N_i^j \geq M} \geq an\log(n)\right) \leq \frac{d}{a\log(n)f(M)}(2 + 2c\log(n)) \leq \frac{C}{af(M)}.$$

And by taking $A = n^2$ and $B = 1$ in equation (2.10) we have for some constant c:

$$\mathbb{P}_0\left(\exists i \leq n, s_i^j \geq n^2\right) \leq \frac{c}{n}.$$

So for any $\varepsilon > 0$ we get, by taking m_ε such that $f(m_\varepsilon) \geq \frac{1}{\varepsilon^3}$ and using Lemma 2.21:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m_\varepsilon} \geq \varepsilon n\log(n)\right) \leq 5\varepsilon + \mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{N_i^j \geq M} \geq \varepsilon^2 n\log(n)\right)$$

$$\leq 5\varepsilon + \frac{c}{n} + C\varepsilon.$$

So there exists a constant C such that for any $\varepsilon > 0$ there exists m_ε such that:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{\ell_i^j \geq m_\varepsilon} \geq \varepsilon n\log(n)\right) \leq C\varepsilon.$$

If $\kappa_j > 1$, we take $M = 0$ and $A = \infty$ in (2.14) we get for some constant C:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j \geq an\log(n)\right) \leq \frac{d}{a\log(n)f(0)} \left(2 + c\int_{t=2}^{\infty} t^{-\kappa_j}\,\mathrm{d}t\right) = \frac{C}{a\log(n)}.$$

And therefore by Lemma 2.21, for any $\varepsilon > 0$

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j \geq \varepsilon n\log(n)\right) \leq 5\varepsilon + \mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j \geq \varepsilon^2 n\log(n)\right)$$

$$\leq 5\varepsilon + \frac{C}{\varepsilon^2 \log(n)}.$$

So we have the result we want $\qquad\square$

Now we have all the tools to get a first limit theorem on the time spent in traps.

Lemma 2.23. *Set* $\alpha \in (0,\infty)^{2d}$ *and let* $\overline{\alpha} := \sum_{i=1}^{2d} \alpha_i$. *Let* $J = \{j \in [\![1,d]\!], 2\overline{\alpha} - \alpha_j - \alpha_{j+d} = \kappa\}$ *and* $\widetilde{\mathcal{T}}_j$ *be the set of vertices* x *such that there exists* $j \in J$ *such that either* $(x, x + e_j) \in \mathcal{T}$ *or* $(x, x - e_j) \in \mathcal{T}$. *Let* $\{x_i^j, y_i^j\}$ *be the* i^{th} *trap in the direction* j *encountered after time* τ_2.

For $\kappa < 1$, *for any* m *there exists a constant* C_m *such that:*

$$n^{-\frac{1}{\kappa}} \sum_{j \in J} \sum_{i \geq 0} \ell_i^j 1_{N_i^j \leq m} 1_{\exists k \leq \tau_{n+1}-1, Y_k \in \{x_i^j, y_i^j\}} \xrightarrow[n\to\infty]{} C_m \mathcal{S}_1^\kappa \text{ in law for } \mathbb{P}_0.$$

For $\kappa = 1$, *for any* m *there exists a constant* $C_m > 0$ *such that:*

$$\frac{1}{n\log(n)} \sum_{j \in J} \sum_{i \geq 0} \ell_i^j 1_{N_i^j \leq m} 1_{\exists k \leq \tau_{n+1}-1, Y_k \in \{x_i^j, y_i^j\}} \xrightarrow[n\to\infty]{} C_m \text{ in probability for } \mathbb{P}_0.$$

Proof. For every configuration $p \in \bigcup_{n \geq 1} I_n$ (I_n is defined in 2.16) let C_p be the expectation of the number of traps of configuration p encountered between times τ_2 and $\tau_3 - 1$ (it is also the expectation of the number of traps of configuration p encountered between times τ_i and $\tau_{i+1} - 1$ for any $i \geq 2$). We clearly have:

$$C_p \leq \mathbb{E}_{\mathbb{P}_0} \left(\sum_{x \in \mathbb{Z}^d} 1_{\exists i \in [\tau_2, \tau_3 - 1], Y_i = x} \right) < \infty.$$

Once we know that a trap is in a direction $j \in J$ and has a configuration p for some partially forgotten random walk, the exact number of back and forths the walk does in this trap is still random, because the exact number of back and forths knowing the transition probabilities of the trap is random and because the transition probabilities of the trap are still random, following the law (cf Lemma 2.17):

$$D_p r^{\kappa-1} (1+k)^{p_x} (1-k)^{p_y} h(r(1+k), r(1-k)) 1_{r \leq \frac{1}{4}} \mathrm{d}r \mathrm{d}k,$$

where $r(1+k) = 1 - \omega(x, y)$, $r(1-k) = 1 - \omega(y, x)$ and the value of p_x, p_y are explicit but irrelevant. Let N be such that $p \in I_N$ (ie the walk exits the trap N times) we also have that there exists a constant C_α that only depends on α such that:

$$|\log(h(r(1+k), r(1-k)))| \leq C_\alpha N r.$$

The number of back and forths is the sum of N iid geometric random variables $(H_1, \ldots, H_N)$ of parameter $1 - q$ with $q = \omega(x, y)\omega(y, x) = 1 - 2r + r^2(1 - k^2)$. This gives us the following bound:

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^N H_i \geq a | q \right) &\leq N \mathbb{P}\left(H_1 \geq \frac{a}{N} | q \right) \\
&\leq N(1-q) q^{\frac{a}{N}} \\
&\leq N \exp\left(\log(1 - 2r + r^2(1 - k^2)) \frac{a}{N} \right) \\
&\leq N \exp\left((-2r + r^2) \frac{a}{N} \right).
\end{aligned}$$

For $r \in \left[\frac{2\kappa n \log(a)}{a}, \frac{1}{2} \right]$ we have $-2r + r^2 \leq -r$ and

$$\begin{aligned}
N \exp\left((-2r + r^2) \frac{a}{N} \right) &\leq N \exp\left(-r \frac{a}{N} \right) \\
&\leq N \exp\left(-\frac{2\kappa N \log(a)}{a} \frac{a}{N} \right) \\
&= N a^{-2\kappa}.
\end{aligned}$$

Now let ℓ^- be equal to twice the number of back and forths: $\ell^- := 2\sum_{i=1}^N H_i$. Now we look at $\mathbb{P}\left(\ell^- \geq a \text{ and } r \leq \frac{2\kappa N \log(a)}{a} \right)$, we want to show that it is equivalent to $Ca^{-\kappa}$ for some constant C. First we want to have a good approximation of $\mathbb{P}\left(2\sum_{i=1}^N H_i \geq a | q \right)$ for large q. Now let $\tilde{H}_1, \ldots, \tilde{H}_n$ be iid exponential random variables of parameter $-\log(q)$ such that for every i, $H_i = \lfloor \tilde{H}_i \rfloor$. And we define $\tilde{\ell}^- = 2\sum_{i=1}^n \tilde{H}_i$. Now it is easy to show by induction on n that:

$$\mathbb{P}_0\left(\tilde{\ell}^- \geq 2a | q \right) = \sum_{j=0}^{N-1} \frac{(-a \log(q))^j}{j!} \exp(\log(q)a).$$

Now we clearly have:
$$\ell^- \le \tilde{\ell}^- \le \ell^- + 2N$$

so
$$\mathbb{P}_0\left(\ell^- \ge a|q\right) \le \mathbb{P}_0\left(\tilde{\ell}^- \ge a|q\right)$$

and
$$\mathbb{P}_0\left(\ell^- \ge a|q\right) \ge \mathbb{P}_0\left(\tilde{\ell}^- \ge a - 2N|q\right).$$

We want to show that $\mathbb{P}\left(\tilde{\ell}^- \ge a|q\right)$ and $\mathbb{P}\left(\tilde{\ell}^- \ge a - 2N|q\right)$ are more or less equal. We clearly have:
$$\mathbb{P}_0\left(\tilde{\ell}^- \ge a - 2N|q\right) \le \mathbb{P}_0\left(\tilde{\ell}^- \ge a|q\right)$$

and we also have:
$$\mathbb{P}_0\left(\tilde{\ell}^- \ge 2a - 2N|q\right) = \sum_{j=0}^{N-1} \frac{(-a\log(p))^j}{j!}\left(1 - \frac{N}{a}\right)^j \exp(\log(q)a)\exp(-\log(p)N)$$
$$\ge \exp(-\log(q)N)\left(1 - \frac{N}{a}\right)^N \sum_{j=0}^{N-1} \frac{(-a\log(q))^j}{j!}\exp(\log(q)a).$$

First we want to show that we can replace $\log(q)$ by $-2r$. We clearly have $\log(q) \le -2r + r^2$. We also have $\log(q) \ge \log(1 - 2r)$ and for $r \in [0, \frac{1}{4}]$, there exists a constant C that does not depend on r such that $\log(1 - 2r) \ge -2r - Cr^2$. So we get:
$$2r - r^2 \le -\log(q) \le 2r + Cr^2.$$

So
$$\exp(-2ar)\exp(-Car^2) \le \exp(a\log(q)) \le \exp(-2ar)\exp(ar^2).$$

So we get:
$$\forall j, \ \frac{(-a\log(q))^j}{j!}\exp(\log(q)a) \le \frac{(2ar)^j}{j!}\exp(-2ar)\left(1 + \frac{Cr}{2}\right)^j \exp(ar^2)$$

and
$$\frac{(-a\log(q))^j}{j!}\exp(\log(q)a) \ge \frac{(2ar)^j}{j!}\exp(-2ar)\left(1 - \frac{r}{2}\right)^j \exp(-Car^2).$$

Now we will define $g^+(a, r)$ and $g^-(a, r)$ by:
$$g^+(a, r) = \left(1 + \frac{Cr}{2}\right)^j \exp(ar^2)\exp\left(rC_\alpha N\right)$$
$$g^-(a, r) = \left(1 - \frac{r}{2}\right)^j \exp(-Car^2)\exp\left(-rC_\alpha N\right)\exp\left((2r - r^2)N\right)\left(1 - \frac{N}{a}\right)^N,$$

where C is the same constant as in the previous inequality and C_α is the same as in (2.4). And for every $r \le \frac{1}{4}, k \in [-1, 1]$ we have:
$$\frac{(-a\log(q))^j}{j!}\exp(\log(q)a)h(r(1 - k), r(1 + k)) \le \frac{(2ar)^j}{j!}\exp(-2ar)g^+(a, r)$$

and
$$\frac{(-a\log(q))^j}{j!}h(r(1 - k), r(1 + k))\exp(\log(q)a) \ge \frac{(2ar)^j}{j!}\exp(-2ar)g^-(a, r).$$

We clearly have that $g^+(a,r)$ is increasing in r while $g^-(a,r)$ is decreasing in r and $g^+(a,0) = 1$ and $g^-(a,0) = \left(1 - \frac{N}{a}\right)^N$.

So, for any $c > 0$, we have the following 2 inequalities:

$$\mathbb{P}_0(\ell^- \geq 2a \text{ and } 1 - q \leq c)$$

$$\leq \mathbb{P}_0(\tilde{\ell}^- \geq 2a \text{ and } 1 - q \leq c)$$

$$\leq \mathbb{P}_0(\tilde{\ell}^- \geq 2a \text{ and } r \leq c) \qquad \text{since } 1 - q \geq 2r - r^2 \geq r$$

$$= \int_{r=0}^{c} \int_{k=-1}^{1} D_p r^{\kappa-1} (1+k)^{p_x} (1-k)^{p_y} h(r(1+k), r(1-k)) \mathbb{P}_0(\tilde{\ell}^- \geq 2a | q) \mathrm{d}k \mathrm{d}r$$

$$\leq \int_{r=0}^{c} \int_{k=-1}^{1} D_p r^{\kappa-1} (1+k)^{p_x} (1-k)^{p_y} \sum_{j=0}^{N-1} \frac{(2ar)^j}{j!} \exp(-2ar) g^+(a,r) \mathrm{d}k \mathrm{d}r$$

$$\leq g^+(a,c) \int_{k=-1}^{1} (1+k)^{p_x} (1-k)^{p_y} \mathrm{d}k \int_{r=0}^{c} D_p r^{\kappa-1} \sum_{j=0}^{N-1} \frac{(2ar)^j}{j!} \exp(-2ar) \mathrm{d}r,$$

and

$$\mathbb{P}_0(\ell^- \geq 2a \text{ and } 1 - q \leq c)$$

$$\geq \mathbb{P}_0(\tilde{\ell}^- \geq 2a - 2N \text{ and } 1 - q \leq c)$$

$$\geq \mathbb{P}_0(\tilde{\ell}^- \geq 2a - 2N \text{ and } 2r \leq c) \qquad \text{since } 1 - q \leq 2r$$

$$= \int_{r=0}^{\frac{c}{2}} \int_{k=-1}^{1} D_p r^{\kappa-1} (1+k)^{p_x} (1-k)^{p_y} h(r(1+k), r(1-k)) \mathbb{P}_0(\tilde{\ell}^- \geq 2a - 2N | q) \mathrm{d}k \mathrm{d}r$$

$$\geq \int_{r=0}^{\frac{c}{2}} \int_{k=-1}^{1} D_p r^{\kappa-1} (1+k)^{p_x} (1-k)^{p_y} \sum_{j=0}^{N-1} \frac{(a2r)^j}{j!} \exp(-2ar) g^-(a,r) \mathrm{d}k \mathrm{d}r$$

$$\geq g^-\left(a, \frac{c}{2}\right) \int_{k=-1}^{1} (1+k)^{p_x} (1-k)^{p_y} \mathrm{d}k \int_{r=0}^{\frac{c}{2}} D_p r^{\kappa-1} \sum_{j=0}^{N-1} \frac{(a2r)^j}{j!} \exp(-2ar) \mathrm{d}r.$$

If we take $c = a^{-\frac{3}{4}}$ we clearly get that when $a \to \infty$, $g^-(a, a^{-\frac{3}{4}}) \to 1$ and $g^+(a, a^{-\frac{3}{4}}) \to 1$. Furthermore, for any constant c':

$$\int_{r=0}^{c'a^{-\frac{3}{4}}} D_p r^{\kappa-1} \sum_{j=0}^{N-1} \frac{(a2r)^j}{j!} \exp(-2ar) \mathrm{d}r$$

$$= (2a)^{-\kappa} \int_{r=0}^{2c'a^{\frac{1}{4}}} D_p C r^{\kappa-1} \sum_{j=0}^{N-1} \frac{r^j}{j!} \exp(-r) \mathrm{d}r$$

$$\sim (2a)^{-\kappa} D_p \sum_{j=0}^{N-1} \frac{\Gamma(j+\kappa)}{j!}$$

$$= (2a)^{-\kappa} D_p' \text{ for some constant } D_p' \text{ that only depends on the configuration } p.$$

Therefore we get:

$$\mathbb{P}_0(\ell^- \geq 2a \text{ and } 1 - q \leq a^{-\frac{3}{4}}) \sim D_p' \left(\int_{k=-1}^{1} (1+k)^{p_x} (1-k)^{p_y} \mathrm{d}k \right) (2a)^{-\kappa}.$$

So there exists a constant $D_{p,\alpha}$ that only depends on α and p such that:

$$\mathbb{P}_0(\ell^- \geq 2a \text{ and } 1-q \leq a^{-\frac{3}{4}}) \sim D_{p,\alpha}(2a)^{-\kappa}.$$

And thus:

$$\mathbb{P}_0(\ell^- \geq a) \sim D_{p,\alpha}a^{-\kappa}.$$

Now let ℓ be the total time spent in the trap. It is equal to ℓ^- plus the number of times the walk enters and exits the trap by the same vertex plus twice the number of times the walk enters and exits the trap by different vertices. This means there exists a constant δ_p that only depends on the configuration such that $\ell = \ell^- + \delta_p$. This, in turn, means that we have also the asymptotic equality:

$$\mathbb{P}_0(\ell \geq a) \sim D_{p,\alpha}a^{-\kappa}.$$

Now, let ℓ_i^p be the time spent in the ith trap with configuration p.

First, if $\kappa < 1$, by Theorem 3.7.2 of [11] we get that for some constant c_p:

$$n^{-\frac{1}{\kappa}}\sum_{i=1}^n \ell_i^p \xrightarrow[n\to\infty]{} c_p\mathcal{S}_1^\kappa \text{ in law for } \mathbb{P}_0.$$

Now we use the fact that the number of traps of configuration p between two renewal times has a finite expectation C_p to show that we have the convergence we want. Let $M_{n,p}$ be the number of traps of configuration p the walk has entered before the nth renewal time. For any $\varepsilon > 0$ and any p we have:

$$\mathbb{P}_0(M_{n,p} \in [(C_p - \varepsilon)n, (C_p + \varepsilon)n]) \xrightarrow[n\to\infty]{} 1.$$

Therefore for any configuration p:

$$n^{-\frac{1}{\kappa}}\sum_{i=(C_p-\varepsilon)n}^{(C_p+\varepsilon)n} \ell_i^p \xrightarrow[n\to\infty]{} (2\varepsilon)^{\frac{1}{\kappa}}c_pS_\kappa \text{ in law for } \mathbb{P}_0.$$

And for any $m \in \mathbb{N}$:

$$n^{-\frac{1}{\kappa}}\sum_{p\in I^m}\sum_{i=(C_p-\varepsilon)n}^{(C_p+\varepsilon)n} \ell_i^p \xrightarrow[n\to\infty]{} (2\varepsilon)^{\frac{1}{\kappa}}\left(\sum_{p\in I^m}(c_p)^\kappa\right)^{\frac{1}{\kappa}}S_\kappa \text{ in law for } \mathbb{P}_0.$$

We write $I^m(J)$ all the configuration of I^m that are in a direction $j \in J$. Now, using the fact that the ℓ_p^i are non negative, for any $n \in \mathbb{N}$ and any $\varepsilon > 0$ small enough, we have:

$$\mathbb{P}_0\left(n^{-\frac{1}{\kappa}}\left|\sum_{p\in I^m(J)}\sum_{i=1}^{M_{n,p}}\ell_i^p - \sum_p\sum_{i=1}^{C_pn}\ell_p^i\right| \geq \eta\right)$$

$$\leq \mathbb{P}_0(\exists p \in I^m(J), N_{n,p} \notin [(C_p-\varepsilon)n, (C_p+\varepsilon)n]) + \mathbb{P}\left(n^{-\frac{1}{\kappa}}\sum_{p\in I^m(J)}\sum_{i=(C_p-\varepsilon)n}^{(C_p+\varepsilon)n}\ell_i^p \geq \eta\right)$$

$$= o(1) + \mathbb{P}_0\left((2\varepsilon)^{\frac{1}{\kappa}}\left(\sum_{p\in I^m(J)}(c_p)^\kappa\right)^{\frac{1}{\kappa}}\mathcal{S}_1^\kappa \geq \eta\right).$$

Since it is true for all ε, we get that

$$n^{-\frac{1}{\kappa}}\left|\sum_{p\in I^m(J)}\sum_{i=1}^{M_{n,p}}\ell_i^p - \sum_{p\in I^m(J)}\sum_{i=1}^{C_pn}\ell_p^i\right| \xrightarrow[n\to\infty]{} 0 \text{ in probability for } \mathbb{P}_0.$$

And since

$$n^{-\frac{1}{\kappa}} \sum_{p \in I^m(J)} \sum_{i=1}^{C_p n} \ell_i^p \xrightarrow[n \to \infty]{} \left(\sum_{p \in I^m(J)} (c_p)^\kappa \right)^{\frac{1}{\kappa}} \mathcal{S}_1^\kappa \text{ in probability for } \mathbb{P}_0,$$

we get:

$$n^{-\frac{1}{\kappa}} \sum_{p \in I^m(J)} \sum_{i=1}^{M_{n,p}} \ell_i^p \xrightarrow[n \to \infty]{} \left(\sum_{p \in I^m(J)} (c_p)^\kappa \right)^{\frac{1}{\kappa}} \mathcal{S}_1^\kappa \text{ in law for } \mathbb{P}_0$$

Now if $\kappa = 1$, we first want to show that we can neglect the values larger than $n \log(n)$. Let p be a configuration, ℓ_i^p the total time spent in the ith trap in the configuration p encountered, C_p the constant such that the number of traps encountered before time $\tau_{n+1} - 1$ is equivalent to $C_p n$, $M_{n,p}$ the number of traps in the configuration p encountered before the time $\tau_{n+1} - 1$ and c_p the constant such that $\mathbb{P}_0(\ell_i^p \geq n) \sim c_p n^{-1}$. We get:

$$\mathbb{P}_0(\exists i \leq M_{n,p}, \ell_i^p \geq n \log(n)) \leq \mathbb{P}_0(\exists i \leq 2C_p n, \ell_i^p \geq n \log(n)) + \mathbb{P}_0(M_{n,p} \geq 2C_p n)$$
$$\leq 2C_p n \frac{c_p}{n \log(n)} + o(1).$$
$$= o(1)$$

Now we can compute the expectation and variance of $\ell_i^p \wedge n \log(n)$:

$$\mathbb{E}_{\mathbb{P}_0}(\ell_i^p \wedge n \log(n)) \sim \int_{t=1}^{n \log(n)} \frac{c_p}{t} \mathrm{d}t$$
$$\sim c_p \log(n).$$

Now for the variance we get:

$$\mathrm{Var}_{\mathbb{P}_0}(\ell_i^p \wedge n \log(n)) \leq \mathbb{E}_{\mathbb{P}_0}((\ell_i^p \wedge n \log(n))^2)$$
$$\sim \int_{t=1}^{n \log(n)} 2t \frac{c_p}{t} \mathrm{d}t$$
$$\sim 2c_p n \log(n).$$

So for n large enough:
$$\mathrm{Var}_{\mathbb{P}_0}(\ell_i^p \wedge n \log(n)) \leq 4c_p n \log(n).$$

First, for any constant c, for n big enough:

$$\mathbb{P}_0 \left(\left| \sum_{i=1}^{cn} \ell_i^p \wedge n \log(n) - cn c_p \log(n) \right| \geq \varepsilon n \log(n) \right)$$
$$\leq \mathbb{P}_0 \left(\left| \sum_{i=1}^{cn} \ell_i^p \wedge n \log(n) - cn \mathbb{E}(\ell_1^p \wedge n \log(n)) \right| \geq \frac{1}{2} \varepsilon n \log(n) \right) \text{ for } n \text{ big enough, by (2.4)}$$
$$\leq cn \frac{4 \mathrm{Var}_{\mathbb{P}_0}(\ell_1^p \wedge n \log(n))}{(\varepsilon n \log(n))^2}$$
$$\leq cn \frac{16 c_p n \log(n)}{(\varepsilon n \log(n))^2}$$
$$= \frac{16 c c_p}{\log(n)} = o(1)$$

This means that we have the following results:

$$\mathbb{P}_0\left(\sum_{i=1}^{(C_p+\varepsilon)n} \ell_i^p \wedge n\log(n) - (C_p+\varepsilon)nc_p\log(n) \geq \varepsilon n\log(n)\right) \xrightarrow[n\to\infty]{} 0$$

and

$$\mathbb{P}_0\left(\sum_{i=1}^{(C_p-\varepsilon)n} \ell_i^p \wedge n\log(n) - (C_p+\varepsilon)nc_p\log(n) \leq -\varepsilon n\log(n)\right) \xrightarrow[n\to\infty]{} 0$$

Then, by definition of C_p we get, for any $\varepsilon \geq 0$:

$$\mathbb{P}_0\left(|M(n,p) - C_p n| \geq \varepsilon n\right) \xrightarrow[n\to\infty]{} 0.$$

Then, using the fact that $\sum_{i=1}^n \ell_i^p \wedge a$ is increasing in n for any a, we get:

$$\mathbb{P}_0\left(\sum_{i=1}^{M(n,p)} \ell_i^p \geq (C_p+\varepsilon)(c_p+\varepsilon)n\log(n)\right)$$

$$\leq \mathbb{P}_0(M(n,p) \geq (C_p+\varepsilon)n) + \mathbb{P}_0\left(\sum_{i=1}^{(C_p+\varepsilon)n} \ell_i^p \geq (C_p+\varepsilon)(c_p+\varepsilon)n\log(n)\right)$$

$$= o(1).$$

Similarly, we have:

$$\mathbb{P}_0\left(\sum_{i=1}^{M(n,p)} \ell_i^p \leq (C_p-\varepsilon)(c_p-\varepsilon)n\log(n)\right)$$

$$\leq \mathbb{P}_0(M(n,p) \geq (C_p-\varepsilon)n) + \mathbb{P}_0\left(\sum_{i=1}^{(C_p-\varepsilon)n} \ell_i^p \geq (C_p-\varepsilon)(c_p-\varepsilon)n\log(n)\right)$$

$$= o(1).$$

Therefore,

$$\frac{1}{n\log(n)} \sum_{i=1}^{M(n,p)} \ell_i^p \xrightarrow[n\to\infty]{} C_p c_p \text{ in probability for } \mathbb{P}_0.$$

Now we just have to sum on all configurations $p \in I^m$ that are in a direction $j \in J$ to get the result we want. $\qquad\square$

2.5 Only the time spent in traps matter

Now to properly show the result we want, we have to show that some quantities and some events are negligible, this is what this section is devoted to.

Lemma 2.24. *Let j be in $[\![1,d]\!]$. Let $\{x_i^j, y_i^j\}$ be the ith trap visited by the walk in the direction j after time τ_2, s_i^j its strength, ℓ_i^j the time spent in this trap and N_i^j the number of times the walk enters the trap:*

$$\ell_i^j = \sum_{k\geq 0} 1_{Y_k \in \{x_i^j, y_i^j\}},$$

$$N_i^j = \sum_{k\geq 0} 1_{Y_k \in \{x_i, y_i\} \text{ and } Y_{k+1} \notin \{x_i^j, y_i^j\}}.$$

Let $\kappa_j = 2 \sum_{i=1}^{2d} \alpha_i - \alpha_j - \alpha_{j+d} \geq \kappa$. Let $M(n,j)$ be the number of traps in the direction j encountered between times τ_2 and $\tau_n - 1$.

If $\kappa < 1$ and $\kappa_j = \kappa$, for any $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that for n large enough:

$$\mathbb{P}_0 \left(\sum_{i=1}^{M(n,j)} \ell_i^j 1_{s_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \varepsilon n^{\frac{1}{\kappa}} \right) \leq \varepsilon.$$

Proof. Let $\gamma \in \left(\kappa, \frac{\kappa + \kappa'}{2} \right)$ be such that $\gamma \leq 1$. Let β be a positive real. Let $\{\overline{x}_i^j, \overline{y}_i^j\}$ be the ith trap visited by the walk in the direction j after time τ_2 such that $\{\overline{x}_i^j \cdot e_1, \overline{y}_i^j \cdot e_1 \geq Y_{\tau_2} \cdot e_1$. Let $\overline{s}_i^j$ be its strength $\overline{\ell}_i^j$ the time spent in this trap and $\overline{N}_i^j$ the number of times the trap is visited.

By Lemma 2.2 the number of traps encountered between 2 renewal times has a finite expectation and since the $(M(2i+1, j) - M(2i, j))_{i \in \mathbb{N}^*}$ are iid and so are the $(M(2i+2, j) - M(2i+1, j))_{i \in \mathbb{N}^*}$, there exists a constant C_j such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} M(n,j) \xrightarrow[n \to \infty]{} C_j.$$

So for any $\varepsilon > 0$, for n large enough:

$$\mathbb{P}_0(M(n,j) \geq 2C_j n) \leq \frac{\varepsilon}{4}.$$

We have for n large enough:

$$\mathbb{P}_0 \left(\sum_{i=1}^{M(n,j)} \ell_i^j 1_{s_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \varepsilon n^{\frac{1}{\kappa}} \right)$$

$$\leq \mathbb{P}_0 \left(\sum_{i=1}^{M(n,j)} \overline{\ell}_i^j 1_{\overline{s}_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \frac{1}{2} \varepsilon n^{\frac{1}{\kappa}} \right) + \mathbb{P}_0 \left(\sum_{i=1}^{M(3,j)} \ell_i^j 1_{s_i \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \frac{1}{2} \varepsilon n^{\frac{1}{\kappa}} \right)$$

$$\leq \mathbb{P}_0 \left(\sum_{i=1}^{M(n,j)} \overline{\ell}_i^j 1_{\overline{s}_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \frac{1}{2} \varepsilon n^{\frac{1}{\kappa}} \right) + \mathbb{P}_0 \left(\tau_3 \geq \frac{1}{2} \varepsilon n^{\frac{1}{\kappa}} \right)$$

$$\leq \mathbb{P}_0 \left(\sum_{i=1}^{M(n,j)} \overline{\ell}_i^j 1_{\overline{s}_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \frac{1}{2} \varepsilon n^{\frac{1}{\kappa}} \right) + \frac{\varepsilon}{4} \text{ for } n \text{ large enough}$$

$$\leq \mathbb{P}_0 \left(\sum_{i=1}^{2C_j n} \overline{\ell}_i^j 1_{\overline{s}_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \frac{1}{2} \varepsilon n^{\frac{1}{\kappa}} \right) + 2\frac{\varepsilon}{4} \text{ for } n \text{ large enough }.$$

Then by Lemma 2.21 we have:

$$\mathbb{P}_0 \left(\sum_{i=1}^{2C_j n} \overline{\ell}_i^j 1_{\overline{s}_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \frac{1}{2} \varepsilon n^{\frac{1}{\kappa}} \right) \leq \frac{\varepsilon}{4} + \mathbb{P}_0 \left(\sum_{i=1}^{2C_j n} \overline{N}_i^j \overline{s}_i^j 1_{\overline{s}_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}} \right).$$

And finally we have:

$$\mathbb{P}_0 \left(\sum_{i=1}^{2C_j n} \overline{N}_i^j \overline{s}_i^j 1_{\overline{s}_i^j \leq \varepsilon' n^{\frac{1}{\kappa}}} \geq \frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}} \right) \leq \mathbb{P}_0 \left(\sum_{i=1}^{2C_j n} (\overline{N}_i^j)^\gamma (\overline{s}_i^j)^\gamma 1_{\overline{s}_i^j \leq \beta n^{\frac{1}{\kappa}}} \geq \left(\frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}} \right)^\gamma \right)$$

$$\leq \left(\frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}} \right)^{-\gamma} \mathbb{E}_{\mathbb{P}_0} \left(\sum_{i=1}^{2C_j n} (\overline{N}_i^j)^\gamma (\overline{s}_i^j)^\gamma 1_{\overline{s}_i^j \leq \beta n^{\frac{1}{\kappa}}} \right)$$

$$= \left(\frac{\varepsilon^2}{40}n^{\frac{1}{\kappa}}\right)^{-\gamma} \sum_{i=1}^{2C_j n} \mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma(\overline{s}_i^j)^\gamma 1_{\overline{s}_i^j \leq \beta n^{\frac{1}{\kappa}}}\right).$$

Then by Lemma 2.20 we get, for some constant c that does not depend on β:

$$\left(\frac{\varepsilon^2}{40}n^{\frac{1}{\kappa}}\right)^{-\gamma} \sum_{i=1}^{2C_j n} \mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma(\overline{s}_i^j)^\gamma 1_{\overline{s}_i \leq \beta n^{\frac{1}{\kappa}}}\right) \leq c \left(\frac{\varepsilon^2}{40}n^{\frac{1}{\kappa}}\right)^{-\gamma} \sum_{i=1}^{2C_j n} \mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma\right)\left(\varepsilon' n^{\frac{1}{\kappa}}\right)^{\gamma-\kappa}$$

$$= c \left(\frac{\varepsilon^2}{40}\right)^{-\gamma}(\varepsilon')^{\gamma-\kappa} n^{-1} \sum_{i=1}^{2C_j n} \mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma\right).$$

And by Lemma 2.19 there exists a constant c that does not depend on β such that:

$$\left(\frac{\varepsilon^2}{40}\right)^{-\gamma}(\varepsilon')^{\gamma-\kappa} n^{-1} \sum_{i=1}^{2C_j n} \mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma\right) \leq c\left(\frac{\varepsilon^2}{40}\right)^{-\gamma}(\varepsilon')^{\gamma-\kappa}.$$

So by taking β small enough we get the result we wanted. $\square$

Lemma 2.25. *Let* $J = \{j \in [\![1,d]\!], \kappa_j > \kappa\}$.
If $\kappa = 1$ *there exists a constant C such that $\mathbb{P}_0$ almost surely:*

$$\frac{1}{n}\sum_{i=0}^{\tau_n-1} 1_{Y_i \in \tilde{\mathcal{T}}_J} \xrightarrow[n\to\infty]{} C.$$

If $\kappa < 1$ *there exists a constant $C > 0$ and a constant $\gamma \in (\kappa, 1]$ such that $\mathbb{P}_0$ almost surely, for n large enough:*

$$n^{-\frac{1}{\gamma}}\sum_{k=0}^{\tau_n-1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \leq C.$$

Proof. For any $j \in J$ we define $\kappa_j = 2\sum_{i=1}^{2d}\alpha_i - \alpha_j - \alpha_{j+d} > \kappa$. Let $\{x_i^j, y_i^j\}$ be the ith trap in the direction j the walk enters after time τ_2 and such that $x_i^j \cdot e_1, y_i^j \cdot e_1 \geq Y_{\tau_2} \cdot e_1$. Let N_i^j be the number of times the walk exits $\{x_i^j, y_i^j\}$ and ℓ_i^j the time the walk spends in this trap. Let $M(i,j)$ be the number of traps in the direction j entered before time τ_i. The $(M(2i+2,j)-M(2i+1,j))_{i\in\mathbb{N}^*}$ are iid and so are the $(M(2i+1,j)-M(2i,j))_{i\in\mathbb{N}^*}$, they also all have the same law (the only issue is that since a trap span over two vertices, there might be a slight overlap between traps of two different 'renewal slabs'). Now, since the number of different vertices the walk encounters between two renewal times has a finite expectation, the $(M(i+1,j)-M(i,j))$ have a finite expectation and therefore there exists a constant C_j such that $\mathbb{P}_0$ almost surely:

$$M(n,j) - C_j n \xrightarrow[n\to\infty]{} -\infty.$$

Now let $\tilde{Y}$ be the partially forgotten walk associated with Y. We get that knowing the environment, the partially forgotten walk and the renewal position Y_{τ_2} the time spend in the $\{x_i^j, y_i^j\}$, the kth time the walk enters this trap is equal to $\varepsilon_{i,k}^j + 2H_{i,k}^j$ where $\varepsilon_{i,k}^j$ is 1 if the walk enters the trap by the same vertex it leaves it and 2 otherwise and $H_{i,k}^j$ is a geometric random variable that counts the number of back and forths. The parameter of $H_{i,k}^j$ is $1-p_i^j$ with $p_i^j := \omega(x_i^j, y_i^j)\omega(y_i^j, x_i^j)$.

First, lets look at the case $\kappa = 1$. Since the $\left(\sum_{j=\tau_{2i}}^{\tau_{2i+1}-1} 1_{Y_i \in \tilde{\mathcal{T}}_J}\right)_{i\in\mathbb{N}^*}$ are iid and so are the $\left(\sum_{j=\tau_{2i+1}}^{\tau_{2i+2}-1} 1_{Y_i \in \tilde{\mathcal{T}}_J}\right)_{i\in\mathbb{N}^*}$, we just have to prove that their expectation is not infinite to

have the result we want. If their expectation were infinite, then we would have that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{M(n,j)}\ell_i^j\xrightarrow[n\to\infty]{}\infty.$$

Therefore we would have $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\ell_i^j\xrightarrow[n\to\infty]{}\infty.$$

But

$$\begin{aligned}
\mathbb{E}_{\mathrm{P}_0^\omega}\left(\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\ell_i^j|\tilde{Y}\right)&=\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\sum_{k=1}^{N_i^j}\mathbb{E}_{\mathrm{P}_0^\omega}\left(\varepsilon_{i,k}^j+2H_{i,k}^j|\tilde{Y}\right)\\
&=\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\sum_{k=1}^{N_i^j}\left(\varepsilon_{i,k}^j+2\frac{p_i^j}{1-p_i^j}\right)\\
&\leq 2\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\sum_{k=1}^{N_i^j}\frac{1}{1-p_i^j}\\
&\leq C\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{nC_j}N_i^j s_i^j,
\end{aligned}$$

where s_i^j is the strength of the trap $\{x_i^j,y_i^j\}$. Now we get:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_0}\left(\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\ell_i^j\right)&\leq\frac{1}{n}\mathbb{E}_{\mathbb{P}_0}\left(C\sum_{j\in J}\sum_{i=1}^{nC_j}N_i^j s_i^j\right)\\
&=C\frac{1}{n}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{j\in J}\sum_{i=1}^{nC_j}N_i^j\int_{t=0}^{\infty}1_{(s_i^j)\geq t}\mathrm{d}t\right)\\
&\leq C\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{nC_j}\mathbb{E}_{\mathbb{P}_0}\left(N_i^j\left(2+\int_{t=2}^{\infty}1_{s_i^j\geq t}\mathrm{d}t\right)\right)\\
&\leq C\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{nC_j}\left(2\mathbb{E}_{\mathbb{P}_0}(N_i^j)+C\int_{t=2}^{\infty}\mathbb{E}_{\mathbb{P}_0}(N_i^j 1_{s_i^j\geq t})\mathrm{d}t\right).
\end{aligned}$$

Now by Lemma 2.20 we know that there exists a constant C such that for any $t\geq 2$:

$$\mathbb{E}_{\mathbb{P}_0}(N_i^j 1_{s_i^j\geq t})\leq Ct^{-\kappa_j}\mathbb{E}_{\mathbb{P}_0}(N_i^j).$$

So there exists a constant C' (the value of this constant will change depending on the line) such that:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_0}\left(\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\ell_i^j\right)&\leq C'\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{nC_j}\mathbb{E}_{\mathbb{P}_0}(N_i^j)\\
&\leq C'\sum_{j\in J}C_j\quad\text{by Lemma 2.19}\\
&\leq C'.
\end{aligned}$$

This means that we cannot have $\frac{1}{n} \sum_{j \in J} \sum_{i=1}^{C_j n} N_i^j \xrightarrow[n \to \infty]{} \infty$ $\mathbb{P}_0$ almost surely. Therefore the random variables $\left(\sum_{j=\tau_{2i}}^{\tau_{2i+1}-1} 1_{Y_i \in \tilde{\mathcal{T}}_J} \right)_{i \in \mathbb{N}^*}$ have finite expectation and so have the random variables $\left(\sum_{j=\tau_{2i+1}}^{\tau_{2i+2}-1} 1_{Y_i \in \tilde{\mathcal{T}}_J} \right)_{i \in \mathbb{N}^*}$. So we have the result we want.

If $\kappa < 1$, we will basically use the same method. First there exists $\gamma \in (\kappa, 1]$ such that $\gamma < \frac{\kappa + \kappa'}{2}$ and for every $j \in J$, $\gamma < \kappa_j$.

We have that:

$$\limsup n^{-\frac{1}{\gamma}} \sum_{k=0}^{\tau_n - 1} 1_{Y_k \in \tilde{\mathcal{T}}_J} = \limsup n^{-\frac{1}{\gamma}} \sum_{i=2}^{n-1} \sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k \in \tilde{\mathcal{T}}_J}.$$

And since:

$$\left(n^{-\frac{1}{\gamma}} \sum_{i=2}^{n} \sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \right)^{\gamma} \le \frac{1}{n} \sum_{j \in J} \sum_{i=1}^{n} \left(\sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \right)^{\gamma}$$

we also have:

$$\limsup n^{-\frac{1}{\gamma}} \sum_{i=2}^{n} \sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \le \left(\limsup \frac{1}{n} \sum_{i=2}^{n} \left(\sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \right)^{\gamma} \right)^{\frac{1}{\gamma}}.$$

Now, since the random variables $\left(\left(\sum_{k=\tau_{2i}}^{\tau_{2i}-1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \right)^{\gamma} \right)_{i \in \mathbb{N}^*}$ are iid and so are the random variables $\left(\left(\sum_{k=\tau_{2i+1}}^{\tau_{2i+1}-1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \right)^{\gamma} \right)_{i \in \mathbb{N}^*}$ we have that there exists a constant $C_\infty \in [0, \infty]$ such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} \sum_{i=2}^{n} \left(\sum_{k=\tau_i}^{\tau_i - 1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \right)^{\gamma} \xrightarrow[n \to \infty]{} C_\infty.$$

Now, by definition of the C_j and since $(a+b)^\gamma \le a^\gamma + b^\gamma$ we have that if $C_\infty = \infty$ then $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} \sum_{j \in J} \sum_{i=1}^{C_j n} \left(N_i^j \right)^\gamma \xrightarrow[n \to \infty]{} \infty.$$

However we have (using the same techniques and notations as in the case $\kappa = 1$):

$$\begin{aligned}
\mathbb{E}_{\mathrm{P}_0^\omega} \left(\frac{1}{n} \sum_{j \in J} \sum_{i=1}^{C_j n} (\ell_i^j)^\gamma | \tilde{Y} \right) &= \frac{1}{n} \sum_{j \in J} \sum_{i=1}^{C_j n} \mathbb{E}_{\mathrm{P}_0^\omega} \left(\left(\sum_{k=1}^{N_i^j} \varepsilon_i^j + 2 H_{i,k}^j \right)^\gamma | \tilde{Y} \right) \\
&\le \frac{1}{n} \sum_{j \in J} \sum_{i=1}^{C_j n} \mathbb{E}_{\mathrm{P}_0^\omega} \left(\sum_{k=1}^{N_i^j} \varepsilon_i^j + 2 H_{i,k}^j | \tilde{Y} \right)^\gamma \\
&\le \frac{1}{n} \sum_{j \in J} \sum_{i=1}^{C_j n} \left(N_i^j \frac{2}{p_i^j} \right)^\gamma \\
&\le C \frac{1}{n} \sum_{j \in J} \sum_{i=1}^{C_j n} \left(N_i^j s_i^j \right)^\gamma.
\end{aligned}$$

Now by the same method as the one for $\kappa = 1$, by using Lemma 2.20 and Lemma 2.19 we get:

$$\mathbb{E}_{\mathbb{P}_0} \left(\frac{1}{n} \sum_{j \in J} \sum_{i=1}^{C_j n} (\ell_i^j)^\gamma \right) \le C.$$

This means that $C_\infty < \infty$ and therefore:

$$\limsup n^{-\frac{1}{\gamma}} \sum_{k=0}^{\tau_n - 1} 1_{Y_k \in \tilde{\mathcal{T}}_J} \leq (C_\infty)^{\frac{1}{\gamma}} < \infty. \qquad \square$$

Lemma 2.26. *Let $A_{\varepsilon,n}^{i_1,i_2}(i)$ be the event that the walk visits at least two trap of strength at least $\varepsilon n^{\frac{1}{\kappa}}$ between times τ_i and $\tau_{i+i_1} - 1$ and that it enters these traps at most i_2 times. We have that for any $i_1, i_2 \geq 1$:*

$$\mathbb{P}_0 \left(\bigcup_{2 \leq i \leq n} A_{\varepsilon,n}^{i_1,i_2}(i) \right) \xrightarrow[n \to \infty]{} 0.$$

Proof. Let $\overline{\alpha} := \sum_{i=1}^{2d} \alpha_i$. Let $M(i)$ be the number of traps visited before time τ_i. We know by Lemma 2.2 that the number $M(i + i_1) - M(i)$ of traps visited between times τ_i and $\tau_{i+i_1} - 1$ has a finite expectation (for $\mathbb{P}_0$) and by Proposition 1.2 the $((M(2i+2) - M(2i+1))_{i \geq 1}$ are iid and so are the $(M(2i+1) - M(2i))_{i \geq 1}$. This means that there is a positive constant C such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} M(n) \xrightarrow[n \to \infty]{} C.$$

Now let $M^{i_2}(i)$ be the number of traps visited at most i_2 times before time τ_i. We know that:

$$\mathbb{P}_0(M^{i_2}(n + i_1) \geq 2Cn) \xrightarrow[n \to \infty]{} 0.$$

Now, for any $\eta > 0$ we have:

$$\mathbb{P}_0 \left(\exists i \leq n, M(i + i_1) - M(i) \geq \eta n \right)$$
$$\leq \sum_{i \leq n} \mathbb{P}_0 \left(M(i + i_1) - M(i) \geq \eta n \right)$$
$$= o(1) + \sum_{2 \leq i \leq n} \mathbb{P}_0 \left(M(i + i_1) - M(i) \geq \eta n \right)$$
$$= o(1) + (n - 1) \mathbb{P}_0 \left(M(2 + i_1) - M(2) \geq \eta n \right)$$
$$= o(1) \text{ since } M(2 + i_1) - M(2) \text{ has a finite expectation.}$$

Now let A_i be the event "the ith trap visited by the walk is of strength at least $\varepsilon n^{\frac{1}{\kappa}}$ and that the walk enters this trap at most i_2 times". We have:

$$\mathbb{P}_0 \left(\exists i \leq 2Cn, \exists j \leq \eta n, A_i \text{ and } A_{i+j} \right)$$
$$\leq \mathbb{P}_0 \left(\exists i \leq \frac{2C}{\eta}, \exists j_1, j_2 \in [\![i\eta n, i\eta n + 2\eta n]\!], j_1 \neq j_2 \text{ and } A_{j_1} \text{ and } A_{j_2} \right)$$
$$\leq \sum_{i=0}^{\frac{2C}{\eta}} \mathbb{P}_0 \left(\exists j_1, j_2 \in [\![i\eta n, i\eta n + 2\eta n]\!], j_1 \neq j_2 \text{ and } A_{j_1} \text{ and } A_{j_2} \right)$$
$$\leq \sum_{i=0}^{\frac{2C}{\eta}} \sum_{j_1 = i\eta n}^{i\eta n + 2\eta n} \sum_{j_2 = i\eta n}^{i\eta n + 2\eta n} \mathbb{P}_0 \left(A_{j_1} \text{ and } A_{j_2} \right) 1_{j_1 \neq j_2}.$$

Now let $(\tilde{Y}_n)_{n \in \mathbb{N}}$ be the partially forgotten walk, by Lemma 2.17 if s_j is the strength of the jth trap visited and N_j is the number of times the walk enters the jth trap, there exists a constant D_j that only depends on its configuration such that for any $B > 2$,

$$\mathbb{P}_0 \left(s_j \geq B | \tilde{Y}, \tilde{\omega} \right) \leq D_j B^{-\kappa} \exp \left(\frac{5(N_i + 2\overline{\alpha})}{B} \right).$$

Let D^{i_2} be the maximum value of $D_j \exp\left(\frac{5(Z_i + 2\overline{\alpha})}{2}\right)$ we can get for configuration of traps entered at most i_2 times. We get that for any j:

$$\mathbb{P}_0(s_j \geq B \text{ and } N_j \leq i_2 | \tilde{Y}, \tilde{\omega}) \leq D^{i_2} B^{-\kappa}.$$

We also know that the strengths of the traps are independent, knowing the partially forgotten walk and the equivalence class of the environment for the trap-equivalent relation. Therefore we have, for any $\eta > 0$:

$$\sum_{i=0}^{\frac{2C}{\eta}} \sum_{j_1 = i\eta n}^{i\eta n + 2\eta n} \sum_{j_2 = i\eta n}^{i\eta n + 2\eta n} \mathbb{P}_0\left(A_{j_1} \text{ and } A_{j_2}\right) 1_{j_1 \neq j_2}$$

$$\leq \sum_{i=0}^{\frac{2C}{\eta}} \sum_{j_1 = i\eta n}^{i\eta n + 2\eta n} \sum_{j_2 = i\eta n}^{i\eta n + 2\eta n} (D^{i_2})^2 \left(\varepsilon n^{\frac{1}{\kappa}}\right)^{-2\kappa}$$

$$\leq 2 \frac{2C}{\eta} (\eta n)^2 (D^{i_2})^2 \varepsilon^{-2\kappa} n^{-2} \text{ for } \eta \text{ small enough}$$

$$= 4C\eta (D^{i_2})^2 \varepsilon^{-2\kappa}.$$

Now, by taking a sequence $(\eta_n)_{n \in \mathbb{N}^*}$ of positive reals such that $\eta_n \xrightarrow[n \to \infty]{} 0$ and such that:

$$\mathbb{P}_0\left(\exists i \leq n, M(i + i_1) - M(i) \geq \eta_n n\right) \xrightarrow[n \to \infty]{} 0,$$

we get:

$$\mathbb{P}_0\left(\bigcup_{2 \leq i \leq n} A_{\varepsilon,n}^{i_1,i_2}(i)\right) \leq \mathbb{P}_0\left(M(n + i_1) \leq 2Cn\right) \text{ or } \left(\exists i \leq n, M(i + i_1) - M(i) \geq \eta_n n\right)$$

$$+ \mathbb{P}_0\left(\exists i \leq 2Cn, \exists j \leq \eta_n n, A_i + \mathbb{P}_0 A_{i+j}\right).$$

Therefore:

$$\mathbb{P}_0\left(\bigcup_{2 \leq i \leq n} A_{\varepsilon,n}^{i_1,i_2}(i)\right) \xrightarrow[n \to \infty]{} 0. \qquad \square$$

Lemma 2.27. *If $\kappa = 1$ there exists a constant C such that $\mathbb{P}_0$ almost surely:*

$$\frac{1}{n} \sum_{i=0}^{\tau_n - 1} 1_{Y_i \notin \tilde{\mathcal{T}}} \xrightarrow[n \to \infty]{} C.$$

If $\kappa < 1$, there exists a constant $C > 0$ and a constant $\beta < \frac{1}{\kappa}$ such that $\mathbb{P}_0$ almost surely, for n large enough:

$$\sum_{x \in \mathbb{Z}^d} \sum_{i=0}^{\tau_n - 1} 1_{Y_i = x} 1_{x \notin \tilde{\mathcal{T}}} \leq Cn^\beta.$$

Proof. Let m be such that $\mathbb{Q}^m$ is well defined. Let $(t_i^m)_{i \in \mathbb{N}}$ be the times at which X^m changes position, with $t_0 := 0$. We have $X_{t_i^m}^m = Y_i$ for all $i \in \mathbb{N}$. Let $(\mathcal{E}_i)_{i \in \mathbb{N}}$ be a sequence of random variables defined by $\mathcal{E}_i = (t_{i+1}^m - t_i^m)\gamma_\omega^m(Y_i)$. By definition of X and Y, $(\mathcal{E}_i)_{i \in \mathbb{N}}$ is a sequence of iid exponential random variables of parameter 1, independent of the walk and the environment.

We will first look at the case $\kappa = 1$.

If $\sum_{i=\tau_2}^{\tau_3 - 1} 1_{Y_i \notin \tilde{\mathcal{T}}}$ has a finite expectation for $\mathbb{P}_0$, since the $\left(\sum_{i=\tau_{2i}}^{\tau_{2i+1}-1} 1_{Y_i \notin \tilde{\mathcal{T}}}\right)_{i \in \mathbb{N}^*}$ are iid and so are the $\left(\sum_{i=\tau_{2i+1}}^{\tau_{2i+2}-1} 1_{Y_i \notin \tilde{\mathcal{T}}}\right)_{i \in \mathbb{N}^*}$ then we have the result we want. On the

other hand, if $\sum_{i=\tau_1}^{\tau_2-1} 1_{Y_i \notin \tilde{\mathcal{T}}}$ has an infinite expectation then, since the random variables $\left(\sum_{i=\tau_i}^{\tau_{i+1}-1} 1_{Y_i \notin \tilde{\mathcal{T}}}\right)_{i \geq 2}$ are non negative, $n^{-1} \sum_{i=\tau_1}^{\tau_n-1} 1_{Y_i \notin \tilde{\mathcal{T}}} \xrightarrow[n \to \infty]{} \infty$ $\mathbb{P}_0$ almost surely.

By the law of large numbers, we get that $\mathbb{P}_0$ almost surely:

$$\exists k \in \mathbb{N}, \forall n \geq k, \sum_{i=0}^{\tau_n-1} \mathcal{E}_i 1_{Y_i \notin \tilde{\mathcal{T}}} \geq \frac{1}{2} \sum_{i=0}^{\tau_n-1} 1_{Y_i \notin \tilde{\mathcal{T}}}.$$

For any point x, if x is not in a trap then, by definition of traps:

$$\frac{1}{\gamma^\omega(x)} \geq \frac{1}{2}.$$

This yields:

$$\sum_{i=0}^{\tau_n-1} \mathcal{E}_i 1_{Y_i \notin \tilde{\mathcal{T}}} \leq 2 \sum_{i=0}^{\tau_n-1} \mathcal{E}_i 1_{Y_i \notin \tilde{\mathcal{T}}} \frac{1}{\gamma^\omega(Y_i)}.$$

And by writing $T_n^m = t_{\tau_n}^m$ we have:

$$\sum_{i=0}^{\tau_n-1} \mathcal{E}_i 1_{Y_i \notin \tilde{\mathcal{T}}} \leq 2 \int_0^{T_n^m} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} \mathrm{d}t.$$

We know by Lemma 2.1 that there exists a constant d_m such that $\mathbb{P}_0$ almost surely:

$$T_n^m - d_m n \xrightarrow[n \to \infty]{} -\infty.$$

We get:

$$\exists k \in \mathbb{N}, \forall n \geq k, \int_0^{T_n^m} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} \mathrm{d}t \leq \int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} \mathrm{d}t.$$

Finally, if $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} \sum_{i=0}^{\tau_n} 1_{Y_i \notin \tilde{\mathcal{T}}} \xrightarrow[n \to \infty]{} \infty.$$

Then $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} \int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} \mathrm{d}t \xrightarrow[n \to \infty]{} \infty.$$

And therefore, since $\mathbb{Q}_0^m$ is absolutely continuous with respect to $\mathbb{P}_0$ we get the same convergence $\mathbb{Q}_0^m$ almost surely. So we would have, since $\frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)}$ is positive:

$$\frac{1}{n} \mathbb{E}_{\mathbb{Q}_0^m} \left(\int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} \mathrm{d}t \right) \xrightarrow[n \to \infty]{} \infty$$

which would mean, since $\mathbb{Q}_0^m$ is a stationary law:

$$\mathbb{E}_{\mathbb{Q}^m} \left(\int_0^1 \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} \mathrm{d}t \right) = \infty.$$

Which is false by Lemma 2.12 so we get the result we want.

Now for the case $\kappa < 1$.

Let $\beta \in \left(\kappa, \frac{\kappa + \kappa'}{2} \right)$ be a real such that $\beta \leq 1$. If $\sum_{x \in \mathbb{Z}^d} \left(\sum_{i=\tau_2}^{\tau_3 - 1} 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}}$ has an infinite expectation (for $\mathbb{P}_0$), since the $\left(\sum_{x \in \mathbb{Z}^d} \left(\sum_{i=\tau_{2j}}^{\tau_{2j+1} - 1} 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}} \right)_{j \in \mathbb{N}^*}$ are iid, we would have that $\mathbb{P}_0$ almost surely:

$$n^{-1} \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=0}^{\tau_n - 1} 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}} \xrightarrow[n \to \infty]{} \infty.$$

By Lemma 3.5 we get that there exists a constant $C > 0$ such that $\mathbb{P}_0$ almost surely:

$$\exists m \in \mathbb{N}, \forall n \geq m, \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=0}^{\tau_{n+1} - 1} \mathcal{E}_i 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}} \geq C \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=0}^{\tau_{n+1} - 1} 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}}.$$

We also have, by writing $T_n^m = t_{\tau_n}^m$:

$$\sum_{x \in \mathbb{Z}^d} \left(\sum_{i=0}^{\tau_{n+1} - 1} \mathcal{E}_i 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}} \leq 4^{\beta} \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=0}^{\tau_{n+1} - 1} \mathcal{E}_i 1_{Y_i = x} \frac{1}{\gamma^{\omega}(x)} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}}$$

$$\leq 4^{\beta} \sum_{x \in \mathbb{Z}^d} \left(\int_0^{T_n^m} \frac{\gamma_{\omega}^m(X_t^m)}{\gamma^{\omega}(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^{\beta}.$$

We know by Lemma 2.1 that there exists a constant d_m such that $\mathbb{P}_0$ almost surely:

$$T_n^m - d_m n \xrightarrow[n \to \infty]{} -\infty.$$

We get:

$$\exists m \in \mathbb{N}, \forall n \geq m, \sum_{x \in \mathbb{Z}^d} \left(\int_0^{T_n^m} \frac{\gamma_{\omega}^m(X_t^m)}{\gamma^{\omega}(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^{\beta} \leq \sum_{x \in \mathbb{Z}^d} \left(\int_0^{d_m n} \frac{\gamma_{\omega}^m(X_t^m)}{\gamma^{\omega}(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^{\beta}.$$

Finally, if $\mathbb{P}_0$ almost surely

$$\frac{1}{n} \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=0}^{\tau_n - 1} 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}} \xrightarrow[n \to \infty]{} \infty$$

then $\mathbb{P}_0$ almost surely

$$\frac{1}{n} \sum_{x \in \mathbb{Z}^d} \left(\int_0^{d_m n} \frac{\gamma_{\omega}^m(X_t^m)}{\gamma^{\omega}(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^{\beta} \xrightarrow[n \to \infty]{} \infty.$$

And therefore, since $\mathbb{Q}_0^m$ is absolutely continuous with respect to $\mathbb{P}_0$ we get that $\mathbb{Q}_0^m$ almost surely:

$$\frac{1}{n} \sum_{x \in \mathbb{Z}^d} \left(\int_0^{d_m n} \frac{\gamma_{\omega}^m(X_t^m)}{\gamma^{\omega}(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^{\beta} \xrightarrow[n \to \infty]{} \infty.$$

So we would have:

$$\frac{1}{n} \mathbb{E}_{\mathbb{Q}_0^m} \left(\sum_{x \in \mathbb{Z}^d} \left(\int_0^{d_m n} \frac{\gamma_{\omega}^m(X_t^m)}{\gamma^{\omega}(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^{\beta} \right) \xrightarrow[n \to \infty]{} \infty.$$

And therefore:

$$\frac{1}{n}\sum_{i=0}^{d_m n}\mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{x\in\mathbb{Z}^d}\left(\int_i^{i+1}\frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)}1_{X_t^m=x}\mathrm{d}t\right)^\beta\right)\xrightarrow[n\to\infty]{}\infty.$$

This would mean, since $\mathbb{Q}_0^m$ is a stationary law that

$$\mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{x\in\mathbb{Z}^d}\left(\int_0^1\frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)}1_{X_t^m=x}\mathrm{d}t\right)^\beta\right)=\infty$$

which is false by Lemma 2.12. Therefore there exists a constant $C>0$ such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{x\in\mathbb{Z}^d}\left(\sum_{i=0}^{\tau_n-1}1_{Y_i=x}\right)^\beta 1_{x\notin\tilde{\mathcal{T}}}\xrightarrow[n\to\infty]{}C.$$

So $\mathbb{P}_0$ almost surely for n large enough:

$$\frac{1}{n}\left(\sum_{x\in\mathbb{Z}^d}\sum_{i=0}^{\tau_n-1}1_{Y_i=x}1_{x\notin\tilde{\mathcal{T}}}\right)^\beta\leq\frac{1}{n}\sum_{x\in\mathbb{Z}^d}\left(\sum_{i=0}^{\tau_n-1}1_{Y_i=x}\right)^\beta 1_{x\notin\tilde{\mathcal{T}}}\leq 2C.$$

And therefore:

$$\sum_{x\in\mathbb{Z}^d}\sum_{i=0}^{\tau_n-1}1_{Y_i=x}1_{x\notin\tilde{\mathcal{T}}}\leq(2Cn)^{\frac{1}{\beta}}.\qquad\square$$

2.6 Proof of the theorems

Now we can finally prove both theorems. We will start by the proof of Theorem 1.9.

Proof of Theorem 1.9. The proof will be divided in three parts, one for each result. The second part and the third one rely on the first part. However, the second part and the third part are independent from one another.

First Part

First we will prove that there exists a constant c such that for any $t\in\mathbb{R}^+$ and any increasing sequence (x_n) such that $x_n\xrightarrow[n\to\infty]{}\infty$, we have the following convergence in law, for $\mathbb{P}_0$:

$$x_n^{-\frac{1}{\kappa}}\tau_{\lfloor x_n t\rfloor}\xrightarrow[n\to\infty]{}ct^{\frac{1}{\kappa}}\mathcal{S}_1^\kappa.$$

The result is obvious for $t=0$. For $t>0$, lemmas 2.27 and 2.25 tell us that we only have to consider the time spent in traps in directions j such that $\kappa_j=\kappa$. Then Lemma 2.22 tells us that with probability larger than $1-\varepsilon$ the time spent in such traps is not more than the time spent in traps where the walk comes back at most m_ε times (for some m_ε) plus at most $\varepsilon x_n^{\frac{1}{\kappa}}$. We also know by Lemma 2.23 that for any m_ε there exists a constant c_ε such that the time spent in traps where the walk comes back at most m_ε times renormalized by $x_n^{-\frac{1}{\kappa}}$ converges in law (for $\mathbb{P}_0$) to $c_\varepsilon t^{\frac{1}{\kappa}}S^\kappa$ so we get the result we want by having ε go to 0 since c_ε is increasing and cannot go to infinity. Since the $(\tau_{i+1}-\tau_i)_{i\geq1}$ are iid (for $\mathbb{P}_0$) by proposition 1.2, we also get that for any sequence $(n_i)_{i\in\mathbb{N}^*}$ with $n_i\geq1$, $\left(i^{-\frac{1}{\kappa}}(\tau_{n_i+it}-\tau_{n_i})\right)_{i\geq1}$ converges in law (for $\mathbb{P}_0$) to $c_1 t^{\frac{1}{\kappa}}\mathcal{S}_1^\kappa$.

Now we want to show that the family of process $\left(t\mapsto x_n^{-\frac{1}{\kappa}}\tau_{\lfloor x_n t\rfloor}\right)_{n\in\mathbb{N}}$ is tight. We will only look at the convergence and tightness for the processes on an interval $[0,A]$. We

use the characterisation of tightness of a sequence of càdlàg functions $(f_n)_{n \in \mathbb{N}}$ given in Theorem 13.3 of [5]:

(i) for each positive ε there exists a c and an n_0 such that:

$$\forall n \geq n_0, \ \mathbb{P}\left(\sup_{t \in [0,T]} |f_n(t)| > c\right) \leq \varepsilon,$$

(ii) for each $\varepsilon > 0$ and $\eta > 0$, there exist a δ, $0 < \delta < T$, and an integer n_0 such that:

$\forall n \geq n_0, \ \mathbb{P}(w_{f_n}(\delta) \geq \eta) \leq \varepsilon$

and

$\forall n \geq n_0, \ \mathbb{P}(v_{f_n}(0, \delta) \geq \eta) \leq \varepsilon$ and $\mathbb{P}(v_{f_n}(T, \delta) \geq \eta) \leq \varepsilon,$

where for any càdlàg function f, w_f and v_f are defined by:

$$w_f(\delta) = \sup\{\min\left(|f(t) - f(t_1)|, |f(t_2) - f(t)|\right) t_1 \leq t \leq t_2 \leq T, t_2 - t_1 \leq \delta\},$$
$$v_f(t, \delta) = \sup\{|f(t_1) - f(t_2)| : t_1, t_2 \in [0, T] \cap (t - \delta, t + \delta)\}.$$

For a sequence of non-decreasing processes (W_n) defined on $[0, T]$, this characterization is implied by the following:

(i) for each positive ε there exists C such that

$\mathbb{P}(W_n(T) \geq C) \leq \varepsilon, \text{ for } n \geq 1,$

(ii) for each $\varepsilon > 0$ there exists a $\delta \in (0, T)$, such that for $n \geq 1$

$(ii.a)$ $\forall x \in [\delta, T - \delta], \ \mathbb{P}(W_n(x + \delta) - W_n(x) \geq \varepsilon \text{ and } W_n(x) - W_n(x - \delta) \geq \varepsilon) \leq \varepsilon$

and

$(ii.b)$ $\mathbb{P}(W_n(\delta) - W_n(0) \geq \varepsilon) \leq \varepsilon$

and

$(ii.c)$ $\mathbb{P}(W_n(T) - W_n(T - \delta) \geq \varepsilon) \leq \varepsilon.$

For the first property, since we know that the sequence $\left(x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n A \rfloor}\right)_{n \in \mathbb{N}}$ converges in law for $\mathbb{P}_0$, the family $\left(x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n A \rfloor}\right)_{n \in \mathbb{N}}$ is tight and therefore for any $\varepsilon > 0$ there exists B_ε such that:

$$\forall n \in \mathbb{N}, \ \mathbb{P}_0\left(x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n A \rfloor} \in [0, B_\varepsilon]\right) \geq 1 - \varepsilon.$$

So:

$$\forall \varepsilon > 0, \exists B_\varepsilon, \forall n \in \mathbb{N}, \mathbb{P}_0\left(\forall t \in [0, A], \ x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n t \rfloor} \in [0, B_\varepsilon]\right) \geq 1 - \varepsilon.$$

Now we will prove the two side conditions ($ii.b$ and $ii.c$). For ($ii.b$), we first choose δ such that $\mathbb{P}_0\left(c_1 \delta^{\frac{1}{\kappa}} \mathcal{S}_1^\kappa \geq \varepsilon\right) \leq \frac{\varepsilon}{2}$. This proves the result for n large enough and then, since the processes we consider are càdlàg, we decrease δ up to the point where we have the result for n small and we get the result we want.

For ($ii.c$), the proof will be essentially the same. Since the increments are iid (except for the first one of which we do not know the law) the law of $x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n A \rfloor} - x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n(A - \delta) \rfloor}$ converges to $c_1 \delta^{\frac{1}{\kappa}} S_\kappa$. So we get that for some δ, for n large enough we have the result we want. For small n we only use the fact that the processes are càdlàg so we get the result we want by decreasing δ.

Now we can prove ($ii.a$). Let $J = \{j \in [\![1, d]\!], \kappa_j = \kappa\}$. First we have, by lemmas 2.27 and 2.25, that for n large enough, the time spent in vertices that are not part of a trap in a direction $j \in J$ before time $\tau_{\lfloor x_n t \rfloor}$ is smaller than $\frac{1}{3} \varepsilon x_n^{\frac{1}{\kappa}}$ with probability at least $1 - \frac{1}{3}\varepsilon$.

Similarly by Lemma 2.23 there exists m_ε such that for n large enough the time spent in traps in direction $j \in J$ such that the walk enters at least m_ε times the trap is lower than $\frac{1}{3}\varepsilon x_n^{\frac{1}{\kappa}}$ with probability at least $1 - \frac{1}{3}\varepsilon$. And finally, there exists β_ε such that for n large enough, by Lemma 2.24, with probability at least $1 - \frac{1}{3}\varepsilon$ the time spent in traps in direction $j \in J$ such that their strength is at most $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ is lower than $\frac{1}{3}\varepsilon x_n^{\frac{1}{\kappa}}$. Condition $(ii.c)$ is not verified if either of the previous three events are not verified which happens with probability at most $1 - \varepsilon$. However if the previous events are verified and there is no i such that there are at least two traps of strength at least $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ visited at most m_ε times between times τ_i and $\tau_{i+2\delta x_n} - 1$ then the main condition is true.

So now we just have to prove that for δ small enough, with high probability there is no i such that there are at least two traps of strength at least $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ visited at most m_ε times between times τ_i and $\tau_{i+2\delta x_n} - 1$. By Lemma 2.26 we have that for any $m \in \mathbb{N}$ the probability that there exists $i \leq x_n$ such that there are two traps of strength at least $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ between times τ_i and $\tau_{i+m} - 1$ goes to 0 when n goes to infinity. So let B_i be the event: "there exists a trap of strength at least $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ visited at most m_ε times between times τ_i and $\tau_{i+1} - 1$". We define the finite sequence (n_i) by:

$$n_1 = \inf\{j \geq 1, B_j\},$$
$$n_{i+1} = \inf\{j \geq n_i + m, B_j\}.$$

We also define $\tilde{n}_i$ by $\tilde{n}_i = \sup\{j, n_j \leq x_i\}$. First we want to prove that $\tilde{n}_i$ cannot be too large. We know that there exists a constant C such that if $M(n)$ is the number of different traps in a direction j visited before time τ_n then for n large enough: $\mathbb{P}_0(M(x_n) \geq Cx_n) \leq \varepsilon$ and by Lemma 2.20 we clearly have that $\mathbb{E}(\tilde{n}_n 1_{M(x_n) \leq Cx_n}) \leq \frac{cC}{\beta^\kappa}$. Therefore if we take $B \geq \frac{cC}{\varepsilon\beta^\kappa}$ we get that for n large enough, $\mathbb{P}_0(\tilde{n}_n \geq B) \leq 2\varepsilon$. Now we want to show that for $\delta > 0$ small enough, $\mathbb{P}_0(\exists i \leq B, n_{i+1} - n_i \leq 2\delta x_n) \leq \varepsilon$ which would yields the desired result. For any i, we have, by Proposition 1.2:

$$\mathbb{P}_0(n_{i+1} - n_i \leq 2\delta x_n) \leq \mathbb{P}_0(n_1 \leq 2\delta x_n).$$

And therefore:

$$\mathbb{P}_0(\exists i \leq \tilde{n}_n, \, n_{i+1} - n_i \leq 2\delta x_n) \leq \mathbb{P}_0(\tilde{n}_n > B) + B\mathbb{P}_0(n_1 \leq 2\delta x_n).$$

We have that there is a constant C such that for n large enough, $\mathbb{P}_0(M(2\delta x_n) \geq 2C\delta x_n) \leq \frac{\varepsilon}{B}$. And then by Lemma 2.20 we have that the expectation of the number of traps of strength at least $\beta x_n^{\frac{1}{\kappa}}$ among the first $2\delta x_n$ traps is lower than $2\delta x_n \frac{c}{\beta^\kappa x_n}$ and therefore for δ small enough, $\mathbb{P}_0(\exists i \leq \tilde{n}_n, n_{i+1} - n_i \leq 2\delta x_n) \leq \varepsilon$. So we have that the sequence of processes is tight.

Now we want to show that its limit is $c_1 \mathcal{S}^\kappa$. Let m be an integer and $(x_i)_{0 \leq i \leq n}$ be reals such that $0 = y_0 < y_1 < \cdots < y_{m-1} < y_m = 1$. We have, since the $(\tau_{i+1} - \tau_i)_{i \geq 1}$ are iid and independent from τ_1:

$$(x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n y_i \rfloor})_{0 \leq i \leq m} \xrightarrow[n \to \infty]{} (\mathcal{S}^\kappa(y_i))_{0 \leq i \leq m}.$$

So we have convergence in the J_1 topology for any increasing sequence x_i that goes to infinity.

Second Part

Let L be defined by:
$$L(t) := \inf\{i, Y_i \cdot e_1 \geq t\}.$$

And let L_n be the renormalized L:

$$L_n(t) := n^{-\frac{1}{\kappa}} L(nt)$$

Notice that:

$$L_{n^\kappa}(t) = \inf\{i, Y^n(i) \cdot e_1 \geq t\}.$$

We have, by definition of τ and L:

$$\forall n \in \mathbb{N}^*, \ L(Y_{\tau_n} \cdot e_1) = \tau_n.$$

We first want to show that the sequence L_n is tight in the M_1 topology. We use the characterisation given in Theorem 12.12.3 of [37]:

(i) for each positive ε there exists c such that:

$$\mathbb{P}\left(\sup_{t \in [0,T]} |f(t)| > c\right) \leq \varepsilon,$$

(ii) for each $\varepsilon > 0$ and $\eta > 0$, there exists a δ, $0 < \delta < T$, and an integer n_0 such that:

$\forall n \geq n_0, \ \mathbb{P}(w_{f_n}(\delta) \geq \eta) \leq \varepsilon$

  and

$\forall n \geq n_0, \ \mathbb{P}(v_{f_n}(0,\delta) \geq \eta) \leq \varepsilon$ and $\mathbb{P}(v_{f_n}(T,\delta) \geq \eta) \leq \varepsilon.$

Where w_f and v_f are defined by:

$$w_f(\delta) = \sup\{\inf_{\alpha \in [0,1]} |f(t) - (\alpha f(t_1) + (1-\alpha)f(t_2))|, t_1 \leq t \leq t_2 \leq T, t_2 - t_1 \leq \delta\},$$

$$v_f(t,\delta) = \sup\{|f(t_1) - f(t_2)| : t_1, t_2 \in [0,T] \cap (t-\delta, t+\delta)\}.$$

First we have:

$$\mathbb{P}_0\left(\sup_{t \in [0,T]} |L_n(t)| > c\right) = \mathbb{P}_0\left(L(nT) > cn^{\frac{1}{\kappa}}\right)$$
$$\leq \mathbb{P}_0\left(\tau_{nT} > cn^{\frac{1}{\kappa}}\right),$$

which is smaller than ε for all n, for c large enough.

Next, since H_n is non-decreasing, we have:

$$\mathbb{P}_0(w_{L_n}(\delta) = 0) = 1.$$

Then, we first use the fact that:

$$v_{L_n}(0,\delta) \leq n^{-\frac{1}{\kappa}} \tau_{n\delta}$$

to get that for δ small enough:

$$\forall n \geq n_0, \ \mathbb{P}_0(v_{L_n}(0,\delta) \geq \eta) \leq \varepsilon.$$

The bound for $v_{L_n}(T,\delta)$ is similar but slightly trickier. For $c = (\mathbb{E}(Y_{\tau_2} - Y_{\tau_1}) \cdot e_1)^{-1}$, we know that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}(Y_{\tau_{cn(T-2\delta)}} \cdot e_1, Y_{\tau_{cn(T+\delta)}} \cdot e_1) \xrightarrow[n\to\infty]{} (T - 2\delta, T + \delta).$$

Therefore, using the fact that L_n is increasing, with probability going to 1:

$$L_n(T) - L_n(T - \delta) \leq n^{-\frac{1}{\kappa}}\left(\tau_{cn(T+\delta)} - \tau_{cn(T-2\delta)}\right).$$

And we have the result we want for δ small enough and n large enough. So we have that the sequence $(L_n)_{n \in \mathbb{N}^*}$ is tight. Now we just have to show that its limit is $C\mathcal{S}^\kappa$ for some constant C. Set $c = (\mathbb{E}(Y_{\tau_2} - Y_{\tau_1}) \cdot e_1)^{-1}$. We will show that $L_n(x)$ is almost equal to $\tau_n(cx)$ which will yield the result. Set $\varepsilon > 0$ and $x \in [0, \infty)$. We want to show that $\mathbb{P}_0(|L_n(x) - \tau_n(cx)| \geq \varepsilon) \xrightarrow[n \to \infty]{} 0$. We will use the following inequality:

$$\mathbb{P}_0(L_n(x) - \tau_n(cx) \geq \varepsilon) \leq \inf_{\delta > 0} \mathbb{P}_0(L_n(x) \geq \tau_n(cx + \delta)) + \mathbb{P}_0(\tau_n(cx + \delta) - \tau_n(cx) \geq \varepsilon).$$

We clearly have, for any $\delta > 0$

$$\limsup_{n \to \infty} \mathbb{P}_0(L_n(x) \geq \tau_n(cx + \delta)) = 0.$$

And for some constant $\tilde{C}$ that does not depend on x or c

$$\mathbb{P}_0(\tau_n(cx + \delta) - \tau_n(cx) \geq \varepsilon) \xrightarrow[n \to \infty]{} \mathbb{P}_0(\tilde{C}\mathcal{S}^\kappa(\delta) \geq \varepsilon).$$

Therefore

$$\mathbb{P}_0(L_n(x) - \tau_n(cx) \geq \varepsilon) \xrightarrow[n \to \infty]{} 0.$$

Similarly we get:

$$\mathbb{P}_0(L_n(x) - \tau_n(cx) \leq -\varepsilon) \xrightarrow[n \to \infty]{} 0.$$

Therefore the limit of L^n is $t \mapsto \tilde{C}\mathcal{S}^\kappa(ct)$ which is equal to $C\mathcal{S}^\kappa$ for some constant C.

Third Part

We will look at a sequence of processes $t \mapsto \overline{\tau}_n(t)$ such that the law of $\overline{\tau}_n$ is the same as that of $t \mapsto x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n t \rfloor}$ and such that almost surely $\overline{\tau}_n \xrightarrow[n \to \infty]{} \overline{\tau}$ in the J_1 topology with the law of $\overline{\tau}$ being that of $\mathcal{S}^\kappa$. We want to show that the law of the inverse of $\overline{\tau}_n$ converges to that of the inverse of $\mathcal{S}^\kappa$. This is a direct consequence of lemmas 3.6 and 3.7. Now if we define $L^\tau(t)$ by $L^\tau(t) = \min\{n \in \mathbb{N}, \tau_n \geq t\}$, we have that in J_1 topology:

$$\frac{1}{x_n} L^\tau\left(x_n^{\frac{1}{\kappa}} t\right) \xrightarrow[n \to \infty]{} \tilde{\mathcal{S}}^\kappa(t)$$

for any increasing sequence x_n such that $x_n \xrightarrow[n \to \infty]{} \infty$. Therefore, for any increasing sequence x_n such that $x_n \xrightarrow[n \to \infty]{} \infty$:

$$\frac{1}{x_n^\kappa} L^\tau(x_n t) \xrightarrow[n \to \infty]{} \tilde{\mathcal{S}}^\kappa(t).$$

Now by Lemma 2.3 there exists $v \in \mathbb{R}^d$ such that $\mathbb{P}_0$ almost surely:

$$\frac{Y_{\tau_{\lfloor t \rfloor}}}{t} \xrightarrow[t \to \infty]{} v.$$

This means that in the J_1 topology, we have the following convergence (in law):

$$\left(t \mapsto \frac{Y_{\tau_{\lfloor x_n t \rfloor}}}{x_n}\right) \xrightarrow[t \to \infty]{} (t \mapsto tv).$$

And therefore, in the J_1 topology,

$$\left(t \mapsto x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n t \rfloor}, t \mapsto \frac{Y_{\tau_{\lfloor x_n t \rfloor}}}{x_n}\right) \xrightarrow[t \to \infty]{} (c_1 \mathcal{S}^\kappa, t \mapsto tv).$$

Now we will look at $(\overline{\tau}_n, d_n)$ where for any n the law of $(\overline{\tau}_n, d_n)$ is the same as the law of $t \mapsto x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n t \rfloor}, t \mapsto \frac{Y_{\tau_{\lfloor x_n t \rfloor}}}{x_n}$ and such that almost surely:

$$(\overline{\tau}_n, d_n) \xrightarrow[n \to \infty]{} (c_1 \mathcal{S}^\kappa, t \mapsto tv).$$

Let $\overline{\tau}$ be such that almost surely $\overline{\tau}_n \xrightarrow[n \to \infty]{} \overline{\tau}$. Let $\Delta_{[0,A]}$ be the distance associated with the infinite norm on $[0, A]$.

If we look at $d_{\overline{\tau}_n^{-1}(t)}$ where $\overline{\tau}_n^{-1}(t) = \inf\{x, \overline{\tau}_n(x) \geq t\}$ we get:

$$\Delta_{[0,A]}\left(d_n(\overline{\tau}_n^{-1}(t)), \overline{\tau}^{-1}(t)v\right) \leq \Delta_{[0,A]}\left(d_n(\overline{\tau}_n^{-1}(t)), \overline{\tau}_n^{-1}(t)v\right) + \Delta_{[0,A]}\left(\overline{\tau}_n^{-1}(t)v, \overline{\tau}^{-1}(t)v\right)$$
$$= \Delta_{[0,A]}\left(d_n(\overline{\tau}_n^{-1}(t)), \overline{\tau}_n^{-1}(t)v\right) + \|v\|\Delta_{[0,A]}\left(\overline{\tau}_n^{-1}(t), \overline{\tau}^{-1}(t)\right).$$

So for any $B, \varepsilon > 0$:

$$\mathbb{P}_0(\Delta_{[0,A]}\left(d_n(\overline{\tau}_n^{-1}(t)), \overline{\tau}^{-1}(t)v\right) \geq \varepsilon)$$
$$\leq \mathbb{P}_0(\overline{\tau}_n^{-1}(A) > B) + \mathbb{P}_0\left(\exists t \in [0, B], \|d_n(t) - tv\| \geq \frac{\varepsilon}{2}\right) + \mathbb{P}_0\left(\Delta_{[0,A]}\left(\overline{\tau}_n^{-1}(t), \overline{\tau}^{-1}(t)\right) \geq \frac{\varepsilon}{2}\right)$$
$$= \mathbb{P}_0(\overline{\tau}_n^{-1}(A) > B) + o(1)$$
$$= \mathbb{P}_0(\overline{\tau}_n(B) < A) + o(1)$$
$$= \mathbb{P}_0(\overline{\tau}(B) < A) + o(1).$$

We clearly have that when B goes to infinity, $\mathbb{P}_0(\overline{\tau}(B) < A)$ goes to 0 so we have that in the J_1 topology:

$$d_n(\overline{\tau}_n^{-1}(t)) \xrightarrow[n \to \infty]{} \overline{\tau}^{-1}(t)v.$$

Since we have that in law (in the following we will write $\tau(x)$ instead of τ_x for the formulas to stay readable):

$$d_n(\overline{\tau}_n^{-1}(t)) = \frac{1}{x_n} Y_{\tau\left(\lfloor x_n(x_n^{-1} L^\tau((x_n)^{\frac{1}{\kappa}} t))\rfloor\right)} = \frac{1}{x_n} Y_{\tau\left(\lfloor \overline{\tau}((x_n)^{\frac{1}{\kappa}} t)\rfloor\right)}$$

we get that in the J_1 topology for any increasing sequence x_n such that $x_n \xrightarrow[n \to \infty]{} \infty$:

$$x_n^{-\kappa} Y_{\tau(\lfloor L^\tau(x_n t)\rfloor)} \xrightarrow[n \to \infty]{} c_1^{-\kappa} \tilde{\mathcal{S}}^\kappa(t)v.$$

Now we only have to show that $Y_{\tau(\lfloor L^\tau(x_n t)\rfloor)}$ and Y_t are almost equal. For every $i > 0$ let R_i be the number of different points visited between times τ_i and $\tau_{i+1} - 1$ and let R_0 be the number of different points visited before time $\tau_i - 1$ (0 if $\tau_i = 0$). The $(R_i)_{i \in \mathbb{N}}$ are independent and the $(R_i)_{i \in \mathbb{N}^*}$ are iid with finite expectation by Lemma 2.2. Let $\varepsilon > 0$ be a constant and let $B > 0$ be such that for x large enough, $\mathbb{P}_0(x^{-\kappa} L^\tau(xA) \geq B) \leq \frac{\varepsilon}{2}$ (taking B such that $\mathbb{P}_0(c_1^{-\kappa} \tilde{\mathcal{S}}^\kappa(A) \geq B) \leq \frac{\varepsilon}{4}$ works). We get that for x large enough:

$$\mathbb{P}_0(\exists t \leq xA, x^{-\kappa}\|Y_{\tau(\lfloor L^\tau(t)\rfloor)} - Y_t\| \geq \varepsilon) \leq \frac{\varepsilon}{2} + \mathbb{P}_0(\exists i \leq Bx^\kappa, R_i \geq \varepsilon x^\kappa)$$
$$\leq \frac{\varepsilon}{2} + \mathbb{P}_0(R_0 \geq \varepsilon x^\kappa) + \mathbb{P}_0(\exists i \in [\![1, Bx^\kappa]\!], R_i \geq \varepsilon x^\kappa)$$
$$\leq \frac{\varepsilon}{2} + o(1) + Bx^\kappa \mathbb{P}_0(R_1 \geq \varepsilon x^\kappa)$$
$$= \frac{\varepsilon}{2} + o(1).$$

So for any $\varepsilon > 0$ we have that for x large enough:

$$\mathbb{P}_0(\exists t \leq xA, x^{-\kappa}\|Y_{\tau(\lfloor L^\tau(xt)\rfloor)} - Y_t\| \geq \varepsilon) \leq \varepsilon.$$

So we get that in the J_1 topology:

$$x^{-\kappa} Y_{\lfloor xt \rfloor} \xrightarrow[n \to \infty]{} \tilde{\mathcal{S}}^\kappa(t) v.$$

Since v and d_α are collinear, we get the result we want. □

Now we can move on to the proof of Theorem 1.11.

Proof of Theorem 1.11. Let $J = \{j \in [\![1, d]\!], \kappa_j = \kappa\}$.

By Lemma 2.27 we get that there exists a constant C such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} \sum_{i=0}^{\tau_n} 1_{Y_i \notin \tilde{\mathcal{T}}} \xrightarrow[n \to \infty]{} C.$$

So we only have to look at the time spent in the traps. By Lemma 2.25 we get that for any $\varepsilon > 0$, for n large enough:

$$\mathbb{P}_0 \left(\frac{1}{n \log(n)} \sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathcal{T}}} 1_{Y_i \notin \tilde{\mathcal{T}}_J} \geq \varepsilon \right) \leq \varepsilon.$$

Therefore we only have to look at the time spent in traps in a direction $j \in J$. For any trap $\{x, y\}$ let $\tilde{N}_x$ be the number of times the walk exits the trap $\{x, y\}$, we have $\tilde{N}_w = \tilde{N}_y$. Let $\varepsilon > 0$ be a positive constant. By Lemma 2.22 there exists a m_ε such that:

$$\mathbb{P}_0 \left(\frac{1}{n \log(n)} \sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathcal{T}}_J} 1_{\tilde{N}_{Y_i} \geq m_\varepsilon} \geq \varepsilon \right) \leq \varepsilon.$$

And by Lemma 2.23 we get that there is a constant C_{m_ε} such that:

$$\frac{1}{n \log(n)} \sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathcal{T}}_J} 1_{\tilde{N}_{Y_i} \leq m_\varepsilon} \xrightarrow[n \to \infty]{} C_{m_\varepsilon} \text{ in probability.}$$

So for n large enough:

$$\mathbb{P}_0 \left(\frac{1}{n \log(n)} \sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathcal{T}}} \in [C_{m_\varepsilon} - 2\varepsilon, C_{m_\varepsilon} + 2\varepsilon] \right) \geq 1 - 2\varepsilon.$$

This means that there exists a constant C_∞ such that:

$$\frac{1}{n \log(n)} \sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathcal{T}}} \xrightarrow[n \to \infty]{} C_\infty \text{ in probability.}$$

And therefore:

$$\frac{1}{n \log(n)} \tau_{n+1} \xrightarrow[n \to \infty]{} C_\infty \text{ in probability.}$$

So we have proved the first part of the theorem.

Now, by Lemma 2.3 we have for some $C > 0$, $\mathbb{P}_0$ almost surely:

$$\frac{Y_{\tau_n} \cdot e_1}{n} \xrightarrow[n \to \infty]{} C.$$

So for any $\varepsilon > 0$, by writing $L(n) := \min\{i, Y_i \cdot e_1 \geq n\}$ and $C^+ = \frac{1}{C(1-\varepsilon)}$:

$$\mathbb{P}_0[L(n) \geq C_\infty C^+(1+\varepsilon)n\log(n)]$$
$$\leq \mathbb{P}_0[L(n) \geq C_\infty C^+(1+\varepsilon)n\log(n) \text{ and } \tau_{C^+n} \leq C_\infty C^+(1+\varepsilon)n\log(n)]$$
$$+ \mathbb{P}_0[\tau_{C^+n} > C_\infty C^+(1+\varepsilon)n\log(n)]$$
$$= \mathbb{P}_0[L(n) \geq C_\infty C^+(1+\varepsilon)n\log(n) \text{ and } \tau_{C^+n} \leq C_\infty C^+(1+\varepsilon)n\log(n)] + o(1)$$
$$\leq \mathbb{P}_0[L(n) \geq \tau_{C^+n}] + o(1)$$
$$= \mathbb{P}_0[Y_{\tau_{C^+n}} \cdot e_1 \leq n] + o(1)$$
$$= \mathbb{P}_0\left[\frac{Y_{\tau_{C^+n}} \cdot e_1}{C^+n} \leq C(1-\varepsilon)\right] + o(1)$$
$$= o(1).$$

The same way we get, by taking $C^- = \frac{1}{C(1+\varepsilon)}$:

$$\mathbb{P}_0(L(n) \leq C_\infty C^-(1-\varepsilon)n\log(n))$$
$$\leq \mathbb{P}_0(L(n) \leq C_\infty C^-(1-\varepsilon)n\log(n) \text{ and } \tau_{C^-n} \geq C_\infty C^-(1-\varepsilon)n\log(n))$$
$$+ \mathbb{P}_0(\tau_{C^-n} < C_\infty C^-(1-\varepsilon)n\log(n))$$
$$= \mathbb{P}_0(L(n) \leq C_\infty C^-(1-\varepsilon)n\log(n) \text{ and } \tau_{C^-n} \geq C_\infty C^-(1-\varepsilon)n\log(n)) + o(1)$$
$$\leq \mathbb{P}_0(L(n) \leq \tau_{C^-n}) + o(1)$$
$$= \mathbb{P}_0(Y_{\tau_{C^-n}} \cdot e_1 \geq n) + o(1)$$
$$= \mathbb{P}_0\left(\frac{Y_{\tau_{C^-n}} \cdot e_1}{C^-n} \geq C(1+\varepsilon)\right) + o(1)$$
$$= o(1).$$

So we get the second result. Now for the last result, we define $L^\tau(n) = \min\{i, \tau_i \geq n\}$ so $\tau_{L^\tau(n)-1} < n \leq \tau_{L^\tau(n)}$. We get, for n big enough:

$$\mathbb{P}_0\left(L^\tau(n) \geq C_\infty^{-1}(1+2\varepsilon)\frac{n}{\log(n)}\right) \leq \mathbb{P}_0\left(\tau_{C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}} \leq n\right).$$

And we have:

$$C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}\log\left(C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}\right) = C_\infty^{-1}(1+\varepsilon)n(1+o(1)).$$

And therefore, using the result of part one:

$$\frac{\tau_{C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}}}{n} \xrightarrow[n\to\infty]{} C_\infty C_\infty^{-1}(1+\varepsilon) = (1+\varepsilon).$$

So we get that:

$$\mathbb{P}_0\left(\tau_{C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}} \leq n\right) \xrightarrow[n\to\infty]{} 0.$$

And therefore:

$$\mathbb{P}_0\left(L^\tau(n) \geq C_\infty^{-1}(1+2\varepsilon)\frac{n}{\log(n)}\right) \xrightarrow[n\to\infty]{} 0.$$

The proof of the lower bound is exactly the same:

$$\mathbb{P}_0\left(L^\tau(n) \leq C_\infty^{-1}(1-\varepsilon)\frac{n}{\log(n)}\right) \leq \mathbb{P}_0\left(\tau_{C_\infty^{-1}(1-\varepsilon)\frac{n}{\log(n)}} \geq n\right).$$

But we have:

$$n^{-1}\tau_{C_\infty^{-1}(1-\varepsilon)\frac{n}{\log(n)}} \xrightarrow[n\to\infty]{} (1-\varepsilon).$$

So

$$\mathbb{P}_0\left(L^\tau(n) \le C_\infty^{-1}(1-\varepsilon)\frac{n}{\log(n)}\right) \xrightarrow[n\to\infty]{} 0.$$

And therefore:

$$\frac{\log(n)}{n}L^\tau(n) \xrightarrow[n\to\infty]{} C_\infty^{-1}.$$

Now, by Lemma 2.3 $\frac{Y_i}{L^\tau(i)} \xrightarrow[i\to\infty]{} D$, $\mathbb{P}_0$ almost surely so we get:

$$\frac{\log(n)}{n}Y_n \xrightarrow[n\to\infty]{} C_\infty^{-1}D. \qquad \square$$

3 Annex

This section is devoted to proving some elementary results on random variables.

This first result is reminiscent of a famous result by Charles de La Vallée Poussin (see for instance p. 19 of [20]) that states that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable iff there exists a convex monotone function Φ with $\Phi(0) = 0$ such that $\sup_{n\in\mathbb{N}} \mathbb{E}(\Phi(X_n)) < +\infty$. By taking $X_n = X$ for all n we get a result similar to our own.

Lemma 3.1. *Let X be a non-negative random variable such that $\mathbb{E}(X) < \infty$. There exists an increasing, positive, concave function ϕ such that $\phi(t)$ goes to infinity when t goes to infinity and:*

$$\mathbb{E}(\Phi(X)) < \infty,$$

where $\Phi(t) = \int_{x=0}^{t} \phi(x)\mathrm{d}x$.

Proof. First we show that there exists a non-decreasing, positive function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(t)$ goes to infinity when t goes to infinity and:

$$\mathbb{E}(Xf(X)) < \infty.$$

To do that we first define the sequence (t_i) by:

$$t_0 = 0$$
$$t_{i+1} = 1 + \inf\left\{x \ge t_i, \mathbb{E}(X1_{X>x}) \le 2^{-(i+1)}\mathbb{E}(X)\right\}.$$

Now we define f by:

$$f(x) = 1 + \sum_{i\ge 0} 1_{x\ge t_i}.$$

We clearly have that f is non-decreasing, positive ($f(t) \ge 2$) and that $f(t)$ goes to infinity when t goes to infinity. As for the expectation we have:

$$\begin{aligned}
\mathbb{E}(Xf(X)) &= \mathbb{E}\left(\sum_{i\ge 0} X1_{X\ge t_i}\right) + \mathbb{E}(X) \\
&= \sum_{i\ge 0} \mathbb{E}\left(X1_{X\ge t_i}\right) + \mathbb{E}(X) \\
&\le \sum_{i\ge 0} 2^{-i}\mathbb{E}(X) + \mathbb{E}(X) \\
&\le 3\mathbb{E}(X) < \infty.
\end{aligned}$$

Now we want to find an increasing concave function ϕ lower than f such that $\phi(t)$ goes to infinity when t goes to infinity. To that effect we will define the sequences (a_i) and (b_i) by:

$$a_0 = 1,$$
$$b_0 = \frac{1}{t_1},$$
$$\forall i \in \mathbb{N}, \ a_{i+1} = a_i + b_i(t_{i+1} - t_i),$$
$$\forall i \in \mathbb{N}, \ \min(b_{i+1} = b_i, \frac{(i+2) - a_i}{t_{i+1} - t_i})$$

and we define ϕ by:

$$\forall i \in \mathbb{N}, \forall x \in [t_i, t_{i+1}), \phi(x) = a_i + b_i(x - t_i).$$

The function ϕ is continuous and its slope is decreasing so it is clearly concave.

We now have to prove that $\lim_{t\to\infty} \phi(t) = \infty$. First we want to show that for every $i \in \mathbb{N}$, $a_i \leq i + 1$. It is obvious for $i \in \{0, 1\}$ and for $i > 0$ we have:

$$a_i \leq a_{i-1} + \frac{(i+1) - a_{i-1}}{t_i - t_{i-1}}(t_i - t_{i-1}) = i + 1.$$

Now we want to show that there can be no i such that $b_i \leq 0$. If there was, we could define j by $j = \min\{i, b_i \leq 0\}$, we would have $j \geq 1$ and:

$$\frac{(j+1) - a_{j-1}}{t_j - t_{j-1}} \leq 0.$$

But since $a_{j-1} \leq j$ it is impossible so all the b_i are positive and therefore ϕ is increasing. Now we will prove that $\lim_{i\to\infty} a_i = \infty$. First we notice that if $b_{i+1} < b_i$ then $b_{i+1} = \frac{(i+2)-a_i}{t_{i+1}-t_i}$ so $a_{i+1} = i + 2$. Therefore, either the b_i are stationary and ϕ is larger than some affine function with positive slope which implies the result we want or the sequence b_i is not stationary and there are infinitely many i such that $a_{i+1} = i + 2$ and therefore we have the result we want.

We still have to show that $\phi \leq f$. We know that ϕ is increasing and we have:

$$\forall i \in \mathbb{N}, \forall x \in [t_i, t_{i+1}), f(x) - \phi(x) = i + 2 - \phi(x) \geq i + 2 - \phi(t_{i+1}) = i + 2 - a_{i+1} \geq 0.$$

So we have the desired result. $\qquad\square$

Lemma 3.2. *Let ϕ be a non-decreasing, positive concave function and $\Phi(x) := \int_{t=0}^{x} \phi(t)\mathrm{d}t$. There exists a constant C_ϕ such that if X is a geometric random variable with success probability $1 - p$:*

$$\frac{1}{2}\Phi\left(\frac{1}{p}\right) \leq \frac{1}{2}\frac{1}{p}\phi\left(\frac{1}{p}\right) \leq \mathbb{E}(\Phi(1 + X)) \leq C_\phi \frac{1}{p}\phi\left(\frac{1}{p}\right) \leq 2C_\phi\Phi\left(\frac{1}{p}\right).$$

Proof. Φ is convex so if X is a geometric random variable with success probability p:

$$\mathbb{E}(\Phi(1 + X)) \geq \Phi(\mathbb{E}(1 + X))$$
$$= \Phi\left(\frac{1}{p}\right)$$
$$= \int_{t=0}^{\frac{1}{p}} \phi(t)\mathrm{d}t$$

$$= \frac{1}{p} \int\limits_{t=0}^{1} \phi\left(t\frac{1}{p}\right) \mathrm{d}t$$

$$\geq \frac{1}{p} \int\limits_{t=0}^{1} t\phi\left(\frac{1}{p}\right) + (1-t)\phi\left(0\right) \mathrm{d}t$$

$$\geq \frac{1}{p} \int\limits_{t=0}^{1} t\phi\left(\frac{1}{p}\right) \mathrm{d}t$$

$$= \frac{1}{2}\frac{1}{p}\phi\left(\frac{1}{p}\right).$$

Now for the upper bound, we will first look at the case where $p \leq \frac{1}{2}$:

$$\mathbb{E}(\Phi(1+X)) = \mathbb{E}\left(\int\limits_{t=0}^{\infty} \phi(t)1_{1+X\geq t}\mathrm{d}t\right)$$

$$= \int\limits_{t=0}^{\infty} \phi(t)\mathbb{P}(X \geq t-1)\mathrm{d}t$$

$$\leq \int\limits_{t=0}^{\infty} \phi(t)(1-p)^{t-1}\mathrm{d}t$$

$$\leq 2\int\limits_{t=0}^{\infty} \phi(t)\exp(t\log(1-p))\mathrm{d}t$$

$$= 2\frac{-1}{\log(1-p)} \int\limits_{t=0}^{\infty} \phi\left(-\frac{t}{\log(1-p)}\right)\exp(-t)\mathrm{d}t$$

$$\leq 2\frac{-1}{\log(1-p)}\left(\phi\left(-\frac{1}{\log(1-p)}\right) + \int\limits_{t=1}^{\infty} \phi\left(-\frac{t}{\log(1-p)}\right)\exp(-t)\mathrm{d}t\right).$$

Now we use the fact that ϕ is concave, this gives us, for $t \geq 1$:

$$\frac{1}{t}\phi\left(-\frac{t}{\log(1-p)}\right) + \left(1-\frac{1}{t}\right)\phi(0) \leq \phi\left(-\frac{1}{\log(1-p)}\right).$$

Since ϕ is positive, we get:

$$\phi\left(-\frac{t}{\log(1-p)}\right) \leq t\phi\left(-\frac{1}{\log(1-p)}\right).$$

So we get:

$$\mathbb{E}(\Phi(1+X)) \leq 2\frac{-1}{\log(1-p)}\left(\phi\left(-\frac{1}{\log(1-p)}\right) + \int\limits_{t=1}^{\infty} t\phi\left(-\frac{1}{\log(1-p)}\right)\exp(-t)\mathrm{d}t\right)$$

$$\leq 2\frac{-1}{\log(1-p)}\left(\phi\left(-\frac{1}{\log(1-p)}\right) + \phi\left(-\frac{1}{\log(1-p)}\right)\right)$$

$$= 4\frac{-1}{\log(1-p)}\phi\left(-\frac{1}{\log(1-p)}\right).$$

Since $-\frac{p}{\log(1-p)} \le 1$ and ϕ is increasing, we get:

$$-\frac{1}{\log(1-p)}\phi\left(-\frac{1}{\log(1-p)}\right) \le \frac{1}{p}\phi\left(\frac{1}{p}\right).$$

And therefore, if $p \le \frac{1}{2}$:

$$\mathbb{E}(\Phi(1+X)) \le 4\frac{1}{p}\phi\left(\frac{1}{p}\right).$$

If $p \ge \frac{1}{2}$ we can couple X with a geometric random variable Y of parameter $\frac{1}{2}$ such that almost surely $Y \ge X$ and since Φ is increasing:

$$\mathbb{E}(\Phi(1+X)) \le \mathbb{E}(\Phi(1+Y)) \le 8\phi(2) \le 8\phi(2)\frac{1}{p}\frac{\phi\left(\frac{1}{p}\right)}{\phi(1)} = 8\frac{\phi(2)}{\phi(1)}\frac{1}{p}\phi\left(\frac{1}{p}\right) \le 16\frac{1}{p}\phi\left(\frac{1}{p}\right).$$

We get the upper bound we wanted.

Now we just have to prove that for any $x \ge 0$, $\frac{1}{2}x\phi(x) \le \Phi(x) \le x\phi(x)$. For the upper bound we have:

$$\Phi(x) = \int_0^x \phi(t)\mathrm{d}t \le \int_0^x \phi(x)\mathrm{d}t = x\phi(x).$$

And for the lower bound we have:

$$\Phi(x) = \int_0^x \phi(t)\mathrm{d}t = \int_0^x \phi\left(\frac{t}{x}x\right)\mathrm{d}t \ge \int_0^x \frac{t}{x}\phi(x)\,\mathrm{d}t = \frac{1}{2}x\phi(x). \qquad \square$$

Lemma 3.3. *Let X be a positive random variable, and let $a = \mathbb{E}(X)$ and $\tilde{X} = X - a$. If $Var(X) \le a^2$ then:*

$$\forall \gamma \in [0,1], \ Var(X^\gamma) \le 2a^{2\gamma}\left(\frac{Var(X)}{a^2}\right).$$

Proof. For any $x \in [-1, \infty)$, let $f_x : [0,1] \to \mathbb{R}$ the function defined by

$$f_x(\gamma) := \gamma \mapsto (1+x)^\gamma.$$

This function is convex and $f_x(1) = 1 + x$ and $f_x'(1) = (1+x)\log(1+x)$ so:

$$\forall \gamma \in [0,1], \ f_x(\gamma) \ge 1+x+(\gamma-1)(1+x)\log(1+x) \ge 1+x-(1-\gamma)(1+x)x \ge 1+\gamma x-(1-\gamma)x^2.$$

By Jensen inequality, we have:

$$\mathbb{E}(X^\gamma) \le a^\gamma.$$

Since $\mathbb{E}(X^\gamma) = a^\gamma\mathbb{E}\left(\left(1+\frac{\tilde{X}}{a}\right)^\gamma\right)$, we also get:

$$\mathbb{E}(X^\gamma) \ge a^\gamma\left(1-(1-\gamma)\frac{Var(X)}{a^2}\right).$$

So if $\mathrm{Var}(X) \le a^2$, then

$$-\mathbb{E}(X^\gamma)^2 \le -a^{2\gamma}\left(1-(1-\gamma)\frac{Var(X)}{a^2}\right)^2 \le -a^{2\gamma}\left(1-2(1-\gamma)\frac{Var(X)}{a^2}\right).$$

We also have:

$$\mathbb{E}(X^{2\gamma}) \le \mathbb{E}(X^2)^\gamma = \left(a^2+\mathrm{Var}(X)\right)^\gamma \le a^{2\gamma}\left(1+\gamma\frac{Var(X)}{a^2}\right).$$

Finally we get:

$$\mathrm{Var}(X^\gamma) \le a^{2\gamma}\left(1+\gamma\frac{Var(X)}{a^2}-1+2(1-\gamma)\frac{Var(X)}{a^2}\right) = a^{2\gamma}(2-\gamma)\frac{Var(X)}{a^2}. \qquad \square$$

Lemma 3.4. *Let $p \in (0, \infty)$ be a positive real, $N \geq 1$ an integer, $h \in (\frac{1}{4}, 1)$ and $q \in (0, \infty)$ with $1 \geq q(1-h) \geq \frac{1}{2}$. Let (ε_i) be a sequence of integer in $\{0, 1\}$. Let $(H_i)_{i \in \mathbb{N}}$ be a sequence of iid random variables following a geometric law of parameter $1 - h$. Let $(\mathcal{E}_{i,j})_{i,j \in \mathbb{N}}$ be a sequence of iid random variables, independent of (H_i) and following an exponential law of parameter p. Now let Z be defined by:*

$$Z = \sum_{i=1}^{N} \sum_{j=1}^{\varepsilon_i + H_i} \mathcal{E}_{i,j} \frac{p}{q}.$$

There exists a universal constant C such that:

$$\forall \gamma \in [0, 1], \ \mathrm{Var}(Z^\gamma) \leq C N^{2\gamma - 1} \leq C N^\gamma.$$

We also have that there are two constant $c_1, c_2 > 0$ that do not depend on γ such that:

$$c_1 N^\gamma \leq \mathbb{E}(Z^\gamma) \leq c_2 N^\gamma.$$

Proof. First we look at the expectation of Z, we get:

$$
\begin{aligned}
\mathbb{E}(Z) &= \sum_{i=1}^{N} \mathbb{E}\left(\sum_{j=1}^{\varepsilon_i + H_i} \frac{1}{p} \frac{p}{q} \right) \\
&= \sum_{i=1}^{N} \frac{1}{q} \left(\varepsilon_i + \frac{h}{1-h} \right) \\
&= \frac{1}{q(1-h)} \sum_{i=1}^{N} \varepsilon_i (1-h) + h.
\end{aligned}
$$

Now we will look at the variance but first we need a small result to simplify the notations, for this result, M will be a non negative random variable and $(X_i)_{i \in \mathbb{N}}$ a sequence of iid real random variables, independent of M. We get:

$$
\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{M} X_i \right) &= \mathbb{E}\left(\left(\sum_{i=1}^{M} X_i \right)^2 \right) - \left(\mathbb{E}\left(\sum_{i=1}^{M} X_i \right) \right)^2 \\
&= \mathbb{E}\left(M \mathbb{E}(X_1^2) + M(M-1)\mathbb{E}(X_1)^2 \right) - \mathbb{E}(M)^2 \mathbb{E}(X_1)^2 \\
&= \mathbb{E}(M)\mathrm{Var}(X_1) + \mathrm{Var}(M)\mathbb{E}(X_1)^2.
\end{aligned}
$$

Now we can compute the variance of Z. First we have:

$$\mathrm{Var}(Z) = \sum_{i=1}^{N} \mathrm{Var}\left(\sum_{j=1}^{\varepsilon_i + H_i} \mathcal{E}_{i,j} \frac{p}{q} \right) = \frac{p^2}{q^2} \sum_{i=1}^{N} \mathrm{Var}\left(\sum_{j=1}^{\varepsilon_i + H_i} \mathcal{E}_{i,j} \right).$$

Then we have:

$$
\begin{aligned}
\frac{p^2}{q^2} \sum_{i=1}^{N} \mathbb{E}(\varepsilon_i + H_i)\mathrm{Var}(\mathcal{E}_{i,1}) &= \frac{p^2}{q^2} \sum_{i=1}^{N} \left(\varepsilon_i + \frac{h}{1-h} \right) \frac{1}{p^2} \\
&= \frac{1}{q^2(1-h)^2} \sum_{i=1}^{N} \varepsilon_i (1-h)^2 + h(1-h),
\end{aligned}
$$

$$
\begin{aligned}
\frac{p^2}{q^2} \sum_{i=1}^{N} \mathrm{Var}((\varepsilon_i + H_i)^2)\mathbb{E}(\mathcal{E}_{i,1})^2 &= \frac{p^2}{q^2} \sum_{i=1}^{N} \frac{h}{1-h} \frac{1}{p^2} \\
&= \frac{1}{q^2(1-h)^2} \sum_{i=1}^{N} h(1-h),
\end{aligned}
$$

So we get, by summing these two equalities:

$$\text{Var}(Z) = \frac{1}{q^2(1-h)^2} \sum_{i=1}^{N} \varepsilon_i(1-h)^2 + 2h(1-h).$$

We have assumed that $h \geq \frac{1}{4}$ and $\frac{1}{2} \leq q(1-h) \leq 1$ therefore we have:

$$\frac{1}{4}N \leq \mathbb{E}(Z) \leq 4N,$$
$$\text{Var}(Z) \leq 20N.$$

Therefore we have:

$$\frac{\text{Var}(Z)}{(\mathbb{E}(Z))^2} \leq 320\frac{1}{N}.$$

So by Lemma 3.3, for $N \geq 320$ we have:

$$\forall \gamma \in [0,1], \ \text{Var}(Z^\gamma) \leq 2\mathbb{E}(Z)^{2\gamma}\left(\frac{\text{Var}(Z)}{\mathbb{E}(Z)^2}\right) \leq 4^{2\gamma}N^{2\gamma}\frac{640}{N}.$$

And if $N \leq 320$ we have:

$$\forall \gamma \in [0,1], \ \text{Var}(Z^\gamma) \leq \mathbb{E}(Z^{2\gamma}) \leq \mathbb{E}(Z^2) \leq (20N + 16N^2).$$

So there exists a constant C such that if $1 \leq N \leq 320$:

$$\text{Var}(Z^\gamma) \leq C\frac{1}{N}.$$

So

$$\forall \gamma \in [0,1], \ \text{Var}(Z^\gamma) \leq CN^{2\gamma-1}.$$

So we have that there exists a constant C such that if $N \geq 1$:

$$\forall \gamma \in [0,1], \ \text{Var}(Z^\gamma) \leq CN^{2\gamma-1} \leq CN^\gamma.$$

Now for the expectation, we first have the upper bound:

$$\mathbb{E}(Z^\gamma) \leq \mathbb{E}(Z)^\gamma \leq (4N)^\gamma.$$

For the lower bound, we will use Holder inequality:

$$\mathbb{E}(Z) = \mathbb{E}\left(Z^{\frac{\gamma}{2-\gamma}}Z^{2\frac{1-\gamma}{2-\gamma}}\right) \leq \mathbb{E}\left(Z^{\frac{\gamma}{2-\gamma}(2-\gamma)}\right)^{\frac{1}{2-\gamma}}\mathbb{E}\left(Z^{2\frac{1-\gamma}{2-\gamma}\frac{2-\gamma}{1-\gamma}}\right)^{\frac{1-\gamma}{2-\gamma}}.$$

This yields:

$$\mathbb{E}(Z)^{2-\gamma} \leq \mathbb{E}(Z^\gamma)\mathbb{E}(Z^2)^{1-\gamma}$$

ie:

$$E(Z^\gamma) \geq \frac{\mathbb{E}(Z)^{2-\gamma}}{\mathbb{E}(Z^2)^{1-\gamma}}.$$

Now we have $E(Z^2) = \text{Var}(Z) + \mathbb{E}(Z)^2$ since $\text{Var}(Z) \leq 80\mathbb{E}(Z)$ and $\mathbb{E}(Z) \geq \frac{1}{4}$ we have $\text{Var}(Z) \leq 320\mathbb{E}(Z)^2$ and therefore: $E(Z^2) \leq 321E(Z)^2$ which yields:

$$E(Z^\gamma) \geq \frac{\mathbb{E}(Z)^{2-\gamma}}{(321\mathbb{E}(Z)^2)^{1-\gamma}} \geq \frac{\mathbb{E}(Z)^\gamma}{321^{1-\gamma}} \geq \frac{\mathbb{E}(Z)^\gamma}{321}. \qquad \square$$

Lemma 3.5. *Let $\beta \in [0,1]$. Let $(N_i)_{i \in \mathbb{N}^*}$ be a sequence of random positive integers and $(A_i)_{i \in \mathbb{N}}$ be a sequence of random finite subsets of $\mathbb{N}$ with the following two properties:*

$$\forall i \geq 0, A_i \subset A_{i+1},$$
$$\#A_i \xrightarrow[i \to \infty]{} \infty.$$

Let $(Z_i)_{i \in \mathbb{N}}$ be independent exponential random variables of parameter 1 independent of $(A_i), (N_i)$.

Then there exists a constant $C > 0$ such that almost surely:

$$\exists m \in \mathbb{N}, \forall n \geq m, \sum_{i \in A_n} \left(\sum_{j=1}^{N_i} Z_i \right)^{\beta} \geq C \sum_{i \in A_n} (N_i)^{\beta}.$$

Proof. Let C be such that $2C - 2^{1-\beta} > 0$ Let $(n_i)_{i \in \mathbb{N}}$ be the sequence defined by:

$$n_i = \min \left\{ i, \# \sum_{i \in A_n} (N_i)^{\beta} \geq 2^i \right\}.$$

We have that if

$$\exists m \in \mathbb{N}, \forall j \geq m, \sum_{i \in A_{n_j}} \left(\sum_{k=1}^{N_i} Z_i \right)^{\beta} \geq 2C \sum_{i \in A_{n_j}} (N_i)^{\beta}$$

and M is such an m then for every $n \geq n_M$, if j is the integer that satisfies $n_j \leq n < n_{j+1}$, we have:

$$\sum_{i \in A_n} (N_i)^{\beta} \leq 2^{j+1}$$

$$\leq 2 \sum_{i \in A_{n_j}} (N_i)^{\beta}$$

$$\leq 2C \sum_{i \in A_{n_j}} \left(\sum_{k=1}^{N_i} Z_i \right)^{\beta}$$

$$\leq 2C \sum_{i \in A_n} \left(\sum_{k=1}^{N_i} Z_i \right)^{\beta}.$$

By Lemma 3.3, for any $i \in \mathbb{N}^*$:

$$\mathrm{Var} \left(\left(\sum_{j=1}^{N_i} \mathcal{E}_{i,j} \right)^{\beta} | (A_k), (N_k) \right) \leq 2(N_i)^{2\beta - 1} \leq 2(N_i)^{\beta}.$$

And by Hölder:

$$\mathbb{E} \left(\left(\sum_{j=1}^{N_i} \mathcal{E}_{i,j} \right)^{\beta} | (A_k), (N_k) \right)$$

$$\geq \mathbb{E} \left(\sum_{j=1}^{N_i} \mathcal{E}_{i,j} | (A_k), (N_k) \right)^{2-\beta} \mathbb{E} \left(\left(\sum_{j=1}^{N_i} \mathcal{E}_{i,j} \right)^{2} | (A_k), (N_k) \right)^{-(1-\beta)}$$

$$= (N_i)^{2-\beta} (N_i^2 + N_i)^{-(1-\beta)}$$

$$\geq (N_i)^{2-\beta} (2N_i^2)^{-(1-\beta)}$$

$$= 2^{\beta - 1} (N_i)^{\beta}.$$

Now we get:

$$\sum_{j \geq 0} \mathbb{P}\left(\sum_{i \in A_{n_j}} \left(\sum_{k=1}^{N_i} Z_i\right)^\beta \leq 2C \sum_{i \in A_{n_j}} (N_i)^\beta\right)$$

$$\leq \sum_{j \geq 0} \mathbb{E}\left(\frac{\operatorname{Var}\left(\sum_{i \in A_{n_j}} \left(\sum_{k=1}^{N_i} Z_i\right)^\beta \big| (A_k), (N_k)\right)}{\left((2C - 2^{1-\beta}) \sum_{i \in A_{n_j}} (N_i)^\beta\right)^2}\right)$$

$$\leq \sum_{j \geq 0} \mathbb{E}\left(\frac{2}{(2C - 2^{1-\beta})^2 \sum_{i \in A_{n_j}} (N_i)^\beta}\right)$$

$$\leq \frac{2}{(2C - 2^{1-\beta})^2} \sum_{j \geq 0} 2^{-j} < \infty.$$

So by Borel-Cantelli we get the result we want $\qquad\square$

Lemma 3.6. *Let f, g be two non-decreasing positive càdlàg functions with $f(0) = g(0) = 0$. Let $A, B > 0$ be constants such that $f(A) \geq B$ and $g(A) \geq B$. Let $\varepsilon, \delta > 0$ be such that:*

$$\forall t \in [0, A + \varepsilon], g(t + \varepsilon) \geq g(t) + \delta$$

and

$$\sup\{|f(t) - g(t)|, t \in [0, A + 2\varepsilon]\} \leq \frac{\delta}{2}.$$

Then:

$$\sup\{|f^{-1}(x) - g^{-1}(t)|, t \in [0, B]\} \leq 2\varepsilon.$$

Proof. Let t be in $[0, B]$. First we have:

$$f\left(g^{-1}(t) + 2\varepsilon\right) \geq g\left(g^{-1}(t) + 2\varepsilon\right) - \frac{\delta}{2} \geq g(g^{-1}(t) + \varepsilon) + \delta - \frac{\delta}{2} \geq t.$$

Therefore $f^{-1}(t) \leq g^{-1}(t) + \varepsilon$. Similarly we have:

$$f\left(g^{-1}(t) - \varepsilon\right) \leq g\left(g^{-1}(t) - \varepsilon\right) + \frac{\delta}{2} \leq g(g^{-1}(t)) - \delta + \frac{\delta}{2} < t.$$

Therefore $f^{-1}(t) \geq g^{-1}(t) - \varepsilon$. So we have the result we want. $\qquad\square$

Lemma 3.7. *Let $t \mapsto \mathcal{S}^\kappa(t)$ be the jump process where $\mathcal{S}^\kappa(1)$ is a completely asymmetric, positive stable law of parameter κ. For any $\varepsilon > 0$ and any $B > 0$ there exist $A > 0$ and $\delta > 0$ such that:*

$$\mathbb{P}(\mathcal{S}^\kappa(A) \geq B) \geq 1 - \varepsilon,$$
$$\mathbb{P}(\exists t \leq A - \varepsilon, \mathcal{S}^\kappa(t + \varepsilon) - \mathcal{S}^\kappa(t) < \delta) \leq \varepsilon.$$

Proof. There clearly exists an A that satisfies the first property. Now we need to find a δ that satisfies the second inequality for this A. We will look at a slightly different property:

$$\exists i \leq \frac{2A}{\varepsilon}, \mathcal{S}^\kappa\left(i\frac{\varepsilon}{2}\right) - \mathcal{S}^\kappa\left((i+1)\frac{\varepsilon}{2}\right) \leq \delta.$$

Since for every $t \leq A - \varepsilon$ there exists $i \leq \frac{2A}{\varepsilon}$ such that: $[i\frac{\varepsilon}{2}, (i+1)\frac{\varepsilon}{2}] \subset [t, t+\varepsilon]$, we have that for any $\delta > 0$:

$$\mathbb{P}(\exists t \leq A - \varepsilon, \mathcal{S}^\kappa(t+\varepsilon) - \mathcal{S}^\kappa(t+\varepsilon) \leq \delta) \leq \mathbb{P}\left(\exists i \leq \frac{2A}{\varepsilon}, \mathcal{S}^\kappa\left(i\frac{\varepsilon}{2}\right) - \mathcal{S}^\kappa\left((i+1)\frac{\varepsilon}{2}\right) \leq \delta\right).$$

And there clearly exists δ such that

$$\mathbb{P}\left(\exists i \leq \frac{2A}{\varepsilon}, \mathcal{S}^\kappa\left(i\frac{\varepsilon}{2}\right) - \mathcal{S}^\kappa\left((i+1)\frac{\varepsilon}{2}\right) \leq \delta\right) \leq \varepsilon.$$

So we get the result we want. $\qquad\square$

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