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A central limit theorem for some generalized martingale arrays

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Abstract

Let $(X_{n,j})$ and $(Y_{n,j})$ be two arrays of real random variables and $f:\mathbb{R}\to\mathbb{R}$ a Borel function. Define $D_n=\sum_j f\Big(\sum_{i=1}^{j-1}Y_{n,i}\Big)\,X_{n,j}$ and $D=Z\,\sqrt{\int_0^1 f^2(B_{G(t)})\,dF(t)}$ where B is a standard Brownian motion, Z a standard normal random variable independent of B, and F and G are distribution functions. Conditions for $D_n\to D$, in distribution or stably, are given. Among other things, such conditions apply to certain sequences of stochastic integrals, when the quadratic variations of the integrand processes converge in distribution but not in probability. An upper bound for the Wasserstein distance between the probability distributions of D_n and D is obtained as well.

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1 The problem

Throughout, all random elements are defined on the same probability space, say $(\Omega,\mathcal{A},P)$, and $(k_n:n\geq 1)$ is a sequence of positive integers such that $\lim_n k_n=\infty$. Moreover, for each $n\geq 1$,

$$(X_{n,j}:j=1,\ldots,k_n)$$

is a vector of real integrable random variables and $\mathcal{F}_{n,1}\subset\ldots\subset\mathcal{F}_{n,k_n}\subset\mathcal{A}$ an increasing sequence of sub-σ-fields. We also let $\mathcal{F}_{n,0}=\{\emptyset,\Omega\}$ and $\mathcal{F}_{n,j}=\mathcal{F}_{n,k_n}$ for $j>k_n$.

The martingale CLT is a classical and basic result. It requires the following conditions (see e.g. [3, Cor. 7] and [6, p. 58–59]):

(a) $\sigma(X_{n,j})\subset\mathcal{F}_{n,j}$ and $E\big(X_{n,j}\mid\mathcal{F}_{n,j-1}\big)=0$ a.s. for all n and j;

(b) $\lim_n E\big\{\max_j|X_{n,j}|\big\}=0$;

(c) $\sum_j X_{n,j}^2\xrightarrow{P}L$, as $n\to\infty$, for some real random variable L;

(d) $\sigma(L)\subset\mathcal{V}$ where $\mathcal{V}=\sigma\Big(\bigcup_j\bigcap_n\sigma\big(\mathcal{F}_{n,j}\cup\mathcal{N}\big)\Big)$ with $\mathcal{N}=\{A\in\mathcal{A}:P(A)=0\}$.

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Theorem 1.1 (Martingale CLT). *Under conditions (a)-(b)-(c)-(d),*

$$\sum_{j=1}^{k_n} X_{n,j} \xrightarrow{d} \sqrt{L}\,Z \qquad \text{as } n\to\infty,$$

where $\xrightarrow{d}$ *denotes convergence in distribution and* Z *is a random variable independent of* L *such that* $Z\sim\mathcal{N}(0,1)$*. Moreover, condition (d) can be replaced by* $\mathcal{F}_{n,j}\subset\mathcal{F}_{n+1,j}$*, for all* n *and* j*, and in this case convergence is stable.*

Suppose now that, in addition to $(X_{n,j})$, we are given a Borel function $f:\mathbb{R}\to\mathbb{R}$ and a further array of real random variables

$$\big(Y_{n,j}: n\geq 1,\, j=1,\dots,k_n\big) \quad \text{such that} \quad \sigma(Y_{n,j})\subset\mathcal{F}_{n,j} \quad \text{for all } n \text{ and } j.$$

Our problem is to investigate the asymptotic behavior of

$$D_n=\sum_{j=1}^{k_n} f\big(S_{n,j-1}\big)\,X_{n,j} \quad \text{where} \quad S_{n,0}=0 \quad \text{and} \quad S_{n,j}=\sum_{i=1}^{j} Y_{n,i}.$$

To motivate this problem, we list some frameworks where D_n plays a role.

(i) D_n arises in parameter estimation and in the analysis of single-path behavior of some processes; see e.g. [1].

(ii) D_n is involved in certain approximation schemes of stochastic differential equations. This issue has been recently the object of intensive research when the equations are driven by fractional Brownian motion; see [4], [5], [8], [11], [14].

(iii) D_n reduces to the *Hermite weighted variation* in the special case

$$Y_{n,j}=B_{\frac{j}{k_n}}-B_{\frac{j-1}{k_n}} \quad \text{and} \quad X_{n,j}=k_n^{-1/2}\mathcal{H}_q\Big(\sqrt{k_n}\,(B_{\frac{j}{k_n}}-B_{\frac{j-1}{k_n}})\Big)$$

where B is Brownian motion and $\mathcal{H}_q$ the Hermite polynomial of degree q. More generally, if B is fractional Brownian motion, Hermite weighted variations have been investigated in [7], [9], [10], [11].

(iv) A possible version of (iii) is

$$Y_{n,j}=k_n^{-1/2}\mathcal{H}_p\Big(\sqrt{k_n}\,(B_{\frac{j}{k_n}}-B_{\frac{j-1}{k_n}})\Big) \quad \text{and} \quad X_{n,j}=k_n^{-1/2}\mathcal{H}_q\Big(\sqrt{k_n}\,(B_{\frac{j}{k_n}}-B_{\frac{j-1}{k_n}})\Big)$$

where $p>0$, $q>0$, $p\neq q$ and B is Brownian motion. To our knowledge, this version has not been studied to date.

To investigate the asymptotic behavior of D_n, an obvious strategy would be applying Theorem 1.1 to $Z_{n,j}=f\big(S_{n,j-1}\big)\,X_{n,j}$. Such a strategy often fails, however, for conditions (c)-(d) are not guaranteed when $X_{n,j}$ is replaced by $Z_{n,j}$. For instance, $\sum_j Z_{n,j}^2$ converges in distribution, but not in probability, to some limit L. This may happen in point (iv) when $p\neq 1$. Or else, $\sum_j Z_{n,j}^2 \xrightarrow{P} L$ but L is not $\mathcal{V}$-measurable. The latter fact can occur even if $f=1$; see Example 3.1.

2 Results

This paper focus on the limiting distribution of D_n. Our main result is Theorem 2.1, which provides a new version of Theorem 1.1 not requiring conditions (c)-(d). While the assumptions replacing (c)-(d) are actually strong, Theorem 2.1 allows to manage some problems not covered by Theorem 1.1. In particular, Theorem 2.1 applies to the approximation of certain stochastic integrals when the quadratic variations of the integrand processes converge in distribution but not in probability. To our knowledge, situations of this type are usually neglected in the literature. Various examples are given in Section 3, where we also obtain an upper bound for the Wasserstein distance between the probability distribution of D_n and that of its limit (Example 3.5). To make the paper more readable, all the proofs are postponed to the final Section 4.

Before stating our results, we introduce some more notation. *From now on*:

- $f:\mathbb{R}\to\mathbb{R}$ is an Holderian function (the definition is recalled at the beginning of Section 4);
- F is a continuous distribution function such that $F(0)=1-F(1)=0$;
- $B=(B_t:t\geq 0)$ is a standard Brownian motion;
- $T=\{T_t:0\leq t\leq 1\}$ is a real cadlag process;
- Z is a standard normal random variable independent of (B,T).

We also introduce the following conditions:

($\star$) There is a constant $u>0$ such that

$$E\Big\{\big(S_{n,j}-S_{n,i}\big)^2\Big\}\leq u\,\frac{j-i}{k_n}\qquad\text{whenever } 0\leq i<j\leq k_n;$$

($\star\star$) $\sup_n\sum_j E(X_{n,j}^2)<\infty$ and there is a constant $v>0$ such that

$$E(X_{n,j}^2\mid\mathcal{F}_{n,j-1})\leq v\,E(X_{n,j}^2)\ \text{ a.s. for all } n \text{ and } j;$$

($\star\star\star$) For each $m\geq 1$, the $2m$-dimensional vectors

$$\Big(S_{n,\lfloor\frac{hk_n}{m}\rfloor},\ \sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}X_{n,j}\ :\ h=0,1,\ldots,m-1\Big)$$

converge in distribution to

$$\Big(T_{\frac{h}{m}},\,u_h\,Z_h:\ h=0,1,\ldots,m-1\Big)$$

where $u_h^2=F((h+1)/m)-F(h/m)$ and $(Z_0,Z_1,\ldots)$ is an i.i.d. sequence of standard normal random variables independent of T.

Condition ($\star$) is automatically true if $(S_{n,j}:0\leq j\leq k_n)$ is a martingale and $E(Y_{n,j}^2)\leq u/k_n$ for all n, j. Condition ($\star\star$) is actually strong. However, ($\star\star$) is trivially true when $\sup_n\sum_j E(X_{n,j}^2)<\infty$ and $X_{n,j}$ is independent of $\mathcal{F}_{n,j-1}$. Moreover, in forthcoming Theorem 2.1 and Corollary 2.2, condition ($\star\star$) may be replaced by another assumption. An analogous remark applies to ($\star\star\star$). While admittedly involved, condition ($\star\star\star$) holds in a few meaningful situations.

Our main result is the following.

Theorem 2.1. *Under conditions (a)-($\star$)-($\star\star$)-($\star\star\star$), one obtains* $D_n\xrightarrow{d}D$ *where*

$$D=Z\sqrt{\int_0^1 f^2(T_t)\,dF(t)}.$$

Moreover, conditions ($\star$)-($\star\star$) can be replaced by

$$E\big(X_{n,j}^4\big)\leq\frac{u}{k_n^2}\quad\text{and}\quad E\Big\{\big(S_{n,j}-S_{n,i}\big)^4\Big\}\leq u\,\Big(\frac{j-i}{k_n}\Big)^2\tag{2.1}$$

for some constant $u>0$ *and all* $0\leq i<j\leq k_n$.

Corollary 2.2. *Let $T_t = B_{G(t)}$ for each $t \in [0,1]$, where G is a distribution function such that $G(0)=0$. Then, $D_n \xrightarrow{d} D$ provided:*

- *Conditions (a)-(b) hold;*
- $\lim_n E\{\max_j |Y_{n,j}|\} = 0$ *and* $E\big(Y_{n,j} \mid \mathcal{F}_{n,j-1}\big) = 0$ *a.s. for all n and j;*
- $\sup_n \sum_j E(X_{n,j}^2) < \infty$ *and there are constants u, $v > 0$ such that*

$$E(Y_{n,j}^2) \leq u/k_n \quad \text{and} \quad E(X_{n,j}^2 \mid \mathcal{F}_{n,j-1}) \leq v\, E(X_{n,j}^2) \ \text{a.s. for all } n \text{ and } j;$$

(this condition may be replaced by (2.1)*);*
- *For all integers $0 \leq h < m$, one obtains*

$$\sum_{j=1+\lfloor \frac{hk_n}{m} \rfloor}^{\lfloor \frac{(h+1)k_n}{m} \rfloor} X_{n,j}^2 \xrightarrow{P} F((h+1)/m) - F(h/m),$$

$$\sum_{j=1+\lfloor \frac{hk_n}{m} \rfloor}^{\lfloor \frac{(h+1)k_n}{m} \rfloor} Y_{n,j}^2 \xrightarrow{P} G((h+1)/m) - G(h/m) \quad \text{and} \quad \sum_{j=1+\lfloor \frac{hk_n}{m} \rfloor}^{\lfloor \frac{(h+1)k_n}{m} \rfloor} X_{n,j}\, Y_{n,j} \xrightarrow{P} 0.$$

To apply Corollary 2.2, we fix two centered orthonormal elements of $L_2(\mathcal{N}(0,1))$, say ϕ and ψ. Thus, $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are Borel functions satisfying

$$\int \phi^2 d\gamma = \int \psi^2 d\gamma = 1 \quad \text{and} \quad \int \phi\, d\gamma = \int \psi\, d\gamma = \int \phi\, \psi\, d\gamma = 0$$

where $\gamma = \mathcal{N}(0,1)$. For instance, as in point (iv) of Section 1, one could take

$$\phi = \frac{\mathcal{H}_p}{\sqrt{\int \mathcal{H}_p^2 d\gamma}} \quad \text{and} \quad \psi = \frac{\mathcal{H}_q}{\sqrt{\int \mathcal{H}_q^2 d\gamma}}$$

where $p > 0$, $q > 0$, $p \neq q$ and $\mathcal{H}_j$ denotes the Hermite polynomial of degree j. Our goal is to find the limiting distribution of D_n when

$$X_{n,j} = k_n^{-1/2} \psi\Big(\sqrt{k_n}\,(B_{\frac{j}{k_n}} - B_{\frac{j-1}{k_n}})\Big) \quad \text{and} \quad Y_{n,j} = k_n^{-1/2} \phi\Big(\sqrt{k_n}\,(B_{\frac{j}{k_n}} - B_{\frac{j-1}{k_n}})\Big). \tag{2.2}$$

Corollary 2.3. *Let $T_t = B_t$ and $F(t) = t$ for all $t \in [0,1]$. If $X_{n,j}$ and $Y_{n,j}$ are given by* (2.2)*, then $D_n \xrightarrow{d} D$.*

Our last result is that, up to requiring some further conditions, $D_n \to D$ stably. Here, stable convergence is meant as follows. We say that $D_n \to D$ *stably with respect to $\mathcal{G}$*, where $\mathcal{G} \subset \mathcal{A}$ is any sub-σ-field, if

$$D_n \xrightarrow{d} D \quad \text{under } P(\cdot \mid H) \quad \text{for each } H \in \mathcal{G} \text{ with } P(H) > 0.$$

Note that $D_n \to D$ stably with respect to $\sigma(V)$, where V is a random variable with values in a separable metric space, if and only if $(D_n, V) \xrightarrow{d} (D, V)$.

Corollary 2.4. *Let $T_0 = 0$ a.s. and $S_{n,j} = T_{\frac{j}{k_n}}$ for all n and j. Assume the conditions of Theorem 2.1 or those of Corollaries 2.2 or 2.3. Then, $D_n \to D$ stably with respect to $\sigma(T)$.*

As an example, under the conditions of Corollary 2.4, one obtains

$$\frac{D_n}{\sqrt{\int_0^1 f^2(T_t)\, dF(t)}} \xrightarrow{d} Z \sim \mathcal{N}(0,1)$$

provided $\int_0^1 f^2(T_t)\,dF(t) > 0$ a.s. We also recall that $T_t = B_{G(t)}$ in case of Corollary 2.2 while $T_t = B_t$ and $F(t) = t$ in case of Corollary 2.3.

In the special case of Brownian motion, Corollaries 2.2, 2.3 and 2.4 improve on some existing results (which concern, more generally, fractional Brownian motion); see [7], [11] and references therein.

3 Examples

We begin by noting that, if condition (d) is dropped, D_n may fail to converge in distribution even if $f = 1$ and conditions (a)-(b)-(c) hold.

Example 3.1 (Example 4 of [6] revisited)**.** Let $f = 1$, $k_n = 3n$, and

$$X_{n,j} = 1_{\{B_1>0\}}\left(B_{\frac{j}{n}} - B_{\frac{j-1}{n}}\right) \quad \text{if } n < j \le 2n.$$

Define also

$$X_{n,j} = B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \quad \text{if } 1 \le j \le n \quad \text{and} \quad X_{n,j} = 0 \quad \text{if } 2n < j \le 3n$$

or

$$X_{n,j} = 0 \quad \text{if } 1 \le j \le n \quad \text{and} \quad X_{n,j} = B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \quad \text{if } 2n < j \le 3n$$

according to whether n is even or odd. Then, conditions (a)-(b)-(c) hold with $\mathcal{F}_{n,j} = \sigma(B_t : t \le j/n)$. However, $D_n = \sum_j X_{n,j}$ does not converge in distribution. Define in fact

$$U = B_1 + 1_{\{B_1>0\}}\,(B_2 - B_1) \quad \text{and} \quad V = 1_{\{B_1>0\}}\,(B_2 - B_1) + B_3 - B_2.$$

Then, $\sum_j X_{n,j} = U$ if n is even, $\sum_j X_{n,j} = V$ if n is odd, but U and V have different probability distributions (for instance, $P(U < 0) \ne P(V < 0)$).

The next example deals with a sequence of integrals driven by a compensated Poisson process.

Example 3.2. Let $\mathcal{B}$ be the Borel σ-field on $\mathbb{R}^+$ and $N = \big\{N(B) : B \in \mathcal{B}\big\}$ a Poisson process on $(\mathbb{R}^+, \mathcal{B})$ with intensity λ. Thus, $N(B)$ is a Poisson random variable with parameter $\lambda(B)$ and $N(B_1), \dots, N(B_k)$ are independent whenever $B_1, \dots, B_k$ are pairwise disjoint. Moreover, λ is a non-atomic Radon measure on $\mathcal{B}$ such that $\lambda(\mathbb{R}^+) = \infty$. Let

$$\Psi = \sum_{l=2}^{\infty} a_l\, C_l$$

where a_l is a constant, C_l is the Charlier polynomial of degree l and parameter 1, and $\sum_{l=2}^{\infty} l!\, a_l^2 < \infty$. Define

$$V_t = N([0,t]) - \lambda([0,t]) \quad \text{and} \quad I_n = (r_{n-1}, r_n]$$

where $0 = r_0 < r_1 < r_2 < \dots$ are such that $\lambda((r_{n-1}, r_n]) = 1$. Finally, let

$$q = \sum_{l=2}^{\infty} l!\, a_l^2 \quad \text{and} \quad S_n(t) = \frac{1}{\sqrt{q\,n}} \sum_{j=1}^{n} 1_{I_j}(t) \sum_{i=1}^{j-1} \Psi\big(N(I_i)\big).$$

Then, Corollary 2.2 yields

$$\frac{1}{\sqrt{n}} \int_0^{r_n} f\big(S_n(t)\big)\,dV_t \xrightarrow{d} Z\sqrt{\int_0^1 f^2(B_t)\,dt}. \tag{3.1}$$

Define in fact

$$k_n = n, \quad Y_{n,j} = (q\,n)^{-1/2}\Psi\big(N(I_j)\big), \quad X_{n,j} = n^{-1/2}\left(V_{r_j} - V_{r_{j-1}}\right) = n^{-1/2}\left(N(I_j) - 1\right).$$

On noting that $S_n(t) = S_{n,j-1}$ for each $t \in I_j$, one obtains

$$\frac{1}{\sqrt{n}}\int_0^{r_n} f\big(S_n(t)\big)\,dV_t = \frac{1}{\sqrt{n}}\sum_{j=1}^n \int_{r_{j-1}}^{r_j} f\big(S_{n,j-1}\big)\,dV_t = \sum_{j=1}^n f\big(S_{n,j-1}\big)\,X_{n,j} = D_n.$$

Moreover, the sequence $L_j := \big(N(I_j)-1\big)\,\Psi\big(N(I_j)\big)$, $j \geq 1$, is i.i.d. with

$$E(L_1) = E\Big\{\,\big(N(I_1)-1\big)\,\Psi\big(N(I_1)\big)\Big\} = \sum_{l=2}^{\infty} a_l\,E\Big\{\,C_1\big(N(I_1)\big)\,C_l\big(N(I_1)\big)\Big\} = 0.$$

Therefore, given $m \geq 1$ and $0 \leq h < m$, the SLLN implies

$$\sum_{j=1+\lfloor\frac{h\,n}{m}\rfloor}^{\lfloor\frac{(h+1)\,n}{m}\rfloor} X_{n,j}Y_{n,j} = \frac{1}{\sqrt{q}\,n}\sum_{j=1+\lfloor\frac{h\,n}{m}\rfloor}^{\lfloor\frac{(h+1)\,n}{m}\rfloor} L_j \overset{a.s.}{\longrightarrow} \frac{1}{\sqrt{q}\,m}\,E(L_1) = 0.$$

All the other conditions of Corollary 2.2 are easily seen to be true with $F(t) = G(t) = t$ for all $t \in [0,1]$. Hence, (3.1) holds. Finally, we note that D_n converges stably with respect to $\sigma(N)$ (and not only in distribution). This is a direct consequence of (3.1), however, which can be shown without involving Corollary 2.4.

We next turn to Wiener integrals.

Example 3.3. For any $h \in L_2([0,1])$, the Wiener integral of h is

$$W_t(h) = \int_0^t h(u)\,dB_u, \quad 0 \leq t \leq 1.$$

In this notation, we fix $f_n,\, g_n \in L_2([0,1])$ and we let

$$J_n = \int_0^1 f\Big(W_t(g_n)\Big)\,dW_t(f_n).$$

We aim to give conditions for J_n to converge in distribution. Suppose

$$\lim_n \int_0^t f_n(u)\,g_n(u)\,du = 0, \quad \lim_n \int_0^t f_n^2(u)\,du = F(t), \quad \lim_n \int_0^t g_n^2(u)\,du = G(t) \qquad (3.2)$$

for each $0 \leq t \leq 1$, where F and G are distribution functions, F is continuous and $F(0) = 1 - F(1) = 0$. Under this condition, define

$$X_{n,j} = \int_{\frac{j-1}{k_n}}^{\frac{j}{k_n}} f_n(u)\,dB_u \quad \text{and} \quad Y_{n,j} = \int_{\frac{j-1}{k_n}}^{\frac{j}{k_n}} g_n(u)\,dB_u.$$

Since $W_{\cdot}(f_n)$ and $W_{\cdot}(g_n)$ have independent increments, the conditions of Corollary 2.2 are easily seen to be true. In addition, the sequence (k_n) can be taken such that $D_n - J_n \overset{P}{\longrightarrow} 0$. Therefore,

$$J_n \overset{d}{\longrightarrow} Z\sqrt{\int_0^1 f^2(B_{G(t)})\,dF(t)}.$$

A last remark is that condition (3.2) does not imply convergence in measure of f_n^2 or g_n^2 (with respect to Lebesgue measure). For instance, if $f_n(t) = \sqrt{2}\,\sin(nt)$ and $g_n(t) = \sqrt{2}\,\cos(nt)$, then (3.2) holds but neither f_n^2 nor g_n^2 converge in measure.

We now consider a situation where the integrand processes are essentially fractional. For instance, they could be fractional Brownian motions.

Example 3.4. Let

$$T_t = \int_0^1 K(s,t)\,dB_s \quad \text{and} \quad J_n = \int_0^1 f(T_t)\,f_n(t)\,dB_t$$

where $K \in L_2([0,1]^2)$ and $f_n \in L_2([0,1])$. Suppose

$$\lim_n \int_0^t f_n(u)\,du = 0 \quad \text{and} \quad \lim_n \int_0^t f_n^2(u)\,du = F(t), \qquad 0 \le t \le 1,$$

for some continuous distribution function F such that $F(1) = 1$. Then,

$$J_n \longrightarrow Z\sqrt{\int_0^1 f^2(T_t)\,dF(t)} \quad \text{stably with respect to } \sigma(T).$$

Define in fact

$$X_{n,j} = \int_{\frac{j-1}{k_n}}^{\frac{j}{k_n}} f_n(t)\,dB_t \quad \text{and} \quad Y_{n,j} = T_{\frac{j}{k_n}} - T_{\frac{j-1}{k_n}}.$$

As in Example 3.3, the sequence (k_n) can be taken such that $D_n - J_n \xrightarrow{P} 0$. Moreover, the conditions of Theorem 2.1 are satisfied. Hence, stable convergence of J_n follows from Corollary 2.4. Finally, we mention how to check the conditions of Theorem 2.1. The basic remark is that $\big(B_{\frac{j}{k_n}} - B_{\frac{j-1}{k_n}} : 1 \le j \le k_n\big)$ is a martingale difference and

$$\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor} E\big\{X_{n,j}\,\big(B_{\frac{j}{k_n}} - B_{\frac{j-1}{k_n}}\big)\big\} = \sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor} \int_{\frac{j-1}{k_n}}^{\frac{j}{k_n}} f_n(t)\,dt \longrightarrow 0$$

for all $m \ge 1$ and $0 \le h < m$. Based on this remark and $\sigma(T_s : s \le t) \subset \sigma(B_s : s \le t)$, the conditions of Theorem 2.1 can be proved by the same argument used in the proof of Corollary 2.2.

Our last example provides an upper bound for the Wasserstein distance between the probability distributions of D_n and D. The required conditions are actually strong. However, in the special case of Brownian motion, such conditions improve on the results known so far; see e.g. Theorem 1.1 of [7]. We also note that, based on these conditions, one could also obtain upper bounds for the total variation distance; see [12] and [13].

Recall that, if X and Y are real integrable random variables, the Wasserstein distance between their probability distributions is $W(X,Y) = \sup_h |E\big(h(X)\big) - E\big(h(Y)\big)|$ where $\sup$ is over the 1-Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$.

Example 3.5. Suppose that:

- f is a bounded Lipschitz function such that $\inf|f| > 0$;
- $k_n = h_n^2$ for some integer h_n;
- For each n, the random vectors $(U_{n,j}, V_{n,j})$, $1 \le j \le k_n$, are i.i.d. with

  $$E(U_{n,1}) = E(V_{n,1}) = E(U_{n,1}V_{n,1}) = 0, \quad E(U_{n,1}^2) = E(V_{n,1}^2) = 1, \quad E\big\{U_{n,1}^4 + V_{n,1}^4\big\} \le u$$

  for some constant u independent of n.

Define $X_{n,j} = \frac{U_{n,j}}{h_n}$, $Y_{n,j} = \frac{V_{n,j}}{h_n}$ and $D = Z\sqrt{\int_0^1 f^2(B_t)\,dt}$. Then, there is a constant c independent of n such that

$$W(D_n, D) \le \frac{c}{k_n^{1/4}} \qquad \text{for each } n \ge 1. \tag{3.3}$$

Inequality (3.3) applies, for instance, to Examples 3.2 and 3.3 (provided f and k_n are as above). Its proof is reported in Section 4.

4 Proofs

In the sequel, as in condition $(\star\star\star)$, we let $u_h = \sqrt{F((h+1)/m) - F(h/m)}$ and we denote by $(Z_0, Z_1, \ldots)$ a i.i.d. sequence of standard normal random variables independent of T. Moreover, $f : \mathbb{R} \to \mathbb{R}$ is Holderian of parameter α. Precisely, there are two constants $\alpha \in (0,1]$ and $b > 0$ such that

$$|f(x) - f(y)| \le b\,|x-y|^\alpha \qquad \text{for all } x,\, y \in \mathbb{R}.$$

We begin with a technical lemma.

Lemma 4.1. *Fix an integer $1 \le m \le k_n$ and assume conditions (a)-$(\star)$-$(\star\star)$ or conditions (a)-(2.1). For each $h = 0, 1, \ldots, m-1$, define*

$$S_{n,j}^{(m)} = S_{n,\lfloor \frac{hk_n}{m} \rfloor} \quad \textit{if} \quad \lfloor \frac{hk_n}{m} \rfloor \le j < \lfloor \frac{(h+1)k_n}{m} \rfloor.$$

Define also

$$D_n^{(m)} = \sum_{j=1}^{k_n} f\big(S_{n,j-1}^{(m)}\big)\, X_{n,j}.$$

Then, there is a constant c, independent of n and m, such that

$$E\big\{(D_n - D_n^{(m)})^2\big\} \le \frac{c}{m^\beta}$$

where $\beta = \alpha$ under (a)-$(\star)$-$(\star\star)$ and $\beta = \alpha/2$ under (a)-(2.1).

Proof. Assume (a)-$(\star)$-$(\star\star)$ and define $r = \sup_n \sum_{j=1}^{k_n} E(X_{n,j}^2)$. Then,

$$\begin{aligned}
E\big\{(D_n - D_n^{(m)})^2\big\} &= \sum_{j=1}^{k_n} E\big\{\big(f(S_{n,j-1}) - f(S_{n,j-1}^{(m)})\big)^2 X_{n,j}^2\big\} \\
&\le v \sum_{j=1}^{k_n} E\big\{\big(f(S_{n,j-1}) - f(S_{n,j-1}^{(m)})\big)^2\big\}\, E(X_{n,j}^2) \\
&\le r\, v \max_{1\le j\le k_n} E\big\{\big(f(S_{n,j-1}) - f(S_{n,j-1}^{(m)})\big)^2\big\} \\
&\le r\, v\, b^2 \max_{1\le j\le k_n} E\big\{|S_{n,j-1} - S_{n,j-1}^{(m)}|^{2\alpha}\big\} \\
&\le r\, v\, b^2 \max_{1\le j\le k_n} E\big\{\big(S_{n,j-1} - S_{n,j-1}^{(m)}\big)^2\big\}^\alpha.
\end{aligned}$$

Hence, to prove the first part of the Lemma, it suffices noting that

$$E\big\{\big(S_{n,j-1} - S_{n,j-1}^{(m)}\big)^2\big\} = E\big\{\big(S_{n,j-1} - S_{n,\lfloor \frac{hk_n}{m}\rfloor}\big)^2\big\} \tag{4.1}$$

$$\le u\,\frac{j-1-\lfloor \frac{hk_n}{m}\rfloor}{k_n} \le \frac{u}{m} \quad \text{whenever } 1 + \lfloor \frac{hk_n}{m} \rfloor \le j \le \lfloor \frac{(h+1)k_n}{m} \rfloor.$$

Similarly, under conditions (a)-(2.1), one obtains

$$\begin{aligned}
E\big\{(D_n - D_n^{(m)})^2\big\} &= \sum_{j=1}^{k_n} E\big\{\big(f(S_{n,j-1}) - f(S_{n,j-1}^{(m)})\big)^2 X_{n,j}^2\big\} \\
&\le \sum_{j=1}^{k_n} \sqrt{E\big\{\big(f(S_{n,j-1}) - f(S_{n,j-1}^{(m)})\big)^4\big\}\, E(X_{n,j}^4)}
\end{aligned}$$

$$\le \frac{b^2u^{1/2}}{k_n}\sum_{j=1}^{k_n} E\big\{(S_{n,j-1}-S_{n,j-1}^{(m)})^4\big\}^{\alpha/2}$$
$$\le \frac{b^2u^{1/2}}{k_n}\sum_{j=1}^{k_n}\Big(\frac{u}{m}\Big)^{\alpha/2} = \frac{b^2u^{1/2}u^{\alpha/2}}{m^{\alpha/2}}$$

where the last inequality follows from (4.1). This concludes the proof. □

Proof of Theorem 2.1. For any real random variables X and Y, the *bounded Lipschitz metric* between their probability distributions is $\rho(X,Y)=\sup_h |E\big(h(X)\big)-E\big(h(Y)\big)|$, where $\sup$ is over the 1-Lipschitz functions $h:\mathbb{R}\to\mathbb{R}$ such that $-1\le h\le 1$. This metric has the property that, for any sequence (X_n) of real random variables, $\lim_n \rho(X_n,X)=0$ if and only if $X_n \xrightarrow{d} X$. Next, assume conditions (a)-($\star$)-($\star\star$)-($\star\star\star$) and define

$$D^{(m)} = Z\sqrt{\sum_{h=0}^{m-1} f^2\big(T_{\frac{h}{m}}\big)\,u_h^2}\qquad \text{for } m\ge 1.$$

Writing $D_n^{(m)}$ as

$$D_n^{(m)} = \sum_{h=0}^{m-1} f\big(S_{n,\lfloor\frac{hk_n}{m}\rfloor}\big)\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor} X_{n,j},$$

condition ($\star\star\star$) implies $D_n^{(m)} \xrightarrow{d} \sum_{h=0}^{m-1} f\big(T_{\frac{h}{m}}\big)\,u_h\,Z_h$. Moreover, it is not hard to see that $\sum_{h=0}^{m-1} f\big(T_{\frac{h}{m}}\big)\,u_h\,Z_h$ has the same distribution as $D^{(m)}$. Hence, for fixed m, condition ($\star\star\star$) yields $D_n^{(m)}\xrightarrow{d} D^{(m)}$ as $n\to\infty$. Observe now that

$$\rho(D_n,D_n^{(m)})\le E|D_n-D_n^{(m)}|\le\sqrt{E\big\{(D_n-D_n^{(m)})^2\big\}}\le\sqrt{\frac{c}{m^\alpha}}$$

where the last inequality is because of Lemma 4.1. Therefore,

$$\limsup_n \rho(D_n,D)\le \limsup_n\big\{\rho(D_n,D_n^{(m)})+\rho(D_n^{(m)},D^{(m)})+\rho(D^{(m)},D)\big\}$$
$$\le \sqrt{c}\,m^{-\alpha/2}+\limsup_n \rho(D_n^{(m)},D^{(m)})+\rho(D^{(m)},D)=\sqrt{c}\,m^{-\alpha/2}+\rho(D^{(m)},D).$$

Finally, since F is continuous and the process $\big\{f^2(T_t):0\le t\le 1\big\}$ has cadlag paths,

$$\int_0^1 f^2(T_t)\,dF(t)=\lim_m\sum_{h=0}^{m-1} f^2\big(T_{\frac{h}{m}}\big)\,u_h^2.$$

Hence, $D^{(m)}\to D$ as $m\to\infty$, and this implies

$$\limsup_n \rho(D_n,D)\le\lim_m\big\{\sqrt{c}\,m^{-\alpha/2}+\rho(D^{(m)},D)\big\}=0.$$

This proves that $D_n\xrightarrow{d} D$ under conditions (a)-($\star$)-($\star\star$)-($\star\star\star$). The proof is exactly the same if ($\star$)-($\star\star$) are replaced by (2.1). □

Proof of Corollary 2.2. It suffices to prove ($\star\star\star$), since all the other conditions of Theorem 2.1 are trivially true. Let $T_t=B_{G(t)}$ for $t\in[0,1]$. With this choice of T, condition ($\star\star\star$) is equivalent to requiring that, for each $m\ge 1$, the vectors

$$\Big(\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor} Y_{n,j}\ ,\ \sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor} X_{n,j}\ :\ h=0,1,\dots,m-1\Big)$$

converge in distribution to

$$\Big(B_{G(\frac{h+1}{m})} - B_{G(\frac{h}{m})}\,,\,u_h\,Z_h:\ h=0,1,\dots,m-1\Big).$$

Fix $m\ge 1$, two vectors $(\alpha_0,\dots,\alpha_{m-1})\in\mathbb{R}^m$ and $(\beta_0,\dots,\beta_{m-1})\in\mathbb{R}^m$, and define

$$M_n=\sum_{h=0}^{m-1}\Big\{\alpha_h\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}Y_{n,j}\;+\;\beta_h\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}X_{n,j}\Big\}\quad\text{and}$$

$$M=\sum_{h=0}^{m-1}\Big\{\alpha_h\big(B_{G(\frac{h+1}{m})}-B_{G(\frac{h}{m})}\big)+\beta_h\,u_h\,Z_h\Big\}.$$

Then, it suffices to show that $M_n\xrightarrow{d}M$. In turn, $M_n\xrightarrow{d}M$ follows from Theorem 1.1. Define in fact

$$U_{n,j}=\alpha_h\,Y_{n,j}+\beta_h\,X_{n,j}\quad\text{whenever}\quad 1+\lfloor\frac{hk_n}{m}\rfloor\le j\le\lfloor\frac{(h+1)k_n}{m}\rfloor$$

$$\text{and}\quad q=\sum_{h=0}^{m-1}\Big\{\alpha_h^2\big(G((h+1)/m)-G(h/m)\big)+\beta_h^2\big(F((h+1)/m)-F(h/m)\big)\Big\}.$$

Then, $M_n=\sum_{j=1}^{k_n}U_{n,j}$ and $M\sim\mathcal{N}(0,q)$ (since (Z_h) is independent of $T=B_G$). Conditions (a)-(b) hold by assumption. Condition (c) holds, with $L=q$, since

$$\sum_{j=1}^{k_n}U_{n,j}^2=\sum_{h=0}^{m-1}\Big\{\alpha_h^2\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}Y_{n,j}^2\;+\;\beta_h^2\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}X_{n,j}^2\;+\;2\alpha_h\beta_h\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}X_{n,j}Y_{n,j}\Big\}.$$

Finally, condition (d) is trivially true for $L=q$ is a constant. Hence, $(U_{n,j})$ satisfies the conditions of Theorem 1.1 and this concludes the proof. □

Proof of Corollary 2.3. If $X_{n,j}$ and $Y_{n,j}$ are given by (2.2), the SLLN yields

$$\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}X_{n,j}^2=\frac{1}{k_n}\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}\psi^2\big(\sqrt{k}_n(B_{\frac{j}{k_n}}-B_{\frac{j-1}{k_n}})\big)\xrightarrow{L_1}\frac{E\{\psi^2(B_1)\}}{m}=\frac{1}{m}.$$

Similarly,

$$\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}Y_{n,j}^2\xrightarrow{L_1}\frac{1}{m}\quad\text{and}\quad\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}X_{n,j}\,Y_{n,j}\xrightarrow{L_1}\frac{E\{\psi(B_1)\,\phi(B_1)\}}{m}=0.$$

Having noted these facts, it is straightforward to check the conditions of Corollary 2.2 with $u=v=1$, $\mathcal{F}_{n,j}=\sigma\big(B_t:t\le j/k_n\big)$ and $F(t)=G(t)=t$ for all $t\in[0,1]$. □

Proof of Corollary 2.4. Since $T_0=0$ a.s., it suffices to show that

$$\Big(D_n,T_{\frac{h}{m}}:h=1,\dots,m\Big)\xrightarrow{d}\Big(D,T_{\frac{h}{m}}:h=1,\dots,m\Big)\tag{4.2}$$

for each $m\ge 1$. To this end, since the conditions of Corollaries 2.2 or 2.3 imply those of Theorem 2.1, we can assume the conditions of Theorem 2.1. Define

$$V_n^{(m)}=\Big(T_{\frac{h+1}{m}}-T_{\frac{h}{m}},\,S_{n,\lfloor\frac{(h+1)k_n}{m}\rfloor}-S_{n,\lfloor\frac{hk_n}{m}\rfloor},\sum_{j=1+\lfloor\frac{hk_n}{m}\rfloor}^{\lfloor\frac{(h+1)k_n}{m}\rfloor}X_{n,j}\;:\;h=0,1,\dots,m-1\Big)$$

$$\text{and}\quad V^{(m)}=\Big(T_{\frac{h+1}{m}}-T_{\frac{h}{m}},\,T_{\frac{h+1}{m}}-T_{\frac{h}{m}},\,u_h\,Z_h\;:\;h=0,1,\dots,m-1\Big).$$

Since $S_{n,j} = T_{\frac{j}{k_n}}$, condition $(\star\star\star)$ implies $V_n^{(m)} \xrightarrow{d} V^{(m)}$ as $n \to \infty$ for all $m \geq 1$. Hence, condition (4.2) follows from the argument used in the proof of Theorem 2.1. □

Proof of inequality (3.3). We just give a sketch of the proof.

Step 1. Since conditions (a)-$(\star)$-$(\star\star)$ hold and f is Lipschitz, Lemma 4.1 applies with $\beta = \alpha = 1$ and $m = h_n$. Hence, there is a constant q independent of n such that

$$W\big(D_n, D_n^{(h_n)}\big) \leq E|D_n - D_n^{(h_n)}| \leq \sqrt{E\big\{\big(D_n - D_n^{(h_n)}\big)^2\big\}} \leq \frac{q}{\sqrt{h_n}}.$$

Step 2. Let b be the Lipschitz constant of f and

$$M_n = \sqrt{\frac{1}{h_n} \sum_{j=0}^{h_n-1} f^2\big(B_{\frac{j}{h_n}}\big)} \quad \text{and} \quad M = \sqrt{\int_0^1 f^2(B_t)\, dt}.$$

Since $D = ZM$ and $M_n + M \geq \inf|f| > 0$, one obtains

$$\begin{aligned} W\big(ZM_n,\, D\big) &\leq E|ZM_n - ZM| \leq E|M_n - M| \leq \frac{1}{\inf|f|}\, E|M_n^2 - M^2| \\ &\leq \frac{2\, b\, \sup|f|}{\inf|f|} \sum_{j=0}^{h_n-1} \int_{\frac{j}{h_n}}^{\frac{j+1}{h_n}} E|B_t - B_{\frac{j}{h_n}}|\, dt \leq \frac{2\, b\, \sup|f|}{\inf|f|}\, \frac{1}{\sqrt{h_n}}. \end{aligned}$$

Step 3. For $k = 0, 1, \ldots, h_n - 1$, define

$$X_{n,k}^* = \frac{1}{\sqrt{h_n}} \sum_{j=1+kh_n}^{(k+1)h_n} U_{n,j} \quad \text{and} \quad Y_{n,k}^* = \frac{1}{\sqrt{h_n}} \sum_{j=1+kh_n}^{(k+1)h_n} V_{n,j}.$$

For each $n \geq 1$, the bivariate random vectors $\big(X_{n,k}^*, Y_{n,k}^*\big)$, $0 \leq k < h_n$, are i.i.d. with mean 0 and covariance matrix I (where I is the identity matrix). Hence, by a result of Bonis [2], there is a constant β independent of n such that

$$W_2\Big[\big(X_{n,k}^*, Y_{n,k}^*\big),\, \mathbf{Z}\Big] \leq \frac{\beta\, \sqrt{E(U_{n,1}^4 + V_{n,1}^4)}}{\sqrt{h_n}} \qquad \text{for each } k = 0, 1, \ldots, h_n - 1,$$

where W_2 is the Wasserstein distance of order 2 and $\mathbf{Z}$ a Gaussian bivariate random vector with mean 0 and covariance matrix I.

Step 4. First note that

$$D_n^{(h_n)} = \frac{f(0)}{\sqrt{h_n}}\, X_{n,0}^* + \frac{1}{\sqrt{h_n}} \sum_{k=1}^{h_n-1} f\left(\frac{\sum_{i=0}^{k-1} Y_{n,i}^*}{\sqrt{h_n}}\right) X_{n,k}^*.$$

Exploiting this formula for $D_n^{(h_n)}$ and the assumptions on f, one obtains

$$W\big(D_n^{(h_n)},\, ZM_n\big) \leq \delta \sqrt{\frac{1}{h_n} \sum_{k=0}^{h_n-1} W_2^2\Big[\big(X_{n,k}^*, Y_{n,k}^*\big),\, \mathbf{Z}\Big]}$$

where δ is a constant idependent of n. (The latter inequality actually requires some algebra, but we omit the explicit calculations). Hence, step 3 implies

$$W\big(D_n^{(h_n)},\, ZM_n\big) \leq \delta \sqrt{\frac{\beta^2 E(U_{n,1}^4 + V_{n,1}^4)}{h_n}} \leq \frac{\delta\beta\, \sqrt{u}}{\sqrt{h_n}}.$$

In view of steps 1 and 2, this concludes the proof. □

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