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A matrix-valued Schoenberg's problem and its applications*

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Abstract

In this paper we present a criterion for positive definiteness of the matrix-valued function $f(t) \coloneqq \exp(-|t|^\alpha[B^+ + B^- \operatorname{sign}(t)])$, where $\alpha \in (0,2]$ and $B^\pm$ are real symmetric and antisymmetric $d\times d$ matrices. We also find a criterion for positive definiteness of its multidimensional generalization $f(t) \coloneqq \exp(-\int_{\mathbb{S}^{d-1}} |\boldsymbol{t}^\top \boldsymbol{s}|^\alpha[B^+ + B^- \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})]\, d\Lambda(\boldsymbol{s}))$ where Λ is a finite measure on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ under a more restrictive assumption that $B^\pm$ commute and are normal. The associated stationary Gaussian random field may be viewed as as a generalization of the univariate fractional Ornstein-Uhlenbeck process. This generalization turns out to be particularly useful for the asymptotic analysis of $\mathbb{R}^d$-valued Gaussian random fields. Another possible application of these findings may concern variogram modelling and general stationary time series analysis.

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1 Introduction

In 1938, Schoenberg [10] posed the problem of determining for which numbers $\alpha > 0$ and norms $\|\cdot\|$ on $\mathbb{R}^n$ the function $\boldsymbol{t} \mapsto \exp\left(-\|\boldsymbol{t}\|^\alpha\right)$ is positive-definite. The complete solution to this problem has been given in 1992 by Koldobsky [7] for the case where $\|\cdot\| = \|\cdot\|_q$. It is clear that if $(\alpha, \|\cdot\|)$ is a pair for which this function is positive definite, then for any $B \geq 0$, the function $\boldsymbol{t} \mapsto \exp\left(-B\|\boldsymbol{t}\|^\alpha\right)$ is also positive definite. However, the question arises whether we can take $B \in \mathbb{C}$ instead of $B \geq 0$? The application of Bochner's theorem shows that the answer is negative. Nonetheless, we can modify this function to make its Fourier transform real. Let $n = 1$, let $\|\cdot\|$ be the absolute value and consider the following function

$$t \mapsto \exp\left(-B|t|^\alpha\right) \mathbb{1}_{t\geq 0} + \exp\left(-\overline{B}|t|^\alpha\right) \mathbb{1}_{t<0}, \qquad t \in \mathbb{R}. \tag{1.1}$$

This family of functions, parameterized by α and B, is of great importance in the theory of stable distributions. This theory provides us with the following answer [11, Remark (7.26)]: (1.1) is positive definite if and only if

$$\alpha \in (0,2], \qquad \operatorname{Re} B \geq 0 \quad \text{and} \quad |\operatorname{Im} B\,| \leq \operatorname{Re} B \cdot \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|. \tag{1.2}$$

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Motivated by applications in the theory of Gaussian processes (more on that below), we aim to extend this result to the case where B is a $d \times d$ matrix. Specifically, we investigate the necessary and sufficient conditions for positive-definiteness of the matrix-valued function f defined analogously to (1.1) by

$$f(t) := \exp\left(-B|t|^\alpha\right) \mathbb{1}_{t\geq 0} + \exp\left(-B^\top |t|^\alpha\right) \mathbb{1}_{t<0}, \qquad t \in \mathbb{R},$$

with $\alpha \in (0,2]$ and B a real $d \times d$ matrix. We will mostly use the following representation of f:

$$f(t) = \exp\left(-|t|^\alpha \left[B^+ + B^- \operatorname{sign}(t)\right]\right), \quad \text{where} \quad B^\pm := \frac{B \pm B^\top}{2}. \tag{1.3}$$

The counterpart of the condition (1.2) in this case is

$$\widetilde{B} := B^+ \sin\left(\frac{\pi\alpha}{2}\right) - iB^- \cos\left(\frac{\pi\alpha}{2}\right) \trianglerighteq 0. \tag{1.4}$$

Here $\trianglerighteq$ denotes positive definiteness. As it turns out, for $\alpha \in [1,2)$ condition (1.4) is both necessary and sufficient for positive definiteness of f, whereas for $\alpha \in (0,1)$ it is necessary, but not sufficient. If $\alpha = 2$, we need to assume that $B^+ \trianglerighteq 0$. This is the subject of Theorem 2.3.

In Theorem 2.2 we present a multivariate extension of this result under a restrictive assumption of B being normal (unitarily diagonalizable). More specifically, we use the theory of multivariate stable laws (see e.g. [9, Chapter 2]) to show that under the same assumption (1.4) for every finite measure Λ on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ the function

$$f(\boldsymbol{t}) = \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \left[B^+ + B^- \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})\right] d\Lambda(\boldsymbol{s})\right), \qquad \boldsymbol{t} \in \mathbb{R}^d$$

is positive definite.

Surprisingly, the condition (1.4) arises also from the study of operator fractional Brownian motions [2, Remark 8]. As it turns out, this is a necessary and sufficient condition for positive definiteness of the following matrix-valued function

$$R(t,s) = B|t|^\alpha + B^\top |s|^\alpha - B|t-s|^\alpha \quad \text{for} \quad t > s$$

and satisfying $R^\top(t,s) = R(s,t)$. This class of multivariate fBm's is not only interesting in its own right, but is also essential in the theory of multivariate Gaussian extremes [3]. The occurrence of the same condition in both problems is not a coincidence. In fact, this observation leads to significant simplifications in the study of multivariate Gaussian extremes, which will be the subject of our upcoming paper on the extremes of locally-stationary $\mathbb{R}^d$-valued random fields. More specifically, the classical Pickands-Piterbarg approach to the asymptotical analysis of high exceedance probabilities of a non-stationary Gaussian process $X(t)$, $t \in \mathbb{R}$ heavily relies on the possibility to find a pair of stationary processes $Y_\pm(t)$, $t \in \mathbb{R}$, which stochastically dominate X from above and from below and are close to X on a given short interval. If X satisfies some weak assumptions, we can take $Y_\pm$ to be the processes associated to the covariance functions $e^{-B_\pm |t|^\alpha}$, $t \in \mathbb{R}$ with specially chosen $B_\pm$, and apply the Slepian inequality. In the case of $\mathbb{R}^d$-valued processes, the same approach with the Gordon inequality instead of Slepian's prompts the consideration of the process associated to (1.3). It remains, however, to show that (1.3) is a covariance function, which is exactly what we study in this paper.

The importance of our results is twofold:

1. they can be used to construct valid covariance functions of $\mathbb{R}^d$-valued Gaussian random fields, and

2. they can be used for cross-variogram and pseudo-variogram modelling, which is important for statistical applications, see [4].

More specifically, positive definiteness of f implies that the following function

$$t \mapsto I - \frac{1}{2}\left[\exp\left(-|t|^{\alpha}\Big[B^{+} + B^{-}\operatorname{sign}(t)\Big]\right) + \exp\left(-|t|^{\alpha}\Big[B^{+} - B^{-}\operatorname{sign}(t)\Big]\right)\right], \qquad t \in \mathbb{R},$$

with I the identity matrix is a cross-variogram and

$$t \mapsto J - \exp\left(-|t|^{\alpha}\Big[B^{+} + B^{-}\operatorname{sign}(t)\Big]\right), \qquad t \in \mathbb{R},$$

with $J_{ij} = 1$ (matrix of all ones) is a pseudo-variogram. Under the assumptions of Theorem 2.2, the same is true for the functions

$$\boldsymbol{t} \mapsto I - \frac{1}{2}\Bigg[\exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^{\top}\boldsymbol{s}\right|^{\alpha}\Big[B^{+} + B^{-}\operatorname{sign}(\boldsymbol{t}^{\top}\boldsymbol{s})\Big]\, d\Lambda(\boldsymbol{s})\right)$$
$$+ \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^{\top}\boldsymbol{s}\right|^{\alpha}\Big[B^{+} - B^{-}\operatorname{sign}(\boldsymbol{t}^{\top}\boldsymbol{s})\Big]\, d\Lambda(\boldsymbol{s})\right)\Bigg], \qquad \boldsymbol{t} \in \mathbb{R}^{d},$$

and

$$\boldsymbol{t} \mapsto J - \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^{\top}\boldsymbol{s}\right|^{\alpha}\Big[B^{+} + B^{-}\operatorname{sign}(\boldsymbol{t}^{\top}\boldsymbol{s})\Big]\, d\Lambda(\boldsymbol{s})\right), \qquad \boldsymbol{t} \in \mathbb{R}^{d}.$$

Finally, let us briefly mention that there are two close relatives of the family of processes corresponding to f: the operator fractional Ornstein-Uhlenbeck process $\boldsymbol{X}(t)$, $t \in \mathbb{R}$ from [3, Section 3.1], associated to the covariance $\operatorname{cov}(\boldsymbol{X}(t), \boldsymbol{X}(s)) = \exp(-|t-s|^{\alpha})$, where α is a symmetric $d \times d$ matrix with eigenvalues belonging to $(0, 2]$, and the multivariate Ornstein-Uhlenbeck process, defined as a solution of a certain stochastic differential equation driven by a Brownian motion. The covariance of the latter is given by $\int_0^t e^{-A(t-s)} B e^{-A(t-s)}\, ds$, where A and B are real $d \times d$ matrices satisfying some additional assumptions. See, for example, [13, 14, 6].

1.1 Brief organization of the paper

Section 2 contains our main results. It begins with a simplified version of the main theorem along with its proof, after which we formulate an extension of this simplified version to multidimensional time. The main result of this contribution is Theorem 2.3. In Section 3 we reproduce for reader's convenience three known theorems (Operator-valued Bochner's Theorem, Bernstein's theorem on completely monotone functions and the Canonical representations of univariate and multivariate stable laws). The proof of the main theorem is presented in Section 4. More technical results are relegated to the Appendix.

1.2 Notation

Throughout the paper, we use the term "positive-definite" to refer to nonnegative-definite functions. To emphasize the case when the inequality is strict, we use the expression "strictly positive-definite."

If f is a matrix-valued function, we write $f \succeq 0$ to indicate that f is a positive-definite function in the following sense: $f^{\top}(t) = f(-t)$ and

$$\sum_{k,m=1}^{n} \boldsymbol{z}_k^{*}\, f(t_k - t_m)\, \boldsymbol{z}_m \geq 0, \qquad \forall\ \{\boldsymbol{z}_k\}_{k=1,\ldots,n} \subset \mathbb{C}^{d},\ \{t_k\}_{k=1,\ldots,n} \subset \mathbb{R}.$$

We will utilize the same notation $f \succeq 0$ is f is a complex-valued positive-definite function. Occasionally, we write the sign $\succ$ to indicate that the positive definiteness is strict.

We also write $A \trianglerighteq 0$ for a matrix A to indicate that A is a positive-definite matrix in the usual sense, namely,

$$A = A^* \quad \text{and} \quad \boldsymbol{z}^* A\, \boldsymbol{z} \geq 0, \qquad \forall\, \boldsymbol{z} \in \mathbb{C}^d.$$

The corresponding strict version will be denoted by $\triangleright$.

Note that we will write "$f \trianglerighteq 0$" to say that a matrix-valued function f is positive-definite *as a matrix*, rather than as a function. The difference between these two notions is crucial for the matrix-valued Bochner's Theorem 3.1.

The Fourier transform of a function f is defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)\, dx\,.$$

The application of $\mathcal{F}$ to matrix-valued functions is performed component-wise.

2 Main results

In order to provide the reader with an intuitive understanding of the main result, we begin by presenting a preliminary, simpler version of the theorem that serves as a warm-up example.

More specifically, assume that B is normal, i.e., there exists a diagonal matrix D and an unitary matrix P such that

$$B = P^* D P. \tag{2.1}$$

The positive definiteness of f is therefore equivalent to that of

$$g(t) \coloneqq \exp\left(-|t|^\alpha \Big[D^+ + D^- \operatorname{sign}(t)\Big]\right).$$

Since D^- is anti-Hermitian, its elements are purely imaginary. We denote them by $-i\lambda_k^- \coloneqq (D^-)_{kk}$, where $\lambda_k^- \in \mathbb{R}$. Similarly, the k-th element of D^+ is denoted by $\lambda_k^+ \coloneqq (D^+)_{kk}$.

Assume further that the matrix B satisfies (1.4). As mentioned in the Introduction, this condition turns out to be necessary for positive-definiteness of f. See Section 4.1 for the proof.

By definition of positive definiteness, $g \succeq 0$ if and only if

$$\sum_{i,j} \boldsymbol{z}_i^\top \exp\left(-|t_i - t_j|^\alpha \Big[D^+ + D^- \operatorname{sign}(t_i - t_j)\Big]\right) \boldsymbol{z}_j$$
$$= \sum_k \sum_{i,j} z_{ik} \exp\left(-|t_i - t_j|^\alpha \Big[\lambda_k^+ - i\lambda_k^- \operatorname{sign}(t_i - t_j)\Big]\right) z_{jk}$$

is non-negative for all possible choices of $\boldsymbol{z}_k \in \mathbb{C}^n$ and $t_i \in \mathbb{R}$. We will show a stronger claim: for all k, the scalar-valued function

$$g_k(t) \coloneqq \exp\left(-|t|^\alpha \Big[\lambda_k^+ - i\lambda_k^- \operatorname{sign}(t)\Big]\right)$$

is positive-definite. This function is known as the characteristic function of the α-stable law, which by the well-known canonical representation theorem (see Theorem 3.4 and Remark 3.5) is positive definite if and only if

$$\lambda_k^+ \geq 0 \quad \text{and} \quad \left|\lambda_k^-\right| \leq \lambda_k^+ \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|. \tag{2.2}$$

If $\alpha \neq 2$, then the assumption $\widetilde{B} \trianglerighteq 0$ implies that these conditions are met. Indeed, if $\widetilde{B} \trianglerighteq 0$, then

$$\lambda_k^+ \sin\left(\frac{\pi\alpha}{2}\right) + \lambda_k^- \cos\left(\frac{\pi\alpha}{2}\right) \geq 0 \quad \text{and} \quad \lambda_k^+ \sin\left(\frac{\pi\alpha}{2}\right) - \lambda_k^- \cos\left(\frac{\pi\alpha}{2}\right) \geq 0,$$

from which the inequalities (2.2) easily follow. If $B \trianglerighteq 0$, then the inequalities (2.2) are also satisfied. We have thus proven the following result.

Theorem 2.1. *If $\alpha \in (0,2)$ and B is a real $d \times d$ normal matrix satisfying (1.4), then the function defined in (1.3) is positive-definite. If $\alpha = 2$ and B is a real $d \times d$ matrix satisfying $B \trianglerighteq 0$, then the function defined in (1.3) is positive-definite.*

By the same proof as above with the use of Theorem 3.6 instead of Theorem 3.4, we obtain the following generalization.

Theorem 2.2 (Multivariate parameter extension of the previous result)**.** *Let $\alpha \in (0,2]$, B is a real $d \times d$ normal matrix satisfying (1.4) or $B^+ \trianglerighteq 0$ if $\alpha = 2$, and Λ is a finite measure on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, then the function defined by*

$$f(\boldsymbol{t}) := \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \left[B_+ + B_- \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})\right] d\Lambda(\boldsymbol{t})\right)$$

is positive definite.

We now proceed to the statement of the general theorem. For $p > 0$ and a matrix A with spectrum in $\mathbb{C} \setminus (-\infty, 0]$, define

$$A^p := \exp\left(p(A-I)\int_0^1 [s(A-I)+I]^{-1}\, ds\right).$$

Theorem 2.3. *Let B be a real $d \times d$ matrix. If the function f defined in (1.3) is positive-definite, then the conditions (1.4) and $B^+ \trianglerighteq 0$ are satisfied. If, on the other hand, the condition (1.4) is satisfied, then*

- *If $\alpha \in (0,1)$ and B is invertible, then f is positive-definite if and only if B additionally satisfies*
$$B^{1/\alpha} + B^{1/\alpha,\top} \trianglerighteq 0. \tag{2.3}$$
- *If $\alpha \in [1,2)$, then f is positive definite.*
- *If $\alpha = 2$ and $B^+ \trianglerighteq 0$, then f is positive definite.*

Remark 2.4. If B is normal, the condition (2.3) follows from (1.4). See Section 4.2 for the proof.

3 Auxiliary results

3.1 Operator-valued Bochner's theorem

The following version of Bochner's theorem is taken from [8, Theorem III.3].

Theorem 3.1 (Operator-valued Bochner Theorem, Neeb 1998)**.** *Let G be a locally compact abelian group, $\widehat{G}$ its character group, and $\mathcal{H}$ a Hilbert space. Then an ultraweakly continuous function $K\colon G \to B(\mathcal{H})$, where $B(\mathcal{H})$ is the set of bounded operators on $\mathcal{H}$, is positive definite if and only if there exists a finite $\mathrm{Herm}^+(\mathcal{H})$-valued measure μ on $\widehat{G}$ such that*

$$K(g) = \int_{\widehat{G}} \chi(g)\, d\mu(\chi).$$

Here $\mathrm{Herm}^+(\mathcal{H})$ is the cone of bounded positive-definite Hermitian operators on $\mathcal{H}$. The Radon measure μ is uniquely determined by K.

We are interested in the particular case of this lemma where $G = \mathbb{R}$ and $\mathcal{H} = \mathbb{R}^d$.[1]

Corollary 3.2 (Matrix-valued Bochner Theorem on $\mathbb{R}$). *A continuous matrix-valued function $f\colon \mathbb{R} \to \mathbb{C}^{d\times d}$ is positive definite $f \succeq 0$ if and only if there exists a matrix-valued measure $\mu \trianglerighteq 0$ on $\mathbb{R}$ such that*

$$f(t) = \int_{\mathbb{R}} e^{-i\xi t}\, d\mu(\xi)\,.$$

3.2 Bernstein's theorem

An infinitely differentiable function $f\colon (0,\infty) \to \mathbb{R}_+$ is said to be completely monotone if for any non-negative integer $n \geq 0$ holds

$$(-1)^n \frac{d^n f}{dt^n}(t) \geq 0, \qquad t > 0.$$

The following version of the celebrated Bernstein's theorem on completely monotone functions is taken from [11, Theorem A.3.6].

Theorem 3.3 (Bernstein 1928). *A real-valued function f is completely monotone if and only if there exists a measure μ on $(0,\infty)$ such that*

$$f(t) = \int_0^\infty e^{-ut}\, d\mu(u)\,.$$

This measure is finite if $\lim_{t\downarrow 0} f(t) < \infty$.

In particular, we will use this theorem with $\exp(-x^\alpha)$ if $\alpha \in (0,1)$, for which μ is finite, and $\exp(-x^{1/\alpha})\, x^{1/\alpha - 1}/\alpha$ if $\alpha \in (1,2)$, for which μ is infinite.

3.3 Canonical representation of stable laws

Two basic sources on α-stable laws are the monograph by Steutel & van Harn [11] and the monograph by Uchaikin & Zolotarev [12]. We will need the following result, which is taken from [11, Theorem 7.11].

Theorem 3.4 (Canonical representation of α-stable laws). *For $\alpha \in (0,2] \setminus \{1\}$, a $\mathbb{C}$-valued function f on $\mathbb{R}^d$ is the characteristic function of a centered non-degenerate stable distribution with exponent α if and only if it is of the form*

$$f(t) = \exp\Big(-|t|^\alpha \Big[\lambda - i\theta\, \mathrm{sign}(t)\Big]\Big)\,, \tag{3.1}$$

where $\lambda > 0$ and θ satisfies

$$|\theta| \leq \lambda \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|. \tag{3.2}$$

Remark 3.5. As remarked in [11, near formula (7.26)], it follows from this theorem combined with some simple considerations that the function defined in (3.1) with $\lambda > 0$ is positive definite *if and only if* the condition (3.2) is satisfied.

If $\lambda = 0$ and $\alpha \neq 1$, the function (3.1) is positive-definite if and only if $\theta = 0$. In case $\alpha = 1$, $\exp(i\theta t)$ is positive definite for any $\theta \in \mathbb{R}$.

The following extension of the previous theorem immediately follows from [9, Theorem 2.3.1].

Theorem 3.6 (Canonical representation of multivariate α-stable laws). *For $\alpha \in (0,2]\setminus\{1\}$, a $\mathbb{C}$-valued function f on $\mathbb{R}$ is the characteristic function of a centered non-degenerate*

[1] See also [1, Theorem 2.10].

stable random vector with exponent α if and only if there exists a finite measure Λ on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ such that

$$f(\boldsymbol{t}) = \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \left[\lambda - i\theta \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})\right] d\Lambda(\boldsymbol{s})\right),$$

where $\lambda > 0$ and θ satisfies (3.2).

Remark 3.7. As above, if $\alpha = 1$, there are no restrictions on θ.

4 Proofs

4.1 Necessary condition

Proof of necessity in Theorem 2.3. Suppose that $f \succeq 0$. For $\boldsymbol{z} \in \mathbb{C}^n$, define

$$f_{\boldsymbol{z}}(t) := \boldsymbol{z}^* f(t)\, \boldsymbol{z}.$$

It is easy to see that these functions satisfy $f_{\boldsymbol{z}} \succeq 0$.[2] Note that

$$\left[\frac{1}{\boldsymbol{z}^*\boldsymbol{z}}\, f_{\boldsymbol{z}}\left(\left(\frac{\boldsymbol{z}^*\boldsymbol{z}}{n}\right)^{1/\alpha} t\right)\right]^n = \left[I - \frac{|t|^\alpha}{n}\boldsymbol{z}^*\left[B^+ + B^- \operatorname{sign}(t)\right]\boldsymbol{z} + O\left(\frac{1}{n^2}\right)\right]^n \xrightarrow[n\to\infty]{} g_{\boldsymbol{z}}(t)$$

with

$$g_{\boldsymbol{z}}(t) = \exp\left(-|t|^\alpha\, \boldsymbol{z}^*\left[B^+ + B^- \operatorname{sign}(t)\right]\boldsymbol{z}\right). \tag{4.1}$$

Since positive-definiteness of scalar-valued functions is preserved under stretching, taking powers and taking limits, we have $g_{\boldsymbol{z}} \succeq 0$.

Since B^- is real antisymmetric, iB^- is Hermitian and therefore $\boldsymbol{z}^* iB^- \boldsymbol{z} \in \mathbb{R}$. By Theorem 3.4 and Remark 3.5, this function is positive-definite if and only if

$$\boldsymbol{z}^* B^+ \boldsymbol{z} \geq 0, \qquad \left|\boldsymbol{z}^* iB^- \boldsymbol{z}\right| \leq \boldsymbol{z}^* B^+ \boldsymbol{z} \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|.$$

Multiplying both sides by $|\cos(\pi\alpha/2)|$ and noting that $\sin(\pi\alpha/2) \geq 0$ for $\alpha \in (0, 2]$, we find that

$$\pm \boldsymbol{z}^* iB^- \boldsymbol{z}\, \cos\left(\frac{\pi\alpha}{2}\right) \leq \boldsymbol{z}^* B^+ \boldsymbol{z}\, \sin\left(\frac{\pi\alpha}{2}\right),$$

which is equivalent to

$$\widetilde{B} := B^+ \sin\left(\frac{\pi\alpha}{2}\right) - iB^- \cos\left(\frac{\pi\alpha}{2}\right) \succeq 0$$

because B is real. □

4.2 Alternative form of the B condition and the eigenvalues of B

Note that

$$i^{\alpha-1} = \sin\left(\frac{\pi\alpha}{2}\right) - i\cos\left(\frac{\pi\alpha}{2}\right).$$

Hence, $\widetilde{B} \trianglerighteq 0$ is equivalent to

$$i^{\alpha-1} B + i^{1-\alpha} B^\top \trianglerighteq 0.$$

Therefore, if λ is an eigenvalue of B, it satisfies

$$\operatorname{Re}\, i^{\alpha-1}\lambda \geq 0 \quad \text{and} \quad \operatorname{Re}\, i^{1-\alpha}\lambda \geq 0$$

[2]Although we will not use it, we want to mention that positive definiteness of scalar-valued projections of a matrix-valued function $f(t, s)$ is in fact equivalent to the positive-definiteness of f itself if $f(t, s) = f(t - s)$. This was originally proved in [5]. See also [1, Theorem 2.10] and the references therein.

because B is real. Rewriting both in terms of their arguments, we obtain

$$\pm\frac{(\alpha-1)\pi}{2}+\arg\lambda\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

It follows that

$$\arg\lambda\in\left[-\frac{\pi\alpha}{2},\frac{\pi\alpha}{2}\right]$$

and therefore

$$\operatorname{Re}\lambda^{1/\alpha}\geq 0. \tag{4.2}$$

If B is normal, (4.2) implies that

$$B^{1/\alpha}+B^{1/\alpha,\top}=P^{\top}\left[D^{1/\alpha}+D^{*,1/\alpha}\right]P\trianglerighteq 0,$$

and the condition (2.3) is satisfied.

4.3 Proof of Theorem 2.3

Proof of Theorem 2.3 in the case $\alpha\in(0,1)$. Assume that B is diagonalizable and $\widetilde{B}\rhd 0$ (strictly). That is, there exists an invertible matrix U and a diagonal matrix D such that $B=U^{-1}DU$.

By Bernstein's theorem 3.3, if $\alpha\in(0,1)$ there exists a finite measure μ on $\mathbb{R}_+$ such that

$$e^{-x^{\alpha}}=\int_0^{\infty}e^{-ux}\,d\mu_{\alpha}(u)\,.$$

By $\widetilde{B}\rhd 0$, the inequality (4.2) is strict and we can plug $x=D^{1/\alpha}|t|$ into this formula. This follows from the fact that the measure μ_{α} is finite and that $\exp(-\lambda^{1/\alpha}t)$ is bounded for each eigenvalue λ of D. Hence, we have

$$e^{-D|t|^{\alpha}}=\int_0^{\infty}e^{-uD^{1/\alpha}t}\,d\mu_{\alpha}(u)\,.$$

Conjugating both sides of the equality with U, we obtain

$$e^{-B|t|^{\alpha}}=\int_0^{\infty}e^{-uB^{1/\alpha}|t|}\,d\mu_{\alpha}(u), \tag{4.3}$$

since $B^{1/\alpha}=U^{-1}D^{1/\alpha}U$. We have thus obtained the following representation of f:

$$f(t)=\int_0^{\infty}\left[e^{-uB^{1/\alpha}|t|}\mathbb{1}_{t\geq 0}+e^{-uB^{1/\alpha,\top}|t|}\mathbb{1}_{t\leq 0}\right]d\mu_{\alpha}(u)\,.$$

Let us compute Fourier transforms of both sides:

$$\mathcal{F}\left[\exp\left(-uB^{1/\alpha}|t|\right)\,\mathbb{1}_{t\geq 0}\right](\xi)=U^{-1}\mathcal{F}\left[\exp\left(-uD^{1/\alpha}|t|\right)\,\mathbb{1}_{t\geq 0}\right](\xi)\,U$$

and by

$$\mathcal{F}\left[e^{-\lambda t}\,\mathbb{1}_{t\geq 0}\right](\xi)=\frac{1}{\lambda-i\xi},\quad\text{for}\quad\operatorname{Re}\lambda>0$$

combined with the fact that $\widetilde{B}\rhd 0$ implies that all eigenvalues λ of B satisfy $|\operatorname{Im}\lambda|<\operatorname{Re}\lambda\cdot|\tan(\pi\alpha/2)|$, we find that

$$\mathcal{F}\left[\exp\left(-uB^{1/\alpha}|t|\right)\,\mathbb{1}_{t\geq 0}\right](\xi)=U^{-1}\left(uD^{1/\alpha}-i\xi\right)^{-1}U=\left(uB^{1/\alpha}-i\xi\right)^{-1}.$$

Hence,

$$\begin{aligned}\mathcal{F}\left[f(t)\right](\xi) &= \int_0^\infty \left[\left(uB^{1/\alpha} - i\xi\right)^{-1} + \left(uB^{1/\alpha,\top} + i\xi\right)^{-1} \right] d\mu_\alpha(u) \\ &= \int_0^\infty u \left(uB^{1/\alpha} - i\xi\right)^{-1} \left[B^{1/\alpha} + B^{1/\alpha,\top}\right] \left(uB^{1/\alpha} - i\xi\right)^{-1,*} d\mu_\alpha(u)\,.\end{aligned}$$

By operator-valued Bochner's theorem 3.1, $f(t) \succeq 0$ if and only if its Fourier transform is a positive definite matrix for each ξ. Setting $\xi = 0$ we obtain

$$\mathcal{F}\left[f(t)\right](0) = \left(B^{1/\alpha}\right)^{-1} \left[B^{1/\alpha} + B^{1/\alpha,\top}\right] \left(B^{1/\alpha}\right)^{-1,\top} \int_0^\infty u^{-1} d\mu_\alpha(u).$$

This matrix is positive-definite if and only if

$$B^{1/\alpha} + B^{1/\alpha,\top} \trianglerighteq 0, \tag{4.4}$$

hence this is a necessary condition for the positive definiteness of f. Note however that if this condition is satisfied, then

$$\left(uB^{1/\alpha} - i\xi\right)^{-1} \left[B^{1/\alpha} + B^{1/\alpha,\top}\right] \left(uB^{1/\alpha,\top} - i\xi\right)^{-1,*} \trianglerighteq 0$$

for each ξ and $\mathcal{F}[f](\xi) \trianglerighteq 0$ pointwise. □

Proof of Theorem 2.3 in the case $\alpha \in [1, 2)$. As in the proof for the case $\alpha \in (0, 1)$, our approach is to find an appropriate representation for the Fourier transform of f. Assume that $\xi \geq 0$. Also assume for now that $\widetilde{B} \triangleright 0$. Then there exists $\theta \in (1 - 1/\alpha, 1/\alpha)$ such that $i^{\alpha\theta}B + i^{-\alpha\theta}B^\top \triangleright 0$. The following formula

$$\mathcal{F}\left[e^{-Bt^\alpha}\mathbb{1}_{t\geq 0}\right](\xi) = \int_0^\infty e^{it\xi I - Bt^\alpha}\, dt = i^\theta \int_0^\infty e^{-i^{\theta-1}t\xi I - Bt^\alpha i^{\alpha\theta}}\, dt \tag{4.5}$$

is proven in the Appendix. Performing a change of variables $s = t^\alpha$, we get

$$\mathcal{F}\left[e^{-Bt^\alpha}\mathbb{1}_{t\geq 0}\right](\xi) = i^\theta \int_0^\infty \frac{s^{1/\alpha - 1}}{\alpha}\, e^{-i^{\theta-1}s^{1/\alpha}\xi I - Bi^{\alpha\theta}s}\, ds\,.$$

By Bernstein's Theorem 3.3 there exists a measure μ on $(0, \infty)$ such that:

$$\frac{s^{1/\alpha-1}}{\alpha}\, e^{-s^{1/\alpha}} = \int_0^\infty e^{-us}\, d\mu(u)\,.$$

Plugging $s \rightsquigarrow i^{\alpha(\theta-1)}\xi^\alpha s$, we obtain

$$i^\theta\, \frac{s^{1/\alpha-1}}{\alpha}\, e^{-i^{\theta-1}s^{1/\alpha}\xi} = i^{1-\alpha+\alpha\theta}\xi^{\alpha-1} \int_0^\infty e^{-ui^{\alpha(\theta-1)}\xi^\alpha s}\, d\mu(u)\,.$$

The last step may be justified by using the fact that $\mathrm{Re}\ i^{\alpha(\theta-1)} > 0$ for $\alpha \in [1, 2)$. Proceeding with the computation above, we find

$$\mathcal{F}\left[e^{-Bt^\alpha}\mathbb{1}_{t\geq 0}\right](\xi) = i^{1-\alpha+\alpha\theta}\xi^{\alpha-1} \int_0^\infty \int_0^\infty e^{-ui^{\alpha(\theta-1)}\xi^\alpha sI - i^{\alpha\theta}Bs}\, ds\, d\mu(u).$$

Now, we can take the integral in s and obtain

$$\mathcal{F}\left[e^{-Bt^\alpha}\mathbb{1}_{t\geq 0}\right](\xi) = \xi^{\alpha-1} \int_0^\infty \left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1} d\mu(u)$$

for $\xi \geq 0$. Similarly,

$$\mathcal{F}\left[e^{-B^\top(-t)^\alpha}\mathbb{1}_{t<0}\right](\xi) = \xi^{\alpha-1}\int_0^\infty \left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1,*} d\mu(u)\,.$$

Combining the last two formulas together, we arrive at

$$\mathcal{F}\left[f(t)\right](\xi) = \xi^{\alpha\theta}\int_0^\infty \left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1}\left[i^{\alpha-1}B + i^{1-\alpha}B^\top\right]\left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1,*} d\mu(u)\,.$$

Note that

$$\left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1}\left[i^{\alpha-1}B + i^{1-\alpha}B^\top\right]\left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1,*} \trianglerighteq 0$$

if and only if

$$i^{\alpha-1}B + i^{1-\alpha}B^\top \trianglerighteq 0,$$

which we have already shown to be true.

If $\widetilde{B} \trianglerighteq 0$, but not necessarily $\widetilde{B} \vartriangleright 0$, let $B_\varepsilon \coloneqq B + \varepsilon I$ for $\varepsilon > 0$ and remark that $\widetilde{B_\varepsilon} \vartriangleright 0$. By the above, we have that

$$f_\varepsilon(t) \coloneqq \exp\left(-|t|^\alpha\left[B_\varepsilon^+ + B_\varepsilon^- \operatorname{sign}(t)\right]\right)$$

is positive definite for each $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$, we find that f is also positive definite. □

Proof of Theorem 2.3 in the case $\alpha = 2$. In this case (1.4) yields $-iB^- \trianglerighteq 0$. By conjugation we also obtain $iB^- \trianglerighteq 0$. Therefore, $\boldsymbol{z}^*(iB^-)\boldsymbol{z} = 0$ for all $\boldsymbol{z} \in \mathbb{C}^d$. Since iB^- is Hermitian, we can conclude that $B^- = 0$, therefore, $B = B^+$ is Hermitian, in particular, it is normal, and the positive definiteness of f follows from Theorem 2.1. □

4.4 Lifting the diagonalizability and strict positive definiteness assumptions

Proof of Theorem 2.3 in the non-diagonalizable case. If $\widetilde{B} \vartriangleright 0$, but B is not diagonalizable, then there exist diagonalizable matrices B_n converging to B as $n \to \infty$ such that the eigenvalues λ of B_n satisfy the strict inequality

$$|\operatorname{Im}\lambda| < \operatorname{Re}\lambda \cdot \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|. \tag{4.6}$$

Hence,

$$e^{-B_n|t|^\alpha} = \int_0^\infty e^{-uB_n^{1/\alpha}|t|}\, d\mu_\alpha(u),$$

which implies (4.3) by passing to a limit as $n \to \infty$. Having deduced (4.3), we can continue the proof the same way as if B were diagonalizable. □

Proof of Theorem 2.3 in the case when the condition $\widetilde{B} \trianglerighteq 0$ *is non-strict.* Take $\varepsilon > 0$ and let $B_\varepsilon \coloneqq (B^{1/\alpha} + \varepsilon I)^\alpha$. The eigenvalues of B_ε satisfy the strict inequality (4.6), and therefore $g_\varepsilon(t) \coloneqq \exp\left(-|t|^\alpha\left[B_\varepsilon^+ + B_\varepsilon^- \operatorname{sign}(t)\right]\right)$ is positive definite if and only if $B_\varepsilon^{1/\alpha} + B_\varepsilon^{1/\alpha,\top} \trianglerighteq 0$.

If $B^{1/\alpha} + B^{1/\alpha,\top} \trianglerighteq 0$, then for all $\varepsilon > 0$ $B_\varepsilon^{1/\alpha} + B_\varepsilon^{1/\alpha,\top} \trianglerighteq 0$ and $g_\varepsilon(t) \succeq 0$. Letting $\varepsilon \downarrow 0$, we obtain $f(t) \succeq 0$ as desired.

If $B^{1/\alpha} + B^{1/\alpha,\top} \trianglerighteq 0$ does not hold, then for all sufficiently small $\varepsilon > 0$ $(B + \varepsilon I)^{1/\alpha} + (B + \varepsilon I)^{1/\alpha,\top} \trianglerighteq 0$ also does not hold, but $\widetilde{B}_\varepsilon \vartriangleright 0$, therefore, f_ε is not positive definite, but then f is not positive definite, because otherwise f_ε would be positive definite as a product of a positive definite matrix-valued function f and a scalar positive definite function $\exp\left(-\varepsilon|t|^\alpha\right)$. □

A Appendix

A.1 Contour rotation in the proof of case $\alpha \in (1, 2]$

Proof of (4.5). Assume that $\operatorname{Re} i^{\alpha\theta}\lambda > 0$ and $\xi \geq 0$. By Cauchy theorem applied to the following contour γ

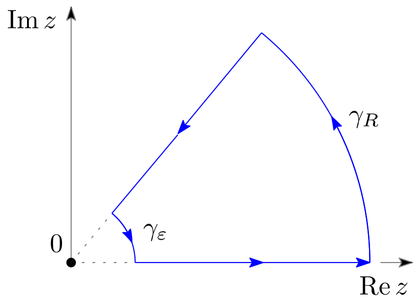

Figure 1: Closed contour γ and two its circular arcs γ_ε and γ_R.
The angle at the origin equals $\arg i^\theta = \pi\theta/2$.

we have that for $0 < \varepsilon < R$ holds

$$\oint_\gamma e^{it\xi - \lambda t^\alpha}\,dt = 0,$$

implying

$$\int_\varepsilon^R e^{it\xi-\lambda t^\alpha}\,dt = \int_{i^\theta\varepsilon}^{i^\theta R} e^{it\xi-\lambda t^\alpha}\,dt - \left[\int_{\gamma_\varepsilon} + \int_{\gamma_R}\right] e^{it\xi-\lambda t^\alpha}\,dt.$$

Since the integral over γ_ε clearly tends to zero as $\varepsilon \to 0$, and the function under the integral is exponentially small on γ_R, we have

$$\lim_{R\to\infty}\int_{\gamma_R} e^{it\xi-\lambda t^\alpha}\,dt = 0.$$

By changing the variable $t \rightsquigarrow i^\theta t$, we obtain

$$\int_0^\infty e^{it\xi-\lambda t^\alpha}\,dt = \int_{i^\theta\mathbb{R}_+} e^{it\xi-\lambda t^\alpha}\,dt = i^\theta\int_0^\infty e^{-i^{\theta-1}t\xi - i^{\alpha\theta}\lambda t^\alpha}\,dt$$

establishing the proof. □

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