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On the capacity for degenerated G-Brownian motion and its application*

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Abstract

In this paper, the capacity $c(\{B_T \in A\})$ of degenerated G-Brownian motion B_T for arbitrary Borel set A is calculated via the viscosity solution of G-heat equation. In particular, unlike the classical Brownian motion and non-degenerated G-Brownian motion which do not weight points, we obtain $c(\{B_T = a\}) > 0$ for each $a \in \mathbb{R}$ in the degenerated case we study here. As an application, we show that $I_A(B_T)$ has no quasi-continuous version unless $A = \emptyset$ or $\mathbb{R}$.

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1 Introduction

Motivated by model uncertainty in finance, Peng [8, 10] introduced the notions of G-expectation $\hat{\mathbb{E}}[\cdot]$ and G-Brownian motion $(B_t)_{t\geq 0}$ via the following G-heat equation:

$$\partial_t u - G\left(\partial^2_{xx} u\right) = 0,\ u(0,x) = \varphi(x), \tag{1.1}$$

where $G(a) = \frac{1}{2}\left(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-\right)$ for $a \in \mathbb{R}$, $\overline{\sigma} > 0$ and $\underline{\sigma} \in [0, \overline{\sigma}]$. Two fixed parameters $\overline{\sigma}^2$ and $\underline{\sigma}^2$ characterize the variance uncertainty of G-Brownian motion, i.e., $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-B_1^2]$. For any bounded and continuous function φ, we have $\hat{\mathbb{E}}[\varphi(x + B_t)] = u(t,x)$, where u is the viscosity solution of (1.1). Due to the nonlinearity of the term $G(\partial^2_{xx} u)$, it is usually difficult to obtain the explicit expression of $\hat{\mathbb{E}}[\varphi(B_t)]$ for non-convex and non-concave function φ.

The G-expectation can also be seen as a upper expectation. Indeed, Denis et al. [2] obtained a representation theorem of G-expectation $\hat{\mathbb{E}}[\cdot]$ by stochastic control method:

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X],\quad \forall X \in Lip(\Omega).$$

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where $\mathcal{P}$ is a family of weakly compact probability measures on $(\Omega, \mathcal{B}(\Omega))$ and $Lip(\Omega)$ is the space of random variables generalized by the G-Brownian motion $(B_t)_{t\geq 0}$. Furthermore, the related G-capacity $c(\cdot)$ can be defined by

$$c(D) = \sup_{P\in\mathcal{P}} P(D), \quad \forall D \in \mathcal{B}(\Omega).$$

In particular, if we consider $A = (-\infty, a]$ or $A = [a, +\infty)$, the value of $c(\{B_T \in A\})$ can be calculated by taking $\varphi(x) = I_{(-\infty,a]}$ or $I_{[a,+\infty)}$ in (1.1), the explicit solution with such φ can be obtained (see Peng et al. [11] and Pei et al. [7]), but there is no result for $A = [a, b]$ as far as we know. Under the non-degenerate case, i.e. $\underline{\sigma} > 0$, Hu et al. [4] proved that $c(\{B_T = a\}) = 0$ for each $(T, a) \in (0, \infty) \times \mathbb{R}$ by finding a kind of viscosity supersolution of G-heat equation (1.1). This property is a natural generalization of the fact in the classical case that Brownian motion does not weight points. Based on this property, [4] further obtained that $I_{[a,b]}(B_T) \in L^1_G(\Omega)$, where $L^1_G(\Omega)$ is the completion of $Lip(\Omega)$ under the norm $\hat{\mathbb{E}}[|\cdot|]$.

In this paper, we firstly calculate the value $c(\{B_T = a\})$ for $a \in \mathbb{R}$ under degenerated case $\underline{\sigma} = 0$ by verifying that the constructed solution is just the viscosity solution of G-heat equation (1.1), and $c(\{B_T = a\}) > 0$ for each $a \in \mathbb{R}$ is easily followed. This result is totally different from the situations of classical Brownian motion as well as the non-degenerated G-Brownian motion. A similar method can also be used to calculate the value of $c(\{B_T \in A\})$ for $A \in \mathcal{B}(\mathbb{R})$, which provide some examples for checking the convergence rate of Peng's central limit theorem (see [6]) when $\underline{\sigma} = 0$. Furthermore, we prove that $I_A(B_T)$ does not belong to $L^1_G(\Omega)$ for any $A \in \mathcal{B}(\mathbb{R})$ unless $A = \emptyset$ or $A = \mathbb{R}$, which is also completely different from the case $\underline{\sigma} > 0$.

This paper is organized as follows. In Section 2, we present some basic notions and results of G-expectation theory. In Section 3, we obtain the G-capacity $c(\{B_T \in A\})$ for each Borel set A under degenerate case. As an application, we prove that $I_A(B_T) \notin L^1_G(\Omega)$ for each non-trivial set $A \in \mathcal{B}(\mathbb{R})$ in Section 4.

2 Preliminaries

We recall some basic notions and results of G-expectation. The readers may refer to Peng [8, 9, 10] for more details.

Let $\Omega = C[0, \infty)$ be the space of real-valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$. Let $B_t(\omega) := \omega_t$, for $\omega \in \Omega$ and $t \geqslant 0$, be the canonical process. Set

$$Lip(\Omega) := \{\varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) : n \in \mathbb{N},\ 0 < t_1 < \cdots < t_n,\ \varphi \in C_{b,Lip}(\mathbb{R}^n)\},$$

where $C_{b,Lip}(\mathbb{R}^n)$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^n$. It is easy to verify that

$$Lip(\Omega) = \left\{\varphi\left(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right) : n \in \mathbb{N},\ 0 < t_1 < \cdots < t_n,\ \varphi \in C_{b,Lip}(\mathbb{R}^n)\right\}.$$

Let

$$G(a) := \frac{1}{2}\left(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-\right), \quad \forall a \in \mathbb{R},$$

where $\overline{\sigma} > 0$ and $\underline{\sigma} \in [0, \overline{\sigma}]$ are fixed parameters characterizing the variance uncertainty.

The G-expectation $\hat{\mathbb{E}}$: $Lip(\Omega) \to \mathbb{R}$ is defined by the following two steps.

Step 1. For each $X = \varphi(B_t - B_s)$ with $0 \leq s \leq t$ and $\varphi \in C_{b,Lip}(\mathbb{R})$, we define

$$\hat{\mathbb{E}}[X] = u(t - s, 0),$$

where u is the viscosity solution of (1.1).

Step 2. For each $X=\varphi\left(B_{t_1},B_{t_2}-B_{t_1},\ldots,B_{t_n}-B_{t_{n-1}}\right)$ with $0<t_1<\cdots<t_n$ and $\varphi\in C_{b,Lip}\left(\mathbb{R}^n\right)$, we define

$$\hat{\mathbb{E}}\left[X\right]=\varphi_0,$$

where φ_0 is obtained via the following procedure:

$$\begin{aligned}
\varphi_{n-1}(x_1,\ldots,x_{n-1})&=\hat{\mathbb{E}}\left[\varphi\left(x_1,\ldots,x_{n-1},B_{t_n}-B_{t_{n-1}}\right)\right],\\
\varphi_{n-2}(x_1,\ldots,x_{n-2})&=\hat{\mathbb{E}}\left[\varphi_{n-1}\left(x_1,\ldots,x_{n-2},B_{t_{n-1}}-B_{t_{n-2}}\right)\right],\\
&\vdots\\
\varphi_1(x_1)&=\hat{\mathbb{E}}\left[\varphi_2\left(x_1,B_{t_2}-B_{t_1}\right)\right],\\
\varphi_0&=\hat{\mathbb{E}}\left[\varphi_1\left(B_{t_1}\right)\right].
\end{aligned}$$

The following is the definition of the viscosity solution of (1.1) (see [1]).

Definition 2.1. *A real-valued continuous function $u\in C\left([0,\infty)\times\mathbb{R}\right)$ is called a viscosity subsolution (resp. supersolution) of (1.1) on $[0,\infty)\times\mathbb{R}$ if $u(0,\cdot)\le\varphi\left(\cdot\right)$ (resp. $u(0,\cdot)\ge\varphi\left(\cdot\right)$), and for all $(t^*,x^*)\in(0,\infty)\times\mathbb{R}$, $\phi\in C^2\left((0,\infty)\times\mathbb{R}\right)$ such that $u\left(t^*,x^*\right)=\phi\left(t^*,x^*\right)$ and $u<\phi$ (resp. $u>\phi$) on $(0,\infty)\times\mathbb{R}\backslash\left(t^*,x^*\right)$, we have*

$$\partial_t\phi\left(t^*,x^*\right)-G\left(\partial^2_{xx}\phi\left(t^*,x^*\right)\right)\le 0 \text{ (resp. } \ge 0).$$

A real-valued continuous function $u\in C\left([0,\infty)\times\mathbb{R}\right)$ is called a viscosity solution of (1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1) on $[0,\infty)\times\mathbb{R}$.

The space $(\Omega,Lip\left(\Omega\right),\hat{\mathbb{E}})$ is called a G-expectation space. The corresponding canonical process $(B_t)_{t\ge0}$ is called a G-Brownian motion. The G-expectation $\hat{\mathbb{E}}$: $Lip\left(\Omega\right)\to\mathbb{R}$ is a typical sublinear expectation, which satisfies the following properties: for each X, $Y\in Lip\left(\Omega\right)$,

- **(i)** Monotonicity: If $X\ge Y$, then $\hat{\mathbb{E}}\left[X\right]\ge\hat{\mathbb{E}}\left[Y\right]$.
- **(ii)** Constant preservation: $\hat{\mathbb{E}}\left[c\right]=c$ for $c\in\mathbb{R}$.
- **(iii)** Subadditivity: $\hat{\mathbb{E}}\left[X+Y\right]\le\hat{\mathbb{E}}\left[X\right]+\hat{\mathbb{E}}\left[Y\right]$.
- **(iv)** Positive homogeneity: $\hat{\mathbb{E}}\left[\lambda X\right]=\lambda\hat{\mathbb{E}}\left[X\right]$ for $\lambda\ge0$.

For every $p\ge1$, we denote by $L^P_G\left(\Omega\right)$ the completion of $Lip\left(\Omega\right)$ under the norm $\left\|X\right\|_p:=\left(\hat{\mathbb{E}}\left[|X|^p\right]\right)^{1/p}$. The G-expectation $\hat{\mathbb{E}}\left[X\right]$ can be extended continuously to $L^1_G\left(\Omega\right)$ under the norm $||\cdot||_1$, and $\hat{\mathbb{E}}$: $L^1_G\left(\Omega\right)\to\mathbb{R}$ still satisfies (i)-(iv).

Denis et al. [2] proved the following representation theorem.

Lemma 2.2. *There exists a weakly compact set of probability measures $\mathcal{P}$ on $(\Omega,\mathcal{B}(\Omega))$ such that*

$$\hat{\mathbb{E}}[X]=\sup_{P\in\mathcal{P}}E_P[X],\quad\forall X\in L^1_G(\Omega),$$

where $\mathcal{B}(\Omega)=\sigma(B_t:t\ge0)$.

The G-expectation $\hat{\mathbb{E}}$ can be regarded as the upper expectation introduced by the set $\mathcal{P}$. It is natural to defined the G-capacity associated to $\mathcal{P}$ as

$$c(D)=\sup_{P\in\mathcal{P}}P(D),\quad\forall D\in\mathcal{B}(\Omega).\tag{2.1}$$

An important property of this capacity is that $c\left(F_n\right)\downarrow c\left(F\right)$ for all closed sets $F_n\downarrow F$ since $\mathcal{P}$ is weakly compact.

A set $A\subset\mathcal{B}(\Omega)$ is polar if $c\left(A\right)=0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set.

Definition 2.3. *A function $X : \Omega \to \mathbb{R}$ is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set $O \subset \Omega$ with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.*

We say that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \to \mathbb{R}$ such that $X = Y$, q.s.

Now we review the characterization of $L_G^p(\Omega)$ for $p \geq 1$.

Lemma 2.4 ([2]). *For each $p \geq 1$, we have*

$$L_G^P(\Omega) = \left\{ X \in L^0(\Omega) : \lim_{N\to\infty} \hat{\mathbb{E}}\left[|X|^p I_{\{|X|\geq N\}}\right] = 0 \text{ and } X \text{ has a quasi-continuous version} \right\},$$

where $L^0(\Omega)$ is the space of all $\mathcal{B}(\Omega)$-measurable real functions.

The following property is important in G-expectation space.

Lemma 2.5 ([2]). *Let $\{X_n\}_{n=1}^\infty \subset L_G^1(\Omega)$ satisfy $X_n \downarrow X$ q.s. Then*

$$\hat{\mathbb{E}}[X_n] \downarrow \hat{\mathbb{E}}[X]. \tag{2.2}$$

Moreover, if $X \in L_G^1(\Omega)$, then

$$\hat{\mathbb{E}}[X_n - X] \downarrow 0.$$

3 Main results

In this section, we consider the degenerated G-Brownian motion, i.e., $\underline{\sigma} = 0$, with the corresponding G-heat equation

$$\partial_t u - \frac{1}{2}\overline{\sigma}^2 \left(\partial_{xx}^2 u\right)^+ = 0,\ u(0,x) = \varphi(x). \tag{3.1}$$

Theorem 3.1. *Let $\underline{\sigma} = 0$ and $\overline{\sigma} > 0$. Define*

$$\Psi(x) = \frac{2}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{r^2}{2}\right) dr. \tag{3.2}$$

For each given $T > 0$, we have

(i) $c(\{B_T = a\}) = \Psi\left(\frac{|a|}{\overline{\sigma}\sqrt{T}}\right), \quad \forall a \in \mathbb{R}.$

(ii) $c(\{B_T \geq a\}) = c(\{B_T \leq -a\}) = \Psi\left(\frac{a}{\overline{\sigma}\sqrt{T}}\right), \quad \forall a \geq 0.$

Proof. (i) We firstly construct the solution of (3.1) with initial condition $\varphi(x) = I_{\{a\}}(x)$ laxly by a similar method as in [3, 7, 11]. Indeed, consider the following ordinary differential equation (ODE for short):

$$\overline{\sigma}^2 \left(p''(x)\right)^+ + xp'(x) = 0, \qquad p(0) = 1, \quad p(\pm\infty) = 0, \tag{3.3}$$

then $u(t,x) := p\left(\frac{x-a}{\sqrt{t}}\right)$ is the solution of the G-heat equation (3.1) with initial condition $u(0,x) = I_{\{a\}}(x)$.

It is not difficult to find that the solution of ODE (3.3) is given by

$$p(x) = \Psi\left(\frac{|x|}{\overline{\sigma}}\right).$$

Furthermore, we obtain

$$u(t,x) = p\left(\frac{x-a}{\sqrt{t}}\right) = \Psi\left(\frac{|x-a|}{\overline{\sigma}\sqrt{t}}\right).$$

Noting that $u(0,x)=I_{\{a\}}(x)$ and $\partial_x u(t,x)$ are discontinuous at $x=a$, we consider the auxiliary functions

$$u_n(t,x):=\Psi\left(\frac{|x-a|}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right),\quad n\in\mathbb{N}.$$

We claim that $u_n(t,x)$ is the viscosity solution of G-heat equation (3.1) with the initial condition $\varphi(x)=\Psi\left(\frac{|x-a|}{\overline{\sigma}\sqrt{\frac{1}{n}}}\right)$.

If $x\neq a$, then $u_n(t,x)$ is a $C^{1,2}$ function on $[0,\infty)\times\mathbb{R}\backslash\{a\}$ and

$$\partial_t u_n(t,x)=-\frac{1}{2}\Psi'\left(\frac{|x-a|}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right)\frac{|x-a|}{\overline{\sigma}\sqrt{\left(\frac{1}{n}+t\right)^3}}$$

and

$$\partial_{xx}^2 u_n(t,x)=\Psi''\left(\frac{|x-a|}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right)\frac{1}{\overline{\sigma}^2\left(\frac{1}{n}+t\right)}.$$

By the definition of $\Psi(x)$, we know that $\Psi''(x)>0$ if $x>0$, and

$$\Psi''(x)=-x\Psi'(x).$$

Then we obtain

$$\partial_t u_n(t,x)-\frac{1}{2}\overline{\sigma}^2\left(\partial_{xx}^2 u_n(t,x)\right)^+=0,$$

which is exactly the classical solution. Obviously, it is the viscosity solution on $[0,\infty)\times\mathbb{R}\backslash\{a\}$.

We only need to verify that $u_n(t,x)$ is the viscosity solution of (3.1) at (t^*,x^*) by Definition 2.1, where $x^*=a$ and $t^*>0$.

Firstly, taking $\psi\in C^2((0,\infty)\times\mathbb{R})$ such that

$$\psi(t,x^*)\geq u_n(t,x^*)=1 \text{ and } \psi(t^*,x^*)=u_n(t^*,x^*)=1,$$

it then follows that $\partial_t\psi(t^*,x^*)=0$. Hence, we obtain

$$\partial_t\psi(t^*,x^*)-\frac{1}{2}\overline{\sigma}^2\left(\partial_{xx}^2\psi(t^*,x^*)\right)^+\leq 0.$$

Thus, u_n is a viscosity subsolution of (3.1).

Secondly, if ψ satisfies

$$\psi(t^*,x^*)=u_n(t^*,x^*) \text{ and } \psi(t^*,x)\leq u_n(t^*,x),$$

then we have

$$\partial_{x+}\psi(t^*,x^*)\leq\partial_{x+}u_n(t^*,x^*) \text{ and } \partial_{x-}\psi(t^*,x^*)\geq\partial_{x-}u_n(t^*,x^*).\tag{3.4}$$

But by the definition of $\Psi(x)$, it is easy to check that

$$\partial_{x-}u_n(t^*,a)=\frac{2}{\bar{\sigma}\sqrt{2\pi\left(\frac{1}{n}+t^*\right)}}>0,$$
$$\partial_{x+}u_n(t^*,a)=-\frac{2}{\bar{\sigma}\sqrt{2\pi\left(\frac{1}{n}+t^*\right)}}<0.$$

So we can not find $\psi \in C^2((0,\infty)\times\mathbb{R})$ satisfying (3.4). Thus, we know that u_n is also a viscosity supersolution of (3.1).

In conclusion, u_n is a viscosity solution of (3.1). Then we obtain

$$\hat{\mathbb{E}}\left[u_n(0,B_T)\right]=u_n(T,0).$$

Noting that $u_n(0,x)\downarrow I_{\{a\}}(x)$, by Lemma 2.5, we can deduce that

$$c(\{B_T\in\{a\}\})=\hat{\mathbb{E}}\left[I_{\{a\}}(B_T)\right]=\lim_{n\to\infty}\hat{\mathbb{E}}\left[u_n(0,B_T)\right]=\lim_{n\to\infty}u_n(T,0)=\Psi\left(\frac{|a|}{\overline{\sigma}\sqrt{T}}\right),$$

which implies the desired result.

(ii) By Proposition 3.1.5 in [10], we know that $(-B_t)_{t\geq 0}$ is also a G-Brownian motion with the same parameters of $(B_t)_{t\geq0}$, thus

$$c(\{B_T\geq a\})=c(\{-B_T\geq a\})=c(\{B_T\leq -a\}),\quad \forall a\geq 0.$$

We only need to prove that

$$c(\{B_T\geq a\})=\Psi\left(\frac{a}{\overline{\sigma}\sqrt{T}}\right),\quad \forall a\geq 0.$$

In fact, we consider

$$u(t,x)=\Psi\left(\frac{a-x}{\overline{\sigma}\sqrt{t}}\right)I_{(-\infty,a)}(x)+I_{[a,\infty)}(x).$$

By the similar arguments in (i), we can show that

$$u_n(t,x)=\Psi\left(\frac{a-x}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right)I_{(-\infty,a)}(x)+I_{[a,\infty)}(x)$$

is the viscosity solution of G-heat equation (3.1) with initial condition

$$u_n(0,x)=\Psi\left(\frac{a-x}{\overline{\sigma}\sqrt{\frac{1}{n}}}\right)I_{(-\infty,a)}(x)+I_{[a,\infty)}(x).$$

Furthermore,

$$c(\{B_T\geq a\})=\lim_{n\to\infty}\hat{\mathbb{E}}\left[u_n(0,B_T)\right]=\lim_{n\to\infty}u_n(T,0)=\Psi\left(\frac{a}{\overline{\sigma}\sqrt{T}}\right).$$ □

Theorem 3.2. *Let $\underline{\sigma}=0$ and $\overline{\sigma}>0$. Then, for each given $T>0$ and $b<0<l$, we have*

(i) $c(\{B_T\in\{b,l\}\})=\sum_{i=-\infty}^{\infty}$ *sgn*$(i)\left[\Psi\left(\frac{|2i(l-b)-b|}{\overline{\sigma}\sqrt{T}}\right)+\Psi\left(\frac{|2i(l-b)+l|}{\overline{\sigma}\sqrt{T}}\right)\right]$, *where sgn*$(x):=I_{[0,\infty)}(x)-I_{(-\infty,0)}(x)$.

(ii) $c(\{B_T\in(-\infty,b]\cup[l,\infty)\})=c(\{B_T\in\{b,l\}\})$.

Proof. Let us consider

$$\begin{aligned}u_n(t,x):=&\sum_{i=-\infty}^{\infty}\operatorname{sgn}(i)\left[\Psi\left(\frac{|2i(l-b)+x-b|}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right)+\Psi\left(\frac{|2i(l-b)+l-x|}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right)\right]I_{(b,l)}(x)\\&+\Psi\left(\frac{|b-x|\wedge|l-x|}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right)I_{(-\infty,b]\cup[l,\infty)}(x).\end{aligned}\tag{3.5}$$

By the similar arguments in the proof of Theorem 3.1, we can show that $u_n(t,x)$ is the viscosity of G-heat equation (3.1) with initial condition $u_n(0,x)$.

Thus (i) is obtained by

$$c(\{B_T\in\{b,l\}\})=\lim_{n\to\infty}u_n(T,0).$$

For (ii), we can analogously prove that

$$u_n(t,x):=\sum_{i=-\infty}^{\infty}\operatorname{sgn}(i)\left[\Psi\left(\frac{|2i(l-b)+x-b|}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right)+\Psi\left(\frac{|2i(l-b)+l-x|}{\overline{\sigma}\sqrt{\frac{1}{n}+t}}\right)\right]I_{(b,l)}(x)$$
$$+I_{(-\infty,b]\cup[l,\infty)}(x)$$

is the viscosity of G-heat equation (3.1) with initial condition $u_n(0,x)$, which completes the proof. □

Remark 3.3. The form of $u_n(t,x)$ in (3.5) is motivated by the passage times of Brownian motion. Let $(W_t)_{t\geq 0}$ be the classical Brownian motion on Wiener probability space $(\tilde{\Omega},\tilde{\mathcal{F}},P^W)$ and τ_x be the passage time defined by

$$\tau_x:=\inf\{t\geq 0:\overline{\sigma}W_t=x\},\quad x\in\mathbb{R}.$$

The stopping time $\tau_{b-x}\wedge\tau_{l-x}$ has the following density for $x\in(b,l)$ (see Section 2.8 in [5]):

$$P^W(\{\tau_{b-x}\wedge\tau_{l-x}\in ds\})=\frac{1}{\sqrt{2\pi\overline{\sigma}^2s^3}}\sum_{i=-\infty}^{\infty}\left\{(2i(l-b)-b+x)\exp\left(-\frac{(2i(l-b)-b+x)^2}{2\overline{\sigma}^2s}\right)\right.$$
$$\left.+(2i(l-b)+l-x)\exp\left(-\frac{(2i(l-b)+l-x)^2}{2\overline{\sigma}^2s}\right)\right\}ds.$$

For $x\in\{b,l\}$, we obtain

$$u_n(t,x)=\int_0^{\frac{1}{n}+t}P^W(\{\tau_{b-x}\wedge\tau_{l-x}\in ds\}).$$

Remark 3.4. The statement (ii) in Theorem 3.2 can be explained in the viewpoint of stochastic control. By [2], we have, for $b<0<l$,

$$c(\{B_T\in(-\infty,b]\cup[l,\infty)\})=\sup_{v\in M^2(0,T;[0,\overline{\sigma}])}P^W\left(\left\{\int_0^T v_s dW_s\in(-\infty,b]\cup[l,\infty)\right\}\right),$$

where $M^2(0,T;[\underline{\sigma},\overline{\sigma}])$ is the space of all $\tilde{\mathcal{F}}_t$-adapted processes $(v_s)_{0\leq s\leq T}$ with $v_s\in[\underline{\sigma},\overline{\sigma}]$. Once the process $\int_0^t v_s dW_s$ firstly hits the point b or l at time $\tau_b\wedge\tau_l$, we then take $v_s\equiv 0$ for $s\geq\tau_b\wedge\tau_l$, in this case, the process $\int_0^t v_s dW_s$ never escapes the interval $(-\infty,b]\cup[l,\infty)$ (it is absorbed at point b or l). Thus, for $b<0<l$,

$$c(\{B_T\in\{b,l\}\})=c(\{B_T\in(-\infty,b]\cup[l,\infty)\}).$$

Theorem 3.5. *Let $\underline{\sigma}=0$ and $\overline{\sigma}>0$. Define*

$$\rho(A):=\inf\{|x|:x\in A\}.$$

For each given $T>0$ and $A\in\mathcal{B}(\mathbb{R})$, then we have

(i) *If $\rho(A)=0$, then $c(\{B_T\in A\})=1$;*

(ii) *If* $A\subset[0,\infty)$ *or* $A\subset(-\infty,0]$, *then* $c(\{B_T\in A\})=\Psi\left(\frac{\rho(A)}{\overline{\sigma}\sqrt{T}}\right)$.

(iii) *If* $\rho(A)\neq 0$, $A\subsetneq[0,\infty)$ *and* $A\subsetneq(-\infty,0]$, *then*

$$\begin{aligned}&c(\{B_T\in A\})\\ &=\sum_{i=-\infty}^{\infty} sgn(i)\left[\Psi\left(\frac{|2i(\rho(A^+)+\rho(A^-))+\rho(A^-)|}{\overline{\sigma}\sqrt{T}}\right)+\Psi\left(\frac{|2i(\rho(A^+)+\rho(A^-))+\rho(A^+)|}{\overline{\sigma}\sqrt{T}}\right)\right],\end{aligned}$$

where $A^+:=\{x:x\in A,\,x\geq 0\}$, $A^-:=\{-x:x\in A,\,x\leq 0\}$.

Proof. If $\rho(A)=0$, there exists a sequence $\{a_n:n\geq 1\}\subset A$ such that $a_n\to 0$. By Theorem 3.1, we get

$$c(\{B_T\in A\})\geq\lim_{n\to\infty}c(\{B_T=a_n\})=\lim_{n\to\infty}\Psi\left(\frac{|a_n|}{\overline{\sigma}\sqrt{T}}\right)=1,$$

which implies (i).

If $A\subset[0,\infty)$, there also exists a sequence $\{x_n:n\geq 1\}\subset A$ such that $x_n\to\rho(A)$. By Theorem 3.1, we know that

$$c(\{B_T\in A\})\geq\lim_{n\to\infty}c(\{B_T=x_n\})=\Psi\left(\frac{\rho(A)}{\overline{\sigma}\sqrt{T}}\right).$$

Noting that $A\subset[\rho(A),\infty)$, by Theorem 3.2, we get

$$c(\{B_T\in A\})\leq c(\{B_T\geq\rho(A)\})=\Psi\left(\frac{\rho(A)}{\overline{\sigma}\sqrt{T}}\right).$$

Thus we have $c(\{B_T\in A\})=\Psi\left(\frac{\rho(A)}{\overline{\sigma}\sqrt{T}}\right)$.

By the similar method for $A\subset(-\infty,0]$, we obtain (ii).

If $\rho(A)\neq 0$, $A\subsetneq[0,\infty)$ and $A\subsetneq(-\infty,0]$, we can find two sequences $\{b_n:n\geq 1\}$ and $\{c_n:n\geq 1\}$ in A such that $b_n<0<c_n$, $-b_n\to\rho(A^-)$ and $c_n\to\rho(A^+)$. By Theorem 3.2 and $A\subset(-\infty,-\rho(A^-)]\cup[\rho(A^+),\infty)$, we get

$$\begin{aligned}c(\{B_T\in A\})&\geq\lim_{n\to\infty}c(\{B_T\in\{b_n,c_n\}\})=c(\{B_T\in\{-\rho(A^-),\rho(A^+)\}\}),\\ c(\{B_T\in A\})&\leq c(\{B_T\in(-\infty,-\rho(A^-)]\cup[\rho(A^+),\infty)\}),\end{aligned}$$

which implies $c(\{B_T\in A\})=c(\{B_T\in\{-\rho(A^-),\rho(A^+)\}\})$. Thus we obtain (iii). □

4 Application to G-expectation

For each $\phi_n\in C_b(\mathbb{R})$ such that $\phi_n\downarrow I_A$, by (2.2), we know that

$$\hat{\mathbb{E}}\left[\phi_n\left(B_T\right)\right]\downarrow c(\{B_T\in A\}).\tag{4.1}$$

The following theorem presents an application of Theorem 3.5.

Theorem 4.1. *Let* $\underline{\sigma}=0$ *and* $\overline{\sigma}>0$. *Then, for each given* $T>0$, $A\in\mathcal{B}(\mathbb{R})$ *with* $A\neq\emptyset$ *and* $A\neq\mathbb{R}$, *we have* $I_A(B_T)\notin L_G^1(\Omega)$.

Proof. Due to $A\neq\emptyset$ and $A\neq\mathbb{R}$, we know that $\partial A\neq\emptyset$. Then one of the following two results must hold.

(i) There exist a point $x_0\in A$ and a sequence $\{x_k:k\geq 1\}\subset A^c$ such that $x_k\to x_0$.

(ii) There exist a point $x_0\in A^c$ and a sequence $\{x_k:k\geq 1\}\subset A$ such that $x_k\to x_0$.

Indeed, considering $x_0 \in \partial A$, if $x_0 \in A$, then (i) holds. Otherwise, (ii) holds if $x_0 \in A^c$. If (i) holds and $I_A(B_T) \in L_G^1(\Omega)$, then

$$h_n(B_T) \vee I_A(B_T) \in L_G^1(\Omega) \text{ and } h_n(B_T) \vee I_A(B_T) - I_A(B_T) \downarrow 0,$$

where

$$h_n(x) = [1 + n(x - x_0)]\, I_{[x_0 - \frac{1}{n}, x_0]}(x) + [1 - n(x - x_0)]\, I_{(x_0, x_0 + \frac{1}{n}]}(x).$$

By (2.2), we have

$$\hat{\mathbb{E}}\left[h_n(B_T) \vee I_A(B_T) - I_A(B_T)\right] \downarrow 0 \text{ as } n \to \infty. \tag{4.2}$$

Moreover, we also know that $h_n(B_T) \vee I_A(B_T) - I_A(B_T) \geq h_n(x_k) I_{\{x_k\}}(B_T)$ for each $k \geq 1$. Then we deduce by Theorem 3.5 that

$$\hat{\mathbb{E}}\left[h_n(B_T) \vee I_A(B_T) - I_A(B_T)\right] \geq \lim_{k \to \infty} h_n(x_k) c(\{B_T = x_k\}) = c(\{B_T = x_0\}) > 0,$$

which contradicts to (4.2). Thus $I_A(B_T) \notin L_G^1(\Omega)$.

If (ii) holds, then A^c satisfies (i). Thus we obtain $I_{A^c}(B_T) \notin L_G^1(\Omega)$, which implies $I_A(B_T) = 1 - I_{A^c}(B_T) \notin L_G^1(\Omega)$. □

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