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On the universality of the Nazarov-Sodin constant*

Andrea Sartori†

Abstract

We study the number of connected components of non-Gaussian random spherical harmonics on the two dimensional sphere $\mathbb{S}^2$. We prove that the expectation of the nodal domains count is independent of the distribution of the coefficients provided it has a finite second moment.

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1 Introduction

1.1 Nodal domains of Laplace eigenfunctions

Let (M,g) be a smooth, compact, connected surface and let Δ_g be the associated Laplace-Beltrami operators. We are interested in the eigenvalue problem

$$\Delta_g f_\lambda + \lambda f_\lambda = 0.$$

Since M is compact, the spectrum of $-\Delta_g$ is a discrete subset of $\mathbb{R}$ with only accumulation point at $+\infty$. The eigenfunctions f_λ are smooth and their nodal set, that is their zero set, is a smooth $1d$ sub-manifold outside a finite set of points [17]. In particular, it is possible to define the nodal domains counting function

$$\mathcal{N}(f_\lambda) := \quad \text{number of connected components of} \quad \{x \in M : f_\lambda(x) = 0\}.$$

Courant's nodal domains theorem [16, Page 21] asserts that there exists some constant $C = C(M) > 0$ such that

$$\mathcal{N}(f_\lambda) \leq C\lambda.$$

On the other hand, it is not possible to obtain any non-trivial lower bound for $\mathcal{N}(\cdot)$. Lewis [26] showed that there exists a sequence of eigenfunction on the two dimension sphere

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†Departement of Mathematics, Tel Aviv University, IL. E-mail: sartori.andrea.math@gmail.com

$\mathbb{S}^2$ with $\mathcal{N}(f_\lambda) \le 3$ and $\lambda \to \infty$. Nevertheless, much of our physical understanding of Laplace eigenfunctions, such as the RWM proposed by Berry [6, 7, 8] and the percolation prediction of Bogomolny and Schmit [11, 12], suggests that, for "generic" Laplace eigenfunctions, we should expect

$$\mathcal{N}(f_\lambda) \ge c\lambda,$$

for some $c = c(M)$.

In order to explore this speculation, Nazarov and Sodin [27] studied the number of nodal domains of random Laplace eigenfunctions on $\mathbb{S}^2$. The eigenvalues on $\mathbb{S}^2$ are given by $\lambda = n(n+1)$ for any integer $n > 0$ and have multiplicity $2n+1$, thus it is possible to define random Laplace eigenfunctions on $\mathbb{S}^2$ as

$$f_n(x) = \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} a_k Y_k(x), \tag{1.1}$$

where a_k are i.i.d standard Gaussian random variables and the Y_k's are an orthonormal base of the spherical harmonics of degree n, that is Laplace eigenfunction with eigenvalue $\lambda = n(n+1)$. In this setting, Nazarov and Sodin [27] found that

$$\mathbb{E}[\mathcal{N}(f)] = c_{NS} n^2 (1 + o_{n\to\infty}(1)),$$

where $c_{NS} > 0$ is the *Nazarov-Sodin* constant, in agreement with the prediction of Bogomolny and Schmit.

The purpose of this note is to explore what happens to the expected number of nodal domains when the a_k's in (1.1) are not Gaussian random variables but i.i.d. with finite second moment. In particular, we are interested in the case when a_i are Bernoulli ± 1 random variables. Since (normalized) Laplace eigenfunctions, such as the Y_k in (1.1), are defined up to sign, the ± 1 case seems to be a very natural model to study "generic" Laplace eigenfunctions.

1.2 Statement of the main result

Given some integer $n > 0$, let $\mathcal{H}_n = \mathcal{H}_n(\mathbb{S}^2) \subset L^2(\mathbb{S}^2)$ be the space of spherical harmonics of degree n on $\mathbb{S}^2$, that is the restriction of homogeneous harmonic polynomials of degree n to $\mathbb{S}^2$. Given an L^2 orthonormal basis $\{Y_k\}_{-n \le k \le n}$ for $\mathcal{H}_n$, we consider the function

$$f_n(x) = c_n \sum_{k=-n}^{n} a_k Y_k(x), \tag{1.2}$$

where

$$\mathbb{E}[a_k] = 0 \qquad\qquad \mathbb{E}[|a_k|^2] = 1 \tag{1.3}$$

and $c_n = (2n+1)^{-1/2}$ is a normalising constant, which has no impact on the zero set, so that $\mathbb{E}[|f_n|^2] = 1$. With the above notation, we prove the following result:

Theorem 1.1. Let f_n be as in (1.2). Suppose that the a_k's are i.i.d. random variables such that $\mathbb{E}[|a_0|^2] < \infty$, then

$$\mathbb{E}[\mathcal{N}(f_n)] = c_{NS} n^2 (1 + o_{n\to\infty}(1)).$$

Even though the study of $\mathcal{N}(\cdot)$ and the related Nazarov-Sodin constant has seen much attention (see Section 1.4 below), Theorem 1.1 seems to be the first result addressing the universality of the nodal domains counting function. In particular, Theorem 1.2 seems to provide further evidence to behavior of "generic" Laplace eigenfunctions conjectured in the physics literature.

1.3 Sketch of the proof

We would like now to briefly discuss the new ideas in the proof of Theorem 1.1. One crucial difference between the Gaussian and the non-Gaussian case is that the distribution of f_n as in (1.2) is *not* base independent (see Claim B.1 below). Therefore, most of the Gaussian tools which are pivotal to the "barrier method" in [27] are not available in the non-Gaussian case. Our proof of Theorem 1.1 takes a different approach which we will now briefly describe.

The proof of Theorem 1.1 comprises essentially of two steps:

1. The first, and main, step is to show the universality, that is independence from the law of the a_k's in (1.2), of the local nodal domains counting function
$$\mathcal{N}\left(x, \frac{1}{n}\right) = \text{number of connected components of } \left\{y \in B\left(x, \frac{1}{n}\right) : f_n(y) = 0\right\},$$
where $B(x, O(n^{-1}))$ is a ball centered at $x \in \mathbb{S}^2$ of radius about n^{-1} (we are purposely being slightly vague here). The counting function counts only nodal components *fully* contained in $B(x, O(n^{-1}))$. This step requires the most care and its proof is essentially split into three parts:
   (a) We show that, for most points $x \in \mathbb{S}^2$, each of the summands in (1.2) is $o(n^{1/2})$. Here we use some quite "elementary" bounds on the L^4-norm of spherical harmonics.
   (b) Using (a), we can apply a Lindeberg-type CLT to f_n to deduce its (asymptotic) Gaussian behavior outside the aforementioned "bad" set of $x \in \mathbb{S}^2$. More precisely, in light of the covariance structure of spherical harmonics (see Section 2.2 below), we will show that, for most $x \in \mathbb{S}^2$, the rescalded field in $B(x, O(n^{-1}))$ is close in C^2-norm to Berry's random waves [6, 7, 8] on the plane:
$$F_x(y) := f_n(\exp_x(y/n)) \approx F_\mu(y),$$
   where F_μ are Berry's random waves, $\exp_x(\cdot)$ is the exponential map at the point $x \in \mathbb{S}^2$ and $y \in B(1)$ (again being slightly imprecise here). At this step, it is important for the scaling factor to be roughly n^{-1}, otherwise the rescaled field would not be comparable with planar random waves.
   (c) We conclude, using the stability of the nodal set under small perturbations, that
$$\mathcal{N}(x, n^{-1}) \xrightarrow{d} \mathcal{N}(F_\mu) \qquad n \to \infty,$$
   where the convergence is in distribution with respect to the product space $\Omega \times \mathbb{S}^2$ (with Ω being the probability space where the random objects are defined).
2. In the second step, we will use the semi-local property of the nodal domains counting functions
$$\mathcal{N}(f_n) = n(1 + o(1)) \int_{\mathbb{S}^2} \mathcal{N}(x, n^{-1}) d\rho(x),$$
where $\rho(\cdot)$ is the uniform measure on $\mathbb{S}^2$, to "reconstruct" $\mathcal{N}(f_n)$ from our understanding of $\mathcal{N}(x, O(n^{-1}))$. This step uses some deterministic bounds on the nodal length, i.e. volume of the zero set, of f_n which follow from the fact that f_n is a polynomial. Finally, using the Faber-Krahn inequality and (c) above, we conclude that
$$\mathbb{E}[\mathcal{N}(f_n)] = n(1 + o(1)) \int_{\mathbb{S}^2} \mathbb{E}[\mathcal{N}(x, n^{-1})] d\rho(x) = n(1 + o(1)) \mathbb{E}[\mathcal{N}(F_\mu)],$$
where the r.h.s. is (asymptotically) $c_{NS} \cdot n$ by the work of Nazarov and Sodin [28].

The proof of Theorem 1.1 uses only “elementary” tools coming from the theory of orthogonal polynomials, as in [34], and the probability language of random functions, as in [9]. In fact, the exposition in this note is fully self contained a part from a result from [27], which we do not include for the sake of brevity of the article, and the well-known Faber-Krahn inequality (Lemma 2.1).

1.4 Discussion and related results

The Nazarov-Sodin constant. Since the pioneering work of Nazarov and Sodin [28], the Nazarov-Sodin constant has been object of much attention. In [27], Nazarov and Sodin extended their work to the number of nodal domains of any ergodic Gaussian stationary field on any (compact and sufficiently smooth) manifold and in [29] they investigated the variance of $\mathcal{N}(f_n)$ in the Gaussian setting. Beliaev, McAuley and Murihead [3, 5, 4] studied, for certain Gaussian random fields F, the number of connected components for the excursion sets $\{F \geq \ell\}$ and level sets $\{F = \ell\}$ for $\ell \neq 0$ and found estimates for the variance and a Central Limit Theorem.

The precise value on the Nazarov-Sodin constant is currently unknown. Estimates based on percolation [11, 12] suggest that

$$\frac{c_{NS}}{4\pi} = \frac{3\sqrt{3}-5}{\pi} \approx 0.0624\ldots,$$

however this seems slightly inconsistent with numerical simulations [2]. Pleijel’s nodal domains Theorem [30], see also [14], asserts that

$$\frac{c_{NS}}{4\pi} \leq \left(\frac{2}{j_0}\right)^2 \approx 0.691,$$

where j_0 is the smallest zero of the 0-th Bessel function. Using some curvature bounds [8] it is possible to show the sharper bound

$$\frac{c_{NS}}{4\pi} \leq \frac{1}{\sqrt{2\pi}} \approx 0.0225.$$

As far as lower bounds are concern, the only result we are aware of is by Ingremeau and Rivera [24],

$$\frac{c_{NS}}{4\pi} \geq 1.39 \times 10^{-4}.$$

The number of nodal domains of Laplace eigenfunction. Finally, we would like to discuss some deterministic lower bounds on the nodal domains counting function. Results in this direction seem few and rare. Ghosh, Reznikov and Sarnak [22, 23] gave a (non-trivial) lower bound for the number of nodal domains for Maass forms on a compact part of the modular surface. Jung and Zelditch [25] gave a lower bound for a large class of negatively curved surfaces.

We conclude by discussing the flat two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. On $\mathbb{T}^2$, by Fourier expansions, every Laplace eigenfunction with eigenvalue $4\pi^2 m$ can be expressed as a linear combination of exponentials $\{\exp(2\pi i\xi \cdot x)\}_{|\xi|^2=m}$, where $\xi \in \mathbb{Z}^2$. In particular, it is possible to form linear combinations of eigenfunctions, with the same eigenvalue, as in (1.2). Using some arithmetic information about the distribution of lattice points, Bourgain [13], and the subsequently Buckley and Wigman [15], showed that, for certain *deterministic* Laplace eigenfunctions

$$\mathcal{N}(f_n) = c_{NS} n^2 (1 + o_{n\to\infty}(1)).$$

In particular, Bourgain’s result [13] holds for linear combination of eigenfunctions, as in (1.2), with $Y_k = \exp(2\pi i\xi\cdot)$ and ± 1 coefficients. In other words, on $\mathbb{T}^2$ when $a_k = \pm 1$, Theorem 1.1 holds *deterministically*.

1.5 Notation

To simplify the exposition we adopt the following standard notation: we write $A \lesssim B$ and $A \gtrsim B$ or $A = O(B)$ to designate the existence of an absolute constant $C > 0$ such that $A \le CB$ and $A \ge CB$. If the said constant $C > 0$ depends on some parameter, β (say), we write $A \lesssim_\beta B$ ecc, if no parameter is specified in the notation, then the constant is absolute. The letters C, c will be used to designate positive constants which may change from line to line. Finally, we write $o_{\beta\to\infty}(1)$ for any function that tends to zero as $\beta \to \infty$. The proof of this result heavily relies on arithmetic information regarding the distribution of lattice points on circles, which cannot be applied in the case of $\mathbb{S}^2$.

2 Tools

2.1 Harmonic-analysis tools

A very useful tool in the study of the nodal set of Laplace eigenfunctions on surfaces is the well-known Faber-Krahn inequality [16, Chapter 4], which we state here in a convenient form for our purpose:

Lemma 2.1 (Faber-Krahn inequality)**.** Let f_n be as in (1.2) and $\Gamma \subset M$ be a nodal domain of f_n with inner radius $r > 0$, that is the radius of the largest geodesic ball inscribed in Γ, then we have

$$r \gtrsim n^{-1}.$$

We will also need the following consequence of elliptic regularity [20, page 336] for harmonic functions. Before stating the result, we need to introduce some notation. Recall that $\mathcal{H}_n$ is the space of spherical harmonics of degree n and let $B(x, r)$ be the spherical disk centered at $x \in \mathbb{S}^2$ of radius $r > 0$. We will need the following result:

Lemma 2.2. Let $f \in \mathcal{H}_n$, $R \ge 1$ be some (large) parameter and $k = 0, 1, 2$, then

$$\|f\|^2_{C^k(B(x,R/n))} \lesssim (nR)^{2k+2} \int_{B(x,10R/n)} |f(y)|^2 dy,$$

where the constant implied in the $\lesssim$-notation does not depend on n, R or x.

For the sake of completeness, we will provide a proof of Lemma 2.2 in Appendix A. We will also need the following bound on the L^4-norm of spherical harmonics, see [33, Theorem 2]:

Lemma 2.3. Let $\{Y_k\}_{k=-n}^n$ be an orthonormal base for $\mathcal{H}_n$, then we have

$$\int_{\mathbb{S}^2} |Y_k|^4 d\rho(x) \lesssim n^{2/3} \log n,$$

where $d\rho(\cdot)$ is the uniform measure on $\mathbb{S}^2$.

Although much sharper bounds are known for the L^4-norm of Laplace eigenfunction, see for example the fundamental work by Sogge [32], Lemma 2.3 will suffice for our purposes and its proof will be provided in Appendix A.

2.2 Orthogonal polynomials tools

We will also need the following two facts about the two points function of f_n as in (1.2). Again, we need to introduce some notation. Let $P_n(\cdot)$ be the classic Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n} \qquad P_n(1) = 1. \tag{2.1}$$

Moreover, let $J_0(\cdot)$ be the 0-th Bessel function

$$J_0(x) = \frac{1}{2\pi}\int_{\mathbb{S}^1} e(x\cdot y)dy, \tag{2.2}$$

where $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle and $e(\cdot) = \exp(2\pi i \cdot)$. We then have the following, see for example [1, Page 454]:

Lemma 2.4 (Two point function)**.** Let $\{Y_k\}_{k=-n}^{n}$ be an orthonormal base for $\mathcal{H}_n$ and $x, y \in \mathbb{S}^2$, then we have

$$\frac{1}{2n+1}\sum_{k=-n}^{n} Y_k(x)Y_k(y) = P_n(\cos\Theta(x,y)),$$

where $\Theta(x,y)$ is the angle between x, y and P_n is as in (2.1).

The following is [34, Theorem 8.21.6]:

Lemma 2.5. Let P_n be as in (2.1) and J_0 be as in (2.2). There exists some absolute constant $C > 0$ such that

$$P_n(\cos(\theta)) = \frac{\theta}{\sin(\theta)} J_0\left(\left(n+\tfrac{1}{2}\right)\theta\right) + E,$$

where

$$E \lesssim \begin{cases} \theta^{1/2}n^{-3/2} & C/n \le \theta \le \pi/2 \\ \theta^2 & 0 \le \theta \le C/n \end{cases}.$$

2.3 Probabilistic tools

We will need a multi-dimensional version of Lindeberg-CLT, see for example [21, Proposition 6.2]:

Lemma 2.6 (CLT)**.** Let $d > 0$ be a positive integer and let $\{V_{n,k}\}_{n,k}$ be a triangular array of $\mathbb{R}^d$-valued random variables, so that the random vectors lying on each of its rows are independent and of zero mean. That is, for any n, k, $V_{n,k} = (V_{n,k}^i)_{i=1}^d$ is a d-dimensional random vector with zero mean, and for every n fixed and every $k_1 \neq k_2$, the vectors V_{n,k_1} and V_{n,k_2} are independent. The random variables $V_{n,k}^i$ are normalized by setting

$$(s_n^i)^2 = \sum_k \mathbb{E}[(V_{n,k}^i)^2],$$

and

$$\tilde{V}_{n,k}^i = (s_n^i)^{-1}V_{n,k}^i.$$

We make the following two assumptions:

1. The covariance matrices
$$(\Sigma_{n,k}))_{ij} = \mathbb{E}[\tilde{V}_{n,k}^i \tilde{V}_{n,k}^j]$$
of the k-th vector of $\{\tilde{V}_{n,k}\}_{n,k}$ satisfy
$$\lim_{n\to\infty}\sum_k \Sigma_{n,k} = \Sigma_0,$$
for some positive definite $d \times d$-positive matrix.
2. One has
$$\max_{i=1,\dots,d}\frac{1}{(s_n^i)^2}\sum_k \mathbb{E}\left[(\tilde{V}_{n,k}^i)^2 \mathbb{1}_{\tilde{V}_{n,k}^i > \varepsilon s_n^i}\right] \to 0, \qquad n\to\infty,$$
for any positive $\varepsilon > 0$, where $\mathbb{1}$ is the indicator function.

Then, we have

$$\sum_k \tilde{V}_{n,k} \xrightarrow{d} N(0, \Sigma_0) \qquad n \to \infty,$$

where the convergence is in distribution, and the rate of convergence depends on the rates of convergence in (1) and (2) only. That is, for every $h : \mathbb{R}^d \to \mathbb{R}$ bounded continuous,

$$\mathbb{E}[h(W_n)] \to \mathbb{E}[h(Z)],$$

where $Z \sim N(0, \Sigma_0)$, with rate of convergence depending on h, and the rate of convergence in (1) and (2).

We will also need the Continuous Mapping Theorem in the following form, see for example [9, Theorem 2.1.]:

Lemma 2.7 (Continuous mapping)**.** Let P_n, P be probability measures on a metric space $(S, \mathcal{S})$, where $\mathcal{S}$ is the Borel σ-field. Suppose that

$$P_n \xrightarrow{d} P \qquad n \to \infty,$$

where the convergence is in distribution, or, in other words, with respect to the weak* topology. Moreover, let $h : S \to S$ be a map and let D_h be the set of discontinuity points of h. If $P(D_h) = 0$, then

$$P_n h^{-1} \xrightarrow{d} P h^{-1} \qquad n \to \infty.$$

Finally, we need the following standard fact about uniform convergence, see [10, Theorem 3.5]

Lemma 2.8. Let X_n be a sequence of random variables such that

$$X_n \xrightarrow{d} X,$$

where the convergence is in distribution, for some random variable X. Suppose that there exists some constants $C, \alpha > 0$, independent of n such that $\mathbb{E}[X_n^{1+\alpha}] \le C$, then

$$\mathbb{E}[X_n] \to \mathbb{E}[X].$$

2.4 Gaussian fields tools

We will need the main result from [28]. Before stating it, we introduce some definitions which will be useful throughout the script. Let Ω be an abstract probability space, with probability measure $\mathbb{P}(\cdot)$ and expectation $\mathbb{E}[\cdot]$. A (real-valued) Gaussian field F is a map $F : \mathbb{R}^2 \times \Omega \to \mathbb{R}$ such that all finite dimensional distributions $(F(x_1, \cdot), \dots, F(x_n, \cdot))$ are multivariate Gaussian vectors and $F(x, \cdot)$ is continuous in x. We say that F is *centered* if $\mathbb{E}[F] \equiv 0$ and *stationary* if its law is invariant under translations $x \to x + \tau$ for $\tau \in \mathbb{R}^2$. The *covariance* function of F is

$$\mathbb{E}[F(x) \cdot F(y)] = \mathbb{E}[F(x - y) \cdot F(0)].$$

Since the covariance is positive definite, by Bochner's theorem, it is the Fourier transform of some measure μ on $\mathbb{R}^2$. So we have

$$\mathbb{E}[F(x)F(y)] = \int_{\mathbb{R}^2} e\left(\langle x - y, s\rangle\right) d\mu(s).$$

The measure μ is called the *spectral measure* of F. Since F is real-valued, μ is symmetric, that is $\mu(-A) = \mu(A)$ for any (measurable) subset $A \subset \mathbb{R}^2$. By Kolmogorov's theorem, μ fully determines F. Thus, from now on, we will simply write $F = F_\mu$ for the centered, stationary Gaussian field with spectral measure μ. For the rest of the script μ will *always* denote the uniform measure on $\mathbb{S}^1$ and also let $\mathcal{N}(F_\mu, R)$ be the number of nodal domain of F_μ fully contained, that is not intersecting the boundary, in $B(R)$, the ball centered at zero of radius $R \geq 1$. We are finally ready to state the main result from [28] which we will need:

Theorem 2.9 (Nazarov-Sodin)**.** Let μ, F_μ and $\mathcal{N}(F_\mu, R)$ be as above. Then, there exists some constant $c_{NS} > 0$ such that

$$\mathbb{E}[\mathcal{N}(F_\mu, R)] = c_{NS}\pi R^2 + o(R^2).$$

2.5 Analysis tools

We will need the following well-known fact about the nodal length, that is the volume of the zero set, of a function f_n as in (1.2).

Lemma 2.10. Let f_n be as in (1.2) and let $\mathcal{L}(f_n) = \mathcal{H}^1(x \in \mathbb{S}^2 : f(x) = 0)$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure, then we have

$$\mathcal{L}(f_n) \lesssim n.$$

Although Lemma 2.10 follows from the much more general result of Donnelly-Fefferman [18], we will give an “elementary” proof in Appendix A. We will also need the following consequence of Thom’s isotopy Theorem, see for example [19, Theorem 3.1] and also [27, Claim 4.3]. We state it in a way convenient for our purposes and will provide a proof, for completeness, in Appendix A:

Lemma 2.11. Let $W \geq 1$ be some parameter, $B(W)$ be the ball of radius W centered at the origin and $\partial B(W)$ be its boundary. Let us write

$$C^1_\star(W) := \Big\{ g \in C^1(B(2W)) \big| |g(x)| + |\nabla g(x)| > 0 \quad \text{for all} \quad x \in B(W) \\ \text{and} \quad |g| + \left| \nabla g - \frac{x \cdot \nabla g}{|x|^2} x \right| > 0 \quad \text{for all} \quad x \in \partial B(W) \Big\},$$

and let $\mathcal{N}(g, W)$ be the number of nodal domain of g fully contained in $B(W)$, that is $\Gamma \cap B(W) = \emptyset$ for all nodal domains of g. Then, $N(\cdot, W)$ is a continuous functional on $C^1_\star(W)$.

We comment that the condition $\left|\nabla g - \frac{x \cdot \nabla g}{|x|^2} x\right| > 0$ assures that the nodal set does not touch the boundary of $B(W)$ tangentially at one point. In particular, it implies that all the nodal domains that do not intersect $\partial B(W)$ are fully contained in $B(W)$ (and well separated from the boundaries).

We will also need the following standard lemma, whose proof will be provided in in Appendix A, see also [28, Lemma 6].

Lemma 2.12 (Bulinskaya’s lemma)**.** Let $F = F_\mu$, with μ be the Lebesgue measure on the unit circle $\mathbb{S}^1$. Then, for any $W \geq 1$, $F \in C^1_*(W)$ almost surely, where is $C^1_*(W)$ as in Lemma 2.11.

3 Convergence in distribution

We are finally ready to begin the proof of Theorem 1.1. In order to state the main result of this section, we need to introduce some notation. Let $\exp_x : T_x\mathbb{S}^2 \cong \mathbb{R}^2 \to \mathbb{S}^2$ be the exponential map, let use define

$$F_x(y) = F_{x,R,n}(y) =:= f(\exp_x(Ry/n)), \tag{3.1}$$

where $R \geq 1$ is some parameter and $y \in B(0,1)$, the ball centered at zero of radius 1. In (3.1), we tacitly assume that n is much larger than R so that the exponential map is a diffeomorphism. Now, we write $\mathcal{N}(F_x)$ for the number of nodal domains of F_x fully contained in $B(0,1)$. Recalling that Ω is the abstract probability space where all the objects in the script are defined, we can think of $\mathcal{N}(F_x)$ as a random variable $\Omega \times \mathbb{S}^2 \to \mathbb{R}$, where $\Omega \times \mathbb{S}^2$ is equipped with the probability measure

$$d\sigma = d\mathbb{P} \otimes d\rho, \tag{3.2}$$

where $d\rho$ is the uniform probability measure on the unit sphere.

The main result of this section is the following:

Proposition 3.1. Let μ, F_μ, $N(F_\mu, R)$ be as in Section 2.4 and $R \geq 1$ some fixed parameter. Then, we have

$$\mathcal{N}(F_x) \xrightarrow{d} \mathcal{N}(F_\mu, R) \qquad n \to \infty,$$

where the convergence happens in the cross space $\Omega \times \mathbb{S}^2$ equipped with the measure $d\sigma$ in (3.2) and the speed of convergence depends on R.

The proof of Proposition 3.1 relies the observation that F_x is suitable close F_μ, as a random function from $(\Omega \times \mathbb{S}^2, d\sigma)$ into $C^1(B(0,1))$, continuously differentiable functions on $B(0,1)$. We rigorously express this claim in the next section after having introduced the relevant notation and background.

3.1 Convergence of random functions

Before stating the main result of this section, we give a brief digression on the convergence of random functions. Let $C^s(V)$ be the space of s-times, $s \geq 0$ integer, continuously differentiable functions on V, a compact subset of $\mathbb{R}^2$. Since $C^s(V)$ is a separable metric space, Prokhorov's Theorem, see [9, Chapters 5 and 6], implies that $\mathcal{P}(C^s(V))$, the space of probability measures on $C^s(V)$, is metrizable via the *Lévy–Prokhorov metric*. This is defined as follows: for a (measurable) subset $A \subset C^s(V)$, denote by $A_{+\varepsilon}$ the ε-neighborhood of A, that is

$$A_{+\varepsilon} := \{p \in C^s(V) \mid \exists\, q \in A,\ \|p - q\| < \varepsilon\} = \bigcup_{p \in A} B(p, \varepsilon),$$

where $\|\cdot\|$ is the C^s-norm and $B(p,\varepsilon)$ is the (open) ball centered at p of radius $\varepsilon > 0$. The *Lévy–Prokhorov metric* $d_P : \mathcal{P}(C^s(V)) \times \mathcal{P}(C^s(V)) \to [0, +\infty)$ is defined for two probability measures μ and ν as:

$$d_P(\mu,\nu) := \inf\left\{\varepsilon > 0 : \mu(A) \leq \nu(A_{+\varepsilon}) + \varepsilon,\ \nu(A) \leq \mu(A_{+\varepsilon}) + \varepsilon\ \forall\, A \subset C^s(V)\right\}.$$

Given an integer $s \geq 1$, $F_x(y)$, as in (3.1), induces a probability measure on $C^s(B(1))$ via the push-forward measure

$$(F_x)_\star \operatorname{Vol}(A) = \sigma(\{(\omega, x) \in \Omega \times \mathbb{S}^2 : F_x(\cdot) \in A\}),$$

where $A \subset C^s(B(1))$ is a measurable subset. Similarly, the push-forward of F_μ defines a probability measure on $C^s(B(1))$ which we denote by $(F_\mu)_\star\mathbb{P}$. We can now measure the distance between F_x and F_μ as the distance between their push-forward measures in $\mathcal{P}(C^s(B(1)))$, the space of probability measures on $C^s(B(1))$, equipped with the Lévy–Prokhorov metric. Therefore, to shorten notation, we will write

$$d_P(F_x, F_\mu) := d_P((F_x)_\star d\sigma, (F_\mu)_\star\mathbb{P}).$$

With the above notation, we have the following result:

Proposition 3.2. Let $R \geq 1$ be some parameter and $F_x(\cdot)$ be as in (3.1), F_μ be as in Section 2.4. Then, we have

$$d_P(F_x, F_\mu) \to 0 \qquad n \to \infty,$$

where d_p is defined with respect to the C^1-norm.

Since the proof of Proposition 3.2 is somehow long and technical, we postpone it to Section 4 below. In the rest of the section we will show how Proposition 3.1 follows from Proposition 3.2.

3.2 Proof of Proposition 3.1

In this section we prove Proposition 3.1 assuming Proposition 3.2.

Proof of Proposition 3.1 given Proposition 3.2. In order to prove Proposition 3.1 we apply Lemma 2.7 to $P_n = (F_x)_\star d\sigma$ and $P = (F_\mu)_\star d\mathbb{P}$ with $h = \mathcal{N}(\cdot)$. By Proposition 3.2, we have

$$P_n \xrightarrow{d} P \qquad n \to \infty.$$

Thanks to Lemma 2.12, applied with $W = 10R$ (say), we may assume that $F_\mu(R\cdot) \in C^1_\star(B(10))$, where $C^1_\star$ is as in Lemma 2.11. Therefore Lemma 2.11 implies that $P(D_h) = 0$. Hence, the assumptions of Lemma 2.7 are satisfied and Proposition 3.1 follows. □

4 Proof of Proposition 3.2

This section is entirely dedicated to the proof of Proposition 3.2. The main tool in the proof will be Lemma 2.6. However, in order to apply the Lindeberg-CLT, we need all the summands in (1.2) to have size, before normalization, $o(n^{1/2})$. This is not always the case as there exists spherical harmonics: writing (θ, ψ) for the spherical coordinates on $\mathbb{S}^2$, the function $g(\theta, \psi) = \sqrt{2n+1} P_n(\cos(\theta))$ satisfies $\max_{\mathbb{S}^2} |g|^2 \gtrsim n$. In order to circumvent this difficulty, we show that the portion of space where spherical harmonics are large is small. This will be the content of the next section.

4.1 Getting rid of large values

This section is dedicated to the proof of the following lemma:

Lemma 4.1. Let $\{Y_k\}_{k=-n}^{n}$ be an orthonormal base for $\mathcal{H}_n$, spherical harmonics of degree n, and let $R, K \geq 1$ be some (large) parameters. Then there exists a subset $\mathcal{B} \subset \mathbb{S}^2$ of volume at most $O(K^4 R^{10} n^{-1/3} \log n)$ such that:

1. We have
$$\sup_{x \in \mathbb{S}^2 \setminus \mathcal{B}} \max_{-n \leq k \leq n} \left(\sup_{B(x, R/n)} |Y_k(\cdot)| \right) \lesssim K^{-1} n^{1/2}$$
2. We have
$$\sup_{x \in \mathbb{S}^2 \setminus \mathcal{B}} \max_{-n \leq k \leq n} \left(\sup_{B(x, R/n)} |n^{-1} \nabla Y_k(\cdot)| \right) \lesssim K^{-1} n^{1/2}.$$

Moreover, the constants implied in the O and $\lesssim$-notation are independent of K, R, n.

Proof. By Lemma 2.2, we have

$$\sup_{B(x,R/n)} |Y_k(x)|^2 \lesssim (nR)^2 \int_{B(x,10R/n)} |Y_k(y)|^2 dy, \tag{4.1}$$
$$\sup_{B(x,R/n)} |\nabla Y_k(x)|^2 \lesssim (nR)^4 \int_{B(x,10R/n)} |Y_k(y)|^2 dy.$$

Thus, in order to prove Lemma 4.1, it is enough to provide upper an bound on $\int |Y_k|^2$. To this end, we first observe that

$$\int_{B(x,10R/n)} |Y_k(y)|^2 dy \lesssim R^2 n^{-2} \int_{B(0,10)} |Y_{k,x}(y)|^2 dy,$$

where $Y_{k,x}(y) = Y_k(\exp_x(Ry/n))$. Moreover, using Lemma 2.3 and switching the order of integration, we have

$$\int_{\mathbb{S}^2} \int_{B(0,10)} |Y_{k,x}(y)|^4 dy d\rho(x) \lesssim n^{\frac{2}{3}} \log n,$$

where $\rho(\cdot)$ is the uniform measure on $\mathbb{S}^2$. Furthermore, we observe that convexity of the L^2-norm (Jensen's inequality) implies

$$\int_{\mathbb{S}^2} \left(\int_{B(0,10)} |Y_{k,x}(y)|^2 d\rho(y) \right)^2 d\rho(x) \lesssim \int_{B(0,10)} \int_{\mathbb{S}^2} |Y_{k,x}(y)|^4 d\rho(x) \lesssim n^{\frac{2}{3}} \log n, \tag{4.2}$$

where, in the first inequality, we have switched the order of integration. Thus, combining Chebischev's bound, (4.1) and (4.2), we have

$$\begin{aligned} &\rho\left(\sup_{B(x,R/n)} |Y_k(x)| > K^{-1} n^{1/2} \right) \lesssim K^4 n^{-2} \int_{\mathbb{S}^2} \sup_{B(x,R/n)} |Y_k(x)|^4 dx \lesssim K^4 R^8 n^{-4/3} \log n \\ &\rho\left(\sup_{B(x,R/n)} |n^{-1} \nabla Y_k(x)|^2 > K^{-1} n^{1/2} \right) \lesssim K^4 R^{10} n^{-4/3} \log n \end{aligned} \tag{4.3}$$

Hence, Lemma 4.1 follows by taking the union bound over the $O(n)$ choices of k in (4.3). □

We are now ready to begin the proof of Proposition 3.2. We begin with a standard reduction step, which shows that it is enough to consider finite dimensional distributions to prove convergence of random functions (in our case).

4.2 Step I: reduction

In order to prove Proposition 3.2 we will first make a reduction step which relays on the following, well-known, result about tightness of sequences of measures on $\mathcal{P}(C^1(V))$. First, a sequence of probability measures $\{\nu_m\}_{m=0}^{\infty}$ on some topological space X is *tight* if for every $\epsilon > 0$, there exists a compact set $K = K(\epsilon) \subset X$ such that

$$\nu_m(X \backslash K) \leq \epsilon,$$

uniformly for all $k \geq 0$. We will need the following lemma, borrowed from [31, Lemma 1], see also [10, Chapters 6 and 7]:

Lemma 4.2 (Tightness)**.** Let V be a compact subset of $\mathbb{R}^2$, and $\{\nu_m\}$ a sequence of probability measures on the space $C^1(V)$ of continuously differentiable functions on V. Then $\{\nu_m\}$ is tight if the following conditions hold:

1. For every multi-index $|\alpha| \leq 1$, there exists some $y \in V$ such that for every $\varepsilon > 0$ there exists $K > 0$ with

   $$\nu_m(g \in C^2(V) : |D^\alpha g(y)| > K) \leq \varepsilon,$$

   where $D^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$.

2. For every $|\alpha| \leq 1$ and $\varepsilon > 0$, we have

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \nu_m \left(g \in C^2(V) : \sup_{|y-y'| \leq \delta} |D^\alpha g(y) - D^\alpha g(y')| > \varepsilon \right) = 0.$$

As a consequence of Lemma 4.2 we have the following:

Lemma 4.3. Let $R \geq 1$ be some parameter and $F_x(\cdot)$ be as in (3.1) and let $\nu_m = (F_x)_\star d\sigma$, where $d\sigma$ is the product measure on $\Omega \times \mathbb{S}^2$ and $(F_x)_\star$ is the push-forward measure. Then, the sequence ν_m is tight.

Proof. In order to check condition (1) in Lemma 4.2, we observe that, by Lemma 2.4, we have

$$\mathbb{E}[|D^\alpha F_x(0)|^2] \lesssim 1.$$

Thus, by Chebichev's bound, we have

$$\sigma\left(|D^\alpha F_x(0)| > K\right) \leq \mathbb{P}\left(|D^\alpha F_x(0)| > K\right) \lesssim K^{-2},$$

which implies condition (1) in Lemma 4.2.

In order to check condition (2) in Lemma 4.2, we first observe that

$$\sup_{|y-y'| \leq \delta} |D^\alpha F_x(y) - D^\alpha F_x(y')| \lesssim \sup_{B(0,1)} |\nabla D^\alpha F_x| \delta. \tag{4.4}$$

By Lemma 2.2, we have

$$\sup_{B(0,1)} |\nabla D^\alpha F_x|^2 \lesssim_R \int_{B(0,2)} |F_x(y)|^2 dy,$$

thus, by Lemma 2.4, we deduce

$$\mathbb{E}[\sup_{B(0,1)} |\nabla D^\alpha F_x|^2] \lesssim_R 1.$$

Again by Chebichev's bound and (4.4), we conclude that

$$\sigma\left(\sup_{|y-y'| \leq \delta} |D^\alpha F_x(y) - D^\alpha F_x(y')| \gtrsim K\delta \right) \leq K^{-2},$$

which, taking $K = \delta^{-1/2}$ say, implies condition (2) in Lemma 4.2. □

In light of Lemma 4.3, in order to prove Proposition 3.2 it is enough to prove convergence of the final-dimensional distributions, that is the following lemma:

Lemma 4.4. Let $R \geq 1$ be some parameter and $F_x(\cdot)$ be as in (3.1), moreover let F_μ be as in Section 2.4. Fix some integer $d > 0$ and let $y_1, \ldots, y_d$ be d fixed points in $B(0,1)$, then

$$(F_x(y_1), \ldots, F_x(y_d)) \xrightarrow{d} (F_\mu(y_1), \ldots, F_\mu(y_d)) \qquad n \to \infty,$$

and for any multi-index $|\alpha| \leq 1$, we also have

$$(D^\alpha F_x(y_1), \ldots, D^\alpha F_x(y_d)) \xrightarrow{d} (D^\alpha F_\mu(y_1), \ldots, D^\alpha F_\mu(y_d)) \qquad n \to \infty.$$

4.3 Step II: convergence of final dimensional distributions

In light of the reduction step, we are left with proving Lemma 4.4, this is the content of this section.

Proof of Lemma 4.4. We are first going to focus on the first claim in Lemma 4.4. By Portmanteau Theorem [9, Theorem 2.1] and [9, Theorem 2.6], we have can fix some bounded continuous function $g : \mathbb{R}^k \to \mathbb{R}$ (say) and we have to show that

$$\int_{\Omega\times\mathbb{S}^2} g((F_x(y_1),\dots,F_x(y_d)))d\sigma \overset{n\to\infty}{\longrightarrow} \int_{\Omega} g(F_\mu(Ry_1),\dots,F_\mu(Ry_d))d\mathbb{P}. \tag{4.5}$$

This would follow from Lemma 2.6 provided, we can check its assumptions, which we are going to do next.

Let us first check the convergence of the relative covariance matrix. First, we observe that, for all $y_1, y_2 \in B(0,1)$ and $x \in \mathbb{S}^2$ and fixed $R \geq 1$, we have

$$\Theta(\exp_x(Ry_1/n), \exp_x(Ry_1/n)) = R\frac{|y_1 - y_2|}{n}(1 + o_{n\to\infty}(1)),$$

where $\Theta(a,b)$ is the angle between $a, b \in \mathbb{S}^2$. Therefore, Lemma 2.4 together with Lemma 2.5 and a straightforward differentiation gives that, uniformly for all $x \in \mathbb{S}^2$, we have

$$\mathbb{E}\left[D^\alpha F_x(y_i) D^\alpha F_x(y_j)\right] \longrightarrow \mathbb{E}\left[D^\alpha F_\mu(Ry_i) D^\alpha F_\mu(Ry_j)\right] \qquad n\to\infty, \tag{4.6}$$

for all $i, j \in \{1, 2, \dots, d\}$ and all multi-indices $|\alpha| \leq 1$. Thus, we have show that the first assumption of Lemma 2.6 holds.

Now, in light of the observation at the beginning of Section 4.1, the second assumption of Lemma 2.6 is *not* satisfied uniformly for all $x \in \mathbb{S}^2$. Thus, we will use Lemma 4.1 to get rid of a “bad” set of $x \in \mathbb{S}^2$, as follows. Let $K = K(n) = \log n$ (say), by Lemma 4.1, applied with such K and $R \geq 1$, there exists some set $\mathcal{B} = \mathcal{B}(n)$ such that

$$\rho(\mathcal{B}) \lesssim_R n^{-1/3}(\log n)^{10},$$

and for all $x \notin \mathcal{B}$ the conclusion of Lemma 4.1 holds. Thus, we may re-write the l.h.s. of (4.5) as

$$\begin{aligned}\int_{\mathbb{S}^2} d\rho(x) \int_\Omega g((F_x(Ry_1),\dots,F_x(Ry_d)))d\mathbb{P} \\ = \int_{\mathbb{S}^2\setminus\mathcal{B}} d\rho(x) \int_\Omega g((F_x(Ry_1),\dots,F_x(Ry_d)))d\mathbb{P} + o_{g,R}(1),\end{aligned}$$

where the error term tends to zero as $n \to \infty$. Thus, it is enough to prove that

$$\int_{\mathbb{S}^2\setminus\mathcal{B}} d\rho(x) \int_\Omega g((F_x(Ry_1),\dots,F_x(Ry_d)))d\mathbb{P} \to \int_\Omega g(F_\mu(Ry_1),\dots,F_\mu(Ry_d))d\mathbb{P}. \tag{4.7}$$

Hence, it is enough to check the second assumption in Lemma 2.6 holds under the conclusion of Lemma 4.1. This is what we are going to show next.

Re-writing the second assumption in Lemma 2.6, we have to show that

$$\sup_{x\in\mathbb{S}^2\setminus\mathcal{B}} \max_{i\in\{0,\dots,d\}} \frac{1}{2n+1}\sum_{k=-n}^{n} \mathbb{E}[|a_i Y_k(\exp_x(Ry_i))\mathbb{1}_{|Y_k(\exp_x(Ry_i))|>\varepsilon(2n+1)^{1/2}}] \to 0 \quad n\to\infty. \tag{4.8}$$

Since there are $2n+1$ summands in (4.8) it enough to prove that,

$$\sup_{x\in\mathbb{S}^2\setminus\mathcal{B}}\max_{i\in\{0,\dots,d\}}\max_k \mathbb{E}[|a_iY_k(\exp_x(Ry_i))\mathbb{1}_{|a_iY_k(\exp_x(Ry_i))|^2>\varepsilon(2n+1)^{1/2}}] \to 0 \qquad n\to\infty,$$

for all $\varepsilon>0$ Since, as discussed above, it is enough to check that the second assumption in Lemma 2.6 holds under the conclusion of Lemma 4.1, we may assume[1] that

$$\sup_{x\in\mathbb{S}^2\setminus\mathcal{B}}\max_k\max_{i\in\{0,\dots,d\}}|Y_k(\exp_x(Ry_i))| \lesssim n^{1/2}(\log n)^{-1}.$$

Therefore, we have

$$\mathbb{E}[|a_iY_k(\exp_x(Ry_i))\mathbb{1}_{|a_iY_k(\exp_x(Ry_i))|^2>\varepsilon(2n+1)^{1/2}}] \le \mathbb{E}[|a_i\mathbb{1}_{|a_i|>c\varepsilon\log n}|^2],$$

for some small numerical constant $c\ge 1$. Since the a_i's have finite second moment their probability distribution decays at infinity. In order words, we may write

$$\mathbb{E}[|a_i\mathbb{1}_{|a_i|>c\varepsilon\log n}|^2] = \int_\Omega |a_i|^2\mathbb{1}_{|a_i|>C\varepsilon\log n}d\mathbb{P}(\omega) = \int_0^\infty t^2\mathbb{1}_{t>c\varepsilon\log n}d\mathbb{P}(|a_i|>t). \tag{4.9}$$

Since the a_i's have finite second moment, by the Dominated Convergence Theorem, we may take the limit $n\to\infty$ inside the integral in (4.9), to see that

$$\max_i \mathbb{E}[|a_i\mathbb{1}_{|a_i|>c\varepsilon\log n}|^2] \to 0 \qquad n\to\infty,$$

for all $\varepsilon>0$ (we can actually take $\varepsilon=(\log n)^{-1/2}$, say). This proves (4.8).

Hence, in light of (4.6) and (4.8), Lemma 2.6 implies (4.7), which, in turn, implies (4.5), concluding the proof of the first claim in Lemma 4.4. In light of the second claim in Lemma 4.1, Lemma 2.4 and Lemma 2.5, the second claim of Lemma 4.4 follows form an identical argument and it is therefore omitted. □

5 Proof of Theorem 1.1

We are finally ready to conclude the proof of Theorem 1.1:

Proof of Theorem 1.1. During the proof, we write $\mathcal{N}(F_x)=\mathcal{N}(F_x,1)$ and $\mathcal{N}(F_\mu)=\mathcal{N}(F_\mu,R)$. First, observe that, by Lemma 2.10, we have

$$\mathcal{L}(f_n)\lesssim n.$$

Therefore, the number of nodal domains with diameter, that is the largest distance between two points on the said domain, larger than R/n is at most C_1n^2/R for some $C_1>0$. Now suppose that Γ is a nodal domain with $\mathrm{Diam}(\Gamma)\le R/n$, then the volume of $x\in\mathbb{S}^2$ such that $\Gamma\cap B(x,R/n)\neq\emptyset$ but Γ is not fully contained in $B(x,R/n)$ is at most $O(\mathcal{L}(\Gamma)\cdot R/n)$, that is the length of Γ times the boundary length of $B(x,R/n)$. Therefore, in light of Lemma 2.10, we obtain

$$\left|\sum_{\mathrm{Diam}(\Gamma)\le R/n}\int \mathbb{1}_{\Gamma\subset B(x,R/n)}dx - \pi\frac{R^2}{n^2}\sum_{\mathrm{Diam}(\Gamma)\le R/n}1\right| \lesssim \frac{R}{n}\sum_\Gamma \mathcal{L}(\Gamma)\lesssim R,$$

where $\mathbb{1}$ is the indicator function, $\Gamma\subset B(x,R/n)$ means that $\Gamma\cap\partial B(x,R/n)=\emptyset$ and we tacitly assumed that $R>100$ (say). Thus, bearing in mind that the number of nodal

[1] Note that here, if necessary, we use the conclusion of Lemma 4.1 with $2R$ in place of R so that the image of the exponential map is contained in $B(x,R/n)\subset\mathbb{S}^2$.

domains with diameter larger than R/n is at most $C_1 n^2/R$, exchanging the order of summation, we have

$$\begin{aligned}\frac{n^2}{\pi R^2}\int \mathcal{N}(F_x)d\rho(x) &= \sum_{\mathrm{Diam}(\Gamma)\le R/n} \frac{n^2}{\pi R^2}\int \mathbb{1}_{\Gamma\in B(x,R/n)}dx \\ &= \sum_{\Gamma} 1 + O\left(\frac{n^2}{R}+R\right) = \mathcal{N}(f_n) + O\left(\frac{n^2}{R}\right), \qquad (5.1)\end{aligned}$$

as R is assumed to be much smaller than n.

Now, by Lemma 2.1, we have

$$\int_{\Omega\times\mathbb{S}^2} \mathcal{N}(F_x)^2 d\sigma \lesssim R^2.$$

Therefore, in light of Proposition 3.1, we may apply Lemma 2.8 with $X_n = \mathcal{N}(F_x)$, $X = \mathcal{N}(F_\mu(R\cdot))$, where μ is the uniform measure on $\mathbb{S}^1$, to see that

$$\int_{\Omega\times\mathbb{S}^2} \mathcal{N}(F_x)d\sigma = \mathbb{E}[\mathcal{N}(F_\mu)](1+o_{n\to\infty}(1)). \qquad (5.2)$$

Combining (5.1) and (5.2), we deduce that

$$\mathbb{E}[\mathcal{N}(f_n)] = \frac{n^2}{\pi R^2}\mathbb{E}[\mathcal{N}(F_\mu)](1+o_{n\to\infty}(1)) + O\left(\frac{n^2}{R^2}\right).$$

Hence, thanks to Theorem 2.9, we have

$$\mathbb{E}[\mathcal{N}(f_n)] = c_{NS}n^2(1+o_{n\to\infty}(1))(1+o_{R\to\infty}(1)) + O\left(\frac{n^2}{R^2}\right),$$

and Theorem 1.1 follows by taking $R\to\infty$ sufficiently slowly compared to n. □

A Proof of auxiliary lemmas

We will now prove Lemma 2.2.

Proof of Lemma 2.2. Let us consider the "harmonic lift" $h(x,t)=\exp(-n(n+1)^{1/2}t)f_n(x) : \mathbb{S}^2\times[-2,2]\to\mathbb{R}$. Note that, $\Delta h = 0$ on (say) any ball contained in the product space $\mathbb{S}^2\times[-2,2]$. Thus, by the mean value property of harmonic functions, for all $y\in\mathbb{S}^2$, we have

$$f_n(y) = h(y,0) \lesssim \mathrm{Vol}(\tilde{B}((y,0),n^{-1}))^{-1}\int_{\tilde{B}((y,0),n^{-1})} h,$$

where $\tilde{B}(\cdot)$ denotes a ball in the product space. Integrating over the auxiliary variable (t), we obtain

$$|f_n(y)| \lesssim n\left(\int_{B(y,n^{-1})}|f_n|^2\right)^{1/2}. \qquad (A.1)$$

Applying (A.1) (squared) to every point in $y\in B(x,R/n)$, we conclude

$$\sup_{B(x,R/n)}|f_n|^2 \lesssim (nR)^2\int_{B(x,10R/n)}|f_n|^2.$$

We now consider bounds on the the derivatives of f_n. Since h (as above) is harmonic, we also have $D^\alpha h$ is harmonic for all multi-indices $|\alpha|\le 1$ say. Therefore, by the mean-value property of harmonic functions and the Divergence Theorem, for all points $w \in \mathbb{S}^2 \times [-1,1]$ and a ball $\tilde{B}(W) = \tilde{B}(w, W)$ with $W > 0$, writing $D^\alpha = D$, we find

$$Dh(w) = \mathrm{Vol}(\tilde{B}(W))^{-1} \int_{\tilde{B}(W)} Dh = \mathrm{Vol}(\partial \tilde{B}(W))^{-1} \int_{\partial \tilde{B}(W)} h \cdot n,$$

where n is the outward pointing unit-norm vector. In particular, we have

$$|Dh(w)| \lesssim \sup_{\tilde{B}(2W)} |h|.$$

Taking $w = (y, 0)$, $W = 1/(2n)$, bearing in mind the factor of n coming from differentiating the exponential in the definition of h, and using (A.1), we obtain

$$|Df_n(y)|^2 \lesssim n^4 \sup_{B(y,1/n)} |f_n|^2.$$

Another covering argument as above concludes the proof of Lemma 2.2 for the C^1-norm. To see Lemma 2.2 for the C^2-norm we simply use the mean-value property and the Divergence Theorem with $D^\beta(D^\alpha h)$, for $|\alpha| \leq 1$ and $|\beta| \leq 1$, to obtain

$$D^\beta(D^\alpha h)(y, 1) \lesssim n^2 \sup_{B((y,1),n^{-1})} |D^\alpha h| \leq n^6 \sup_{B(y,1/n)} |f_n|^2,$$

and repeat the covering argument. □

We will now prove Lemma 2.3 following [33, Theorem 2], we claim no originality.

Proof of Lemma 2.3. Let us suppose that u is a function which maximizes $\int_{\mathbb{S}^2} u^4$ among all functions $u \in \mathcal{H}_n$ with $\|u\|_{L^2} = 1$ (note that u exists since $\mathcal{H}_n$ is a finite dimensional vector space). Now, let us consider the integral kernel of the spectral projector operator $\pi_n : L^2(\mathbb{S}^2) \to \mathcal{H}^n$ which, in light of Lemma 2.4, is given by

$$\sum_{k=-n}^{n} Y_k(x) Y_k(y) = (2n+1) P_n(\langle x, y \rangle) := \varphi_n(\langle x, y \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard inner-product on $\mathbb{S}^2$ so that $\cos(\Theta(x,y)) = \langle \cdot, \cdot \rangle$. Then, by our choice of u, we claim that

$$\pi_n(u^3)(y) := \varphi_n \star u^3(y) = \int_{\mathbb{S}^2} \varphi_n(\langle x, y \rangle) u(y)^3 d\rho(y) = cu(x), \tag{A.2}$$

for some constant $c > 0$. Indeed, for $f \in L^2(\mathbb{S}^2)$, we have

$$\int_{\mathbb{S}^2} f \cdot u = \int_{\mathbb{S}^2} \pi_n(f) \cdot u + \int_{\mathbb{S}^2} (f - \pi_n(f)) \cdot u = \int_{\mathbb{S}^2} \pi_n(f) \cdot u, \tag{A.3}$$

as $u \in \mathcal{H}_n$ and $(f - \pi_n(f))$ is orthogonal to it. Thus, since the integral on the r.h.s. of (A.3) is maximized for $\pi_n(f) = cu$ and $f = u^3$ by definition maximizes the integral on the l.h.s. of (A.3), we conclude (A.2).

In order to find $c > 0$ in (A.2), we multiply both sides by $u(x)$ and integrate with respect to $x \in \mathbb{S}^2$. Thus, bearing in mind that $\|u\|_{L^2} = 1$, we find

$$c = \|u\|_{L^4}^4.$$

Therefore, (A.2) and Hölder inequality (for the L^1-norm) implies

$$\|u\|_{L^4}^4 \|u\|_{L^\infty} = \|\varphi_n \star u^3\|_{L^\infty} \leq \|\varphi_n\|_{L^4} \|u^3\|_{L^{3/4}} = \|\varphi_n\|_{L^4} \|u\|_{L^4}^3.$$

In particular, we obtain the pair of bounds

$$\|u|_{L^4}\|u\|_{L^\infty} \leq \|\varphi_n\|_{L^4} \qquad\qquad \|u\|_{L^4}^4 \leq \|u\|_{L^\infty}^2. \tag{A.4}$$

Hence, if $\|u\|_{L^\infty} \leq n^{1/3}$, then (A.4) implies the bound $\|u\|_{L^4}^4 \leq n^{2/3}$. If $\|u\|_{L^\infty} > n^{1/3}$, since a straightforward computation using Lemma 2.5 implies $\|\varphi_n\|_{L^4} \lesssim n^{1/2}\log n$, (A.4) implies $\|u\|_{L^4}^4 \lesssim n^{2/3}\log n$, as required. □

We will now prove Lemma 2.10:

Proof of Lemma 2.10. First, since spherical harmonics are restrictions of homogeneous polynomials to the sphere, by passing to polar coordinates, we may identify f_n with a bi-variate trigonometric polynomial g_n (say) so that

$$\mathcal{H}^1(x \in \mathbb{S}^2 : f_n(x) = 0) \asymp \mathcal{H}^1(x \in [-1/2, 1/2]^2 : g_n(x) = 0).$$

Therefore, it is enough to prove

$$\mathcal{H}^1(x \in [-1/2, 1/2]^2 : g_n(x) = 0) \lesssim n. \tag{A.5}$$

Now we claim that there exists either an horizontal line ℓ_h (say) or a vertical line ℓ_v (say) such that

$$\mathcal{H}^1(x \in [-1/2, 1/2]^2 : g_n(x) = 0) \lesssim |\{x \in \ell_h : g(x) = 0\}| + |\{x \in \ell_v : g(x) = 0\}|. \tag{A.6}$$

Since the zero set of g_n is an union of smooth curves, it is, in particular, rectifiable (approximable by line segments). So it is enough to show that for any line segment $\mathcal{C}$ with length ℓ claim (A.6) holds. Indeed, let us write $N(x_1)$ for the number of intersections of $\mathcal{C}$ with the vertical line going through $x_1 \in [-1/2, 1/2]$ (the lower side of the square $[-1/2, 1/2]^2$) and, similarly, $N(x_2)$ for the number of intersections of $\mathcal{C}$ with the horizontal line going through $x_2 \in [-1/2, 1/2]$ (the left side of the square $[-1/2, 1/2]^2$). Then, we have

$$\int N(x_1)dx_1 + \int N(x_2)dx_2 \geq P_1(\mathcal{C}) + P_2(\mathcal{C}) \geq \frac{\ell}{10},$$

where P_1, P_2 are the length of the projection of $\mathcal{C}$ on the X and Y-axis respectively, and (A.6) follows. Since g_n is a bi-variate polynomial of degree at most n, its restriction to any vertical or horizontal line is an uni-variate polynomial of degree n, thus, by (A.6), we have

$$\mathrm{Vol}(x \in [-1/2, 1/2]^2 : g_n(x) = 0) \lesssim n,$$

as required. □

We are now going to prove Lemma 2.11, the proof follows [28, Claim 4.2], we claim no originality.

Proof of Lemma 2.11. Let $\{g_k\}_{k=1}^\infty \in C^1_\star$ be a sequence of functions converging to some $g \in C^1_\star(W)$ with respect to the C^1-topology. Since $g \in C^1_\star$, g has finitely many nodal domains in $B(2W)$. In particular, it has finitely many nodal domains which do not touch $\partial B(W)$ and, in light of the definition of $C^1_\star$, there exists some $a > 0$ such that

$$\min \mathrm{Dist}(\Gamma, \partial B(W)) > a,$$

where the minimum is take over all nodal domains Γ of g contained in $B(W)$. Moreover, there exists some b such that

$$|g| + |\nabla g| > b.$$

Now, we claim that each connected component $\Gamma(t)$ of the level set $|g| \leq t$ contains precisely one nodal domain Γ and the $\Gamma(t)$'s are disjoint, provided we choose $t = t(a,b)$ sufficiently small. Let's start by showing that the $\Gamma(t)$'s are disjoint. If they were not, they would meet at a point x (say) where $|\nabla g(x)| = 0$, since $x \in B(2W)$, this implies that $|g(x)| > b$ and thus it does not belong to $\Gamma(b/2)$.

Let us now snow that each connected component $\Gamma(t)$ of the level set $|g| \leq t$ contains precisely one nodal domain Γ, provided we choose $t = t(a,b)$ sufficiently small. Let us first show that each $\Gamma(t)$ contains at least one nodal domain. Indeed, since g is continuous, taking t sufficiently small depending on a, we may assure that

$$\min \mathrm{Dist}(\Gamma(t), \partial B(W)) > a/2.$$

So all $\Gamma(t)$ are sufficiently well separated from the boundaries. Now, let $x \in \partial\Gamma(t)$, then $|g(x)| = t$ and, taking $t < b/2$, we see that $|\nabla g(x)| > b/2$, thus moving in the direction of $\nabla g(x)$ if $g(x) = -t$ and in the direction $\nabla g(x)$ if $g(x) = t$, we will find a point (within $\Gamma(t)$ for appropriately small t depending on a, b) such that $g(x) = 0$.

We are left to show that $\Gamma(t)$ contains at most one nodal domain. Since $|g| + |\nabla g| > b$, there are exists some $c = c(b) > 0$ such that

$$\min \mathrm{Dist}(\Gamma_i, \Gamma_j) > c.$$

By continuity of g, we may choose t sufficiently small such that $\Gamma(t) \subset \Gamma_{+(c/2)}$, the $(c/2)$-neighborhood of Γ (and all this neighborhood are well within $B(W)$, provided we also choose t small compared to a). Thus, $\Gamma(t)$ contains precisely one nodal domain.

To conclude the proof we take k sufficiently large so that each nodal domains of g_k is contained in $\Gamma(t)$, for some $\Gamma(t)$, which implies, as $\mathcal{N}(\cdot)$ is integer values, that

$$\mathcal{N}(g_k, W) = \mathcal{N}(g, W),$$

as required. □

We are now going to prove Lemma 2.12:

Proof of Lemma 2.12. During the proof, we write a point $x \in \mathbb{R}^2$ as $x = (x_1, x_2)$. First, we observe that since F is stationary the distribution of $(F(x), \nabla F(x))$ is independent of the particular point $x \in B(2W)$. Moreover, by the spectral representation of the covariance function, for $i, j = 1, 2$, we have

$$\mathbb{E}[\partial_{x_i} F(x) \cdot F(x)] = 2\pi i \int_{\mathbb{S}^1} \lambda_1 d\lambda = 0,$$

and, similarly,

$$\mathbb{E}[\partial_{x_i} F(x) \partial_{x_j} \cdot F(x)] = 4\pi^2 \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Therefore, the covariance matrix of $(F, \nabla F)$ is non-degenerate, which implies that

$$|F| + |\nabla F| > 0,$$

almost surely. To see the condition at the boundary, we observe that, since the distribution of $(F, \nabla F)$ is invariant under rotations, it is enough to consider the point $x = (1, 0)$ (say).Thus, since the distribution of $(F, \partial_{x_2} F)$ is non-degenerate, the condition

$$|F| + \left| \nabla F - \frac{x \cdot \nabla F}{|x|^2} x \right| > 0,$$

holds almost surely, as required. □

B Not base-independent

We prove that the distribution of f_n as a random function on $C^0(\mathbb{S}^2)$ (say) is not base independent:

Claim B.1. *Given and orthogonal $\{Y_k\}$ for $\mathcal{H}_n$ let $v_n = (f_n)_\star \mathbb{P}$ (on $C^0(\mathbb{S}^2)$, say) be as in Section 3.1 with*

$$f_n(x) = c_n \sum_{k=-n}^{n} a_k Y_k,$$

where the a_k's are i.i.d. Bernulli ± 1. There exists two (different) orthonormal basis $\{Y_k\}_{k=-n}^{n}$ and $\{\tilde{Y_k}\}_{k=-n}^{n}$ such that their associated pushfoward measures, v_n and $\tilde{v_n}$ (say), have different distributions.

Proof. Suppose, by contradiction that $v_n = \tilde{v_n}$, in the sense of distributions. Then, their finite dimensional distributions also agree, in particular, taking $x = (0, 0)$ to be the north pole, in polar coordinates on the sphere, we should have

$$\mathbb{P}(f_n(x) \leq t) = \mathbb{P}(\tilde{f_n}(x) \leq t), \tag{B.1}$$

for all $t \in \mathbb{R}$. Let us now take $Y_k(\theta, \psi) = \exp(ik\psi) P_n^k(\cos\theta)$, where (θ, ψ) are polar coordinates on $\mathbb{S}^2$ and

$$P_n^k(x) = \left(\frac{n(n-k)!}{(n+k)!}\right)^{1/2} \frac{(-1)^k}{2^k k!} (1-x^2)^{k/2} \frac{d^{n+k}}{dx^{n+k}} (x^2-1)^n,$$

are the (normalized) associated Legendre polynomials. Moreover, let us take

$$\tilde{Y_k} = Y_k \quad k \neq 0, 1, \qquad \tilde{Y_0} = \frac{1}{\sqrt{2}}(Y_0 + Y_1), \qquad \tilde{Y_1} = \frac{1}{\sqrt{2}}(Y_0 - Y_1).$$

By definition, at the north pole $Y_k(x) = 0$ for all $k \neq 0$ and $Y_0(x) = \sqrt{2n+1}$, thus, bearing in mind the normalization constant in the definition of f_n, we obtain

$$\mathbb{P}(f_n(x) \leq t) = \mathbb{P}(a_0 \leq t),$$

and

$$\mathbb{P}(\tilde{f_n}(x) \leq t) = \mathbb{P}(2^{-1/2}(a_0 + a_1) \leq t),$$

which contradicts (B.1). □

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