Electron. Commun. Probab. **28** (2023), article no. 28, 1–8.
https://doi.org/10.1214/23-ECP532
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Transience of simple random walks with linear entropy growth

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Abstract

Using the technique of evolving sets, we explore the connection between entropy growth and transience for simple random walks on connected infinite graphs with bounded degree. In particular we show that for a simple random walk starting at a vertex x_0, if the entropy after n steps, E_n is at least Cn where the C is independent of x_0, then the random walk is transient. We also give an example which demonstrates that the condition of C being independent of x_0 is necessary.

Keywords: simple random walk; entropy; transience.
MSC2020 subject classifications: 60J10.
Submitted to ECP on February 13, 2023, final version accepted on July 10, 2023.
Supersedes arXiv:2302.07496v2.

1 Introduction

Let $X_1, X_2, \ldots$ be simple random walk on an infinite graph of maximum degree d and let p be its transition matrix. Let V denote the set of vertices of this graph. The main result of this paper is Theorem 1.1 given below which gives a connection between the entropy growth of the random walk and its transience.

Theorem 1.1. *Suppose that $X_0 = x_0$. Let E_n be the entropy of X_n, i.e., $E_n = -\sum_{x\in V} \mathbf{P}(X_n = x) \log\big(\mathbf{P}(X_n = x)\big)$. If*

$$E_n \geq Cn \tag{1.1}$$

for some C independent of x_0, then the random walk is transient.

This problem was suggested to us by Itai Benjamini.

Note that the entropy defined in the above theorem is a finite sum since each vertex has finite degree and hence the support of X_n is finite. So the entropy makes perfect sense and there is no question of convergence. Henceforth whenever the inequality (1.1) holds for some C independent of the starting position, we will say that the *linear entropy condition* holds. To see why the bounded degree assumption is necessary, note that if we append b_k pendant edges to vertex k in $\mathbf{Z}^+$ (the graph of positive integers) then the resulting graph is still recurrent, and if the sequence b_k grows fast enough then the walk satisfies the linear growth condition. In Section 6 we give an example (told to us by Gady Kozma) to show that the theorem fails to hold if the C in (1.1) is not independent of the starting position of the random walk.

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2 Entropy and the probability of escape in n steps

2.1 Background and notation

For any set $V' \subset V$, define $p^n(x, V') := \sum_{v \in V'} p^n(x, v)$, with a similar definition for $p(V', x)$. For real numbers a and b, define $a \wedge b := \min(a, b)$. Our proof of Theorem 1.1 uses a set valued process called the evolving set process, which was used in [4] to obtain bounds on mixing times of Markov chains in terms of isoperimetric inequaltiies. The notion of evolving sets is related to strong stationary duals introduced by Fill and Diaconis [1]. Before we introduce evolving sets, let's make the following observation. Recall that d is the maximum degree of G.

Lemma 2.1. *Assume that the linear entropy growth condition holds and that $X_0 = x_0$. Then for any set $A \subset V$ and integer $n \geq 1$, we have*

$$p^n(x_0, A^c) \geq \frac{Cn - \log 2|A|}{n \log d}. \tag{2.1}$$

Proof. Fix $A \subset V$ and define $A^c := \mathrm{support}(X_n) \setminus A$. For $i \in V$ define $p_i := p^n(x_0, i)$ and define

$$q := p^n(x_0, A^c) = \sum_{i \in A^c} p_i$$

Note that, by concavity of the function $x \mapsto -x \log x$, the average value of $-p_i \log p_i$ over the set A is at most $-\frac{1-q}{|A|} \log \frac{1-q}{|A|}$. Furthermore, the average value of $-p_i \log p_i$ over the set A^c is at most $-\frac{q}{|A^c|} \log \frac{q}{|A^c|}$. Hence the entropy satisfies

$$\begin{aligned} E_n &\leq -(1-q) \log \frac{1-q}{|A|} - q \log \frac{q}{|A^c|} \\ &= [-(1-q)\log(1-q) - q \log q] + (1-q) \log |A| + q \log |A^c| \,. \end{aligned} \tag{2.2}$$

(Equation (2.2) also follows from the entropy chain rule.) By concavity of $x \mapsto -x \log x$, the term in square brackets is at most $\log 2$. Combining this with the linear entropy growth condition gives

$$Cn \leq E_n \leq \log 2 + q(\log |A^c| - \log |A|) + \log |A|.$$

Rearranging terms gives

$$q \geq \frac{Cn - \log |A| - \log 2}{\log |A^c| - \log |A|} \,.$$

Since $|A^c| \leq |\mathrm{support}(X_n)| \leq d^n$ and $\log |A| \geq 0$, the lemma follows. □

Suppose that π is a stationary measure for the transition kernel p. For integers $t \geq 0$ define $Q_t(x, y) = \pi(x) p^t(x, y)$. For $B \subset V$, define $Q_t(B, y) := \sum_{x \in B} Q_t(x, y)$. Then we have the following corollary.

Corollary 2.2. *For any finite set $S \subset V$ we have*

$$Q_n(S, A^c) \geq \pi(S) \frac{Cn - \log 2|A|}{n \log d}.$$

3 The evolving set process

Definition (Evolving Sets). We now define a process on subsets of V, which is similar to the evolving sets of [4], except that here the sets are only defined at certain times $T_0 \leq T_1 \leq T_2 \leq \cdots$. Fix (the starting state) $S \subset V$. We define a nonnegative integer valued process $T_0, T_1, \ldots$ and a set valued process $S_{T_0}, S_{T_1}, \ldots$ inductively as follows. Let

$U_1, U_2, \dots$ be i.i.d Uniform$[0,1]$ random variables. Set $T_0 = 0$ and $S_{T_0} = S$. Now assuming T_{m-1} and $S_{T_{m-1}}$ are given, define $T_m = T_{m-1} + L_m$, where $L_1, L_2, \dots$ is a positive integer valued process such that L_j is a function of $S_{T_{j-1}}$. Then define

$$S_{T_m} = \{y : Q_{L_m}(S_{T_{m-1}}, y) \geq U_m \pi(y)\}.$$

The construction used here is similar to the one in [4]. However, here we consider many steps of Markov chain at a time. We shall call the process $S_{T_0}, S_{T_1}, \dots$ the *intermittant evolving set process.*

Note that the intermittant evolving set process depends on the sequence of random variables $L_1, L_2, \dots$ (the "time gaps"). The results in this section are true for any choice of time gaps. (From Section 4 onwards, we choose a particular sequence $\left(L_j\right)_{j\geq 1}$.) For every integer $t \geq 0$, define

$$a(t) = \max\{i \ : \ T_i \leq t\}.$$

Then we have the following lemma.

Lemma 3.1. *For every nonnegative integer t, we have*

$$Q_t(S, y) = \mathbf{E}\big[Q_{t-T_{a(t)}}(S_{T_{a(t)}}, y)\big]. \tag{3.1}$$

Proof. For integers n with $0 \leq n \leq t$, define $a_n(t) := a(t) \wedge n$. Note that $a_0(t) = 0$. Furthermore, since $T_t \geq t$, we have $a(t) \leq t$ and hence $a_t(t) = a(t)$. Thus, it is enough to show that for every i with $0 \leq i < t$, we have

$$\mathbf{E}\big[Q_{t-T_{a_{i+1}(t)}}(S_{T_{a_{i+1}(t)}}, y)\big] = \mathbf{E}\big[Q_{t-T_{a_i(t)}}(S_{T_{a_i(t)}}, y)\big]\,. \tag{3.2}$$

Fix i with $0 \leq i < t$. We shall prove (3.2) by conditioning on the values of T_i and S_{T_i}. That is, we will show that for every j with $1 \leq j \leq t$ and $S \subset V$, we have

$$\mathbf{E}\big[Q_{t-T_{a_{i+1}(t)}}(S_{T_{a_{i+1}(t)}}, y) \mid T_i = j, S_{T_i} = S\big] = \mathbf{E}\big[Q_{t-T_{a_i(t)}}(S_{T_{a_i(t)}}, y) \mid T_i = j, S_{T_i} = S\big]\,. \tag{3.3}$$

So now fix j and S, and suppose that $T_i = j$ and $S_{T_i} = S$. Recall that L_{i+1} is a function S_{T_i}. If $T_{i+1} = j + L_{i+1} > t$ then $a_i(t) = a_{i+1}(t) = i$ and hence (3.3) is trivially true, so assume $T_{i+1} \leq t$. Then $a_{i+1}(t) = i+1$ and $a_i(t) = i$. So in this case we just have to show that

$$\mathbf{E}\big[Q_{t-T_{i+1}}(S_{T_{i+1}}, y) \mid T_i = j, S_{T_i} = S\big] = Q_{t-j}(S, y)\,. \tag{3.4}$$

But

$$\begin{aligned}
\mathbf{E}\big[Q_{t-T_{i+1}}(S_{T_{i+1}}, y) \mid T_i = j, S_{T_i} = S\big] &= \sum_{x\in V} \mathbf{P}\big[x \in S_{T_{i+1}} \mid T_i = j, S_{T_i} = S\big] Q_{t-T_{i+1}}(x, y) \\
&= \sum_{x\in V} \mathbf{P}\big[x \in S_{T_{i+1}} \mid T_i = j, S_{T_i} = S\big] \pi(x) p^{t-T_{i+1}}(x, y) \\
&= \sum_{x\in V} Q_{L_{i+1}}(S, x) p^{t-T_{i+1}}(x, y),
\end{aligned} \tag{3.5}$$

where the last line holds by the definition of the intermittant evolving set process. Note that for each $s \in S$, we have

$$\begin{aligned}
\sum_{x\in V} Q_{L_{i+1}}(s, x) p^{t-T_{i+1}}(x, y) &= \sum_{x\in V} \pi(s) p^{L_{i+1}}(s, x) p^{t-T_{i+1}}(x, y) \\
&= \pi(s) p^{t+L_{i+1}-T_{i+1}}(s, y) \\
&= Q_{t-T_i}(s, y),
\end{aligned} \tag{3.6}$$

where the last line holds because $T_{i+1} = T_i + L_{i+1}$. Finally, note that combining (3.6) with (3.5) yields (3.4). □

Corollary 3.2. *If the initial state of the intermittant evolving set process is $S = \{x_0\}$, then*

$$P^t(x_0, y) = \frac{1}{\pi(x_0)} \mathbf{E}\big[Q_{t-T_{a(t)}}(S_{T_{a(t)}}, y)\big].$$

4 Relating transience and evolving sets

Corollary 3.2 is useful since it will help us to prove an upper bound on $\sum_{t \geq 0} P^t(x_0, y)$, which implies transience. In Lemma 4.1 below we relate this sum to $\sum_{i=0}^{\infty} \mathbf{E}\big[\sqrt{\pi(S_{T_i})}\big]$, for a specific choice of π and $(L_j)_{j \geq 1}$. Henceforth we will fix π and $(L_j)_{j \geq 1}$ to be the following:

$$\pi(x) = \text{degree}(x); \qquad L_m = 2\left\lceil \frac{\log(8 \cdot \pi(S_{T_{m-1}}))}{C} \right\rceil \text{ for } m \geq 1, \tag{4.1}$$

where C is the constant appearing in the entopy growth condition.

Lemma 4.1. *Let π and L_m be as in (4.1). Then for any $y \in V$, we have*

$$\sum_{t=0}^{\infty} P^t(x_0, y) \leq 8d \left\lceil \frac{1}{C} \right\rceil \sum_{i=0}^{\infty} \mathbf{E}\Big[\sqrt{\pi(S_{T_i})}\Big]. \tag{4.2}$$

Proof. Note that for any $m \geq 0$, $S' \subset V$ and $y \in V$, we have $Q_m(S', y) \leq Q_m(V, y) = \pi(y)$. The equality in the preceding line is due to the fact that π is a stationary measure. Moreover, $Q_m(\emptyset, y) = 0$. Therefore, we have

$$Q_{t-T_{a(t)}}(S_{T_{a(t)}}, y) \leq Q_{t-T_{a(t)}}(V, y)\mathbf{1}\big(S_{T_{a(t)}} \neq \emptyset\big) = \pi(y)\mathbf{1}\big(S_{T_{a(t)}} \neq \emptyset\big). \tag{4.3}$$

Using (4.3) above and Corollary 3.2, we get

$$\begin{aligned}
\sum_{t=0}^{\infty} P^t(x_0, y) &= \frac{1}{\pi(x_0)} \sum_{t=0}^{\infty} \mathbf{E}\big[Q_{t-T_{a(t)}}(S_{T_{a(t)}}, y)\big] \\
&\leq \frac{\pi(y)}{\pi(x_0)} \sum_{t=0}^{\infty} \mathbf{E}\big[\mathbf{1}\big(S_{T_{a(t)}} \neq \emptyset\big)\big] \\
&\leq d \sum_{t=0}^{\infty} \mathbf{E}\big[\mathbf{1}\big(S_{T_{a(t)}} \neq \emptyset\big)\big],
\end{aligned} \tag{4.4}$$

where the last inequality follows from the fact that $1 \leq \pi(z) \leq d$ for any z. Next, note that when $T_m \leq t < T_{m+1}$, we have $a(t) = m$ and hence

$$\sum_{t=0}^{\infty} \mathbf{1}\big(S_{T_{a(t)}} \neq \emptyset\big) = \sum_{i=0}^{\infty} (T_{i+1} - T_i)\mathbf{1}\big(S_{T_i} \neq \emptyset\big) = \sum_{i=0}^{\infty} L_{i+1}\mathbf{1}\big(S_{T_i} \neq \emptyset\big).$$

Taking the value of L_i from (4.1) and combining with (4.4) gives

$$\sum_{t=0}^{\infty} P^t(x_0, y) \leq d \cdot \mathbf{E}\left[\sum_{i=0}^{\infty} 2\left\lceil \frac{\log(8 \cdot \pi(S_{T_i}))}{C} \right\rceil \mathbf{1}\big(S_{T_i} \neq \emptyset\big)\right]. \tag{4.5}$$

Now observe the following fact about the natural logarithm which will be useful for bounding the above.

$$4\sqrt{x} \geq \lceil \log(8x) \rceil \quad \text{for } x \geq 1. \tag{4.6}$$

To prove this, first note that $4\sqrt{x} \geq \log(8x) + 1$ for $x \geq 1$. This is true since the inequality holds for $x = 1$ and $\frac{d}{dx}(4\sqrt{x} - \log(8x) - 1) = \frac{2}{\sqrt{x}} - \frac{1}{x} \geq 0$ when $x \geq 1$. Since $\lceil \log(8x) \rceil \leq \log(8x) + 1$, the previous inequality gives us (4.6).

Thus (4.5) and (4.6) imply that

$$\sum_{t=0}^{\infty} P^t(x_0,y) \leq 8d\left\lceil \frac{1}{C} \right\rceil \mathbf{E}\left[\sum_{i=0}^{\infty} \sqrt{\pi(S_{T_i})}\mathbf{1}\big(S_{T_i}\neq\emptyset\big)\right]$$
$$= 8d\left\lceil \frac{1}{C} \right\rceil \sum_{i=0}^{\infty} \mathbf{E}\Big[\sqrt{\pi(S_{T_i})}\Big]. \qquad \square$$

5 Decay of $\mathbf{E}\sqrt{\pi(S_{T_i})}$

We will show that $\mathbf{E}\sqrt{\pi(S_{T_i})}$ decays exponentially in i, by proving the following theorem.

Theorem 5.1. *For all $m \geq 1$, there exists a constant $0 \leq \alpha < 1$ depending only on C and d such that*

$$\mathbf{E}\Big[\sqrt{\pi(S_{T_m})}\;\Big|\; S_{T_{m-1}}\Big] \leq \alpha \cdot \sqrt{\pi(S_{T_{m-1}})}. \tag{5.1}$$

We will come back to the proof of theorem 5.1 after we prove a lemma.

Lemma 5.2. *Let R be a nonnegative random variable and suppose that $\mathbf{E}(R)=1$. Then*

$$\mathbf{E}\sqrt{R} \leq 1 - \frac{1}{8}\mathbf{E}\Big(R\,\mathbf{1}(R\geq 4)\Big).$$

Proof. Let $f(x)=\sqrt{x}$. Since f is concave and $f'(1)=\frac{1}{2}$, we have

$$\sqrt{x} \leq 1 + \frac{1}{2}(x-1) \;\text{ for all } x \geq 0.$$

Since $f'(4)=\frac{1}{4}<\frac{1}{3}$, concavity of f implies that

$$\sqrt{x} \leq 1 + \frac{1}{3}(x-1) \;\text{ for all } x \geq 4.$$

It follows that

$$\sqrt{R} \leq 1 + \frac{1}{2}(R-1) - \frac{1}{6}(R-1)\mathbf{1}(R\geq 4).$$

Taking expectations gives

$$\begin{aligned}\mathbf{E}\sqrt{R} &\leq 1 - \frac{1}{6}\mathbf{E}\Big((R-1)\,\mathbf{1}(R\geq 4)\Big)\\ &\leq 1 - \frac{1}{8}\mathbf{E}\Big(R\,\mathbf{1}(R\geq 4)\Big),\end{aligned}$$

where the last line follows from the fact that

$$(R-1)\,\mathbf{1}(R\geq 4) \geq \frac{3}{4}R\,\mathbf{1}(R\geq 4). \qquad \square$$

Proof of theorem 5.1: We consider one "superstep" of the evolving set process, that is, the update from time T_m to time T_{m+1}. Suppose that $S_{T_m}=S$ and let $\widetilde{S}$ be the value of $S_{T_{m+1}}$, namely

$$\widetilde{S} = \{y : Q_{L_m}(S,y) \geq U_m\pi(y)\},$$

where U_m is a uniform random variable. Define $R := \frac{\pi(\widetilde{S})}{\pi(S)}$. Note that $\mathbf{E}R=1$, since

$$\begin{aligned}\mathbf{E}\pi(\widetilde{S}) &= \sum_{y\in V}\pi(y)\mathbf{P}(y\in\widetilde{S})\\ &= \sum_{y\in V}\pi(y)\frac{Q_{L_m}(S,y)}{\pi(y)}\\ &= Q_{L_m}(S,V) = \pi(S).\end{aligned}$$

(This is analogous to the martingale property of the evolving sets of [4].) Thus, we can apply lemma 5.2. With this in mind, we aim for a lower bound on $\mathbf{E}\Big(R\mathbf{1}(R \geq 4)\Big)$.

Let S_u be the value of $\widetilde{S}$ when $U_m = u$. That is,

$$S_u = \{y : Q_{L_m}(S, y) \geq u\pi(y)\}.$$

Let $\mathcal{S}$ be the collection of sets $\{S_u : u \in [0,1]\}$. Note that the collection $\mathcal{S}$ is nested. Furthermore, note that $\{S' \in \mathcal{S} : \pi(S') < 4\pi(S)\}$ is a finite subset of $\mathcal{S}$, so it has a largest element, which we denote by S_*. This set S_* has the property that $\pi(S_*) < 4\pi(S)$, but any set in $\mathcal{S}$ that strictly contains S_* has measure at least $4\pi(S)$. We have

$$\begin{aligned} \mathbf{E}\Big(R\mathbf{1}(R \geq 4)\Big) &= \frac{1}{\pi(S)}\mathbf{E}\Big(\pi(\widetilde{S})\mathbf{1}(\widetilde{S} \supsetneq S_*)\Big) \\ &\geq \frac{1}{\pi(S)}\mathbf{E}(\pi(\widetilde{S} - S_*)) \end{aligned} \tag{5.2}$$

But

$$\begin{aligned} \mathbf{E}(\pi(\widetilde{S} - S_*)) &= \int_0^1 \sum_{y \in S_*^c} \mathbf{1}(y \in S_u)\pi(y)\, du \\ &= \int_0^1 \sum_{y \in S_*^c} \mathbf{1}\Big(\frac{Q_{L_m}(S,y)}{\pi(y)} \geq u\Big)\pi(y)\, du. \end{aligned} \tag{5.3}$$

Since the indicator is nonnegative, we can apply Fubini and write the quantity (5.3) as

$$\begin{aligned} \sum_{y \in S_*^c}\Big[\int_0^1 \mathbf{1}\Big(\frac{Q_{L_m}(S,y)}{\pi(y)} \geq u\Big)\, du\Big]\pi(y) &= \sum_{y \in S_*^c}\Big[\frac{Q_{L_m}(S,y)}{\pi(y)}\Big]\pi(y) \\ &= Q_{L_m}(S, S_*^c) \\ &\geq \pi(S)\frac{CL_m - \log(2|S_*|)}{L_m \log d}, \end{aligned} \tag{5.4}$$

by Corollary 5.3. Now, using the fact that $\pi(S_*) < 4\pi(S)$ and the definition of L_m gives

$$\begin{aligned} \log(2|S_*|) &\leq \log(8\pi(S)) \\ &= \tfrac{1}{2}C \cdot \frac{2\log(8\pi(S))}{C} \\ &\leq \tfrac{1}{2}CL_m. \end{aligned}$$

It follows that the quantity (5.4) is at least

$$\pi(S)\frac{C}{2\log d},$$

Combining this with (5.2) gives

$$\mathbf{E}\Big(R\mathbf{1}(R \geq 4)\Big) \geq \frac{C}{2\log d}\,.$$

Hence Lemma 5.2 implies that the Theorem holds with the constant $\alpha = 1 - \frac{C}{16\log d}$. $\square$

By induction, we have the following corollary.

Corollary 5.3. *Let π, d, C, S_{T_j} be as in section 3. Then,*

$$\mathbf{E}\Big[\sqrt{\pi(S_{T_m})}\Big] \leq \alpha^m \cdot \pi(x_0) \tag{5.5}$$

where $\alpha < 1$ is a constant that depends only on C and d.

The main theorem of this paper can now be easily proved using Lemma 4.1 and Corollary 5.3 as follows.

Proof of theorem 1.1: To show that any vertex x_0 is transient, it is sufficient to show that $\sum_{t=0}^{\infty} P^t(x_0, x_0) < \infty$. By Lemma 4.1 and Corollary 5.3 we get

$$\begin{aligned}\sum_{t=0}^{\infty} P^t(x_0, x_0) &\le 8d\left\lceil \frac{1}{C} \right\rceil \sum_{i=0}^{\infty} \mathbf{E}\Big[\sqrt{\pi(S_{T_i})}\Big] \\ &\le 8d\left\lceil \frac{1}{C} \right\rceil \sum_{i=0}^{\infty} \alpha^i \pi(x_0) < \infty\end{aligned}$$

since $0 \le \alpha < 1$. □

6 Necessity of the uniformity of C in the linear entropy growth condition

The following example, which was conveyed to us by Gady Kozma, shows that the constant C in Theorem 1.1 cannot be allowed to depend on the starting position of the walk. Start with $\mathbf{Z}^+$ and then add an edge from each vertex $n \ge 1$ to the root of T_n, where T_n is a full binary tree of height $2{\uparrow\uparrow}n$ (i.e., a tower of powers of height n). Call this graph G. Then we will see that although the random walk on G is recurrent, the

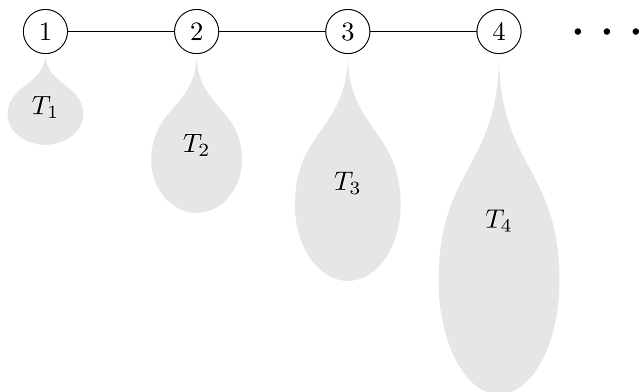

Figure 1: The Graph G.

entropy of the random walk grows at least linearly for any starting point. First, note that simple random walk on G, which we will denote by $(X_t)_{t\ge 0}$, is recurrent, since the effective resistance between any vertex and infinity is infinite. (See [3] for background on random walks and electrical networks; or for an alternative argument, note that if $T_1 < T_2 < \cdots$ are the times when the walk is at one of the vertices $n \in \mathbf{Z}^+$, then X_{T_t} is a simple random walk on the positive integers, which is recurrent.)

It remains to show is that for simple random walk on G, the entropy grows at least linearly. For $n \ge 1$ let V_n be the union of $\{n\}$ and vertex set of T_n, and define $S_n = V_1 \cup \cdots \cup V_n$. Let x_0 be the starting state of the walk and define

$$l = \min\{n : x_0 \in S_n\}\,.$$

We shall find a linear lower bound on the entropy of X_t that holds whenever t is sufficiently large. For $t \geq 8|S_l|$, define

$$k_0 := \max\left\{k : 2|S_k|^2 \leq \frac{t}{4}\right\}.$$

A calculation shows that the heights of the trees T_n grow sufficiently fast so that for sufficiently large t we have $\mathrm{diameter}(T_{k_0+2}) \geq t$. Let $T_{k_0} = \min\{t : X_t = k_0\}$ be the hitting time of k_0. Since for any tree with i vertices, the maximum expected hitting time of any vertex is at most $2i^2$ (see Chapter 10 of [2]), we have

$$\mathbf{E}T_{k_0} \leq 2|S_{k_0}|^2 \leq \frac{t}{4}\,,$$

and hence

$$\mathbf{P}\left(T_{k_0} \geq \frac{t}{2}\right) \leq \frac{1}{2}$$

by Markov's inequality. After the walk hits k_0 the probability that it goes to $k_0 + 2$ in the next two steps, and then crosses the edge into T_{k_0+2} in the step after that, is also bounded away from zero. Finally, the probability that the remaining steps up to time t are all spent in T_{k_0+2} is also bounded away from zero, because when the walk is in T_{k_0+2}, the distance from $k_0 + 2$ behaves like a $\frac{2}{3}\uparrow, \frac{1}{3}\downarrow$ random walk until the walk hits the $k_0 + 2$ again. Putting this all together, we see that the probability that the walk performs at least $\frac{t}{2} - 3$ steps in T_{k_0+2} is bounded away from zero, and hence the probability that X_t is a vertex in T_{k_0+2} at distance at least $\frac{t}{12}$ from $k_0 + 2$ is bounded away from zero.

Since for any d the walk is equally likely to be at any of the 2^d vertices at distance d from $k_0 + 2$ in T_{k_0+2}, this implies that if t is sufficiently large then $E_t \geq Ct$ for a universal constant C. Since $E_t > 0$ for all $t > 0$ there is a constant C_{x_0}, depending on the starting state x_0, such that

$$E_t \geq C_{x_0} t$$

for all $t \geq 0$.

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Acknowledgments. We are grateful to Itai Benjamini for sharing this problem with us and we thank Gady Kozma for sharing the counterexample of Section 6.