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Stochastic differential equations in a scale of Hilbert spaces. Global solutions

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Abstract

A stochastic differential equation with coefficients defined in a scale of Hilbert spaces is considered. The existence, uniqueness and path-continuity of infinite-time solutions are proved by an extension of the Ovsyannikov method. These results are applied to a system of equations describing non-equilibrium stochastic dynamics of (real-valued) spins of an infinite particle system on a typical realization of a Poisson or Gibbs point process in $\mathbb{R}^n$. The paper improves the results of the work by the second named author "Stochastic differential equations in a scale of Hilbert spaces", Electron. J. Probab. 23, where finite-time solutions were constructed.

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1 Introduction

The purpose of this work is to study an infinite-dimensional stochastic differential equation (SDE)

$$d\xi(t) = f(\xi(t))dt + \Phi(\xi(t))dW(t), \tag{1.1}$$

with the coefficients f and Φ defined in a scale of densely embedded expanding Hilbert spaces $(X_\alpha)_{\alpha\in\mathcal{A}}$, where $\mathcal{A} \subset\mathbb{R}$ is an interval, and W is a cylindrical Wiener process on a fixed Hilbert space $\mathcal{H}$. That is, f and Φ are Lipschitz continuous maps $X_\alpha \to X_\beta$ and $X_\alpha \to H_\beta := HS(\mathcal{H}, X_\beta)$, $\beta > \alpha$, respectively, but are not in general well-defined in any fixed X_α, with the corresponding Lipschitz constants $L_{\alpha\beta}$ becoming infinite as $|\alpha - \beta| \to 0$. Here $HS(\mathcal{H}, X_\beta)$ stands for the space of Hilbert-Schmidt operators $\mathcal{H} \to X_\beta$.

Equation (1.1) cannot be treated by methods of the classical theory of SDEs in Banach spaces (see e.g. [11] and [16]), because its coefficients are singular in any fixed X_α. Some progress has been achieved in the case where

$$L_{\alpha\beta} \sim (\beta - \alpha)^{-q} \text{ as } |\alpha - \beta| \to 0, \tag{1.2}$$

with $q = \frac{1}{2}$. Under this condition, a strong solution with initial value in X_α exists in X_β up to a finite time $T_{\alpha\beta} \sim (\beta - \alpha)^{1/2}$, see [8]. This work generalizes the Ovsyannikov method for ordinary differential equations, see e.g. [18, 4, 9], in which setting it is sufficient to assume that $q = 1$.

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It has been noticed in [9] that, in case of $0 < q < 1$, a solution of the differential equation

$$\frac{d}{dt}u(t) = f(u(t)),\ u(0) \in X_\alpha,$$

with f as in (1.1), exists in any X_β, $\beta > \alpha$, with infinite lifetime. In the present paper, we build upon the ideas of [9], which enable us to generalize the results of [8] and prove the existence and uniqueness of a global strong solution ξ of equation (1.1) in any X_β, $\beta > \alpha$, with initial value $\xi(0) \in X_\alpha$, provided (1.2) holds with $0 < q < \frac{1}{2}$. Moreover, we show that ξ is p-integrable for any $p < q^{-1}$ and has a continuous modification, which also solves (1.1).

The structure of the paper is as follows. In Section 2 we introduce the framework and notations and formulate our main existence and uniqueness result. In Section 3 we obtain technical estimates, which play crucial role in what follows. Sections 4 and 5 are devoted to the proof of our main existence and uniqueness result and derivation of a growth estimate of the solution, respectively.

In Section 6, we consider an infinite system of coupled SDEs in $S := \mathbb{R}$ of the form

$$d\sigma_x(t) = f_x(\bar{\sigma})dt + \Phi_x(\bar{\sigma})dW_x(t),\ x \in \gamma,\ \bar{\sigma} = (\sigma_x)_{x\in\gamma}, \tag{1.3}$$

where $\gamma \subset M = \mathbb{R}^d$ is a locally finite (countable) set (configuration) and $W = (W_x)_{x\in\gamma}$ is a collection of independent Wiener processes in S. We assume that the drift and diffusion coefficients f_x and Φ_x have the form

$$f_x(\bar{\sigma}) = \sum_{y\in\gamma} \varphi_{xy}(\sigma_x, \sigma_y),\quad \Phi_x(\bar{\sigma}) = \sum_{y\in\gamma} \Psi_{xy}(\sigma_x, \sigma_y), \tag{1.4}$$

where the mappings $\varphi_{xy} : S \times S \to S$ and $\Psi_{xy} : S \times S \to S$ satisfy uniform Lipschitz conditions and have finite range, that is, $\varphi_{xy} \equiv \Psi_{xy} \equiv 0$ whenever $|x - y| \geq r$ for a fixed $r > 0$. The latter condition implies that, for any $x \in \gamma$, both sums in (1.4) have finite number n_x of non-zero elements. The numbers n_x, $x \in \gamma$, can be interpreted as vertex degrees of the geometric graph γ_r with the vertex set γ and the set of edges $\{\{x, y\} : x, y \in \gamma, |x - y| < r\}$. A natural approach to the study of system (1.3) is to consider it as a single equation in a Hilbert or Banach space of sequences. However, for general configurations γ, the corresponding vertex degrees of the graph γ_r are unbounded, which implies that the system (1.3) cannot be controlled in a single Banach space. This is in contrast to the case where $\gamma = \mathbb{Z}^d$ (or any bounded degree graph), which has been well-studied, see e.g. [17] and more recent developments in, [1, 2, 21] and references therein. However, under mild conditions on the density of γ (holding for e.g. Poisson and Gibbs point processes in $\mathbb{R}^n$), it is possible to apply the approach discussed above and construct a solution in the scale of Hilbert spaces S^γ_α of weighted sequences $(\sigma_x)_{x\in\gamma}$ such that $\sum_{x\in\gamma} |\sigma_x|^2 e^{-\alpha|x|} < \infty,\ \alpha > 0$. This approach was first used in [8] where local solutions of the above system were constructed.

The study of the system (1.3) is motivated by applications in statistical mechanics. It can be used for constructing and studying stochastic dynamics of countable systems of particles randomly distributed in M (which can in general be any metric space). Each particle is characterized by its position $x \in M$ and an internal parameter (spin) $\sigma_x \in S$. Two spins σ_x and σ_y are allowed to interact via a pair potential if the distance between x and y is no more than a fixed interaction radius r, that is, they are neighbors in the geometric graph γ_r. Equilibrium states of such systems are described by “annealed” and “quenched” Gibbs measures on $\Gamma(M, S)$ and S^γ, respectively. Here $\Gamma(M, S)$ is the space of configurations $\{(x, \sigma_x)\}_{x\in\gamma}$ with marks (see e.g. [15]) and S^γ stands for the Cartesian power of S.

The questions of the existence, uniqueness and multiplicity of the above Gibbs measures were considered in [7, 12, 13, 14]. The methods of the present paper and forthcoming work [10] will allow to construct non-equilibrium stochastic dynamics associated with these measures and study its ergodic properties. In particular, the results of Section 6 are used in a forthcoming paper [10] for the construction of a mixed-type jump diffusion dynamics in $\Gamma(M,S)$.

The study of such dynamics is motivated by a variety of applications, in particular in modelling of non-crystalline (amorphous) substances, e.g. ferrofluids and amorphous magnets, see e.g. [29], [27, Section 11], [6] and [12, 13]. Observe that the configuration space $\Gamma(M,S)$ possesses a fibration-like structure over the space $\Gamma(M)$ of position configurations γ, with the fibres identified with S^γ, see [12]. Thus the construction of spin dynamics of a quenched system (in S^γ) is complementary to that of the dynamics in $\Gamma(M)$. The latter has been discussed by many authors, see e.g. [25, 26, 20, 5, 3, 19] and references given there.

Finally, in Section 7 we give two further examples of the maps satisfying condition (1.2).

For a discussion of the relationship of our construction with the results on SDEs in nuclear spaces ([22, 23]) see [8].

2 Setting and main results

In this section we introduce the general framework we will be using below. Let us consider a family $\mathfrak{B}$ of Banach spaces B_α indexed by $\alpha \in \mathcal{A} := [\alpha_*, \alpha^*]$ with fixed $0 \le \alpha_*, \alpha^* < \infty$, and denote by $\|\cdot\|_{B_\alpha}$ the corresponding norms. When speaking of these spaces and related objects, we will always assume that the range of indices is $[\alpha_*, \alpha^*]$, unless stated otherwise.

Definition 2.1. *The family $\mathfrak{B}$ is called a scale if*

$$B_\alpha \subset B_\beta \text{ and } \|u\|_{B_\beta} \le \|u\|_{B_\alpha} \text{ for any } \alpha < \beta,\ u \in B_\alpha,\ \alpha, \beta \in \mathcal{A},$$

where the embedding means that B_α is a dense vector subspace of B_β.

We will use the following notations:

$$\overline{B} := \bigcup_{\alpha \in [\alpha_*, \alpha^*)} B_\alpha, \quad \underline{B} := \bigcap_{\alpha \in (\alpha_*, \alpha^*]} B_\alpha.$$

Definition 2.2. *For two scales $\mathfrak{B}_1$, $\mathfrak{B}_2$ (with the same index set) and a constant $q > 0$ we introduce the class $\mathcal{GL}_q(\mathfrak{B}_1, \mathfrak{B}_2)$ of (generalized Lipschitz) maps $g\colon \overline{B}_1 \to \overline{B}_2$ such that*

(1) $g(B_{1,\alpha}) \subset B_{2,\beta}$ for any $\alpha < \beta$;

(2) there exists constant $L > 0$ such that

$$\|g(u) - g(v)\|_{B_{2,\beta}} \le \frac{L}{|\beta - \alpha|^q} \|u - v\|_{B_{1,\alpha}} \tag{2.1}$$

for any $\alpha < \beta$ and $u, v \in B_{1,\alpha}$.

We will write $\mathcal{GL}_q(\mathfrak{B}) := \mathcal{GL}_q(\mathfrak{B}_1, \mathfrak{B}_2)$ if $\mathfrak{B}_1 = \mathfrak{B}_2 =: \mathfrak{B}$.

Remark 2.3. $g \in \mathcal{GL}_q(\mathfrak{B}_1, \mathfrak{B}_2)$ generates a map $\underline{B}_1 \to \underline{B}_2$.

Observe that (2.1) implies the linear growth condition

$$\|g(u)\|_{B_{2,\beta}} \le \frac{K}{|\beta - \alpha|^q} \left(1 + \|u\|_{B_{1,\alpha}}\right),\ u \in B_{1,\alpha}, \tag{2.2}$$

for some constant K and any $\alpha < \beta$. Without loss of generality we assume that $K = L$.

In what follows, we will use the following three main scales:

(1) the scale $\mathfrak{X}$ of separable Hilbert spaces X_α;
(2) the scale $\mathfrak{H}$ of spaces

$$H_\alpha \equiv HS(\mathcal{H}, X_\alpha) := \{\text{Hilbert-Schmidt operators } \mathcal{H} \to X_\alpha\}, \tag{2.3}$$

for a fixed separable Hilbert space $\mathcal{H}$;
(3) the scale $\mathfrak{Z}_T^p$ of Banach spaces $Z_{\alpha,T}^p$ of predictable random processes $u : [0,T] \to X_\alpha$ with finite norm

$$\|u\|_{Z_{\alpha,T}^p} := \sup_{t\in[0,T]} \left(\mathbb{E}\,\|u(t)\|_{X_\alpha}^p\right)^{1/p},$$

defined on a suitable filtered probability space $\mathcal{P} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

Our aim is to construct a strong solution of equation (1.1), that is, a solution of the stochastic integral equation

$$u(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t \Phi(u(s))dW(s),\ t \in [0,T], \tag{2.4}$$

with coefficients acting in the scale $\mathfrak{X}$. Here $W(t),\ t \leq T$, is a fixed cylindrical Wiener process in $\mathcal{H}$ (cf. (2.3)) defined on the probability space $\mathcal{P}$. We suppose that u_0 is an $\mathcal{F}_0$-measurable p-integrable X_α-valued random variable, for a fixed $\alpha \in [\alpha_*, \alpha^*)$ and $p \geq 2$.

The following theorem states the main result of this paper.

Theorem 2.4 (Existence and uniqueness)**.** *Assume that $f \in \mathcal{GL}_q(\mathfrak{X})$ and $\Phi \in \mathcal{GL}_q(\mathfrak{X}, \mathfrak{H})$ for some $q \in (0, \frac{1}{2})$. Then, for any $p \in \left[2, q^{-1}\right)$ and $T > 0$, the following holds:*

(1) equation (2.4) has a unique solution $u \in Z_{\alpha^*,T}^2$;
(2) $u \in Z_{\beta,T}^p$ for any $\beta > \alpha$;
(3) for any $\beta > \alpha$, the solution $u \in Z_{\beta,T}^p$ has a continuous X_β-valued modification that satisfies (2.4).

The proof is given in Section 4 below. We will show that the map $u \mapsto \mathcal{T}(u)$, where

$$\mathcal{T}(u)(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t \Phi(u(s))dW(s),\ t \in [0,T], \tag{2.5}$$

has a unique fixed point in $Z_{\beta,T}^p$ for any $\beta > \alpha$, by Picard iterative process.

From now on, we keep random variable u_0 fixed and assume without loss of generality that it takes values in X_{α_*} (otherwise, we can always re-define the parameter set $\mathcal{A}$). We also fix $p \in \left[2, q^{-1}\right)$ and an arbitrary $T > 0$ and write Z_β^p instead of $Z_{\beta,T}^p$.

Remark 2.5. Let us observe that if $\xi \in Z_\alpha^p$ and $\alpha < \beta$ then $\Phi(\xi(t)),\ t > 0$, is a predictable H_β-valued process because $\Phi : X_\alpha \to H_\beta$ is continuous by inequality (2.1). Inequality (2.2) shows that

$$\sqrt{\mathbb{E}\int_0^T \|\Phi(\xi(s))\|_{H_\beta}^2\, ds} \leq C_1\left(1 + \|\xi\|_{Z_\alpha^2}\right) \leq C_2\left(1 + \|\xi\|_{Z_\alpha^p}\right) < \infty$$

for some constants $C_1, C_2 > 0$, because $\xi \in Z_\alpha^p \subset Z_\alpha^2$. Thus the stochastic integral

$$\int_0^t \Phi(\xi(s))dW(s),\ t \leq T, \tag{2.6}$$

is unambiguously defined as a square integrable and almost surely continuous X_β-valued martingale, see e.g. [16], [28].

Remark 2.6. The solution u satisfies (2.4) as element of the space $Z_{\alpha^*,T}^2$. That is, in a more explicit form, the first part of Theorem 2.4 states that $\sup_{0\leq t\leq T}\mathbb{E}\|u(t) - \mathcal{T}(u)(t)\|_{X_{\alpha^*}} = 0$.

3 Main estimates

In this section, we derive certain estimates of the map $\mathcal{T}$ defined by formula (2.5). We fix $q \in (0, \frac{1}{2})$ and arbitrary $p \in \left[2, q^{-1}\right)$.

Theorem 3.1. *Assume that $f \in \mathcal{GL}_q(\mathfrak{X})$ and $\Phi \in \mathcal{GL}_q(\mathfrak{X}, \mathfrak{H})$. Then $\mathcal{T} \in \mathcal{GL}_q(\mathfrak{Z}^p)$.*

Proof. Let us fix $\alpha < \beta$. Observe that $f(u(s)) \in X_\beta$ and $\Phi(u(s)) \in H_\beta$ for any $u \in Z^p_\alpha$ and $s \in [0, T]$, and the integrals in the right-hand side of (2.5) are well-defined in X_β.

We first prove the inclusion $\mathcal{T}(Z^p_\alpha) \subset Z^p_\beta$. Let $\xi \in Z^p_\alpha$ and introduce the notation $\widehat{\mathcal{T}}(\xi)(t) := \mathcal{T}(\xi)(t) - u_0 = \int_0^t f(\xi(s))ds + \int_0^t \Phi(\xi(s))dW(s)$. Then, using the Hölder inequality and well-known formula for the moments of the Ito integral (see e.g. [11, 28]) we obtain

$$\mathbb{E}\left[||\widehat{\mathcal{T}}(\xi)(t)||^p_{X_\beta}\right] \le 2^{p-1}t^{p-1}\int_0^t \mathbb{E}\,||f(\xi(s))||^p_{X_\beta}ds + 2^{p-1}p\left[\frac{p}{2(p-1)}\right]^{p/2} t^{p/2-1}\int_0^t \mathbb{E}\,||\Phi(\xi(s))||^p_{H_\beta}ds. \quad (3.1)$$

An application of estimate (2.2) to the right-hand side of (3.1) above shows that the estimate

$$\mathbb{E}\left[||\widehat{\mathcal{T}}(\xi)(t)||^p_{X_\beta}\right] \le \frac{\hat{L}(T)}{(\beta-\alpha)^{pq}}\int_0^t \mathbb{E}\,||\xi(s)||^p_{X_\alpha}ds, \quad (3.2)$$

holds with $\hat{L}(T) = (T^{p-1} + p\left[\frac{p}{2(p-1)}\right]^{p/2} T^{p/2-1})2^{p-1}L^p$, so that

$$||\mathcal{T}(\xi)||_{Z^p_\beta} \le ||u_0||_{Z^p_\beta} + ||\widehat{\mathcal{T}}(\xi)||_{Z^p_\beta} \le ||u_0||_{Z^p_\beta} + \frac{\sqrt[p]{\hat{L}(T)T}}{(\beta-\alpha)^q}||\xi||_{Z^p_\alpha} < \infty, \quad (3.3)$$

because $||u_0||_{Z^p_\beta} < \infty$ for any $\beta \ge \alpha$.

Now we shall show that condition (2.1) of Definition 2.2 holds. Introducing notations $\bar{f}(s) := f(\xi_1(s)) - f(\xi_2(s))$ and $\bar{\Phi}(s) := \Phi(\xi_1(s)) - \Phi(\xi_2(s))$, $s \in [0, T]$, and applying arguments as above together with estimate (2.1) we see that

$$\mathbb{E}\left[||\mathcal{T}(\xi_1)(t) - \mathcal{T}(\xi_2)(t)||^p_{X_\beta}\right] \le \frac{\hat{L}(T)}{(\beta-\alpha)^{pq}}\int_0^t \mathbb{E}\,||\xi_1(s) - \xi_2(s)||^p_{X_\alpha}ds, \quad (3.4)$$

so that

$$||\mathcal{T}(\xi_1) - \mathcal{T}(\xi_2)||_{Z^p_\beta} \le \frac{\sqrt[p]{\hat{L}(T)T}}{(\beta-\alpha)^q}||\xi_1 - \xi_2||_{Z^p_\alpha}, \quad (3.5)$$

and the proof is complete. □

Corollary 3.2. *For any $\alpha < \beta$ and all $n \in \mathbb{N}$ we have $\mathcal{T}^n : Z^p_\alpha \to Z^p_\beta$, where $\mathcal{T}^n$ stands for the n-th composition power of $\mathcal{T}$.*

Lemma 3.3. *For any $n \in \mathbb{N}$, $\alpha < \beta$ and $\xi, \eta \in Z^p_\alpha$ we have the estimate*

$$||\mathcal{T}^n(\xi) - \mathcal{T}^n(\eta)||^p_{Z^p_\beta} \le \frac{n^{npq}}{n!}\left(\frac{\hat{L}(T)T}{(\beta-\alpha)^{pq}}\right)^n ||\xi - \eta||^p_{Z^p_\alpha}. \quad (3.6)$$

Proof. We fix a partition of the interval $[\alpha, \beta]$ in n intervals $[\psi_k, \psi_{k+1}]$, $k = 0, \dots, n-1$, $\psi_0 = \alpha$, $\psi_n = \beta$, of equal length $\frac{\beta-\alpha}{n}$. Then, iterating estimate (3.4) with intervals

$[\psi_k, \psi_{k+1}]$ in place of $[\alpha, \beta]$, we obtain

$$\begin{aligned}\mathbb{E}\left[||\mathcal{T}(\mathcal{T}^{n-1}(\xi))(t) - \mathcal{T}(\mathcal{T}^{n-1}(\eta))(t)||^p_{X_\beta}\right] \\ \leq \frac{\hat{L}(T)n^{pq}}{(\beta-\alpha)^{pq}} \int_0^t \mathbb{E}\,||\mathcal{T}^{n-1}(\xi(s)) - \mathcal{T}^{n-1}(\eta(s))||^p_{X_\alpha} ds \\ \leq \cdots \leq \left[\frac{\hat{L}(T)n^{pq}}{(\beta-\alpha)^{pq}}\right]^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbb{E}\,||\xi(s) - \eta(s)||^p_{X_\alpha} ds dt_{n-1} \cdots dt_1, \quad (3.7)\end{aligned}$$

and the result follows. □

Corollary 3.4. *Fix an arbitrary $\delta \in (\alpha, \beta)$. Setting $\eta = \mathcal{T}(\xi) \in Z^p_\delta$ and iterating estimate (3.6) with α replaced by δ we see that, for any $m > n$,*

$$||\mathcal{T}^n(\xi) - \mathcal{T}^m(\xi)||_{Z^p_\beta} \leq ||\xi - \mathcal{T}(\xi)||_{Z^p_\delta} \sum_{k=n}^{m-1} \frac{\sqrt[p]{\hat{L}(T)^k T^k}}{(\beta-\delta)^{kq}} \frac{k^{kq}}{\sqrt[p]{k!}}. \quad (3.8)$$

Remark 3.5. Observe that the inclusion $Z^p_\alpha \subset Z^2_\alpha$ implies that the estimate (3.6) and preceding statements hold with p replaced by any $p' \in [2, p)$. In particular, $\mathcal{T} \in \mathcal{GL}_q(\mathfrak{Z}^2)$.

Finally, we prove regularity of the right-hand side of (3.8). In what follows, we will use the notation

$$E^{(p)}(t, \varepsilon, q) := 1 + \sum_{n=1}^{\infty} \frac{t^n}{\varepsilon^{nq}} \frac{n^{nq}}{\sqrt[p]{n!}} \quad (3.9)$$

Observe that for $p = 1$ and $q = 0$ the right-hand side of (3.9) reduces to an exponential series, so that $E^{(1)}(c, \varepsilon, 0) = e^c$.

Lemma 3.6. *For any $t, p, \varepsilon > 0$ and $q \in [0, \frac{1}{p})$ we have $E^{(p)}(t, \varepsilon, q) < \infty$.*

Proof. Analyzing the ratio of terms of series (3.9) we see that

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{t^{(n+1)}}{\varepsilon^{q(n+1)}} \frac{(n+1)^{q(n+1)}}{((n+1)!)^{1/p}}}{\frac{t^n}{\varepsilon^{qn}} \frac{n^{qn}}{(n!)^{1/p}}} &= \lim_{n\to\infty} \frac{t}{\varepsilon^q} (n+1)^{qn+q-\frac{1}{p}} \frac{1}{n^{qn}} \\ &= \lim_{n\to\infty} \frac{t}{\varepsilon^q} \left(1 + \frac{1}{n}\right)^{qn} (n+1)^{q-\frac{1}{p}} = \frac{t}{\varepsilon^q} e^q \lim_{n\to\infty} (n+1)^{q-\frac{1}{p}} = 0,\end{aligned}$$

provided $q - \frac{1}{p} < 0$, which proves the result. □

Corollary 3.7. *Setting $t = \sqrt[p]{\hat{L}(T)T}$ and $\varepsilon = \beta - \alpha$ we obtain equality*

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{\sqrt[p]{\hat{L}(T)^k T^k}}{(\beta-\alpha)^{kq}} \frac{k^{kq}}{\sqrt[p]{k!}} = 0 \quad (3.10)$$

for any $\alpha < \beta$ and $q < p^{-1}$.

4 Fixed point theorem and proof of the main result

We will now prove the fixed point theorem for the map $\mathcal{T}$ defined by (2.5), which immediately allows us to establish the existence and uniqueness of a solution of equation (2.4). We suppose without loss of generality that $u_0 \in Z^p_{\alpha_*}$.

Theorem 4.1. *Suppose that $p \in \left[2, q^{-1}\right)$. Then the map $\mathcal{T} : \overline{Z}^p \to \overline{Z}^p$ has a unique fixed point u. Moreover, $u \in \underline{Z}^p$, the equality $\mathcal{T}(u) = u$ holds in any Z^p_α, $\alpha > \alpha_*$ and for an arbitrary $\xi \in Z^p_\alpha$ we have*

$$\lim_{n\to\infty} \mathcal{T}^n(\xi) = u,$$

where the convergence takes place in Z^p_β for all $\beta > \alpha$.

Proof. Let us fix $\xi \in Z^p_{\alpha_*}$. Estimate (3.8) and Corollary 3.7 show that the sequence $\{\mathcal{T}^n(\xi)\}_{n=1}^\infty$ is Cauchy in Z^p_β and thus converges in Z^p_β, for any $\beta > \alpha_*$. Thus there exists $u \in Z^p_\beta$ such that

$$\lim_{n\to\infty} \mathcal{T}^n(\xi) = u,$$

where the convergence takes place in Z^p_β. Observe that for any $\beta_1 < \beta_2$ the space $Z^p_{\beta_1}$ is a dense vector subspace of $Z^p_{\beta_2}$, which implies that u is independent of the choice of β, and so $u \in \underline{Z}^p = \cap_{\beta>\alpha_*} Z^p_\beta$.

To show that u is a fixed point, we can now fix arbitrary $\beta > \alpha_*$ and choose any $\delta \in (\alpha_*, \beta)$. Observe that $\mathcal{T}^n(\xi) \in Z^p_\delta$ and $\mathcal{T} : Z^p_\delta \to Z^p_\beta$ is continuous. Therefore, passing to the limit in both sides of the equality

$$\mathcal{T}(\mathcal{T}^n(\xi)) = \mathcal{T}^{n+1}(\xi) \in Z^p_\beta$$

we can conclude that

$$\mathcal{T}(u) = u \text{ in } Z^p_\beta \text{ for any } \beta > \alpha_*,$$

which also implies that for all $t \le T$ we have $\mathcal{T}(u)(t) = u(t)$ almost everywhere.

Suppose now that there exists another element $v \in Z^p_{\beta'}$, $\beta' > \alpha_*$, such that $\mathcal{T}(v) = v$. Assume without loss of generality that $\beta \ge \beta'$ and fix $\delta > \beta$. Then $v \in Z^p_\beta \subset Z^p_\delta$ and so we have

$$||u - v||^p_{Z^p_\delta} = ||\mathcal{T}^n(u) - \mathcal{T}^n(v)||^p_{Z^p_\delta} \le \frac{n^{pnq}}{n!} \left(\frac{\hat{L}(T)T}{(\delta-\beta)^{pq}}\right)^n ||u - v||^p_{Z^p_\beta} \to 0,\ n \to \infty, \quad (4.1)$$

by (3.6) and (3.10), which implies that $v = u$ in Z^p_δ and thus in Z^p_β.

Observe that similar arguments show that for any $\alpha > \alpha_*$ and $\xi \in Z^p_\alpha$ the sequence $\{\mathcal{T}^n(\xi)\}_{n=1}^\infty$ converges in any Z^p_β for all $\beta > \alpha$ and $\lim_{n\to\infty} \mathcal{T}^n(\xi) \in Z_\beta$ is a fixed point of the map $\mathcal{T}$. The uniqueness of the fixed point implies that $\lim_{n\to\infty} \mathcal{T}^n(\xi) = u \in \underline{Z}^p$. The proof is complete. □

Remark 4.2. Taking into account Remark 3.5 we see that the statement of Theorem 4.1 holds with p replaced by any $p' \in [2, p)$. This implies in particular that the fixed point u is unique in $\underline{Z}^{p'}$.

Remark 4.3. In particular, for any $\alpha < \beta$ and $\xi \in X_\alpha$ we have $\lim_{n\to\infty} \|\mathcal{T}^n(\xi)(t) - u(t)\|_{X_\beta} = 0$ and $\|\mathcal{T}(u)(t) - u(t)\|_{X_\alpha} = 0$ a.s., $t \le T$.

Proof of Theorem 2.4. The first two statements follow immediately from Theorem 4.1 and Remark 4.2 above. Indeed, the unique fixed point u of the map $\mathcal{T}$ gives the solution of equation (2.4).

To prove the third statement, consider the X_β-valued process

$$\mathcal{T}(u)(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t \Phi(u(s))dW(s),\ t \in [0, T],$$

which is, according to the first part of the theorem, a modification of the process u. By the general properties of stochastic integrals, this process is a.s. continuous, see Remark 2.5.

If $p > 2$, the existence of a continuous modification of u can be also shown directly by an application of Kolmogorov's continuity theorem in a rather standard way. Indeed, using (2.4) and the arguments similar to those in the proof of (3.3) we obtain for any $\alpha \in (\alpha_*, \beta)$ that

$$\begin{aligned}\mathbb{E}||u(t)-u(s)||^p_{X_\beta} &\leq \mathbb{E}\left[\left\|\int_s^t f(u(\tau))d\tau + \int_s^t \Phi(u(\tau))dW(\tau),\right\|^p_{X_\alpha}\right] \\ &\leq \frac{C(t-s)}{(\beta-\alpha)^{pq}}\,\|u\|^p_{Z^p_\alpha}\,|t-s|^{p/2},\ 0\leq s<t\leq T,\end{aligned}$$

where $C(\tau) = (\tau^{p/2} + p[\frac{p}{2(p-1)}]^{p/2})2^{p-1}L^p \leq C(T)$, $0 < \tau < T$. So we have the estimate

$$\mathbb{E}||u(t)-u(s)||^p_{X_\beta} \leq k(u,T)\,|t-s|^{p/2}$$

with $k(u,T) = \frac{C(T)}{(\beta-\alpha)^{pq}}\,\|u\|^p_{Z^p_\alpha}$, which implies the existence of a continuous X_β-valued modification η.

Observe now that by (3.5) we have for any $\beta' > \beta$

$$\|\eta - \mathcal{T}(\eta)\|^p_{Z^p_{\beta'}} = ||\mathcal{T}(u) - \mathcal{T}(\eta)||^p_{Z^p_{\beta'}} \leq \frac{\sqrt[p]{\hat{L}(T)T}}{(\beta'-\beta)^{pq}}\,\|u-\eta\|^p_{Z^p_\beta} = 0,\ t\in[0,T].$$

Since β and β' are arbitrary, the proof is complete. □

Remark 4.4. Observe that $\|u-\eta\|_{Z^p_\beta} = 0$, so the processes u and η coincide as elements of Z^p_β.

5 Estimate of the solution

In this section, we derive a norm estimate of the solution u from Theorem 2.4.

Lemma 5.1. *For any $\beta > \alpha$ we have*

$$||u||_{Z^p_\beta} \leq E^{(p)}\left(\sqrt[p]{\hat{L}(T)T}, \frac{\beta-\alpha}{2}, q\right)\left[1 + \left(\mathbb{E}||u_0||^p_{X_\alpha}\right)^{1/p}\right].$$

Proof. By (3.8) with $\xi = u_0$ and $n = 0$ we obtain

$$||u_0 - \mathcal{T}^m(u_0)||_{Z^p_\beta} \leq \sum_{n=0}^{m-1}\frac{n^{nq}}{\sqrt[p]{n!}}\left[\frac{\sqrt[p]{\hat{L}(T)T}}{(\beta-\delta)^q}\right]^n ||u_0 - \mathcal{T}(u_0)||_{Z^p_\delta}$$

for any $\delta \in (\alpha, \beta)$. An application of (2.2) to the right-hand side of the equality

$$\mathbb{E}||u_0 - \mathcal{T}(u_0)(t)||^p_{X_\delta} = \mathbb{E}\left[\left\|\int_0^t f(u_0)ds + \int_0^t \Phi(u_0)dW(s)\right\|^p_{X_\delta}\right]$$

gives us an estimate similar to (3.5), namely

$$||u_0 - \mathcal{T}(u_0)||_{Z^p_\delta} \leq \frac{\sqrt[p]{\hat{L}(T)T}}{(\delta-\alpha)^q}\left(\mathbb{E}\left(1 + ||u_0||^p_{X_\alpha}\right)\right)^{1/p}.$$

Thus, taking into account that $\left(\mathbb{E}\left(1+||u_0||^p_{X_\alpha}\right)\right)^{1/p} \leq 1 + \left(\mathbb{E}||u_0||^p_{X_\alpha}\right)^{1/p}$ and setting $\delta = \frac{\beta+\alpha}{2}$, we obtain for any $m \in \mathbb{N}$

$$||u_0 - \mathcal{T}^m(u_0)||_{Z^p_\beta} \leq \sum_{n=1}^{m}\frac{n^{nq}}{\sqrt[p]{n!}}\left[\frac{\sqrt[p]{\hat{L}(T)T}}{(\frac{\beta-\alpha}{2})^q}\right]^n\left[1 + \left(\mathbb{E}||u_0||^p_{X_\alpha}\right)^{1/p}\right].$$

Passing to the limit as $m \to \infty$ (cf. Theorem 4.1) we obtain the bound

$$||u_0 - u||_{Z^p_\beta} \le \sum_{n=1}^{\infty} \frac{n^{nq}}{\sqrt[p]{n!}} \left[\frac{\sqrt[p]{\hat{L}(T)T}}{(\frac{\beta-\alpha}{2})^q} \right]^n \left[1 + \left(\mathbb{E}||u_0||^p_{X_\alpha}\right)^{1/p}\right].$$

Therefore

$$\begin{aligned} ||u||_{Z^p_\beta} &\le \left(\mathbb{E}||u_0||^p_{X_\beta}\right)^{1/p} + \sum_{n=1}^{\infty} \frac{n^{nq}}{\sqrt[p]{n!}} \left[\frac{\sqrt[p]{\hat{L}(T)T}}{(\frac{\beta-\alpha}{2})^q} \right]^n \left[1 + \left(\mathbb{E}||u_0||^p_{X_\alpha}\right)^{1/p}\right]. \\ &\le \left(1 + \sum_{n=1}^{\infty} \frac{n^{nq}}{\sqrt[p]{n!}} \left[\frac{\sqrt[p]{\hat{L}(T)T}}{(\frac{\beta-\alpha}{2})^q} \right]^n\right) \left[1 + (\mathbb{E}||u_0||^p_{X\alpha})^{1/p}\right] \\ &= E^{(p)}\left(\sqrt[p]{\hat{L}(T)T}, \frac{\beta-\alpha}{2}, q\right) \left[1 + \left(\mathbb{E}||u_0||^p_{X_\alpha}\right)^{1/p}\right], \end{aligned}$$

which completes the proof. □

Remark 5.2. Observe that we have $\sqrt[p]{\hat{L}(T)T} \le a(T) = a_p \max\left(T, T^{1/2}\right)$, where $a_p = 2^{(p-1)/p}(p^{1/p}[\frac{p}{(p-1)}]^{1/2} + 1)L$.

6 Stochastic spin dynamics of a quenched particle system

In this section, we apply the results of Section 2 to system (1.3), which is motivated by the study of stochastic dynamics of interacting particle systems and serves as our main example. We follow the scheme of paper [8], adapted to our present setting, which allows to show the existence of solutions with arbitrary large lifetime and their path-continuity. Let $\gamma \subset \mathbb{R}^d$ be a locally finite set (configuration) representing a collection of point particles. Each particle with position $x \in \gamma$ is characterized by an internal parameter (spin) $\sigma_x \in S{:=}\mathbb{R}$. We fix an interaction radius $r > 0$ and assume that the number

$$n_x \equiv n_{x,r}(\gamma) := \#\left\{y \in \gamma : |x - y| < r\right\} \tag{6.1}$$

satisfies the following regularity condition.

Condition 6.1. *There exist constants* $q \in (0, \frac{1}{2})$ *and* $a \equiv a(\gamma, r, q) > 0$ *such that*

$$n_x \le a\left(1 + |x|\right)^q,\ x \in \mathbb{R}^d. \tag{6.2}$$

Remark 6.2. Condition (6.2) holds if γ is a typical realization of a Poisson or Gibbs (Ruelle) point process in $\mathbb{R}^d$. For such configurations, the following stronger (logarithmic) bound holds:

$$n_{x,r}(\gamma) \le c(\gamma)\left[1 + \log(1 + |x|)\right]^{1/2} r^d,$$

see e.g. [30] and [24, p. 1047]. Thus (6.2) holds for any $q > 0$.

Consider the Cartesian power $S^\gamma = \{\bar{u} = (u_x)_{x\in\gamma},\ u_x \in S\}$. Our dynamics will live in the scale of Hilbert spaces

$$X_\alpha = S^\gamma_\alpha := \left\{\bar{u} \in S^\gamma : \|\bar{u}\|_\alpha := \sqrt{\sum_{x\in\gamma} |u_x|^2 e^{-\alpha|x|}} < \infty\right\},\ 0 < \alpha_* \le \alpha \le \alpha^*.$$

where the parameters α_* and α^* are chosen in an arbitrary way and fixed. We set

$$\mathcal{H} = S_0^\gamma := \left\{ \bar{u} \in S^\gamma : \|\bar{u}\|_0 := \sqrt{\sum_{x\in\gamma} |u_x|^2} < \infty \right\}$$

and define the corresponding spaces $\mathcal{GL}_q(\mathfrak{X})$ and $\mathcal{GL}_q(\mathfrak{X}, \mathfrak{H})$, cf. Definition 2.2. Observe that $W(t) := (W_x(t))_{x\in\gamma}$ is a cylindrical Wiener process in $\mathcal{H}$.

We assume that the families of mappings $\{\varphi_{xy}\}_{x,y\in\gamma}$ and $\{\Psi_{xy}\}_{x,y\in\gamma}$ from (1.4) satisfy the following condition.

Condition 6.3.

- finite range: $\varphi_{xy} = \Psi_{xy} \equiv 0$ if $|x-y| \geq r$, where r is the interaction radius from (6.1);
- uniform Lipschitz continuity:

$$|\varphi_{xy}(z_1', z_2') - \varphi_{xy}(z_1'', z_2'')| \leq C\left(|z_1' - z_1''| + |z_2' - z_2''|\right),$$
$$|\Psi_{xy}(z_1', z_2') - \Psi_{xy}(z_1'', z_2'')| \leq C\left(|z_1' - z_1''| + |z_2' - z_2''|\right)$$

  for some constant $C > 0$ and all $x, y \in \gamma$ and $z_1', z_2', z_1'', z_2'' \in S$.

Define a map $\overline{\varphi} : S^\gamma \to S^\gamma$ and a linear operator $\widehat{\Psi}(\bar{u}) : S^\gamma \to S^\gamma$ by the formulae

$$\overline{\varphi}_x(\bar{u}) = \sum_{y\in\gamma} \varphi_{xy}(u_x, u_y) \text{ and } \left(\widehat{\Psi}(\bar{u})\ \bar{\sigma}\right)_x = \sum_{y\in\gamma} \Psi_{xy}(u_x, u_y)\sigma_x,$$

$x \in \gamma$, $\bar{u} \in S^\gamma$, respectively.

The proof of the following result is similar to that of Lemma 5.4 in [8], where the case of $q = 1/2$ is considered.

Lemma 6.4. *We have $\overline{\varphi} \in \mathcal{GL}_q(\mathfrak{X})$ and $\widehat{\Psi} \in \mathcal{GL}_q(\mathfrak{X}, \mathfrak{H})$.*

Now we can return to the discussion of system (1.3). We can write it in the form (1.1) with $f = \overline{\varphi}$, $\Phi = \widehat{\Psi}$ and $W(t) = (W_x(t))_{x\in\gamma}$, and apply the results of the previous sections to its integral counterpart. We summarize those results in the following theorem, which follows directly from Theorem 2.4.

Theorem 6.5. *Assume that Conditions 6.1 and 6.3 hold. Then, for any $\alpha > 0$, $\bar{\sigma}(0) \in X_\alpha$, $p \in [2, q^{-1})$ and $T > 0$, system (1.3) has a unique strong solution $u \in Z_\beta^p$ for any $\beta > \alpha$. This solution has a continuous modification that satisfies (1.3).*

This result implies of course that, for each $x \in \gamma$, equation (1.3) has a path-continuous strong solution, which is unique in the class of predictable square-integrable processes.

Remark 6.6. For a configuration γ as in Remark 6.2, the statement of the theorem above holds for any $p \geq 2$.

7 Further examples

In this section we give two examples of linear maps of the class $\mathcal{GL}_q(\mathfrak{B})$.

Example 1. Consider the scale $\mathfrak{B}$ of Banach spaces $B_\alpha := L^p(\mathbb{R}, e^{-\alpha|x|}dx)$, $p > 1$, $\alpha \in [\alpha_*, \alpha^*]$, and the integral operator

$$Au(x) = \int K(x,y)u(y)dy,\ x \in \mathbb{R},$$

with kernel K satisfying the bound

$$|K(x,y)| \leq ae^{-\frac{\beta^*}{p}|x-y|}\left(1 + |y|\right)^\delta,\ \delta > 0, \tag{7.1}$$

for some $\beta^* \in (\alpha_*, \alpha^*)$, $a > 0$ and a.a. $x, y \in \mathbb{R}$.

Remark 7.1. It is clear that $K(x,y)$ can grow to infinity along the main diagonal $x=y$, which implies that A is in general unbounded in any weighted L^p.

Proposition 7.2. *Assume that (7.1) holds. Then* $A \in \mathcal{GL}_q(\mathfrak{B})$ *with* $q = \frac{p\delta}{p-1}$.

Remark 7.3. For an implementation of any version of Ovsyannikov-type method, we need $0 < q \le 1$, which implies $\delta \le \frac{p-1}{p} < 1$.

Proof. We start with the following estimate of the norm of operator A in B_β, $\beta < \alpha^*$:

$$\begin{aligned}\|Au\|^p_{B_\beta} &\le \int \left[\int |K(x,y)u(y)|dy\right]^p e^{-\beta|x|}dx \\ &\le a^p \int \left[\int e^{-\frac{\beta^*}{p}|x-y|} (1+|y|)^\delta |u(y)|\, dy\right]^p e^{-\beta|x|}dx \\ &= a^p \int \left[\int e^{-\varepsilon|x-y|} (1+|y|)^\delta |u(y)|\, e^{-\frac{\beta}{p}|x-y|}dy\right]^p e^{-\beta|x|}dx,\end{aligned}$$

where $\varepsilon = \frac{\beta^*-\beta}{p}$. Observe that $e^{-\frac{\beta}{p}|x-y|}e^{-\frac{\beta}{p}|x|} \le e^{-\frac{\beta}{p}|y|}$, so that

$$\|Au\|^p_{B_\beta} \le a^p \int \left[\int e^{-\varepsilon|x-y|} (1+|y|)^\delta |u(y)|\, e^{-\frac{\beta}{p}|y|}dy\right]^p dx.$$

For θ such that $\theta^{-1} + p^{-1} = 1$ we have

$$e^{-\varepsilon|x-y|} (1+|y|)^\delta |u(y)|\, e^{-\frac{\beta}{p}|y|} = \left[e^{-\frac{\varepsilon}{\theta}|x-y|} (1+|y|)^\delta e^{-\frac{\beta-\alpha}{p}|y|}\right] \times \left[e^{-\frac{\varepsilon}{p}|x-y|} |u(y)|\, e^{-\frac{\alpha}{p}|y|}\right]$$

for any $\alpha < \beta$. Then, by Holder's inequality,

$$\begin{aligned}&\|Au\|^p_{B_\beta} \\ &\le a^p \int \left[\left\{\int \left[e^{-\frac{\varepsilon}{\theta}|x-y|} (1+|y|)^\delta e^{-\frac{\beta-\alpha}{p}|y|}\right]^\theta dy\right\}^{p/\theta} \times \int \left[e^{-\frac{\varepsilon}{p}|x-y|} |u(y)|\, e^{-\frac{\alpha}{p}|y|}\right]^p dy\right] dx \\ &\qquad \le a^p b c^{p/\theta+1} \|u\|^p_{B_\alpha},\end{aligned}$$

where

$$b = \sup_{s\ge 0} (1+s)^{\theta\delta} e^{-\frac{\theta}{p}(\beta-\alpha)s} \text{ and } c = \int e^{-\varepsilon|y|}dy.$$

It remains to compute constant b. Equating to 0 the derivative $\frac{\partial}{\partial s}(1+s)\, e^{-\frac{1}{p\delta}(\beta-\alpha)s}$ we obtain

$$b = \frac{C}{(\beta-\alpha)^{\theta\delta}},\ C = \left(p\delta e^{\frac{\alpha^*-\alpha_*}{p\delta}-1}\right)^{\theta\delta}.$$

It is clear that estimate (2.1) holds with $q = \theta\delta = \frac{p\delta}{p-1}$. □

Example 2. A somewhat similar example is given by the spaces of sequences

$$B_\alpha := \left\{(u_k)_{k\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} |u_k|^p e^{-\alpha|k|} < \infty\right\},\ p > 1,$$

and the linear map given an infinite matrix $A = (A_{kj})_{k,j\in\mathbb{Z}}$ is with elements satisfying the bound

$$|A_{kj}| \le a e^{-\frac{\beta^*}{p}|k-j|} (1+|j|)^\delta,\ k,j \in \mathbb{Z}.$$

The proof of the inclusion $A \in \mathcal{GL}_q(\mathfrak{B})$, $q = \frac{p\delta}{p-1}$, is similar to that of Proposition 7.2. Similar to the previous example, we have in general

$$|A_{kk}| \to \infty,\ k \to \infty,$$

so that operator A is unbounded in any weighted l^p.

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