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Time-reversal of multiple-force-point chordal $\mathrm{SLE}_\kappa(\underline{\rho})$

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Abstract

Chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ is a natural variant of the chordal SLE curve. It is a family of random non-crossing curves on the upper half plane from 0 to ∞, whose law is influenced by additional force points on $\mathbb{R}$. When there are force points away from the origin, the law of $\mathrm{SLE}_\kappa(\underline{\rho})$ is not reversible, unlike the ordinary chordal SLE_κ. Zhan (2019) gives an explicit description of the law of the time reversal of $\mathrm{SLE}_\kappa(\underline{\rho})$ when all force points lie on the same sides of the origin, and conjectured that a similar result holds in general. We prove his conjecture. Specifically, based on Zhan's result, using the techniques from the Imaginary Geometry developed by Miller and Sheffield (2013), we show that when $\kappa \in (0,8)$, the law of the time reversal of non-boundary filling $\mathrm{SLE}_\kappa(\underline{\rho})$ process is absolutely continuous with respect to $\mathrm{SLE}_\kappa(\underline{\hat{\rho}})$ for some $\underline{\hat{\rho}}$ determined by $\underline{\rho}$, with the Radon-Nikodym derivative being a product of conformal derivatives.

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1 Introduction

The Schramm-Loewner Evolution (SLE_κ) with $\kappa > 0$ is an important family of random non-self-crossing curves introduced by Schramm [Sch00]. They have been proved or conjectured to described a large class of two-dimensional lattice models at criticality. We refer the reader to [Law08, Sch11, Smi06] for basic properties of SLE and their relation to 2D lattice models.

The most basic version of SLE is the chordal SLE_κ curve, which is a random curve between two boundary points of a simply connected domain characterized by conformal invariance and the domain Markov property. It was conjectured by Rohde and Schramm [RS05] that chordal SLE_κ with $\kappa \in (0,8]$ satisfies reversibility. Namely, modulo a time reparametrization the time reversal of a chordal SLE_κ curve is also a chordal SLE_κ. The conjecture was first proved for $\kappa \in (0,4]$ by Zhan [Zha08b] using the so-called commutation coupling. The $\kappa \in (4,8)$ case was proved by Miller and Sheffield [MS16c] using the imaginary geometry theory. The chordal SLE_8 is the scaling limit of UST Peano

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curve with half free and half wired boundary conditions [LSW11] and therefore is also reversible.

Chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curves are important variants of chordal SLE. They are still curves between two boundary points of a simply connected domain, but their laws depend on some additional marked points called force points. They were introduced by Lawler, Schramm and Werner [LSW03] in the theory of conformal restriction, and play a fundamental role in imaginary geometry as flow lines emanating from a boundary point [MS16a]. In [MS16b, MS16c], it was proved that chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ for $\kappa \in (0,8)$ with at most two force points lying infinitesimally close to the starting point satisfy the reversibility. When there are force points away from the origin, the law of chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ is not reversible anymore. Recently, Zhan [Zha22] gave an explicit description of the law of the time reversal of $\mathrm{SLE}_\kappa(\underline{\rho})$ when $\kappa \in (0,4]$ and all force points lie on the same side of the origin, and when $\kappa \in (4,8)$, all force points lie on the same side, and the curve is not boundary touching on this side. In the same paper, he conjectured that a similar result holds for general chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ with $\kappa \in (0,8)$ as long as the curve is non-boundary filling; see [Zha22, Conjecture 1.3]. In this paper we prove his conjecture.

To state our main result, we introduce the necessary notations to describe chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curves with their precise definition postponed to Section 2.1. Let $\kappa \in (0,8]$. Fix the force points $x^{k,L} < \cdots < x^{1,L} < x^{0,L} = 0^- < x^{0,R} = 0^+ < x^{1,R} < \cdots < x^{\ell,R}$ and for each force point $x^{i,q}$, $q \in \{L,R\}$, we assign a weight $\rho^{i,q} \in \mathbb{R}$, such that

$$\sum_{i=0}^{j} \rho^{i,L} > (-2)\vee\left(\frac{\kappa}{2}-4\right) \text{ for all } 0 \le j \le k \text{ and } \sum_{i=0}^{j} \rho^{i,R} > (-2)\vee\left(\frac{\kappa}{2}-4\right) \text{ for all } 0 \le j \le \ell. \tag{1.1}$$

We refer to the vectors of force points and weights as $\underline{x} = (\underline{x}^L;\underline{x}^R)$ and $\underline{\rho} = (\underline{\rho}^L;\underline{\rho}^R)$. Given an $\mathrm{SLE}_\kappa(\underline{\rho})$ process η from 0 to ∞ in the upper half plane $\mathbb{H}$ with force points $\underline{x}$, for each $i \ge 1$ and $q \in \{L,R\}$, let $D_\eta^{i,q}$ be the connected component of $\mathbb{H}\backslash\eta$ containing $x^{i,q}$, and $\sigma_\eta^{i,q}, \xi_\eta^{i,q}$ be the first and the last point on $\partial D_\eta^{i,q}$ traced by η. Consider the conformal map $\psi_\eta^{i,q} : D_\eta^{i,q} \to \mathbb{H}$ sending $(\sigma_\eta^{i,q}, x_\eta^{i,q}, \xi_\eta^{i,q})$ to $(0,\pm 1,\infty)$ where we take the $+$ sign when $q = R$ and take the $-$ sign when $q = L$.

We now introduce a family of measures on curves describing the time reversal of chordal $\mathrm{SLE}_\kappa(\underline{\rho})$.

Definition 1.1. *Suppose $\underline{x}$ and $\underline{\rho}$ satisfy (1.1). We associate a power parameter $\alpha^{i,q} \in \mathbb{R}$ for each $x^{i,q}$ with $\alpha^{0,L} = \alpha^{0,R} = 0$. Define $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ with force points $\underline{x}$ to be the measure on continuous curves in $\overline{\mathbb{H}}$ from 0 to ∞ which is absolutely continuous with respect to $\mathrm{SLE}_\kappa(\underline{\rho})$ with Radon-Nikodym derivative*

$$\frac{d\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})}{d\mathrm{SLE}_\kappa(\underline{\rho})}(\eta) = \prod_{q\in\{L,R\}}\prod_{i\ge 1} |x^{i,q} \cdot (\psi_\eta^{i,q})'(x^{i,q})|^{\alpha^{i,q}}. \tag{1.2}$$

Let us recall some statements on the time reversal of chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ processes from existing literature. The first one is about the time reversal of $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ processes, which is shown in [MS16b, Theorem 1.1] and [MS16c, Theorem 1.2]. Let $J : \mathbb{H} \to \mathbb{H}$ be the map $J(z) = -1/z$. For a curve η, we write $\mathcal{R}(\eta)$ for its time reversal.

Theorem A. Let $\kappa \in (0,8]$ and $\rho^L, \rho^R > -2$ such that $\rho^L, \rho^R \ge \frac{\kappa}{2} - 4$ if $\kappa \in (4,8]$. Let η be an $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ process in $\mathbb{H}$ from 0 to ∞ with force points $0^-;0^+$. Then modulo time parametrization, $\mathcal{R}(J\circ\eta)$ is the $\mathrm{SLE}_\kappa(\rho^R;\rho^L)$ process in $\mathbb{H}$ from 0 to ∞ with force points $0^-;0^+$.

We comment that the $\kappa = 8$ case above is not stated in [MS16c, Theorem 1.2], yet it readily follows from the reversibility of chordal SLE_8 and [MS16b, Theorem 1.1] along with SLE duality [Zha08a, MS16a] (see Proposition 3.6).

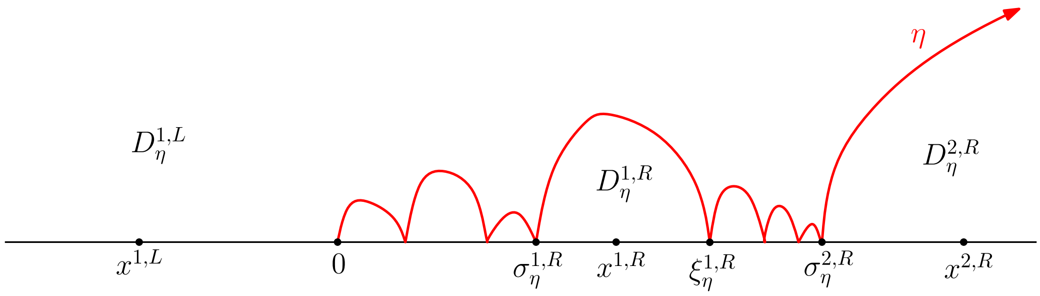

Figure 1: An $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ processes with force points $x^{1,L},0^-;0^+,x^{1,R},x^{2,R}$. By definition $\sigma^{1,L}_\eta=0^-$, $\xi^{1,L}_\eta=\xi^{2,R}_\eta=\infty$. The conformal map $\psi^{i,R}_\eta:D^{i,R}_\eta\to\mathbb{H}$ sends $(\sigma^{i,R}_\eta,x^{i,R},\xi^{i,R})$ to $(0,1,\infty)$ for $i=1,2$, and the conformal map $\psi^{1,L}_\eta:D^{1,L}_\eta$ sends $(\sigma^{1,L}_\eta,x^{1,L},\xi^{1,L})$ to $(0,-1,\infty)$. Then the measure $\widetilde{\mathrm{SLE}}_\kappa(\rho^{0,L},\rho^{1,L};\rho^{0,R},\rho^{1,R},\rho^{2,R};\alpha^{1,L};\alpha^{1,R},\alpha^{2,R})$ is absolutely continuous w.r.t. $\mathrm{SLE}_\kappa(\rho^{0,L},\rho^{1,L};\rho^{0,R},\rho^{1,R},\rho^{2,R})$ with Radon-Nikodym derivative $|x^{1,L}(\psi^{1,L}_\eta)'(x^{1,L})|^{\alpha^{1,L}}|x^{1,R}(\psi^{1,R}_\eta)'(x^{1,R})|^{\alpha^{1,R}}|x^{2,R}(\psi^{2,R}_\eta)'(x^{2,R})|^{\alpha^{2,R}}$.

When all the force points lie on the same side of 0, the following theorem is shown in Theorem 1.2 and Section 3.2 of [Zha22] via the construction of the reversed curve.

Theorem B. Let $\ell\ge 0$. Fix $\rho^{0,R},\dots,\rho^{\ell,R}\in\mathbb{R}$, such that $\kappa\in(0,4]$, $\min_{0\le j\le\ell}\sum_{i=0}^{j}\rho^{i,R}>-2$ if $\kappa\in(0,4]$, and $\min_{0\le j\le\ell}\sum_{i=0}^{j}\rho^{i,R}\ge\frac{\kappa}{2}-2$ if $\kappa\in(4,8)$. Let $\underline{\rho}^R$ be the vector of $\rho^{i,R}$ and η be a chordal $\mathrm{SLE}_\kappa(\underline{\rho}^R)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $0^+=x^{0,R}<x^{1,R}<\cdots<x^{\ell,R}$. Let $x^{\ell+1,R}=+\infty$ and $\rho^{\ell+1,R}=-\sum_{i=0}^{\ell}\rho^{i,R}$. For $0\le i\le\ell$, let $\hat{x}^{i,L}=J(x^{\ell+1-i,R})$, $\hat\rho^{i,L}=-\rho^{\ell+1-i,R}$. Here we use the convention $J(\pm\infty)=0^\mp$. For $1\le i\le\ell$, let $\hat\alpha^{i,L}=\frac{\hat\rho^{i,L}(\kappa-4)}{2\kappa}$. Let $\underline{\hat{x}}^L$, $\underline{\hat\rho}^L$, $\underline{\hat\alpha}^L$ be the vector of $\hat{x}^{i,q}$, $\hat\rho^{i,q}$ and $\hat\alpha^{i,q}$. Then up to reparametrization, the law of $\mathcal{R}(J\circ\eta)$ is equal to $\frac{1}{Z}\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat\rho}^L;\underline{\hat\alpha}^L)$ with force points $\underline{\hat{x}}^L$ for some normalizing constant $Z\in(0,\infty)$.

In [Zha22], the time reversal of $\mathrm{SLE}_\kappa(\underline{\rho}^R)$ is described in terms of reversed intermediate $\mathrm{SLE}_\kappa(\underline{\rho})$ ($\mathrm{iSLE}^r_\kappa(\underline{\rho})$) process, which agrees with $\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat\rho}^L;\underline{\hat\alpha}^L)$ when normalized to be a probability measure. The $\mathrm{iSLE}^r_\kappa(\underline{\rho})$ process is described explicitly using a Loewner evolution based on Appell-Lauricella multiple hypergeometric function. The constant Z can be traced via [Zha22, (3.16),(3.19)] and [Zha22, Remark 3.6], and can be expressed by a hypergeometric function (in fact a product of the gamma functions) depending only on $\kappa,\underline{\rho}^R$ but not on the location of the force points $\underline{x}^R$.

Our main result is the following.

Theorem 1.2. *Let $\kappa\in(0,8]$. Fix $\underline{x}$, $\underline{\rho}$ with (1.1), and let η be a chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 0 to ∞ with force points $\underline{x}$. Let $x^{k+1,L}=-\infty$, $x^{\ell+1,R}=+\infty$, $\rho^{k+1,L}=-\sum_{i=0}^{k}\rho^{i,L}$ and $\rho^{\ell+1,R}=-\sum_{i=0}^{\ell}\rho^{i,R}$. For $0\le i\le\ell$, let $\hat{x}^{i,L}=J(x^{\ell+1-i,R})$, $\hat\rho^{i,L}=-\rho^{\ell+1-i,R}$. For $0\le i\le k$, let $\hat{x}^{i,R}=J(x^{k+1-i,L})$, $\hat\rho^{i,R}=-\rho^{k+1-i,R}$. For $i\ge 1$ and $q\in\{L,R\}$, let $\hat\alpha^{i,q}=\frac{\hat\rho^{i,q}(\kappa-4)}{2\kappa}$. Let $\underline{\hat{x}}$, $\underline{\hat\rho}$, $\underline{\hat\alpha}$ be the vector of $\hat{x}^{i,q}$, $\hat\rho^{i,q}$ and $\hat\alpha^{i,q}$. Then up to reparametrization, the law of $\mathcal{R}(J\circ\eta)$ is equal to $\frac{1}{Z}\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat\rho};\underline{\hat\alpha})$ with force points $\underline{\hat{x}}$ for some normalizing constant $Z:=Z(\underline{\rho})\in(0,\infty)$.*

The $\kappa=4$ case of Theorem 1.2 is covered by [WW17, Theorem 1.1.6] by realizing $\mathrm{SLE}_4(\underline{\rho})$ curves as level lines of Gaussian free field with appropriate boundary conditions. In this case, the reversed curve is just $\mathrm{SLE}_\kappa(\underline{\hat\rho})$ with no weighting, i.e., $\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat\rho};\underline{\hat\alpha})$ with $\hat\alpha^{i,q}=0$.

Based on Theorem A and Theorem B, our proof is mainly relying on the techniques from the Imaginary Geometry [MS16a, MS16b, MS16c, MS17]. For $\kappa\in(0,4)$, we first

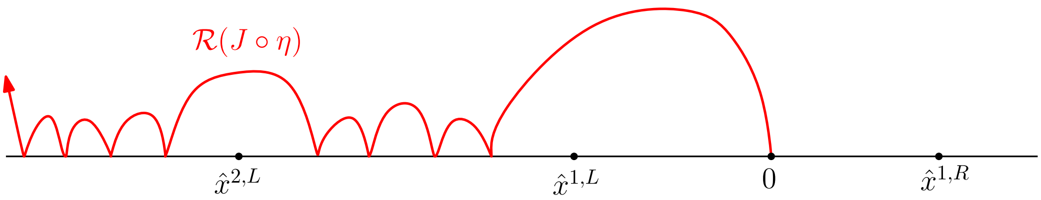

Figure 2: An example of Theorem 1.2. Let $\alpha^{i,q}=0$ in Figure 1 so that η is an $\mathrm{SLE}_\kappa(\rho^{0,L},\rho^{1,L};\rho^{0,R},\rho^{1,R},\rho^{2,R})$ process with force points $0^-,x^{1,L},0^+,x^{1,R},x^{2,R}$. Let $J(z)=-1/z$, $\hat{x}^{1,L}=J(x^{2,R})$, $\hat{x}^{2,L}=J(x^{1,R})$, $\hat{x}^{1,R}=J(x^{1,L})$. Then the time reversal of $J\circ\eta$ is an $\widetilde{\mathrm{SLE}}_\kappa(\rho^{0,R}+\rho^{1,R}+\rho^{2,R},-\rho^{2,R},-\rho^{1,R};\rho^{0,L}+\rho^{1,L},-\rho^{1,L};\frac{\rho^{2,R}(4-\kappa)}{2\kappa},\frac{\rho^{1,R}(4-\kappa)}{2\kappa};\frac{\rho^{1,L}(4-\kappa)}{2\kappa})$ process with force points $0^-,\hat{x}^{1,L},\hat{x}^{2,L},0^+,\hat{x}^{1,R}$. When $\rho^{1,L}=\rho^{1,R}=\rho^{2,R}=0$, the law of $J\circ\eta$ is $\mathrm{SLE}_\kappa(\rho^{0,R};\rho^{0,L})$ with force points at 0^- and 0^+, as shown in [MS16b].

extend a commutation relation between two $\mathrm{SLE}_\kappa(\underline{\rho})$-type processes (i.e., two $\mathrm{SLE}_\kappa(\underline{\rho})$ with possibly different $\underline{\rho}$ values) from the theory of GFF flow lines to the setting of two $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$-type processes (Proposition 2.1), from which we are able to add a force point located at 0^- in Theorem B (Lemma 3.2). Using this extended result with the commutation relation, we can construct a pair of $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$-type processes (η_1,η_2), such that conditioned on one curve, the time reversal of the other curve is the ordinary $\mathrm{SLE}_\kappa(\underline{\rho})$ process with only one degenerate force point (i.e. $0^\pm$) on the left or right side. Then from the SLE resampling property [MS16b, Theorem 4.1], the two conditional laws uniquely characterize the joint law of the reversal of (η_1,η_2), which finishes the proof for $\kappa\in(0,4)$. For $\kappa\in(4,8]$, we apply the $\kappa\in(0,4)$ result along with the SLE duality [Zha08a, Dub09, MS16a], which states that for $\kappa>4$, the boundaries of SLE_κ-type processes are $\mathrm{SLE}_{\frac{16}{\kappa}}$-type processes (see Proposition 3.6).

For the $\kappa>8$ regime, it has been shown in [MS17, Theorem 1.19] that the time reversal of chordal $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ with force points at $0^-;0^+$ is $\mathrm{SLE}_\kappa(\tilde{\rho}^R;\tilde{\rho}^L)$ where $\rho^L,\rho^R\in(-2,\frac{\kappa}{2}-2)$ and $\tilde{\rho}^q=\frac{\kappa}{4}-2-\rho^q$ for $q\in\{L,R\}$. The time reversal of $\mathrm{SLE}_\kappa(\underline{\rho})$ is not known when $\kappa>8$ and there are force points located at $\mathbb{R}\backslash\{0\}$.

We comment that the reversibility of SLE processes can also be inferred from the conformal welding of Liouville quantum gravity surfaces (see e.g. [DMS21, AHS20, ASY22]). For instance, by viewing the welding interface from the opposite direction, Theorem A is a direct consequence of [AHS20, Theorem 2.2]. The time reversal of $\mathrm{SLE}_\kappa(\rho^-;\rho^+,\rho_1)$ with force points $0^-;0^+,1$ has also been discussed in [ASY22, Section 7.1] via the conformal welding of quantum triangles. We expect that this method can also be used to describe the time reversal of other types of SLE curves, such as radial SLE with force points and SLE on the annulus.

In Section 2.1, we recap the $\mathrm{SLE}_\kappa(\underline{\rho})$ processes along with its coupling with the GFF as imaginary geometry flow lines in [MS16a]. In Section 2.2, we establish a commutation relation for $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ processes and recap the SLE *resampling properties.* Finally in Section 3, we prove Theorem 1.2.

2 Preliminaries

In this paper we work with non-probability measures and extend the terminology of ordinary probability to this setting. For a finite or σ-finite measure space $(\Omega,\mathcal{F},M)$, we say X is a random variable if X is an $\mathcal{F}$-measurable function with its *law* defined via the push-forward measure $M_X=X_*M$. In this case, we say X is *sampled* from M_X and write $M_X[f]$ for $\int f(x)M_X(dx)$. *Weighting* the law of X by $f(X)$ corresponds to working with the measure $d\tilde{M}_X$ with Radon-Nikodym derivative $\frac{d\tilde{M}_X}{dM_X} = f$, and *conditioning* on some event $E \in \mathcal{F}$ (with $0 < M[E] < \infty$) refers to the probability measure $\frac{M[E\cap\cdot]}{M[E]}$ over the space $(E, \mathcal{F}_E)$ with $\mathcal{F}_E = \{A \cap E : A \in \mathcal{F}\}$.

Throughout this paper, for a continuous simple curve η from 0 to ∞ in $\mathbb{H} \cup \mathbb{R}$, we shall refer to the subset of $\mathbb{H}\backslash\eta$ consisted of connected components whose boundaries contain a subinterval of $(-\infty, 0)$ (resp. $(0,\infty)$) as the left (resp. right) part of $\mathbb{H}\backslash\eta$. For $n \geq 0$, $\underline{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ and $a \in \mathbb{R}$, we write $a + \underline{x}$ for $(a + x_0, x_1, \dots, x_n)$ and $a\underline{x}$ for $(ax_0, \dots, ax_n)$. The formal notation is used for weights and latter is for the locations of force points under dilation and SLE duality purposes (see Proposition 3.6).

2.1 $\mathrm{SLE}_\kappa(\underline{\rho})$ process and the imaginary geometry

Fix $\kappa > 0$. We start with the SLE_κ process on the upper half plane $\mathbb{H}$. Let $(B_t)_{t\geq 0}$ be the standard Brownian motion. The SLE_κ is the probability measure on continuously growing curves $(K_t)_{t\geq 0}$ in $\overline{\mathbb{H}}$, whose mapping out function $(g_t)_{t\geq 0}$ (i.e., the unique conformal transformation from $\mathbb{H}\backslash K_t$ to $\mathbb{H}$ such that $\lim_{|z|\to\infty} |g_t(z) - z| = 0$) can be described by

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - W_s} ds, \; z \in \mathbb{H}, \tag{2.1}$$

where $W_t = \sqrt{\kappa} B_t$ is the Loewner driving function. For the force points $x^{k,L} < \dots < x^{1,L} < x^{0,L} = 0^- < x^{0,R} = 0^+ < x^{1,R} < \dots < x^{\ell,R}$ and the weights $\rho^{i,q} \in \mathbb{R}$, the $\mathrm{SLE}_\kappa(\underline{\rho})$ process is the probability measure on curves $(K_t)_{t\geq 0}$ in $\overline{\mathbb{H}}$ growing the same as ordinary SLE_κ (i.e., satisfies (2.1)) except that the Loewner driving function $(W_t)_{t\geq 0}$ are now characterized by

$$\begin{aligned} W_t &= \sqrt{\kappa} B_t + \sum_{q\in\{L,R\}} \sum_i \int_0^t \frac{\rho^{i,q}}{W_s - V_s^{i,q}} ds; \\ V_t^{i,q} &= x^{i,q} + \int_0^t \frac{2}{V_s^{i,q} - W_s} ds, \; q \in \{L, R\}. \end{aligned} \tag{2.2}$$

It has been proved in [MS16a] that the $\mathrm{SLE}_\kappa(\underline{\rho})$ process a.s. exists, is unique and generates a continuous curve until the *continuation threshold*, the first time t such that $W_t = V_t^{j,q}$ with $\sum_{i=0}^{j} \rho^{i,q} \leq -2$ for some j and $q \in \{L, R\}$.

Now we recap the definition of the Gaussian Free Field. Let $D \subsetneq \mathbb{C}$ be a domain. We construct the GFF on D with *Dirichlet boundary conditions* as follows. Consider the space of smooth functions on D with finite Dirichlet energy and zero value near ∂D, and let $H(D)$ be its closure with respect to the inner product $(f,g)_\nabla = \int_D (\nabla f \cdot \nabla g) \, dx \, dy$. Then the (zero boundary) GFF on D is defined by

$$h = \sum_{n=1}^{\infty} \xi_n f_n \tag{2.3}$$

where $(\xi_n)_{n\geq 1}$ is a collection of i.i.d. standard Gaussians and $(f_n)_{n\geq 1}$ is an orthonormal basis of $H(D)$. The sum (2.3) a.s. converges to a random distribution independent of the choice of the basis $(f_n)_{n\geq 1}$. For a function g defined on ∂D with harmonic extension f in D and a zero boundary GFF h, we say that $h + f$ is a GFF on D with boundary condition specified by g. See [DMS21, Section 4.1.4] for more details.

Next we introduce the notion of *GFF flow lines*. We restrict ourselves to the range $\kappa \in (0, 4)$. Heuristically, given a GFF h, $\eta(t)$ is a flow line of angle θ if

$$\eta'(t) = e^{i(\frac{h(\eta(t))}{\chi} + \theta)} \text{ for } t > 0, \text{ where } \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}. \tag{2.4}$$

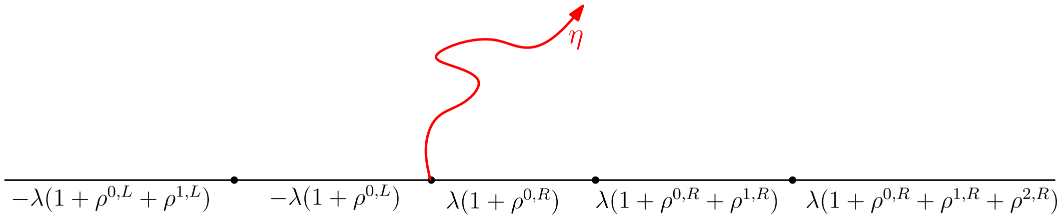

Figure 3: An $\mathrm{SLE}_\kappa(\rho^{0,L},\rho^{1,L};\rho^{0,R},\rho^{1,R},\rho^{2,R})$ process coupled with the GFF h with illustrated boundary conditions as the (zero angle) flow line of h. The θ angle flow line of h then has the law as $\mathrm{SLE}_\kappa(\rho^{0,L}-\frac{\theta\chi}{\lambda},\rho^{1,L};\rho^{0,R}+\frac{\theta\chi}{\lambda},\rho^{1,R},\rho^{2,R})$ process.

To be more precise, let $(K_t)_{t\geq 0}$ be the hull at time t of the $\mathrm{SLE}_\kappa(\underline{\rho})$ process described by the Loewner flow (2.1) with $(W_t, V_t^{i,q})$ solving (2.2), and let $\mathcal{F}_t$ be the filtration generated by $(W_t, V_t^{i,q})$. Let $\mathfrak{h}_t^0$ be the bounded harmonic function on $\mathbb{H}$ with boundary values

$$-\lambda\left(1+\sum_{i=0}^{j}\rho^{i,L}\right) \text{ on } (V_t^{j+1,L},V_t^{j,L}), \quad \text{and } \lambda\left(1+\sum_{i=0}^{j}\rho^{i,R}\right) \text{ on } (V_t^{j,R},V_t^{j+1,R}) \tag{2.5}$$

and $-\lambda$ on $(V_t^{0,L},W_t)$, λ on $(W_t,V_t^{0,R})$ where $\lambda=\frac{\pi}{\sqrt{\kappa}}$, $x^{k+1,L}=-\infty, x^{\ell+1,R}=+\infty$. Set $\mathfrak{h}_t(z)=\mathfrak{h}_t^0(g_t(z))-\chi\arg g_t'(z)$. Let $\tilde{h}$ be a zero boundary GFF on $\mathbb{H}$ and

$$h=\tilde{h}+\mathfrak{h}_0. \tag{2.6}$$

Then as proved in [MS16a, Theorem 1.1], there exists a coupling between h and the $\mathrm{SLE}_\kappa(\underline{\rho})$ process (K_t), such that for any $\mathcal{F}_t$-stopping time τ before the continuation threshold, K_τ is a local set for h and the conditional law of $h|_{\mathbb{H}\setminus K_\tau}$ given $\mathcal{F}_\tau$ is the same as the law of $\mathfrak{h}_\tau+\tilde{h}\circ g_\tau$.

For $\kappa<4$, the $\mathrm{SLE}_\kappa(\underline{\rho})$ coupled with the GFF h as above is referred as a *flow line* of h from 0 to ∞, and we say an $\mathrm{SLE}_\kappa(\underline{\rho})$ curve is a flow line of angle θ if it can be coupled with $h+\theta\chi$ in the above sense. For $\kappa'>4$, the $\mathrm{SLE}_{\kappa'}(\underline{\rho})$ curve coupled with a GFF $-h$ as above is referred as a counterflow lines of h.

So far we have discussed $\mathrm{SLE}_\kappa(\underline{\rho})$ processes on the upper half plane, and for general simply connected domains, the definition can be extended via conformal mappings. Namely, let $x,y\in\partial D$, $\underline{\tilde{x}}\subset\partial D$ be the force points and $\psi: D\to\mathbb{H}$ be a conformal map with $\psi(x)=0,\psi(y)=\infty$. Then a sample from the chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ process in D from x to y is obtained by first taking an curve η from $\mathrm{SLE}_\kappa(\underline{\rho})$ with force points $\psi(\underline{\tilde{x}})$ and then output $\psi^{-1}(\eta)$. Observe that the term $x^{i,q}\cdot(\psi_\eta^{i,q})'(x^{i,q})$ in (1.2) is invariant under dilations of $\mathbb{H}$, which implies that for $a>0$ and an $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ process η with force points $\underline{x}$, the law of $\psi\circ\eta$ is $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ with force points $a\underline{x}$. This implies that the notion of $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ can also be extended to general simply connected domains by the same way. Moreover, if η is a flow line of some GFF h, then $\psi^{-1}(\eta)$ is the flow line of $h\circ\psi-\chi\arg\psi'$ in D from $\psi^{-1}(0)$ to $\psi^{-1}(\infty)$.

To simplify our language, we are going to extend the notion of $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ processes to certain non-simply connected domains. Let $D\subset\mathbb{C}$ be some domain and $x,y\in\partial D$, such that the boundary ∂D consist of two non-crossing simple curves η_D^L,η_D^R running from x to y which possibly intersect and bounce-off each other. Let $\underline{x}^L:=[(x^{k,L},\dots,x^{1,L},x^{0,L})]\subset\eta_D^L$ and $\underline{x}^R:=[(x^{0,R},x^{1,R},\dots,x^{\ell,R})]\subset\eta_D^R$ with $x^{0,L}=x^-$ and $x^{0,R}=x^+$, such that for $i\geq 1$ and $q\in\{L,R\}$, none of the $x^{i,q}$'s lies on $\eta_D^L\cap\eta_D^R$. Further assume η_D^L visits $\underline{x}^L$ in the order of $x^{0,L},\dots,x^{k,L}$, and η_D^R visits $\underline{x}^R$ in the order of $x^{0,R},\dots,x^{\ell,R}$. On each connected component $\tilde{D}$ of D, let $x_{\tilde{D}}$ (resp. $y_{\tilde{D}}$) be the first (resp. last) point on $\partial\tilde{D}$

traced by η_D^L, and let $i_{\tilde D}^L$ and $j_{\tilde D}^L$ (resp. $i_{\tilde D}^R$ and $j_{\tilde D}^R$) be the largest and smallest integer such that $\partial\tilde D\cap\eta_D^L$ (resp. $\partial\tilde D\cap\eta_D^R$) is between $x^{i_{\tilde D},L}$ and $x^{j_{\tilde D},L}$ (resp. $x^{i_{\tilde D},R}$ and $x^{j_{\tilde D},R}$). Let $\mu_{\tilde D}$ be the measure $\widetilde{\mathrm{SLE}}_\kappa(\sum_{i=0}^{i_{\tilde D}^L}\rho^{i,L},\rho^{i_{\tilde D}^L+1,L},\dots,\rho^{j_{\tilde D}^L-1,L};\sum_{i=0}^{i_{\tilde D}^R}\rho^{i,R},\rho^{i_{\tilde D}^R+1,R},\dots,\rho^{j_{\tilde D}^R-1,R};$ $\alpha^{i_{\tilde D}^L+1,L},\dots,\alpha^{j_{\tilde D}^L-1,L};\alpha^{i_{\tilde D}^R+1,R},\dots,\alpha^{j_{\tilde D}^R-1,R})$ in $\eta_{\tilde D}$ for curves running from $x_{\tilde D}$ to $y_{\tilde D}$ with force points $x_{\tilde D}^-,x^{i_{\tilde D}^L+1,L},\dots,x^{j_{\tilde D}^L-1,L};x_{\tilde D}^+,x^{i_{\tilde D}^R+1,R},\dots,x^{j_{\tilde D}^R-1,R}$. Sample $(\eta_{\tilde D})_{\tilde D}$ from the product measure $\prod_{\tilde D}\mu_{\tilde D}$. Concatenate all the $\eta_{\tilde D}$'s, and define the law of this curve from x to y in by $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ in D with force points $\underline{x}$.

We remark that our definition above is natural in the following sense. Temporarily assume $\underline{\alpha}$ is zero. Let $\psi^L:\mathbb{C}\backslash\eta_D^L\to\mathbb{C}\backslash(-\infty,0)$ and $\psi^R:\mathbb{C}\backslash\eta_D^R\to\mathbb{C}\backslash(0,\infty)$ be the conformal maps sending x to 0 and y to ∞. Let $V_0^{i,L}=\psi^L(x^{i,L})$ and $V_0^{i,R}=\psi^R(x^{i,R})$. Consider a GFF h on D with boundary conditions such that $h\circ(\psi^L)^{-1}-\chi\arg((\psi^L)^{-1})'$ agrees with (2.5) on $(-\infty,0)$ and $h\circ(\psi^R)^{-1}-\chi\arg((\psi^R)^{-1})'$ agrees with (2.5) on $(0,\infty)$ with $t=0$. In each connected component $\tilde D$ construct the flow line $\eta_{\tilde D}$ of h from $x_{\tilde D}$ to $y_{\tilde D}$, and the $\mathrm{SLE}_\kappa(\underline{\rho})$ process in D can be understood as the concatenation of all the $\eta_{\tilde D}$'s. For non-zero $\underline{\alpha}$ we can further weight by the corresponding conformal derivatives.

The $\mathrm{SLE}_\kappa(\underline{\rho})$ curve η satisfies the following Domain Markov property. Let τ be some stopping time for η. On the event that τ is less than the continuation threshold, the conditional law of $\eta(t+\tau)_{t\ge0}$ given $\eta([0,\tau])$ is an $\mathrm{SLE}_\kappa(\underline{\rho})$ on $\mathbb{H}\backslash K_\tau$ with force points $\underline{x_\tau}$, where $x_\tau^{i,L}=\inf\{x:x\in\{x^{i,L}\cup(\eta([0,\tau])\cap\mathbb{R})\}\}$ and $x_\tau^{i,R}=\sup\{x:x\in\{x^{i,R}\cup(\eta([0,\tau])\cap\mathbb{R})\}\}$, and if two force points $x^{i,q}$ and $x^{j,q}$ are equal, they could be merged into a single force point of weight $\rho^{i,q}+\rho^{j,q}$.

2.2 The coupling of the two flow lines

One important implication of the flow line coupling of SLE and the GFF is that, for two $\mathrm{SLE}_\kappa(\underline{\rho})$ processes η_1 and η_2 coupled within the same imaginary geometry, one can easily read off the conditional laws of η_1 given η_2 and η_2 given η_1. Suppose η_1 and η_2 are flow lines of h, then given η_1, the conditional law of η_2 is the same as the law of the flow line (with some angle) of the GFF in $\mathbb{H}\backslash\eta_1$ with the *flow line boundary conditions* (see [MS16a, Figure 1.10] for more explanation) induced by η_1, and vice versa for the law of η_1 given η_2.

Now we state the following commutation relation between $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ processes. See Figure 4 for an illustration. Suppose $(\Omega,\mathcal{F})$ is a σ-finite measure space and $X:\Omega\to A$ is a random variable with law μ. Also suppose $(\nu_x)_{x\in A}$ is a family of σ-finite measures on $(\Omega,\mathcal{F})$. By first sampling X from μ and then Y from ν_X, we refer to a sample (X,Y) from the measure $\nu_x(dy)\mu(dx)$ on $(\Omega,\mathcal{F})$.

Proposition 2.1. *Let $\kappa\in(0,4)$. Fix $x^{k,L}<\dots<x^{1,L}<x^{0,L}=0^-<x^{0,R}=0^+<x^{1,R}<\dots<x^{\ell,R}$, $\rho^{i,q}\in\mathbb{R}$, $\alpha^{i,q}\in\mathbb{R}$ for $q\in\{L,R\}$ and $\rho>-2$. Let $\tilde\rho=(\rho+2+\underline{\rho}^L;-\rho-2+\underline{\rho}^R)$. Suppose that $\underline{\rho},\underline{\tilde\rho}$ both satisfy* (1.1)*. The following three laws on pairs of curves (η_1,η_2) agree:*

- *Sample η_1 in $\mathbb{H}$ from 0 to ∞ as $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ with force points $\underline{x}$. Then sample an $\widetilde{\mathrm{SLE}}_\kappa(\rho;\underline{\tilde\rho}^R;0;\underline{\alpha}^R)$ process η_2 in the right part of $\mathbb{H}\backslash\eta_1$ with force points $(0^-;\underline{x}^R)$;*
- *Sample η_2 in $\mathbb{H}$ from 0 to ∞ as $\widetilde{\mathrm{SLE}}_\kappa(\underline{\tilde\rho};\underline{\alpha})$ with force points $\underline{x}$. Then sample an $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho}^L;\rho;\underline{\alpha}^L;0)$ process η_1 in the left part of $\mathbb{H}\backslash\eta_2$ with force points $(\underline{x}^L;0^+)$;*
- *Sample η_1 in $\mathbb{H}$ from 0 to ∞ as $\mathrm{SLE}_\kappa(\underline{\rho})$ with force points $\underline{x}$. Then sample an $\mathrm{SLE}_\kappa(\rho;\underline{\tilde\rho}^R)$ process in the right part of $\mathbb{H}\backslash\eta_1$ with force points $(0^-;\underline{x}^R)$. For $i\ge1$ and $j=1,2$, let $D_{\eta_j}^{i,q}$ be the connected component of $\mathbb{H}\backslash\eta_j$ with $x^{i,q}$ on the boundary, $\sigma_{\eta_j}^{i,q}$ (resp. $\xi_{\eta_j}^{i,q}$) be the first (resp. last) point on $\partial D_{\eta_j}^{i,q}$ traced by η_j, and $\psi_{\eta_j}^{i,q}:D_{\eta_j}^{i,q}\to\mathbb{H}$ be the conformal map sending $(\sigma_{\eta_j}^{i,q},x^{i,q},\xi_{\eta_j}^{i,q})$ to $(0,\pm1,\infty)$ where*

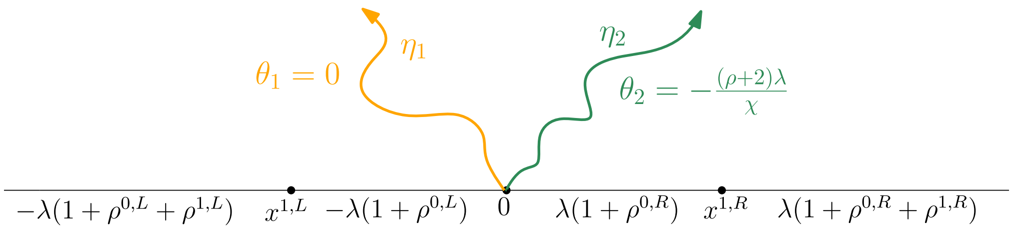

Figure 4: Let h be the GFF with illustrated boundary conditions and (η_1,η_2) be the angle $(0,-\frac{(\rho+2)\lambda}{\chi})$ flow lines of h. Then conditioned on η_1, η_2 is an $\mathrm{SLE}_\kappa(\rho;-\rho-2+\rho^{0,R},\rho^{1,R})$ process, and conditioned on η_2, η_1 is an $\mathrm{SLE}_\kappa(\rho^{0,L},\rho^{1,L};\rho)$ process. By Lemma 2.2, these two conditional laws uniquely characterize the joint law (η_1,η_2). In Proposition 2.1, we prove that once we weight the law of (η_1,η_2) by $|x^{1,L}(\psi^{1,L}_{\eta_1})'(x^{1,L})|^{\alpha^{1,L}}\cdot|x^{1,R}(\psi^{1,R}_{\eta_2})'(x^{1,R})|^{\alpha^{1,R}}$, a sample of (η_1,η_2) can be produced by first sampling an $\widetilde{\mathrm{SLE}}_\kappa(\rho^{0,L},\rho^{1,L};\rho^{0,R},\rho^{1,R};\alpha^{1,L};\alpha^{1,R})$ process η_1 and then an $\widetilde{\mathrm{SLE}}_\kappa(\rho;-\rho-2+\rho^{0,R},\rho^{1,R};0;\alpha^{1,R})$ curve η_2 in the right part of $\mathbb{H}\backslash\eta_1$ (which possibly induces a weighting on the law of η_1), or equivalently first sampling an $\widetilde{\mathrm{SLE}}_\kappa(\rho+2+\rho^{0,L},\rho^{1,L};-\rho-2+\rho^{0,R},\rho^{1,R};\alpha^{1,L};\alpha^{1,R})$ process η_2 and then an $\widetilde{\mathrm{SLE}}_\kappa(\rho^{0,L},\rho^{1,L};\rho;\alpha^{1,L};0)$ process η_1 in the left part of $\mathbb{H}\backslash\eta_2$.

wetake the $+$ sign when $q=R$ and the $-$ sign when $q=L$. Now weight the law of (η_1,η_2) by

$$\prod_{i=1}^{k}\left|x^{i,L}\cdot(\psi^{i,L}_{\eta_1})'(x^{i,L})\right|^{\alpha^{i,L}}\cdot\prod_{i=1}^{\ell}\left|x^{i,R}\cdot(\psi^{i,R}_{\eta_2})'(x^{i,R})\right|^{\alpha^{i,R}}.$$

We remark that the topological configuration of the two curves (η_1,η_2) could be rather complicated, as they may intersect other, and both intersect the boundaries $(-\infty,0)$ and $(0,\infty)$, and we shall apply the definition of $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ processes in non-simply connected domains as specified in the previous section.

Proof. If $\alpha^{i,q}=0$ for all possible i and q, then as argued in [MS16a, Section 6], the pairs (η_1,η_2) generated from the three ways can all be realized as sampling the angle $(\theta_1,\theta_2)=(0,-\frac{(\rho+2)\lambda}{\chi})$ flow lines of the GFF with boundary conditions as (2.5) and (2.6) and therefore the claim follows. Let $\mathcal{P}$ be the corresponding law of these two flow lines, which agrees with the law of (η_1,η_2) constructed as in the third way before we do the weighting.

Now for general $\alpha^{i,q}$ we let $\mathcal{L}$ be the law on the pairs (η_1,η_2) constructed as in the first way of the statement (i.e., first sample η_1 and then η_2). For each $1\le i\le\ell$, let $j^{i,R}_{\eta_1}$ be the smallest $j\ge1$ such that $x^{j,R}\in D^{i,R}_{\eta_1}$, and let $y^{i,R}_{\eta_1}=\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}(x^{i,R})$. Then by definition of the $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ processes, we have

$$\frac{d\mathcal{L}}{d\mathcal{P}}(\eta_1,\eta_2)=\left(\prod_{q\in\{L,R\}}\prod_{i\ge1}\left|x^{i,q}\cdot(\psi^{i,q}_{\eta_1})'(x^{i,q})\right|^{\alpha^{i,q}}\right)\cdot\prod_{i=1}^{\ell}\left|y^{i,R}_{\eta_1}\cdot(\psi^{i,R}_{\eta_2|\eta_1})'(y^{i,R}_{\eta_1})\right|^{\alpha^{i,R}}\tag{2.7}$$

where $\psi^{i,R}_{\eta_2|\eta_1}$ is the corresponding conformal map $\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}(D^{i,R}_{\eta_2})\to\mathbb{H}$ with $(\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}(\sigma^{i,R}_{\eta_2}),y^{i,R}_{\eta_1},\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}(\xi^{i,R}_{\eta_2}))$ mapped to $(0,1,\infty)$. Now we observe that $\psi^{i,R}_{\eta_2}=\psi^{i,R}_{\eta_2|\eta_1}\circ\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}$, and by definition we have

$$\psi^{i,R}_{\eta_1}(z)=\frac{\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}(z)}{\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}(x^{i,R})}=\frac{\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}(z)}{y^{i,R}_{\eta_1}}$$

which implies that

$$(\psi^{i,R}_{\eta_2})'(x^{i,R}) = (\psi^{i,R}_{\eta_2|\eta_1})'(\psi^{j^{i,R}_{\eta_1},R}_{\eta_1}(x^{i,R}))\cdot(\psi^{j^{i,R}_{\eta_1},R}_{\eta_1})'(x^{i,R}) = (\psi^{i,R}_{\eta_2|\eta_1})'(y^{i,R}_{\eta_1})\cdot y^{i,R}_{\eta_1}\cdot(\psi^{i,R}_{\eta_1})'(x^{i,R})$$

and therefore (2.7) can be rewritten as

$$\frac{d\mathcal{L}}{d\mathcal{P}}(\eta_1,\eta_2) = \prod_{i=1}^{k}\left|x^{i,L}\cdot(\psi^{i,L}_{\eta_1})'(x^{i,L})\right|^{\alpha^{i,L}}\cdot\prod_{i=1}^{\ell}\left|x^{i,R}\cdot(\psi^{i,R}_{\eta_2})'(x^{i,R})\right|^{\alpha^{i,R}}. \tag{2.8}$$

Using a similar argument, one can show that if we let $\tilde{\mathcal{L}}$ be the law on the pairs (η_1,η_2) constructed as in the second way of the statement, then $\frac{d\tilde{\mathcal{L}}}{d\mathcal{P}}$ is also the same as (2.8). Therefore the claim follows. □

Proposition 2.1 gives three equivalent ways to characterize the joint law of (η_1,η_2). On the other hand, at least when $\alpha^{i,q}=0$, the two conditional laws $\eta_1|\eta_2$ and $\eta_2|\eta_1$ as in Proposition 2.1 uniquely determines the joint law of (η_1,η_2).

Lemma 2.2. *Let $\kappa\in(0,4)$, $\underline{\rho},\rho,\underline{x}$ be the same as Proposition 2.1. Suppose (η_1,η_2) are random non-crossing curves in $\mathbb{H}$ from 0 to ∞ sampled from some probability measure, such that conditioned on η_1, η_2 is an $\mathrm{SLE}_\kappa(\rho;\tilde{\underline{\rho}}^R)$ in the right part of $\mathbb{H}\backslash\eta_1$, and conditioned on η_2, η_1 is an $\mathrm{SLE}_\kappa(\underline{\rho}^L;\rho)$ in the left part of $\mathbb{H}\backslash\eta_2$. Then the joint law of (η_1,η_2) is the same as in Proposition 2.1 with $\alpha^{i,q}=0$ for all i,q.*

Proof. When η_1 a.s. does not intersect η_2 (i.e., $\rho\ge\frac{\kappa}{2}-2$), the claim follows from the same argument as in [MS16b, Section 4]. For the remaining case, we may first separate the starting and ending points of (η_1,η_2) as in the first step of [MS16b, Proof of Theorem 4.1] and then apply the same argument in [MSW19, Appendix A]. See also Appendix A for an alternative proof based on Markov chain irreducibility results in [MT09] and Lemma 3.1. □

We are going to use the following variant of Lemma 2.2, which follows from exactly the same Markov chain remixing argument in [MS16b, Theorem 4.1] and [MSW19, Appendix A].

Lemma 2.3. *Let $\kappa\in(0,4)$, $\underline{\rho},\rho,\underline{x}$ be the same as Proposition 2.1. For $\varepsilon>0$, let $D_\varepsilon=\bigcup_{q\in\{L,R\}}\bigcup_{i\ge1}B(x^{i,q},\varepsilon)$. Fix ε sufficiently small such that $0\notin D_\varepsilon$. Let $\mathcal{P}$ be the joint law of (η_1,η_2) as described in Lemma 2.2, and $\mathcal{P}_\varepsilon$ be the probability measure given by conditioning $\mathcal{P}$ on the event $E_\varepsilon:=\{\eta_1\cap\overline{D}_\varepsilon=\eta_2\cap\overline{D}_\varepsilon=\emptyset\}$. Now suppose $(\tilde\eta_1,\tilde\eta_2)$ is a sample from some probability measure on curves in $\mathbb{H}$ running from 0 to ∞, such that the conditional law of $\tilde\eta_2$ given $\tilde\eta_1$ is the $\mathrm{SLE}_\kappa(\rho;\tilde{\underline{\rho}}^R)$ in the right part of $\mathbb{H}\backslash\tilde\eta_1$ conditioned on not hitting $\overline{D}_\varepsilon$, and the conditional law of $\tilde\eta_1$ given $\tilde\eta_2$ is the $\mathrm{SLE}_\kappa(\underline{\rho}^L;\rho)$ in the left part of $\mathbb{H}\backslash\tilde\eta_2$ conditioned on not hitting $\overline{D}_\varepsilon$. Then the joint law of $(\tilde\eta_1,\tilde\eta_2)$ is the same as $\mathcal{P}_\varepsilon$ defined above.*

3 Proof of Theorem 1.2

In this section, we prove our main result Theorem 1.2. We start with the $\kappa\in(0,4)$ case, where we first extend Theorem B to $\mathrm{SLE}_\kappa(\rho;\underline{\rho}^R)$ curves (i.e. adding a force point at 0^- in Theorem B) and then apply the SLE resampling properties (Lemma 2.2). For $\kappa\in(4,8)$ case, we shall use the *SLE duality*, and the $\kappa=4$ case is covered in [WW17, Theorem 1.1.6].

We begin with the following variant of [MS17, Lemma 3.9], which roughly states that flow lines of the GFF can stay arbitrarily close to a given curve. Let $\underline{x},\underline{\rho},\eta$ be as in Theorem 1.2. Recall from [MS16a, Remark 5.3] and [MW17, Lemma 2.1] that, for

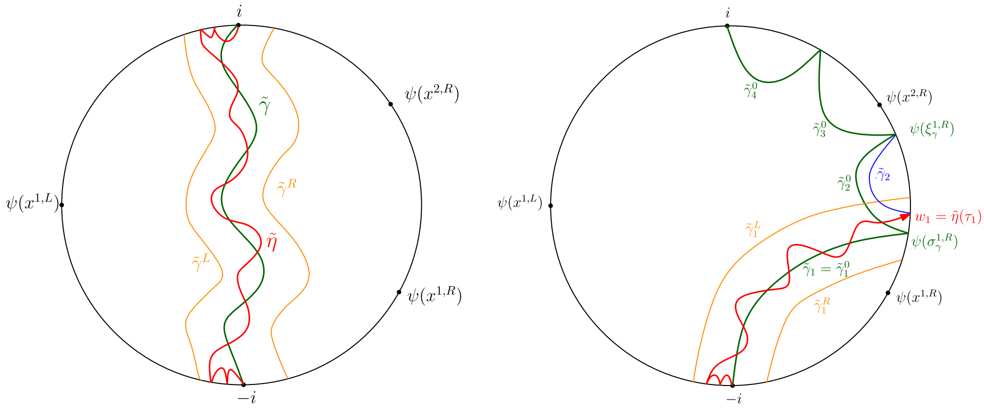

Figure 5: An illustration of Lemma 3.1, where we show that flow lines of the GFF can stay arbitraily close to some given curve. **Left:** the curve $\tilde{\gamma}$ intersects $\partial\mathbb{D}$ only at $\pm i$, and we construct the curves $\tilde{\gamma}^L, \tilde{\gamma}^R$ within the ε-neighborhood of $\tilde{\gamma}$. Let U_γ be the region between $\tilde{\gamma}^L$ and $\tilde{\gamma}^R$, $\tilde{h}_\gamma$ be some GFF in U_γ with the same boundary conditions as $\tilde{h}$ on $\partial\mathbb{D} \cap \partial U_\gamma$ and flow line boundary conditions on $\tilde{\gamma}^L, \tilde{\gamma}^R$ such that the flow line $\tilde{\eta}_\gamma$ of $\tilde{h}_\gamma$ from $-i$ to i a.s. has positive distance to $\tilde{\gamma}^L \cup \tilde{\gamma}^R$. Then by the GFF absolute continuity argument, the flow line $\tilde{\eta}$ is contained in U_γ with positive probability. **Right:** $\tilde{\gamma}$ consists of 4 arcs in $\mathbb{D}$, namely $\tilde{\gamma}_1^0, \dots, \tilde{\gamma}_4^0$. Let $\tilde{\gamma}_1 = \tilde{\gamma}_1^0$ and construct $\tilde{\gamma}_1^L, \tilde{\gamma}_1^R, U_{\tilde{\gamma}_1}$ analogously. Let I_1 be the component of $U_{\tilde{\gamma}_1} \cap \mathbb{D}$ with $\psi(\sigma_\gamma^{1,R})$ on the boundary. By the same argument, the event where $\tilde{\eta}$ hits I_1 at w_1 at some time τ_1 without hitting $\tilde{\gamma}_1^L \cup \tilde{\gamma}_1^R$ has nonzero probability. Then conditioned on $\tilde{\eta}([0,\tau_1])$, we construct the curve $\tilde{\gamma}_2$ from w_1 to $\psi(\xi_\gamma^{1,R})$ staying close to $\tilde{\gamma}_2^0$ and iterate the same argument.

fixed $0 \le j \le \ell$, if $\rho^{0,R} + \cdots + \rho^{j,R} \ge \frac{\kappa}{2} - 2$, then η a.s. does not hit $(x^{j,R}, x^{j+1,R})$, while if $\rho^{0,R} + \cdots + \rho^{j,R} < \frac{\kappa}{2} - 2$, then η has positive probability of hitting $(x^{j,R}, x^{j+1,R})$. Let $J_{\underline{\rho}}$ be the collection of $0 \le j \le \ell$ such that $\rho^{0,R} + \cdots + \rho^{j,R} < \frac{\kappa}{2} - 2$. For a simple curve γ in $\overline{\mathbb{H}}$ from 0 to ∞ with $\gamma \cap \{x^{1,R}, \dots, x^{\ell,R}\} = \emptyset$, let $J_\gamma = \{0 \le j \le \ell : \gamma \cap (x^{j,R}, x^{j+1,R}) \ne \emptyset\}$. We say that γ is admissible w.r.t. $(\underline{\rho}^R, \underline{x}^R)$ if $J_\gamma \subset J_{\underline{\rho}}$. In other words, γ is admissible if it does not hit the intervals $(x^{j,R}, x^{j+1,R})$ where the $\mathrm{SLE}_\kappa(\underline{\rho})$ process η a.s. does not hit. Similarly, we can define the notion of admissibility for $(\underline{\rho}^L, \underline{x}^L)$. We define the domain $D_\gamma^{i,R}$ to be the connected component of $\mathbb{H}\backslash\gamma$ containing $x^{i,R}$, and the points $\sigma_\gamma^{j,R}$ and $\xi_\gamma^{j,R}$ analogously. Consider the conformal map $\psi : \mathbb{H} \to \mathbb{D}$ sending $(0, 1, \infty)$ to $(-i, 1, i)$.

Lemma 3.1. *Let γ be an admissible curve w.r.t. $(\underline{\rho}, \underline{x})$ and $\tilde{\gamma} = \psi(\gamma)$. Let η be an $\mathrm{SLE}_\kappa(\underline{\rho})$ process in $\mathbb{H}$ with force points $\underline{x}$, and $\tilde{\eta} = \psi(\eta)$. For $\varepsilon > 0$, define the event E_ε where (i) $\tilde{\eta}$ stays in the ε-neighborhood of $\tilde{\gamma}$ and (ii) for any $1 \le j \le \ell$, $|\psi(\xi_\eta^{j,R}) - \psi(\xi_\gamma^{j,R})| < \varepsilon$ and $|\psi(\sigma_\eta^{j,R}) - \psi(\sigma_\gamma^{j,R})| < \varepsilon$. Then for any $\varepsilon > 0$, the event E_ε happens with positive probability.*

Proof. We first comment that for each admissible γ, we may construct some admissible curve γ_ε such that (i) $\sigma_{\gamma_\varepsilon}^{j,R} = \sigma_\gamma^{j,R}$ and $\xi_{\gamma_\varepsilon}^{j,R} = \xi_\gamma^{j,R}$ for all $1 \le j \le \ell$ (ii) $\psi(\gamma_\varepsilon)$ is contained in the ε-neighborhood of $\tilde{\gamma}$ and (iii) $\gamma_\varepsilon \cap (\mathbb{R} \cup \{\infty\}) = \{0, \xi_{\gamma_\varepsilon}^{1,R}, \sigma_{\gamma_\varepsilon}^{1,R}, \dots, \xi_{\gamma_\varepsilon}^{\ell,R}, \sigma_{\gamma_\varepsilon}^{\ell,R}\}$. From this point of view, without loss of generality we may assume that $\gamma \cap (\mathbb{R} \cup \{\infty\}) = \{0, \xi_\gamma^{1,R}, \sigma_\gamma^{1,R}, \dots, \xi_\gamma^{\ell,R}, \sigma_\gamma^{\ell,R}\}$.

Let h be a GFF on $\mathbb{H}$ with boundary conditions (2.6) such that η is the flow line of h, and $\tilde{h} = h \circ \psi^{-1} - \chi \arg(\psi^{-1})'$. Then $\tilde{\eta}$ is the flow line of $\tilde{h}$. Assume ε is sufficiently small such that for $1 \le j \le \ell$, $\psi(x^{j,R})$ is not in the ε-neighborhood of $\tilde{\gamma}$. We begin with the case where $\gamma \cap \mathbb{R} = \{0\}$. We choose some simple path $\tilde{\gamma}^L$ (resp. $\tilde{\gamma}^R$) in $\overline{\mathbb{D}}\backslash\tilde{\gamma}$ connecting the points $e^{(3\pi/2-\varepsilon/6)i}$ and $e^{(\pi/2+\varepsilon/6)i}$ (resp. $e^{(3\pi/2+\varepsilon/6)i}$ and $e^{(\pi/2-\varepsilon/6)i}$) such that $\tilde{\gamma}^L \cup \tilde{\gamma}^R$ is contained in the ε-neighborhood of γ, and let U_γ be the component of $\mathbb{D}\backslash(\tilde{\gamma}^L \cup \tilde{\gamma}^R)$ between $\tilde{\gamma}^L$ and $\tilde{\gamma}^R$. Then as in the proof of [MS17, Lemma 3.9], we may construct a GFF $\tilde{h}_\gamma$ in U_γ with the same boundary conditions on $\partial\mathbb{D} \cap \partial U_\gamma$ as $\tilde{h}$ and flow line boundary conditions (see e.g. [MS16a, Figure 1.10]) on $\tilde{\gamma}^L \cup \tilde{\gamma}^R$ such that the flow line of $\tilde{h}_\gamma$ from $-i$ to i has the law as $\mathrm{SLE}_\kappa(\rho^{0,L}, -\rho^{0,L}, \sum_{j=0}^k \rho^{j,L}; \rho^{0,R}, -\rho^{0,R}, \sum_{j=0}^\ell \rho^{j,R})$ with force points $(0^-, e^{(3\pi/2-\varepsilon/6)i}, e^{(\pi/2+\varepsilon/6)i}; 0^+, e^{(3\pi/2+\varepsilon/6)i}, e^{(\pi/2-\varepsilon/6)i})$ and thus a.s. has positive distance from $\tilde{\gamma}^L \cup \tilde{\gamma}^R$. We may choose some (non-random) constant $\zeta > 0$ small such that this distance is at least ζ with probability greater than 1/2. Therefore it follows from the same argument of [MS17, Lemma 3.9] that, if we set $U_\gamma^\zeta := \{z \in U_\gamma; \mathrm{dist}(z, \tilde{\gamma}^L \cup \tilde{\gamma}^R) > \zeta\}$, by the GFF absolute continuous property [MS16a, Proposition 3.4], the law of $\tilde{h}_\gamma$ is absolutely continuous w.r.t. $\tilde{h}$ when restricted to the domain U_γ^ζ. Since flow lines are a.s. determined by and local sets of the GFF [MS16a, Theorem 1.2], and the flow line of $\tilde{h}_\gamma$ is contained within U_γ^ζ with probability at least 1/2, it follows that, with positive probability $\tilde{\eta}$ is contained in U_γ^ζ and thus in the ε-neighborhood of $\tilde{\gamma}$, which finishes the case when $\gamma \cap \mathbb{R} = \{0\}$.

For rest of the case, we write $\tilde{\gamma} = \bigcup_{i=1}^m \tilde{\gamma}_i^0$ where each $\tilde{\gamma}_i^0 : [0,1] \to \overline{\mathbb{D}}$ is a subarc of $\tilde{\gamma}$ intersecting $\partial\mathbb{D}$ only at the endpoints, and they are aligned in the order traced by $\tilde{\gamma}$. Let $\tilde{\gamma}_1 = \tilde{\gamma}_1^0$. We construct the simple curve $\tilde{\gamma}_1^L$ (resp. $\tilde{\gamma}_1^R$) in $\overline{\mathbb{D}}\backslash\tilde{\gamma}_1$ within the ε/m neighborhood of $\tilde{\gamma}_1$ connecting a point on $\partial\mathbb{D}$ on the left (resp. right) side of $\tilde{\gamma}_1$ and a point on $\partial\mathbb{D}$ on the same side of $\tilde{\gamma}_1$. Let $U_{\tilde{\gamma}_1}$ be the component of $\mathbb{D}\backslash(\tilde{\gamma}_1^L \cup \tilde{\gamma}_1^R)$ between $\tilde{\gamma}_1^L$ and $\tilde{\gamma}_1^R$, and I_1 be the component of $\partial U_{\tilde{\gamma}_1} \cap \partial\mathbb{D}$ with $\tilde{\gamma}_1(1)$ on its boundary. Then as above and [MS17, Lemma 3.9], we may construct a GFF $\tilde{h}_{\tilde{\gamma}_1}$ on $U_{\tilde{\gamma}_1}$ with same boundary conditions as $\tilde{h}$ on $\partial U_{\tilde{\gamma}_1} \cap \partial\mathbb{D}$ and flow line condition on $\tilde{\gamma}_1^L \cup \tilde{\gamma}_1^R$, such that the flow line of $\tilde{h}_{\tilde{\gamma}_1}$ a.s. has positive distance to $\tilde{\gamma}_1^L \cup \tilde{\gamma}_1^R$. Again using the same GFF absolute continuity of $\tilde{h}_{\tilde{\gamma}_1}$ w.r.t. $\tilde{h}$ as above and in [MS17, Lemma 3.9], the event E_1 where the flow line $\tilde{\eta}$ first hits I_1 at w_1 at time τ_1 without hitting $\tilde{\gamma}_1^L \cup \tilde{\gamma}_1^R$ has positive probability. On E_1, we choose a simple curve $\tilde{\gamma}_2 : [0,1] \to \overline{\mathbb{D}}$ such that $\tilde{\gamma}_2(0) = w_1$, $\tilde{\gamma}_2(1) = \tilde{\gamma}_2^0(1)$, $\tilde{\gamma}_2((0,1)) \cap \partial\mathbb{D} = \emptyset$ and $\tilde{\gamma}_2$ stays within the $2\varepsilon/m$-neighborhood of $\tilde{\gamma}_2^0$. Define $\tilde{\gamma}_2^L, \tilde{\gamma}_2^R, I_2$ analogously (with ε/m replaced by $2\varepsilon/m$). It follows from the same argument that, conditioned on E_1 and $\tilde{\eta}|_{[0,\tau_1]}$, the event E_2 where $\tilde{\eta}$ first hits I_2 at w_2 at time τ_2 without hitting $\tilde{\gamma}_2^L \cup \tilde{\gamma}_2^R$ has positive probability. Now we can conclude the proof by iterating this process, except that at the final step we construct the curve $\tilde{\gamma}_m$ and apply the argument from the $\gamma \cap \mathbb{R} = \{0\}$ case. □

Lemma 3.2. *Theorem 1.2 holds for* $\kappa \in (0,4)$, $k = 0$ *and* $\rho^{0,L} \leq 0$. *That is, under* $J(z) = -1/z$, *the law of time reversal of* $\mathrm{SLE}_\kappa(\rho^{0,L}; \underline{\rho}^R)$ *processes in* $\mathbb{H}$ *with force points* $(0^-; \underline{x}^R)$ *agrees with the* $\frac{1}{Z}\widetilde{\mathrm{SLE}}_\kappa(\hat{\underline{\rho}}; \hat{\underline{\alpha}})$ *described in Theorem 1.2 for some constant* $Z = Z(\underline{\rho}, \underline{x}^R) \in (0,\infty)$.

Proof. We sample an $\mathrm{SLE}_\kappa(\underline{\rho}^R)$ curve η_2 in $\mathbb{H}$ from 0 to ∞ with force points $\underline{x}^R$, and conditioned on η_2, we sample an $\mathrm{SLE}_\kappa(-\rho^{0,L} - 2; \rho^{0,L})$ process η_1 from 0 to ∞ in the left part of $\mathbb{H}\backslash\eta_2$ with force points 0^- and 0^+. Then it follows from the construction in Proposition 2.1 that conditioned on η_1, η_2 is an $\mathrm{SLE}_\kappa(\rho^{0,L}; \underline{\rho}^R)$ process in the right part of $\mathbb{H}\backslash\eta_2$ with force points $(0^-; \underline{x}^R)$.

For $i = 1, 2$, let $\hat{\eta}_i = \mathcal{R}(J \circ \eta_i)$. Recall the notion of $\hat{x}^{i,q}, \hat{\rho}^{i,q}, \hat{\alpha}^{i,q}$ in the statement of Theorem 1.2. Now by Theorem B, we know that law of $\hat{\eta}_2$ is the probability measure proportional to $\widetilde{\mathrm{SLE}}_\kappa(\hat{\underline{\rho}}^L; \hat{\underline{\alpha}}^L)$ process from 0 to ∞ with force points $\hat{\underline{x}}^L$. Moreover, by Theorem A, the conditional law of $\hat{\eta}_1$ given $\hat{\eta}_2$ is the $\mathrm{SLE}_\kappa(\hat{\rho}^{0,R}; -\hat{\rho}^{0,R} - 2)$ process in the right part of $\mathbb{H}\backslash\hat{\eta}_2$ with force points 0^- and 0^+. Therefore it follows from Proposition 2.1 that the conditional law of $\hat{\eta}_2$ given $\hat{\eta}_1$ is a constant (possibly depending on $\hat{\eta}_1$) times

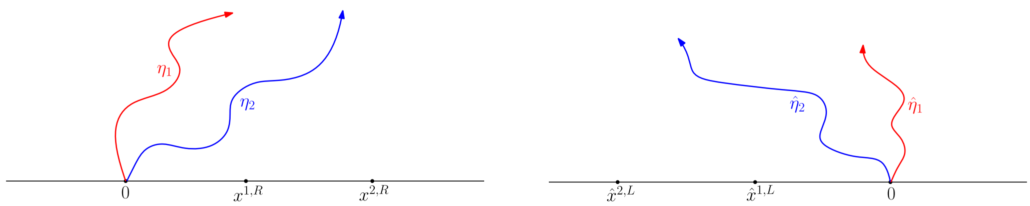

Figure 6: An illustration of the proof of Lemma 3.2 for $\ell = 2$. **Left:** We sample an $\mathrm{SLE}_\kappa(\rho^{0,R}, \rho^{1,R}, \rho^{2,R})$ process η_2, and an $\mathrm{SLE}_\kappa(-\rho^{0,L} - 2; \rho^{0,L})$ process η_1 in the left part of $\mathbb{H}\backslash\eta_2$. Then the conditional law of η_2 given η_1 is the $\mathrm{SLE}_\kappa(\rho^{0,L}; \rho^{0,R}, \rho^{1,R}, \rho^{2,R})$ process in the right part of $\mathbb{H}\backslash\eta_1$ with force points $0^-; 0^+, x^{1,R}, x^{2,R}$. **Right:** For $i = 1, 2$, $\hat\eta_i$ is the time reversal of $J \circ \eta_i$, and $\hat x^{1,L} = -1/x^{2,R}$, $\hat x^{2,L} = -1/x^{1,R}$. By Theorem B, $\hat\eta_2$ has the law $\frac{1}{Z}\widetilde{\mathrm{SLE}}_\kappa(\rho^{0,R} + \rho^{1,R} + \rho^{2,R}, -\rho^{2,R}, -\rho^{1,R}; \frac{\rho^{2,R}(4-\kappa)}{2\kappa}, \frac{\rho^{1,R}(4-\kappa)}{2\kappa})$ with force points $0^-, \hat x^{1,L}, \hat x^{2,L}$. By Theorem A, conditioned on $\hat\eta_2$, $\hat\eta_1$ is the $\mathrm{SLE}_\kappa(\hat\rho^{0,R}; -\hat\rho^{0,R} - 2)$ process in the right part of $\mathbb{H}\backslash\hat\eta_2$ with force points 0^- and 0^+. Therefore by Proposition 2.1, the conditional law of $\hat\eta_2$ given $\hat\eta_1$ is $\frac{1}{Z(\eta_1)}\widetilde{\mathrm{SLE}}_\kappa(\rho^{0,R} + \rho^{1,R} + \rho^{2,R}, -\rho^{2,R}, -\rho^{1,R}; \rho^{0,L}; \frac{\rho^{2,R}(4-\kappa)}{2\kappa}, \frac{\rho^{1,R}(4-\kappa)}{2\kappa}; 0)$ in the left part of $\mathbb{H}\backslash\hat\eta_1$ with force points $0^-, \hat x^{1,L}, \hat x^{2,L}; 0^+$, and the claim follows by comparing the two figures. In Lemma 3.4, we further show that the constant $Z(\eta_1)$ is actually independent of η_1.

$\widetilde{\mathrm{SLE}}_\kappa(\hat{\underline{\rho}}^L; \hat\rho^{0,R}; \hat{\underline{\alpha}}^L)$ in the left part of $\mathbb{H}\backslash\hat\eta_1$ with force points $(\hat{\underline{x}}^L; 0^+)$. This justifies the reversibility of an $\mathrm{SLE}_\kappa(\rho^{0,L}; \underline{\rho}^R)$ process in the right part of $\mathbb{H}\backslash\eta_1$ with force points $(0^-; \underline{x}^R)$.

Now consider the conformal map $\psi : \mathbb{H} \to \mathbb{D}$ sending $(0, 1, \infty)$ to $(-i, 1, i)$, and let $\tilde\eta_j = \psi \circ \eta_j$ for $j = 1, 2$. For $\varepsilon > 0$, let A_ε be the partial annulus $\{z : \frac{\pi}{2} - \varepsilon < \arg z < \frac{3\pi}{2} + \varepsilon;\ 1 - \varepsilon < |z| < 1\}$. Fix $\varepsilon_0 > 0$ small such that A_{ε_0} contains none of points in $\psi(\underline{x}^R)$ other than $\psi(x^{0,R}) = -i$ and let $\varepsilon < \varepsilon_0$. Then by Lemma 3.1, the event E'_ε where $\tilde\eta_1$ is contained in the domain A_ε has positive probability. On the event E'_ε, let σ_ε be the last point on the arc $\{z : \frac{3\pi}{2} < \arg z < 2\pi;\ |z| = 1\}$ hit by $\tilde\eta_1$, and ξ_ε be the first point on the arc $\{z : 0 < \arg z < \frac{\pi}{2};\ |z| = 1\}$ hit by $\tilde\eta_1$. Let D_1^ε be the connected component of $\mathbb{D}\backslash\eta_1$ with 1 on the boundary, and $\psi_\varepsilon : D_1^\varepsilon \to \mathbb{D}$ sending $(\sigma_\varepsilon, 1, \xi_\varepsilon)$ to $(-i, 1, i)$. Then $\psi_\varepsilon \circ \tilde\eta_2$ is an $\mathrm{SLE}_\kappa(\rho^{0,L}; \underline{\rho}^R)$ process in $\mathbb{D}$ with force points $((-i)^-; \psi_\varepsilon \circ \psi(\underline{x}^R))$ (with $\psi_\varepsilon((-i)^+)$ identified as $(-i)^+$), and the law of its time reversal is proportional to the $\widetilde{\mathrm{SLE}}_\kappa(\hat{\underline{\rho}}^L; \hat\rho^{0,R}; \hat{\underline{\alpha}}^L; 0)$ process. Therefore as we condition on E'_ε and send $\varepsilon \to 0$, D_1^ε converges to $\mathbb{D}$ in Caratheodory topology, and $\psi_\varepsilon \circ \psi(\underline{x}^R)$ converges to $\psi(\underline{x}^R)$. To conclude the proof, we look at the time reversal result of $\psi_\varepsilon \circ \eta_2$ in $\mathbb{D}$, send $\varepsilon \to 0$ and apply the continuity of $\mathrm{SLE}_\kappa(\underline{\rho})$ processes w.r.t. the location of force points from [MS16a, Section 2]. To be more precise, suppose η and $(\eta^\varepsilon)_{\varepsilon>0}$ are $\mathrm{SLE}_\kappa(\underline{\rho})$ processes in $\mathbb{D}$ from $-i$ to i with force points $\underline{y}$ and $\underline{y}^\varepsilon$, such that $y^{0,L,\varepsilon} = (-i)^-$, $y^{0,R,\varepsilon} = (-i)^+$ and $y^{j,q,\varepsilon} \to y^{j,q}$ as $\varepsilon \to 0$. Let h and $(h^\varepsilon)_{\varepsilon>0}$ be the corresponding GFF on $\mathbb{D}$ such that η and $(\eta^\varepsilon)_{\varepsilon>0}$ are the flow lines of h and $(h^\varepsilon)_{\varepsilon>0}$ from $-i$. For $\delta > 0$, let $D'_\delta := \bigcup_{q\in\{L,R\}}\bigcup_{j\geq 1} B(y^{j,L}, \delta)$. By [MS16a, Proposition 3.4, Remark 3.5], the total variation distance between $h^\varepsilon|_{\mathbb{D}\backslash D'_\delta}$ and $h|_{\mathbb{D}\backslash D'_\delta}$ goes to 0 as $\varepsilon \to 0$ for fixed δ. Since flow lines are deterministic functions and local sets of the GFF [MS16a, Theorem 1.2], it follows that the law of η^ε conditioned on not hitting D'_δ converges in total variation distance to that of η conditioned on not hitting D'_δ as $\varepsilon \to 0$. From this argument, the law of the time reversal of an $\mathrm{SLE}_\kappa(\rho^{0,L}; \underline{\rho}^R)$ process conditioned on having distance δ to $x^{j,R}$ for $j \geq 1$ agrees with $\widetilde{\mathrm{SLE}}_\kappa(\hat{\underline{\rho}}^L; \hat\rho^{0,R}; \hat{\underline{\alpha}}^L; 0)$ conditioned on the same event (up to a multiplicative constant), and the claim follows by taking $\delta \to 0$. $\square$

Corollary 3.3. *Let $\underline{\rho}, \underline{x}, \underline{\hat{x}}, \underline{\hat{\rho}}$ be as in Lemma 3.2, and γ be a curve in $\overline{\mathbb{H}}$ from 0 to ∞ which is admissible w.r.t. $(2+\rho^{0,L}+\underline{\rho}^R, \underline{x}^R)$. Let H_γ^R be the right part of $\mathbb{H}\backslash\gamma$. Sample an $\mathrm{SLE}_\kappa(\underline{\rho})$ process η_0 in H_γ^R with force points $\underline{x}$. Then there exists some constant $Z(\gamma, \underline{\rho}, \underline{x})$ such that the law of the time reversal of $J(\eta_0)$ is equal to $\frac{1}{Z(\gamma,\underline{\rho},\underline{x})}$ times $\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat{\rho}};\underline{\hat{\alpha}})$ in $J(H_\gamma^R)$ with the force points $\underline{\hat{x}}$.*

Proof. We apply Lemma 3.2 within (finitely many) connected components whose boundaries contain the force points $\underline{x}$, and apply Theorem A for rest of the connected components (where there are no constants). Then the constant $Z(\gamma, \underline{\rho}, \underline{x})$ is now a (finite) product of the corresponding constants in each of the connected components. □

The next lemma states that in some sense, the constant Z in Lemma 3.2 does not depend on the choice of $\underline{x}$. This follows by comparing the two ways of viewing the marginal law of the $\hat{\eta}_1$ above: directly applying Lemma 3.2, and applying Proposition 2.1 to the pair $(\hat{\eta}_1, \hat{\eta}_2)$.

Lemma 3.4. *In the setting of Corollary 3.3, the constant $Z(\gamma, \underline{\rho}, \underline{x})$ does not depend on γ.*

Proof. Let η_1, η_2, $\hat{\eta}_1$, $\hat{\eta}_2$ be as in the proof of Lemma 3.2 and $\mathbb{P}$ be the corresponding background probability measure. Let $\hat{\mathcal{P}}$ be the probability measure describing the law of (η_1^0, η_2^0) where η_2^0 is an $\mathrm{SLE}_\kappa(\underline{\hat{\rho}}^L)$ with force points $\underline{\hat{x}}^L$ and η_1^0 is an $\mathrm{SLE}_\kappa(\rho^{0,L}, -\rho^{0,L}-2)$ in the right part of $\mathbb{H}\backslash\eta_2^0$ with force points $(0^-;0^+)$. Then by applying Lemma 3.2 (to η_2) and Proposition 2.1, the law of $(\hat{\eta}_1, \hat{\eta}_2)$ is absolutely continuous w.r.t. $\hat{\mathcal{P}}$ with Radon-Nikodym derivative

$$\frac{1}{Z(\underline{\rho}^R;\underline{x}^R)}\prod_{i=1}^{\ell}|\hat{x}^{i,L}\cdot\psi'_{\hat{\eta}_2}(\hat{x}^{i,L})|^{\frac{\hat{\rho}^{i,R}(\kappa-4)}{2\kappa}}. \tag{3.1}$$

On the other hand, since the conditional law of η_2 given η_1 is $\mathrm{SLE}_\kappa(\rho^{0,L};\underline{\rho}^R)$, it follows from the definition of the constant $Z(\gamma, \underline{\rho}, \underline{x})$ that the conditional law of $\hat{\eta}_2$ given $\hat{\eta}_1$ is $\frac{1}{Z(\eta_1,\underline{\rho},\underline{x})}\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat{\rho}};\underline{\hat{\alpha}})$ in the left part of $\mathbb{H}\backslash\hat{\eta}_1$. Moreover, we know from Proposition 2.1 that the marginal law of η_1 is $\mathrm{SLE}_\kappa(-\rho^{0,L}-2;2+\rho^{0,L}+\underline{\rho}^R)$ with force points $0^-;\underline{x}^R$. By Lemma 3.2, there exists some constant $Z_1 := Z((-\rho^{0,L}-2;2+\rho^{0,L}+\underline{\rho}^R),\underline{x}^R)$ such that the marginal law of the curve $\hat{\eta}_1$ is $1/Z_1$ times the $\widetilde{\mathrm{SLE}}_\kappa(\rho^{0,L}+2+\underline{\hat{\rho}}^L;-\rho^{0,L}-2;\underline{\hat{\alpha}}^L;0)$ with force points $(\underline{\hat{x}}^L;0^+)$. Together with Proposition 2.1, we infer that the law of $(\hat{\eta}_1, \hat{\eta}_2)$ is absolutely continuous w.r.t. $\hat{\mathcal{P}}$ with Radon-Nikodym derivative

$$\frac{1}{Z(\eta_1,\underline{\rho},\underline{x})Z((-\rho^{0,L}-2;2+\rho^{0,L}+\underline{\rho}^R),\underline{x}^R)}\prod_{i=1}^{\ell}|\hat{x}^{i,L}\cdot\psi'_{\hat{\eta}_2}(\hat{x}^{i,L})|^{\frac{\hat{\rho}^{i,R}(\kappa-4)}{2\kappa}}. \tag{3.2}$$

It then follows by comparing (3.1) with (3.2) that $Z(\eta_1, \underline{\rho}, \underline{x}) = \frac{Z(\underline{\rho}^R;\underline{x}^R)}{Z((-\rho^{0,L}-2;2+\rho^{0,L}+\underline{\rho}^R),\underline{x}^R)}$ a.s.. We condition on the positive probability event E_ε for (η_1,γ) as in Lemma 3.1. Then the domain $D_{\eta_1}^{j,R}$ is converging in Caratheodory topology to $D_\gamma^{j,R}$ as $\varepsilon\to 0$, and the claim follows from a similar argument as in the end of the proof of Lemma 3.2 via $\mathrm{SLE}_\kappa(\underline{\rho})$ continuity over the location of force points. □

Proposition 3.5. *Theorem 1.2 holds for $\kappa\in(0,4)$.*

Proof. The proof is organized as follows. We first construct a pair of reversed curves $(\hat{\eta}_1, \hat{\eta}_2)$ by Proposition 2.1, and then apply Lemma 3.2 to get the conditional laws of $\eta_i := \mathcal{R}(\eta_i)$ given η_j for $1\le i\ne j\le 2$. Finally we apply Lemma 2.3 to identify the law of the forward curves (η_1, η_2) with the usual $\mathrm{SLE}_\kappa(\underline{\rho})$.

Let $\hat\eta_1$ be an $\mathrm{SLE}_\kappa(\underline{\hat\rho})$ process in $\mathbb{H}$ from 0 to ∞ with force points $\underline{\hat x}$. Pick $0<a<2$ such that the weights $\underline{\tilde\rho}=(a+\underline{\hat\rho}^L;-a+\underline{\hat\rho}^R)$ also satisfy the bound (1.1). Conditioned on $\hat\eta_1$, sample an $\mathrm{SLE}_\kappa(a-2;\underline{\tilde\rho}^R)$ process $\hat\eta_2$ in the right component of $\mathbb{H}\backslash\hat\eta_1$ with force points $(0^-;\underline{\hat x}^R)$. For $\varepsilon>0$, let $D_\varepsilon=\bigcup_{q\in\{L,R\}}\bigcup_{i\geq 1}B(x^{i,L},\varepsilon)$, $\hat D_\varepsilon=J(D_\varepsilon)$ and suppose ε is sufficiently small such that $0,\infty\notin D_\varepsilon$. Let $\hat F_\varepsilon$ be the event where $(\hat\eta_1,\hat\eta_2)$ are disjoint from $\overline{\hat D}_\varepsilon$. Define the conformal maps $\psi^{i,q}_{\hat\eta_j}$ as in Proposition 2.1. Let $\hat{\mathcal{P}}$ be the law of $(\hat\eta_1,\hat\eta_2)$, and define the probability measure $\mathcal{Q}_\varepsilon$ on pairs of non-crossing simple curves from 0 to ∞ by

$$\frac{d\mathcal{Q}_\varepsilon}{d\hat{\mathcal{P}}}(\hat\eta_1,\hat\eta_2)=\frac{1}{Z_\varepsilon}1_{\hat F_\varepsilon}\cdot\prod_{i=1}^k|\hat x^{i,L}\cdot(\psi^{i,L}_{\hat\eta_1})'(\hat x^{i,L})|^{\frac{\hat\rho^{i,L}(\kappa-4)}{2\kappa}}\cdot\prod_{j=1}^{\ell}|\hat x^{j,R}\cdot(\psi^{j,R}_{\hat\eta_2})'(\hat x^{j,R})|^{\frac{\hat\rho^{j,R}(\kappa-4)}{2\kappa}}\tag{3.3}$$

where Z_ε is the normalizing constant making $\mathcal{Q}_\varepsilon$ a probability measure. Observe that on the event $\hat F_\varepsilon$, by Koebe's 1/4 theorem, there exists some constant M depending only on ε and $\underline{x}$ such that $1/M<|(\psi^{i,q}_{\hat\eta_j})'(\hat x^{i,q})|<M$, which implies that the constant Z_ε is well-defined. Moreover, by Proposition 2.1, under the measure $\mathcal{Q}_\varepsilon$, conditioned on $\hat\eta_1$, $\hat\eta_2$ is an $\widetilde{\mathrm{SLE}}_\kappa(a-2;\underline{\tilde\rho}^R;\underline{\hat\alpha}^R)$ process in the right component of $\mathbb{H}\backslash\hat\eta_1$ with force points $(0^-;\underline{x}^R)$ conditioned on not hitting $\overline{\hat D}_\varepsilon$, while conditioned on $\hat\eta_2$, $\hat\eta_1$ is an $\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat\rho}^L;a-2;\underline{\hat\alpha}^L)$ process in the left component of $\mathbb{H}\backslash\hat\eta_2$ with force points $(\underline{\hat x}^L;0^+)$ conditioned on not hitting $\overline{\hat D}_\varepsilon$, where $\hat\alpha^{i,q}=\frac{\hat\rho^{i,q}(\kappa-4)}{2\kappa}$.

For $i=1,2$, let $\eta_i=\mathcal{R}(J\circ\hat\eta_i)$. Then by Lemma 3.2, under the measure $\mathcal{Q}_\varepsilon$, given η_1, the conditional law of η_2 is the $\mathrm{SLE}_\kappa(-a+\underline{\rho}^L;a-2)$ process in the left component of $\mathbb{H}\backslash\eta_1$ with force points $(\underline{x}^L;0^+)$ conditioned on not hitting $\overline{D}_\varepsilon$, and the conditional law of η_1 given η_2 is the $\mathrm{SLE}_\kappa(a-2;\underline{\rho}^R)$ process in the right component of $\mathbb{H}\backslash\eta_2$ with force points $(0^-;\underline{x}^R)$ conditioned on not hitting $\overline{D}_\varepsilon$. Let $\tilde\eta_1$ be an independent $\mathrm{SLE}_\kappa(\underline{\rho})$ process in $\mathbb{H}$ from 0 to ∞ with force points $\underline{x}$, and sample an $\mathrm{SLE}_\kappa(-a+\underline{\rho}^L;a-2)$ process $\tilde\eta_2$ in the left component of $\mathbb{H}\backslash\tilde\eta_1$ with force points $(\underline{x}^L;0^+)$. Therefore (η_1,η_2) and $(\tilde\eta_1,\tilde\eta_2)$ satisfy the same resampling properties as in Lemma 2.3. Then by Lemma 2.3, the joint law of (η_1,η_2) agrees with that of $(\tilde\eta_1,\tilde\eta_2)$ conditioned on $\{\tilde\eta_1\cap\overline{D}_\varepsilon=\tilde\eta_2\cap\overline{D}_\varepsilon=\emptyset\}$. In particular, the marginal law of η_1 under $\mathcal{Q}_\varepsilon$ is the $\mathrm{SLE}_\kappa(\underline{\rho})$ process conditioned on not hitting $\overline{D}_\varepsilon$ and weighted by the probability where the $\mathrm{SLE}_\kappa(-a+\underline{\rho}^L;a-2)$ process in the left component of $\mathbb{H}\backslash\eta_1$ is disjoint from $\overline{D}_\varepsilon$.

On the other hand, by Proposition 2.1, a sample $(\hat\eta_1,\hat\eta_2)$ from $\mathcal{Q}_\varepsilon$ can be produced by (i) sampling $\hat\eta_1$ from $\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat\rho};\underline{\hat\alpha})$ process conditioned on not hitting $\overline{\hat D}_\varepsilon$ (ii) weighting the law of $\hat\eta_1$ by $Z_0(\hat\eta_1)$, where $Z_0(\hat\eta_1)$ is the measure of an $\widetilde{\mathrm{SLE}}_\kappa(a-2;\underline{\tilde\rho}^R;0;\underline{\hat\alpha}^R)$ process in the right component of $\mathbb{H}\backslash\hat\eta_1$ with force points $(0^-;\underline{x}^R)$ being disjoint from $\overline{\hat D}_\varepsilon$ and (iii) sampling an $\widetilde{\mathrm{SLE}}_\kappa(a-2;\underline{\tilde\rho}^R;0;\underline{\hat\alpha}^R)$ process $\hat\eta_2$ in the right component of $\mathbb{H}\backslash\hat\eta_1$ with force points $(0^-;\underline{x}^R)$ conditioned on not hitting $\overline{\hat D}_\varepsilon$. Meanwhile, by Lemma 3.2, $Z_0(\hat\eta_1)$ is equal to $Z(\eta_1,(-a+\underline{\rho}^L;a-2),\underline{x}^R)$ times the probability of an $\mathrm{SLE}_\kappa(-a+\underline{\rho}^L;a-2)$ process in the left component of $\mathbb{H}\backslash\eta_1$ being disjoint from $\overline{D}_\varepsilon$. By Lemma 3.1, the latter probability is positive for any fixed η_1, while by Lemma 3.4, the constant $Z(\eta_1,(-a+\underline{\rho}^L;a-2),\underline{x}^R)$ is independent of η_1. Therefore by comparing the marginal laws of η_1 and $\hat\eta_1$, we conclude that under $J(z)=-1/z$, the time reversal of an $\mathrm{SLE}_\kappa(\underline{\rho})$ process with force point $\underline{x}$ conditioned on not hitting $\overline{D}_\varepsilon$ agree with the $\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat\rho};\underline{\hat\alpha})$ process with force point $\underline{\hat x}$ conditioned on not hitting $\overline{\hat D}_\varepsilon$. Since $\varepsilon>0$ can be arbitrarily small, the claim therefore follows. $\square$

For $\kappa\in(4,8)$, the argument is based on the following *SLE duality* argument, which follows from [Zha08a, Theorem 5.1] and [MS16a, Theorem 1.4, Proposition 7.30].

Proposition 3.6. *Let $\kappa \in (4,8]$, $\tilde\kappa = \frac{16}{\kappa}$ and $\underline{\rho}$ satisfying* (1.1) *and $x^{k,L} < \cdots < x^{0,L} = 0^- < x^{0,R} = 0^+ < \cdots < x^{\ell,R}$. Let $\rho^{k+1,L} = -\sum_{i=0}^k \rho^{i,L}$ and $\rho^{\ell+1,R} = -\sum_{i=0}^\ell \rho^{i,R}$. Let $\hat\rho^{i,L} = -\rho^{\ell+1-i,R}$ for $0 \le i \le \ell$ and $\hat\rho^{j,R} = -\rho^{k+1-j,L}$ for $0 \le j \le k$. Let η' be an $\mathrm{SLE}_\kappa(\underline{\rho})$ process in $\mathbb{H}$ from 0 to ∞. Then the left boundary η_L of η' is an $\mathrm{SLE}_{\tilde\kappa}(\frac{\tilde\kappa}{2}-2+\frac{\tilde\kappa}{4}\underline{\hat\rho}^L; \tilde\kappa-4+\frac{\tilde\kappa}{4}\underline{\hat\rho}^R)$ process from ∞ to 0 with force points $\underline{\hat x} := (+\infty, x^{\ell,R}, \dots, x^{1,R}; -\infty, x^{k,L}, \dots, x^{1,L})$, and the right boundary η_R of η' is an $\mathrm{SLE}_{\tilde\kappa}(\tilde\kappa-4+\frac{\tilde\kappa}{4}\underline{\hat\rho}^L; \frac{\tilde\kappa}{2}-2+\frac{\tilde\kappa}{4}\underline{\hat\rho}^R)$ process from ∞ to 0 with force points $\underline{\hat x}$. Moreover, conditioned on η_L and η_R, η' is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$ process independently in each connected component of $\mathbb{H}\backslash(\eta_L \cup \eta_R)$ between η_L and η_R, and conditioned on η_L, η_R is an $\mathrm{SLE}_{\tilde\kappa}(\tilde\kappa-4+\frac{\tilde\kappa}{4}\underline{\hat\rho}^L; -\frac{\tilde\kappa}{2})$ process in $\mathbb{H}\backslash\eta_L$ to the right of η_L from ∞ to 0 with force points $(+\infty, x^{\ell,R}, \dots, x^{1,R}; -\infty)$.*

Proposition 3.7. *Theorem 1.2 holds for $\kappa \in (4,8]$.*

Proof. Let η' be an $\mathrm{SLE}_\kappa(\underline{\rho})$ process in $\mathbb{H}$ from 0 to ∞ with force point $\underline{x}$, η_L, η_R be its left and right boundary, and $\mathcal{R}(\eta')$ be the time-reversal of η'. Let $\tilde\kappa = \frac{16}{\kappa}$, $\tilde x^{i,L} = x^{\ell+1-i,R}$ for $0 \le i \le \ell$ and $\tilde x^{j,R} = x^{k+1-j,L}$ for $0 \le j \le k$, where $x^{k+1,L} = -\infty$ and $x^{\ell+1,R} = +\infty$. Let $\hat\rho^{i,q}$ be as in the statement of Proposition 3.6. Note that $\underline{\hat x} = J(\underline{\tilde x})$. Then by Proposition 3.5 and Proposition 3.6, the law of the left boundary $\mathcal{R}(\eta_L)$ of $\mathcal{R}(\eta')$ is proportional to the $\widetilde{\mathrm{SLE}}_{\tilde\kappa}(\tilde\kappa-4+\frac{\tilde\kappa}{4}\underline{\rho}^L; \frac{\tilde\kappa}{2}-2+\frac{\tilde\kappa}{4}\underline{\rho}^R; \underline{\alpha}^L; \underline{\alpha}^R)$ from 0 to ∞ with force points $\underline{x}$ and $\alpha^{i,q} = \frac{\tilde\kappa}{4}\rho^{i,q} \cdot \frac{\tilde\kappa-4}{2\tilde\kappa} = -\rho^{i,q}\frac{4-\kappa}{2\kappa}$. Likewise, conditioned on $\mathcal{R}(\eta_L)$, the law of the right boundary $\mathcal{R}(\eta_R)$ of $\mathcal{R}(\eta')$ is $\frac{1}{Z(\eta_L)}\widetilde{\mathrm{SLE}}_{\tilde\kappa}(-\frac{\tilde\kappa}{2}; \tilde\kappa-4+\frac{\tilde\kappa}{4}\underline{\rho}^R; 0; \underline{\alpha}^R)$ process from 0 to ∞ to the right of $\mathbb{H}\backslash\mathcal{R}(\eta_L)$ with force points $(0^-; \underline{x}^R)$. Moreover, by Lemma 3.4, since $-\frac{\tilde\kappa}{2} < 0$, the constant $Z(\eta_L)$ does not depend on η_L.

On the other hand, let $\tilde\eta'$ be an $\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat\rho}; \underline{\hat\alpha})$ process in $\mathbb{H}$ from ∞ to 0 with force points $\underline{\tilde x}$, and $\hat\alpha^{i,q} = \frac{\hat\rho^{i,q}(\kappa-4)}{2\kappa}$. Note that by definition, for each point $x^{i,q}$, its assigned power parameter is precisely $\alpha^{i,q}$. Then by Proposition 3.6, the law of the left and right boundaries $(\tilde\eta_L, \tilde\eta_R)$ of $\tilde\eta'$ can be produced by the following procedure:

(i) Sample an $\mathrm{SLE}_{\tilde\kappa}(\tilde\kappa-4+\frac{\tilde\kappa}{4}\underline{\rho}^L; \frac{\tilde\kappa}{2}-2+\frac{\tilde\kappa}{4}\underline{\rho}^R)$ process η_1 from 0 to ∞ with force points $\underline{x}$, and given η_1, sample an $\mathrm{SLE}_{\tilde\kappa}(-\frac{\tilde\kappa}{2}; \tilde\kappa-4+\frac{\tilde\kappa}{4}\underline{\rho}^R)$ process η_2 to the right of $\mathbb{H}\backslash\eta_1$ with force points $(0^-, \underline{x}^R)$;

(ii) For $i \ge 1$ and $j = 1,2$, let $D^{i,q}_{\eta_j}$ be the connected component of $\mathbb{H}\backslash\eta_j$ with $x^{i,q}$ on the boundary, $\sigma^{i,q}_{\eta_j}$ (resp. $\xi^{i,q}_{\eta_j}$) be the first (resp. last) point on $\partial D^{i,q}_{\eta_j}$ traced by η_j, and $\psi^{i,q}_{\eta_j}: D^{i,q}_{\eta_j} \to \mathbb{H}$ be the conformal map sending $(\sigma^{i,q}_{\eta_j}, x^{i,q}, \xi^{i,q}_{\eta_j})$ to $(0, \pm 1, \infty)$ where we take the $+$ sign when $q = R$;

(iii) Weight the law of (η_1, η_2) by

$$\prod_{i=1}^k |x^{i,L} \cdot (\psi^{i,L}_{\eta_1})'(x^{i,L})|^{\alpha^{i,L}} \cdot \prod_{i=1}^\ell |x^{i,R} \cdot (\psi^{i,R}_{\eta_2})'(x^{i,R})|^{\alpha^{i,R}}.$$

We conclude by Proposition 2.1 that up to a finite multiplicative constant, the law of $(\mathcal{R}(\eta_L), \mathcal{R}(\eta_R))$ agrees with that of $(\tilde\eta_L, \tilde\eta_R)$. Moreover, by Proposition 3.6 and Theorem A, conditioned on $(\mathcal{R}(\eta_L), \mathcal{R}(\eta_R))$, $\mathcal{R}(\eta')$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$ process independently in each connected component of $\mathbb{H}\backslash(\mathcal{R}(\eta_L) \cup \mathcal{R}(\eta_R))$ between $\mathcal{R}(\eta_L)$ and $\mathcal{R}(\eta_R)$, which agrees with the conditional law of $\tilde\eta'$ given $(\tilde\eta_L, \tilde\eta_R)$. Therefore the law of $\mathcal{R}(\eta')$ agrees up to a multiplicative constant with that of $\tilde\eta'$, which finishes the proof of Theorem 1.2 for $\kappa \in (4,8]$). □

Proof of Theorem 1.2. The theorem follows directly by applying Proposition 3.5 for $\kappa \in (0,4)$, Proposition 3.7 for $\kappa \in (4,8]$ along with [WW17, Theorem 1.1.6] for $\kappa = 4$. □

A Proof of Lemma 2.2

In this section, we give an alternative proof of Lemma 2.2 based on Lemma 3.1 and the irreducibility of Markov chain argument from [MT09]. Let $(X, \mathcal{F})$ be a state space where $\mathcal{F}$ is a σ-algebra. For a Markov chain $\{X_n\}_{n\geq 0}$ and a measure φ on $(X, \mathcal{F})$, if for any $x \in X$ and $A \in \mathcal{F}$, $\mathbb{P}(X_n \in A \text{ for some } n | X_0 = x) > 0$ whenever $\varphi(A) > 0$, then $\{X_n\}_{n\geq 0}$ is said to be *φ-irreducible*. By [MT09, Theorem 4.0.1], there exists a unique *maximal irreducibility* measure ψ on $(X, \mathcal{F})$ such that $\{X_n\}_{n\geq 0}$ is *ψ-irreducible*. Then [MT09, Proposition 10.1.1, Theorem 10.0.1] tells us that a ψ-irreducible Markov chain with an invariant probability measure is recurrent, and thus admit a unique invariant measure.

Without loss of generality, we take a conformal map $\mathbb{H} \to \mathbb{D}$ and assume that we are in the setting where η_1, η_2 are continuous curves in $\mathbb{D}$ from $-i$ to i such that given one curve η_i, the other curve is the $\mathrm{SLE}_\kappa(\underline{\rho}^j)$ process in $\mathbb{D}\backslash\eta_i$ (recall the definition of $\mathrm{SLE}_\kappa(\underline{\rho})$ processes in non-simply connected domains in Section 2.1) as in the statement of Lemma 2.2. By an identical argument of the first step of the proof of [MS16b, Theorem 4.1] (i.e., draw counterflowlines η_1' by SLE duality, run η_1, η_1' for a small amount of time and look at the remaining parts of η_1, η_2), we may work on the case where the starting and ending points of η_1, η_2 are distinct. To be more precise, let $a, b, c, d \in \partial\mathbb{D}$ be 4 points in counterclockwise order, $x^{0,L}, \ldots, x^{k,L}, d^L$ be some marked points on the $\widehat{da}$ arc of $\partial\mathbb{D}$, and $x^{0,R}, \ldots, x^{\ell,R}$ be some marked points on the $\widehat{bc}$ arc of $\partial\mathbb{D}$ with $x^{0,R} = a^+$. Let X be the space of non-crossing continuous curves (γ_1, γ_2) connecting (a, b) with (d, c) in $\overline{\mathbb{D}}$ such that γ_1 (resp. γ_2) is disjoint from $\underline{x}^R \cup \{c\}$ (resp. $\underline{x}^L \cup \{d\}$) and does not trace any segment of the arc $\widehat{bc}$ (resp. $\widehat{da}$), and $\mathcal{F}$ be the Borel σ-algebra on X generated by Hausdorff topology. We are going to show that there exists at most one probability measure μ on $(X, \mathcal{F})$ such that, for a sample (η_1, η_2) from μ, conditioned on η_1, η_2 is an $\mathrm{SLE}_\kappa(\rho - 2; \underline{\rho}^R)$ curve in the right component of $\mathbb{D}\backslash\eta_1$ with force points $b^-; \underline{x}^R$, and conditioned on η_2, η_1 is an $\mathrm{SLE}_\kappa(\underline{\rho}^L, \frac{\kappa}{2} - 2 - \overline{\rho}^L; \rho - 2, \kappa - 2 - \rho)$ in the left component of $\mathbb{D}\backslash\eta_2$ with force points $\underline{x}^L, d^L; a^R, c^R$, where $\overline{\rho}^L = \sum_{i=0}^k \rho^{i,L}$, and a^R (resp. c^R) is the left most point of $\eta_2 \cap \widehat{ab}$ (resp. $\eta_2 \cap \widehat{cd}$). See Figure 7 for an illustration.

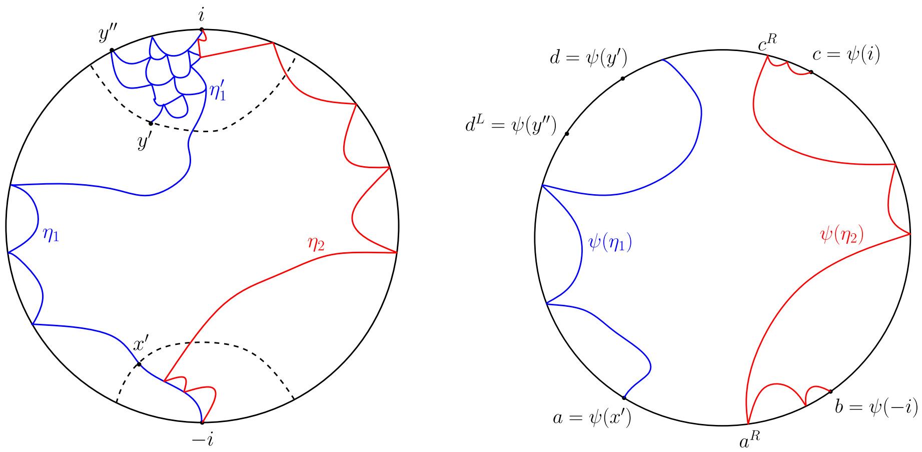

Figure 7: First step of the proof of Lemma 2.2, where we first run η_1 and its associated counterflowline η_1' until they hit $\partial B(-i, \varepsilon)$ and $\partial B(i, \varepsilon)$ as in [MS16b, Proof of Theorem 4.1] and map back to $\mathbb{D}$ by a conformal map ψ. Given $\psi(\eta_1)$, $\psi(\eta_2)$ is an $\mathrm{SLE}_\kappa(\rho - 2; \underline{\rho}^R)$ process in the right component of $\mathbb{D}\backslash\psi(\eta_1)$, while given $\psi(\eta_2)$, $\psi(\eta_1)$ is an $\mathrm{SLE}_\kappa(\underline{\rho}^L, \frac{\kappa}{2} - 2 - \overline{\rho}^L; \rho - 2, \kappa - 2 - \rho)$ in the left component of $\mathbb{D}\backslash\psi(\eta_2)$ with the force points on the right located at a^R, c^R. Note that since η_1 a.s. merges into η_1', $\psi(\eta_1)$ a.s. terminates between c and d.

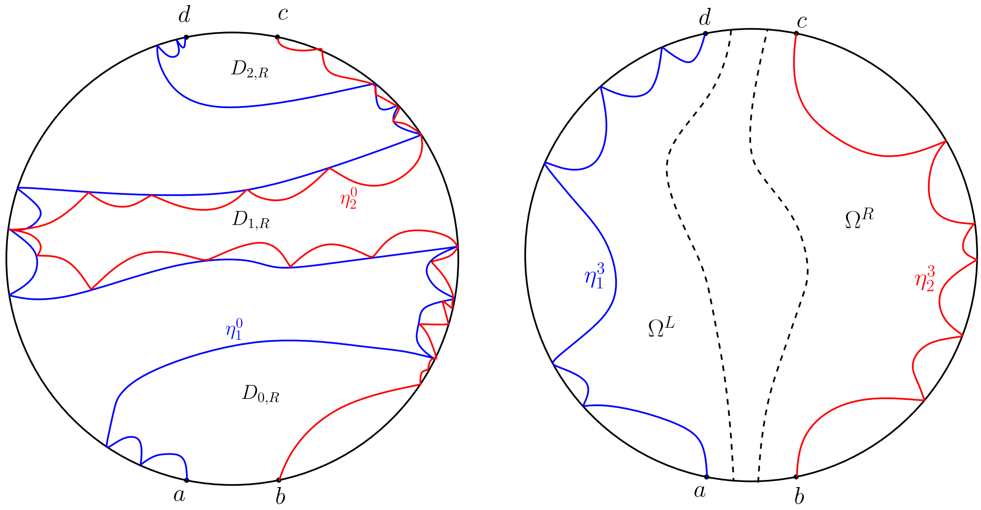

Figure 8: The Markov chain resampling in Proof of Lemma 2.2. **Left:** Initial phase for the two given curves (η_1^0, η_2^0), with $D_{0,R}, D_{1,R}, D_{2,R}$ being the connected components of $\mathbb{D}\backslash\eta_1^0$ whose boundary intersects both the da arc and bc arc. **Right:** By applying Lemma 3.1 in each of $D_{0,R}, D_{1,R}, D_{2,R}$, when we sample η_2^1 in the right component of $\mathbb{D}\backslash\eta_1^0$, with positive probability η_2^1 is disjoint from the ad arc. It then follows by applying Lemma 3.1 twice more $\mathbb{P}(X_3 \in X_\Omega | X_0 = (\eta_1^0, \eta_2^0)) > 0$.

We construct a Markov chain on $(X, \mathcal{F})$ as follows. Let $X_0 = (\eta_1^0, \eta_2^0) \in X$, and for given $n \geq 0$ and $X_n = (\eta_1^n, \eta_2^n)$, we first uniformly pick $i \in \{1, 2\}$ and sample η_i^{n+1} in $\mathbb{D}\backslash\eta_{3-i}^n$ from the conditional law induced by μ as described in the previous paragraph. Let $\eta_{3-i}^{n+1} = \eta_{3-i}^n$ and set $X_{n+1} = (\eta_1^{n+1}, \eta_2^{n+1})$. Pick a', b' on the arc $\widehat{ab}$, and c', d' on the arc $\widehat{cd}$ and draw two disjoint simple curves (γ^L, γ^R) in $\mathbb{D}$ connecting (a', b') with (d', c'). Let Ω^L be left component of $\mathbb{D}\backslash\gamma^L$, and Ω^R be right component of $\mathbb{D}\backslash\gamma^R$. Let $X_\Omega := \{(\gamma_1, \gamma_2) \in X : \gamma_1 \subset \overline{\Omega^L}, \gamma_2 \subset \overline{\Omega^R}\}$. We are going to show that $\{X_n\}_{n\geq 0}$ is φ-irreducible for $\varphi = \mu|_{X_\Omega}$ and thus admits a unique invariant probability measure, which concludes the proof by [MT09].

Given $(\eta_1^0, \eta_2^0) \in X$, let $D_{0,R}, \ldots, D_{m,R}$ be the connected components of $\mathbb{D}\backslash\eta_1^0$ whose boundary has nonempty intersection with both $\widehat{da}$ and $\widehat{bc}$. Note that the number of such components is finite by the continuity of η_1^0. Then by applying Lemma 3.1 in each of $D_{0,R}, \ldots, D_{m,R}$, when we sample η_2^1 in the right component of $\mathbb{D}\backslash\eta_1^0$ from the conditional law induced by μ, there is a positive probability such that η_2^1 is disjoint from the arc $\widehat{da}$. Under this event, by Lemma 3.1, when we sample η_1^2 from the corresponding conditional law in the left component of $\mathbb{D}\backslash\eta_2^1$, there is positive chance that η_1^2 is disjoint from the arc $\widehat{bc}$ and stays in the domain $\overline{\Omega^L}$. (Note that although η_1^2 merges with the arc $\widehat{cd}$ before reaching the target c, Lemma 3.1 extends to this setting and is still applicable.) Applying Lemma 3.1 once more, under this event, when we sample η_2^3 from the corresponding conditional law in the right component of $\mathbb{D}\backslash\eta_1^2$, there is a positive probability that η_2^3 is contained in $\overline{\Omega^R}$. Therefore we conclude that for any $(\eta_1^0, \eta_2^0) \in X$, $\mathbb{P}(X_3 \in X_\Omega | X_0 = (\eta_1^0, \eta_2^0)) > 0$. Note that this also implies that $\mu(X_\Omega) > 0$. See also Figure 8.

Finally, from the GFF flow line local absolute continuity [MS16a, Proposition 3.4] and [MS16a, Theorem 1.2], given any curves γ_2, $\tilde{\gamma}_2$ in $\overline{\Omega^R}$, when we sample η_1, $\tilde{\eta}_1$ in the left component of $\mathbb{D}\backslash\gamma_2$ and $\mathbb{D}\backslash\tilde{\gamma}_2$ according to the conditional law described by μ, when restricted to the event η_1, $\tilde{\eta}_1$ are contained in $\overline{\Omega^L}$, the laws of η_1 and $\tilde{\eta}_1$ are mutually absolutely continuous w.r.t. each other. In particular, this implies that for any $A \in \mathcal{F}$ with $\mu|_{X_\Omega}(A) > 0$, $\mathbb{P}(X_5 \in A | X_0 = (\eta_1^0, \eta_2^0)) > 0$. This justifies the irreducibility of $\{X_n\}_{n\geq 0}$ and thus concludes the proof. $\square$

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