

Example of a Dirichlet process whose zero energy part has finite p -th variation

László Bondici* Vilmos Prokaj†

Abstract

Let B^H be a fractional Brownian motion on \mathbb{R} with Hurst parameter $H \in (0, 1)$ and let F be its pathwise antiderivative (so F is a differentiable random function such that $F'(x) = B_x^H$) with $F(0) = 0$. Let B be a standard Brownian motion, independent of B^H . We show that the zero energy part $A_t = F(B_t) - \int_0^t F'(B_s)dB_s$ of $F(B)$ has positive and finite p -th variation in a special sense for $p_0 = \frac{2}{1+H}$. We also present some simulation results about the zero energy part of a certain median process which suggest that its $4/3$ -th variation is positive and finite.

Keywords: semimartingale property; semimartingale function; Dirichlet process; p -th variation.

MSC2020 subject classifications: 60H05; 60J65; 60J55.

Submitted to ECP on May 29, 2023, final version accepted on October 1, 2023.

Supersedes arXiv:2206.11980.

1 Introduction

In a recent paper ([5]) we showed that a certain median process lacks the semimartingale property. This median process has a decomposition into a sum of a martingale and a process with zero quadratic variation. Such a process is called a Dirichlet process in [3], and a process with zero quadratic variation is said to have zero energy. Since the median process above is not a semimartingale, the zero energy part of the decomposition can not be of finite total variation. The proof provided in [5] is indirect, it does not compute the total variation of the zero energy part. This paper tries to make the first steps for this computation. We consider a simpler case where the computation can be carried out and we also present some simulation result for the above median process. These results suggest that if we compute the $4/3$ -th variation along a specially selected sequence of partitions, then it has a finite limit. The exponent $4/3$ is the same as in [9]. They consider a process X obtained from a Brownian motion B with the formula $X_t = \int_0^t \mathbb{1}_{(B_s \leq B_t)} ds$, that is X is the amount of time spent so far below the current value of B .

Without going into too much details, short term increments of these processes are obtained roughly by substituting a Brownian motion B into a continuously differentiable random function F whose first derivative f has non-zero and finite quadratic variation. The increment of the zero energy part of $F(B)$ during an upcrossing of the interval $(0, \delta)$ is $F(\delta) - F(0) - \int_0^\tau f(B_s)dB_s$ where τ is the exit time of $(-\delta, \delta)$ and B is conditioned to

*Alfréd Rényi Institute of Mathematics, Budapest, Hungary and ELTE Eötvös Loránd University, Budapest, Hungary. E-mail: bondici@renyi.hu

†ELTE Eötvös Loránd University, Budapest, Hungary. E-mail: vilmos.prokaj@ttk.elte.hu

Example of a Dirichlet process whose zero energy part has finite p -th variation

$B_\tau = \delta$. By the mean value theorem there is a $\xi \in (0, \delta)$ such that $F(\delta) - F(0) = f(\xi)\delta$ and we get that the increment is roughly $\delta(f(\xi) - f(0))$. Here the stochastic integral is simply approximated with $f(0)(B_\tau - B_0) = f(0)\delta$.

Similarly, the increment of the zero energy part during a downcrossing of $(0, \delta)$ is roughly $\delta(f(\delta) - f(\xi))$. Since the number of down and upcrossing up to time t differs only by one and is proportional to $1/\delta$, we get that the contribution of crossings of $(0, \delta)$ is roughly proportional to $\frac{1}{\delta}|\delta(f(\delta) - f(0))|^p$. Similar computation can be done for intervals of form $(r\delta, (r+1)\delta)$. The fact that the quadratic variation of f is finite roughly means that for most of the increments $f((r+1)\delta) - f(r\delta)$ is of order $\delta^{1/2}$ so finally we arrive at the conclusion that in order to have finite p -th variation it is needed that $\frac{1}{\delta}\delta^{3/2p} = \delta$, that is $p = 4/3$.

This argument is far from being rigorous. The aim of this paper to make this argument precise for the simplest case when f is a fractional Brownian motion. The increments of a fractional Brownian motion with Hurst index H over an interval of length δ is of size δ^H , so the last step of the above heuristic computation is $\frac{1}{\delta}\delta^{(1+H)p} = \delta$, that is $p = 2/(1+H)$.

Theorem 1.1. Let B^H be a fractional Brownian motion (fBM) on \mathbb{R} with Hurst parameter H and let B be a standard Brownian motion (BM), independent of B^H . Denote F the pathwise antiderivative of B^H with $F(0) = 0$ and

$$A_t = F(B_t) - \int_0^t F'(B_s)dB_s.$$

For $p_0 = \frac{2}{1+H}$ the p_0 -th variation of A on any $[0, t]$ exists and equals to ct with $c = \mathbb{E}(|A_1|^{p_0})$.

Especially, for $p < p_0$ the p -th variation is infinite almost surely on any non degenerate interval, while for $p > p_0$ it is identically zero.

In the previous claim the p -th variation $(V_t^{(p)})_{t \geq 0}$ is defined similarly to the quadratic variation, that is, for any t and any (deterministic) sequence of subdivisions $(t_i^{(n)})$ of $[0, t]$ whose mesh goes to zero we have that

$$\sum_i |A_{t_{i+1}^{(n)}} - A_{t_i^{(n)}}|^p \rightarrow V_t^{(p)} \quad \text{in probability.}$$

We start with a somewhat simpler claim.

Theorem 1.2. Using the assumptions and the notation of Theorem 1.1, for $\delta > 0$ let

$$\tau_0^\delta = 0, \quad \tau_{k+1}^\delta = \inf\{t > \tau_k^\delta : |B_t - B_{\tau_k^\delta}| \geq \delta\}.$$

Then with $c = \mathbb{E}(|A_{\tau_1^\delta}|^{p_0})$ we have

$$\sum_{k: \tau_k^\delta < t} |A_{\tau_{k+1}^\delta} - A_{\tau_k^\delta}|^{p_0} \rightarrow ct \quad \text{in probability as } \delta \rightarrow 0.$$

2 Proof of Theorem 1.2

Using the scaling property of the (fractional) Brownian motion we have that

$$\sum_{k: \tau_k^\delta < t} |A_{\tau_{k+1}^\delta} - A_{\tau_k^\delta}|^{p_0} \stackrel{d}{=} \delta^2 \sum_{k: \tau_k^1 < t/\delta^2} |A_{\tau_{k+1}^1} - A_{\tau_k^1}|^{p_0}. \quad (2.1)$$

For details see Lemma 2.4 below. Since $\mathbb{E}(\tau_1^1) = 1$, we have by the law of large numbers that $\lim_{k \rightarrow \infty} \frac{\tau_k^1}{k} = 1$ a.s. This implies that we can replace the right hand side of (2.1) with

$$\delta^2 \sum_{k < t/\delta^2} |A_{\tau_{k+1}^1} - A_{\tau_k^1}|^{p_0}$$

Example of a Dirichlet process whose zero energy part has finite p -th variation

and investigate its limiting behavior as $\delta \rightarrow 0$. The difference is that now the number of summands is deterministic, therefore the summands are identically distributed by their definition, although not independent. It is also clear that we can further simplify the expression; to prove Theorem 1.2 it is enough to show that

$$\frac{1}{n} \sum_{k < n} |A_{\tau_{k+1}} - A_{\tau_k}|^{p_0} \rightarrow \mathbb{E} \left(|A_{\tau_1}|^{p_0} \right) \quad \text{in probability.} \quad (2.2)$$

As δ is now fixed to 1, we drop it from the notation. In this form it is a weak law of large numbers and we prove it by showing that the variance of this sum is $o(n^2)$. For this we use the strong mixing property of the increments of the fractional Brownian motion which follows easily from the decay of the correlation (see [4]). We formulate it in Lemma 2.2 below.

We finish the proof of Theorem 1.2 by showing the next proposition.

Proposition 2.1. *Let B^H be a fBM on \mathbb{R} with Hurst parameter H and let B be a standard BM, independent of B^H . With $B_t^{(k)} = B_{\tau_k+t} - B_{\tau_k}$, $t \geq 0$ and $B^{(H,k)}(x) = B^H(x + B_{\tau_k}) - B^H(B_{\tau_k})$, $x \in \mathbb{R}$, let $\xi_k = (B^{(k)}, B^{(H,k)})$. Then $(\xi_k)_{k \geq 0}$ is strictly stationary and strong mixing in the sense that*

$$\text{cov}(g(\xi_0), g(\xi_k)) \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

for any measurable $g : C[0, \infty) \times C(\mathbb{R}) \rightarrow \mathbb{R}$ functional for which $g(\xi_0)$ is a square integrable random variable.

From Proposition 2.1 we have the weak law of large numbers for square integrable functionals $g(\xi_k)$. Indeed, using the stationarity we can estimate the variance

$$\text{Var} \left(\frac{1}{n} \sum_{k=0}^{n-1} g(\xi_k) \right) \leq \frac{2}{n} \sum_{k=0}^{n-1} |\text{cov}(g(\xi_0), g(\xi_k))|.$$

Here $|\text{cov}(g(\xi_0), g(\xi_k))| \rightarrow 0$, hence its arithmetic mean sequence does the same.

It is possible to show that $|A_{\tau_1}|^{p_0}$ is square integrable, but we do not need this result. Indeed, if we know the L^2 and hence the L^1 convergence of the averages $\frac{1}{n} \sum_{k=0}^{n-1} g(\xi_k)$ for bounded g , then we have the same limiting relation for integrable functionals as well. So to finish the proof it is enough to show that A_{τ_1} is square integrable,

$$A_{\tau_1} = F(B_{\tau_1}) - \int_0^{\tau_1} F'(B_s) dB_s.$$

Here $F(B_{\tau_1})$ has the same law as $\int_0^1 B_x^H dx$ which has a normal law, so this part is obviously square integrable. For the Itô integral part we can use the isometry combined with the occupation time formula ([8, Chapter VI., Corollary (1.6)]) to obtain that

$$\begin{aligned} \mathbb{E} \left(\left(\int_0^{\tau_1} F'(B_s) dB_s \right)^2 \right) &= \mathbb{E} \left(\int_0^{\infty} (F'(B_s))^2 \mathbb{1}_{(s \leq \tau_1)} ds \right) = \int_{\mathbb{R}} \mathbb{E} \left((F'(x))^2 \right) \mathbb{E}(L_{\tau_1}^x) dx \\ &\leq \sup_{|x| \leq 1} \mathbb{E} \left((B_x^H)^2 \right) \mathbb{E} \left(\int_{\mathbb{R}} L_{\tau_1}^x dx \right) = \sup_{|x| \leq 1} \mathbb{E} \left((B_x^H)^2 \right) \mathbb{E}(B_{\tau_1}^2) = 1. \end{aligned}$$

It remains to check that Proposition 2.1 holds. We do this using the next lemma whose proof involves only elementary computation, hence it is left for the reader.

Lemma 2.2. *There is a constant depending only on H , such that for $x, x', y \in \mathbb{R}$ and for a fractional Brownian motion B^H with Hurst index H*

$$\text{cov}((T_y B^H)_x, B_{x'}^H) \leq C \left(\frac{|x||x'|}{|y|^{2(1-H)}} \right), \quad \text{if } \frac{\max(|x|, |x'|)}{|y|} \leq \frac{1}{2},$$

where T_y is the translation with y , that is $(T_y B^H)_x = B_{y+x}^H - B_y^H$.

Example of a Dirichlet process whose zero energy part has finite p -th variation

This lemma extends easily with the monotone class argument to a much wider set of functionals involving scaling as well. In what follows we need it in the following form

Corollary 2.3. *Let B^H be a fractional Brownian motion with Hurst index H . For $c > 0$ let $(S_c B^H)_x = |c|^{-H} B_{cx}^H$. Then for a measurable functional $g : C(\mathbb{R}) \rightarrow \mathbb{R}$,*

$$\text{cov}(g(S_{c_n} T_{y_n} B^H), g(S_{c'_n} T_{y'_n} B^H)) \rightarrow 0,$$

provided that $g(B^H)$ is square integrable and $c_n c'_n = o((y_n - y'_n)^2)$.

Proof. Using the monotone class argument it is enough to prove for functionals of the form $g(B^H) = g(B_{x_1}^H, \dots, B_{x_k}^H) = g(B_{\underline{x}}^H)$, where g is a bounded continuous function on \mathbb{R}^k . For this case it is enough to show that the $(S_{c_n} T_{y_n} B^H)_{\underline{x}}$ and $(S_{c'_n} T_{y'_n} B^H)_{\underline{x}}$ are asymptotically independent, so eventually it is enough to check that the covariances $\text{cov}((S_{c_n} T_{y_n} B^H)_{x_i}, (S_{c'_n} T_{y'_n} B^H)_{x_j})$ $1 \leq i, j \leq k$ are vanishing in the limit.

Note that $\text{cov}((S_{c_n} T_{y_n} B^H)_{x_i}, (S_{c'_n} T_{y'_n} B^H)_{x_j}) = (c_n c'_n)^{-H} \text{cov}(T_{y_n - y'_n} B_{c_n x_i}^H, B_{c'_n x_j}^H)$ and $|c_n x_i|, |c'_n x_j| < \frac{1}{2}|y_n - y'_n|$ for n large enough. Then, from Lemma 2.2

$$\begin{aligned} \text{cov}((S_{c_n} T_{y_n} B^H)_{x_i}, (S_{c'_n} T_{y'_n} B^H)_{x_j}) &= (c_n c'_n)^{-H} \text{cov}(T_{y_n - y'_n} B_{c_n x_i}^H, B_{c'_n x_j}^H) \\ &\leq C (c_n c'_n)^{-H} \cdot \left(\frac{c_n |x_i| c'_n |x_j|}{|y_n - y'_n|^{2(1-H)}} \right) \leq C \cdot |x_i| |x_j| \cdot \left(\frac{c_n c'_n}{(y_n - y'_n)^2} \right)^{1-H}, \end{aligned}$$

which tends to zero by the assumption. The proof is complete. \square

Proof of Proposition 2.1. For the strict stationarity we need to show that $(\xi_k)_{k \geq 0}$ and $(\xi_{k+\ell})_{k \geq 0}$ have the same law for each $\ell \geq 0$. By the special structure of the sequence $(\xi_k)_{k \geq 0}$ is obtained from ξ_0 in the same way as $(\xi_{k+\ell})_{k \geq 0}$ is obtained from ξ_ℓ . So it is to show that ξ_0 and ξ_ℓ has the same law. It follows easily from the strong Markov property of the Brownian motion B that $(\tau_\ell, B_{\tau_\ell})$ is independent of $B^{(\ell)}$ and B^H . By the stationarity of the increments of fractional Brownian motion B^H , $(B_{x+y}^H - B_y^H)_{x \in \mathbb{R}}$ has the same law as B^H for any y . Now using the value of B_{τ_ℓ} as y yields that $B^{(\ell)}$ and $B^{(H, \ell)}$ are independent, and has the same joint law as B and B^H .

For the strong mixing property it is enough to consider functionals of the form $g(B, B^H) = g_1(B)g_2(B^H)$ with g_1, g_2 bounded and then use monotone class argument. For this special case it is enough to show that

$$\mathbb{E}(g_1(B)g_1(B^{(k)})) \rightarrow \mathbb{E}^2(g_1(B)) \quad \text{and} \quad \mathbb{E}(g_2(B^H)g_2(B^{(H,k)}) \mid \sigma(B)) \xrightarrow{P} \mathbb{E}^2(g_2(B^H)),$$

as k tends to infinity. Then using the boundedness of g_1, g_2 the result easily follows.

Using the translation notation from Lemma 2.2, $\mathbb{E}(g_2(B^H)g_2(B^{(H,k)}) \mid \sigma(B))$ is easy to express,

$$\mathbb{E}(g_2(B^H)g_2(B^{(H,k)}) \mid \sigma(B)) = \mathbb{E}(g_2(B^H)g_2(T_x B^H)) \Big|_{x=B_{\tau_k}}.$$

So this part follows from Corollary 2.3 and the fact that $|B_{\tau_k}| \xrightarrow{P} \infty$.

Concerning $g_1(B)$, we can consider bounded functionals of the form $g_1(B_{\underline{t}})$, where $\underline{t} \in [0, \infty)^d$ and $B_{\underline{t}}$ is the vector variable obtained from B by sampling the values at the time points $\underline{t} = (t_1, \dots, t_d)$. Let $\underline{t} \wedge \tau_k = (t_1 \wedge \tau_k, \dots, t_d \wedge \tau_k)$, then $g_1(B_{\underline{t} \wedge \tau_k}) \xrightarrow{P} g_1(B_{\underline{t}})$ so

$$\mathbb{E}(g_1(B_{\underline{t}})g_1(B_{\underline{t}}^{(k)})) - \mathbb{E}(g_1(B_{\underline{t} \wedge \tau_k})g_1(B_{\underline{t}}^{(k)})) \rightarrow 0$$

and as $g_1(B_{\underline{t} \wedge \tau_k})$ is independent of $g_1(B_{\underline{t}}^{(k)})$ we also have that

$$\mathbb{E}(g_1(B_{\underline{t} \wedge \tau_k})g_1(B_{\underline{t}}^{(k)})) = \mathbb{E}(g_1(B_{\underline{t} \wedge \tau_k}))\mathbb{E}(g_1(B_{\underline{t}}^{(k)})) \rightarrow \mathbb{E}^2(g_1(B_{\underline{t}})) \quad \square$$

Example of a Dirichlet process whose zero energy part has finite p -th variation

Lemma 2.4. Let B^H be a fBM on \mathbb{R} with Hurst parameter H and let B be a standard BM, independent of B^H . For $\delta > 0$ let $B_t^{(\delta)} = \frac{1}{\delta} B_{t\delta^2}$ and $B_x^{(H,\delta)} = \delta^{-H} B_{x\delta}^H$. Then

1. $(B, B^H) \stackrel{d}{=} (B^{(\delta)}, B^{(H,\delta)})$,
2. $\tau_k^\delta(B) = \delta^2 \tau_k^1(B^{(\delta)})$ and
3. $A_t(B, B^H) = \delta^{1+H} A_{t/\delta^2}(B^{(\delta)}, B^{(H,\delta)})$.

Proof. The first two point follows from the scaling invariance of the (fractional) Brownian motion and from the definition of the stopping time sequence.

For the last point

$$A_t = F(B_t) - \int_0^t F'(B_u) dB_u.$$

Here F is the pathwise antiderivative of B^H such that $F(0) = 0$. Then $B_t = \delta B_{t/\delta^2}^{(\delta)}$. If $F^{(\delta)}$ denotes the pathwise antiderivative of $B^{(H,\delta)}$ with $F^{(\delta)}(0) = 0$, then for positive x

$$F(x) = \int_0^x B_y^H dy = \int_0^x \delta^H B_{y/\delta}^{(H,\delta)} dy = \delta^{H+1} F^{(\delta)}(x/\delta),$$

and similarly for negative x . From these computations $F(B_t) = \delta^{1+H} F^{(\delta)}(B_t/\delta^2)$

For the stochastic integral note that $F'(x) = B_x^H = \delta^H B_{x/\delta}^{(H,\delta)} = \delta^H (F^{(\delta)})'(x/\delta)$, so

$$\int_0^t F'(B_u) dB_u = \delta^H \int_0^t (F^{(\delta)})'(B_u/\delta^2) \delta dB_u^{(\delta)} = \delta^{1+H} \int_0^{t/\delta^2} (F^{(\delta)})'(B_u^{(\delta)}) dB_u^{(\delta)}. \quad \square$$

3 Proof of Theorem 1.1

The proof of Theorem 1.1 goes along similar lines as that of Theorem 1.2. For a given interval $I = [a, b] \subset [0, \infty)$ let's define

$$B_t^{(I)} = \frac{B_{(b-a)t+a} - B_a}{(b-a)^{1/2}}, \quad B_x^{(H,I)} = \frac{B_{(b-a)^{1/2}x+B_a} - B_{B_a}^H}{(b-a)^{H/2}} \quad (3.1)$$

The key point here is again the scaling property of the (fractional) Brownian motion: for $0 \leq s < t$, $I = [s, t]$

$$A_t - A_s = F(B_t) - F(B_s) - \int_s^t F'(B_u) dB_u.$$

If $F^{(I)}$ is the random function with $B^{(H,I)}$ as its derivative and $F^{(I)}(0) = 0$, then for $x > 0$

$$\begin{aligned} F^{(I)}(x) &= \int_0^x B_y^{(H,I)} dy = \int_0^x \frac{B_{(t-s)^{1/2}y+B_s} - B_{B_s}^H}{(t-s)^{H/2}} dy \\ &= (t-s)^{-(H+1)/2} (F(B_s + x\sqrt{t-s}) - F(B_s)) - x(t-s)^{-H/2} F'(B_s) \end{aligned}$$

and similarly for $x < 0$. So if we write $B_1^{(I)}$ in place of x we get

$$F^{(I)}(B_1^{(I)}) = (t-s)^{-(H+1)/2} (F(B_t) - F(B_s) - (B_t - B_s)F'(B_s)).$$

Similar for the stochastic integral

$$\begin{aligned} \int_0^1 (F^{(I)})'(B_u^{(I)}) dB_u^{(I)} &= \int_s^t \frac{F'(B_u) - F'(B_s)}{(t-s)^{H/2}} \frac{dB_u}{(t-s)^{1/2}} \\ &= (t-s)^{-(H+1)/2} \left(\int_s^t F'(B_u) dB_u - F'(B_s)(B_t - B_s) \right), \end{aligned}$$

so $|A_t - A_s|^{2/(H+1)} = (t-s)g(B^{([s,t])}, B^{(H,[s,t])})$ with a suitable functional g .

The proof of Theorem 1.2 is based on the next two claims which we prove below.

Example of a Dirichlet process whose zero energy part has finite p -th variation

Proposition 3.1. For a non-degenerated interval $I \subset [0, \infty)$ denote $\xi_I = (B^{(I)}, B^{(H,I)})$. Then the law of ξ_I does not depend on I and

$$\text{cov}(g(\xi_{I_n}), g(\xi_{J_n})) \rightarrow 0 \quad \text{if} \quad \max(|I_n|, |J_n|) \rightarrow 0 \quad \text{and} \quad \liminf_n \text{dist}(I_n, J_n) > 0,$$

provided that $g(\xi_{[0,1]})$ is square integrable.

Corollary 3.2. Suppose that (π_n) is a sequence of subdivisions of $[0, t]$, such that the mesh $\max_{I \in \pi_n} |I| \rightarrow 0$ and g is a functional such that $g(\xi_{[0,1]})$ is integrable. Then

$$\sum_{I \in \pi_n} |I| g(\xi_I) \rightarrow t \mathbb{E}(g(\xi_{[0,1]})) \quad \text{in } L^1.$$

Theorem 1.2 follows from Corollary 3.2 if we apply it to

$$g(\xi_{[0,1]}) = \left| F(B_1) - \int_0^1 F'(B_s) dB_s \right|^{p_0}, \quad F'(x) = B_x^H, \quad F(0) = 0.$$

We check that $g(\xi)$ is integrable by showing that $g(\xi_{[0,1]}) \in L^{2/p_0}$. Note that $2/p_0 < 2$ so it is enough to show that the next random variable is square integrable

$$F(B_1) - \int_0^1 F'(B_s) dB_s, \quad F'(x) = B_x^H, \quad F(0) = 0.$$

Since by trivial estimations $\mathbb{E}(F(x)^2) < x^4$ we obviously have that $F(B_1) \in L^2$. For the second term we apply again Itô isometry followed by the occupation time formula, and

$$\begin{aligned} \mathbb{E} \left(\left(\int_0^1 F'(B_s) dB_s \right)^2 \right) &= \int_{\mathbb{R}} \mathbb{E}((B_x^H)^2) \mathbb{E}(L_1^x) dx = \int_{\mathbb{R}} |x|^{2H} \mathbb{E}(L_1^x) dx \\ &\leq \int_{\mathbb{R}} (x^2 + 1) \mathbb{E}(L_1^x) dx = \mathbb{E} \left(\int_0^1 (B_s^2 + 1) ds \right) = 3/2. \end{aligned}$$

Proof of Proposition 3.1. We start with the law of ξ_I . Since the increments of the fractional Brownian motion B^H are stationary, $(B_{x+y}^H - B_y^H)_{x \in \mathbb{R}}$ has the same law as B^H for any deterministic y . Then by the independence of B and B^H the conditional law of $(B_{x+B_a}^H - B_{B_a}^H)_{x \in \mathbb{R}}$ given B does not depend on B , that is $(B_{x+B_a}^H - B_{B_a}^H)_{x \in \mathbb{R}}$ is independent of B with the same law as B^H . But then by the scaling invariance of B^H the same is true for $B^{(H,I)}$. Finally, by the Markov property and scaling invariance of B we get that $B^{(I)}$ is also a Brownian motion which is obviously independent of $B^{(H,I)}$. So ξ_I has the same law as (B, B^H) which is $\xi_{[0,1]}$ by definition.

To show the asymptotics of the covariance it is enough to consider again functionals of the form $g(B, B^H) = g_1(B)g_2(B^H)$ where g_1, g_2 are bounded. As in the proof of Proposition 2.1 it is enough to show that $\mathbb{E}(g_2(B^{(H,I_n)})g_2(B^{(H,J_n)}) \mid \sigma(B)) \xrightarrow{p} \mathbb{E}^2(g_2(B^H))$ and that $\mathbb{E}(g_1(B^{(I_n)})g_1(B^{(J_n)})) \rightarrow \mathbb{E}^2(g_1(B))$ whenever $(I_n = [a_n, b_n], J_n = [c_n, d_n])_{n \geq 1}$ is a sequence such that $\max(|I_n|, |J_n|) \rightarrow 0$ and $\inf_n \text{dist}(I_n, J_n) > 0$.

Using the independence of B and B^H we get that

$$\mathbb{E} \left(g_2(B^{(H,I_n)})g_2(B^{(H,J_n)}) \mid \sigma(B) \right) = \mathbb{E} \left(g_2(S_{|I_n|^{1/2}} T_x B^H) g_2(S_{|J_n|^{1/2}} T_y B^H) \right) \Big|_{x=B_{a_n}, y=B_{c_n}}$$

where we used the notation of Corollary 2.3. By assumption $\frac{|I_n|^{1/2}|J_n|^{1/2}}{(B_{c_n} - B_{a_n})^2} \xrightarrow{p} 0$ as $n \rightarrow \infty$. Then Corollary 2.3 shows that $\mathbb{E}(g_2(B^{(H,I_n)})g_2(B^{(H,J_n)}) \mid \sigma(B)) \xrightarrow{p} \mathbb{E}^2(g_2(B^H))$.

For $\mathbb{E}(g_1(B^{(I_n)})g_1(B^{(J_n)})) = \mathbb{E}(g_1(S_{|I_n|} T_{a_n} B)g_1(S_{|J_n|} T_{c_n} B))$ we can also use Corollary 2.3 with $H = 1/2$. \square

Example of a Dirichlet process whose zero energy part has finite p -th variation

Proof of Corollary 3.2. First assume that $g(\xi_{[0,1]})$ is square integrable, and for a sequence of partitions π_n of $[0, t]$ let $f_n = \sum_{I, J \in \pi_n} \mathbf{1}_{I \times J} \text{cov}(g(\xi_I), g(\xi_J))$. Then

$$\text{Var}\left(\sum_{I \in \pi_n} |I|g(\xi_I)\right) = \int_{[0, t]^2} f_n \rightarrow 0, \quad \text{if } \max_{I \in \pi_n} |I_n| \rightarrow 0,$$

since the sequence of function (f_n) is dominated by $\text{Var } g(\xi_{[0,1]})$ and tends to zero everywhere but the diagonal of $[0, t]^2$ by Proposition 3.1. Also by Proposition 3.1 the expectation $\mathbb{E}(\sum_{I \in \pi_n} |I|g(\xi_I)) = t\mathbb{E}(g(\xi_{[0,1]}))$ does not depend on n , hence we have the claim for g provided that $g(\xi_{[0,1]}) \in L^2$.

For general g , when $g(\xi_{[0,1]})$ is integrable, we can use truncation. □

4 The zero energy part of the median process

Let B be a Brownian motion and suppose that $(D_t(x))_{t \geq 0, x \in [0, 1]}$ satisfies

$$dD_t(x) = \sigma(D_t(x))dB_t, \quad \sigma(x) = x \wedge (1 - x), \quad D_0(x) = x, \quad x \in [0, 1]. \quad (4.1)$$

This two parameter process was analyzed in [6] in detail. It was shown that the solution $D_t(x)$ can be viewed as a conditional distribution function, and this justifies the (conditional) quantile name for the process $q_t = D_t^{-1}(\alpha)$, $\alpha \in (0, 1)$, and particularly the (conditional) median name for the process $m_t = D_t^{-1}(1/2)$. In [5] it was proved that $q_{s+t} = D_s^{-1}(q_{s,t})$ with $q_{s,t} = D_{s,t}^{-1}(\alpha)$, where $D_{s,t}(x)$ is the solution of (4.1) with B replaced by $B_{s,t} = B_{s+t} - B_s$ “a Brownian motion that starts evolving at time s ”, and $(D_s^{-1})'(x)$ is a random function with infinite total variation in its space variable on $[0, 1]$.

In [5] it was also proved that the quantile process q_t is not a semimartingale, so neither the median process m_t is it. The following decomposition formula holds for q_t (cf. Subsection 5.2 in [5])

$$\bar{A}_t = q_t + \int_0^t (D_s^{-1})'(\alpha)\sigma(\alpha)dB_s.$$

This \bar{A} is a process of zero energy, that is, the quadratic variation of \bar{A} exists and $[\bar{A}] \equiv 0$. If \bar{A} would have finite total variation, then q_t would be a semimartingale, so \bar{A} should have infinite total variation.

We will refer to the process \bar{A} as the zero energy part of the quantile process. In the following we prove that the local martingale part in the previous decomposition of q_t is a true martingale and we will investigate the following main question: whenever the total variation of \bar{A} is infinite, and the quadratic variation is identically 0, is there any $p \in (1, 2)$ for which the p -th variation of \bar{A} is positive and finite? We are not able to give a mathematically rigorous answer to this latter question, but we formulate some heuristic arguments which are supported by some simulation results.

4.1 Space inverse of a stochastic flow

In this subsection we revise a method for obtaining the space inverse of a stochastic flow (which is given by an Itô diffusion) at a given time point. We will use this in the next subsection (in the proof of the martingale property of the local martingale part of q_t). We prove only for the case of the unit diffusion coefficient but with suitable transformations this result can be extended.

Let B be a Brownian motion and let $(G_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a stochastic flow which satisfies the following equation

$$dG_t(x) = \mu(G_t(x))dt + dB_t, \quad G_0(x) = x, \quad x \in \mathbb{R}, \quad (4.2)$$

where μ is a bounded measurable function. Suppose that on an almost sure event the mapping $(t, x) \mapsto G_t(x)$ is continuous and the mappings $x \mapsto G_t(x)$ are homeomorphisms

Example of a Dirichlet process whose zero energy part has finite p -th variation

of \mathbb{R} for all t . We now define a process which produces as its terminal value on an almost sure event the space inverse of $x \mapsto G_T(x)$ for a certain $T > 0$. For this we will use the time reversion of the Brownian motion.

For a fixed time horizon $T > 0$ let $\tilde{B}_u^T = B_{T-u} - B_T$ be the time reversed Brownian motion on $[0, T]$. Let $(H_t^T(x))_{t \in [0, T], x \in \mathbb{R}}$ be the solution of the following equation

$$dH_t^T(x) = -\mu(H_t^T(x))dt + d\tilde{B}_t^T, \quad H_0^T(x) = x, \quad x \in \mathbb{R}. \quad (4.3)$$

Suppose again that on an almost sure event the mapping $(t, x) \mapsto H_t^T(x)$ is continuous and the mappings $x \mapsto H_t^T(x)$ are homeomorphisms of \mathbb{R} for all t .

Recall the following result from [1] (see also in [2]; a different approach and a generalization of this result can be found in [10] and [11])

Theorem 4.1 (Davie (2007), Theorem 1.1). *Consider the following equation*

$$X_t = X_0 + B_t + \int_0^t f(s, X_s)ds, \quad t \geq 0, \quad (4.4)$$

where $X_t \in \mathbb{R}^d$, B is a standard d -dimensional Brownian motion and f is a bounded Borel function from $[0, \infty) \times \mathbb{R}^d$ to \mathbb{R}^d .

For a.e. Brownian path B , there is a unique continuous $X : [0, \infty) \rightarrow \mathbb{R}^d$ satisfying (4.4).

It will be pointed out in the next proof the role of this theorem. From this we know that on an almost sure event there are unique continuous functions $\hat{G}(x_0) : [0, \infty) \rightarrow \mathbb{R}$ and $\hat{H}(x_0) : [0, \infty) \rightarrow \mathbb{R}$ satisfying $d\hat{G}_t(x_0) = \mu(\hat{G}_t(x_0))dt + dB_t$, $\hat{G}_0(x_0) = x_0$ and $d\hat{H}_t(x_0) = -\mu(\hat{H}_t(x_0))dt + dB_t$, $\hat{H}_0(x_0) = x_0$. Taking into account all rational x_0 and then using the continuity of $G_t(x)$ and $H_t^T(x)$, we obtain that on an almost sure event $\Omega' \subset \Omega$ the mappings $(t, x) \mapsto G_t(x)$ and $(t, x) \mapsto H_t^T(x)$ are unique. Then we have the following

Proposition 4.2. *On an almost sure event, for all $u \in [0, T]$ and for all $x \in \mathbb{R}$ we have the following: $G_{T-u}(x) = H_u^T(G_T(x))$ and $H_{T-u}^T(x) = G_u(H_T^T(x))$.*

Epecially, we have $H_T^T(G_T(x)) = x$ and $G_T(H_T^T(x)) = x$, so $G_T^{-1} = H_T^T$.

Proof. We restrict ourselves to Ω' . Let $0 \leq s < t \leq T$. For $u \in [0, T]$ let $\tilde{G}_u^T(x) = G_{T-u}(x)$ be the time reversed process of $(G(x))_{t \in [0, T]}$. Using (4.2) we obtain

$$\begin{aligned} \tilde{G}_t^T(x) - \tilde{G}_s^T(x) &= -(G_{T-s}(x) - G_{T-t}(x)) = -\int_{T-t}^{T-s} \mu(G_u(x))du - (B_{T-s} - B_{T-t}) = \\ &= -\int_s^t \mu(G_{T-u}(x))du + (B_{T-t} - B_T - (B_{T-s} - B_T)) = -\int_s^t \mu(\tilde{G}_u^T(x))du + (\tilde{B}_t^T - \tilde{B}_s^T). \end{aligned}$$

Substituting $G_T^{-1}(x)$ in place of x yields

$$\tilde{G}_t^T(G_T^{-1}(x)) - \tilde{G}_s^T(G_T^{-1}(x)) = -\int_s^t \mu(\tilde{G}_u^T(G_T^{-1}(x)))du + (\tilde{B}_t^T - \tilde{B}_s^T).$$

As $\tilde{G}_0^T(G_T^{-1}(x)) = G_T(G_T^{-1}(x)) = x$, we have obtained that $\tilde{G}_u^T(G_T^{-1}(x))$ solves the equation (4.3) of $H^T(x)$. However, from [12] we know that (4.3) has a unique strong solution, but it is not obvious that $(\tilde{G}_u^T(G_T^{-1}(x)))_{u \in [0, T]}$ is adapted to the filtration

generated by \tilde{B}^T , hence we need the previously cited uniqueness result, Theorem 4.1. It follows that on Ω' we have $H_u^T(x) = \tilde{G}_u^T(G_T^{-1}(x))$, so $H_u^T(G_T(x)) = G_{T-u}(x)$.

A similar argument for H^T yields $G_u(H_T^T(x)) = H_{T-u}^T(x)$. \square

4.2 The local martingale part of the quantile process is a true martingale

Now we turn to the case of the quantile process. Using the previous results we derive a process which produces a transformed version of D_T^{-1} as its terminal value and we prove that the local martingale part of the quantile process is a true martingale.

Proposition 4.3. *Let B be a Brownian motion and suppose that $(D_t(x))_{t \geq 0, x \in [0,1]}$ satisfies (4.1). Then $\int_0^\cdot (D_s^{-1})'(\alpha)\sigma(\alpha)dB_s$ is a true martingale.*

Proof. In the proof we do not use the exact form of σ from the equation (4.1) of $D(x)$, but we only use that it is a Lipschitz continuous function with Lipschitz constant L and it is bounded on $C = (0, 1)$. We also provide the particular results for that special case.

As the first step, consider the Lamperti transform of $D(x)$, so let $\psi : C \rightarrow \mathbb{R}$ be such that $\psi' = 1/\sigma$, where C is a connected component of $\mathbb{R} \setminus \{\sigma = 0\}$. In the special case of $\sigma(x) = x \wedge (1 - x)$ and $C = (0, 1)$ we can use $\psi(x) = \text{sign}(1 - 2x) \ln(1 - |1 - 2x|)$.

Let $G_t(x) = \psi \circ D_t \circ \psi^{-1}(x)$, so $dG_t(x) = -\frac{1}{2}(\sigma' \circ \psi^{-1})(G_t(x))dt + dB_t$, $G_0(x) = x$, $x \in \mathbb{R}$, and let $H_t^T(x)$ be the solution of $dH_t^T(x) = \frac{1}{2}(\sigma' \circ \psi^{-1})(H_t^T(x))dt + d\tilde{B}_t^T$, $H_0^T(x) = x$, $x \in \mathbb{R}$. In the special case we have $dG_t(x) = \frac{1}{2} \text{sign}(G_t(x))dt + dB_t$, $G_0(x) = x$, $x \in \mathbb{R}$ and $dH_t^T(x) = -\frac{1}{2} \text{sign}(H_t^T(x))dt + d\tilde{B}_t^T$, $H_0^T(x) = x$, $x \in \mathbb{R}$.

In order to be able to calculate $(D_T^{-1})'$, let s be a scale function (or Zvonkin transform, [13]) for $H^T(x)$. This removes the drift, and s satisfies $s'(x)(\sigma' \circ \psi^{-1})(x) + s''(x) = 0$. For such a function we have $s' = \frac{c}{\sigma \circ \psi^{-1}}$ with some $c \in \mathbb{R}$, and the transformed process $(F_t^T(x) = s \circ H_t^T(x))_{t \geq 0, x \in \mathbb{R}}$ satisfies

$$dF_t^T(x) = \frac{c}{\sigma \circ (s \circ \psi)^{-1}}(F_t^T(x))d\tilde{B}_t^T, \quad F_0^T(x) = x, \quad x \in \mathbb{R}. \tag{4.5}$$

In the special case a possible s can be $s(x) = \text{sign}(x)(\exp\{|x|\} - 1)$, and with this choice

$$dF_t^T(x) = (1 + |F_t^T(x)|)d\tilde{B}_t^T, \quad F_0^T(x) = x, \quad x \in \mathbb{R}. \tag{4.5'}$$

For $(F_T^T)' = f_T^T$ we use Lemma 16 from [6]. Since $(\frac{c}{\sigma \circ (s \circ \psi)^{-1}})' = -\sigma' \circ (s \circ \psi)^{-1}$, we know that $\frac{c}{\sigma \circ (s \circ \psi)^{-1}}$ is a Lipschitz continuous function (with the same Lipschitz constant L as σ), so F_t^T is differentiable in its space variable, and the space derivative f_t^T satisfies $df_t^T(x) = -\sigma' \circ (s \circ \psi)^{-1}(F_t^T(x))f_t^T(x)d\tilde{B}_t^T$, $f_0^T(x) = 1$, $x \in \mathbb{R}$. From this we obtain that $f_t^T(x) = \exp\{N_t^T(x) - \frac{1}{2}[N^T(x)]_t\}$ with $N_t^T(x) = -\int_0^t \sigma' \circ (s \circ \psi)^{-1}(F_u^T(x))d\tilde{B}_u^T$, and this yields $\mathbb{E}(|f_t^T|^p) \leq \exp\{L^2 p(p - \frac{1}{2})t\}$, where L is the Lipschitz constant for σ .

To be able to apply Proposition 4.2, we need to guarantee that on an almost sure event $(t, x) \mapsto G_t(x)$ and $(t, x) \mapsto H_t^T(x)$ are continuous, $G_t(x)$ and $H_t^T(x)$ are homeomorphisms of \mathbb{R} for all t and their drifts are bounded. The latter property follows from the Lipschitz continuity of σ . Moreover, since also $\frac{c}{\sigma \circ (s \circ \psi)^{-1}}$ is a Lipschitz continuous function, we know that (Theorems 37 and 46 from Chapter V in [7]) the above properties hold for $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$ and $(F_t^T(x))_{t \geq 0, x \in \mathbb{R}}$, and hence hold for $(G_t(x))_{t \geq 0, x \in \mathbb{R}}$ and $(H_t^T(x))_{t \geq 0, x \in \mathbb{R}}$ (as they are transformations). So by Proposition 4.2 we know that almost surely $H_T^T = G_T^{-1} = \psi \circ D_T^{-1} \circ \psi^{-1}$, so $D_T^{-1} = (s \circ \psi)^{-1} \circ F_T^T \circ \psi$, and hence

$$(D_T^{-1})' = \frac{c \cdot \sigma^2 \circ (s \circ \psi)^{-1} \circ F_T^T \circ \psi \cdot f_T^T \circ \psi}{\sigma}.$$

Now suppose that $|\sigma| \leq K$ on C . Then

$$\begin{aligned} \mathbb{E}\left(\left((D_u^{-1})'(\alpha)\sigma(\alpha)\right)^2\right) &= c^2 \mathbb{E}(\sigma^4 \circ (s \circ \psi)^{-1} \circ F_u^T \circ \psi(\alpha) \cdot f_u^T \circ \psi(\alpha)) \\ &\leq c^2 (\mathbb{E}(\sigma^8 \circ (s \circ \psi)^{-1} \circ F_u^T \circ \psi(\alpha)))^{1/2} \left(\mathbb{E}\left((f_u^T)^2 \circ \psi(\alpha)\right)\right)^{1/2} \leq c^2 K^4 \exp\left\{\frac{3}{2}L^2 u\right\}, \end{aligned}$$

Example of a Dirichlet process whose zero energy part has finite p -th variation

and hence $\sup_{0 \leq u \leq t} \mathbb{E} \left((D_u^{-1})'(\alpha) \right)^2 \leq \frac{c^2 K^4}{\sigma^2(\alpha)} \exp\left\{ \frac{3}{2} L^2 t \right\} < \infty$, so in the case of $\sigma(x) = x \wedge (1-x)$ and $C = (0, 1)$ it follows that the local martingale part of the quantile process, $\int_0^\cdot (D_s^{-1})'(\alpha) \sigma(\alpha) dB_s$, is a true martingale. \square

4.3 Simulation framework and results

In this subsection we restrict ourselves to the case $\alpha = 1/2$, so $q_t = m_t$ is the median process, and with $\psi(x) = \text{sign}(1-2x) \ln(1-|1-2x|)$ and $F^T(x)$ satisfying (4.5') we have

$$m_T = (s \circ \psi)^{-1}(F_T^T(0)). \quad (4.6)$$

Now we present the simulation framework and the results which supports the following

Conjecture 4.4. *With the above notations, for $p = 4/3$, the stochastic limit of*

$$\sum_{k=0}^{n-1} |\bar{A}_{t_{k+1}} - \bar{A}_{t_k}|^p, \quad n \rightarrow \infty \quad (4.7)$$

is positive and finite (where $0 = t_0 < t_1 < \dots < t_n = T$ denotes, with simplified indices, a sequence of partitions of $[0, T]$ which have grid size tending to 0).

Instead of the sum in the above conjecture, we investigate the following sum

$$\sum_{k=0}^{n-1} |\mathbb{E}(\bar{A}_{t_{k+1}} - \bar{A}_{t_k} \mid \mathcal{F}_{t_k})|^p, \quad (4.8)$$

where \mathcal{F} is the natural filtration of the Brownian motion B . If for $p > 1$ the p -th variation (4.7) tends to 0 in expectation, then so does the conditional version (4.8), so if the conditional version has a positive and finite limit, then (4.7) can not tend to 0. Moreover, using the Proposition 4.3, the martingale term in the definition of \bar{A} can be eliminated

$$\mathbb{E}(\bar{A}_{t_{k+1}} - \bar{A}_{t_k} \mid \mathcal{F}_{t_k}) = \mathbb{E}(m_{t_{k+1}} - m_{t_k} \mid \mathcal{F}_{t_k}) = \mathbb{E}(m_{t_{k+1}} \mid \mathcal{F}_{t_k}) - m_{t_k}.$$

4.3.1 Calculating the median and calculating the conditional expectation

For calculating the value of the median we can use (4.6), since for m_T it is enough to calculate $F_T^T(0)$ (and transform its value). In order to do this, we want to use the discretised version of the equation (4.5')

$$F_{t_{i+1}}^T = F_{t_i}^T + (1 + |F_{t_i}^T|) \Delta \tilde{B}_{t_i}^T, \quad F_{t_0} = 0, \quad (4.9)$$

where $0 = t_0 < t_1 < \dots < t_k = T$ is an equidistant grid of $[0, T]$ with mesh size $\Delta t = \frac{T}{k}$, and $\Delta \tilde{B}_{t_i}^T$ are correspondingly rescaled independent Rademacher variables (with expectation 0 and variance Δt , so $\mathbb{P}(\Delta \tilde{B}_{t_i}^T = \sqrt{\Delta t}) = \mathbb{P}(\Delta \tilde{B}_{t_i}^T = -\sqrt{\Delta t}) = \frac{1}{2}$). We restrict ourselves to the interval $[0, 1]$, so $T \in [0, 1]$. For the calculation of the median we use the sequence $(t_i = \frac{i}{n})_{i=0, \dots, k}$ for some suitable values of n (typically powers of 10).

Between the time reversed Brownian motions we have the following relationship: for $T_1 < T_2$ and $u \in [0, T_2]$, we have

$$\tilde{B}_u^{T_2} = \tilde{B}_{(u-(T_2-T_1)) \vee 0}^{T_1} + \tilde{B}_{u \wedge (T_2-T_1)}^{T_2}, \quad (4.10)$$

which easily follows from the definition of \tilde{B}^T . Since $\tilde{B}_{(u-(T_2-T_1)) \vee 0}^{T_1}$ can be calculated from the increments of B in the interval $[0, T_1]$, and $\tilde{B}_{u \wedge (T_2-T_1)}^{T_2}$ is an increment of B in $[T_1, T_2]$, they are independent.

Example of a Dirichlet process whose zero energy part has finite p -th variation

Suppose that we want to calculate the values of the median in two consecutive time points, in T_1 and $T_2 = T_1 + \Delta t$, on the same trajectory (until T_1). Suppose that we use a sequence $(\Delta \tilde{B}^{T_1})$ for m_{T_1} ; in order to calculate m_{T_2} , we append one independent rescaled Rademacher variable to the beginning of the sequence $(\Delta \tilde{B}^{T_1})$, and we calculate m_{T_2} along this extended sequence. In this way we can guarantee that we remain on the same trajectory.

Now we describe how to estimate the conditional expectation of m . Let T_1 and $T_2 = T_1 + \Delta t$ be again two consecutive time points. We want to calculate one realization of $\mathbb{E}(m_{T_1+\Delta t} | \mathcal{F}_{T_1})$ on the same trajectory as we did in the case of m_{T_1} . Since for $T_1 < T_2$ and $t \in [T_2 - T_1, T_2]$ we have $F_t^{T_2}(x) = F_{t-(T_2-T_1)}^{T_1}(F_{T_2-T_1}^{T_2}(x))$, we obtain that $F_{T_2}^{T_2}(0) = F_{T_1}^{T_1}(F_{\Delta t}^{T_2}(0))$, and from (4.10) we know that $F_{\Delta t}^{T_2}(0)$ is independent of $F_{T_1}^{T_1}(x)$.

Suppose that m_{T_1} is calculated using the sequence $(\Delta \tilde{B}^{T_1})$. To calculate the conditional expectation $\mathbb{E}(m_{T_1+\Delta t} | \mathcal{F}_{T_1})$ along this trajectory, we can do the following. We use the discretised equation (4.9) of F with the same driving sequence $\Delta \tilde{B}^{T_1}$, but with different initial values: we approximate the distribution of $F_{\Delta t}^{T_2}(0)$ using a finer grid of $[0, \Delta t]$ (using (4.9) with independent random increments), and we calculate the values of $F_{T_1}^{T_1}$ from these points as initial values. Finally, we take the average of $F_{T_1}^{T_1}$'s.

4.3.2 One single increment and p -th variation for some values of p

Next we summarize the simulation results. We present the results regarding one single increment on the time interval $[1, 1 + \Delta\tau]$ for different values of p , so we investigate

$$|\mathbb{E}(\bar{A}_{1+\Delta\tau} - \bar{A}_1 | \mathcal{F}_1)|^p = |\mathbb{E}(m_{1+\Delta\tau} | \mathcal{F}_1) - m_1|^p.$$

The values of $\Delta\tau$ range from 10^{-2} to 10^{-7} . To approximate the values of m_1 and $\mathbb{E}(m_{1+\Delta\tau} | \mathcal{F}_1)$ we set the grid mesh to 10^{-6} ($n = 10^6$ gridpoints).

We have simulated 10^3 realizations, and the results can be seen on the figure below. This is a log-log (base 10) plot. On the x -axis we indicated $\Delta\tau$, while on the y -axis we marked the average of the above mentioned increments. Around the mean we can also see the 95% confidence interval of the expected value. We also give the slope and intercept values for the fitted lines, and the sum of squared residuals (SSR).

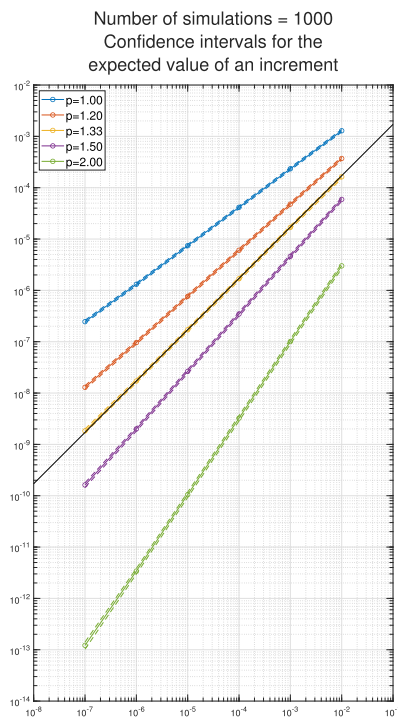
This figure (Figure 1) and the fitted lines (Table 1) suggest the following: there is a linear relationship between the logarithm of $\Delta\tau$ and the logarithm of the expected value of one single increment:

$$\log(\mathbb{E}(|\mathbb{E}(\bar{A}_{1+\Delta\tau} - \bar{A}_1 | \mathcal{F}_1)|^p)) = c_0 + c_1 \log(\Delta\tau), \text{ so } \mathbb{E}(|\mathbb{E}(\bar{A}_{1+\Delta\tau} - \bar{A}_1 | \mathcal{F}_1)|^p) = \tilde{c}_0(\Delta\tau)^{c_1}$$

By a scaling argument we can suppose that this relationship is valid not only on the interval $[1, 1 + \Delta\tau]$ but also on other intervals. The slope of the thick line is $c_1 = 1$, and is very close to the points which belong to the case $p = 4/3$. As the number of intervals in $[0, 1]$ with length $\Delta\tau$ is roughly $\frac{1}{\Delta\tau}$, this has the following consequences:

- for $p = 4/3$, the sum $\sum_{k=0}^{n-1} \mathbb{E}(|\mathbb{E}(\bar{A}_{t_{k+1}} - \bar{A}_{t_k} | \mathcal{F}_{t_k})|^p)$ has a positive and finite limit;
- for $p < 4/3$, the above sum is unbounded from above;
- for $p > 4/3$, the above sum tends to 0.

Example of a Dirichlet process whose zero energy part has finite p -th variation



p	slope	intercept	SSR
1.00	0.7455	-1.3994	0.0002
1.20	0.8935	-1.6443	0.0002
1.33	0.9920	-1.8034	0.0003
1.50	1.1149	-1.9980	0.0004
2.00	1.4820	-2.5593	0.0008

Table 1: Fitted lines

Figure 1: One single increment

References

- [1] A. M. Davie, *Uniqueness of solutions of stochastic differential equations*, Int. Math. Res. Not. IMRN (2007), no. 24, 26. MRMR2377011
- [2] F. Flandoli, *Regularizing properties of Brownian paths and a result of Davie*, Stoch. Dyn. **11** (2011), no. 2–3, 323–331. MR2836529
- [3] H. Föllmer, *Dirichlet processes*, in: Stochastic Integrals (Proc. Sympos., Univ. Durham, Durham, 1980), Lecture Notes in Math., vol. 851, Springer, Berlin, 1981, pp. 476–478. MR621001
- [4] G. Maruyama, *Infinitely divisible processes*, Teor. Veroyatnost. i Primenen. **15** (1970), 3–23. MR0285046
- [5] V. Prokaj and L. Bondici, *On the lack of semimartingale property*, Stochastic Process. Appl. **146** (2022), 335–359. MR4374939
- [6] V. Prokaj, M. Rásonyi, and W. Schachermayer, *Hiding a constant drift*, Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), no. 2, 498–514. MR2814420
- [7] P. E. Protter, *Stochastic integration and differential equations*, Stochastic Modelling and Applied Probability, vol. 21, Springer-Verlag, Berlin, 2005, Second edition. Version 2.1, Corrected third printing. MR2273672
- [8] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, third ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999. MR1725357
- [9] L. C. G. Rogers and J. B. Walsh, *$A(t, B_t)$ is not a semimartingale*, in: Seminar on Stochastic Processes, 1990 (Vancouver, BC, 1990) (E. Çinlar, P.J. Fitzsimmons, and R.J. Williams, eds.), Progr. Probab., vol. 24, Birkhäuser Boston, Boston, MA, 1991, pp. 275–283. MR1118450
- [10] A. V. Shaposhnikov, *Some remarks on Davie’s uniqueness theorem*, Proc. Edinb. Math. Soc. (2) **59** (2016), no. 4, 1019–1035. MR3570126

Example of a Dirichlet process whose zero energy part has finite p -th variation

- [11] A. V. Shaposhnikov, *Correction to the paper "some remarks on Davie's uniqueness theorem"*, ArXiv e-prints (2017), 1–9. MR0817985
- [12] A. Ju. Veretennikov, *Strong solutions and explicit formulas for solutions of stochastic integral equations*, Mat. Sb. (N.S.) **111(153)** (1980), no. 3, 434–452, 480. MR568986
- [13] A. K. Zvonkin, *A transformation of the phase space of a diffusion process that will remove the drift*, Mat. Sb. (N.S.) **93(135)** (1974), 129–149, 152. MR0336813

Electronic Journal of Probability

Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS¹)
- Easy interface (EJMS²)

Economical model of EJP-ECP

- Non profit, sponsored by IMS³, BS⁴, ProjectEuclid⁵
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund⁶ (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

¹LOCKSS: Lots of Copies Keep Stuff Safe <http://www.lockss.org/>

²EJMS: Electronic Journal Management System: <https://vtex.lt/services/ejms-peer-review/>

³IMS: Institute of Mathematical Statistics <http://www.imstat.org/>

⁴BS: Bernoulli Society <http://www.bernoulli-society.org/>

⁵Project Euclid: <https://projecteuclid.org/>

⁶IMS Open Access Fund: <https://imstat.org/shop/donation/>