

Hölder continuity of the convex minorant of a Lévy process

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Abstract

We characterise the Hölder continuity of the convex minorant of most Lévy processes. The proof is based on a novel connection between the path properties of the Lévy process at zero and the boundedness of the set of r -slopes of the convex minorant.

Keywords: Lévy process; convex minorant; Hölder continuity.

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1 Introduction and main results

The Hölder continuity¹ of continuous random functions is a classical topic, analysed extensively for Brownian motion and related processes, see e.g. [12] for fractional Brownian motion. Typically, such results make use of Kolmogorov’s extension theorem (see [6, Thm 3.23]). The convex minorant of a Lévy process is a continuous random function, which may but need not be smooth [2, 1], motivating the question of its Hölder continuity. However, as the increments of the convex minorant (and their moments) are not tractable and its local behaviour varies greatly with the characteristics of the Lévy process [2], Kolmogorov’s extension theorem is not the right tool. In this paper we establish sufficient and necessary conditions for the Hölder continuity of the convex minorant of a Lévy process, using a generalisation of the 0–1 law in [2, Thm 3.1], the characterisation of small time behaviour of the Lévy path in [4, Thm 2.1] and an elementary lemma by Khinchine (see Lemma 2.3 below). We prove for example that, in the absence of a Brownian component, the critical Hölder exponent is the reciprocal of the Blumenthal–Gettoor index for most infinite variation Lévy processes (complete results are given in Table 1 below). A short YouTube [3] video describes the results and the structure of our proofs.

The convex minorant C of the path of the Lévy process X on $[0, T]$ is the pointwise largest convex function dominated by X . By [2, Prop. 1.3], C is Lipschitz (i.e., 1-Hölder) continuous if and only if X is of finite variation. In what follows we assume that X is of infinite variation. Then C is not Lipschitz on $[0, T]$ but, by convexity, is Lipschitz on every

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¹Given $r \in (0, 1]$ the function $f : [0, T] \rightarrow \mathbb{R}$ is r -Hölder continuous by definition if its Hölder constant $\sup_{0 \leq x < y \leq T} |f(y) - f(x)| / (y - x)^r$ is finite.

interval $[\varepsilon, T - \varepsilon]$, $\varepsilon > 0$, with Lipschitz constant $\max\{|C'_\varepsilon|, C'_{T-\varepsilon}\}$ given in terms of the right-derivative C' of C . Note that the rate at which $\max\{|C'_\varepsilon|, C'_{T-\varepsilon}\}$ tends to infinity as $\varepsilon \downarrow 0$, analysed in [1], is insufficient to characterise the r -Hölder continuity of C on $[0, T]$ for $r \in (0, 1)$, since C'_ε may fluctuate between functions that are not asymptotically equivalent as $\varepsilon \downarrow 0$, see [1, Rem. 2.14(a)] (analogous behaviour is observed for $C'_{T-\varepsilon}$).

Let σ and ν be the Gaussian coefficient and Lévy measure of X , respectively (see [10] for background on Lévy processes). Let β denote the Blumenthal–Gettoor index of X given by

$$\beta := \inf \{p > 0 : J_p < \infty\}, \quad \text{where } J_p := \int_{(-1,1)} |x|^p \nu(dx). \tag{1.1}$$

Note that, when $\sigma = 0$, $\beta \in [1, 2]$ since X is of infinite variation. (If $\sigma > 0$, we may have $\beta \in [0, 1)$.) Since $J_2 < \infty$, we have $\bar{\sigma}^2(u) := \int_{(-u,u)} x^2 \nu(dx) < \infty$ for $u > 0$ and $\bar{\sigma}^2(u) \rightarrow 0$ as $u \rightarrow 0$ (but the convergence may be arbitrarily slow). Define also

$$\lambda_2 := \inf \left\{ \lambda > 0 : \int_0^1 e^{-\lambda^2/(2\bar{\sigma}^2(u))} \frac{du}{u} < \infty \right\} \quad \& \quad I_2 := \int_0^1 \mathbb{E}[\min\{(X_t/\sqrt{t})^2, 1\}] \frac{dt}{t}. \tag{1.2}$$

Theorem 1.1, summarised in Table 1 below, characterises, for most Lévy process of infinite variation, the set of $r \in (0, 1)$ for which C is r -Hölder continuous. The characterisation is given in terms of J_β , except for $\beta = 2$, where λ_2 and I_2 are required.

Table 1: The critical level $r \in (0, 1)$ for r -Hölder continuity of the convex minorant is $1/2$ in the presence of a Brownian component and $1/\beta$ in its absence, where $\beta \in [1, 2]$ is the Blumenthal–Gettoor index.

Infinite variation Lévy process X		$r \in (0, 1)$	Is C r -Hölder continuous?	
$\sigma^2 > 0$		$r < 1/2$	Yes	
		$r \geq 1/2$	No	
$\sigma^2 = 0$	$\beta = 1$		$r \in (0, 1)$	Yes
	$\beta \in (1, 2)$	$J_\beta = \infty$	$r < 1/\beta$	Yes
			$r \geq 1/\beta$	No
		$J_\beta < \infty$	$r \leq 1/\beta$	Yes
			$r > 1/\beta$	No
	$\beta = 2$	$\lambda_2 = \infty$	$r < 1/2$	Yes
			$r \geq 1/2$	No
		$I_2 < \infty$	$r \leq 1/2$	Yes
			$r > 1/2$	No
		$\lambda_2 < \infty$ and $I_2 = \infty$	$r < 1/2$	Yes
$r > 1/2$			No	
		$r = 1/2$	Inconclusive	

1.1 r -Hölder continuity and sets of r -slopes

The convex minorant C is piecewise linear with countably many maximal intervals of linearity (see, e.g. [5]). Denote the corresponding sequences of horizontal lengths and vertical heights by $(\ell_n)_{n \in \mathbb{N}}$ and $(\xi_n)_{n \in \mathbb{N}}$, respectively. Thus, over the n -th interval of linearity (where C has slope ξ_n/ℓ_n), C is clearly r -Hölder with Hölder constant $|\xi_n|/\ell_n^r$. Our main objective is to characterise when a Lévy processes has a convex minorant that is r -Hölder continuous for $r \in (0, 1)$. It turns out that, for a large class of Lévy processes, the a.s. finiteness of $k_r := \sup_{n \in \mathbb{N}} |\xi_n|/\ell_n^r$ implies that C is r -Hölder a.s. It is important to

note that neither 0 nor T are the endpoints of an interval of linearity of C since X is of infinite variation [2, Sec. 1.1.2], implying that, even though C is always “locally r -Hölder” on $(0, T)$ by convexity (i.e. r -Hölder on every compact subinterval of $(0, T)$), it may fail to be r -Hölder on $[0, T]$.

For any $r \in (0, 1)$, define the set of r -slopes by $\mathcal{S}_r := \{\xi_n/\ell_n^r : n \in \mathbb{N}\}$, which is either a.s. bounded ($k_r < \infty$) or a.s. unbounded ($k_r = \infty$) by [2, Cor. 3.2]. By Lemma 2.1 below, we have:

$$k_r = \sup_{s \in \mathcal{S}_r} |s| \leq \sup_{0 \leq u < t \leq T} \frac{|C_t - C_u|}{(t - u)^r} \leq \left(\sum_{s \in \mathcal{S}_r} |s|^{1/(1-r)} \right)^{1-r} =: K_r \quad \text{a.s.} \quad (1.3)$$

Note that the upper bound K_r on the r -Hölder constant in (1.3) is in fact the L^p -norm of C' for $p = 1/(1 - r)$. Furthermore, the first inequality in (1.3) may be strict, see Example 1.7 below.

The utility of (1.3) lies in the fact that it controls the Hölder continuity of convex minorant C , since C is r -Hölder if $K_r < \infty$ and it is *not* r -Hölder if $k_r = \infty$. Our main result, Theorem 1.1, shows that, for all Lévy processes and $r \in (0, 1) \setminus \{1/2\}$ (and even $r = 1/2$ if either $\sigma^2 > 0$ or $\beta \in [1, 2)$), k_r and K_r are simultaneously finite or infinite: $\mathbb{P}(\{K_r = \infty\} \cap \{k_r < \infty\}) = 0$, yielding Table 1. Since, by Proposition 2.2 below, for any Lévy process X and any $r \in (0, 1)$, we have $\mathbb{P}(k_r = \infty) \in \{0, 1\}$ and $\mathbb{P}(K_r = \infty) \in \{0, 1\}$, the main function of Theorem 1.1 is thus to rule out the possibility of having $k_r < \infty = K_r$ a.s. Only in the extreme case $\sigma^2 = 0$, $\beta = 2$ and $r = 1/2$, does there exist a Lévy processes for which $k_r < \infty = K_r$ a.s., making (1.3) inconclusive in determining if C is $\frac{1}{2}$ -Hölder, see Proposition 1.5 and Example 1.6 below.

Theorem 1.1. *Let X be a Lévy process of infinite variation and $r \in (0, 1)$.*

- (i) *If $\sigma^2 > 0$, then $k_r = \infty$ a.s. for $r \in [1/2, 1)$ and $K_r < \infty$ a.s. for $r \in (0, 1/2)$.*
- (ii) *If $\sigma^2 = 0$ and $\beta \in [1, 2)$, then the following equivalences hold:*

$$k_r < \infty \text{ a.s.} \iff J_{1/r} < \infty \iff K_r < \infty \text{ a.s.}$$

- (iii) *If $\sigma^2 = 0$ and $\beta = 2$, then $k_r = \infty$ a.s. for $r \in (1/2, 1)$ and $K_r < \infty$ a.s. for $r \in (0, 1/2)$. If $r = 1/2$, the following implications hold:*
 - (a) $\lambda_2 = \infty \implies k_{1/2} = \infty$ a.s.,
 - (b) $\lambda_2 \in (0, \infty) \implies k_{1/2} < \infty = K_{1/2}$ a.s.,
 - (c) $\lambda_2 = 0 \implies k_{1/2} < \infty$ a.s.,
 - (d) $I_2 < \infty \implies K_{1/2} < \infty$ a.s.

We note that the *only* inconclusive cases in Theorem 1.1 are (iii-b) and (iii-c). In fact, Example 1.6 below presents a Lévy process as in Theorem 1.1(iii-b). The proof of Theorem 1.1 rests on Propositions 1.2–1.5, which we introduce next.

Proposition 1.2. *Let X be a Lévy process of infinite variation. If $\sigma^2 > 0$, then $k_r = \infty$ a.s. for $r \in [1/2, 1)$ and $K_r < \infty$ a.s. for $r \in (0, 1/2)$. If $\sigma^2 = 0$, then $k_r = \infty$ a.s. for $r \in (1/\beta, 1)$ and $K_r < \infty$ a.s. for $r \in (0, 1/\beta)$.*

By the inequalities in (1.3), Proposition 1.2 characterises Hölder continuity of C when either $\beta = 1$ or $\sigma^2 > 0$, implying the rows one, two and three in Table 1. Moreover, Proposition 1.2 reveals that the critical level of the Hölder exponent is $r = 1/\beta$ with $\beta \in (1, 2]$, considered next. Define

$$I_\beta := \int_0^1 \mathbb{E} \left[\min\{(|X_t|/t^{1/\beta})^{\beta/(\beta-1)}, 1\} \right] \frac{dt}{t} \in (0, \infty], \quad \beta \in (1, 2]. \quad (1.4)$$

Proposition 1.3. *Let X be a Lévy process of infinite variation with $\sigma^2 = 0$ and $\beta \in (1, 2]$. Then the following holds: (i) $J_\beta = \infty \iff k_{1/\beta} = \infty$ a.s.; (ii) $I_\beta < \infty \iff K_{1/\beta} < \infty$ a.s.*

Implicit in Proposition 1.3 is the fact that $I_\beta < \infty$ implies $J_\beta < \infty$. Checking the finiteness of the integral I_β in Proposition 1.3(ii) may appear hard as it is given in (1.4) in terms of the truncated moments of the marginals of X . We now give sufficient conditions for $I_\beta < \infty$ in terms of the Lévy measure ν , implying, in particular, that $J_\beta < \infty \iff I_\beta < \infty$ when $\beta \in (1, 2)$. Define

$$\bar{\nu}(u) := \nu(\mathbb{R} \setminus (-u, u)) \quad \text{for } u \in (0, 1]. \tag{1.5}$$

Note that, by Fubini's theorem, $J_p = \int_{(-1,1)} |x|^p \nu(dx) = \int_0^1 \bar{\nu}(t^{1/p}) dt - \bar{\nu}(1)$ for any $p > 0$. In particular, the condition $J_\beta < \infty$ is equivalent to $\int_0^1 \bar{\nu}(t^{1/\beta}) dt < \infty$.

Proposition 1.4. *Let X be an infinite variation Lévy process with $\sigma^2 = 0$. If $\beta \in (1, 2)$, then $J_\beta < \infty \iff I_\beta < \infty$. If $\beta = 2$, then $\int_0^1 \log(1/t) \bar{\nu}(\sqrt{t}) dt < \infty$ implies $I_2 < \infty$.*

By Proposition 1.4, for any process with $\beta \in (1, 2)$, Proposition 1.3 characterises the $(1/\beta)$ -Hölder continuity of C . Moreover, since $\int_0^1 \bar{\nu}(\sqrt{t}) dt < \infty$ for any Lévy measure ν , Proposition 1.4 shows that $I_2 < \infty$ for many Lévy measures.

If X has no Brownian component (i.e. $\sigma^2 = 0$) but satisfies $\beta = 2$, it is possible to have $k_{1/2} < \infty$ and $K_{1/2} = \infty$ a.s., rendering (1.3) insufficient to ascertain whether C is $\frac{1}{2}$ -Hölder continuous. Indeed, the phenomenon $k_{1/2} < \infty = K_{1/2}$ occurs whenever the constant λ_2 in (1.2) lies in $(0, \infty)$. In fact, we have the following.

Proposition 1.5. *Suppose that $\sigma^2 = 0$ and $\beta = 2$. Then (i) $\lambda_2 = \infty$ implies $k_{1/2} = \infty$ a.s., (ii) $\lambda_2 \in (0, \infty)$ implies $k_{1/2} < \infty = K_{1/2}$ a.s., and (iii) $\lambda_2 = 0$ implies $k_{1/2} < \infty$ a.s.*

Observe that $\lambda_2 = 0$ whenever $\Lambda(e^{-e}) < \infty$ where $\Lambda(x) := \int_0^x (\log \log(1/t)) \bar{\nu}(\sqrt{t}) dt$ (see [4, Rem. 2(iii)]) since, by Fubini's theorem, $\bar{\sigma}^2(x) \leq \int_0^x \bar{\nu}(\sqrt{t}) dt \leq \Lambda(x) / \log \log(1/x)$ for $x \in (0, e^{-e})$ and $\lim_{x \downarrow 0} \Lambda(x) = 0$. Note that the condition $\int_0^1 \log(1/t) \bar{\nu}(\sqrt{t}) dt < \infty$ in Proposition 1.4 is stronger than $\Lambda(e^{-e}) < \infty$, so that, even in the case where $\Lambda(e^{-e}) < \infty$, we cannot establish $I_2 < \infty$ via Proposition 1.4. We suspect, however, that $I_2 < \infty$ whenever $\lambda_2 = 0$.

We conclude this subsection with two examples.

Example 1.6 ([4, Rem 2(iii)]). Let $\nu(dx) = \frac{1}{2} x^{-3} \log(1/x)^{-1} (\log \log(1/x))^{-2} \mathbb{1}_{(0, e^{-e})}(x) dx$, so that $\bar{\sigma}^2(x) = \frac{1}{2} (\log \log(1/x))^{-1} \mathbb{1}_{(0, e^{-e})}(x)$. A direct calculation gives $\lambda_2 = 1 \in (0, \infty)$. In this case, we clearly have $k_{1/2} < \infty = K_{1/2}$ a.s. by Proposition 1.5, making (1.3) insufficient to determine whether C is $\frac{1}{2}$ -Hölder.

Example 1.7. In this example we construct a piecewise linear convex function for which the first inequality in (1.3) is strict. Fix $r \in (0, 1)$, $T > 0$ and consider the piecewise linear convex function $C : t \mapsto -T^r \min\{t, T/2\} - rT^r \max\{t - T/2, 0\}$ on $[0, T]$, which has two faces, both of length $T/2$ and with heights $-T^r$ and $-rT^r$, respectively. Then

$$\sup_{0 \leq u < t \leq T} \frac{|C_t - C_u|}{(t - u)^r} = \frac{|C_T - C_0|}{T^r} = 1 + r > 2^r = k_r,$$

where the inequality follows since $g : x \mapsto 2^x - 1 - x$ is convex and $g(0) = g(1) = 0$.

1.2 Strategy for the proofs and connections with the literature

For any $r \in (0, 1)$, we give sufficient as well as necessary conditions for the convex minorant C to be r -Hölder continuous in terms of the set of r -slopes $\mathcal{S}_r := \{\xi_n / \ell_n^r : n \in \mathbb{N}\}$ (see (1.3) and Lemma 2.1). In Proposition 2.2 we generalise the 0–1 law of [2, Thm 3.1] and use it to characterise the finiteness of K_r in terms of the truncated moments of the marginals of X . Through Khintchine's characterisation of the two-sided upper functions of X [7], given in Lemma 2.3 below, and the 0–1 law in [2, Cor 3.2], we find that $k_r < \infty$ a.s. if and only if the a.s. constant (by Blumenthal's 0–1 law [11, Prop. 40.4]) $\lambda_{1/r} =$

$\limsup_{t \downarrow 0} |X_t|/t^r$ is finite, see Corollary 2.5 below. The final ingredient in the proofs of Propositions 1.2 and 1.3 are the characterisations of the limit $\lambda_{1/r}$ given in [10, Sec. 47] and [4], respectively. In Section 3 we discuss a possible extension of Proposition 1.2 and its connection to the characterisation in [13] of the limits $\limsup_{t \downarrow 0} |X_t|/h(t)$ for a non-decreasing h .

2 Proofs

We begin with an elementary deterministic lemma that yields the inequalities in (1.3).

Lemma 2.1. *Let f be an absolutely continuous, piecewise linear function with infinitely many faces, defined on the interval $[a, b]$. Given any enumeration of the maximal intervals of linearity of f , let $(l_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ be the corresponding sequences of horizontal lengths and vertical heights, respectively, of those line segments. Then for any $r \in (0, 1)$ we have*

$$\sup_{n \in \mathbb{N}} |h_n|l_n^{-r} \leq \sup_{a \leq u < t \leq b} \frac{|f(t) - f(u)|}{(t - u)^r} \leq \left(\sum_{n \in \mathbb{N}} (|h_n|l_n^{-r})^{1/(1-r)} \right)^{1-r}.$$

Proof. Fix $r \in (0, 1)$ and let $p = 1/(1-r) > 1$. Let (g_n, d_n) , $n \in \mathbb{N}$, be the maximal intervals of linearity of f where the slope of f over (g_n, d_n) equals h_n/l_n and $\sum_{n \in \mathbb{N}} (d_n - g_n) = b - a$. The lower bound is obvious since it is attained by restricting the supremum to the values $(u, t) = (g_n, d_n)$. To establish the upper bound, first note that f' exists on the set $\bigcup_{n \in \mathbb{N}} (g_n, d_n)$ of measure $b - a$ and

$$\int_a^b |f'(t)|^p dt = \sum_{n \in \mathbb{N}} \int_{g_n}^{d_n} |f'(t)|^p dt = \sum_{n \in \mathbb{N}} \frac{|h_n|^p}{l_n^p} \int_{g_n}^{d_n} dt = \sum_{n \in \mathbb{N}} \frac{|h_n|^p}{l_n^{p-1}}.$$

By Hölder's inequality with p and $q = p/(p - 1) = 1/r > 1$, it follows

$$|f(t) - f(u)| \leq \int_u^t |f'(x)| dx = \int_a^b \mathbb{1}_{[u, t]}(x) |f'(x)| dx \leq (t - u)^{1/q} \left(\int_a^b |f'(x)|^p dx \right)^{1/p}.$$

Thus, we have

$$\begin{aligned} \sup_{a \leq u < t \leq b} \frac{|f(t) - f(u)|}{(t - u)^r} &\leq \left(\int_a^b |f'(x)|^p dx \right)^{1/p} \\ &= \left(\sum_{n \in \mathbb{N}} \frac{|h_n|^p}{l_n^{p-1}} \right)^{1/p} = \left(\sum_{n \in \mathbb{N}} (|h_n|l_n^{-r})^{1/(1-r)} \right)^{1-r}. \quad \square \end{aligned}$$

The proofs of Propositions 1.2 & 1.3 hinge on two key tools. First is the 0–1 law in Proposition 2.2, generalising [2, Thm 3.1] to unbounded functionals of the faces of C , and second is Khintchine's characterisation of the upper functions of $|X|$ at zero given in Lemma 2.3 below. Recall that since X is of infinite activity, its convex minorant C is a piecewise linear function whose maximal intervals of linearity have corresponding sequences of horizontal lengths $(l_n)_{n \in \mathbb{N}}$ and vertical heights $(\xi_n)_{n \in \mathbb{N}}$ given by the formulae in [5, Thm 3.1]. (If X is of finite activity, the intervals of linearity of the stick-breaking representation in [5, Thm 3.1] are *not maximal* and, in fact, all but finitely many faces have the same slope).

Proposition 2.2. *Let $\phi : \mathbb{R} \times (0, \infty) \rightarrow [0, \infty)$ be measurable. Then the sum $\sum_{n \in \mathbb{N}} \phi(\xi_n, l_n)$ is either a.s. finite or a.s. infinite. Moreover, we have*

$$\sum_{n \in \mathbb{N}} \phi(\xi_n, l_n) < \infty \quad \text{a.s.} \iff \int_0^1 \mathbb{E}[\min\{\phi(X_t, t), 1\}] \frac{dt}{t} < \infty. \quad (2.1)$$

Proof. Note that $\sum_{n \in \mathbb{N}} a_n < \infty$ if and only if $\sum_{n \in \mathbb{N}} \min\{a_n, 1\} < \infty$ for any sequence $(a_n)_{n \in \mathbb{N}}$ in $[0, \infty)$. Thus, $\sum_{n \in \mathbb{N}} \phi(\xi_n, \ell_n) < \infty$ a.s. $\iff \sum_{n \in \mathbb{N}} \min\{\phi(\xi_n, \ell_n), 1\} < \infty$ a.s. and the equivalence in (2.1) follows from [5, Thm 3.1] and the 0–1 law [2, Thm 3.1] applied to the bounded function $(t, x) \mapsto \min\{\phi(x, t), 1\}$. \square

The following characterisation due to Khintchine [7] is central in relating the upper fluctuations of $|X|$ and the faces of C . Recall that, for any positive measurable function $h : (0, \infty) \rightarrow (0, \infty)$, $\limsup_{t \downarrow 0} |X_t|/h(t)$ is a.s. a constant on $[0, \infty]$ by Blumenthal’s 0–1 law [11, Prop. 40.4].

Lemma 2.3 (Khintchine). *Suppose X is not compound Poisson with drift. Let $h : (0, \infty) \rightarrow (0, \infty)$ be measurable and increasing at 0 and fix $R > 0$. The following statements hold.*

- (i) *If $\int_0^1 \mathbb{P}(|X_t|/h(t) > R/4)t^{-1}dt < \infty$, then $\limsup_{t \downarrow 0} |X_t|/h(t) \leq R$ a.s.*
- (ii) *If $\int_0^1 \mathbb{P}(|X_t|/h(t) > 8R)t^{-1}dt = \infty$, then $\limsup_{t \downarrow 0} |X_t|/h(t) \geq R$ a.s.*

Remark 2.4. For completeness and accessibility, we give a short elementary proof of Lemma 2.3 in Appendix A below. It is based on Khintchine’s proof of a closely related result in the Russian text [7, Fundamental lemma]. It is not essential for the results in this paper, but it is natural to enquire whether Lemma 2.3 holds with the constants $R/4$ and $8R$ in the integral conditions substituted by R .

Corollary 2.5. *Suppose X is not compound Poisson with drift. Let $h : (0, \infty) \rightarrow (0, \infty)$ be measurable and increasing at 0. Define the set of h -slopes $\mathcal{S}_h := \{\xi_n/h(\ell_n) : n \in \mathbb{N}\}$ and set $k_h := \sup_{s \in \mathcal{S}_h} |s|$. Then $\mathbb{P}(k_h = \infty) \in \{0, 1\}$. Moreover, $k_h < \infty$ a.s. if and only if $\limsup_{t \downarrow 0} |X_t|/h(t) < \infty$ a.s.*

Proof. Suppose there exists $R \in (0, \infty)$ such that $\int_0^1 \mathbb{P}(|X_t|/h(t) > R)t^{-1}dt < \infty$. Then [2, Cor. 3.2] (applied to $f(t, x) = |x|/h(t)$) implies that $\mathcal{S}_h \cap (\mathbb{R} \setminus [-R, R])$ is a.s. a finite set and hence $k_h < \infty$ a.s. Similarly, since $\limsup_{t \downarrow 0} |X_t|/h(t)$ is a.s. constant, Lemma 2.3(i) implies $\limsup_{t \downarrow 0} |X_t|/h(t) \leq 4R$.

Next assume that for all $R \in (0, \infty)$ we have $\int_0^1 \mathbb{P}(|X_t|/h(t) > R)t^{-1}dt = \infty$. Then [2, Cor. 3.2] (applied to $f(t, x) = |x|/h(t)$) implies that $\mathcal{S}_h \cap (\mathbb{R} \setminus [-R, R])$ is a.s. an infinite set for any $R > 0$. Hence $k_h \geq R$ a.s. for any $R > 0$, implying that $k_h = \infty$ a.s. Similarly, Lemma 2.3(ii) implies $\limsup_{t \downarrow 0} |X_t|/h(t) \geq R/8$ a.s. for any $R > 0$ and hence $\limsup_{t \downarrow 0} |X_t|/h(t) = \infty$.

Since $\int_0^1 \mathbb{P}(|X_t|/h(t) > R)t^{-1}dt$ is either finite for some R or infinite for all R , it follows that $\mathbb{P}(k_h = \infty)$ is either 0 or 1, respectively. Moreover, the former (resp. latter) case implies that $\limsup_{t \downarrow 0} |X_t|/h(t)$ is finite (resp. infinite) a.s. \square

Proof of Proposition 1.2. First note that, for any $p > 0$ the sum $\sum_{n \in \mathbb{N}} \ell_n^p$ is finite a.s. (with mean T^p/p) by [2, Thm 3.1]. Pick any $r' > r$ and note that $|\xi_n|/\ell_n^r \leq k_{r'} \ell_n^{r'-r}$ for every $n \in \mathbb{N}$, implying

$$K_r^{1/(1-r)} = \sum_{s \in \mathcal{S}_r} |s|^{1/(1-r)} = \sum_{n \in \mathbb{N}} (|\xi_n|/\ell_n^r)^{1/(1-r)} \leq k_{r'}^{1/(1-r)} \sum_{n \in \mathbb{N}} \ell_n^{(r'-r)/(1-r)}. \tag{2.2}$$

In particular, $K_r < \infty$ whenever $k_{r'} < \infty$ for some $r' > r$.

Assume first that $\sigma^2 > 0$, then [10, Prop. 47.11] yields $\limsup_{t \downarrow 0} |X_t|/\sqrt{t \log \log(1/t)} = \sqrt{2}|\sigma| > 0$. Thus, the limit $\lambda_{1/r} = \limsup_{t \downarrow 0} |X_t|/t^r$ equals 0 (resp. ∞) a.s. for $r \in (0, 1/2)$ (resp. $r \in [1/2, 1)$). Then, by Corollary 2.5, we have $k_r = \infty$ for all $r \in [1/2, 1)$ and $k_{(r+1/2)/2} < \infty$ for $r \in (0, 1/2)$ since $(r + 1/2)/2 < 1/2$. In the latter case, $r < (r + 1/2)/2$ and hence $K_r < \infty$ by (2.2).

Next assume $\sigma^2 = 0$. By [10, Prop. 47.24], $\lambda_{1/r}$ equals 0 (resp. ∞) a.s. for $r \in (0, 1/\beta)$ (resp. $r \in (1/\beta, 1)$) where β is the Blumenthal–Gettoor index defined in (1.1). As before,

by Corollary 2.5, we have $k_r = \infty$ for all $r \in (1/\beta, 1)$ and $k_{(r+1/\beta)/2} < \infty$ for $r \in (0, 1/\beta)$ since $(r+1/\beta)/2 < 1/\beta$. In the latter case, $r < (r+1/\beta)/2$ and hence $K_r < \infty$ by (2.2). \square

Recall that, by Fubini's theorem, $J_p = \int_{(-1,1)} |x|^p \nu(dx) = \int_0^1 \bar{\nu}(t^{1/p}) dt - \bar{\nu}(1)$ for $p > 0$.

Proof of Proposition 1.3. Let $r = 1/\beta \in [1/2, 1)$. By [4, Thm 2.1], $\int_0^1 \bar{\nu}(t^r) dt$ is finite (resp. infinite) if and only if $\lambda_{1/r} = \limsup_{t \downarrow 0} |X_t|/t^r$ is finite (resp. infinite) a.s. Thus, by Corollary 2.5, $\int_0^1 \bar{\nu}(t^r) dt = \infty$ if and only if $k_r = \infty$. By Proposition 2.2 (with $\phi(x, t) = (|x|/t^r)^{1/(1-r)}$): $I_\beta = \int_0^1 \mathbb{E}[\min\{|X_t|/t^r, 1\}^{1/(1-r)}] t^{-1} dt$ is finite if and only if $K_r^{1/(1-r)} = \sum_{n \in \mathbb{N}} |\xi_n|^{1/(1-r)} / \ell_n^{r/(1-r)}$ is finite a.s., completing the proof. \square

The following elementary result is required to establish Proposition 1.4.

Lemma 2.6. *Let $f : (0, 1) \rightarrow [0, \infty)$ be a non-increasing function. Then the condition $\int_0^1 f(x) dx < \infty$ implies $\int_0^1 x^{p-1} f(x)^p dx < \infty$ for any $p \geq 1$.*

Proof. Observe that, for all $x \in (0, 1)$, we have

$$\begin{aligned} x f(x) &\leq g(x) := \sum_{n \in \mathbb{N}} 2^{1-n} f(2^{-n}) \mathbb{1}_{[2^{-n}, 2^{1-n})}(x) \\ &= 4 \sum_{n \in \mathbb{N}} (2^{-n}/2) f(2^{1-n}/2) \mathbb{1}_{[2^{-n}, 2^{1-n})}(x) \leq 4 \frac{x}{2} f\left(\frac{x}{2}\right). \end{aligned}$$

Thus, defining $w_n := 2^{1-n} f(2^{-n})$ for $n \in \mathbb{N}$, we have

$$\sum_{n \in \mathbb{N}} w_n \log 2 = \int_0^1 g(x) \frac{dx}{x} \leq 4 \int_0^1 \frac{x}{2} f\left(\frac{x}{2}\right) \frac{dx}{x} = 4 \int_0^{1/2} f(x) dx < \infty.$$

In particular, $W := \sup_{n \in \mathbb{N}} w_n < \infty$ and hence,

$$\int_0^1 x^{p-1} f(x)^p dx \leq \int_0^1 g(x)^p \frac{dx}{x} = \sum_{n \in \mathbb{N}} w_n^p \log 2 \leq \sum_{n \in \mathbb{N}} w_n W^{p-1} \log 2 < \infty. \quad \square$$

Proof of Proposition 1.4. Recall that $I_\beta < \infty$ implies $J_\beta < \infty$ by Proposition 1.3, so it suffices to prove the converse. Define

$$\varpi(u) := \int_{(-1,1) \setminus (-u,u)} x \nu(dx) \in \mathbb{R} \quad \text{for } u \in (0, 1].$$

We will show that, under our assumptions, the following integrals are finite:

$$\begin{aligned} \text{(i)} \quad &\int_0^1 t \bar{\nu}(t^{1/\beta})^2 dt < \infty, \quad \text{(ii)} \quad \int_0^1 t^{1-2/\beta} \varpi(t^{1/\beta})^2 dt < \infty, \\ \text{(iii)} \quad &\int_0^1 \mathbb{E}[\min\{|X_t|/t^{1/\beta}, 1\}^2] \frac{dt}{t} < \infty, \quad \text{(iv)} \quad I_\beta = \int_0^1 \mathbb{E}[\min\{|X_t|/t^{1/\beta}, 1\}^{\beta/(\beta-1)}] \frac{dt}{t} < \infty. \end{aligned}$$

Let $r = 1/\beta$ and note that $\int_0^1 \bar{\nu}(t^r) dt < \infty$ by assumption. Thus, Lemma 2.6 (with $f(x) = \bar{\nu}(x^r)$) gives (i) $\int_0^1 t \bar{\nu}(t^r)^2 dt < \infty$. Since $1/(1-r) = \beta/(\beta-1) \geq 2$ and $\min\{|x|, 1\}^p \leq \min\{|x|, 1\}^q$ for $p \geq q$, (iii) implies (iv). It remains to show that (ii) and (iii) hold.

Let us establish (ii) $\int_0^1 t^{1-2r} \varpi(t^r)^2 dt < \infty$. Denote $\bar{\nu}_1(x) := \bar{\nu}(x) - \bar{\nu}(1)$ for $x \in (0, 1]$. By Fubini's theorem, we have

$$|\varpi(u)| \leq \int_{(-1,1)} \mathbb{1}_{\{u \leq |x| < 1\}} \int_0^{|x|} dy \nu(dx) = \int_0^1 \bar{\nu}_1(\max\{y, u\}) dy = u \bar{\nu}_1(u) + \int_u^1 \bar{\nu}_1(y) dy.$$

Hence, the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ yields

$$\frac{1}{2}t^{1-2r}\varpi(t^r)^2 \leq t\bar{\nu}(t^r)^2 + t^{1-2r}\left(\int_{t^r}^1 \bar{\nu}(y)dy\right)^2, \quad t \in (0, 1].$$

Since (i) $\int_0^1 t\bar{\nu}(t^r)^2 dt < \infty$, to establish (ii) we need only show that $\int_0^1 t^{1-2r}\left(\int_{t^r}^1 \bar{\nu}(y)dy\right)^2 dt$ is finite. Since $\min\{a, b\}^2 \leq ab$ and $r = 1/\beta < 1$, Fubini's theorem gives

$$\begin{aligned} 2(1-r)\int_0^1 t^{1-2r}\left(\int_{t^r}^1 \bar{\nu}(y)dy\right)^2 dt &= 2(1-r)\int_0^1 \int_0^1 \left(\int_0^{\min\{x,y\}^{1/r}} t^{1-2r} dt\right) \bar{\nu}(x)\bar{\nu}(y) dx dy \\ &= \int_0^1 \int_0^1 \min\{x, y\}^{2/r-2} \bar{\nu}(x)\bar{\nu}(y) dx dy \\ &\leq \left(\int_0^1 x^{1/r-1} \bar{\nu}(x) dx\right)^2 = \left(r\int_0^1 \bar{\nu}(t^r) dt\right)^2 < \infty. \end{aligned}$$

It remains to establish (iii) $\int_0^1 \mathbb{E}[\min\{|X_t|/t^r, 1\}^2]t^{-1} dt < \infty$. Let γ be the drift parameter of X corresponding to the cutoff function $x \mapsto \mathbb{1}_{\{|x|<1\}}$ (see [10, Def. 8.2]) and recall $\bar{\sigma}^2(u) = \int_{(-u,u)} x^2 \nu(dx)$ for $u > 0$. Apply [1, Lem. A.1] (with $\varepsilon = K = t^r$) to obtain

$$\int_0^1 \mathbb{E}[\min\{|X_t|, t^r\}^2] \frac{dt}{t^{1+2r}} \leq \int_0^1 [t^2(\gamma - \varpi(t^r))^2 + t\bar{\sigma}^2(t^r) + t^{2r+1}\bar{\nu}(t^r)] \frac{dt}{t^{1+2r}}.$$

Since the integrals $\int_0^1 \gamma^2 t^{1-2r} dt = \gamma^2/(2 - 2r)$, $\int_0^1 t^{1-2r}\varpi(t^r)^2 dt$ and $\int_0^1 \bar{\nu}(t^r) dt$ are all finite (recall $r = 1/\beta < 1$), it remains to show that $\int_0^1 t^{-2r}\bar{\sigma}^2(t^r) dt < \infty$.

By Fubini's theorem, we obtain

$$\bar{\sigma}^2(x) = \int_{(-x,x)} \left(\int_0^{|u|} 2y dy\right) \nu(du) = 2\int_0^x y(\bar{\nu}(y) - \bar{\nu}(x)) dy \leq 2\int_0^x y\bar{\nu}(y) dy, \quad x \in (0, 1].$$

Consider the case $r = 1/\beta \in (1/2, 1)$. Fubini's theorem gives

$$\int_0^1 t^{-2r}\bar{\sigma}^2(t^r) dt \leq 2\int_0^1 \int_0^{t^r} t^{-2r} y\bar{\nu}(y) dy dt = 2\int_0^1 \frac{y^{1/r-2} - 1}{2r - 1} y\bar{\nu}(y) dy,$$

which is finite since $\int_0^1 y^{1/r-1}\bar{\nu}(y) dy = r\int_0^1 \bar{\nu}(t^r) dt < \infty$. Now consider the case $r = 1/\beta = 1/2$. Again by Fubini's theorem, we obtain

$$\begin{aligned} \int_0^1 t^{-1}\bar{\sigma}^2(\sqrt{t}) dt &\leq 2\int_0^1 \int_0^{\sqrt{t}} t^{-1} y\bar{\nu}(y) dy dt \\ &= 4\int_0^1 \log(1/y) y\bar{\nu}(y) dy = \int_0^1 \log(1/x) \bar{\nu}(\sqrt{x}) dx < \infty. \quad \square \end{aligned}$$

Proof of Proposition 1.5. Recall that $\lambda_2 = \limsup_{t \downarrow 0} |X_t|/\sqrt{t} \in [0, \infty]$ by [4, Thm 2.1]. If $\lambda_2 = \infty$, then $k_{1/2} = \infty$ by Corollary 2.5. If $\lambda_2 \in [0, \infty)$ then $k_{1/2} < \infty$ by Corollary 2.5. Finally, assume $\lambda_2 \in (0, \infty)$. Then $\int_0^1 t^{-1}\mathbb{P}(|X_t|/\sqrt{t} > R) dt = \infty$ for $R < \lambda_2/4$ by Lemma 2.3(i). Thus, for any $\varepsilon \in (0, \lambda_2/4)$, $\mathcal{S}_{1/2}$ has infinitely many points with magnitude on the interval $[\varepsilon, \infty)$ by [2, Cor. 3.2], implying $K_{1/2} = \infty$. \square

3 Concluding remarks

It is natural to consider the question of whether the convex minorant C is h -Hölder continuous, i.e., if $\sup_{0 \leq u < t \leq T} |C_t - C_u|/h(t-u) < \infty$, for an appropriate general concave increasing function $h : (0, \infty) \rightarrow (0, \infty)$. In this context, it is also easy to see that

$$\sup_{0 \leq u < t \leq T} \frac{|C_t - C_u|}{h(t-u)} \geq k_h = \sup_{n \in \mathbb{N}} \frac{|\xi_n|}{h(\ell_n)} = \sup_{s \in \mathcal{S}_h} |s|,$$

where the finiteness of k_h can be completely characterised via Corollary 2.5 and part (a) of the main theorem in [13] in terms of the Lévy measure ν (see Corollary 3.1 in Subsection 3.1 below for details).

It is not, however, immediately clear how to construct a tractable upper bound, say K_h , satisfying $K_h < \infty$ whenever $k_h < \infty$. Indeed, a crucial step in proving Lemma 2.1 (and hence (1.3)) is the application of Hölder’s inequality to establish that the r -Hölder constant of C is bounded by the L^p -norm of the derivative C' for $p = 1/(1-r)$. This step is not easily extendable to a general concave function h since there is no sufficiently sharp extension of Hölder’s inequality (see, e.g. [9, 8]). Thus, it appears that a generalisation of our results beyond the case where h is a power function would require analysing the integral $\int_u^t |C'_v| dv$ for all $0 \leq u < t \leq T$ by other means. For instance, the results in [1] obtain upper and lower functions for $|C'|$ at 0 and T , yielding upper and lower bounds on $\int_u^t |C'_v| dv$ for $u < t$ close to either 0 or T . Note however, that there may exist a large gap between the upper and lower functions of C' , see [1, Rem. 2.14(a)], showing that this question is nontrivial.

3.1 When is k_h finite?

The following corollary is a direct consequence of Corollary 2.5 and part (a) of the main theorem in [13]. Recall the definition of $\bar{\nu}$ and ϖ in (1.5) and let γ be the drift parameter of X (for the cutoff function $x \mapsto \mathbb{1}_{(-1,1)}(x)$, see [10, Def. 8.2]) and $\bar{\sigma}^2(u) = \int_{(-u,u)} x^2 \nu(dx)$ for $u > 0$.

Corollary 3.1. *Suppose X is not compound Poisson with drift. Then, for any function h increasing at 0 with $h(0) = 0$, the variable $k_h < \infty$ a.s. (resp. $k_h = \infty$ a.s.) if and only if $\limsup_{t \downarrow 0} |X_t|/h(t)$ is a.s. finite (resp. infinite). Moreover, the following statements hold.*

- (i) *If $\sigma^2 > 0$, then $k_h < \infty$ a.s. if and only if $\liminf_{t \downarrow 0} h(t)/\sqrt{t \log \log(1/t)} > 0$ a.s.*
- (ii) *If $\sigma^2 = 0$ and $\limsup_{x \downarrow 0} (x^{-2} \bar{\sigma}^2(x) + x^{-1} |\gamma - \varpi(x)|) / \bar{\nu}(x) < \infty$, then the random variable $k_h < \infty$ a.s. if and only if $\int_0^1 \bar{\nu}(h(t)) dt < \infty$.*
- (iii) *Suppose $\sigma^2 = 0$ and $\limsup_{x \downarrow 0} (x^{-2} \bar{\sigma}^2(x) + x^{-1} |\gamma - \varpi(x)|) / \bar{\nu}(x) = \infty$. Then there exists a non-decreasing function h^* such that $\limsup_{t \downarrow 0} |X_t|/h^*(t) \in (0, \infty)$ a.s. (h^* constructed in the paragraph below). Moreover, the following implications hold*

$$\limsup_{t \downarrow 0} h^*(t)/h(t) < \infty \implies k_h < \infty \text{ a.s.}$$

$$\liminf_{t \downarrow 0} h^*(t)/h(t) = \infty \implies k_h = \infty \text{ a.s.}$$

Wee and Kim [13] proved that $\limsup_{t \downarrow 0} |X_t|/h^*(t) \in (0, \infty)$ a.s. for a non-decreasing function h^* if and only if $\sigma^2 = 0$ and $\liminf_{x \downarrow 0} \bar{\nu}(x)/(\bar{\nu}(x) + x^{-2} \bar{\sigma}^2(x) + x^{-1} |\gamma - \varpi(x)|) = 0$. In the following two cases, which are exhaustive by [13, Lem. 3.3], we describe a construction of the function h^* , implicitly given in the proof of [13, Thm 3.4].

- (a) Suppose $\liminf_{x \downarrow 0} (\bar{\nu}(x) + x^{-1} |\gamma - \varpi(x)|) / (x^{-2} \bar{\sigma}^2(x)) = 0$. Choose a sequence $u_n \downarrow 0$, such that $u_{n+1}^{-2} \bar{\sigma}^2(u_{n+1}) > 2u_n^{-2} \bar{\sigma}^2(u_n)$ for $n \in \mathbb{N}$ and $\sum_{n \in \mathbb{N}} \log(n) (\bar{\nu}(u_n) + u_n^{-1} |\gamma - \varpi(u_n)|) / (u_n^{-2} \bar{\sigma}^2(u_n)) < \infty$. Let $t_n = \log(n) / (u_n^{-2} \bar{\sigma}^2(u_n))$ for all $n \in \mathbb{N}$, and define $h^*(t) := u_n \log(n)$ for $t_{n+1} < t \leq t_n$ and $n \in \mathbb{N}$.
- (b) Suppose $\liminf_{x \downarrow 0} (\bar{\nu}(x) + x^{-2} \bar{\sigma}^2(x)) / (x^{-1} |\gamma - \varpi(x)|) = 0$. Choose a sequence $u_n \downarrow 0$, such that $u_{n+1}^{-1} |\gamma - \varpi(u_{n+1})| \geq 2u_n^{-1} |\gamma - \varpi(u_n)|$ for $n \in \mathbb{N}$ and $\sum_{n \in \mathbb{N}} (\bar{\nu}(u_n) + u_n^{-2} \bar{\sigma}^2(u_n)) / (u_n^{-1} |\gamma - \varpi(u_n)|) < \infty$. Let $t_n = 1 / (u_n^{-1} |\gamma - \varpi(u_n)|)$ and define $h^*(t) := u_n$ for $t_{n+1} < t \leq t_n$ and $n \in \mathbb{N}$.

A Proof of Lemma 2.3

We present a short proof of Lemma 2.3, based on the proof of [7, Fundamental lemma].

Proof of Lemma 2.3. Fix $0 < s < t$ and $0 < y < x$, then $\{|X_t| \geq x\} \subset \{|X_s| \geq y\} \cup \{|X_t - X_s| \geq x - y\}$. Since $X_t - X_s \stackrel{d}{=} X_{t-s}$, this yields

$$\mathbb{P}(|X_t| \geq x) \leq \mathbb{P}(|X_s| \geq y) + \mathbb{P}(|X_{t-s}| \geq x - y). \tag{A.1}$$

In particular, taking $s = t/2$ and $y = x/2$ gives $P(t, x) := \mathbb{P}(|X_t| \geq x) \leq 2P(t/2, x/2)$. Without loss of generality, we assume throughout that h is non-decreasing on $(0, 1]$.

Part (i). It suffices to show that, given $R > 0$, the condition $\int_0^1 P(t, Rh(t))t^{-1}dt < \infty$ implies $\limsup_{t \downarrow 0} |X_t|/h(t) \leq 4R$ a.s. The proof is split in three steps.

Step 1. We first show that $P(t, 2Rh(t)) \rightarrow 0$ as $t \downarrow 0$ and, in particular, there exists some $\varepsilon > 0$ such that $P(t, 2Rh(t)) < 1/2$ for $t \in (0, \varepsilon)$. Denote $p(t) = P(t, Rh(t))$ for $t > 0$. Since h is non-decreasing, (A.1) implies $P(t, 2Rh(t)) \leq p(s) + p(t - s)$. Integrating the previous inequality over $[t/2, t]$ with respect to the measure $s^{-1}ds$ yields

$$\begin{aligned} P(t, 2Rh(t)) \log 2 &\leq \int_{t/2}^t p(s) \frac{ds}{s} + \int_{t/2}^t p(t - s) \frac{ds}{s} \\ &\leq \int_{t/2}^t p(s) \frac{ds}{s} + \int_{t/2}^t p(t - s) \frac{ds}{t - s} = \int_0^t p(s) \frac{ds}{s} < \infty. \end{aligned}$$

Thus, the limit $\lim_{t \downarrow 0} \int_0^t p(s)s^{-1}ds = 0$ implies $\lim_{t \downarrow 0} P(t, 2Rh(t)) = 0$.

Step 2. Define $\bar{X}_t := \sup_{s \in [0, t]} X_s$ for $t \geq 0$. We will show that $\mathbb{P}(\bar{X}_t > 4Rh(t)) \leq 2\mathbb{P}(X_t > 2Rh(t))$ for $t \in (0, \varepsilon)$ where ε is as in Step 1. Fix $n \in \mathbb{N}$, set $t_k := tk/n$ for $k \in \{1, \dots, n\}$ and define the events

$$A_k := \{X_{t_i} \leq 4Rh(t) \text{ for all } i \in \{1, \dots, k - 1\}\} \cap \{X_{t_k} > 4Rh(t)\}, \quad k \in \{1, \dots, n\}.$$

Since the increments of X are independent and stationary, we have

$$\begin{aligned} \mathbb{P}(X_t > 2Rh(t)|A_k) &\geq \mathbb{P}(X_t - X_{t_k} > -2Rh(t)|A_k) = \mathbb{P}(X_t - X_{t_k} > -2Rh(t)) \\ &\geq \mathbb{P}(|X_t - X_{t_k}| < 2Rh(t)) = \mathbb{P}(|X_{t-t_k}| < 2Rh(t)) \\ &\geq \mathbb{P}(|X_{t-t_k}| < 2Rh(t - t_k)). \end{aligned}$$

By step 1, $t - t_k < \varepsilon$ for $t \in (0, \varepsilon)$ and hence $\mathbb{P}(X_t > 2Rh(t)|A_k) > 1/2$ for $k \in \{1, \dots, n\}$.

Define $M_t^{(n)} := \max_{1 \leq k \leq n} X_{t_k}$, then $\{M_t^{(n)} > 4Rh(t)\} = \bigcup_{k=1}^n A_k$. Since the sets A_k are disjoint, for any $t \in (0, \varepsilon)$ we have

$$\mathbb{P}(M_t^{(n)} > 4Rh(t)) = \sum_{k=1}^n \mathbb{P}(A_k) \leq 2 \sum_{k=1}^n \mathbb{P}(A_k)\mathbb{P}(X_t > 2Rh(t)|A_k) \leq 2\mathbb{P}(X_t > 2Rh(t)).$$

Since X is càdlàg, $M_t^{(2^n)} \uparrow \bar{X}_t$ a.s. as $n \rightarrow \infty$. Hence, the monotone convergence theorem yields $\mathbb{P}(\bar{X}_t > 4Rh(t)) = \lim_{n \rightarrow \infty} \mathbb{P}(M_t^{(2^n)} > 4Rh(t)) \leq 2\mathbb{P}(X_t > 2Rh(t))$.

Step 3. Define $p_n := \mathbb{P}(\sup_{t \in [2^{-n}, 2^{1-n}]} (X_t/h(t)) > 4R)$ for $n \in \mathbb{N}$ and let n_ε be the smallest positive integer larger than $1 + \log(1/\varepsilon)/\log 2$, where ε is as in Step 1. Since h is non-decreasing, Step 2 and (A.1) imply that for all $n \geq n_\varepsilon$ and $t \in [2^{-n}, 2^{1-n}]$ we have

$$\begin{aligned} p_n &\leq \mathbb{P}\left(\sup_{t \in [2^{-n}, 2^{1-n}]} X_t > 4Rh(2^{-n})\right) = \mathbb{P}(\bar{X}_{2^{-n}} > 4Rh(2^{-n})) \leq \mathbb{P}(\bar{X}_t > 4Rh(2^{-n})) \\ &\leq 2\mathbb{P}(X_t > 2Rh(2^{-n})) \leq 2\mathbb{P}(X_t > 2Rh(t/2)) \leq 4\mathbb{P}(X_{t/2} > Rh(t/2)). \end{aligned}$$

Integrating the previous inequality over $t \in [2^{-n}, 2^{1-n}]$ and summing over $n \geq n_\varepsilon$ gives

$$\begin{aligned} \sum_{n=n_\varepsilon}^{\infty} p_n \frac{\log 2}{4} &= \sum_{n=n_\varepsilon}^{\infty} \int_{2^{-n}}^{2^{1-n}} \frac{p_n}{4} \frac{dt}{t} \leq \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{1-n}} \mathbb{P}(X_{t/2} > Rh(t/2)) \frac{dt}{t} \\ &= \int_0^2 \mathbb{P}(X_t > Rh(t)) \frac{dt}{t} < \infty. \end{aligned}$$

The Borel–Cantelli lemma implies $\sup_{t \in [2^{-m-1}, 2^{-m}]} (X_t/h(t)) \leq 4R$ for all but finitely many n , implying $\limsup_{t \rightarrow 0} X_t/h(t) \leq 4R$ a.s. By symmetry, $\limsup_{t \rightarrow 0} (-X_t)/h(t) \leq 4R$ a.s., proving part (i).

Part (ii). It suffices to show that, given $R > 0$, if $\int_0^1 P(t, 8Rh(t))t^{-1}dt = \infty$ (recall $P(t, x) = \mathbb{P}(|X_t| > x)$) then $\limsup_{t \downarrow 0} |X_t|/h(t) \geq R$ a.s. The proof requires three steps.

Step 1. Define $M(t) := \sup_{s \in (0, t]} (|X_s|/h(s))$. We will show that

$$B_n := \left\{ \sup_{t \in [2^{-n-1}, 2^{-n}]} |X_t - X_{2^{-n-1}}| > 2Rh(2^{-n}), M(2^{-n-1}) \leq R \right\} \subset \{M(2^{-n}) > R\}. \tag{A.2}$$

To see (A.2) note that, on the event B_n there exists some $t \in [2^{-n-1}, 2^{-n}]$ satisfying

$$M(2^{-n}) \geq |X_t| \geq |X_t - X_{2^{-n-1}}| - |X_{2^{-n-1}}| > 2Rh(2^{-n}) - Rh(2^{-n-1}) \geq Rh(2^{-n}) \geq Rh(t).$$

Step 2. We claim $\sum_{n \in \mathbb{N}} q_n = \infty$, where $q_n := \mathbb{P}(\sup_{t \in [0, 2^{-n-1}]} |X_t| > 2Rh(2^{-n}))$. For $t \leq 2^{-n-1}$, apply (A.1) twice to get $4q_n \geq 4P(t, 2Rh(2^{-n})) \geq P(4t, 8Rh(2^{-n}))$. Hence, for any $t \in [2^{-n-2}, 2^{-n-1}]$, we have $4q_n \geq P(4t, 8Rh(4t))$. Integrating the previous inequality on $[2^{-n-2}, 2^{-n-1}]$ with respect to $t^{-1}dt$ yields

$$(4 \log 2)q_n \geq \int_{2^{-n-2}}^{2^{-n-1}} P(4t, 8Rh(4t)) \frac{dt}{t} = \int_{2^{-n}}^{2^{-n+1}} P(t, 8Rh(t)) \frac{dt}{t}, \quad \text{for all } n \in \mathbb{N}.$$

Thus, the condition $\int_0^1 P(t, 8Rh(4t))t^{-1}dt = \infty$ implies $\sum_{n \in \mathbb{N}} q_n = \infty$.

Step 3. Define $r_n := \mathbb{P}(M(2^{-n}) > R)$ for $n \in \mathbb{N} \cup \{0\}$. By Step 1, the event $B_n \subset \{M(2^{-n-1}) \leq R\}$ in (A.2) satisfies

$$q_n(1 - r_{n+1}) = \mathbb{P}(B_n) \leq \mathbb{P}(M(2^{-n-1}) \leq R, M(2^{-n}) > R) = r_n - r_{n+1}.$$

This further implies that, for any $k \geq 0$ and $n \in \mathbb{N}$,

$$0 \leq 1 - r_n \leq (1 - r_{n+1})(1 - q_n) \leq (1 - r_{n+k+1}) \prod_{i=n}^{n+k} (1 - q_i).$$

Since $\sum_{n \in \mathbb{N}} q_n = \infty$, it follows that $\prod_{i=n}^{n+k} (1 - q_i) \rightarrow 0$ as $k \rightarrow \infty$. Indeed, if $q_\infty := \limsup_{k \rightarrow \infty} q_k > 0$, then the limit is obvious since $1 - q_i \leq 1 - q_\infty/2 < 1$ for infinitely many $i \geq n$ and, if $q_\infty = 0$, the result follows from [6, Lem. 5.8]. Hence, we have $r_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and thus $\limsup_{t \downarrow 0} |X_t|/h(t) = \lim_{t \downarrow 0} M(t) \geq R$ a.s. \square

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