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Largest component of subcritical random graphs with given degree sequence*

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Abstract

We study the size of the largest component of two models of random graphs with prescribed degree sequence, the configuration model (CM) and the uniform model (UM), in the (barely) subcritical regime. For the CM, we give upper bounds that are asymptotically tight for certain degree sequences. These bounds hold under mild conditions on the sequence and improve previous results of Hatami and Molloy on the barely subcritical regime. For the UM, we give weaker upper bounds that are tight up to logarithmic terms but require no assumptions on the degree sequence. In particular, the latter result applies to degree sequences with infinite variance in the subcritical regime.

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1 Introduction

Let $[n] \coloneqq \{1, \dots, n\}$ be a set of n vertices. Let $\mathbf{d}_n = (d_1, \dots, d_n)$ be a degree sequence with $m \coloneqq \sum_{i\in[n]} d_i$ an even positive integer. Without loss of generality, we will assume that $d_1 \leq \cdots \leq d_n$. Additionally, we may assume that $d_1 \geq 1$; if there are elements with degree 0 we can remove them and study the remainder of the sequence. Let $\Delta = \Delta_n$ be the maximum degree of $\mathbf{d}_n$.

The *configuration model*, denoted by $\mathbb{CM}_n = \mathbb{CM}_n(\mathbf{d}_n)$ and introduced by Bollobás in [1], is the random multigraph on $[n]$ generated by assigning d_i *half-edges* (or stubs)

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to vertex i, and then pairing the half-edges uniformly at random. The *uniform model*, denoted by $\mathbb{G}_n = \mathbb{G}_n(\mathbf{d}_n)$, is the random simple graph on $[n]$ obtained by choosing a simple graph uniformly at random among all graphs on $[n]$ where vertex i has degree d_i. Throughout the paper, all the results on the uniform model will assume that the sequence $\mathbf{d}_n$ is *graphical*; that is, there exists at least one graph on $[n]$ with such degree sequence. We will use the Landau notation for functions o, O to denote the asymptotic behavior of functions on n, as n goes to infinity.

For any graph G on $[n]$, let $L_1(G)$ denote the order of a largest component. A central problem in random graph theory is to find a parameter of the model $\alpha = \alpha(n)$ and a value $\alpha_0 = \alpha_0(n)$ such that L_1 undergoes a phase transition at $\alpha = \alpha_0$. The set of parameters is then divided into three regimes: *subcritical*, where $\alpha \le (1-\epsilon)\alpha_0$ for some $\epsilon > 0$; *critical*, where $\alpha = (1+o(1))\alpha_0$; and *supercritical*, where $\alpha \ge (1+\epsilon)\alpha_0$ for some $\epsilon > 0$. The problem of determining the critical window is to find a value $\alpha_1 = \alpha_1(n)$ with $\alpha_1 = o(\alpha_0)$, such that L_1 has the same asymptotic order for any α such that $(\alpha - \alpha_0)/\alpha_1 = O(1)$. Then the critical regime is further divided into three parts: *barely subcritical*, where $(\alpha-\alpha_0)/\alpha_1 \to -\infty$; *critical window*, where $(\alpha - \alpha_0)/\alpha_1 = O(1)$; and *barely supercritical*, where $(\alpha - \alpha_0)/\alpha_1 \to \infty$.

The main goal of this paper is to study the largest component phase transition $L_1(\mathbb{CM}_n)$ and $L_1(\mathbb{G}_n)$ in the subcritical and the barely subcritical regimes. Generally speaking, the study of configuration model is simpler due to the existence of an explicit model with good independence properties. In contrast, most of the results existing for $\mathbb{G}_n$ arise from $\mathbb{CM}_n$ by observing that the probability that $\mathbb{CM}_n$ generates a simple graph is sufficiently large.

In order to understand the phase transition, define

$$Q = Q_n(\mathbf{d}_n) \coloneqq \frac{1}{m}\sum_{i\in[n]} d_i(d_i-2)\ , \tag{1.1}$$

$$R = R_n(\mathbf{d}_n) \coloneqq \frac{1}{m}\sum_{i\in[n]} d_i(d_i-2)^2\ . \tag{1.2}$$

The parameter Q should be understood as the drift (expected change per step) of an exploration process that will be defined later; R will control the variance of this drift.

In the first part of the paper, we will focus on the case $Q_n \le 0$. It is easy to check that the bound on Q_n implies $\Delta_n = O(\sqrt{n})$ and $m \le 2n$. Also note the implicit bound on the maximum degree $\Delta_n = O(n^{1/3}R_n^{1/3})$ obtained by just considering the contribution of a vertex of maximum degree to R_n.

Let D_n be the degree of a uniform random vertex and let $\hat{D}_n$ be its size-biased distribution; that is, for $k \ge 1$,

$$\mathbb{P}(\hat{D}_n = k) = \frac{kn\mathbb{P}(D_n = k)}{m}\ . \tag{1.3}$$

For $b, h \in \mathbb{N}$, let $\mathcal{L}(b,h) := \{b + hk : k \in \mathbb{Z}\}$ be the integer lattice containing b with step h. For a random variable X on $\mathbb{Z}$, the *step* of X is the largest integer h such that $\mathbb{P}(X \in \mathcal{L}(b,h)) = 1$ for some $b \in \mathbb{N}$.

We will study the $\mathbb{CM}_n$ under the following conditions on the degree sequence:

Assumption 1.1. *There exists a discrete random variable D supported on $\mathbb{Z}_{\ge 0}$ such that*

(i) $D_n \to D$ *in distribution;*

(ii) $Q_n \to 0$*;*

(iii) $\mathbb{P}(D \notin \{0,2\}) > 0$;

(iv) *for all n, the random variables D and D_n have the same step;*

(v) $\mathbb{E}(D_n^4) \le \Delta_n^{1/2}$.

Remark 1.2. Conditions (i)-(iii) are usual in this setting. In particular, they imply that R_n is bounded away from zero, which will be often used in the proofs.

Condition (iv) simply asks that the limiting degree distribution D has the same step as the random variables D_n which converge to it. This restriction is not particularly strong and forbids no limiting degree sequence, only the way in which we converge to it.

Condition (v) is the most restrictive one. As $Q_n = o(1)$, we have $\mathbb{E}[D_n^2] = O(1)$, which implies $\mathbb{E}[D_n^4] = O(\Delta_n^2)$. Thus, this condition can be understood as a “polynomial limitation” on the contribution of large degree vertices to the fourth moment. It would be interesting to see up to which point a condition on the fourth moment is needed.

Our first result upper bounds the size of the largest component when Q is not too large with respect to R.

Theorem 1.3. *Let $\epsilon > 0$. Let $\mathbf{d}_n$ be a degree sequence satisfying Theorem 1.1 and $\Delta|Q| = o(R)$. If $Q \le -\omega(n)n^{-1/3}R^{2/3}$ for some $\omega(n) \to \infty$, then*

$$\mathbb{P}\left(L_1(\mathbb{CM}_n(\mathbf{d}_n)) \le (1+\epsilon)\frac{2R}{Q^2}\log\left(\frac{|Q|^3 n}{R^2}\right)\right) = 1 - o(1)\ . \tag{1.4}$$

Remark 1.4. As noted in [11], under the condition $|Q|\Delta = o(R)$ the critical window is $|Q| = O(n^{-1/3}R^{2/3})$. Therefore, Theorem 1.3 bounds the largest component in the whole barely subcritical regime. See Section 1.1 for further discussion.

Let $\eta \coloneqq \eta_n = \hat{D}_n - 2$ and note that

$$Q = \mathbb{E}[\eta] \quad \text{and} \quad R = \mathbb{E}\left[\eta^2\right]\ . \tag{1.5}$$

Consider its moment generating function

$$\varphi(\theta) \coloneqq \varphi_n(\theta) = \mathbb{E}\left[e^{\theta\eta_n}\right]\ . \tag{1.6}$$

Theorem 1.3 is in fact a consequence of a more general result that does not require a bound of Q in terms of R.

Theorem 1.5. *Let $\epsilon > 0$. Let $\mathbf{d}_n$ be a degree sequence satisfying Theorem 1.1 and $\Delta_n \le n^{1/6}$. Let $\theta_0 \in (0,1)$ be the smallest solution θ of $\varphi'(\theta) = 0$. Define*

$$T_n \coloneqq \frac{1}{|\log\varphi(\theta_0)|}\cdot\log\left(\frac{|\log\varphi(\theta_0)|^{3/2}}{\varphi''(\theta_0)^{1/2}}\mathbb{E}\left[D_n e^{\theta_0 D_n}\right] n\right)\ . \tag{1.7}$$

If $\theta_0 m \ge \omega(n)T_n$ for some $\omega(n) \to \infty$, then

$$\mathbb{P}(L_1(\mathbb{CM}_n(\mathbf{d}_n)) \le (1+\epsilon)T_n) = 1 - o(1)\ . \tag{1.8}$$

Remark 1.6. The value θ_0 exists and is bounded as $n \to \infty$. We have $\varphi(0) = 1$ and $\varphi'(0) = Q < 0$. Recall that η is supported in $\{-1, 0, 1, \dots\}$. By Theorem 1.1, items (ii) and (iii), $\{D \ge 3\}$ happens with positive probability and so if we define $p := \mathbb{P}(D \ge 3)$, then $\mathbb{P}(\eta \ge 1) \ge p$ and $\varphi(\theta) \ge (1-p)e^{-\theta} + pe^{\theta}$. So $\varphi(\theta) \to \infty$ as $\theta \to \infty$ and it must at some point have positive derivative. Thus, there exists $\theta_0 \in (0,1)$ such that $\varphi'(\theta_0) = 0$.

It is interesting to understand if these results also hold in the uniform setting. It is well known that $\mathbb{CM}_n$ conditioned on being simple is distributed as $\mathbb{G}_n$ [1]. Here we use a version of this result that needs no assumption on the maximum degree.

Theorem 1.7 (Bollobás [1]; Janson [15, Theorem 1.1])**.** *Let $\mathbf{d}_n$ be a degree sequence satisfying $m = \Theta(n)$ and $\mathbb{E}\left[D_n^2\right] = O(1)$. Then*

$$\liminf_{n\to\infty} \mathbb{P}(\mathbb{CM}_n(\mathbf{d}_n) \text{ is simple}) > 0 \ , \tag{1.9}$$

and conditioned on being simple, $\mathbb{CM}_n$ has the same law as $\mathbb{G}_n$. Therefore, any result that holds with probability $1 - o(1)$ for $\mathbb{CM}_n(\mathbf{d}_n)$, also holds with probability $1 - o(1)$ for $\mathbb{G}_n(\mathbf{d}_n)$.

As Theorem 1.3 and Theorem 1.5 assume that $Q_n \leq 0$, we have that $\mathbb{E}\left[D_n^2\right] = O(1)$ and we can use Theorem 1.7 to transfer their conclusions to $\mathbb{G}_n$, provided that their hypothesis are satisfied.

The second part of our paper focuses on the size of the largest component in the (barely) subcritical regime of $\mathbb{G}_n$ without further assumptions on the degree sequence. The lack of a tractable model for $\mathbb{G}_n$ hampers its analysis and the upper bounds obtained are weaker than the ones obtained for $\mathbb{CM}_n$ and probably not of the right order.

Let S_* be a smallest set of vertices of largest degree that satisfies

$$\sum_{u\in[n]\setminus S_*} d_u(d_u - 2) \leq 0 \tag{1.10}$$

and define

$$m_* = \sum_{v\in S_*} d_v \ . \tag{1.11}$$

We may assume that S_* is formed by vertices of largest degrees. Note that if $Q \leq 0$, then $S_* = \emptyset$ and $m_* = 0$.

For any $m_0 \geq 0$ and $Q_0 \leq 0$, we call $\mathbf{d}_n$ an (m_0, Q_0)-*subcritical* degree sequence if there exists $S \subseteq [n]$ with

$$\sum_{v\in S} d_v \leq m_0 \quad \text{and} \quad \frac{1}{m}\sum_{w\in[n]\setminus S} d_w(d_w - 2) \leq Q_0 \ . \tag{1.12}$$

Our most general result on $\mathbb{G}_n$ is the following.

Theorem 1.8. *Let $\mathbf{d}_n$ be an (m_0, Q_0)-subcritical degree sequence for some parameters satisfying $m_0 \geq 3m_*$, $m_0|Q_0| \geq (\Delta|Q_0| + R)\log\left(\frac{nQ_0^2}{\Delta|Q_0|+R}\right)$ and $Q_0^2 n \geq \omega(n)m_0$ for some $\omega(n) \to \infty$. Then, there exists an absolute constant $C > 0$ such that*

$$\mathbb{P}\left(L_1(\mathbb{G}_n(\mathbf{d}_n)) \leq Cm_0/|Q_0|\right) = 1 - o(1) \ .$$

Remark 1.9 (Infinite degree variance)**.** The main strength of Theorem 1.8 is that it applies to degree sequences with subcritical behaviour but infinite degree variance. To our knowledge, the only results available in this setting are of the form $L_1(\mathbb{G}_n(\mathbf{d}_n)) = o(n)$ [2, 18]. Note that, even if such results were available for $\mathbb{CM}_n$, Theorem 1.7 is not strong enough to transfer them to $\mathbb{G}_n$.

Remark 1.10 (The $Q \leq 0$ case)**.** To compare it with previous work, let us get more explicit results for the case $Q \leq 0$ (i.e. $\mathbb{E}[D_n^2] \leq 2\mathbb{E}[D_n]$). In this case, $m_* = 0$ and we can choose $Q_0 \coloneqq Q$ and $m_0 \coloneqq (\Delta + R/|Q_0|)\log\left(\frac{nQ_0^2}{\Delta|Q_0|+R}\right)$. Also note that $Q \leq 0$ implies $R = O(\Delta)$.

(a) If $|Q|$ is bounded away from zero, then all conditions in Theorem 1.8 are satisfied and

$$L_1(\mathbb{G}_n(\mathbf{d}_n)) = O(\Delta \log n) \ . \tag{1.13}$$

(b) If $|Q| = o(1)$, then we split depending on how R and $\Delta|Q|$ compare to each other.

(b.1) If $\Delta|Q| = O(R)$, then for any $Q \leq -\omega(n)n^{-1/3}R^{1/3}$,

$$L_1(\mathbb{G}_n(\mathbf{d}_n)) = O\left(\frac{R}{Q^2}\log\left(\frac{nQ^2}{R}\right)\right) , \tag{1.14}$$

obtaining a weaker version of Theorem 1.3, that is valid for all degree sequences.

(b.2) If $R = O(\Delta|Q|)$, then for any $Q \leq -\omega(n)n^{-1/2}\Delta^{1/2}$,

$$L_1(\mathbb{G}_n(\mathbf{d}_n)) = O\left(\frac{\Delta}{|Q|}\log\left(\frac{nQ}{\Delta}\right)\right) . \tag{1.15}$$

Remark 1.11. If $Q_0^2 n = O(m_0)$, then the behaviour of $\mathbb{G}_n$ is no longer (barely) subcritical. It is interesting to study the size of the largest component in this case.

We finally provide the existence of infinitely many degree sequences showing the tightness of some of our upper bounds (see Theorem 1.13).

Proposition 1.12. *For any $Q < 0$, $\Delta = o(\sqrt{n})$ and $\log n = o(\Delta)$, there exists a degree sequence $\widehat{\mathbf{d}}_n$ with $\Delta_n(\widehat{\mathbf{d}}_n) = \Delta$, $Q_n(\widehat{\mathbf{d}}_n) \sim Q$ and $R = R_n(\widehat{\mathbf{d}}_n) \sim \Delta$, such that*

$$\mathbb{P}\left(L_1(\mathbb{G}_n(\widehat{\mathbf{d}}_n)) \geq (1 + o(1))\frac{2R}{Q^2}\log\left(\frac{n}{R^2}\right)\right) = 1 - o(1) .$$

The degree sequence in the previous theorem is obtained by having roughly $(1 + Q)n/\Delta^2$ vertices of degree Δ and the rest of degree 1.

Remark 1.13. We can compare the lower bound in Theorem 1.12 with our upper bounds. The degree sequence $\widehat{\mathbf{d}}_n$ satisfies $R \sim \Delta$, so $\Delta|Q| = O(R)$. In the case $\Delta < n^{1/2-\delta}$ for some constant $\delta > 0$, the proposition gives a family of degree sequences for which Eq. (1.13) is of the right order.

While Theorem 1.12 is only stated for Q bounded away from zero, one could similarly define degree sequences $\widehat{\mathbf{d}}_n$ for which $Q = o(1)$, in which case $\Delta|Q| = o(R)$. Provided that $Q \leq -\omega(n)n^{-1/3}R^{2/3}$ for some $\omega(n) \to \infty$, one obtains the lower bound in Eq. (5.8) that coincides asymptotically with Theorem 1.3 and, up to logarithmic terms, with Eq. (1.14). (See Theorem 5.3.)

1.1 Previous work

The foundational paper of Erdős and Rényi [8] located the phase transition for the existence of a linear order component in a uniformly chosen graph on n vertices and m edges, $\mathbb{G}(n,m)$, showing that the order of the largest component undergoes a double jump at $m = n/2$, in particular $L_1(\mathbb{G}(n,m)) = O(\log n)$ if $m \leq cn$ and $c < 1/2$, $L_1(\mathbb{G}(n,m)) = \Theta(n^{2/3})$ if $m = n/2$, and $L_1(\mathbb{G}(n,m)) = \Theta(n)$ if $m \leq cn$ and $c > 1/2$. This result can be easily transferred to the Binomial random graph $\mathbb{G}(n,p)$ with $p = 2m/n$, which has become the reference model for random graphs. The size of the largest component in all regimes is well understood, see e.g. Sections 4 and 5 in [12].

The study of the phase transitions in random graphs with given degree sequences was pioneered by Molloy and Reed [20]. The so-called *Molloy-Reed criterion* determines the phase transition at $Q = 0$, provided that the degree sequence satisfies a number of technical conditions. The criterion has been extended to degree sequences with bounded degree variance [16] and uniformly integrable sequences [2], providing the asymptotic value of L_1 in the supercritical regime $Q > 0$ in terms of the survival probability of an associated branching process, similarly as in the $\mathbb{G}(n,p)$ case. Interestingly, the criterion is no longer valid for general degree sequences due to the presence of high degree vertices (hubs) or an extremely large number of degree 2 vertices. Joos et al. [18] gave an extended criterion that determines whether any given degree sequence typically produces a linear order component.

While the behaviour of the largest component in the supercritical regime resembles the simpler Erdős-Rényi model, this does not happen in the subcritical one, when $Q < 0$. Trivially, we have $L_1(\mathbb{G}_n) \geq \Delta + 1$ which could be much larger than logarithmic. In [20] the authors showed that $L_1(\mathbb{CM}_n) = O(\Delta^2 \log n)$ for subcritical sequences. More precise results are known for power-law degree sequences. Durrett [6] conjectured[1] that if $\mathbb{P}(D_n = k) \sim ck^{-\gamma}$ for some $\gamma > 3$ and $c > 0$, then $L_1(\mathbb{CM}_n) = O(\Delta)$. In this setting, $\gamma > 3$ implies $\mathbb{E}\left[D_n^{\gamma-1}\right] = O(1)$. Pittel [25] showed that $L_1(\mathbb{CM}_n) = O(\Delta \log n)$ for subpower-law distributions. Janson [14] proved a strong version of the conjecture: if $\mathbb{P}(D_n \geq k) = O(k^{1-\gamma})$ for some $\gamma > 3$, then

$$L_1(\mathbb{CM}_n) = \frac{\Delta}{|Q|} + o(n^{1/(\gamma-1)}) \tag{1.16}$$

For power-law distributions, we have $\Delta = \Theta(n^{1/(\gamma-1)})$ with high probability, and the second term is negligible. From the intuitive point of view, the largest component is obtained by starting a subcritical branching process with expected offspring $1 + Q$ from each vertex adjacent to the vertex of largest degree. The expected total progeny of such process is $1/|Q|$. One can interpret the result of Theorem 1.8 in a similar spirit: in the largest component there might be at most $O(m_0)$ edges and from each of these edges a piece of size $O(1/|Q_0|)$ hangs. For $Q < 0$, (1.16) can be compared to the weaker bound Eq. (1.13) that holds regardless of the shape of the degree sequence tail. As shown in Theorem 1.10, for general degree sequences Eq. (1.13) cannot be improved.

The critical regime has attracted a lot of interest in recent years [3, 4, 11, 13, 19, 26] with several papers specialising on the finite second or finite third moment cases. Here we focus on the results known for the barely subcritical regime.

Riordan [26] showed that if $\Delta = O(1)$ then Eq. (1.4) holds, even more, one has asymptotic equality and control on the second order term. Theorem 1.5 can be seen as a generalization of the upper bound in [26] to a wider class of sequences that allows $\Delta \to \infty$ as $n \to \infty$.

Hatami and Molloy [11] studied the critical window under some mild conditions on the degree sequence. They showed that $|Q| = O(n^{-1/3}R^{2/3})$ is the critical window of $\mathbb{CM}_n$. Regarding the barely subcritical regime, for $Q \leq -\omega(n)n^{-1/3}R^{2/3}$ with $\omega(n) \to \infty$, they showed that

$$L_1(\mathbb{CM}_n) = O\left(\sqrt{\frac{n}{|Q|}}\right) . \tag{1.17}$$

One can check that Eq. (1.17) coincides in order with Eq. (1.4) at the boundary of the critical window $|Q| = \Theta(n^{-1/3}R^{2/3})$, while Eq. (1.4) improves Eq. (1.17) in the whole barely subcritical regime, provided that Theorem 1.1 holds.

Under infinite variance, the probability of $\mathbb{CM}_n$ being simple can be exponentially small in n. Thus, only results that hold with exponentially high probability can be transferred from $\mathbb{CM}_n$ to $\mathbb{G}_n$, see e.g. [2]. Another approach is to study $\mathbb{G}_n$ directly using the *switching method* [18]. With both strategies, the best bound given in the subcritical regime is $L_1(\mathbb{G}_n) = o(n)$. Theorem 1.8 provides the first explicit general bound to L_1 at subcriticallity for infinite variance degree sequences. As discussed in Theorem 1.13, this bound cannot be substantially improved without further assumptions.

[1] In fact, this was conjectured for a slightly different model where the degrees are i.i.d. copies of D_n conditioned on their sum being even.

It thus remains as an open question to determine the exact size of the largest component in the (barely) subcritical regime. Hofstad, Janson and Łuczak conjectured that $L_1(\mathbb{CM}_n)$ is concentrated in this regime [13]. Supported by the result of Riordan for constant maximum degree, we conjecture that the upper bound in Eq. (1.8) is asymptotically tight for all degree sequences that satisfy some mild assumptions. Note that certain conditions on the degree sequence are needed; for instance, in Theorem 5.2 we exhibit a degree sequence for which the second moment does not converge to the second moment of the limit distribution, and $L_1(\mathbb{CM}_n)$ is non-concentrated.

1.2 Structure of the paper

The paper is organized as follows. In Section 2 we prove Theorem 1.3 and Theorem 1.5. This is done by reducing the problem to computing the probability that a sum of independent random variables (that approximates the exploration process) takes a particular value, in a similar way as in [23, 26]. Later, in Section 4, building upon results of Mukhin [22], we develop a novel local limit theorem to estimate such a probability. In Section 3 we prove Theorem 1.8 on the uniform model. As we consider graphs with infinite variance, we cannot apply Theorem 1.7 to transfer the results from the configuration to the uniform model. Instead, we take a more combinatorial approach using the switching method, similar to the arguments used in [18]. Finally, in Section 5 we prove Theorem 1.12.

2 Barely subcritical regime for the configuration model

2.1 Exploration process

In this section we introduce a process that given a vertex $v \in [n]$ explores $\mathbb{CM}_n$ starting by the component containing v. We set a total order of the half-edges as follows. For every vertex v, consider an arbitrary order of its d_v half-edges. Then, the half-edges are ordered, first by its corresponding vertex (using the total order on $[n]$) and then by the order given within the half-edges incident to a vertex.

The process constructs $\mathbb{CM}_n$ by, at each step, pairing two unpaired half-edges, resulting in a new edge of the graph. We will denote by $\mathcal{F}_t$ the history of the process at time t. With a slight abuse of notation, we treat $\mathcal{F}_t$ as the subgraph formed by the partial matching at time t and the vertices that are incident to the edges in the matching. Given the total order of the half-edges, v and the subgraph $\mathcal{F}_t$, we can recover the whole history of the process at time t, which justifies the abuse of notation.

The main random variable we would like to track is $X_t = X_t(v)$, defined as the number of unmatched half-edges incident to $V(\mathcal{F}_t)$ when the process started at v. Note that if $X_t = 0$, there are no unpaired half-edges and thus $\mathcal{F}_t$ is a union of components of $\mathbb{CM}_n$ containing the component of v.

The *exploration process of* $\mathbb{CM}_n$ *starting at* $v \in [n]$ is defined as follows:

1) Let $\mathcal{F}_0$ be the single-vertex graph on $\{v\}$ and $X_0 = d_v$.
2) While $V(\mathcal{F}_t) \neq [n]$,
   - 2a) If $X_t = 0$, choose a uniformly unmatched half-edge and let u be the vertex incident to it. Let $\mathcal{F}_{t+1}$ be constructed from $\mathcal{F}_t$ by adding $\{u\}$ as an isolated vertex, and let $X_{t+1} = d_u$.
   - 2b) Otherwise, choose the smallest unmatched half-edge e incident to $V(\mathcal{F}_t)$ and pair it with a half-edge f chosen uniformly at random from all the unmatched ones. Let u be the vertex incident to f.
     - i) If $u \notin V(\mathcal{F}_t)$, let $\mathcal{F}_{t+1}$ be constructed from $\mathcal{F}_t$ by adding vertex u and edge ef and let $X_{t+1} = X_t + d_u - 2$.

ii) Otherwise, let $\mathcal{F}_{t+1}$ be constructed from $\mathcal{F}_t$ by adding edge ef and let $X_{t+1} = X_t - 2$.

Note that X_t is measurable with respect to $\mathcal{F}_t$. We define the following parameters:

$$\begin{aligned} \eta_{t+1} &\coloneqq X_{t+1} - X_t \ , \\ M_t &\coloneqq X_t + \sum_{u \notin V(\mathcal{F}_t)} d_u \ , \\ Q_t &= \frac{1}{M_t - 1} \sum_{u \notin V(\mathcal{F}_t)} d_u(d_u - 2) \ , \\ R_t &= \frac{1}{M_t - 1} \sum_{u \notin V(\mathcal{F}_t)} d_u(d_u - 2)^2 \ . \end{aligned} \tag{2.1}$$

It is straightforward to check that if $X_t > 0$, then

$$\mathbb{E}\left[\eta_{t+1} \mid \mathcal{F}_t\right] = Q_t \quad \text{and} \quad \mathbb{E}\left[(\eta_{t+1})^2 \mid \mathcal{F}_t\right] = R_t \ . \tag{2.2}$$

2.2 Stochastic domination and random sums

Recall the definition of $T = T_n$ given in Eq. (1.7).

Define the random variable β as follows: for every $\ell \in L \coloneqq \{-1, 0, 1, \dots, n-3\}$,

$$\mathbb{P}(\beta = \ell) \coloneqq \begin{cases} \frac{m}{m-4T}\,\mathbb{P}(\eta = -1) - \frac{4T}{m-4T} & \text{if } \ell = -1 \ , \\ \frac{m}{m-4T}\,\mathbb{P}(\eta = \ell) & \text{if } \ell \geq 0 \ . \end{cases} \tag{2.3}$$

The random variable β is defined in such a way it stochastically dominates η_t during the first $2T$ steps of the exploration process, regardless of the particular exploration.

Let Q_β, R_β, $\varphi_\beta(\theta)$, θ_0^β, and T_β be defined as in (1.5)–(1.7) replacing η by β. By the choice of β all main parameters are asymptotically equal to the original ones, as the following result demonstrates.

Lemma 2.1. *For every $k \geq 0$, we have $\varphi_\beta^{(k)}(\theta) = (1 + o(\theta_0))\varphi^{(k)}(\theta) + o(\theta_0)$. Moreover, $Q_\beta = (1 + o(1))Q$, $R_\beta = (1 + o(1))R$ and $T_\beta = (1 + o(1))T$.*

We postpone the proof of the lemma until the end of the section.

Let $(\beta_t)_{t\geq 1}$ be a sequence of iid copies of β. For $s \in N$, define the stochastic process $W_t = W_t^s$ by $W_0 = s$ and for $t \geq 0$

$$W_{t+1} = W_t + \beta_t = s + \sum_{i=1}^{t} \beta_i \ . \tag{2.4}$$

Define the stopping time

$$\tau_W^s \coloneqq \inf\{t : W_t^s = 0\} \ .$$

In order to study τ_W^s we will need a precise control on the probability that W_t takes a particular value. Thus, we introduce the following local limit theorem, that will be proved in Section 4.

Theorem 2.2. *Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables with step h and taking values on $\mathcal{L}(v_0, h)$ for some $v_0 \in \mathbb{N}$. Define $S_n = \sum_{i=1}^n X_i$. Suppose that $\mu = \mathbb{E}(X_1) = 0$, $\sigma^2 = \mathrm{Var}(X_1)$ and $\gamma = \mathbb{E}|X_1|^3$. Then,*

$$\sup_{w \in \mathcal{L}(nv_0, h)} \left| \mathbb{P}(S_n = w) - \frac{h}{\sqrt{2\pi n \sigma^2}} \exp\left(-\frac{w^2}{2n\sigma^2}\right) \right| \leq \frac{32h\gamma}{\sigma^4 n} + \frac{\pi^{3/2}\gamma}{h\sigma^2 n H_h(X_1)}. \tag{2.5}$$

We are now ready to prove the main technical result of this part, which bounds the probability that the stopping time takes a particular value.

Lemma 2.3. *For every $t \geq T_\beta$ and $s = s(n)$ we have*

$$\mathbb{P}(\tau_W^s = t) \leq 2h \cdot s e^{\theta_0^\beta s} \left(\varphi_\beta''(\theta_0^\beta)\right)^{-1/2} \frac{(\varphi_\beta(\theta_0^\beta))^t}{t^{3/2}} , \tag{2.6}$$

where h is the step of D.

Moreover, for every $\epsilon > 0$ we have that

$$\mathbb{P}(\tau_W^s \geq (1+\epsilon)T_\beta) = o\left(\frac{s e^{\theta_0^\beta s}}{\mathbb{E}\left[D_n e^{\theta_0^\beta D_n}\right]} \cdot \frac{T_\beta}{n}\right) . \tag{2.7}$$

Proof. The dependence on s is implicit in all the notation below. Recall that $L \coloneqq \{-1, 0, 1, \ldots, n-3\}$. Define the following sequences

$$\begin{aligned} I_t &= \{\mathbf{b} = (b_1, \ldots, b_t) \in L^t : s + b_1 + \cdots + b_t = 0\} \\ \hat{I}_t &= \{\mathbf{b} = (b_1, \ldots, b_t) \in I_t : s + b_1 + \cdots + b_i > 0, \forall i \in [t-1]\} \end{aligned} \tag{2.8}$$

We can write

$$\mathbb{P}(W_t = 0) = \sum_{\mathbf{b} \in I_t} \prod_{i=1}^t \mathbb{P}(\beta_i = b_i) \quad \text{and} \quad \mathbb{P}(\tau_W^s = t) = \sum_{\mathbf{b} \in \hat{I}_t} \prod_{i=1}^t \mathbb{P}(\beta_i = b_i) \tag{2.9}$$

A variant of Spitzer's lemma [23, Lemma 9] implies that

$$\mathbb{P}(\tau_W^s = t) \leq \frac{s}{t} \mathbb{P}(W_t = 0) \tag{2.10}$$

We use exponential tilting to bound the probability that $W_t = 0$, as in [23, 26]. Consider the random variable β_θ defined for $\ell \in L$ by

$$\mathbb{P}(\beta_\theta = \ell) = \frac{e^{\theta \ell} \mathbb{P}(\beta = \ell)}{\varphi_\beta(\theta)} . \tag{2.11}$$

Let $(\beta_{\theta,t})_{t \geq 1}$ be a sequence of iid copies of β_θ. Define the stochastic process $W_{\theta,t}$ by $W_{\theta,0} = s$ and for $t \geq 0$

$$W_{\theta,t+1} = s + \sum_{i=1}^t \beta_{\theta,i} . \tag{2.12}$$

Algebraic manipulations give

$$\mathbb{P}(W_t = 0) = (\varphi_\beta(\theta))^t e^{\theta s} \mathbb{P}(W_{\theta,t} = 0) . \tag{2.13}$$

By definition of θ_0^β, $\mathbb{E}\left[\beta_{\theta_0^\beta}\right] = \mathbb{E}\left[\beta e^{\beta \theta_0^\beta}\right] = 0$. We may write $W_{\theta_0^\beta, t} = s + S_t$, where $S_t = \sum_{i=1}^t Y_i$ and $(Y_i)_{i \in [t]}$ is a collection of iid copies of $\beta_{\theta_0^\beta}$. In particular, we have

$$\begin{aligned} \mu &= \mathbb{E}[Y_1] = \varphi_\beta'(\theta_0^\beta) = 0 , \\ \sigma^2 &= \mathbb{E}\left[Y_1^2\right] = \frac{\varphi_\beta''(\theta_0^\beta)}{\varphi_\beta(\theta_0^\beta)} , \\ \gamma &= \mathbb{E}\left[|Y_1|^3\right] \leq 2 + \mathbb{E}\left[Y_1^3\right] = \frac{2\varphi_\beta(\theta_0^\beta) + \varphi_\beta'''(\theta_0^\beta)}{\varphi_\beta(\theta_0^\beta)} . \end{aligned} \tag{2.14}$$

where in the inequality, we have used that $Y_1 \geq -1$.

We will apply Theorem 2.2 and show that the error term is negligible with respect to the Gaussian probability. Recall that h is the step of the limiting distribution D, which by Theorem 1.1(iv) is also the step of the distribution of Y_1. Since h and $H_h(Y_1)$ (as defined in (4.16)) are constants, the order of the first error term in (2.5) is at most the order of the second one, and it suffices to bound the latter. Theorem 1.1(iii) implies that $\sigma^2 \geq \mathbb{P}(\hat{D}_n \neq 2) > 0$ for large n, and that $\gamma = O(\Delta^{1/2})$. Therefore, for any $t \geq T_\beta$

$$\frac{\gamma}{\sigma^2 t} = O\left(\frac{1}{\sqrt{\sigma^2 t}} \cdot \sqrt{\frac{\Delta}{T_\beta}}\right) = o\left(\frac{1}{\sqrt{\sigma^2 t}}\right).$$

where we used that $\Delta = o(T)$ and $T \sim T_\beta$ by Theorem 2.1.

Since $\mathbb{P}(Y_1 = -1) > 0$, we may choose $v_0 = -1$. Thus, for sufficiently large n, we conclude that for any $w \in \mathcal{L}(-t, h)$,

$$\mathbb{P}(S_t = w) \leq \frac{2h}{\sqrt{2\pi t \sigma^2}} \tag{2.15}$$

We can now use (2.15) with $w = -s$ to obtain

$$\mathbb{P}(W_{\theta_0,t} = 0) = \mathbb{P}(S_t = -s) \leq \frac{2h}{\sqrt{2\pi t}} \left(\frac{\varphi_\beta(\theta_0^\beta)}{\varphi_\beta''(\theta_0^\beta)}\right)^{1/2} \tag{2.16}$$

Let us show that $\varphi_\beta(\theta_0^\beta)$ is close to 1. On the one hand, we will use the inequality $e^x \leq 1 + xe^x$ for all $x \in \mathbb{R}$, with equality if and only if $x = 0$. Since $\mathbb{P}(\beta = 0) \neq 1$, by the choice of θ_0^β

$$\varphi_\beta(\theta_0^\beta) = \mathbb{E}\left[e^{\theta_0^\beta \beta}\right] < 1 + \theta_0^\beta \mathbb{E}\left[\beta e^{\theta_0^\beta \beta}\right] = 1 \tag{2.17}$$

On the other hand, using $e^x \geq 1 + x$ for $x \in \mathbb{R}$, that θ_0^β is bounded as $n \to \infty$ and $\mathbb{E}[\beta] = o(1)$, we obtain

$$\varphi_\beta(\theta_0^\beta) \geq 1 + \theta_0^\beta \mathbb{E}[\beta] = 1 + o(1) . \tag{2.18}$$

We conclude that $\varphi_\beta(\theta_0^\beta) \sim 1$ and thus, we use have the asymptotic equivalence

$$|\log \varphi_\beta(\theta_0^\beta)| \sim 1 - \varphi_\beta(\theta_0^\beta). \tag{2.19}$$

Combining Eqs. (2.10), (2.13), (2.16) and (2.17), we obtain

$$\mathbb{P}(\tau_W^s = t) \leq 2h \cdot s e^{\theta_0^\beta s} \left(\varphi_\beta''(\theta_0^\beta)\right)^{-1/2} \frac{(\varphi_\beta(\theta_0^\beta))^t}{t^{3/2}} .$$

proving the first part of the lemma.

For the second statement of the lemma, it suffices to prove it for small enough ϵ, so we may assume $\epsilon \in (0, 1)$. Observe that $\mathbb{P}(\tau_W^s = t) \neq 0$ implies that $s = hk - t$ for some $k \in \mathbb{Z}$. Since $v_0 = -1$ and h are coprime, there are at most $\lceil T/h \rceil$ values $t \in [T]$ such that $\mathbb{P}(\tau_W^s = t) \neq 0$.

As our bound on $\mathbb{P}(\tau_W^s = t)$ is decreasing in t, and $h = o(T_\beta)$ we have

$$\mathbb{P}(\tau_W^s \geq (1+\epsilon)T_\beta) = \sum_{\substack{t \geq (1+\epsilon)T_\beta \\ t \in \mathcal{L}(-1,h)}} \mathbb{P}(\tau_W^s = t) \leq \sum_{t \geq (1+\epsilon)T_\beta - h} \frac{\mathbb{P}(\tau_W^s = t)}{h} \leq \sum_{t \geq (1+\epsilon/2)T_\beta} \frac{\mathbb{P}(\tau_W^s = t)}{h} .$$

Using (2.17) for the geometric sum and (2.19), it follows that

$$\mathbb{P}(\tau_W^s \geq (1+\epsilon)T_\beta) \leq 2se^{\theta_0^\beta s}\left(\varphi_\beta''(\theta_0^\beta)\right)^{-1/2}\frac{(\varphi_\beta(\theta_0^\beta))^{(1+\epsilon/2)T_\beta}}{T_\beta^{3/2}}\sum_{\ell\geq 0}(\varphi_\beta(\theta_0^\beta))^\ell$$
$$\sim 2se^{\theta_0^\beta s}\left(\varphi_\beta''(\theta_0^\beta)\right)^{-1/2}\frac{(\varphi_\beta(\theta_0^\beta))^{(1+\epsilon/2)T_\beta}}{|\log\varphi_\beta(\theta_0^\beta)|T_\beta^{3/2}}\ . \tag{2.20}$$

Using the definition of T in (1.7) and Theorem 2.1, we obtain that

$$T_\beta \sim \frac{1}{|\log\varphi_\beta(\theta_0^\beta)|}\log\left(|\log\varphi_\beta(\theta_0^\beta)|^{3/2}(\varphi_\beta''(\theta_0^\beta))^{-1/2}\mathbb{E}\left[D_n e^{\theta_0^\beta D_n}\right]n\right)$$
$$\sim \frac{1}{|\log\varphi_\beta(\theta_0^\beta)|}\log\left(T_\beta^{-3/2}(\varphi_\beta''(\theta_0^\beta))^{-1/2}\mathbb{E}\left[D_n e^{\theta_0^\beta D_n}\right]n\right)\ .$$

Using that $(\varphi_\beta(\theta_0^\beta))^{(1+\epsilon/2)T_\beta} = o\left(\left(T_\beta^{-3/2}(\varphi_\beta''(\theta_0^\beta))^{-1/2}\mathbb{E}\left[D_n e^{\theta_0^\beta D_n}\right]n\right)^{-1}\right)$, noting that $|\log\varphi_\beta(\theta_0^\beta)|^{-1} = O(T_\beta)$, and plugging the bounds in (2.20) we obtain

$$\mathbb{P}(\tau_W^s \geq (1+\epsilon)T_\beta) = o\left(\frac{se^{\theta_0^\beta s}}{\mathbb{E}\left[D_n e^{\theta_0^\beta D_n}\right]}\cdot\frac{T_\beta}{n}\right)\ .$$

□

2.3 Proof of Theorem 1.5

Fix $\epsilon > 0$ sufficiently small and $v \in [n]$. Define the following stopping time with respect to the filtration given by $\mathcal{F}_t$ starting at v,

$$\tau_X(v) := \inf\{t:\ X_t(v) = 0\}\ .$$

Note that $\tau_X(v)$ is the number of edges in the component of v, denoted by $\mathcal{C}(v)$. Moreover, for every $t \leq 2T \wedge \tau_X(v)$ the random variable β stochastically dominates η_t. Thus, $X_t(v)$ is stochastically dominated by $W_t^{d_v}$.

Let $\delta = \epsilon/3$. It follows from Theorems 2.1 and 2.3 that

$$\mathbb{P}(\tau_X(v) \geq (1+2\delta)T) \leq \mathbb{P}(\tau_X \geq (1+\delta)T_\beta) \leq \mathbb{P}(\tau_W^s \geq (1+\delta)T_\beta) = o\left(\frac{d_v e^{\theta_0^\beta d_v}}{\mathbb{E}\left[D_n e^{\theta_0^\beta D_n}\right]}\cdot\frac{T}{n}\right)\ . \tag{2.21}$$

Let Z be the number of components of order at least $(1+\epsilon)T$. For any $\epsilon > 0$, we can write

$$Z = \sum_{\mathcal{C}}\mathbf{1}_{|\mathcal{C}|\geq(1+\epsilon)T} = \sum_{v\in[n]}\frac{\mathbf{1}_{|\mathcal{C}(v)|\geq(1+\epsilon)T}}{|\mathcal{C}(v)|} \leq \frac{1}{T}\sum_{v\in[n]}\mathbf{1}_{|\mathcal{C}(v)|\geq(1+\epsilon)T}\ , \tag{2.22}$$

where the first sum is over the connected components of $\mathbb{CM}_n$.

Since $\mathcal{C}(v)$ is a connected subgraph, it has at least $|\mathcal{C}(v)|-1$ edges. Thus, the probability of $|\mathcal{C}(v)| \geq k$ is bounded from above by the probability $\tau_X(v) \geq k-1$. Using Eq. (2.21) we obtain

$$\mathbb{E}[Z] \leq \frac{1}{T}\sum_{v\in[n]}\mathbb{P}(\tau_X(v) \geq (1+\epsilon)T-1) = o\left(\frac{1}{\mathbb{E}\left[D_n e^{\theta_0^\beta D_n}\right]n}\right)\sum_{v\in[n]}d_v e^{\theta_0^\beta d_v} = o(1)\ .$$

Theorem 1.5 follows by Markov's inequality on Z.

2.4 Proof of Theorem 1.3

Recall that $\Delta|Q| = o(R)$ and that $\varphi(\theta)$ is the moment generating function of η. Thus, $\varphi(0) = 1$, $\varphi'(0) = Q$, $\varphi''(0) = R$ and $\varphi^{(k)}(0) \le \Delta^{k-3}R$ for all $k \ge 3$. This implies that the radius of convergence of φ (and so of any of its derivatives) is at least $2|Q|/R$. So, for any θ with $|\theta| < 2|Q|/R$, we have

$$\varphi'(\theta) = \varphi'(0) + \theta\varphi''(0) + O(\theta^2\varphi'''(0)) = Q + \theta R + o(Q)\ .$$

By the choice of θ_0, we have $\varphi'(\theta_0) = 0$ and $\theta_0 \sim |Q|/R$, so we can also write

$$\varphi(\theta_0) = \varphi(0) + \theta_0\varphi'(0) + \frac{\theta_0^2\varphi''(0)}{2} + o(1) \sim 1 - \frac{Q^2}{2R}\ .$$

and

$$|\log\varphi(\theta_0)| \sim \frac{Q^2}{2R}\ .$$

By considering the Taylor expansion of φ around θ_0, similar arguments give that $\varphi''(\theta_0) \sim R$.

Finally, observe that for any $\delta > 0$,

$$\mathbb{E}\left[De^{D\theta_0}\right] \le \mathbb{E}\left[De^{(1+\delta)\Delta Q/R}\right] = \mathbb{E}\left[De^{o(1)}\right] = O(1)\ . \tag{2.23}$$

Using all previous estimations, we can write

$$T \sim \frac{2R}{Q^2}\log\left(\frac{|Q|^3 n}{R^2}\right)\ .$$

It is straightforward to check that, in this case, the condition $\theta_0 m \ge \omega(n)T_n$ is equivalent to $Q \le -\omega(n)n^{-1/3}R^{2/3}$.

Note that the condition $\Delta_n \le n^{1/6}$ is only required in Theorem 2.1 in the case $R = O(\Delta|Q|)$. Thus, the desired result follows from Theorem 1.5 without further restrictions on the degree sequence.

2.5 Proof of Theorem 2.1

The first part of the lemma follows directly from

$$\begin{aligned}\varphi_\beta^{(k)}(\theta) = \mathbb{E}(\beta^k e^{\theta\beta}) &= \frac{m}{m-4T}\mathbb{E}(\eta^k e^{\theta\eta}) - \frac{4T}{m-4T}(-1)^k e^{-\theta} \qquad (2.24)\\ &= (1+O(T/m))\varphi^{(k)}(\theta) + O(T/m)\\ &= (1+o(\theta_0))\varphi^{(k)}(\theta) + o(\theta_0). \qquad (2.25)\end{aligned}$$

where in the last line we used the hypothesis $T = o(m\theta_0)$ in Theorem 1.5.

Using $e^x \ge 1 + x$, we have

$$0 = \mathbb{E}[\eta e^{\theta_0\eta}] \ge \mathbb{E}[\eta(1+\theta_0\eta)] = Q + \theta_0 R$$

and, as R is bounded away from zero, by Theorem 1.1(iii)

$$0 \le \theta_0 \le \frac{|Q|}{R} = O(Q) = o(1). \tag{2.26}$$

Since $Q_\beta = \mathbb{E}(\beta)$ and $R_\beta = \mathbb{E}(\beta^2)$, it follows from (2.25) with $\theta = 0$ and, $k = 1$ and $k = 2$, respectively, and from (2.26) that

$$Q_\beta = (1+o(1))Q \quad \text{and } R_\beta = (1+o(1))R. \tag{2.27}$$

For the second part, we split into two cases. If $\Delta|Q| = o(R)$, we are in the setting of Theorem 1.3. In such case

$$|\log \varphi_\beta(\theta_0^\beta)| \sim \frac{Q_\beta^2}{2R_\beta} \sim \frac{Q^2}{2R} \sim |\log \varphi(\theta_0)|,$$

(see Section 2.4 for the first and third equivalences) and the result follows from the first part of the lemma.

Otherwise $R = O(\Delta|Q|)$. As R is bounded away from zero by Theorem 1.1(iii) and $\Delta_n \leq n^{1/6}$, it follows that $|Q|$ is of order at least $n^{-1/6}$. Again all we need to show is that $|\log \varphi_\beta(\theta_0^\beta)| \sim |\log \varphi(\theta_0)|$ and then the rest will follow by the first part of the lemma. We do this by bounding $\varphi(\theta_0) - \varphi_\beta(\theta_0^\beta)$.

By construction, β stochastically dominates η and it follows that $\varphi_\beta^{(k)}(\theta) \geq \varphi^{(k)}(\theta)$ for all $k \geq 0$ and $\theta \geq 0$. In particular,

$$0 < \theta_0^\beta \leq \theta_0 \tag{2.28}$$

Combining (2.24) for $k = 0$, (2.26) and (2.28),

$$\varphi_\beta(\theta_0^\beta) - \varphi(\theta_0^\beta) = \frac{4T}{m - 4T}(\varphi(\theta_0^\beta) - e^{-\theta_0^\beta}) \leq \frac{4T\theta_0}{m}(1 + o(1)) = O\left(\frac{T|Q|}{mR}\right). \tag{2.29}$$

As φ_β'' is an increasing function with $\varphi_\beta''(0) = (1 + o(1))R$ and $\varphi_\beta'(\theta_0^\beta) = 0$, the fundamental theorem of calculus implies

$$(\theta_0 - \theta_0^\beta)R \leq (1 + o(1)) \int_{\theta_0^\beta}^{\theta_0} \varphi_\beta''(t)dt = (1 + o(1))\varphi_\beta'(\theta_0) = O\left(\frac{T}{m}\right). \tag{2.30}$$

where the last equality follows from (2.25).

We have $|\varphi'(t)| \leq |Q|$ for all $t \in [0, \theta_0]$; indeed, φ' is increasing with $\varphi'(0) = Q$ and $\varphi'(\theta_0) = 0$, Similarly as before, using (2.30) we conclude that

$$\varphi(\theta_0) - \varphi(\theta_0^\beta) = \int_{\theta_0^\beta}^{\theta_0} \varphi'(t)dt \leq (\theta_0 - \theta_0^\beta)|Q| = O\left(\frac{T|Q|}{mR}\right). \tag{2.31}$$

Combining (2.29) and (2.31),

$$\varphi_\beta(\theta_0^\beta) - \varphi(\theta_0) = O\left(\frac{T|Q|}{mR}\right). \tag{2.32}$$

Recall that $\varphi'(0) = Q < 0$. Using the inequality $e^x \leq 1 + 2x$ for $x \in [0, 1]$, we have

$$\varphi'\left(\frac{|Q|}{5\Delta}\right) \leq \mathbb{E}\left[\eta\left(1 + \frac{2|Q|\eta}{5\Delta}\right)\right] = Q + \frac{2|Q|R}{5\Delta} < 0$$

where the final inequality holds because $Q < 0$ implies $R < 2\Delta$. It follows that $\theta_0 \geq \frac{|Q|}{5\Delta}$.

By using the fact that $\varphi'(\theta) < 0$ for all $\theta \in [0, \theta_0)$ and Taylor expansion of $\varphi(\theta)$ around $\theta = 0$, we obtain

$$\begin{aligned}\varphi(\theta_0) \leq \varphi\left(\frac{|Q|}{5\Delta}\right) &\leq 1 - \frac{Q^2}{5\Delta} + \sum_{k\geq 2} \frac{\mathbb{E}[\eta^k]}{k!} \cdot \left(\frac{|Q|}{5\Delta}\right)^k \\ &\leq 1 - \frac{Q^2}{5\Delta} + \frac{2Q^2}{25\Delta} \sum_{\ell \geq 0} \left(\frac{|Q|}{5}\right)^\ell \\ &\leq 1 - \frac{Q^2}{10\Delta}. \end{aligned} \tag{2.33}$$

where in the third inequality we used that $\mathbb{E}[\eta^k] \le 2\Delta^{k-1}$ for all $k \in \mathbb{N}$, since $Q < 0$.

Using the bound (2.33) in the definition of T gives the simple upper bound $T \le \frac{10\Delta}{Q^2}\log(n)$. By our bounds on Δ and Q, $T^2|Q| = O(n^{5/6}\log n) = o(mR)$. Thus, substituting this into (2.32), we have

$$\varphi_\beta(\theta_0^\beta) - \varphi(\theta_0) = o\left(\frac{1}{T}\right). \tag{2.34}$$

By definition, $|\log\varphi(\theta_0)| \ge 1/T$. Therefore,

$$\varphi(\theta_0) \le e^{-1/T} = 1 - \frac{1+o(1)}{T} \tag{2.35}$$

Combining (2.34) and (2.35),

$$|\log\varphi_\beta(\theta_0^\beta)| \sim 1 - \varphi_\beta(\theta_0^\beta) \sim 1 - \varphi(\theta_0) \sim |\log\varphi(\theta_0)|,$$

concluding the proof of the lemma.

3 Subcritical regime for the uniform model

3.1 Exploration process

We will use the exploration process described in [18] that, given $V_0 \subset [n]$, reveals the components of $\mathbb{G}_n$ one by one starting with the components containing V_0.

We first describe the exploration process on a fixed graph where each vertex has an order in its adjacency list. Precisely, an *input* is a pair (G, Π), with G a graph on $[n]$ and $\Pi = (\pi_v)_{v\in[n]}$ a collection of permutations where π_v has length d_v and induces a natural order on the edges incident to v. The process constructs a sequence of sets $V_0 \subset V_1 \subset \dots$ such that at time t all the edges in $G[V_t]$ have been revealed. Let $E(A, B)$ be the set of edges between sets A and B; we write $E(v, B)$ if $A = \{v\}$. Similarly as before, we define $X_t = X_t(v) = |E(V_t, [n] \setminus V_t)|$ to be the number of edges between the explored and unexplored parts. If $X_t = 0$, V_t is a set of vertices forming a union of components, including the ones intersecting V_0. We also define $M_t = \sum_{w\in[n]\setminus V_t} d_w$, and we let L_t be the number of leaves $v \in [n] \setminus V_t$, that is $d_v = 1$.

The *exploration process of* (G, Π) *starting at* $V_0 \subset [n]$ is defined as follows:

1) Let $X_0 = |E(V_0, [n] \setminus V_0)|$.
2) While $V_t \ne [n]$,
   - 2a) If $X_t = 0$, choose a vertex u in $[n] \setminus V_t$ with probability proportional to its degree and let $w_{t+1} = u$, i.e. $\mathbb{P}(w_{t+1} = u) = \frac{d_u}{M_t}$. Let $V_{t+1} = V_t \cup \{w_{t+1}\}$ so that $X_{t+1} = d_{w_{t+1}}$.
   - 2b) Otherwise, choose v_{t+1} the smallest vertex incident to at least one edge in $[n] \setminus V_t$ and let e_{t+1} be the smallest edge in $E(v_{t+1}, [n] \setminus V_t)$. Let w_{t+1} be the endpoint of e_{t+1} in $[n] \setminus V_t$. Reveal the existence of all edges in $E(w_{t+1}, V_t)$. Let $V_{t+1} = V_t \cup \{w_{t+1}\}$ so that $X_{t+1} = X_t - 1 + d_{w_{t+1}} - |E(w_{t+1}, V_t)|$.

There are two main differences between this exploration process and the one defined in Section 2.1: we explore vertex by vertex instead of edge by edge, and we start from a set instead of a single vertex.

We will run the exploration process on an input (G, Π) chosen uniformly at random from all the inputs where G is a graph on $[n]$ with degree sequence $\mathbf{d}_n$. This is equivalent to sampling $G \sim \mathbb{G}_n$ and, independently, letting $\Pi = \Pi(\mathbf{d}_n)$ be a collection of uniformly and independent permutations of lengths $(d_v)_{v\in[n]}$. We will use the *principle of deferred decisions* exposing the restriction of $\pi_{v_{t+1}}$ to $E(v_{t+1}, [n] \setminus V_t)$ at time t. Let $(\mathcal{F}_t)_{t\ge0}$

be the filtration of the space of inputs given by the history of the process just after exposing the order on $E(v_{t+1}, [n] \setminus V_t)$. The random objects X_t, V_t, M_t, L_t, v_{t+1} and e_{t+1} are $\mathcal{F}_t$-measurable, while w_{t+1} is $\mathcal{F}_{t+1}$-measurable. We will use $\mathbb{P}_t(\cdot) := \mathbb{P}(\cdot \mid \mathcal{F}_t)$ and $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \mid \mathcal{F}_t]$ to denote respectively the probability and expected value conditioned to $\mathcal{F}_t$.

3.2 Deterministic properties of the process

Let $\mathbf{d}_n$ be an (m_0, Q_0)-subcritical degree sequence, as defined in (1.12). We will assume that $\mathbf{d}_n$, m_0 and Q_0 satisfy the conditions in Theorem 1.8, namely:

(C1) $m_0 \geq 3m_*$ (where m_* is defined in (1.11));

(C2) $m_0|Q_0| \geq (\Delta|Q_0| + R)\log\left(\frac{nQ_0^2}{\Delta|Q_0|+R}\right)$;

(C3) $Q_0^2 n \geq \omega(n)m_0$ for some $\omega(n) \to \infty$.

First of all, we may assume that $m_0 = o(n)$, as otherwise since $|Q_0| \leq 1$, there is nothing to prove. Let n_i be the number of vertices $v \in [n]$ with $d_v = i$. By (1.10), we have that

$$\sum_{\substack{w\in[n]\setminus S_* \\ d_w \geq 3}} d_w \leq \sum_{\substack{w\in[n]\setminus S_* \\ d_w \geq 3}} d_w(d_w - 2) \leq n_1 .$$

It follows that

$$\sum_{w\in[n]\setminus S_*} d_w \leq n_1 + 2n_2 + \sum_{\substack{w\in[n]\setminus S_* \\ d_w \geq 3}} d_w \leq 2(n_1 + n_2) \leq 2n.$$

By condition (C1) and since $m_0 = o(n)$, we conclude that $m \leq 3n$.

Define

$$T := \frac{m_0}{|Q_0|} \geq \frac{\Delta|Q_0| + R}{Q_0^2} \log\lambda \tag{3.1}$$

where,

$$\lambda := \frac{nQ_0^2}{\Delta|Q_0| + R} \geq \frac{\omega(n)}{|Q_0|} \to \infty , \tag{3.2}$$

by (C3).

Also by (C3) and since $n = \Theta(m)$, we have that $T = o(|Q_0|m)$; we will use this bound repeatedly throughout the proof.

Let S be the set that certifies (m_0, Q_0)-subcriticality as defined in (1.12). As remarked, we may assume that S is formed by vertices of largest degrees. Condition (C2) implies that $\Delta = o(m_0)$. So, we may also assume that

$$\sum_{v\in S} d_v \geq \frac{m_0}{2} , \tag{3.3}$$

as increasing the set S can only decrease Q_0.

Throughout the proof, we will assume that

$$\Delta' := \max_{w\in[n]\setminus S} d_w \geq 2 . \tag{3.4}$$

Otherwise there are at most m_0 vertices of degree at least 2 and any component has order at most $O(m_0)$ and we are done.

Given $v \in [n]$, we will run the exploration process by starting from $V_0 = S \cup \{v\}$. We now give some basic deterministic properties of the process with such initial set.

For any $t \in \mathbb{N}$, let $n_i(t)$ be the number of vertices $v \in [n] \setminus S_t$ with $d_v = i$.

Lemma 3.1. *If $V_0 = S \cup \{v\}$, then:*

1. $\sum_{u\in V_0} d_u \le 2|Q_0|T$;
2. $\Delta' = o\left(\frac{n_1}{\Delta}\right)$.

Moreover, for every $t = O(T)$ we have:

3. $n_1(t) \ge n_1/2$;
4. $M_t \ge m/3$.

Proof. For Item 1, just observe that $d_v \le \Delta \le m_0 = |Q_0|T$.

For Item 2, since condition (C1) holds, $\Delta' \ge 2$ and by Eqs. (1.10), (1.11) and (3.3), we have

$$0 \ge \sum_{v\in[n]\setminus S_*} d_v(d_v-2) \ge -n_1 + (\Delta'-2)\sum_{w\in S\setminus S_*} d_w \ge -n_1 + (\Delta'-2)(m_0/2 - m_*) \ ,$$

From here it follows that $\Delta' = O(n_1/m_0) = o(n_1/\Delta)$.

For Item 3, by (1.12) and (3.4) we can write

$$|Q_0|m \le -\sum_{v\in[n]\setminus S} d_v(d_v-2) \le n_1 - \sum_{\substack{v\in[n]\setminus S\\ d_v\ge 3}} d_v(d_v-2) \le n_1$$

Since $T = o(|Q_0|m)$, we have $n_1(t) \ge n_1 - 1 - t \ge n_1/2$.

For Item 4, by Eq. (1.10) we have

$$0 \ge -n_1 + \sum_{\substack{v\in[n]\setminus S_*\\ d_v\ge 3}} d_v = m - m_* - 2(n_1+n_2) \ .$$

Counting only the contribution of vertices of degree 1 or 2 to M_t, by the previous inequality we obtain

$$M_t \ge n_1 + 2n_2 - 2t \ge \frac{m-m_*}{2} - 2t \ge \frac{m}{3}$$

since $m_* \le m_0$ which we assumed is $o(m)$ and $t \le T = o(m)$.

□

3.3 Bounding the increments

We chose V_0 to contain all the large degree vertices so we have control on the maximum degree outside the initial set (Theorem 3.1). This allows us to estimate the probability of connecting to a vertex w at step t via switchings.

Lemma 3.2. *For any $v \in [n]$, any $t = O(T)$ and any $w \in [n]\setminus V_t$, we have*

$$\mathbb{P}_t(w_{t+1} = w) \le (1+o(1))\frac{d_w}{M_t} \ . \tag{3.5}$$

Moreover, if $d_w = 1$

$$\mathbb{P}_t(w_{t+1} = w) \ge (1+o(1))\frac{d_w}{M_t} \ . \tag{3.6}$$

Proof. The proof uses an edge-switching argument. A *switching* is a local operation that transforms an input into another one. Given an input (G,Π) and two oriented edges (a,b) and (c,d) with $ab, cd \in E(G)$ and $ac, bd \notin E(G)$, we obtain the new input by deleting the edges ab and cd, and adding the edges ac and bd. Note that this operation preserves the degree of each vertex and does not modify the permutations of the adjacency lists. Also note that we must ensure that ac and bd are non-edges, as otherwise the operation would not produce an input. We will restrict to switchings that do not modify the edges within V_t in order to switch between inputs in $\mathcal{F}_t$.

Fix $w \in [n] \setminus V_t$. If $X_t = 0$, then w is chosen with probability d_w/M_t, so we may assume that $X_t > 0$. Let v_{t+1}, e_{t+1} and w_{t+1} as described in the process. Given $\mathcal{F}_t$, v_{t+1} and e_{t+1} are fixed, while w_{t+1} is a random vertex. Let $\mathcal{A} \subseteq \mathcal{F}_t$ be the set of inputs with $w_{t+1} = w$ and $\mathcal{B} = \mathcal{F}_t \setminus \mathcal{A}$. We will estimate the number of switchings between $\mathcal{A}$ and $\mathcal{B}$ to prove the lemma.

We first proof Eq. (3.5). To switch from $\mathcal{B}$ to $\mathcal{A}$, we need to switch the edges (v_{t+1}, w_{t+1}) and (w, u) for $u \in N(w)$ and there are at most d_w such switchings for each input in $\mathcal{B}$. To switch from $\mathcal{A}$ to $\mathcal{B}$, it suffices to select the edges (v_{t+1}, w) and (x, y) with $x \notin N(v_{t+1}) \cup V_t$ and $y \notin N(w)$. By Theorem 3.1, there are at most $\Delta\Delta' + \Delta'\Delta = o(n_1) = o(M_t)$ oriented edges (x, y) with $x \in [n] \setminus V_t$ that violate the previous condition. Thus, there are at least $(1 + o(1))M_t$ switchings for each input in $\mathcal{A}$. It follows that

$$\mathbb{P}_t(w_{t+1} = w) = \frac{|\mathcal{A}|}{|\mathcal{A}| + |\mathcal{B}|} \leq \frac{|\mathcal{A}|}{|\mathcal{B}|} \leq (1 + o(1))\frac{d_w}{M_t} \; .$$

We now prove Eq. (3.6). Suppose that $d_w = 1$. To switch from $\mathcal{A}$ to $\mathcal{B}$ we must choose the oriented edge (v_{t+1}, w) and an oriented edge (x, y) with $x \in [n] \setminus V_t$, otherwise we would alter the edges within V_t. It follows that there are at most M_t switchings for each input in $\mathcal{A}$. To switch from $\mathcal{B}$ to $\mathcal{A}$, we must choose the oriented edge (v_{t+1}, w_{t+1}) and the unique oriented edge (w, u), where u is the only neighbour of w. Observe that if either $v_{t+1}w$ or $w_{t+1}u$ is an edge of the graph, the switching is invalid. Instead of giving a lower bound for the number of switchings of a fixed input in $\mathcal{B}$, we will give a lower bound for the average number of switchings over $\mathcal{B}$. For each $z \in [n] \setminus (V_t \cup \{w\})$, let $\mathcal{B}_z$ be the set of inputs in $\mathcal{B}$ with $w_{t+1} = z$. Given an input (G, Π) and $x \in [n] \setminus V_t$ with $d_x = 1$, we say that the input is x-*good* if $v_{t+1}x, zy \notin E(G)$, where y is the only neighbour of x; otherwise we call the input x-*bad*. Since $d_x = 1$ and $d_z \leq \Delta'$, by Theorem 3.1, there are at most $\Delta + \Delta'\Delta = o(n_1) = o(n_1(t))$ vertices x for which a given input is x-bad. We can generate a random input in $\mathcal{B}_z$, by first choosing one uniformly at random and then permuting the labels of the vertices of degree 1 in $[n] \setminus (V_t \cup \{z\})$. Thus, the probability that a random input in $\mathcal{B}_z$ is w-bad is $o(1)$. If an input is w-good, switching (v_{t+1}, w_{t+1}) with (w, u) yields an input in $\mathcal{A}$. It follows that

$$\mathbb{P}_t(w_{t+1} = w) = \frac{|\mathcal{A}|}{|\mathcal{A}| + |\mathcal{B}|} = \frac{1}{1 + |\mathcal{B}|/|\mathcal{A}|} \geq \frac{1}{1 + (1 + o(1))M_t} = (1 + o(1))\frac{1}{M_t} \qquad \square$$

Define $\eta_t = d_{w_t} - 2$. Next result bounds the first and second moments of η_t. In this lemma we crucially use that $T = o(|Q_0|m)$. Namely, by (1.12), the initial drift of X_t is at most $|Q_0|m + 1$ and in T steps its asymptotic order will not change (as it can only increase by one at each step). If $Q_0^2 n = O(m_0)$, then $|Q_0|m = O(T)$ and the initial drift is not governing the evolution of X_t throughout the exploration process, as noted in Theorem 1.11.

Lemma 3.3. *For any $v \in [n]$ and any $t = O(T)$, we have*

$$\mathbb{E}_t \left[\eta_{t+1}\right] \leq \frac{Q_0}{2} \quad \textit{and} \quad \mathbb{E}_t \left[(\eta_{t+1})^2\right] \leq 4R \; . \tag{3.7}$$

Proof. Note that $\sum_{i=1}^t d_{w_i}(d_{w_i}-2) \geq -t$. Using Theorem 3.1 and $t = O(T) = o(|Q_0|m)$,

$$\sum_{w\in[n]\setminus V_t} d_w(d_w-2) = \sum_{w\in[n]\setminus V_0} d_w(d_w-2) - \sum_{i=1}^{t} d_{w_i}(d_{w_i}-2) \leq Q_0 m + t + 1 \leq \frac{Q_0 m}{2} \leq 0\ . \tag{3.8}$$

Applying Theorem 3.2 and $M_t \leq m$,

$$\mathbb{E}_t\left[\eta_{t+1}\right] = \sum_{w\in[n]\setminus V_t} (d_w-2)\mathbb{P}_t(w_{t+1}=w) \leq \frac{1+o(1)}{M_t} \sum_{w\in[n]\setminus V_t} d_w(d_w-2) \leq Q_0/2\ .$$

Similarly, we can bound the second moment. By Theorem 3.1 and Eq. (3.5),

$$\mathbb{E}_t\left[(\eta_{t+1})^2\right] = \sum_{w\in[n]\setminus V_t} (d_w-2)^2\mathbb{P}_t(w_{t+1}=w) \leq \frac{(1+o(1))}{M_t} \sum_{w\in[n]\setminus V_t} (d_w-2)^2 d_w \leq 4R$$

□

3.4 Proof of Theorem 1.8

Let $\gamma \coloneqq 80$. Define the stopping time

$$\tau_X = \tau_X(v) = \inf\{t : X_t = 0\} \wedge (\gamma T + 1)\ ,$$

where X_t is obtained by starting the process with $V_0 = S \cup \{v\}$. We omit the floor and ceiling functions in this section for ease of notation.

Instead of studying X_t, we focus on the stochastic process $(Z_t)_{t\geq 0}$ defined by $Z_0 = 2|Q_0|T$ and for $t \in \mathbb{N}$

$$Z_{t+1} \coloneqq Z_t + \eta_{t+1} = 2|Q_0|T + \sum_{i=0}^{t} \eta_{i+1}\ . \tag{3.9}$$

Observe that Z_t is $\mathcal{F}_t$-measurable. For any $t < \tau_X$, we can bound the increments $X_{t+1} - X_t \leq d_{w_{t+1}} - 2 = \eta_{t+1}$. Therefore, for every $t \leq \tau_X(v)$ we have $X_{t+1} \leq Z_{t+1}$.

Define the stopping time

$$\tau_Z = \tau_Z(v) \coloneqq \inf\{t : Z_t = 0\} \wedge (\gamma T + 1)\ , \tag{3.10}$$

where Z_t is obtained by starting the process with $V_0 = S \cup \{v\}$. Hence, $\tau_X(v) \leq \tau_Z(v)$ and it suffices to bound the latter from above.

Write $\mu_{t+1} \coloneqq (\eta_{t+1} - \mathbb{E}_t\left[\eta_{t+1}\right])\mathbf{1}_{t<\tau_Z}$ and $S_{t+1} \coloneqq \sum_{i=0}^t \mu_{i+1}$. For every $t < \tau_Z$, we can write

$$Z_{t+1} = 2|Q_0|T + S_{t+1} + \sum_{i=0}^{t} \mathbb{E}_i\left[\eta_{i+1}\right]\ . \tag{3.11}$$

Since $\mathbb{E}_i\left[\mu_{i+1}\right] = 0$ for all $i \geq 0$, S_t is a martingale with respect to $\mathcal{F}_t$ with $S_0 = 0$. We will use the following Bennett-type concentration inequality for martingales due to Freedman.

Lemma 3.4 ([9]). *Let $(S_t)_{t\geq 0}$ be a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ with $S_0 = 0$ and increments $\mu_{t+1} = S_{t+1} - S_t$. Suppose there exists $c > 0$ such that $\max_{t\geq 0}|\mu_{t+1}| \leq c$ almost surely. For $t \geq 0$, define*

$$V(t+1) \coloneqq \sum_{i=0}^{t} \mathbb{E}_i\left[(\mu_{i+1})^2\right]\ .$$

Then, for every $\alpha, \beta > 0$

$$\mathbb{P}\left(S_t \geq \alpha \text{ and } V(t) \leq \beta \text{ for some } t \geq 1\right) \leq \exp\left(\frac{-\alpha^2}{2(\beta + c\alpha)}\right) .$$

Deterministically, we have $\max_{t\geq 0} |\mu_{t+1}| \leq \Delta =: c$. Moreover, by Theorem 3.3 for all $t \geq 0$,

$$V(t) \leq \sum_{i=0}^{t-1} \mathbb{E}_i\left[(\eta_{i+1})^2 \mathbf{1}_{i<\tau_Z}\right] \leq 4R(t \wedge \gamma T). \tag{3.12}$$

Choose $\alpha = (\gamma/3)|Q_0|T$ and $\beta = 4R\gamma T$. Thus, for all $t \geq 0$, $V(t) \leq \beta$ deterministically and, since $|Q_0| \leq 1$, $2(\beta + c\alpha) \leq 8\gamma(R + \Delta|Q_0|)T$. By Theorem 3.4, uniformly on the choice of $v \in [n]$

$$\mathbb{P}(S_t \geq \alpha \text{ for some } t) \leq \exp\left(-\frac{\gamma T|Q_0|^2}{72(R + \Delta|Q_0|)}\right) = O\left(1/\lambda\right) . \tag{3.13}$$

using (3.1) and since $\gamma \geq 72$.

By Eq. (3.7) we have $\sum_{i=0}^{\gamma T-1} \mathbb{E}_i\left[\eta_{i+1}\right] \leq (\gamma/2)|Q_0|T$. Combining it with Eq. (3.13), we obtain uniformly on $v \in [n]$

$$\mathbb{P}\left(\tau_Z(v) > \gamma T\right) = \mathbb{P}\left(Z_t > 0 \text{ for all } t \leq \gamma T\right) \leq \mathbb{P}\left(S_{\gamma T} > (\gamma/2 - 2)|Q_0|T\right) = O\left(1/\lambda\right) . \tag{3.14}$$

since $\gamma \geq 12$.

Observe that if $|\mathcal{C}(v)| > (\gamma + 2)T$, then $\tau_Z(v) \geq \tau_X(v) > \gamma T$. As in Eq. (2.22), letting Z be the number of components of size larger than $(\gamma + 2)T$ and by Eq. (3.14)

$$\mathbb{E}\left[Z\right] \leq \frac{1}{(\gamma + 2)T} \sum_{v\in[n]} \mathbb{P}(|\mathcal{C}(v)| > (\gamma + 2)T) \leq \frac{1}{T} \sum_{v\in[n]} \mathbb{P}(\tau_Z(v) > \gamma T) = O\left(1/\log \lambda\right) = o(1) .$$

Markov's inequality concludes the proof.

4 A local limit theorem

A local limit theorem estimates the probability distribution of a suitably rescaled sum of independent random variables, by the density function of an infinitely divisible random variable, in our case a Gaussian. Local limit theorems are a useful tool to determine the component size in random graphs [23, 26]. For our application, we will need the step distribution to allow for the existence of very large degrees, as well as the fact that the degree sequence may be supported on an lattice with step different than 1. This prevents us from using classical results such as Berry-Esseen Theorem (see [7, Theorem 3.4.9]). Our goal is to develop a precise local limit theorem which will allow us to deal with our step distributions. Our result is based on previous local limit theorems by Doney [5] and Mukhin [22, 21] from which we derive more explicit error bounds. In particular the main result of this section is following,

Theorem 4.1. *Let* $X_1, X_2, \dots, X_t$ *be independent and identically distributed random variables with step* h *and taking values on* $\mathcal{L}(v_0, h)$ *for some* $v_0 \in \mathbb{N}$*. Define* $S_t = \sum_{i=1}^t X_i$*. Suppose that* $\mu = \mathbb{E}(X_1) = 0$*,* $\sigma^2 = \mathrm{Var}(X_1)$ *and* $\gamma = \mathbb{E}|X_1|^3$*, and let* $\varphi(s)$ *be the characteristic function of* X_1*. Then,*

$$\sup_{w\in\mathcal{L}(tv_0,h)} \left|\mathbb{P}(S_t = w) - \frac{h}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{w^2}{2t\sigma^2}\right)\right| \leq \frac{32h\gamma}{\sigma^4 t} + \frac{h}{\pi}\int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} |\varphi(s)|^t ds. \tag{4.1}$$

To prove this theorem, we will require the following Fourier inverse theorems.

Theorem 4.2 (Continuous Fourier Inverse Theorem [7, Theorem 3.3.5]). *Suppose that X is a random variable with characteristic function $\varphi_X(s)$. Suppose further that $\varphi_X(s)$ is integrable, then X is a continuous random variable with density function $f(y)$ defined by*

$$f(y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-isy}\varphi_X(s)ds.$$

Theorem 4.3 (Discrete Fourier Inverse Theorem [7, Exercise 3.3.2 (ii)]). *Let X be a random variable with characteristic function $\varphi_X(s)$. Suppose that there exists $h > 0$ such that $\mathbb{P}(X \in h\mathbb{Z}) = 1$. Then for any $a \in h\mathbb{Z}$,*

$$\mathbb{P}(X = a) = \frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-isa}\varphi_X(s)ds.$$

As we are interested in local limit theorems it will be useful to note that the characteristic function of the standard normal distribution is given by $N(s) = e^{-\frac{s^2}{2}}$. We will use next lemma that bounds the pointwise difference between the characteristic function of a sum of independent random variables (suitably renormalized), with $N(s)$.

Lemma 4.4 ([24, Page 109, Lemma 1]). *Let $X_1, \dots, X_t$ be independent random variables such that for every $i \in [t]$, $\mathbb{E}(X_i) = 0$, $\sigma_i^2 = \mathrm{Var}(X_i)$ and $\gamma_i = \mathbb{E}|X_i|^3$. Define*

$$B_t := \sum_{i=1}^t \sigma_i^2, \qquad L_t := B_t^{-3/2}\sum_{i=1}^t \gamma_i, \qquad T_t := B_t^{-1/2}\sum_{i=1}^t X_i\ .$$

Let $f_t(s)$ be the characteristic function of T_t. Then,

$$|f_t(s) - N(s)| \le 16L_t|s|^3 e^{-\frac{s^2}{3}} \;\; \textit{for} \;\; |s| \le \frac{1}{4L_t}. \tag{4.2}$$

Proof of Theorem 4.1. Let $\varphi(s)$ be the characteristic function of X_1 and $\psi_t(s)$ the characteristic function of S_t. By basic properties of characteristic functions, it is easy to see that $\psi_t(s) = \varphi(s)^t$. By Theorems 4.2 and 4.3 we may deduce that

$$\begin{aligned}\mathbb{P}(S_t = w) - &\frac{h}{\sqrt{2\pi t\sigma^2}}\exp\left(-\frac{w^2}{2t\sigma^2}\right)\\ &= \frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-isw}(\psi_t(s) - N(s\sigma\sqrt{t}))ds - \frac{h}{\pi}\int_{\frac{\pi}{h}}^{\infty} e^{-isw}N(s\sigma\sqrt{t})ds.\end{aligned}$$

Therefore, by applying various forms of the triangle inequality we obtain the bound

$$\begin{aligned}|\mathbb{P}(S_t = w) - &\frac{h}{\sqrt{2\pi t\sigma^2}}\exp\left(-\frac{w^2}{2t\sigma^2}\right)\Bigg|\\ &\le \frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\psi_t(s) - N(s\sigma\sqrt{t})|ds + \frac{h}{\pi}\int_{\frac{\pi}{h}}^{\infty} N(s\sigma\sqrt{t})ds.\end{aligned} \tag{4.3}$$

To bound the first integral in (4.3) we split it into three parts. For $\epsilon > 0$ (which we shall pick later) we have

$$\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\psi_t(s) - N(s\sigma\sqrt{t})|ds \le \int_{-\epsilon}^{\epsilon} |\psi_t(s) - N(s\sigma\sqrt{t})|ds + 2\int_{\epsilon}^{\frac{\pi}{h}} N(s\sigma\sqrt{t})ds + 2\int_{\epsilon}^{\frac{\pi}{h}} |\varphi(s)|^t ds \tag{4.4}$$

This bound is useful because both $\psi_t(s)$ and $N(s\sigma\sqrt{t})$ only contribute a non-trivial amount to the left hand side of (4.4) for s very close to 0.

In bounding the first integral of (4.4) we will use Theorem 4.4, which is applicable in our setting. Rescaling we can write (4.2) as

$$|\psi_t(s) - N(s\sigma\sqrt{t})| \leq 16\gamma t|s|^3 e^{-\frac{t^2\sigma^2 s}{3}} \ \text{ for } \ |s| \leq \frac{\sigma^2}{4\gamma}.$$

So, for any $\epsilon \leq \sigma^2/(4\gamma)$ we have

$$\begin{aligned}\int_{-\epsilon}^{\epsilon} |\psi_t(s) - N(s\sigma\sqrt{t})|ds &\leq 16\gamma t \int_{-\epsilon}^{\epsilon} |s|^3 e^{-\frac{s^2\sigma^2 t}{3}} ds \\ &\leq 16\gamma t \int_{-\infty}^{\infty} |s|^3 e^{-\frac{s^2\sigma^2 t}{3}} ds \\ &= \frac{16\gamma}{\sigma^4 t} \int_{-\infty}^{\infty} |s|^3 e^{-\frac{s^2}{3}} ds = \frac{144\gamma}{\sigma^4 t}. \end{aligned} \tag{4.5}$$

The next step is to bound the second term of (4.4). Note that we can combine this with bounding the second term of (4.3), so we need to give an upper bound on

$$\int_{\epsilon}^{\infty} N(s\sigma\sqrt{t})ds = \frac{1}{\sigma\sqrt{t}} \int_{\epsilon\sigma\sqrt{t}}^{\infty} e^{-\frac{s^2}{2}} ds = \frac{\sqrt{2\pi}}{\sigma\sqrt{t}} \mathbb{P}(\mathcal{N}(0,1) > \epsilon\sigma\sqrt{t}). \tag{4.6}$$

We use the Chernoff's bound for the standard normal distribution, $\mathbb{P}(\mathcal{N}(0,1) > x) \leq e^{-\frac{x^2}{2}}$ and the simple inequality $e^{-x} \leq x^{-1/2}$ for $x > 0$, to obtain

$$\int_{\epsilon}^{\infty} N(s\sigma\sqrt{t})ds = \frac{\sqrt{2\pi}}{\sigma\sqrt{t}} \mathbb{P}(\mathcal{N}(0,1) > \epsilon\sigma\sqrt{t}) \leq \frac{2\sqrt{\pi}}{\epsilon\sigma^2 t} \tag{4.7}$$

Choosing $\epsilon = \sigma^2/(4\gamma)$ and combining (4.3), (4.4), (4.5) and (4.7), we find that for any $w \in \mathcal{L}(tv_0, h)$,

$$\begin{aligned}\left|\mathbb{P}(S_t = w) - \frac{h}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{w^2}{2t\sigma^2}\right)\right| &\leq \left(\frac{72}{\pi} + \frac{16}{\sqrt{\pi}}\right) \frac{h\gamma}{\sigma^4 t} + \frac{h}{\pi} \int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} |\varphi(s)|^t ds \\ &\leq \frac{32h\gamma}{\sigma^4 t} + \frac{h}{\pi} \int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} |\varphi(s)|^t ds, \end{aligned} \tag{4.8}$$

concluding the proof of the theorem. □

For the remainder of this section we will focus on bounding the integral term in the RHS of (4.1), thus proving Theorem 2.2. To this end we introduce the parameter $H_M(X)$, which generalises a similar parameter introduced by Mukhin [21, 22]. For an integer-valued random variable X, we define $X^* = X - X'$ to be the *symmetrisation* of X, where X' is an independent copy of X. Furthermore, for $\alpha \in \mathbb{R}$ define $\langle\alpha\rangle$ to be the distance from α to the nearest integer. Then for a random variable X and $m \in \mathbb{R}$ we define the following parameter

$$H(X, m) := \mathbb{E}\langle X^* m\rangle^2.$$

The parameter $H(X, m)$ measures in a certain sense how close is X^* to be a random variable supported on a lattice with step $1/|m|$ centered at the origin.

Additionally, for $m \in \mathbb{R}$ we define

$$D(X, m) := \inf_{\alpha\in\mathbb{R}} \mathbb{E}\langle(X - \alpha)m\rangle^2 = \inf_{\alpha\in\mathbb{R}} \mathbb{E}\langle mX - \alpha\rangle^2. \tag{4.9}$$

Here $D(X, m)$ measures how close is X to be a random variable supported on a translation of the lattice with step $1/|m|$

The following lemma from [21] will be useful.

Lemma 4.5. *If $\varphi(s)$ is the characteristic function of the random variable X then*

$$4H\left(X,\frac{s}{2\pi}\right)\le 1-|\varphi(s)|\le 2\pi^2 H\left(X,\frac{s}{2\pi}\right) \tag{4.10}$$

We provide a full proof of the statement, for the sake of completeness.

Proof. We look at the characteristic function of X^*, $\varphi^*(s)$. Note that X^* is by definition symmetric around the origin and hence so is $\varphi^*(s)$. Writing $\mathcal{D}(X^*)$ for the domain of X^* which is discrete, we have

$$\varphi^*(s)=\frac{\varphi^*(s)+\varphi^*(-s)}{2}=\sum_{x\in\mathcal{D}(X^*)}\frac{e^{isx}+e^{-isx}}{2}\mathbb{P}(X^*=x)=\sum_{x\in\mathcal{D}(X^*)}\cos(sx)\mathbb{P}(X^*=x). \tag{4.11}$$

As $\cos(x)$ is symmetric around π and periodic with period 2π, we have the identity

$$\cos(x)=\cos\left(2\pi\left\langle\frac{x}{2\pi}\right\rangle\right).$$

We can use this identity to rewrite (4.11) as

$$\varphi^*(s)=\sum_{x\in\mathcal{D}(X^*)}\cos\left(2\pi\left\langle\frac{sx}{2\pi}\right\rangle\right)\mathbb{P}(X^*=x). \tag{4.12}$$

Consider the following bounds on $\cos(x)$ valid for $x\in[0,\pi]$,

$$1-\frac{x^2}{2}\le\cos(x)\le 1-\frac{2x^2}{\pi^2}. \tag{4.13}$$

As $2\pi\left\langle\frac{y}{2\pi}\right\rangle\in[0,\pi]$ for any $y\in\mathbb{R}$, we can use (4.13) in combination with (4.12) to deduce that

$$1-2\pi^2\mathbb{E}\left\langle\frac{X^*s}{2\pi}\right\rangle^2\le\varphi^*(s)\le 1-8\mathbb{E}\left\langle\frac{X^*s}{2\pi}\right\rangle^2 \tag{4.14}$$

Finally, by definition of X^*, note that $\varphi^*(s)=\varphi(s)\varphi(-s)=|\varphi(s)|^2$. As $|\varphi(s)|\in[0,1]$ we may deduce that

$$1-2\pi^2H\left(X,\frac{s}{2\pi}\right)\le|\varphi(s)|\le 1-4H\left(X,\frac{s}{2\pi}\right) \tag{4.15}$$

which may easily be rearranged to give the statement of the lemma. □

For $M\in\mathbb{N}$ and an integer-valued random variable X we define

$$H_M(X):=\inf_{\frac{1}{4M}\le m\le\frac{1}{2M}}H(X,m) \tag{4.16}$$

The following results of Mukhin bound $H(X,m)$ in terms of $D(X,m)$ and of $H_M(X)$ for an appropriately chosen M.

Lemma 4.6 ([22, Lemmas 1 and 5]). *For any integer-valued random variable X, and $m\in\mathbb{R}$*

$$D(X,m)\le H(X,m)\le 4D(X,m)\,. \tag{4.17}$$

Moreover, for any $q\in[0,1]$ and $d\in\mathbb{R}$, we have

$$H(X,qd)\ge\frac{q^2}{4}\inf_{1\le\lambda\le 2}H(X,\lambda d)=\frac{q^2}{4}H_{1/4d}(X). \tag{4.18}$$

Note that (4.18) trivially holds for any $q \in [1,2]$, as the value $H(X, qd)$ is included in the infimum and $q^2/4 \le 1$. Letting $d = 1/4M$ and $q = 4mM$ and using the previous observation, (4.18) can be written as: for any $2mM \le 1$

$$H(X,m) \ge 4M^2 m^2 H_M(X). \tag{4.19}$$

Proof of Theorem 2.2. We will apply Theorem 4.1 and use Theorem 4.5 to give an explicit upper bound on the integral term in (4.1) as follows. Recall that X_1 is integer-valued with step h. By Theorem 4.5 and using $\ln(1/x) \ge 1 - x$ for $x > 0$,

$$\int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} |\varphi(s)|^t ds \le \int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} e^{-t(1-|\varphi(s)|)} ds \le \int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} e^{-4tH(X,\frac{s}{2\pi})} ds. \tag{4.20}$$

Now, note that the upper limit of the integral in (4.20) is π/h. So, as $(\pi/h)/(2\pi) = 1/(2h)$ we may apply Lemma 4.6 in the form of (4.19) with $M = h$ and $m = s/2\pi \le 1/2h$ (so $2mM \le 1$) to deduce that

$$\begin{aligned}\int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} |\varphi(s)|^t ds &\le \int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} e^{-\frac{4th^2 H_h(X)s^2}{\pi^2}} ds \le \int_{\frac{\sigma^2}{4\gamma}}^{\infty} e^{-\frac{4th^2 H_h(X)s^2}{\pi^2}} ds \\ &= \frac{\pi^{3/2}}{2h(tH_h(X))^{1/2}} \mathbb{P}\left(\mathcal{N}(0,1) > \frac{\sigma^2 h (tH_h(X))^{1/2}}{\sqrt{2}\pi\gamma}\right) \\ &\le \frac{\pi^{3/2}}{2h(tH_h(X))^{1/2}} e^{-\frac{\sigma^4 h^2 t H_h(X)}{4\pi^2\gamma^2}}, \end{aligned} \tag{4.21}$$

where the final inequality follows by the Chernoff's bound. This allows us to deduce, once again using the inequality $e^{-x} < x^{-1/2}$, that this integral is bounded above as

$$\int_{\frac{\sigma^2}{4\gamma}}^{\frac{\pi}{h}} |\varphi(s)|^t ds \le \frac{\pi^{5/2}\gamma}{h^2\sigma^2 t H_h(X)},$$

concluding the proof of the theorem. □

To give an explicit upper bound on the error probability, we need to deduce that $H_h(X_1)$ is bounded from below. For $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{Z}^k$, define

$$w(\mathbf{x}) := \max_{i \ne \ell \ne j} \frac{|x_i - x_\ell|}{\gcd(|x_i - x_\ell|, |x_j - x_\ell|)}.$$

Then the fact that $H_h(X_1)$ is bounded from below is implied by the following lemma,

Lemma 4.7. *Let X be an integer valued random variable with step h and with atoms $x_1, \dots, x_k$. Then there exists an absolute constant $C > 0$ such that*

$$H_h(X) \ge C \cdot \frac{\min_{i \in [k]} \mathbb{P}(X = x_i)}{k(w(\mathbf{x})h)^2} \tag{4.22}$$

Proof. By (4.17), we have that

$$H_h(X) \ge \min_{1/4h \le d \le 1/2h} D(X,d)$$

For all $\mathbf{x} \in \mathbb{Z}^k$ and $\beta, d \in \mathbb{R}$, define

$$S(\mathbf{x}, \beta, d) := \sum_{i=1}^{k} \langle \beta + x_i d \rangle \tag{4.23}$$

Cauchy-Schwartz's inequality implies that

$$\begin{aligned} D(X,d) &:= \inf_{\alpha\in\mathbb{R}} \mathbb{E}\langle (X-\alpha)d\rangle^2 \\ &= \inf_{\beta\in\mathbb{R}} \sum_{i=1}^{k} \langle \beta + x_i d\rangle^2 \mathbb{P}(X=x_i) \\ &\geq \min_{i\in[k]} \mathbb{P}(X=x_i) \inf_{\beta\in\mathbb{R}} \sum_{i=1}^{k} \langle \beta + x_i d\rangle^2 \\ &\geq \frac{\min_{i\in[k]} \mathbb{P}(X=x_i)}{k} \inf_{\beta\in\mathbb{R}} S(\mathbf{x},\beta,d)^2. \end{aligned} \tag{4.24}$$

It thus suffices to bound the infimum of S when $\mathbf{x}$ and d are fixed. The derivative of S with respect to β satisfies the following properties:

(i) it is well defined for all β such that $\langle \beta + x_i d\rangle \notin \{0, 1/2\}$ for all $i \in [k]$;
(ii) it is constant between any two consecutive values at which the derivative is undefined;
(iii) it takes integer values in $\{-k, \dots, k\}$ anywhere where it is defined.

Thus, the minimum of S is attained at β_0, for which the derivative is not defined. By relabelling the x_i, we may assume that $\langle \beta_0 + x_k d\rangle \in \{0, 1/2\}$. If $\langle \beta_0 + x_k d\rangle = 1/2$, then $\inf_{\beta\in\mathbb{R}} S(\mathbf{x},\beta,d)^2 \geq 1/2$ and plugging it in (4.24) we obtain

$$H_h(X) \geq D(X,d) \geq \frac{\min_{i\in[k]} \mathbb{P}(X=x_i)}{2k}$$

and as $h, w(\mathbf{x}) \geq 1$, (4.22) holds.

So we may assume that $\langle \beta_0 + x_k d\rangle = 0$, and, in fact, we can choose $\beta_0 = -x_k d$. For $i \in [k-1]$, define $y_i := x_i - x_k$. Since X has step h, the x_i are not all contained in a non-trivial arithmetic progression and $\gcd(y_1, y_2, \dots, y_{k-1}) = h$. By a simple extension of Bézout's Lemma there exist $\lambda_i \in \mathbb{Z}$ with $|\lambda_i| \leq w(\mathbf{x})h$ for all $i \in [k-1]$ and $\lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_{k-1} y_{k-1} = h$. Now, using the identities $\langle m\beta\rangle \leq |m|\langle\beta\rangle$ for any $m \in \mathbb{Z}$ and $\langle \beta_1 + \beta_2\rangle \leq \langle\beta_1\rangle + \langle\beta_2\rangle$, we obtain

$$\inf_{\beta\in\mathbb{R}} S(\mathbf{x},\beta,d) = \sum_{i=1}^{k} \langle \beta_0 + x_i d\rangle = \sum_{i=1}^{k-1} \langle y_i d\rangle \geq \sum_{i=1}^{k-1} \frac{\langle \lambda_i y_i d\rangle}{|\lambda_i|} \geq \frac{\left\langle \sum_{i=1}^{k-1} \lambda_i y_i d \right\rangle}{w(\mathbf{x})h} = \frac{\langle hd\rangle}{w(\mathbf{x})h} \tag{4.25}$$

Observing that $\langle hd\rangle = hd$ for all $d \leq 1/2h$, we obtain,

$$\begin{aligned} H_h(X) &\geq \min_{1/4h \leq d \leq 1/2h} D(X,d) \\ &\geq \min_{1/4h \leq d \leq 1/2h} \frac{\min_{i\in[k]} \mathbb{P}(X=x_i)}{k} \left(\frac{\langle hd\rangle}{w(\mathbf{x})h} \right)^2 \\ &= \frac{\min_{i\in[k]} \mathbb{P}(X=x_i)}{16k(w(\mathbf{x})h)^2}. \end{aligned}$$

□

5 Proof of Theorem 1.12

In this section we prove Theorem 1.12. This show that the upper bound in Theorem 1.8 cannot be vastly improved; in particular the order is tight for some sequences with $Q < 0$.

Given $\epsilon \in (0,1)$ and $\Delta = \Delta(n) = o(\sqrt{n})$ with $\log n = o(\Delta)$, define

$$\ell = \left\lfloor (1-\epsilon)\frac{n}{\Delta^2} \right\rfloor . \tag{5.1}$$

Consider the degree sequence $\widehat{\mathbf{d}}_n$ that contains $n-\ell$ vertices of degree 1 and ℓ vertices of degree Δ. We may assume that the sum of the degrees is even, otherwise we may add another vertex of degree 1. For the sake of simplicity, we will omit the floor in the definition of ℓ. Straightforward computations show that $m = (1+O(\frac{1}{\Delta}))n$ and $Q \sim -\epsilon$.

Let $L \subseteq [n]$ denote the set of vertices of degree Δ. Let $\mathbb{G}_*$ be the random subgraph induced by $\mathbb{CM}_n(\widehat{\mathbf{d}}_n)$ on L. Let $\mathbb{G}(L,p)$ be the Erdős-Rényi random graph on the vertex set L, where each edge in $\binom{L}{2}$ is chosen independently with probability p. Let $\mathbb{P}_*(\cdot)$ and $\mathbb{P}_p(\cdot)$ be the probability measures on (multi)graphs with vertex set L associated to $\mathbb{G}_*$ and $\mathbb{G}(L,p)$, respectively, and let $\mathbb{E}_*\left[\cdot\right]$ and $\mathbb{E}_p\left[\cdot\right]$ the expected value operator defined in these probability spaces.

We briefly sketch the proof. Most of the half-edges in $\widehat{\mathbf{d}}_n$ are incident to vertices of degree 1. So typically, all vertices in L will pair most of their half-edges with the ones incident to $V \setminus L$ and the order of the largest component in $\mathbb{CM}_n(\widehat{\mathbf{d}}_n)$ will be of order at least $\Delta L_1(\mathbb{G}_*)$. To estimate $L_1(\mathbb{G}_*)$, we will show that $\mathbb{G}_*$ behaves like $\mathbb{G}(L,p_*)$ with $p_* := \frac{\Delta^2}{n} = \frac{(1-\epsilon)}{\ell}$. Classic results on the subcritical regime of random graphs will give lower bounds for $L_1(\mathbb{G}(L,p_*))$ that also apply to $L_1(\mathbb{G}_*)$. We will finally use Eq. (1.9) to transfer the lower bound on the largest component from $\mathbb{CM}_n(\widehat{\mathbf{d}}_n)$ to $\mathbb{G}_n(\widehat{\mathbf{d}}_n)$.

Precisely, we will show that certain small subgraphs in $\mathbb{G}_*$ appear with the same probability as in $\mathbb{G}(L,p_*)$. Let Z_s be the number of isolated trees of size s in $\mathbb{G}_*$. Paley-Zygmund's inequality implies

$$\mathbb{P}_*(Z_s > 0) \geq \frac{\mathbb{E}_*\left[Z_s\right]^2}{\mathbb{E}_*\left[Z_s^2\right]} . \tag{5.2}$$

Lemma 5.1. *For every $s = O(\log \ell)$ we have,*

$$\begin{aligned}\mathbb{E}_*\left[Z_s\right] &= (1+o(1))\mathbb{E}_{p_*}\left[Z_s\right]\\ \mathbb{E}_*\left[Z_s^2\right] &= (1+o(1))\mathbb{E}_{p_*}\left[Z_s^2\right] .\end{aligned}$$

Proof. Choose $S \subset L$ with $|S| = s$ and any tree T with $V(T) = S$. Let A_T be the event that S induces an isolated copy of T in L, which can be defined for $\mathbb{G}_*$ and $\mathbb{G}(L,p)$.

Fix an arbitrary ordering of $E(T)$, $e_1, \dots, e_{s-1}$. A *realisation* of T is a set of pairs of half-edges $\{a_1b_1, \dots, a_{s-1}b_{s-1}\}$ such that the endpoints of e_i are the vertices incident to a_i and b_i. Let $k(T)$ be the number of realisations of T. If $d_1, \dots, d_s$ is the degree sequence of T, then $\sum_{i=1}^s d_i = 2(s-1)$ and

$$k(T) = \prod_{i=1}^{s} \frac{\Delta!}{(\Delta-d_i)!} = \Delta^{2(s-1)} \prod_{i=1}^{s}\left(1+O\left(\frac{s}{\Delta}\right)\right) = (1+o(1))\Delta^{2(s-1)} , \tag{5.3}$$

since $s^2 = o(\Delta)$.

The event A_T admits a partition into $k(T)$ subevents $A_T^1, \dots, A_T^{k(T)}$ depending on the realisation of T. For $i \in [k(T)]$, $\mathbb{P}_*(A_T^i)$ is equal to the probability that $\mathbb{CM}_n$ satisfies:

(P1) the i-th realisation of T is in $\mathbb{CM}_n$;

(P2) for every $u \in S$ and every incident half-edge a not in the i-th realisation, a is paired in $\mathbb{CM}_n$ to a half-edge incident to $V \setminus L$.

For $i \in [k]$, consider the i-th realisation of T, let $(a_1^i b_1^i, \dots, a_{s-1}^i b_{s-1}^i)$ be a sequence of pairings (a *pairing* is a pair of half-edges that have been matched in $\mathbb{CM}_n$) corresponding to $E(T)$ and let $(\bar{a}_s^i \bar{b}_s^i, \dots, \bar{a}_r^i \bar{b}_r^i)$ (where $r = s(\ell - s) + \binom{s}{2}$) be a sequence of all pairs that are not a pairing and have at least one half-edge in S (we will assume $\bar{a}_j^i$ is always incident to S). Let B_j be the event that $a_l^i b_l^i$ is a pairing for all $l \leq j \wedge (s-1)$ and $\bar{a}_l^i \bar{b}_l^i$ is not a pairing for all $s \leq l \leq j$.

We can write

$$\mathbb{P}_*(A_T) = \sum_{i=1}^{k(T)} \prod_{j=1}^{s-1} \mathbb{P}(a_j^i b_j^i \in E(\mathbb{CM}_n) \mid B_{j-1}) \prod_{j=s}^{r} \mathbb{P}(\bar{a}_j^i \bar{b}_j^i \notin E(\mathbb{CM}_n) \mid B_{j-1}) \tag{5.4}$$

We first estimate the probability of (P1) given by the first product in (5.4). Each term on the first product in Eq. (5.4) is $\frac{1}{m - O(s)} = \left(1 + O(\frac{s}{m} + \frac{1}{\Delta})\right) \frac{1}{n}$; so the first product is

$$\prod_{j=1}^{s-1} \mathbb{P}(a_j^i b_j^i \in E(\mathbb{CM}_n) \mid B_{j-1}) = \left(1 + O\left(\frac{s^2}{m} + \frac{s}{\Delta}\right)\right) \frac{1}{n^{s-1}} \tag{5.5}$$

In order to estimate the probability of (P2), which is given by the second product in Eq. (5.4), we compute the probability that each half-edge a incident to S is not paired with half-edges in L. There are exactly $s\Delta - 2(s-1)$ such events, and each has probability $1 - \frac{\ell\Delta - O(s\Delta)}{m + O(s\Delta)} = \left(1 - \frac{\ell\Delta}{n}\right)\left(1 + O(\frac{s\Delta + \ell}{n})\right)$. Thus, we have

$$\begin{aligned}
\prod_{j=s}^{r} \mathbb{P}(\bar{a}_j^i \bar{b}_j^i \notin E(\mathbb{CM}_n) \mid B_{j-1}) &= \left(1 - \frac{\ell\Delta}{n}\right)^{s\Delta - 2(s-1)} \left(1 + O\left(\frac{s\Delta + \ell}{n}\right)\right)^{s\Delta} \\
&= \left(1 - \frac{\Delta^2}{n}\right)^{s\ell} e^{O(s/\ell + s/\Delta)} \left(1 + O\left(\frac{s^2}{\ell} + \frac{s}{\Delta}\right)\right) \\
&= (1 + o(1))\,(1 - p_*)^{r-s+1} \ ,
\end{aligned} \tag{5.6}$$

where we used that $(1 - x/N)^y = (1 - y/N)^x e^{O((x^2 y + y^2 x)/N^2)}$ with $x = \ell, y = \Delta, N = n/\Delta$, and that $s = o(\Delta), s^2 = o(\ell)$.

Plugging Eqs. (5.3), (5.5) and (5.6) into Eq. (5.4), we obtain

$$\mathbb{P}_*(A_T) = (1 + o(1)) p_*^{s-1} (1 - p_*)^{r-s+1} = (1 + o(1)) \mathbb{P}_{p_*}(A_T) \ .$$

Adding over all sets $S \subset L$ with $|S| = s$ and over all trees T with $V(T) = S$, we obtain the first part of the lemma.

For the second part, choose $S, S' \subset [m]$ with $|S| = |S'| = s$ and any pair of trees T and T' with $V(T) = S$ and $V(T') = S'$. Note that $\mathbb{P}_*(A_T, A_{T'}) = 0$ unless $S = S'$ and $T = T'$, or $S \cap S' = \emptyset$. Suppose we are in the latter case, and let $r = 2s(m - 2s) + \binom{2s}{2}$. Following similar computations as the ones we did for a single tree, we obtain

$$\mathbb{P}_*(A_T, A_{T'}) = (1 + o(1)) \mathbb{P}_{p_*}(A_T, A_{T'}) \ ,$$

and adding over all pairs of sets and trees supported on these sets, the second part also follows. $\square$

The moments of Z_s in $\mathbb{G}(L, p)$ are well-studied in random graph theory. Let $I_\lambda = \lambda - 1 - \ln \lambda$ be the large deviation rate function for Poisson random variables with mean $\lambda > 0$. For $\lambda = 1 - \epsilon$, any $a < (I_\lambda)^{-1}$ and $s_0 = \lfloor a \log \ell \rfloor$, we have $\mathbb{E}_{p_*}\left[Z_{s_0}^2\right] = (1 + o(1)) \mathbb{E}_{p_*}\left[Z_{s_0}\right]^2$ (see e.g. Lemma 2.12(i) in [10]). Combining this with Theorem 5.1 and Eq. (5.2), $\mathbb{G}_*$ has with high probability an isolated tree of size s_0. As every vertex in L has degree Δ, there are exactly $\Delta s_0 - 2(s_0-1)$ vertices of degree 1 that attach to the given tree. Therefore, there exists a component in $\mathbb{CM}_n(\widehat{\mathbf{d}}_n)$ of order $(1+o(1))\Delta s_0$.

Observe that $I_\lambda = \frac{\epsilon^2}{2} + O(\epsilon^3)$ and since $Q \sim -\epsilon$, we have $I_\lambda \sim \frac{Q^2}{2}$. As $\mathbb{E}\left[D^2\right] = O(1)$ and $R \sim \Delta$, we can use Theorem 1.7 to deduce that

$$L_1(\mathbb{G}_n(\widehat{\mathbf{d}}_n)) \geq (1+o(1))\Delta s_0 \geq (1+o(1))\frac{2R}{Q^2}\log\left(\frac{n}{R^2}\right),$$

with probability $1-o(1)$. This concludes the proof of the proposition.

Remark 5.2 (Concentration of $L_1(\mathbb{G}_n(\mathbf{d}_n))$)**.** Theorem 1.12 imposes the condition $\Delta = o(\sqrt{n})$, or equivalently $\ell \to \infty$ as $n \to \infty$. If Δ is of order $\sqrt{n}$, it is easy to check that the probability that $\mathbb{G}_* = H$ is bounded away from 0 for every H of order ℓ. Since the size of the largest component is asymptotically equal to $\Delta L_1(\mathbb{G}_*)$, $L_1(\mathbb{CM}_n(\widehat{\mathbf{d}}_n))$ and $L_1(\mathbb{G}_n(\widehat{\mathbf{d}}_n))$ are not concentrated.

Remark 5.3 (The case $Q = o(1)$)**.** The largest component of Erdős-Rényi is well-studied in the barely subcritical regime (see e.g. Theorem 5.6 in [17]). If $p = \frac{1-\epsilon(\ell)}{\ell}$ with $\epsilon(\ell) > 0$ and $\ell^{-1/3} \ll \epsilon(\ell) \ll 1$, then

$$L_1(\mathbb{G}(\ell,p)) \sim 2\epsilon^2 \log(\epsilon^3 \ell)\,. \tag{5.7}$$

Let $\ell = (1-\epsilon(n))\frac{n}{\Delta^2}$ and define the degree sequence $\widehat{\mathbf{d}}_n$ as before. Again, $Q \sim -\epsilon$ and $R \sim \Delta$. In particular $\Delta|Q| = o(R)$ holds.

Set $p = \frac{1-\epsilon(n)}{\ell}$. The same argument as in the proof of Theorem 1.12 and Eq. (5.7) gives

$$\begin{aligned} L_1(\mathbb{G}_n(\widehat{\mathbf{d}}_n)) &\geq \Delta L_1(\mathbb{G}(\ell,p)) \\ &\geq (1+o(1))\frac{2\Delta}{\epsilon^2}\log\left(\frac{\epsilon^3 n}{\Delta^2}\right) \\ &\geq (1+o(1))\frac{2R}{Q^2}\log\left(\frac{|Q|^3 n}{R^2}\right)\,. \end{aligned} \tag{5.8}$$

The condition $\epsilon(\ell) \gg \ell^{-1/3}$ for Eq. (5.7) is equivalent to $Q \leq -\omega(n) n^{-1/3} R^{2/3}$, for some $\omega(n) \to \infty$.

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