

On the boundary at infinity for branching random walk*

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Abstract

We prove that a supercritical branching random walk on a transient Markov chain converges almost surely under rescaling to a random measure on the Martin boundary of the underlying Markov chain. Several open problems and conjectures about this limiting measure are presented.

Keywords: Martin boundary; branching random walk; Markov chains.

MSC2020 subject classifications: 60J10; 60J80; 60J45.

Submitted to ECP on January 13, 2023, final version accepted on October 15, 2023.

1 Introduction

The asymptotic behavior of Markov chains on graphs is at the core of potential theory, which blends together tools from probability, algebra and analysis. This asymptotic behaviour is often understood via an appropriate *boundary theory*, with the relevant boundaries coming in both topological and measure-theoretic varieties. On the one hand, the *Poisson boundary* is a measure-theoretic object encoding all possible (positive probability) limiting behaviours of the random walk and which, by a foundational theorem of Blackwell [11], may be defined equivalently either in terms of the invariant σ -algebra or the space of bounded harmonic functions on the Markov chain. On the other hand, the *Martin boundary* is a *topological* boundary that both recovers the measure-theoretic Poisson boundary and also includes information about possible *singular* (probability zero) behaviours of the walk at infinity.

While the Poisson and Martin boundaries of a Markov chain are rather abstract and difficult to study in general, they are known to coincide with concrete geometric compactifications in several classes of examples including hyperbolic groups [3, 4, 27], planar graphs [21, 5, 26], lamplighter groups [28, 15, 39], and Diestel-Leader graphs [47]. In general, however, the structure of the Martin boundary might be richer than what one would guess geometrically [43]. For random walks on groups, various important and beautiful works have established characterizations and criteria for the non-triviality of the Poisson and Martin boundaries in terms of speed, entropy, heat kernel decay and volume growth [6, 28, 42, 2, 16]. See [46, 40, 48] for overviews of the literature.

*TH was partially supported by ERC starting grant 804166 (SPRS). EC was supported by the project “Programma per Giovani Ricercatori Rita Levi Montalcini” awarded by the Italian Ministry of Education. EC acknowledges partial support by “INdAM–GNAMPA Project 2019” and “INdAM–GNAMPA Project 2020”.

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In this work we aim to initiate a systematic investigation of the boundary theory of *branching random walks*. The main goal of this work is to point out a new direction for research, connecting branching random walk, a well-studied (and only partially understood) Markov chain, with the Martin boundary for the underlying random walk, which is itself a Markov chain.

Let $M = (S, P)$ be a Markov chain with countable state space S and transition matrix P , and let μ be an *offspring distribution*, that is, a probability measure supported on the non-negative integers. Branching random walk (BRW) on M is a Markov process governed by P and μ in the following sense. Choose any state $x \in S$ and at time 0 place one alive particle at x . Inductively, at each time step all alive particles reproduce independently of one another according to μ and then die, so that each particle is replaced by an independent random number of offspring with law μ . Subsequently, as part of the same time step, all (alive) particles take one step of the Markov chain, independently of one another, according to the transition kernel P . This defines a Markov process $(B_n)_{n \geq 0}$ with state space equal to $\{\text{finitely supported functions } S \rightarrow \mathbb{N}\}$, where $B_n(x)$ is the number of (alive) particles at the state x at time n . One can also equivalently consider the branching random walk as a tree-indexed Markov chain where the underlying genealogical tree of the particles is itself a random Galton-Watson tree [8, 9]. In particular, it follows from the classical theory of branching processes that the process dies almost surely if the mean $\bar{\mu} = \sum n\mu(n)$ satisfies $\bar{\mu} < 1$ and survives forever with positive probability when $\bar{\mu} > 1$. A detailed account of what is known about branching random walk on \mathbb{Z} , a rich subject that is largely disjoint from the kind of questions we consider here, can be found in [44].

Previous works on the behaviour at infinity of branching random walk have focused either on the geometric properties of the *trace* (that is, the graph spanned by the edges crossed by the process) [7, 12, 13, 25] or the set of points accumulated to by the process in some geometric compactification. The latter approach has been carried out primarily in the setting of trees and hyperbolic spaces, where a detailed and sophisticated theory has now been developed [35, 30, 24, 36, 34, 33, 12, 23, 22, 45].

Our main theorem states that branching random walks always converge under suitable rescaling to a random measure on the Martin boundary. We are optimistic that this random measure is both an interesting object of study in its own right and should lead to new perspectives and clarity on the limit-set theory in the hyperbolic case. We write \mathbb{P}_x and \mathbb{E}_x for probabilities and expectations taken with respect to the law of the branching random walk started with a single particle at x and write \mathbf{P}_x and \mathbf{E}_x for probabilities and expectations taken with respect to the underlying Markov chain started at x .

Theorem 1.1. *Let $M = (S, P)$ be a transient, irreducible Markov chain, let $(X_n)_n$ denote a trajectory of M , and let X_∞ be its limit point on the Martin boundary. Let μ be an offspring distribution with mean $\bar{\mu} > 1$ satisfying the $L \log L$ condition $\sum_{n=1}^\infty \mu(n)n \log n < \infty$, and let $B = (B_n)_{n \geq 0}$ be a branching random walk on M started at some state o . Then $(\bar{\mu})^{-n} B_n$ almost surely converges weakly to a random measure \mathbf{W} on the Martin compactification \mathcal{M} of M that is supported on the active part of the Martin boundary of M and satisfies*

$$\mathbb{E}_o \mathbf{W}(A) = \mathbf{P}_o(X_\infty \in A) \tag{1.1}$$

for every Borel set $A \subseteq \mathcal{M}$.

A selection of open problems regarding this random measure is discussed in Section 4.

The so-called ‘ $L \log L$ condition’ $\sum_{n=1}^\infty \mu(n)n \log n < \infty$ appearing here is needed for the limiting measure \mathbf{W} to be non-zero. Indeed, if μ is supercritical and $(|B_n|)_{n \geq 0}$ is the

underlying branching process of the branching random walk $(B_n)_{n \geq 0}$ then the limit

$$W := \lim_{n \rightarrow \infty} (\bar{\mu})^{-n} |B_n|$$

exists almost surely by the martingale convergence theorem. The Kesten-Stigum theorem [32, 38] states that the following are equivalent:

1. $W > 0$ with positive probability.
2. $W > 0$ almost surely on the event that $(|B_n|)_{n \geq 0}$ survives forever.
3. $\mathbb{E}W = 1$.
4. $(\bar{\mu})^{-n} |B_n|$ is uniformly integrable.
5. μ satisfies the $L \log L$ condition.

Since any measure arising as the weak limit of $(\bar{\mu})^{-n} B_n$ must have total mass W , it follows that if μ does *not* satisfy the $L \log L$ condition then $(\bar{\mu})^{-n} B_n$ almost surely converges weakly to the zero measure. (In particular, the conclusions of Theorem 1.1 other than the formula (1.1) hold vacuously in this case.)

Remark 1.2. The (space-time) Martin boundary of *branching processes* (and hence of the Markov chain governing the *number* of particles in a branching random walk) has been investigated in [37] (see also [41]), whose results imply in particular that the random variable $W = \lim_{n \rightarrow \infty} (\bar{\mu})^{-n} |B_n|$ generates the entire tail σ -algebra of the branching process $(|B_n|)_{n \geq 0}$. For more recent results we refer the reader to [1] and references therein. While it is natural to guess from this that the random measure \mathbf{W} encodes all the tail information of the branching random walk $(B_n)_{n \geq 0}$, we observe in Section 4 that this is not the case even in very simple examples.

Remark 1.3. The hypothesis of irreducibility of Theorem 1.1 can be replaced by the weaker assumption that $\sup_n P^n(o, x) > 0$ for every x .

Remark 1.4. While we were in the final stages of preparing this paper, Kaimanovich and Woess [29] posted a preprint to the arXiv establishing similar results on the convergence of branching random walks to random measures; their results allow one to consider compactifications other than the Martin compactification and branching Markov chains with spatially dependent branching mechanism. While the two works were conducted largely independently, both have their origins in discussions between EC, VK, and WW that took place during the workshop “*Random walks on graphs and potential theory*” at the University of Warwick in 2015 where VK suggested to study the weak limits of the empirical measures of branching random walks. We are optimistic that the distinct perspectives offered by the two works will prove complementary to each other in the future development of the subject.

2 Background on the Martin boundary

We now briefly recall the definition and basic properties of the Martin boundary, referring the reader to [46] and [14] for further details. Let P be the transition matrix of an irreducible, transient Markov chain $M = (S, P)$ on a countable state space S . Recall that a function $h : S \rightarrow \mathbb{R}$ that is either non-negative or bounded¹ is said to be **harmonic** if $h(x) = \sum_{y \in S} P(x, y)h(y)$ for every $x \in S$ and is said to be **superharmonic** if $h(x) \geq \sum_{y \in S} P(x, y)h(y)$ for every $x \in S$. The space of all non-negative superharmonic

¹Throughout this section, we use the assumption that our functions are either non-negative or bounded to guarantee that all sums appearing are well-defined in $(-\infty, \infty]$.

functions on S is denoted by S^+ . Choose a reference vertex $o \in S$. For each $x, y \in S$, the **Martin kernel** $K(x, y)$ is the non-negative superharmonic function defined by

$$K(x, y) = \frac{G(x, y)}{G(o, y)},$$

where $G(x, y) = \sum_{n \geq 0} P^n(x, y)$ is the Green function. The **Martin compactification** $\mathcal{M} = \mathcal{M}(M)$ is defined to be the unique smallest compactification of the discrete set S for which all the Martin kernels $K(x, \cdot)$ extend continuously. A detailed explanation of how this property uniquely determines \mathcal{M} can be found in [46, Section 24], where it is also shown that \mathcal{M} is a Polish space. Thus, a sequence of states $(y_n)_{n \geq 0}$ converges in the Martin compactification if and only if the Martin kernels $K(x, y_n)$ converge for each $x \in S$. Note that this does indeed give a compact space since $G(o, y) \geq \sup_n P^n(o, x)G(x, y)$ for every $x, y \in S$ and hence $0 \leq K(x, y) \leq (\sup_n P^n(o, x))^{-1} < \infty$ for every $x, y \in S$. The **Martin boundary** of M is the space $\partial\mathcal{M} = \mathcal{M} \setminus S$. Note that if $\xi \in \partial\mathcal{M}$ then we can define the Martin kernel $K(\cdot, \xi) := \lim_{n \rightarrow \infty} K(\cdot, y_n)$ where $(y_n)_{n \geq 0}$ is any sequence of states converging to ξ ; the choice of sequence does not affect the limit thus obtained.

The most important properties of the Martin boundary are as follows:

1. Convergence of the random walk [46, Theorem 24.10]: If $(X_n)_{n \geq 0}$ is a trajectory of the Markov chain M then X converges almost surely to a limit point $X_\infty \in \partial\mathcal{M}$. Moreover, the Martin kernel $K(x, \cdot)$ is the Radon-Nikodym derivative of the law of X_∞ under \mathbf{P}_x with respect to the law of X_∞ under \mathbf{P}_o in the sense that

$$\mathbf{E}_x [F(X_\infty)] = \mathbf{E}_o [F(X_\infty)K(x, X_\infty)]$$

for every $x \in S$ and every continuous function $F : \partial\mathcal{M} \rightarrow [0, \infty)$.

2. Representation of positive harmonic functions [46, Theorem 24.7]: If $h : S \rightarrow [0, \infty)$ is a non-negative harmonic function then there exists a finite Borel measure ν_h on $\partial\mathcal{M}$ such that

$$h(x) = \int K(x, \xi) d\nu_h(\xi)$$

for every $x \in S$. Conversely, if ν is a finite Borel measure on $\partial\mathcal{M}$ then $\int K(x, \xi) d\nu(\xi)$ is a non-negative harmonic function on S . The function h is bounded if and only if ν_h may be taken to be absolutely continuous with respect to the law of X_∞ with essentially bounded Radon-Nikodym derivative; in particular, if $h : S \rightarrow \mathbb{R}$ is a bounded harmonic function then there exists a bounded Borel function $\phi : \partial\mathcal{M} \rightarrow \mathbb{R}$ such that

$$h(x) = \mathbf{E}_o [\phi(X_\infty)K(x, X_\infty)] = \mathbf{E}_x [\phi(X_\infty)]$$

for every $x \in S$ [46, Theorem 24.12]. In both cases, the representation is not unique in general but can be made unique by restricting the boundary data to be supported on the *minimal* Martin boundary [46, Theorem 24.9].

3. Probabilistic Fatou Theorem [14, Theorem 10]: If $\phi : \partial\mathcal{M} \rightarrow \mathbb{R}$ satisfies $\mathbf{E}_o |\phi(X_\infty)| < \infty$ and $h(x) = \mathbf{E}_x \phi(X_\infty)$ denotes the harmonic extension of ϕ to S then

$$h(X_n) \rightarrow \phi(X_\infty)$$

almost surely as $n \rightarrow \infty$.

Remark 2.1. Let us warn the reader that the Martin boundary can be quite ill-behaved in general, and in particular that the harmonic extension of a continuous function need not be continuous on the Martin compactification.

3 Proof of the convergence theorem

For the remainder of this section we fix an irreducible, countable-state Markov chain $M = (S, P)$ and a reference vertex $o \in S$, fix an offspring measure μ satisfying the $L \log L$ condition, and let B_n be a branching random walk on M with offspring measure μ , started with a single particle at o . We write $\bar{B}_n = (\bar{\mu})^{-n} B_n$ for each $n \geq 0$ and, given a function $f : S \rightarrow \mathbb{R}$, write $\langle f, \bar{B}_n \rangle := \sum_{x \in S} f(x) \bar{B}_n(x)$.

Remark 3.1. When (as in Theorem 1.1) we write that a sequence “almost surely converges weakly” we mean that for almost all realizations of the branching Markov chain the considered sequence converges weakly to the specified limit.

Lemma 3.2. *If $h : S \rightarrow \mathbb{R}$ is harmonic and is either bounded or non-negative then $(\langle h, \bar{B}_n \rangle)_{n \geq 0}$ is a martingale with respect to the filtration generated by $(B_n)_{n \geq 0}$. Similarly, if $h : S \rightarrow \mathbb{R}$ is superharmonic and is either bounded or non-negative then $(\langle h, \bar{B}_n \rangle)_{n \geq 0}$ is a supermartingale with respect to the filtration generated by $(B_n)_{n \geq 0}$.*

Proof of Lemma 3.2. Let \mathcal{F}_n be the σ -algebra generated by $(B_i)_{i=0}^n$. If $h : S \rightarrow \mathbb{R}$ is either bounded or non-negative then we have by linearity of expectation that

$$\begin{aligned} \mathbb{E}[\langle h, \bar{B}_{n+1} \rangle \mid \mathcal{F}_n] &= \sum_{x \in S} h(x) (\bar{\mu})^{-n-1} \mathbb{E}[B_{n+1}(x) \mid \mathcal{F}_n] \\ &= (\bar{\mu})^{-n} \sum_{x, y \in S} B_n(y) P(y, x) h(x) = \langle Ph, \bar{B}_n \rangle \end{aligned}$$

where in the second equality we have used that $\mathbb{E}[B_{n+1}(x) \mid \mathcal{F}_n] = \bar{\mu} \sum_{y \in S} B_n(y) P(y, x)$ by linearity of expectation and the definition of branching random walk. The claim follows since $Ph = h$ when h is harmonic and $Ph \leq h$ when h is superharmonic. \square

Corollary 3.3. *If $h : S \rightarrow \mathbb{R}$ is superharmonic and is either bounded or non-negative then the sequence $\langle h, \bar{B}_n \rangle$ converges almost surely as $n \rightarrow \infty$.*

Lemma 3.4. *Let $V \subseteq \partial\mathcal{M}$ be closed and let $h(x) := \mathbf{P}_x(X_\infty \in V)$ be the harmonic extension of $\mathbb{1}_V$ to S . Then*

$$\sup\{\mathbf{W}(V) : \mathbf{W} \text{ is a subsequential weak limit of } \bar{B}_n\} \leq \lim_{n \rightarrow \infty} \langle h, \bar{B}_n \rangle$$

almost surely.

Proof of Lemma 3.4. Since $\partial\mathcal{M}$ is metrisable, each closed set can be written as the intersection of a decreasing sequence of open sets, namely the $1/k$ neighbourhoods of the set in some compatible metric. Let V be a closed subset of $\partial\mathcal{M}$ and let $V = \bigcap_k U_k$ for some decreasing sequence of open sets $(U_k)_{k \geq 1}$ containing V . Since $\{X_\infty \in V\} = \bigcap_k \{X_n \text{ ever visits } U_k\}$, we have by continuity of measure that for each $\varepsilon > 0$ there exists an open neighbourhood U of V in \mathcal{M} such that

$$\mathbf{P}_o(X_n \text{ ever visits } U) \leq \mathbf{P}_o(X_\infty \in V) + \varepsilon.$$

Fix one such choice of ε and U and let $f : S \rightarrow [0, 1]$ be the function

$$f(x) := \mathbf{P}_x(X_n \text{ ever visits } U).$$

Note that f is harmonic on $S \setminus U$, superharmonic on $S \cap U$, and satisfies $f \geq h$ and $f \geq \mathbb{1}_U$ everywhere. Since $\langle f, \bar{B}_n \rangle$ is a non-negative supermartingale, it converges a.s., and since U is open we have by the Portmanteau Theorem that

$$\begin{aligned} \sup\{\mathbf{W}(V) : \mathbf{W} \text{ is a subsequential weak limit of } \bar{B}_n\} &\leq \sup\{\mathbf{W}(U) : \mathbf{W} \text{ is a subsequential weak limit of } \bar{B}_n\} \\ &\leq \limsup_{n \rightarrow \infty} \langle \mathbb{1}_U, \bar{B}_n \rangle \leq \limsup_{n \rightarrow \infty} \langle f, \bar{B}_n \rangle = \lim_{n \rightarrow \infty} \langle f, \bar{B}_n \rangle. \end{aligned}$$

On the other hand, since $f-h$ is non-negative, superharmonic, and satisfies $f(o)-h(o) \leq \varepsilon$ we also have that

$$\mathbb{E}_o \left[\lim_{n \rightarrow \infty} \langle f-h, \overline{B}_n \rangle \right] \leq f(o) - h(o) \leq \varepsilon$$

and hence by Markov's inequality that

$$\begin{aligned} \mathbb{P} \left(\sup \{ \mathbf{W}(V) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \} \geq \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle + \delta \right) \\ \leq \mathbb{P} \left(\lim_{n \rightarrow \infty} \langle f, \overline{B}_n \rangle \geq \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle + \delta \right) \leq \frac{\varepsilon}{\delta} \end{aligned}$$

for every $\delta > 0$. Since $\varepsilon, \delta > 0$ were arbitrary it follows that

$$\sup \{ \mathbf{W}(V) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \} \leq \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle \text{ almost surely}$$

as claimed. □

We next apply this one-sided estimate for closed sets to deduce an *equality* holding for each open set, closed set, and continuous function.

Lemma 3.5.

1. Let $A \subseteq \partial\mathcal{M}$ be either open or closed and let $h(x) := \mathbf{P}_x(X_\infty \in A)$ be the harmonic extension of $\mathbb{1}_A$ to S . The event "every subsequential limit \mathbf{W} of \overline{B}_n satisfies $\mathbf{W}(A) = \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle$ " has probability one.
2. Let $\phi : \partial\mathcal{M} \rightarrow \mathbb{R}$ be continuous and let h be the harmonic extension of ϕ to S . The event "every subsequential limit \mathbf{W} of \overline{B}_n satisfies $\int \phi(\xi) d\mathbf{W}(\xi) = \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle$ " has probability one.

Proof of Lemma 3.5. We begin with claim 1. We first prove that if $A = U \subseteq \partial\mathcal{M}$ is open then

$$\sup \{ \mathbf{W}(U) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \} \leq \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle \tag{3.1}$$

almost surely, where h is the harmonic extension of $\mathbb{1}_U$ to S . Since $\partial\mathcal{M}$ is metrisable, the open set U may be written as a countable union of closed sets $U = \bigcup_{m \geq 1} V_m$. (For example, if d is a compatible metric on $\partial\mathcal{M}$ then U can be written as the union of the sequence of closed sets $V_m = \{ \xi \in \partial\mathcal{M} : d(\xi, U^c) \geq 1/m \}$.) Fix one such representation $U = \bigcup_{m \geq 1} V_m$ and for each $m \geq 1$ let $h_m(x) := \mathbf{P}_x(X_\infty \in V_m)$ be the harmonic extension of $\mathbb{1}_{V_m}$ to S . Since $h_m \leq h$ for each $m \geq 1$, we have by continuity of measure and Lemma 3.4 that

$$\begin{aligned} \sup \{ \mathbf{W}(U) : \mathbf{W} \text{ a subsequential weak limit of } \overline{B}_n \} \\ = \sup_m \sup \{ \mathbf{W}(V_m) : \mathbf{W} \text{ a subsequential weak limit of } \overline{B}_n \} \\ \leq \sup_m \lim_{n \rightarrow \infty} \langle h_m, \overline{B}_n \rangle \leq \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle \end{aligned}$$

almost surely as claimed. Now, if U is open then $U^c = \partial\mathcal{M} \setminus U$ is closed and the harmonic extension of $\mathbb{1}_{U^c}$ to S is given by $1-h$. Thus, it follows from Lemma 3.4 that

$$\sup \{ \mathbf{W}(U^c) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \} \leq \lim_{n \rightarrow \infty} \langle 1-h, \overline{B}_n \rangle \tag{3.2}$$

almost surely, and since $\int 1 d\mathbf{W}(\xi) = \lim_{n \rightarrow \infty} \langle 1, \overline{B}_n \rangle = W < \infty$ almost surely for every subsequential weak limit \mathbf{W} of \overline{B}_n , we can rearrange (3.2) to deduce that

$$\inf \{ \mathbf{W}(U) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \} \geq \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle \tag{3.3}$$

almost surely. Together (3.1) and (3.3) imply the open case of the claim, and the closed case follows by taking complements as in the proof of (3.2).

We now turn to the second claim. We may assume without loss of generality that ϕ takes values in $[0, 1]$. For each $t \in [0, 1]$ let $V_t = \{\xi \in \partial\mathcal{M} : \phi(\xi) \geq t\}$ and let $h_t(x) := \mathbf{P}_x(X_\infty \in V_t)$ be the harmonic extension of $\mathbb{1}_{V_t}$ to S , so that $\phi = \int_0^1 \mathbb{1}_{V_t} dt$ and $h = \int_0^1 h_t dt$. We have by Lemma 3.4 that

$$\begin{aligned} & \sup \left\{ \int \phi(\xi) d\mathbf{W}(\xi) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \right\} \\ & \leq \int_0^1 \sup \{ \mathbf{W}(V_t) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \} dt \\ & \leq \int_0^1 \lim_{n \rightarrow \infty} \langle h_t, \overline{B}_n \rangle dt \leq \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle, \end{aligned} \tag{3.4}$$

almost surely, where we have used Fatou’s lemma in the final inequality. Since ϕ takes values in $[0, 1]$ we can define a continuous function $(1 - \phi) : \partial\mathcal{M} \rightarrow [0, 1]$, which has harmonic extension to S given by $1 - h$, and we have by symmetry that

$$\sup \left\{ \int (1 - \phi(\xi)) d\mathbf{W}(\xi) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \right\} \leq \lim_{n \rightarrow \infty} \langle 1 - h, \overline{B}_n \rangle. \tag{3.5}$$

Since $\int 1 d\mathbf{W}(\xi) = \lim_{n \rightarrow \infty} \langle 1, \overline{B}_n \rangle = W$ for every subsequential weak limit \mathbf{W} of \overline{B}_n , we can rearrange (3.5) to deduce that

$$\inf \left\{ \int \phi(\xi) d\mathbf{W}(\xi) : \mathbf{W} \text{ is a subsequential weak limit of } \overline{B}_n \right\} \geq \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle \tag{3.6}$$

almost surely. The claim follows from (3.4) and (3.6). □

The open-set case of Lemma 3.5 has the following immediate corollary. Recall that the *active part of the Martin boundary* $\partial\mathcal{M}_{\text{act}}$ is defined to be the support of the law of X_∞ under \mathbf{P}_o , that is, the smallest closed subset of $\partial\mathcal{M}$ for which $\mathbf{P}_o(X_\infty \in \partial\mathcal{M}_{\text{act}}) = 1$.

Corollary 3.6. *The event “every subsequential limit of \overline{B}_n is supported on $\partial\mathcal{M}_{\text{act}}$ ” has probability one.*

Proof. The complement $\partial\mathcal{M}_{\text{act}}^c$ is open and has harmonic extension $h(x) = \mathbf{P}_x(X_\infty \in \partial\mathcal{M}_{\text{act}}^c) \equiv 0$ by definition. The claim therefore follows immediately from Lemma 3.5. □

We next use the separability of $\partial\mathcal{M}$ to exchange the order of the quantifiers in the second item of Lemma 3.5.

Lemma 3.7. *The event² “for every continuous function $\phi : \partial\mathcal{M} \rightarrow [0, 1]$ with harmonic extension h to S , the sequence $\langle h, \overline{B}_n \rangle$ converges and every subsequential limit \mathbf{W} of \overline{B}_n satisfies $\int \phi(\xi) d\mathbf{W}(\xi) = \lim_{n \rightarrow \infty} \langle h, \overline{B}_n \rangle$ ” has probability one.*

Proof. The Martin compactification $S \cup \partial\mathcal{M}$ is compact and metrisable, and since S is dense in $S \cup \partial\mathcal{M}$ it is separable also. Since subsets of separable metric spaces are separable, the boundary $\partial\mathcal{M}$ is also separable. As such, the space $C(\partial\mathcal{M})$ of continuous functions on $\partial\mathcal{M}$ is separable also. (Indeed, if d is a metric compatible with the topology

²Although we call this an event, it is not clear that it is Borel measurable. Still, the claim makes sense as a statement about the set being co-null, and we avoid dwelling on the point further. In fact this set is co-analytic (i.e., the complement of a continuous image of a Borel set on a Polish space), and is therefore universally measurable (i.e., measurable with respect to the completion of any Borel measure) by Lusin’s theorem [31, Theorem 21.10]. Similar remarks apply to several other ‘events’ we consider throughout the rest of the proof.

of $\partial\mathcal{M}$ and Q is a countable dense subset of $\partial\mathcal{M}$ then the \mathbb{Q} -algebra generated by the continuous functions of the form $\{d(\cdot, q) : q \in Q\}$ is a countable dense subset of $C(\partial\mathcal{M})$.

Let A be a countable dense subset of $C(\partial\mathcal{M})$. We have by Corollary 3.3 and Lemma 3.5 that the event $\Omega = \{W = \lim_{n \rightarrow \infty} \langle 1, \bar{B}_n \rangle\}$ exists and is finite, and for each function $\phi \in A$ with harmonic extension h the sequence $(\langle h, \bar{B}_n \rangle)_{n \geq 0}$ converges and every subsequential weak limit \mathbf{W} of \bar{B}_n satisfies $\int \phi(\xi) d\mathbf{W}(\xi) = \lim_{n \rightarrow \infty} \langle h, \bar{B}_n \rangle$ has probability one.

Now, if ψ is an arbitrary continuous function on $\partial\mathcal{M}$, then for each $\varepsilon > 0$ there exists $\phi \in A$ with $|\phi - \psi| \leq \varepsilon$, so that if h and h' denote the harmonic extensions of ϕ and ψ to S then $|h - h'| \leq \varepsilon$. Thus, on the event Ω we have that

$$\begin{aligned} & \sup \left\{ \int \psi(\xi) d\mathbf{W}(\xi) : \mathbf{W} \text{ a subsequential limit of } \bar{B}_n \right\} \\ & \leq \varepsilon W + \sup \left\{ \int \phi(\xi) d\mathbf{W}(\xi) : \mathbf{W} \text{ a subsequential limit of } \bar{B}_n \right\} \\ & = \varepsilon W + \lim_{n \rightarrow \infty} \langle h', \bar{B}_n \rangle \leq 2\varepsilon W + \liminf_{n \rightarrow \infty} \langle h, \bar{B}_n \rangle, \end{aligned}$$

where the final inequality follows since $|\langle h - h', \bar{B}_n \rangle| \leq \varepsilon \langle 1, \bar{B}_n \rangle$ for every $n \geq 0$. We also have symmetrically that

$$\begin{aligned} & \inf \left\{ \int \psi(\xi) d\mathbf{W}(\xi) : \mathbf{W} \text{ a subsequential limit of } \bar{B}_n \right\} \\ & \geq -\varepsilon W + \inf \left\{ \int \phi(\xi) d\mathbf{W}(\xi) : \mathbf{W} \text{ a subsequential limit of } \bar{B}_n \right\} \\ & = -\varepsilon W + \lim_{n \rightarrow \infty} \langle h', \bar{B}_n \rangle \geq -2\varepsilon W + \limsup_{n \rightarrow \infty} \langle h, \bar{B}_n \rangle, \end{aligned}$$

yielding that the chain of inequalities

$$\begin{aligned} -2\varepsilon W + \limsup_{n \rightarrow \infty} \langle h, \bar{B}_n \rangle & \leq \inf \left\{ \int \psi(\xi) d\mathbf{W}(\xi) : \mathbf{W} \text{ a subsequential limit of } \bar{B}_n \right\} \\ & \leq \sup \left\{ \int \psi(\xi) d\mathbf{W}(\xi) : \mathbf{W} \text{ a subsequential limit of } \bar{B}_n \right\} \\ & \leq 2\varepsilon W + \liminf_{n \rightarrow \infty} \langle h, \bar{B}_n \rangle, \end{aligned}$$

holds pointwise on the event Ω . The claim follows since ψ and $\varepsilon > 0$ were arbitrary. \square

We may now conclude the proof of the main theorem. Note that we have not yet used the $L \log L$ condition. This will be needed only to verify the identity (1.1); if the $L \log L$ condition does *not* hold then \bar{B}_n converges to the zero measure!

Proof of Theorem 1.1. We have by Corollary 3.6 and Lemma 3.7 that the event “every subsequential weak limit of \bar{B}_n is supported on $\partial\mathcal{M}_{\text{act}}$, and if $\phi : \partial\mathcal{M} \rightarrow \mathbb{R}$ is continuous then $\int \phi(\xi) d\mathbf{W}_1(\xi) = \int \phi(\xi) d\mathbf{W}_2(\xi)$ for any two subsequential weak limits $\mathbf{W}_1, \mathbf{W}_2$ of \bar{B}_n ” holds almost surely. (Again, it is very important that the quantifier over all continuous functions and subsequential limits is *inside* the event being defined!) The Riesz-Markov-Kakutani representation theorem implies that measures on a compact metric space are determined by their integrals against continuous functions, so that all subsequential weak limits of \bar{B}_n are identical almost surely. Since the space of measures on \mathcal{M} is a compact metrisable space under the weak topology, it follows that \bar{B}_n converges almost surely to a (random) limit measure \mathbf{W} as claimed.

Now, if the offspring measure μ satisfies the $L \log L$ condition then we have by the Kesten-Stigum theorem that the martingale $(\langle 1, \bar{B}_n \rangle)_{n \geq 0}$ is uniformly integrable. As such,

if the $L \log L$ condition holds then $(\langle h, \bar{B}_n \rangle)_{n \geq 0}$ is a uniformly integrable martingale for each bounded harmonic function h and we have by Lemma 3.5 that if $\phi : \partial\mathcal{M} \rightarrow \mathbb{R}$ is continuous with harmonic extension h then

$$\mathbb{E} \left[\int \phi(\xi) d\mathbf{W}(\xi) \right] = \mathbb{E} \left[\lim_{n \rightarrow \infty} \langle h, \bar{B}_n \rangle \right] = \langle h, \bar{B}_0 \rangle = \mathbf{E}_o[\phi(X_\infty)].$$

This implies that the expectation of \mathbf{W} and the law of X_∞ determine the same measure on $\partial\mathcal{M}$, which is equivalent to the claimed equality (1.1). \square

Remark 3.8. As pointed out to us by an anonymous referee, the proof of Theorem 1.1 can be rephrased as follows: First, using the Riesz–Markov–Kakutani representation theorem, one can define for each $n \geq 1$ a random measure \mathbf{W}_n on $\partial\mathcal{M}$ by

$$\int \phi(\xi) d\mathbf{W}_n(\xi) = \langle h_\phi, \bar{B}_n \rangle$$

for each continuous function $\phi : \partial\mathcal{M} \rightarrow \mathbb{R}$, where h_ϕ denotes the harmonic extension of ϕ to S . Since the total measure of \mathbf{W}_n converges a.s. to W , the random family of random measures $\{\mathbf{W}_n\}$ is norm-tight almost surely, and it therefore follows from general principles that \mathbf{W}_n converges weakly to some limiting measure \mathbf{W} if and only if $\mathbf{W}_n(\phi)$ converges for every ϕ in some countable dense set of continuous functions on $\partial\mathcal{M}$, a fact that follows immediately from Lemma 3.2.

4 Open problems

As mentioned in Remark 1.2, since W generates the tail σ -algebra of the branching process $(|B_n|)_{n \geq 0}$ [37, 41] and X_∞ generates the invariant σ -algebra of the underlying random walk, it is tempting to wonder whether the random measure \mathbf{W} generates the tail σ -algebra of the branching random walk. Unfortunately this is not the case even in some very simple examples. Indeed, consider branching random walk on \mathbb{Z}^3 with deterministic branching governed by the degenerate offspring measure $\mu(16) = 1$. Since the Martin boundary of \mathbb{Z}^3 consists of a single point $\{1\}$ and $W = 16$, \mathbf{W} is deterministically equal to a point mass of mass 16 at 1 . On the other hand, if we start with a single particle at the origin, then the number of particles at $(n, 0, 0)$ at time n is itself a branching process with Binomial(16, 1/6) offspring distribution. Since this offspring distribution is supercritical but has positive probability to die after a single step, it survives forever with probability strictly between 0 and 1. As such, $\{B_n(n, 0, 0) > 0 \text{ for infinitely many } n\}$ is a non-trivial tail event that is not in the σ -algebra generated by \mathbf{W} . Similarly, $\{\max\{x : B_n(x, 0, 0) > 0\} > \max\{x : B_n(-x, 0, 0) > 0\} \text{ for all sufficiently large } n\}$ is a non-trivial *invariant event* for the branching random walk that does not belong to the σ -algebra generated by \mathbf{W} . This examples suggests that the following problem is non-trivial.

Problem 4.1. Characterise the σ -algebra generated by the limiting measure \mathbf{W} .

As discussed in the introduction, much of the existing literature on the boundary behaviour of branching random walk focuses on the set of accumulation points of the walk [35, 30, 45], and it would be interesting to reinterpret these analyses through the lens of Theorem 1.1. To do this, an important first step would be to compare the set of limit points with the support of \mathbf{W} . Again, simple examples show that these are *not* always the same: If one performs a supercritical branching random walk on the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ then every vertex is visited infinitely often, so that the set of limit points is the entire Martin boundary, which has more than one point, see [2]. Meanwhile, the measure \mathbf{W} is supported on the active part of the Martin boundary,

which in this case is the single point $\{1\}$ since $\mathbb{Z}_2 \wr \mathbb{Z}$ is Liouville. Still, we expect that in ‘sufficiently nice’ examples, such as hyperbolic groups, the two sets will coincide.

Problem 4.2. Give conditions under which the support of \mathbf{W} coincides almost surely with the set of accumulation points of B .

For hyperbolic groups there are many further questions one could ask about the limiting measure \mathbf{W} , mirroring what is known about the limit set. For example one can try to compute the Hausdorff dimension of the measure, or study the tail probability that an unusually large mass is placed on a small ball.

Problem 4.3. Study the fine properties of the limiting measure \mathbf{W} for branching random walk on a hyperbolic group.

Much of the interest in branching random walk on *nonamenable groups* stems from the fact that there are *two* phase transitions: The transition between certain death and possible survival when $\bar{\mu} = 1$ and the transition between transience and recurrence when $\bar{\mu} = 1/\rho$, where ρ is the spectral radius of the random walk: when $\bar{\mu} \leq 1/\rho$ each point is visited at most finitely often almost surely while when $\bar{\mu} > 1/\rho$ every point is visited infinitely often almost surely on the event that the walk survives forever [9, 10, 20]. It is likely that the answers to each of the questions above are different in the two cases $\bar{\mu} \leq 1/\rho$ and $\bar{\mu} > 1/\rho$, with the possibility of particularly interesting behaviour at or near the boundary case $\bar{\mu} = 1/\rho$.

It is also natural to wonder how this phase transition manifests itself in the behaviour of the limiting measure \mathbf{W} . One plausible scenario is as follows: For random walks on nonamenable groups, the random measure \mathbf{W} is always singular with respect to harmonic measure (i.e. the law of X_∞), but is, in some qualitative sense, ‘more singular’ in the transient regime $\bar{\mu} \leq 1/\rho$ than in the recurrent regime $\bar{\mu} > 1/\rho$. While we are not confident enough to make a formal conjecture, it is also plausible that \mathbf{W} is almost surely supported on a strict subset of $\partial\mathcal{M}_{\text{act}}$ whenever $\bar{\mu} \leq 1/\rho$. This is true for hyperbolic groups by the results of [45]. The results of [25] also suggest that the support of \mathbf{W} is always totally disconnected in the transient regime.

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Acknowledgments. This work was initiated while TH was a senior research associate at the University of Cambridge, during which time he was supported by ERC starting grant 804166 (SPRS). EC was supported by the project “Programma per Giovani Ricercatori Rita Levi Montalcini” awarded by the Italian Ministry of Education. EC also acknowledges partial support by “INdAM–GNAMPA Project 2019” and “INdAM–GNAMPA Project 2020”.