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Intrinsic ultracontractivity and uniform convergence to the Q-process for symmetric Markov processes*

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Abstract

In this paper, we explore the relationship between exponential convergence to a unique quasi-stationary distribution, the existence of an exponentially ergodic Q-process for a symmetric Markov process and ultracontractivity of its associated semigroup. In particular, it is shown that intrinsic ultracontractivity implies uniform convergence to the Q-process under suitable assumptions. Another goal is to specify some parameters related to the underlying quasi-stationary distribution and Q-process.

Keywords: quasi-stationary distribution; Q-process; intrinsic ultracontractivity; Dirichlet form.
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1 Introduction

Let E be a locally compact separable metric space and m a positive Radon measure on E with full support. Let also $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form on $L^2(E;m)$, $(-\mathcal{A}, D(\mathcal{A}))$ be its generator and $\{P_t := e^{-\mathcal{A}t}\}_{t\geq 0}$ be its associated semigroup. It is then well-known, cf. [9, Theorem 7.2.1], that $(\mathcal{E}, D(\mathcal{E}))$ is associated with a m-symmetric Hunt process $X = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in E_\Delta}, \zeta)$ with the state space $(E_\Delta, \mathcal{B}(E_\Delta))$ and lifetime $\zeta := \inf\{t \geq 0 : X_t = \Delta\}$. Here $\inf \emptyset = \infty$ by convention; $E_\Delta := E \cup \Delta$ is the one-point compactification of E, $\mathcal{B}(E_\Delta) := \mathcal{B}(E) \cup \{F \cup \Delta : F \in \mathcal{B}(E)\}$; Ω is the space of all right continuous maps ω from $[0, \infty]$ into E_Δ possessing the left limits such that $\omega(t) = \Delta$ for any $t \geq \zeta$ and $\omega(\infty) = \Delta$; X_t is defined by $X_t(\omega) = \omega(t), \omega \in \Omega, t \geq 0$; $\{\mathcal{F}_t\}_{t\geq 0}$ is the minimum completed admissible filtration that makes X is a strong Markov process and quasi-left-continuous; $\mathbb{P}_x$ is a probability measure on the space of trajectories emanating from the point x. It is noteworthy that this Hunt process is unique up to a properly exception set $\mathcal{N}$, see [9, Theorem 4.2.8]. Extend all functions f on E to E_Δ by setting $f(\Delta) = 0$ and define

$$\mathcal{P}_t f(x) := \mathbb{E}_x f(X_t) \mathbb{1}_{\{t<\zeta\}} = \mathbb{E}_x f(X_t), x \in E \setminus \mathcal{N}, f \in B_+(E),$$

where $\mathbb{E}_x$ denotes the expectation corresponding the probability measure $\mathbb{P}_x$, $B_+(E)$ is the set of non-negative Borel functions on E. Due to [9, Theorem 7.2.1], the relation

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between $\{P_t\}$ and $\{\mathcal{P}_t\}$ is given by the equation $P_t f = \mathcal{P}_t f$ m-a.e. for all $t > 0$ and f belonging to the set $B_b(E)$ of bounded Borel functions on E.

This paper is devoted to investigating the relationship between exponential convergence to a unique quasi-stationary distribution, the existence of an exponentially ergodic Q-process (defined as the process X conditioned to never exit E, see e.g., [4, Theorem 3.1] or [10, Remark 3.5]. More precisely, if the following limits are well defined,

$$Q_x(A) = \lim_{t\to\infty} \mathbb{P}_x(A|\zeta > t),\ A \in \mathcal{F}_s,\ s \geq 0,$$

and they determine a family of probability measures $\{Q_x\}_{x\in E}$ such that the collection $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{Q_x\}_{x\in E})$ is an E-valued homogeneous Markov process, then this Markov process is called a Q-process of X.) for X and ultracontractivity of $\{P_t\}$. Quasi-stationary distributions and Q-process capture the asymptotic behavior of Markov processes conditioned on long-term survival, which is a central topic in studying killed Markov processes. The reader can refer to [3, 4, 5, 10, 11, 13, 14, 15, 18, 19, 21] and references there for more details about quasi-stationary distributions and Q-process. Among them, [11, 15] developed the link between quasi-stationary distributions and ultracontractivity. The existence and exponential ergodicity of the Q-process for general almost surely absorbed Markov processes have been studied in [4, 5]. In [4], the authors show that the necessary and sufficient conditions for exponential convergence to a unique quasi-stationary distribution in the total variation norm also ensure the existence and exponential ergodicity of the Q-process. Conversely, they prove in [5] that, if the conditioned process converges uniformly to a conservative Markov process which is itself ergodic (this property is referred to as *uniform convergence to the Q-process*), then the original process admits a unique quasi-stationary distribution and the conditioned process converges toward it exponentially fast, uniformly in its initial distribution. However, most of the results in [4, 5] are generally discussed from the perspective of probability theory and it is not reflected in [5] that under which conditions on X uniform convergence to the Q-process will occur. From an analytical point of view and under suitable assumptions, this paper reveals that ultracontractivity of $\{P_t\}$ suggests exponential convergence to a unique quasi-stationary distribution in the total variation norm (but not necessarily uniformly in its initial distribution) and the existence of an exponentially ergodic Q-process for X. In particular, it is shown that intrinsic ultracontractivity implies uniform convergence to the Q-process. What is more, as we will see, the Q-process is nothing but the consequence of the original process being transformed by the bottom eigenfunction of $\mathcal{A}$. Another goal is to specify some parameters related to the underlying Q-process and quasi-stationary distributions. These parameters play an important role in numerical simulation and they are almost given in the forms of existence in the previous results.

The article is organized as follows. The next section introduces the preliminary mathematical framework. Section 3 is subject to building the main conclusions. Section 4 presents a concrete example to apply our conclusions.

2 Preliminaries

Let us start with the appropriate mathematical framework. The following notations will be used.

Notation 2.1. $\mathcal{P}(E)$ denotes the set of probability measures on $(E, \mathcal{B}(E))$; The norm in the usual Banach space $L^p(E; m)$ is written as $\|\cdot\|_{p,m}, 1 \leq p \leq \infty$; The scalar product and norm in $L^2(E; m)$ are written as $(\cdot,\cdot)_m, \|\cdot\|_m$ respectively; $(f, g)_m$ or $m(fg)$ will be used to denote the integration of fg with respect to measure m if $fg \in L^1(E; m)$; We occasionally write some symbols by selectively omitting their subscripts when there is no confusion, e.g., writing $(\cdot,\cdot)_m$ for $(\cdot,\cdot)$, writing $\|\cdot\|_{p,m}$ for $\|\cdot\|_p$.

Throughout the paper, let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form on $L^2(E;m)$, $(-\mathcal{A}, D(\mathcal{A}))$ be its generator and $\{P_t := e^{-\mathcal{A}t}\}$ be its associated semigroup. And we also suppose that P_t has a joint continuous integral kernel $p(t,x,y)$ on $(0,\infty)\times E\times E$ so that

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy)$$

for all $t>0$, $x\in E$ and $f\in L^2(E;m)$. By identifying with two Hunt processes that possess a common properly exceptional set outside which their transition functions coincide, the function $p(t,x,y)$ can be viewed as a transition density function (with respect to measure m) of the Hunt process $X=(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\{X_t\}_{t\geq 0},\{\mathbb{P}_x\}_{x\in E_\Delta})$ generated by $(\mathcal{E},D(\mathcal{E}))$. See e.g., [1, 2, 9]. In order to investigate simply the relationship between exponential convergence to a unique quasi-stationary distribution, the existence of an exponentially ergodic Q-process for X and ultracontractivity of $\{P_t\}$, the square root of $p(t,x,x)$ will be denoted by $b_t(x)$ and the following conditions will be imposed on $\{P_t\}$ and $p(t,x,y)$.

Hypothesis 2.1. We will introduce the following assumptions.

- **H1** $p(t,x,y)>0$ for all $t>0, x,y\in E$.
- **H2** The trace $\mathrm{Tr}(P_t)$ of P_t is finite for all $t>0$.
- **H3** $b_t\in L^1(E;m)\cap L^\infty(E;m)$ for all $t>0$.
- **H4** For any $t>0$, there exists a $c_t<\infty$ so that $p(t,x,y)\geq c_t b_t(x) b_t(y)$ for all $x,y\in E$.

In what follows, let's make some comments on Hypothesis 2.1 and then recall the definition of quasi-stationary distribution.

Remark 2.1. (i) **H1** suggests that $\{P_t\}$ is positivity improving and irreducible.

(ii) Under assumptions **H1** and **H2**, $\{P_t\}$ is said to be *ultracontractive* if and only if P_t is bounded from $L^2(E;m)$ to $L^\infty(E;m)$ for all $t>0$. While the later is equivalent to saying that there is a constant $a_t<\infty$ so that $0\leq p(t,x,y)\leq a_t$ for all $t>0$ and all m-almost everywhere $x,y\in E$, see [7, 8], [6, Lemma 2.1.2] or [12, Definition 2.1].

(iii) Utilizing Chapman–Kolmogorov equation, Cauchy-Schwarz inequality and the symmetry of $p(t,x,y)$, it is easy to find $p(t,x,y)\leq b_t(x)b_t(y), t>0, x,y\in E$. As a result, $b_t\in L^\infty(E;m)$ for all $t>0$ suggests that $\{P_t\}$ is ultracontractive.

(iv) The ultracontractivity of $\{P_t\}$ implies that it is also trace class if $m(E)<\infty$.

(v) Under assumptions **H1** and **H2**, $\{P_t\}$ is said to be *intrinsically ultracontractive* if and only if **H4** holds, see [7, Theorem 3.2] or [8, Theorem 2.1].

(vi) Quasi-stationarity and quasi-ergodicity of general Markov processes fulfilling the assumptions similar to **H1**, **H2** and **H3** have been studied in [22], see [22, (A1) and (A2)].

(vii) The assumptions **H1**, **H2**, **H4** imply **H3** (see Lemma 2.4 below).

Definition 2.2. *$\mu\in\mathcal{P}(E)$ is called a quasi-stationary distribution of X if*

$$\mathbb{P}_\mu(X_t\in\cdot|\zeta>t)=\mu(\cdot),\quad t\geq 0,$$

where $\mathbb{P}_\mu(\cdot)=\int_E \mathbb{P}_x(\cdot)\mu(dx)$ is the probability taken for X with an initial distribution μ.

The symmetry and Markovian property guarantee that P_t extends continuously as a map from $L^2(E;m)\cap L^p(E;m)$ to a positive one-parameter contraction semigroup on $L^p(E;m)$ for each $p\in[1,\infty]$, refer to [6, Theorem 1.4.1]. Before stating our main results, let us prepare two key lemmas. These two lemmas allow P_tf and $p(t,x,y)$ to be expanded by eigenfunctions of $\mathcal{A}$ for each $t>0$, $f\in L^p(E;m)$ and $p\in[1,\infty]$, which is useful for studying the long term behavior of P_t or X.

Lemma 2.3. *Assume **H1**, **H2** and **H3**.*

(i) *$\mathcal{A}$ has purely discrete spectrum with eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with $0 \le \lambda_1 < \lambda_2 \le \cdots \uparrow +\infty$ and there exists a complete orthonormal basis $\{\varphi_n\}$ of $L^2(E;m)$, where each λ_n is counted according to multiplicity, $\varphi_n \in D(\mathcal{A})$ is a continuous function on E so that $\mathcal{A}\varphi_n = \lambda_n \varphi_n, n \ge 1$ and φ_1 can be chosen to be strictly positive on E.*

(ii) *For all $\varepsilon > 0$, $t > 0$ and $x, y \in E$, $p(t,x,y)$ has the following representation*

$$p(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y),$$

where the series is absolutely and uniformly convergent on $[\varepsilon,\infty) \times E \times E$.

(iii) *b_t is a continuous function in $L^2(E;m)$ and $|\varphi_n(x)| \le e^{\lambda_n t/2} b_t(x)$ for all $n \ge 1$, $x \in E$ and $t > 0$.*

(iv) *For each $x \in E$, $b_t(x)$ and $e^{\lambda_1 t/2} b_t(x)$ are the analytic, logarithmically convex, monotonically decreasing functions of t.*

(v) *For each $\varepsilon > 0$, $p \in [1,\infty]$, $t > 0$, $f \in L^p(E;m)$ and $x \in E$, $P_t f(x)$ has the expansion*

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy) = \sum_{i=1}^{\infty} e^{-\lambda_i t}(\varphi_i, f)\varphi_i(x),$$

where the series is absolutely and uniformly convergent on $[\varepsilon,\infty) \times E$. Moreover, $P_t f$ has a bounded continuous version on E for each $t > 0$ and $f \in L^p(E;m)$.

(vi) *For each $p \in [1,\infty]$ and $f \in L^p(E;m)$, $\rho(e^{\lambda_1 t} P_t f)$ is uniformly convergent to $(\varphi_1, f)\|\varphi_1\|_{1,\rho} < \infty$ in $\rho \in \mathcal{P}(E)$ as $t \to \infty$.*

Proof. (i)–(iv) may be found in [7, Lemma 2.1 and Corollary 2.2] and [8, Section 2] except for the absolute and uniform convergence in (ii). While the latter can be verified by **H2**, **H3**, (iii) and (iv). Indeed, for a fixed $0 < \epsilon < 1$ and $\varepsilon > 0$, we see that for all $x, y \in E$ and $t > \varepsilon$,

$$\sum_{i=1}^{\infty} e^{-\lambda_i t}|\varphi_i(x)\varphi_i(y)| \le \sum_{i=1}^{\infty} e^{-(1-\epsilon)\lambda_i t}\|b_{\epsilon t}\|_{\infty}^2 \le \sum_{i=1}^{\infty} e^{-(1-\epsilon)\lambda_i \varepsilon}\|b_{\epsilon\varepsilon}\|_{\infty}^2 < \infty.$$

To confirm (v), we first prove that the first equation holds in $f \in L^p(E;m)$. By treating separately f^+ and f^-, we can assume $f \ge 0$. For the case of $f \in L^{\infty}(E;m)$, the first equation follows by taking a nonnegative sequence $\{f_n\} \subset L^2(E;m)$ such that $f_n(x) \uparrow f(x)$ for m-a.e $x \in E$ and using the dominated convergence or monotone convergence. Now for $f \in L^p(E;m)$ and let $f_n = \min\{f, n\}$. Then $f_n(x) \uparrow f(x)$ for m-a.e $x \in E$ and $\|P_t f_n - P_t f\|_p \le \|f_n - f\|_p \to 0$ as $n \to \infty$ for $t > 0$. The latter implies that there is a subsequence of $P_t f_n(x)$ that converges to $P_t f(x)$ for m-a.e $x \in E$. Employing the monotone convergence theorem again, the first equation holds in $f \in L^p(E;m)$. To verify the second equation, all we must show is that integration and summation in the second equation may be interchanged. Letting $p_n(t,x,y)$ be the nth partial sum of $p(t,x,y)$, then the second equation is completed by the dominated convergence since for fixed $t > 0, x \in E$ and $f \in L^p(E;m)$,

$$|p_n(t,x,\cdot)f(\cdot)| \le b_t(x) b_t(\cdot)|f(\cdot)| \in L^1(E;m), \quad n \ge 1.$$

While the desired convergence is an immediate consequence of the following inequality,

$$\sum_{i=1}^{\infty} e^{-\lambda_i t}|(\varphi_i, f)\varphi_i(x)| \le \sum_{i=1}^{\infty} e^{-\lambda_i t}\|\varphi_i\|_{p^*}\|f\|_p\|\varphi_i\|_{\infty}$$
$$\le \sum_{i=1}^{\infty} e^{-(1-\epsilon)\lambda_i t}\|b_{\epsilon t}\|_{p^*}\|b_{\epsilon t}\|_{\infty}\|f\|_p < \infty,\ 0 < \epsilon < 1,\ t > 0,$$

which together with the continuity of $\{\varphi_n\}$ finishes the second part, where $1/p+1/p^*=1$ and we have used the relation that for all $t>0$, $r\in[1,\infty]$, $i\geq 1$,

$$|\varphi_i|\leq e^{\lambda_i t/2}b_t\in L^1(E;m)\cap L^\infty(E;m)\subset L^r(E;m).$$

To build (vi), we first observe that $\{\varphi_n\}\subset L^p(E;m)$, $p\in[1,\infty]$ by **H3** and (iii). For a fixed $0<\epsilon\leq(\lambda_2-\lambda_1)/(2\lambda_2)$, then (vi) is true since **H2** and (iv) yield

$$\begin{aligned}|\rho(e^{\lambda_1 t}P_tf)-(\varphi_1,f)\rho(\varphi_1)|&=\big|\rho\big(\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)t}(\varphi_i,f)\varphi_i\big)\big|\\
\leq&\big|\rho\big(\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)t}\|\varphi_i\|_{p^*}\|f\|_p|\varphi_i|\big)\big|\\
\leq&\big|\rho\big(\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)t}\|e^{\lambda_i\epsilon t/2}b_{\epsilon t}\|_{p^*}\|f\|_pe^{\lambda_i\epsilon t/2}b_{\epsilon t}\big)\big|\\
\leq&\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1-\epsilon\lambda_i)t}\|b_{\epsilon t}\|_{p^*}\|f\|_p\|b_{\epsilon t}\|_\infty\to 0 \text{ as } t\to\infty,\ \rho\in\mathcal{P}(E).\end{aligned}$$

□

Lemma 2.4. *Assume **H1**, **H2** and **H4**.*

(i) *$\varphi_1\in L^1(E;m)\cap L^\infty(E;m)$ and there exists a decreasing functions $\tilde{c}_t<\infty$ such that $|\varphi_n(x)|\leq\tilde{c}_te^{(\lambda_n-\lambda_1)t}\varphi_1(x)$ for all $n\geq1$, $x\in E$ and $t>0$.*

(ii) *For any $\varepsilon>0$ there is a $t(\varepsilon)$ such that for all $t>t(\varepsilon)$,*

$$1-\varepsilon\leq q(t,x,y):=\frac{e^{\lambda_1 t}p(t,x,y)}{\varphi_1(x)\varphi_1(y)}\leq1+\varepsilon,\quad x,y\in E.$$

It turns out that for all $t>t(\varepsilon)$,

$$(1-\varepsilon)m(\varphi_1)\varphi_1\leq e^{\lambda_1 t}P_t\mathbb{1}\leq(1+\varepsilon)m(\varphi_1)\varphi_1.$$

Proof. (i) As mentioned in Remark 2.1, **H4** is equivalent to saying that $\{P_t\}$ is intrinsically ultracontractive. By [7, Theorem 3.2] or [8, Theorem 2.1], there exist two t-dependent constants c_1 and c_2 so that

$$c_1\varphi_1(x)\varphi_1(y)\leq p(t,x,y)\leq c_2\varphi_1(x)\varphi_1(y),\quad x,y\in E. \tag{2.1}$$

Now recall that $p(t,x,y)$ is a m-transition density function by assumption and φ_1 is a strictly positive function on E by Lemma 2.3(i). Integrating the left-hand inequality of (2.1) with respect to y over E and over a compact set $K\subset E$ respectively gives $\varphi_1\in L^1(E;m)$ and $\varphi_1\in L^\infty(E;m)$. To establish the second assertion, we set

$$\tilde{\mathcal{A}}=U^{-1}(\mathcal{A}-\lambda_1)U \text{ and } P_t^{\varphi_1}=e^{-\tilde{\mathcal{A}}t},\ t\geq0,$$

where U is a unitary map from $L^2(E;\varphi_1^2\cdot m)$ to $L^2(E;m)$ defined by $Uf(x)=\varphi_1(x)f(x)$. Then $P_t^{\varphi_1}$ is bounded from $L^2(E;\varphi_1^2\cdot m)$ to $L^\infty(E;\varphi_1^2\cdot m)$ for all $t>0$ by **H4**, and the relation between $P_t^{\varphi_1}$ and P_t is given by

$$P_t^{\varphi_1}f=e^{\lambda_1 t}\varphi_1^{-1}P_t(\varphi_1 f),\ t\geq0,\ f\in L^2(E;\varphi_1^2\cdot m). \tag{2.2}$$

Therefore (i) is completed by taking $\tilde{c}_t=\|P_t^{\varphi_1}\|_{\infty,2}$ (the operator norm of $P_t^{\varphi_1}$ from $L^2(E;\varphi_1^2\cdot m)$ to $L^\infty(E;\varphi_1^2\cdot m)$) and substituting f with φ_n/φ_1 into (2.2), as long as one notice that φ_n is the eigenfunction of P_t corresponding to the eigenvalue $e^{-\lambda_n t}$, $n\geq1$.

(ii) It is a joint result of (i), Lemma 2.3(ii) and **H2**. In fact, for all $\varepsilon\in(0,\frac12)$ we have

$$|q(t,x,y)-1|\leq(\tilde{c}_{\varepsilon t})^2\sum_{i=2}^\infty e^{-(1-2\varepsilon)(\lambda_i-\lambda_1)t},\quad x,y\in E,\quad t>0.\ \square$$

3 Main results

By virtue of Lemma 2.3 and Lemma 2.4, we summarise the relationship between exponential convergence to a unique quasi-stationary distribution, the existence of an exponentially ergodic Q-process for X and ultracontractivity of $\{P_t\}$ in this section.

Theorem 3.1. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form and X be its associated Hunt process presented in Section 2.*

(i) *Assume **H1**, **H2** and **H3**. Then X possesses a unique quasi-stationary distribution $\mu := \varphi_1 \cdot m/\|\varphi_1\|_1$ which attracts exponentially all $\rho \in \mathcal{P}(E)$ (but not necessarily uniformly in $\rho \in \mathcal{P}(E)$), i.e.,*

$$\begin{aligned}&\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)t}\|\mathbb{P}_\rho(X_t \in \cdot|\zeta > t) - \mu\|_{TV}\\ &= \sup_{F\in\mathcal{B}(E)} \left|\sum_{i=2}^{\kappa+1} \frac{[(\varphi_i, \mathbb{1}_F) - (\varphi_i, \mathbb{1})\mu(F)]\rho(\varphi_i)}{\|\varphi_1\|_1\rho(\varphi_1)}\right|,\end{aligned}$$

where κ is the multiplicity of the second eigenvalue λ_2.

(ii) *Assume **H1**, **H2** and **H4**. The attraction in (i) is uniform in $\rho \in \mathcal{P}(E)$.*

Proof. (i) Let $\mathbb{1} = \mathbb{1}_E$. Due to Lemma 2.3(v), we find that for all $F \in \mathcal{B}(E)$ and $\rho \in \mathcal{P}(E)$,

$$\begin{aligned}\mathbb{P}_\rho(X_t \in F|\zeta > t) - \mu(F) &= \frac{\rho(P_t\mathbb{1}_F) - \rho(\mu(F)P_t\mathbb{1})}{\rho(P_t\mathbb{1})}\\ &= \frac{\rho\big(\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)t}[(\varphi_i, \mathbb{1}_F) - (\varphi_i, \mathbb{1})\mu(F)]\varphi_i\big)}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}.\end{aligned}$$

Thus if κ is the multiplicity of λ_2, multiplying both sides by $e^{(\lambda_2-\lambda_1)t}$ deduces

$$\begin{aligned}&e^{(\lambda_2-\lambda_1)t}\big(\mathbb{P}_\rho(X_t \in F|\zeta > t) - \mu(F)\big)\\ &- \frac{\rho\big(\sum_{i=2}^{\kappa+1}[(\varphi_i, \mathbb{1}_F) - (\varphi_i, \mathbb{1})\mu(F)]\varphi_i\big)}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ &= \frac{\rho\big(\sum_{i=\kappa+2}^\infty e^{-(\lambda_i-\lambda_2)t}[(\varphi_i, \mathbb{1}_F) - (\varphi_i, \mathbb{1})\mu(F)]\varphi_i\big)}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}.\end{aligned} \tag{3.1}$$

Notice that Lemma 2.3(vi) and Lemma 2.3(iii) respectively yield

$$\begin{aligned}&\rho(e^{\lambda_1 t}P_t\mathbb{1}) \to m(\varphi_1)\rho(\varphi_1) \text{ as } t \to \infty,\\ &\rho\Big(\sum_{i=\kappa+2}^\infty e^{-(\lambda_i-\lambda_2)t}|(\varphi_i, \mathbb{1}_F) - (\varphi_i, \mathbb{1})\mu(F)||\varphi_i|\Big)\\ &\le \sum_{i=\kappa+2}^\infty 2e^{-(\lambda_i-\lambda_2)t}m(|\varphi_i|)\rho(|\varphi_i|)\\ &\le \sum_{i=\kappa+2}^\infty 2e^{-(\lambda_i-\lambda_2-\epsilon\lambda_i)t}\|b_{\epsilon t}\|_1\|b_{\epsilon t}\|_\infty, \ \ \forall\, 0 < \epsilon \le (\lambda_{\kappa+2} - \lambda_2)/(2\lambda_{\kappa+2}).\end{aligned}$$

Recall that b_t is decreasing by Lemma 2.3(iv) and $\{P_t\}$ is of trace class by **H2**. Therefore the desired results are valid by taking successively the absolute value, the supremum with respect to $F \in \mathcal{B}(E)$ and then the limit as $t \to \infty$ on both sides of (3.1).

(ii) As a matter of fact, for any $0<\varepsilon<1$ and $0<\epsilon<(\lambda_{\kappa+2}-\lambda_2)/(2\lambda_{\kappa+2})$, Lemma 2.4 suggests that there is a $t(\varepsilon)$ such that for all $t>t(\varepsilon)$, $F\in\mathcal{B}(E)$ and $\rho\in\mathcal{P}(E)$,

$$\begin{aligned}&\frac{\rho\big(\sum_{i=\kappa+2}^{\infty}e^{-(\lambda_i-\lambda_2)t}|(\varphi_i,\mathbb{1}_F)-(\varphi_i,\mathbb{1})\mu(F)||\varphi_i|\big)}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}\\ \le&\frac{\sum_{i=\kappa+2}^{\infty}2e^{-(\lambda_i-\lambda_2)t}m(|\varphi_i|)\rho(|\varphi_i|)}{(1-\varepsilon)m(\varphi_1)\rho(\varphi_1)}\\ \le&\sum_{i=\kappa+2}^{\infty}2(\tilde{c}_{\epsilon t})^2e^{-[\lambda_i-\lambda_2-2\epsilon(\lambda_i-\lambda_1)]t}/(1-\varepsilon).\end{aligned}$$

Recall that $\tilde{c}_t$ is decreasing by Lemma 2.4 and $\{P_t\}$ is of trace class by **H2**, the last series of the above inequality goes to 0 as $t\to\infty$. □

Under the assumptions **H1**, **H2** and **H3**, it is easy to check by [9, Section 6.3] that the Doob's φ_1-transform $P_t^{\varphi_1}$ of P_t defined by (2.2) is a $\varphi_1^2\cdot m$-symmetric Markov semigroup on $L^2(E;\varphi_1^2\cdot m)$ and generates a conservative $\varphi_1^2\cdot m$-symmetric right process $X^{\varphi_1}=(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\{X_t\}_{t\ge0},\{\mathbb{P}_x^{\varphi_1}\}_{x\in E},\zeta)$ on E, where $\mathbb{P}_x^{\varphi_1}$ satisfies for all $s\ge0$ and $F\in\mathcal{F}_s$,

$$\mathbb{P}_x^{\varphi_1}(F)=\mathbb{E}_x^{\varphi_1}(\mathbb{1}_F)=e^{\lambda_1 s}\mathbb{E}_x[\mathbb{1}_F\varphi_1(X_s)]/\varphi_1(x).$$

In [9, Section 6.3], X^{φ_1} is also said to be the transformed process by the supermartingale multiplicative functional $L_t:=e^{\lambda_1 t}\varphi_1(X_t)/\varphi_1(X_0)$. The next theorem shows that the exponentially ergodic Q-process of X exists and coincides with X^{φ_1}. Moreover, more properties of X^{φ_1} are obtained.

Theorem 3.2. *Let $(\mathcal{E},D(\mathcal{E}))$ be a regular Dirichlet form and X be its associated Hunt process presented in Section 2. Assume **H1**, **H2** and **H3**.*

(i) *The Q-process of X not only exists but also coincides with X^{φ_1}. That is,*

$$\forall s\ge0,\forall x\in E:\quad\lim_{t\to\infty}\sup_{F\in\mathcal{F}_s}|\mathbb{P}_x(F|\zeta>t)-\mathbb{P}_x^{\varphi_1}(F)|=0.$$

Besides, X^{φ_1} admits a $\varphi_1^2\cdot m$-symmetric transition function $P_t^{\varphi_1}$ and a $\varphi_1^2\cdot m$-symmetric transition density function

$$q(t,x,y)=\frac{e^{\lambda_1 t}p(t,x,y)}{\varphi_1(x)\varphi_1(y)}.$$

(ii) *The Q-process of X generates the Dirichlet form $(\mathcal{E}^{\varphi_1},D(\mathcal{E}^{\varphi_1}))$,*

$$\begin{cases}\mathcal{E}^{\varphi_1}(u,v)=\mathcal{E}(\varphi_1u,\varphi_1v)-\lambda_1(\varphi_1u,\varphi_1v)\\ D(\mathcal{E}^{\varphi_1})=\{u\in L^2(E;\varphi_1^2\cdot m):u\varphi_1\in D(\mathcal{E})\}.\end{cases}$$

(iii) *X^{φ_1} possesses a stationary distribution $\nu:=\varphi_1^2\cdot m$ which attracts exponentially those probability measures ρ which fulfill $\|\varphi_n/\varphi_1\|_{1,\rho}\le\gamma_\rho(t)e^{\lambda_n t}$, $n\ge1$, $t\in(0,\infty)$ for some decreasing finite function γ_ρ, i.e.,*

$$\lim_{t\to\infty}e^{(\lambda_2-\lambda_1)t}\|\mathbb{P}_\rho^{\varphi_1}(X_t\in\cdot)-\nu\|_{TV}=\sup_{F\in\mathcal{B}(E)}\left|\sum_{i=2}^{\kappa+1}(\varphi_1\mathbb{1}_F,\varphi_i)\rho(\varphi_i\varphi_1^{-1})\right|.$$

For instance, if ρ has compact support in E or ρ has a density k with respect to m so that $k/\varphi_1\in L^p(E;m)$ for some $1\le p\le\infty$, then ρ is attracted to ν exponentially in the above sense. Accordingly, X^{φ_1} is exponentially ergodic and ν is the unique stationary distribution for it.

Proof. (i) Since X^{φ_1} is the transformed process of X by $\{L_t\}_{t\geq 0}$, the second part of item (i) can be derived directly from the definition of X^{φ_1}. It remains to prove the first part. On the one hand, it is known from [9, Section 6.3] that for all $s \geq 0$ and $F \in \mathcal{F}_s$,

$$\mathbb{P}_x^{\varphi_1}(F) = \mathbb{E}_x^{\varphi_1}(\mathbb{1}_F) = e^{\lambda_1 s}\mathbb{E}_x[\mathbb{1}_F \varphi_1(X_s)]/\varphi_1(x).$$

On the other hand, for all $s \geq 0$, $F \in \mathcal{F}_s$ and $t \geq s$, the Markov property yields that

$$\mathbb{P}_x(F \cap \{\zeta > t\}) = \mathbb{E}_x[\mathbb{1}_F \mathbb{P}_{X_s}(t - s < \zeta)].$$

Consequently, for all $s \geq 0$, $F \in \mathcal{F}_s$ and $t \geq s$, we obtain

$$\begin{aligned} &\mathbb{P}_x(F|\zeta > t) - \mathbb{P}_x^{\varphi_1}(F) \\ =&\frac{e^{\lambda_1 s}\mathbb{E}_x\{\mathbb{1}_F[e^{\lambda_1(t-s)}\mathbb{P}_{X_s}(t-s<\zeta) - m(\varphi_1)\varphi_1(X_s)]\}}{e^{\lambda_1 t}P_t\mathbb{1}(x)} \\ +&\frac{[m(\varphi_1)\varphi_1(x) - e^{\lambda_1 t}P_t\mathbb{1}(x)]e^{\lambda_1 s}\mathbb{E}_x\{\mathbb{1}_F\varphi_1(X_s)\}}{e^{\lambda_1 t}P_t\mathbb{1}(x)\varphi_1(x)}. \end{aligned} \tag{3.2}$$

Now recall by Lemma 2.3(vi) that

$$\begin{aligned} &\lim_{t\to\infty}\sup_{x\in E}|e^{\lambda_1 t}\mathbb{P}_x(t<\zeta) - m(\varphi_1)\varphi_1(x)| \\ =&\lim_{t\to\infty}\sup_{x\in E}|e^{\lambda_1 t}P_t\mathbb{1}(x) - m(\varphi_1)\varphi_1(x)| = 0. \end{aligned} \tag{3.3}$$

Therefore, the first part follows by (3.2) (3.3) and the dominated convergence.

(ii) Notice that for all $u \in L^2(E; \varphi_1^2 \cdot m)$ we get

$$\begin{aligned} (u - P_t^{\varphi_1}u, u)_{\varphi_1^2\cdot m} &= (u\varphi_1 - e^{\lambda_1 t}P_t(u\varphi_1), u\varphi_1)_m \\ &= e^{\lambda_1 t}(u\varphi_1 - P_t(u\varphi_1), u\varphi_1)_m + (1 - e^{\lambda_1 t})(u\varphi_1, u\varphi_1)_m. \end{aligned}$$

Hence the result.

(iii) For all $F \in \mathcal{B}(E)$ and $\rho \in \mathcal{P}(E)$ fulfilling $\|\varphi_n/\varphi_1\|_{1,\rho} \leq \gamma_\rho(t)e^{\lambda_n t}$, $n \geq 1$, $t \in (0,\infty)$ for some monotone decreasing finite function γ_ρ, we find

$$\begin{aligned} &\mathbb{P}_\rho^{\varphi_1}(X_t \in F) - \nu(F) = \rho\big(e^{\lambda_1 t}\varphi_1^{-1}P_t(\varphi_1\mathbb{1}_F)\big) - \nu(F) \\ =&\rho\big(\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)t}(\mathbb{1}_F\varphi_1, \varphi_i)\varphi_i/\varphi_1\big). \end{aligned}$$

Multiplying both sides by $e^{(\lambda_2-\lambda_1)t}$ deduces

$$\begin{aligned} &\left|e^{(\lambda_2-\lambda_1)t}\big(\mathbb{P}_\rho^{\varphi_1}(X_t\in F) - \nu(F)\big) - \sum_{i=2}^{\kappa+1}(\varphi_1\mathbb{1}_F, \varphi_i)\rho(\varphi_i\varphi_1^{-1})\right| \\ =&\left|\rho\big(\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)t}(\mathbb{1}_F\varphi_1,\varphi_i)\varphi_i/\varphi_1\big)\right| \\ \leq&\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)t}\|\varphi_1\|\|\varphi_i\|\|\varphi_i/\varphi_1\|_{1,\rho} \\ \leq&\sum_{i=\kappa+2}^{\infty} e^{-[(1-\epsilon)\lambda_i-\lambda_2]t}\gamma_\rho(\epsilon t), \quad \forall 0<\epsilon<(\lambda_{\kappa+2}-\lambda_2)/(2\lambda_{\kappa+2}), \end{aligned}$$

which together with the procedures of proving Theorem 3.1(i) concludes item (iii). □

From Theorem 3.1(ii), it is not difficult to find the relationship between intrinsic ultracontractivity and exponential convergence to a unique quasi-stationary distribution. In what follows, we indicate the relationship between intrinsic ultracontractivity and uniform convergence to the Q-process.

Theorem 3.3. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form and X be its associated Hunt process presented in Section 2. Assume **H1**, **H2** and **H4**.*

(i) *The stationary distribution ν of the Q-process X^{φ_1} of X attracts exponentially and uniformly all $\rho \in \mathcal{P}(E)$ in the sense of Theorem 3.2(iii).*

(ii) *The following limits are valid,*

$$\forall s > 0 : \lim_{t\to\infty} \sup_{x\in E} \|\mathbb{P}_x^{\varphi_1}(X_s \in \cdot) - \mathbb{P}_x(X_s \in \cdot|\zeta > t)\|_{TV} = 0,$$
$$\lim_{t\to\infty} \sup_{\rho_1,\rho_2\in\mathcal{P}(E)} \|\mathbb{P}_{\rho_1}^{\varphi_1}(X_t \in \cdot) - \mathbb{P}_{\rho_2}^{\varphi_1}(X_t \in \cdot)\|_{TV} = 0. \tag{3.4}$$

Proof. (i) It is clear by Lemma 2.4(i) that $\|\varphi_n/\varphi_1\|_{1,\rho} \le \tilde{c}_t e^{(\lambda_n-\lambda_1)t}$ for all $n \ge 1$, $t \in (0,\infty)$ and $\rho \in \mathcal{P}(E)$. The rest of the argument is similar to the proof of Theorem 3.2(iii).

(ii) Notice that $\{X_s \in F\} \in \mathcal{F}_s$ and

$$\mathbb{P}_x(s < \zeta) = P_s\mathbb{1}(x), \quad \mathbb{E}_x\{\varphi_1(X_s)\} = P_s\varphi_1(x) = e^{-\lambda_1 s}\varphi_1(x),$$

for all $F \in \mathcal{B}(E)$, $s > 0$ and $x \in E$. In light of (3.2), Lemma 2.3(v), Lemma 2.4 and Fubini's theorem, we get that for any $\varepsilon \in (0,1)$ and $\epsilon \in (0,\frac{1}{2})$ there is a $t(\varepsilon)$ such that for all $t > \max\{s, t(\varepsilon)\}$, $F \in \mathcal{B}(E)$ and $x \in E$,

$$\begin{aligned}
&|\mathbb{P}_x(X_s \in F|\zeta > t) - \mathbb{P}_x^{\varphi_1}(X_s \in F)| \\
\le& \frac{e^{\lambda_1 s}\mathbb{E}_x|e^{\lambda_1(t-s)}P_{t-s}\mathbb{1}(X_s) - m(\varphi_1)\varphi_1(X_s)|}{e^{\lambda_1 t}P_t\mathbb{1}(x)} \\
+& \frac{|m(\varphi_1)\varphi_1(x) - e^{\lambda_1 t}P_t\mathbb{1}(x)|e^{\lambda_1 s}\mathbb{E}_x\{\varphi_1(X_s)\}}{e^{\lambda_1 t}P_t\mathbb{1}(x)\varphi_1(x)} \\
\le& \frac{e^{\lambda_1 s}\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)(t-s)}m(|\varphi_i|)\mathbb{E}_x|\varphi_i(X_s)|}{e^{\lambda_1 t}P_t\mathbb{1}(x)} \\
+& \frac{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)t}m(|\varphi_i|)|\varphi_i(x)|}{e^{\lambda_1 t}P_t\mathbb{1}(x)} \\
\le& \frac{e^{\lambda_1 s}\sum_{i=2}^\infty e^{-(1-\epsilon)(\lambda_i-\lambda_1)(t-s)}m(|\varphi_i|)\tilde{c}_{\epsilon(t-s)}\mathbb{E}_x\varphi_1(X_s)}{(1-\varepsilon)m(\varphi_1)\varphi_1(x)} \\
+& \frac{\sum_{i=2}^\infty e^{-(\lambda_i-\lambda_1)t}m(|\varphi_i|)|\varphi_i(x)|}{(1-\varepsilon)m(\varphi_1)\varphi_1(x)} \\
\le& \sum_{i=2}^\infty (\tilde{c}_{\epsilon(t-s)})^2 e^{-(1-2\epsilon)(\lambda_i-\lambda_1)(t-s)}/(1-\varepsilon) + \sum_{i=2}^\infty (\tilde{c}_{\epsilon t})^2 e^{-(1-2\epsilon)(\lambda_i-\lambda_1)t}/(1-\varepsilon),
\end{aligned}$$

which together with the fact that $\tilde{c}_t$ is decreasing by Lemma 2.4(i) and $\{P_t\}$ is of trace class by **H2** finishes the first limit. The second limit is an immediate consequence of item (i) and following inequality

$$\begin{aligned}
&\|\mathbb{P}_{\rho_1}^{\varphi_1}(X_t \in \cdot) - \mathbb{P}_{\rho_2}^{\varphi_1}(X_t \in \cdot)\|_{TV} \\
\le& \|\mathbb{P}_{\rho_1}^{\varphi_1}(X_t \in \cdot) - \nu\|_{TV} + \|\mathbb{P}_{\rho_2}^{\varphi_1}(X_t \in \cdot) - \nu\|_{TV}.
\end{aligned}$$
□

Remark 3.4. We remark that the strong ergodicity (3.4) of the Q-process implies its exponential ergodicity, see [5, Theorem 2.4] or [17, Theorem 16.0.2]. Hence both convergences in Theorem 3.3(ii) hold exponentially. In fact, this result can be seen directly in the proof of Theorem 3.3(ii). In addition, the exponential convergence rates can be determined by the gap $\lambda_2 - \lambda_1$ between the second eigenvalue and the first eigenvalue. That is, for any $\epsilon > 0$, there exist constants $C_\epsilon, C > 0$ such that for all $t, s \geq 0$,

$$\sup_{x\in E} \|\mathbb{P}_x^{\varphi_1}(X_s \in \cdot) - \mathbb{P}_x(X_s \in \cdot|\zeta > t)\|_{TV} \leq C_\epsilon e^{-(1-\epsilon)(\lambda_2-\lambda_1)(t-s)},$$
$$\sup_{\rho_1,\rho_2\in\mathcal{P}(E)} \|\mathbb{P}_{\rho_1}^{\varphi_1}(X_t \in \cdot) - \mathbb{P}_{\rho_2}^{\varphi_1}(X_t \in \cdot)\|_{TV} \leq Ce^{-(\lambda_2-\lambda_1)t}.$$

As a byproduct of Theorem 3.3(ii), we arrive at the following corollary and the proof of it may be found in [5, Corollary 2.2, Corollary 2.3 and Theorem 2.4]. In particular, this corollary means that X is conditionally ergodic.

Corollary 3.5. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form and X be its associated Hunt process presented in Section 2. Assume **H1**, **H2** and **H4**.*

(i) *There exist three positive constants c_3, γ, γ' such that, for all $T > 0$, all probability measure μ_T on $[0, T]$ and all bounded measurable functions $f : E \to \mathbb{R}$,*

$$\left|\mathbb{E}_x\Big(\int_0^T f(X_t)\mu_T(dt)|T < \zeta\Big) - \nu(f)\right|$$
$$\leq c_3\|f\|_\infty \int_0^T \big(e^{-\gamma' t} + e^{-\gamma(T-t)}\big)\mu_T(dt).$$

(ii) *There exists a positive constant c_4 such that, for all $T > 0$ and all bounded measurable functions $f : E \to \mathbb{R}$,*

$$\left|\mathbb{E}_x\Big(\frac{1}{T}\int_0^T f(X_t)dt|T < \zeta\Big) - \nu(f)\right| \leq c_4\frac{\|f\|_\infty}{T}.$$

4 Example

In this section, we present a typical example which will meet respectively Theorems 3.1-3.3 under various cases.

Example 4.1. Let $E = (r_1, r_2)$, $-\infty < r_1 < r_2 < \infty$ be a one-dimensional interval and λ be the Lebesgue measure. Let also m and k be positive Radon measures on E with $\text{supp}[m] = E$ fulfilling

$$m(r_1, r_2) + k(r_1, r_2) < \infty. \tag{4.1}$$

Consider the following functional spaces, functional and operator:

$$\mathcal{F}^R = \{u \in L^2(E; m) \cap L^2(E; k) : u \text{ is absolutely continuous and } (u', u')_\lambda < \infty\},$$
$$D(\mathcal{E}) = \{u \in \mathcal{F}^R : u(r_i) := \lim_{x\to r_i} u(x) = 0, \quad i = 1, 2\},$$
$$D(\mathcal{A}) = \{u \in L^2(E; m) \cap L^2(E; k) : u \text{ is absolutely continuous, a version of } du/dx$$
$$\text{is of bounded variation on each finite subinterval of } E,\ (1/2)(d(du/dx) - udk)$$
$$\text{is absolutely continuous with respect to } m \text{ and the derivative belongs to } L^2(E; m),$$
$$u(r_1) = u(r_2) = 0\},$$
$$\mathcal{E}(u, v) = \frac{1}{2}(u', v')_\lambda + (u, v)_k, \quad \forall u, v \in D(\mathcal{E}),$$
$$\mathcal{A}u = -\frac{\frac{1}{2}d\frac{du}{dx} - udk}{dm}, \quad \forall u \in D(\mathcal{A}).$$

(i) $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form on $L^2(E;m)$, $(-\mathcal{A}, D(\mathcal{A}))$ and $\{P_t := e^{-\mathcal{A}t}\}$ are its associated generator and semigroup respectively.

(ii) $(\mathcal{E}, D(\mathcal{E}))$ satisfies the Sobolev type inequality: for all $1 \le p \le \infty$ we have

$$\|u\|_{p,m}^2 \le 2(r_2 - r_1) m(r_1, r_2)^{\frac{2}{p}} \mathcal{E}(u,u), \;\; u \in D(\mathcal{E}).$$

(iii) $\{P_t\}$ is a ultracontractive and trace class semigroup on $L^2(E;m)$ and admits a strictly positive, joint continuous integral kernel $p(t,x,y)$ on $(0,\infty) \times E \times E$.

(iv) $\{P_t\}$ is a intrinsically ultracontractive and trace class semigroup on $L^2(E;m)$ and admits a strictly positive, joint continuous integral kernel $p(t,x,y)$ on $(0,\infty)\times E \times E$ provided that k is a zero measure or supp$[k]$ is compact in E.

To see Example 4.1(i)-(iv), we first note that inequality (4.1) is equivalent to saying that the boundaries r_1, r_2 of the one-dimensional generalized diffusion operator $\mathcal{A}$ are regular. Then items (i)(iv) are proved by [9, Example 1.2.2 and Example 1.3.1] and [20, Theorem 2.11 or Example 5.1] respectively. It remains to show (ii) since the fact that $\{P_t\}$ is a ultracontractive and trace class semigroup can be guaranteed by (ii) and [6, Theorem 2.1.4, Theorem 2.4.2] and the rest assertions may be found in [20] or [16]. To see (ii), we observe that

$$|u(x)|^2 = \left|\int_{r_1}^{x} u'(z)dz\right|^2 \le (r_2 - r_1)(u', u')_\lambda, \;\; u \in D(\mathcal{E}), \;\; x \in E,$$

which suggests that $D(\mathcal{E})$ is continuously imbedded in $L^\infty(E;\lambda)$ and

$$\begin{aligned} &\|u\|_{\infty,\lambda}^2 \le (r_2 - r_1)(u', u')_\lambda, \\ &\|u\|_{p,m}^2 \le (r_2 - r_1) m(r_1, r_2)^{\frac{2}{p}} (u', u')_\lambda, \;\; u \in D(\mathcal{E}). \end{aligned}$$

It's well known, cf. [9, Theorem 7.2.2], [20] or [16], that $(\mathcal{E}, D(\mathcal{E}))$ shall give rise to a generalized diffusion process X on E. Besides, with the help of Remark 2.1, Example 4.1 (iii) and (iv) correspond to assumptions **H1**, **H2**, **H3** and **H1**, **H2**, **H4** respectively. Now we are in a position to apply our results.

Theorem 4.2. *Let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form given by Example 4.1 and X be its associated process, then Theorem 3.1(i) and Theorem 3.2 can be applied to X. Provided further that k is a zero measure or supp$[k]$ is compact in E, then Theorem 3.1(ii), Theorem 3.3 and Corollary 3.5 can be applied to X.*

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