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On the Feller–Dynkin and the martingale property of one-dimensional diffusions*

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Abstract

We show that a one-dimensional regular continuous Markov process X with scale function $\mathfrak{s}$ is a Feller–Dynkin process precisely if the space transformed process $\mathfrak{s}(\mathsf{X})$ is a martingale when stopped at the boundaries of its state space. As a consequence, the Feller–Dynkin and the martingale property are equivalent for regular diffusions on natural scale with open state space. By means of a counterexample, we also show that this equivalence fails for multidimensional diffusions. Moreover, for Itô diffusions we discuss relations to Cauchy problems.

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1 The Feller–Dynkin and the martingale property of diffusions

1.1 The setting

Let $J \subset \mathbb{R}$ be a non-empty closed, open or half open possibly unbounded interval. We denote the interior of J by J°, the closure of J in $[-\infty, \infty]$ by $\operatorname{cl}(J)$ and its boundary $\operatorname{cl}(J)\backslash J^\circ$ by ∂J. Using the classical sextuple notation of Blumenthal and Getoor, let

$$\mathbf{M} \triangleq (\Omega, \mathcal{F}, \mathcal{F}_t \colon t \geq 0, \mathsf{X}_t \colon t \geq 0, \theta_t \colon t \geq 0, \mathbb{P}_x \colon x \in J)$$

be a (path-)continuous (temporally homogeneous) conservative strong Markov process (called *diffusion* in the following) with state space $(J, \mathcal{B}(J))$. Throughout this paper we assume that $\mathbf{M}$ is *regular*, i.e., $\mathbb{P}_x(\tau_y < \infty) > 0$ for every $x \in J^\circ$ and $y \in J$, where

$$\tau_y \triangleq \inf(t \in \mathbb{R}_+ \colon \mathsf{X}_t = y).$$

We denote the *scale function* of $\mathbf{M}$ by $\mathfrak{s}$ and its *speed measure* by $\mathfrak{m}$. For precise definitions and properties we refer to the classical monographs [2, 23, 25]. The regular diffusion $\mathbf{M}$ is said to be on *natural scale* in case its scale function is the identity, i.e., $\mathfrak{s} = \mathrm{Id}$. Finally, let us recall the boundary terminology used in this paper (which is taken

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from the monograph [2]). We fix a reference point $c \in J^\circ$ and define for $x \in J^\circ$

$$\mathfrak{u}(x) \triangleq \begin{cases} \int_{(c,x]} \mathfrak{m}((c,z])\mathfrak{s}(dz), & \text{for } x \geq c, \\ \int_{(x,c]} \mathfrak{m}((z,c])\mathfrak{s}(dz), & \text{for } x \leq c, \end{cases}$$
$$\mathfrak{v}(x) \triangleq \begin{cases} \int_{(c,x]} (\mathfrak{s}(y) - \mathfrak{s}(c))\mathfrak{m}(dy), & \text{for } x \geq c, \\ \int_{(x,c]} (\mathfrak{s}(c) - \mathfrak{s}(y))\mathfrak{m}(dy), & \text{for } x \leq c. \end{cases} \tag{1.1}$$

Moreover, for $b \in \partial J$ we write $\mathfrak{u}(b) \triangleq \lim_{J^\circ \ni x \to b} \mathfrak{u}(x)$ and $\mathfrak{v}(b) \triangleq \lim_{J^\circ \ni x \to b} \mathfrak{v}(x)$. A boundary point $b \in \partial J$ is called

regular if $\mathfrak{u}(b) < \infty$ and $\mathfrak{v}(b) < \infty$,
exit if $\mathfrak{u}(b) < \infty$ and $\mathfrak{v}(b) = \infty$,
entrance if $\mathfrak{u}(b) = \infty$ and $\mathfrak{v}(b) < \infty$,
natural if $\mathfrak{u}(b) = \infty$ and $\mathfrak{v}(b) = \infty$.

These definitions are independent of the choice of the reference point $c \in J^\circ$. Regular and exit boundaries are called *closed* or *accessible*, and entrance and natural boundaries are called *open* or *inaccessible*. As already indicated by the names, open boundaries are *not* in the state space J while closed ones are. We call a regular boundary point b *absorbing* if $\mathfrak{m}(\{b\}) = \infty$ and *reflecting* if $\mathfrak{m}(\{b\}) < \infty$.

1.2 Equivalence of the Feller–Dynkin and the martingale property

As $\mathbb{M}$ is a (strong) Markov process, we can define a semigroup $(T_t)_{t\geq 0}$ via

$$T_t f(x) \triangleq \mathbb{E}_x\big[f(\mathsf{X}_t)\big], \quad (t, f, x) \in \mathbb{R}_+ \times C_b(J) \times J.$$

Let $C_0(J)$ be the space of continuous functions from J into $\mathbb{R}$ that are vanishing at infinity. We endow $C_0(J)$ with the sup-norm which renders it into a Banach space. As J is an interval, the space $C_0(J)$ has a simple representation given by

$$C_0(J) = \Big\{ f \in C(J) \colon \lim_{x \to b} f(x) = 0 \text{ for all } b \in \partial J \backslash J \Big\}. \tag{1.2}$$

The process X is called a *Feller–Dynkin (FD) process* if the semigroup $(T_t)_{t\geq 0}$ is a strongly continuous semigroup on $C_0(J)$. We define the stopping time

$$\zeta \triangleq \inf(t \in \mathbb{R}_+ \colon \mathsf{X}_t \not\in J^\circ).$$

It is well-known ([25, Corollary V.46.15]) that the stopped process $\mathsf{Y} \triangleq \mathfrak{s}(\mathsf{X}_{\cdot \wedge \zeta})$ is a local $\mathbb{P}_x$-martingale for all $x \in J^\circ$. The following theorem is our main result.

Theorem 1.1. *The following are equivalent:*

(i) X *is an FD process.*
(ii) Y *is a* $\mathbb{P}_x$*-martingale for every* $x \in J^\circ$*.*
(iii) *Every open boundary point is natural.*

As an immediate consequence of Theorem 1.1, we obtain the following:

Corollary 1.2. *Suppose that* $\mathbb{M}$ *is on natural scale and that all regular boundaries are absorbing. Then,* X *is an FD process if and only if it is a* $\mathbb{P}_x$*-martingale for all* $x \in J^\circ$*.*

Proof. We recall that exit boundaries are always absorbing in the sense that they cannot be left by the diffusion, see [2, Problem 14, p. 370]. Hence, under the assumption that all regular boundaries are absorbing, we have a.s. $\mathsf{X} = \mathsf{X}_{\cdot\wedge\zeta}$. Thanks to this observation, the claim follows directly from Theorem 1.1. □

Remark 1.3. In case M is on natural scale but allowed to have a reflecting boundary point, the equivalence from Corollary 1.2 *fails*. Indeed, consider Brownian motion reflected at the origin, which is *no* martingale but an FD process. This example also motivates the necessity to consider the *stopped* process $\mathsf{Y} = \mathfrak{s}(\mathsf{X}_{\cdot\wedge\zeta})$ in Theorem 1.1.

Remark 1.4. Urusov and Zervos [26] proved that (iii) in Theorem 1.1 is equivalent to the martingale properties of the so-called *r-excessive local martingales*. By virtue of Theorem 1.1, their result provides another characterization of the FD property in terms of martingale properties.

Remark 1.5. A standard example for a non-FD process is the three-dimensional Bessel process (denoted Bes^3; [23, Section VI.3]) and its inverse is a standard example for a strict local martingale. These examples are connected via Theorem 1.1 as $\mathfrak{s}(x) = -1/x$ for $x > 0$ is a scale function of Bes^3. We emphasize that the state space of Bes^3 is necessarily $(0, \infty)$ as it is otherwise no *regular* diffusion.

Our contribution in Theorem 1.1 is the equivalence of (i) and (ii), which we think is quite surprising. In Section 1.3 below we comment in detail on related literature. In the proof of Theorem 1.1, which is given in Section 2 below, we will see that Y is a true martingale if X needs a long time to get close to open boundary points and that X is an FD process if it needs a long time to get away from them. It seems to be a coincidence that these properties are equivalent. Indeed, as we discuss in Section 1.4 below, the equivalence of the FD and the martingale property is a one-dimensional phenomenon.

It is well-known ([16, Theorem 33.9] or [25, Theorem V.47.1]) that any regular diffusion on natural scale is a time change of Brownian motion. On page 280 of their monograph [24], Rogers and Williams write the following: *Deciding whether or not the FD property is preserved under probabilistic operations such as time-substitution is generally a very difficult problem.* In the same spirit, it is well-known that the semimartingale property, but not necessarily the martingale property, is preserved by changes of time. Thanks to these observations, the equivalence of (i) and (ii) in Theorem 1.1 can be seen as follows: The time change related to the diffusion X preserves the martingale property of the underlying Brownian motion precisely when it preserves its FD property.

1.3 Comments on related literature

The question when a non-negative Itô diffusion with dynamics

$$d\mathsf{X}_t = \sigma(\mathsf{X}_t)dW_t, \quad W = \text{Brownian motion},$$

is a true martingale is, e.g., interesting for mathematical finance, where the martingale property decides about the absence and existence of certain arbitrage opportunities. Motivated by such an application, Delbaen and Shirakawa [9] proved an analytic integral test for the martingale property. Later, Kotani [19] and Hulley [14] gave answers for general regular diffusions on natural scale via integral tests depending on the speed measure. More precisely, the equivalence of (ii) and (iii) in Theorem 1.1 is their result.

The quite different question whether an Itô diffusion with drift is an FD process was studied by Feller [12] and Clément and Timmermans [7] from an analytic perspective, and by Azencott [1] from a more probabilistic point of view. We emphasis that Azencott was also interested in higher dimensional settings. These references provide the equivalence of (i) and (iii) in Theorem 1.1 for certain Itô diffusions. In their monograph, Ethier and Kurtz ([11, Theorem 8.1.1]) present several Itô type generators of strongly continuous semigroups on $C(\mathrm{cl}(J))$ and they attribute these results to Mandl [20]. For Itô diffusions with sufficiently regular coefficients, these yield the implication (iii) $\Rightarrow$ (i) from Theorem 1.1 and also a representation of the generator, see [11, Corollary 8.1.2]. Furthermore, Kallenberg ([16, Theorem 33.13]) proved the following related result: Form $\overline{J}$ via attaching entrance boundaries of X to J. Then, X extends to an FD process on $\overline{J}$. As $\overline{J} = J$ in case all open boundary points are natural, this theorem also implies the implication (iii) $\Rightarrow$ (i) from Theorem 1.1.

Among many other things, for a regular second order differential operator $\mathcal{L}$ on $C_c^\infty(J^\circ)$, Eberle [10] (for $1 \le p < \infty$) and Wu and Zhang [28] (for $p = \infty$) studied whether the closure of $\mathcal{L}$ generates a strongly continuous semigroup on $L^p(\mathfrak{m})$ (with a suitable topology), where $\mathfrak{m}$ denotes the speed measure associated to $\mathcal{L}$. In this case, $\mathcal{L}$ is said to be *L^p-unique*. Eberle proved that all infinite boundaries are natural if and only if L^p-uniqueness holds for all $p > 1$, and Wu and Zhang proved that the same condition is equivalent to L^∞-uniqueness. These results are related to Theorem 1.1 in the sense that, roughly speaking, uniqueness of FD semigroups can be viewed as some limit of L^p-uniqueness as $p \to \infty$, see [10, Remark, p. 3].

Our main observation is the equivalence of (i) and (ii) in Theorem 1.1. The purpose of this paper is to report this phenomenon and, as we find it not intuitive, to explain it via a complete and (mainly) self-contained proof, which borrows and connects many ideas from [1, 14, 19].

1.4 A counterexample for the multidimensional case

It is natural to ask whether the equivalence of (i) and (ii) from Theorem 1.1 also holds in a multidimensional setting. In this section we give an example, inspired by a comment on page 238 in [1], which shows that this is not the case. In other words, the equivalence of the FD and the martingale property is a one-dimensional phenomenon.

Take $d \ge 2$, define $\Omega \triangleq C(\mathbb{R}_+, \mathbb{R}^d)$ and denote the coordinate process by $\mathsf{X} = (\mathsf{X}_t)_{t\ge 0}$. Let $\mathcal{F}$ and $(\mathcal{F}_t)_{t\ge0}$ be the σ-field and the (right-continuous) filtration generated by X. Furthermore, let $\mathbb{W}_x$ be the d-dimensional Wiener measure with starting point $x \in \mathbb{R}^d$. Let $D \subset \mathbb{R}^d$ be a nonempty domain of finite Lebesgue measure. A point $o \in \partial D$ is called *irregular* if $\mathbb{W}_o(\tau_D = 0) = 0$ with $\tau_D \triangleq \inf(t > 0 \colon \mathsf{X}_t \notin D)$. Irregularity can also be defined via the Dirichlet problem, see [22, Theorem 4.2.2]. The set of irregular points is denoted by $\mathcal{I}$. Note that $\mathbb{W}_o(\tau_D = 0) = 1$ for all $o \in \partial D \backslash \mathcal{I}$ by Blumenthal's zero-one law.

Example 1.6. (i) If $D \equiv \{x \in \mathbb{R}^d \colon 0 < \|x\| < 1\}$, then $\mathcal{I} = \{0\}$.

(ii) An example for a domain with a connected boundary containing an irregular point is *Lebesgue's thorn*, see [17, Example 4.2.17].

Define

$$D' \triangleq \mathrm{cl}(D)\backslash\mathcal{I}, \qquad \mathbb{P}_x \triangleq \mathbb{W}_x \circ \mathsf{X}^{-1}_{\cdot\wedge\tau_D} \colon\ x \in D'.$$

Finally, we set

$$\mathbf{M} \triangleq (\Omega, \mathcal{F}, \mathcal{F}_t \colon t \ge 0, \mathsf{X}_t \colon t \ge 0, \theta_t \colon t \ge 0, \mathbb{P}_x \colon x \in D'),$$

where $(\theta_t)_{t\ge0}$ is the usual shift operator on Ω, i.e., $\theta_s\omega(t) = \omega(t+s)$ for $\omega \in \Omega$ and $s, t \in \mathbb{R}_+$.

Theorem 1.7. *$\mathbf{M}$ is a strong Markov process with state space $(D', \mathcal{B}(D'))$ and X is a $\mathbb{P}_x$-martingale for every $x \in D$. Moreover, $\mathbf{M}$ is an FD process if and only if $\mathcal{I} = \varnothing$.*

Discussion. To see the connection of Theorems 1.1 and 1.7, notice that an irregular boundary point can be viewed as a multidimensional version of an entrance boundary: Brownian motion started in an irregular point $o \in \partial D$ enters D immediately and stays there for some time. Further, as $\mathcal{I}$ is a polar set ([22, Theorem 2.6.3]), Brownian motion never hits $\mathcal{I}$ when started in D'. Hence, roughly speaking, Theorem 1.7 shows that $\mathbb{M}$ is an FD process if and only if there are no entrance boundary points, which is also the equivalence of (i) and (iii) in Theorem 1.1. We point to the difference that entrance boundaries are necessarily infinite for diffusions on natural scale, while irregular points are elements of $\mathbb{R}^d$. This is related to the well-known fact ([22, Proposition 2.3.2]) that there are *no* irregular points in the one-dimensional case. Indeed, for $d = 1$ the system $\mathbb{M}$ is also known to be an FD process (this is confirmed by Theorem 1.1). In contrast to the FD property, irregular points do not affect the martingale property.

To get an intuition for the influence of the dimension d, let us discuss the example of a Brownian motion in the punctured domain $\mathbb{R}^d\backslash\{0\}$. This example is related to part (i) of Example 1.6 although, strictly speaking, it is not covered by Theorem 1.7. Let W be a d-dimensional Brownian motion starting in $x \neq 0$. A straightforward application of Itô's formula yields that

$$\|W\| = \|x\| + B + \int_0^\cdot \frac{(d-1)dt}{2\|W_t\|},$$

where $B \triangleq \int_0^\cdot \langle W_t, dW_t\rangle / \|W_t\|$ is a one-dimensional Brownian motion by Lévy's characterization. This easy computation recovers the well-known fact that $\|W\|$ is a d-dimensional Bessel process, denoted Bes^d, with initial value $\|x\| > 0$. Notice that the dimension d enters the picture through the drift coefficient. As $d \geq 2$, the origin is an entrance boundary for Bes^d ([20, Section II.6.3]) and latter is *no* FD process by Theorem 1.1. Since $\|W\|$ is a d-dimensional Bessel process, it is easy to see that this failure of the FD property transfers to Brownian motion in $\mathbb{R}^d\backslash\{0\}$.

Proof of Theorem 1.7. The strong Markov property of $\mathbb{M}$ can be proved as in [15, Section 3.9, pp. 102 – 103].

The martingale property follows from those of Brownian motion and the optional stopping theorem. To see this, first note that $\mathsf{X}^{-1}_{\cdot\wedge\tau_D}(\mathcal{F}_t) \subset \mathcal{F}_{t\wedge\tau_D}$ for all $t \in \mathbb{R}_+$. Then, the optional stopping theorem yields that for all $s < t$ and $G \in \mathcal{F}_s$ we have $\mathsf{X}_t, \mathsf{X}_s \in L^1(\mathbb{P}_x)$ and

$$\mathbb{E}^{\mathbb{P}_x}\big[\mathsf{X}_t \mathbb{1}_G\big] = \mathbb{E}^{\mathbb{W}_x}\big[\mathsf{X}_{t\wedge\tau_D}\mathbb{1}_{\{\mathsf{X}_{\cdot\wedge\tau_D}\in G\}}\big] = \mathbb{E}^{\mathbb{W}_x}\big[\mathsf{X}_{s\wedge\tau_D}\mathbb{1}_{\{\mathsf{X}_{\cdot\wedge\tau_D}\in G\}}\big] = \mathbb{E}^{\mathbb{P}_x}\big[\mathsf{X}_s \mathbb{1}_G\big].$$

This is the martingale property.

If $\mathcal{I} = \varnothing$, then $\mathbb{M}$ is an FD process by [18, Theorem 4.1.9]. We now show the converse direction, i.e., we assume that $\mathcal{I} \neq \varnothing$ and we take $o \in \mathcal{I}$. Thanks to [22, Proposition 4.2.14], there exists a compact set $K \subset D$ such that

$$\limsup_{\substack{x\to o\\ x\in D}} \mathbb{W}_x(T_K < \tau_D) > 0, \quad T_K \triangleq \inf(t > 0 \colon \mathsf{X}_t \in K). \tag{1.3}$$

Furthermore, by [6, (X), p. 148] and the assumption that D has finite Lebesgue measure, there exists an $\alpha > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^{\mathbb{W}_x}\big[e^{\alpha\tau_D}\big] < \infty. \tag{1.4}$$

Using Galmarino's test ([15, p. 86]) and the Cauchy–Schwarz inequality, for all $x \in D$ we obtain

$$\begin{aligned}\mathbb{W}_x(T_K < \tau_D) &= \mathbb{W}_x(T_K(\mathsf{X}_{\cdot\wedge\tau_D}) < \tau_D)\\ &\leq \mathbb{E}^{\mathbb{W}_x}\big[e^{\alpha(\tau_D - T_K(\mathsf{X}_{\cdot\wedge\tau_D}))/2}\big]\\ &\leq \sup_{z\in\mathbb{R}^d} \mathbb{E}^{\mathbb{W}_z}\big[e^{\alpha\tau_D}\big]^{\frac{1}{2}}\mathbb{E}^{\mathbb{P}_x}\big[e^{-\alpha T_K}\big]^{\frac{1}{2}}.\end{aligned} \tag{1.5}$$

By [8, Remark 1], we have

$$\mathbf{M} \text{ is an FD process } \Rightarrow \lim_{\substack{x\to o\\ x\in D}} \mathbb{E}^{\mathbb{P}_x}\big[e^{-\alpha T_K}\big] = 0. \tag{1.6}$$

Finally, (1.3) – (1.6) yield that $\mathbf{M}$ is no FD process. □

Remark 1.8. Let us also sketch a more direct idea of proof for the failure of the FD property in case $\mathcal{I} \neq \varnothing$. We only consider the example of Brownian motion in $\mathbb{R}^d\backslash\{0\}$. Take a continuous function $f\colon \mathbb{R} \to [0,1]$ such that $f(0) = 0$ and $f(x) > 0$ for all $x \neq 0$, let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence with $x_n \to 0$ and take $t > 0$. It is well-known that the map $x \mapsto \mathbb{W}_x$ is continuous from $\mathbb{R}^d$ into the set of probability measures on $(\Omega, \mathcal{F})$ with the topology of convergence in distribution. Hence,

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{W}_{x_n}}\big[f(\|\mathsf{X}_t\|)\big] = \mathbb{E}^{\mathbb{W}_0}\big[f(\|\mathsf{X}_t\|)\big] > 0,$$

which shows that Brownian motion in $\mathbb{R}^d\backslash\{0\}$ is no FD process.

1.5 Equivalence of Cauchy problems in Itô diffusion settings

It is well-known that the FD and the martingale property have close relations to existence and uniqueness properties of Cauchy problems. Thanks to Theorem 1.1, we can connect these relations.

Suppose that $J = (l, r)$ for $-\infty \leq l < r \leq \infty$ and that

$$\mathfrak{s}(x) \triangleq \int_c^x \exp\Big(-\int_c^\xi \frac{2b(z)dz}{\sigma^2(z)}\Big)d\xi, \qquad \mathfrak{m}(dx) \triangleq \frac{dx}{\mathfrak{s}'(x)\sigma^2(x)},$$

where $c \in J$ is an arbitrary reference point and $b\colon J \to \mathbb{R}$ and $\sigma\colon J \to \mathbb{R}\backslash\{0\}$ are continuous functions. Moreover, we set

$$Sf \triangleq bf' + \tfrac{\sigma^2}{2}f'' \;\text{ for }\; f \in D(S) \triangleq \big\{f \in C_0(J) \cap C^2(J)\colon Sf \in C_0(J)\big\}.$$

Remark 1.9. In case X is an FD process, it is known that $(S, D(S))$ is its infinitesimal generator, see [11, Corollary 8.1.2].

We start with a consequence of a main result from [3] which relates (ii) and (iii) from Theorem 1.1 to existence and uniqueness of a classical solution to a certain Cauchy problem with boundary datum of linear growth.

Theorem 1.10. *Suppose that $J = (0,\infty), b \equiv 0$ and that σ is locally Hölder continuous with exponent $1/2$. Then,* (i) – (iii) *from Theorem 1.1 are equivalent to the following:*

(iv) *For every continuous function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ of linear growth, i.e., $|g(x)| \leq C(1 + |x|)$ with $C > 0$, and any finite time horizon $T > 0$ the Cauchy problem*

$$\begin{cases} \frac{du}{dt} + \frac{\sigma^2}{2}u'' = 0, & \text{on } (0,\infty)\times[0,T),\\ u(0,t) = g(0), & t \in [0,T],\\ u(x,T) = g(x), & x \in \mathbb{R}_+,\end{cases}$$

has a unique solution $u\colon \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ such that $u \in C^{2,1}((0,\infty)\times[0,T))$.

Proof. The equivalence of (iii) and (iv) follows from [3, Theorem 2]. □

In case (iii) fails, it has been shown in [4, 5] that for appropriate boundary data the associated Cauchy problem still has a solution which is unique among all solutions with certain non-standard boundary behavior.

The proof of Theorem 1.10 in [3] uses PDE theory in combination with uniform integrability properties, which stem from the martingale property of X, i.e., item (ii) from Theorem 1.1.

Next, we provide another characterization of (i) – (iii) from Theorem 1.1 in terms of properties of Cauchy problems.

Theorem 1.11. (i) – (iii) *from Theorem 1.1 are equivalent to each of the following:*

(v) *For all $g \in D(S)$ the Cauchy problem*

$$\frac{du}{dt} = Su, \quad u(0) = g,$$

has a unique solution $u\colon \mathbb{R}_+ \to C_0(J)$ which is a continuously differentiable function such that $u(t) \in D(S)$ for all $t > 0$.

(vi) *For all $g \in D(S)$ there exists a continuous function $u\colon \mathbb{R}_+ \to C_0(J)$ such that $u(0) = g, u(t) \in D(S)$ for all $t > 0$, $Su\colon (0,\infty) \to C_0(J)$ is continuous, and*

$$u(t) - u(\varepsilon) = \int_\varepsilon^t Su(s)ds$$

for all $t > \varepsilon > 0$.

Proof. If (iii) holds, [11, Corollary 8.1.2] and [21, Theorem 4.1.3] yield (v). Obviously, (v) implies (vi). Suppose that (vi) holds. As $C^2_c(J)$ is dense in $C_0(J)$ and $C^2_c(J) \subset D(S)$, the operator $(S, D(S))$ is densely defined. Furthermore, it follows from [7, Proposition 1] that $(S, D(S))$ is closed and dissipative. Hence, (vi) and [11, Proposition 1.3.4] yield that $(S, D(S))$ is the generator of a strongly continuous semigroup on $C_0(J)$. Now, it follows verbatim as in the proof of [7, Lemma 3] that there exist two positive monotone solutions u_l and u_r to $u = Su$ such that $\lim_{x\to l} u_l(x) = \lim_{x\to r} u_r(x) = 0$. As in the proof of Lemma 2.3 below, if l is not natural then there exists a positive increasing solution u^*_l to $u = Su$ with $\lim_{x\to l} u^*_l(x) > 0$. However, since $u^*_l = cu_l$ for $c > 0$ (see [15, p. 129]), this yields a contradiction. The same argument shows that r is natural. The proof is complete. ☐

It is interesting to observe that for the Cauchy problem from (iv) uniqueness fails in case (i) – (iii) from Theorem 1.1 fail, see the proof of [3, Theorem 2]. In other words, existence is not the decisive property in Theorem 1.10. This is quite different for the Cauchy problem from (v). In case it has a solution (for all initial data), then (vi) holds and (i) – (iii) from Theorem 1.1 hold, too.

2 Proof of Theorem 1.1

As the scale function $\mathfrak{s}$ is continuous and strictly increasing, $\mathfrak{s}\colon J \to \mathfrak{s}(J)$ is a homeomorphism and, by virtue of [23, Exercise VII.3.18], $\mathfrak{s}(\mathsf{X})$ is a regular diffusion with state space $J^* \triangleq \mathfrak{s}(J)$, scale function Id and speed measure $\mathfrak{m} \circ \mathfrak{s}^{-1}$. We notice the following implications: If $f \in C_0(J)$ then $f \circ \mathfrak{s}^{-1} \in C_0(J^*)$, and if $f \in C_0(J^*)$ then $f \circ \mathfrak{s} \in C_0(J)$, see (1.2). Thus, X and $\mathfrak{s}(\mathsf{X})$ are simultaneously FD processes. We elaborate this claim in more detail. Suppose that X is an FD process and take $f \in C_0(J^*)$ and $t > 0$. Then, $g \triangleq f \circ \mathfrak{s} \in C_0(J)$ and

$$\big(J \ni x \mapsto \mathbb{E}_x\big[f(\mathfrak{s}(\mathsf{X}_t))\big] = \mathbb{E}_x\big[g(\mathsf{X}_t)\big]\big) \in C_0(J).$$

Therefore, $\mathbb{E}_{\mathfrak{s}^{-1}(\cdot)}[f(\mathfrak{s}(\mathsf{X}_t))] \in C_0(J^*)$, which shows that $\mathfrak{s}(\mathsf{X})$ is an FD process. The converse implication (i.e., that X is an FD process in case $\mathfrak{s}(\mathsf{X})$ is an FD process) follows the same way. With this observation in mind, we can and will w.l.o.g. assume that X is on natural scale, i.e., we assume that $\mathfrak{s} = \mathrm{Id}$.

Lemma 2.1. (i) X *is an FD process if and only if*

$$\lim_{x\to b}\mathbb{E}_x\big[e^{-\alpha\tau_y}\big]=0 \text{ for all } y\in J^\circ,\alpha>0 \text{ and any infinite } b\in\partial J. \tag{2.1}$$

(ii) $\mathsf{X}_{\cdot\wedge\zeta}$ *is a* $\mathbb{P}_x$*-martingale for all* $x\in J^\circ$ *if and only if*

$$\lim_{y\to b} y\mathbb{E}_x\big[e^{-\alpha\tau_y}\big]=0 \text{ for all } x\in J^\circ,\alpha>0 \text{ and any infinite } b\in\partial J. \tag{2.2}$$

Part (i) of Lemma 2.1 is a version of [1, Proposition 3.1] and [8, Remark 1] for our framework. In [1] the result is shown for multidimensional Itô diffusions with (locally) Hölder coefficients and in [8] it is shown in a general martingale problem framework. The general idea for its proof given below is taken from [1]. The argument in [1] for the *only if* implication uses analytic tools. The proof given below borrows the supermartingale argument from [8]. Part (ii) can be extracted from [14], although it has not been stated there in this form. Furthermore, (ii) can be deduced from [26, Theorem 2.2], or [5, Theorem 2.3], together with [19, Theorem 1]. For completeness, we give a full proof for which we borrow arguments from the proof of [14, Theorem 3.9].

As every regular diffusion is already a Feller process, it is an FD process if and only if $T_t f$ vanishes at infinity for all $f\in C_0(J)$ and $t>0$. Thus, X should be an FD process precisely if it stays some time close to open boundaries. Part (i) of Lemma 2.1 quantifies this intuition. At this point we stress that regular diffusions on natural scale always stay some time close to finite open boundaries. This explains why only the infinite boundaries are mentioned in (2.1). As we have seen in Section 1.4, this is quite different in the multidimensional setting for which finite points make a difference.

To get an idea for part (ii), consider $J=(0,\infty)$ and note that for all $y\geq x$ the stopped process $\mathsf{X}_{\cdot\wedge\tau_y}$ is a bounded local $\mathbb{P}_x$-martingale and consequently, a $\mathbb{P}_x$-martingale. The condition (2.2) can be viewed as a criterion for the uniform $\mathbb{P}_x$-integrability of $\{\mathsf{X}_{t\wedge\tau_y}: y\geq x\}$ for every $t>0$, which is necessary and sufficient for the $\mathbb{P}_x$-martingale property of X. To get an intuition for this, recall the criterion of de la Vallée Poussin: A family $\Pi\subset L^1$ is uniformly integrable if and only if there exists a convex monotone function $H\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $\sup_{X\in\Pi}\mathbb{E}[H(|X|)]<\infty$ and $H(x)/x\to\infty$ as $x\to\infty$. The condition (2.2) mirrors this criterion with $H(x)=1/\mathbb{E}_y[e^{-\alpha\tau_x}]$ for $x>y$.

At first glance (2.2) seems to be stronger than (2.1). For example, suppose that $\mathfrak{g}(x,y)\triangleq\mathbb{E}_x[e^{-\alpha\tau_y}]$ is symmetric in $x,y\in J^\circ$. Then, (2.2) clearly implies (2.1). It turns out that this situation is quite special: $\mathfrak{g}$ is symmetric if and only if the diffusion X behaves like a Brownian motion up to a constant scale factor in the interior of its state space J.[1] In case M is a Brownian motion, it is easy to show that $\mathbb{E}_x[e^{-\alpha\tau_y}]=e^{-\sqrt{2\alpha}|x-y|}$ for all $x,y\in\mathbb{R}$ and both (2.1) and (2.2) are satisfied. We have the following general relation:

Lemma 2.2. (2.1) $\Leftrightarrow$ (2.2) $\Leftrightarrow$ *all infinite boundaries are natural.*

[1]Let g_1 and g_2 be the functions from (2.3) in the proof of Lemma 2.2. Symmetry of $\mathfrak{g}$ means that $g_1g_2=1$. Let B be the Wronskian, i.e., const. $\equiv B=g_1^+g_2-g_1g_2^+$, see [15, p. 130]. Using the product rule we obtain

$$0=(g_1g_2)^+=g_1^+g_2+g_1g_2^+=\begin{cases}2g_1^+g_2-B=2g_1^+/g_1-B,\\ 2g_1g_2^++B=2g_2^+/g_2+B,\end{cases}$$

which means that $g_1^+=Bg_1/2$ and $g_2^+=-Bg_2/2$. Using these identities, $dg_i^+=2\alpha g_i d\mathfrak{m}, i=1,2$, and integration by parts, we obtain for all $a,b\in J^\circ$ with $a<b$ that

$$\begin{aligned}0&=\int_{(a,b]}d(g_1g_2)^+=\int_{(a,b]}g_1^+dg_2+\int_{(a,b]}g_2dg_1^++\int_{(a,b]}g_2^+dg_1+\int_{(a,b]}g_1dg_2^+\\&=2\int_a^b g_1^+g_2^+dx+4\alpha\int_{(a,b]}g_1g_2d\mathfrak{m}=-\frac{B^2}{2}(b-a)+4\alpha\mathfrak{m}((a,b]).\end{aligned}$$

Consequently, $\mathfrak{m}(dx)=$ const. dx on $(J^\circ,\mathcal{B}(J^\circ))$. Hence, X behaves like a Brownian motion up to a constant scale factor in the interior of J.

Lemma 2.2 shows that X approaches infinite boundaries slow enough to be a martingale precisely when it needs long enough to get away from them to be an FD process. This connection seems to be a surprising coincidence. Lemma 2.2 is known in different formulations, see [14, Propositions 3.12 and 3.13], [15, Table 1, p. 130], [19, Lemma 3] or [26, Theorem 2.2]. Below we give a complete and mainly analytic proof, which borrows ideas from these references. It would also be interesting to find a proof for the equivalence (2.1) ⇔ (2.2) without an excursion via natural boundaries. As we are not aware of such an argument, we think that the analytic character of the proof for Lemma 2.2 supports the impression that the equivalence of the FD and the martingale property is quite surprising.

Proof of Theorem 1.1. Lemmata 2.1 and 2.2 imply Theorem 1.1. □

Proof of Lemma 2.2. Fix $\alpha > 0$ and a reference point $y \in J^\circ$. Using the notation of Itô and McKean ([15, pp. 128]), for $x \in J^\circ$ we define

$$\begin{aligned} g_1(x) &\triangleq \begin{cases} \mathbb{E}_x\big[e^{-\alpha\tau_y}\big], & x \le y,\\ 1/\mathbb{E}_y\big[e^{-\alpha\tau_x}\big], & y < x,\end{cases}\\ g_2(x) &\triangleq \begin{cases} 1/\mathbb{E}_y\big[e^{-\alpha\tau_x}\big], & x \le y,\\ \mathbb{E}_x\big[e^{-\alpha\tau_y}\big], & y < x.\end{cases}\end{aligned} \tag{2.3}$$

It is well-known ([25, Proposition V.50.3]) that g_1 and g_2 are strictly convex, continuous, strictly monotone, and positive and finite (throughout J°). More precisely, g_1 is strictly increasing and g_2 is strictly decreasing. Furthermore, g_1 and g_2 both solve the differential equation

$$\frac{1}{2\alpha}\frac{d}{d\mathfrak{m}}\frac{d^+g}{dx} = g,$$

that is for $z, y \in J^\circ$ with $z < y$

$$\frac{d^+g}{dx}(y) - \frac{d^+g}{dx}(z) = 2\alpha\int_{(z,y]} g(u)\mathfrak{m}(du).$$

Case 1: ∞ is a boundary point of J. Clearly, for $b = \infty$ the property (2.1) means that $g_2(\infty) \triangleq \lim_{x\to\infty} g_2(x) = 0$, and (2.2) means that

$$\lim_{x\to\infty}\frac{x}{g_1(x)} = 0. \tag{2.4}$$

We now translate (2.4) to a property of $g_1^+ \triangleq d^+g_1/dx$. As g_1 is convex, we have

$$\frac{g_1(x) - g_1(z)}{x - z} \le g_1^+(x), \quad x, z \in J^\circ, x > z,$$

which shows that (2.4) implies $g_1^+(\infty) \triangleq \lim_{x\to\infty} g_1^+(x) = \infty$. Conversely, L'Hopital's rule (see [27, Theorem 3] for a suitable version with right derivatives) yields that (2.4) is implied by $g_1^+(\infty) = \infty$. Thus, (2.4) is equivalent to $g_1^+(\infty) = \infty$. We claim the following:

$$g_1^+(\infty) = \infty \;\Rightarrow\; g_2(\infty) = 0 \;\Rightarrow\; \infty \text{ is natural } \;\Rightarrow\; g_1^+(\infty) = \infty. \tag{2.5}$$

These implications yield the equivalences in Lemma 2.2 for the boundary point ∞.

Proof of 1st implication in (2.5)*:* By [25, Theorem V.50.7] (or [15, p. 130]), the Wronskian is constant, i.e., $g_2 g_1^+ - g_1 g_2^+ \equiv$ constant $\triangleq B$. Now, $g_2 g_1^+ \le B$ shows that $g_1^+(\infty) = \infty \Rightarrow g_2(\infty) = 0$.

Proof of 2nd implication in (2.5)*:*

Lemma 2.3. *If ∞ is not natural, then there exists a continuous and decreasing function $g\colon J^\circ \to [1,\infty)$ such that $\frac{1}{2\alpha}\frac{d}{d\mathfrak{m}}\frac{d^+g}{dx} = g$ and $\lim_{x\to\infty} g(x) \triangleq g(\infty) = 1$.*

Proof. We mimic the proof of [17, Lemma 5.5.26] (see also [20, Section II.2]). Assume that ∞ is not natural. Set $u_0 = 1$ and

$$u_n(x) \triangleq \int_x^\infty \int_{(y,\infty)} u_{n-1}(z)\mathfrak{m}(dz)dy = \int_{(x,\infty)} (z-x)u_{n-1}(z)\mathfrak{m}(dz),$$

for $x \in J^\circ$ and $n = 1, 2, \dots$. We stress that $u_1, u_2, \dots$ are well-defined, continuous and decreasing, because ∞ is *not* natural. Induction shows that

$$u_n \le \frac{u_1^n}{n!}, \quad n = 1, 2, \dots. \tag{2.6}$$

Indeed, the case $n = 1$ is clear and if the inequality holds for $n \in \mathbb{N}$, then

$$\begin{aligned} u_{n+1} &= \int_\cdot^\infty \int_{(y,\infty)} u_n(z)\mathfrak{m}(dz)dy \le \frac{1}{n!}\int_\cdot^\infty u_1^n(y)\mathfrak{m}((y,\infty))dy \\ &= \frac{-1}{n!}\int_\cdot^\infty u_1^n(y)u_1(dy) = \frac{u_1^{n+1}}{(n+1)!}. \end{aligned}$$

Using (2.6), we also get

$$\Big|\frac{d^+u_n}{dx}\Big| \le \frac{u_1^{n-1}}{(n-1)!}\mathfrak{m}((\,\cdot\,,\infty)), \quad n = 1, 2, \dots. \tag{2.7}$$

Thanks again to (2.6), $g \triangleq \sum_{n=0}^\infty (2\alpha)^n u_n$ defines a continuous and decreasing function. We also see that $1 + 2\alpha u_1 \le g \le e^{2\alpha u_1}$ and consequently, $g(\infty) = 1$. Moreover, using (2.7), we get

$$\begin{aligned} \frac{d^+g}{dx} &= \sum_{n=1}^\infty (2\alpha)^n \frac{d^+u_n}{dx} = \sum_{n=1}^\infty (2\alpha)^n(-1)\int_{(\,\cdot\,,\infty)} u_{n-1}(z)\mathfrak{m}(dz) \\ &= -2\alpha \int_{(\,\cdot\,,\infty)} \sum_{n=0}^\infty (2\alpha)^n u_n(z)\mathfrak{m}(dz) = -2\alpha\int_{(\,\cdot\,,\infty)} g(z)\mathfrak{m}(dz). \end{aligned}$$

For $y, z \in J^\circ$ with $y < z$ this shows that

$$\frac{d^+g}{dx}(z) - \frac{d^+g}{dx}(y) = 2\alpha\int_{(y,z]} g(x)\mathfrak{m}(dx),$$

which is nothing else than $\frac{1}{2\alpha}\frac{d}{d\mathfrak{m}}\frac{d^+g}{dx} = g$. In summary, g has all claimed properties. □

Assume that ∞ is not natural and take g as in Lemma 2.3. Then, the uniqueness theorem [2, Theorem 16.69] implies that $g = c\,g_2$ for a constant $c > 0$. Thus, $g_2(\infty) > 0$ and we conclude that $g_2(\infty) = 0 \Rightarrow \infty$ is natural.

Proof of 3rd implication in (2.5)*:* Assume that $g_1^+(\infty) < \infty$. Then, using the subdifferential inequality, we obtain for every $a \in J^\circ$ that

$$\int_{(a,\infty)} (z-a)\mathfrak{m}(dz) \le \int_{(a,\infty)} \frac{g_1(z)\mathfrak{m}(dz)}{g_1^+(a)} = \frac{g_1^+(\infty) - g_1^+(a)}{2\alpha g_1^+(a)} < \infty.$$

Consequently, ∞ cannot be natural. We conclude that ∞ is natural $\Rightarrow g_1^+(\infty) = \infty$.

Case 2: $-\infty$ is a boundary point of J. In this case (2.1) means that $g_1(-\infty) \triangleq \lim_{x\to-\infty} g_1(x) = 0$, and (2.2) means that $\lim_{x\to-\infty} \frac{x}{g_2(x)} = 0$. As in the previous case, we see that

$$\lim_{x\to-\infty} \frac{x}{g_2(x)} = 0 \quad \Leftrightarrow \quad g_2^+(-\infty) = -\infty.$$

The following implications also follow as in the previous case:

$$g_2^+(-\infty) = -\infty \;\Rightarrow\; g_1(-\infty) = 0 \;\Rightarrow\; -\infty \text{ is natural } \;\Rightarrow\; g_2^+(-\infty) = -\infty.$$

Hence, the equivalence in Lemma 2.2 holds for the boundary point $-\infty$. The proof is complete. □

Proof of Lemma 2.1 (i). First, assume that X is an FD process. Fix $y \in J^\circ, \alpha > 0$ and let $g \in C_0(J)$ be such that $g(J) \subset [0,1]$ and $g(y) = 1$. Furthermore, define

$$R_\alpha g \triangleq \int_0^\infty e^{-\alpha s} T_s g \, ds.$$

It is well-known ([23, Section III.2.6]) that $R_\alpha g \in C_0(J)$ and that $e^{-\alpha \cdot} R_\alpha g(\mathsf{X})$ is a $\mathbb{P}_x$-supermartingale for every $x \in J$. Moreover, as $t \mapsto T_t$ is continuous in the origin, we also see that $R_\alpha g(y) > 0$. The optional stopping theorem yields that

$$R_\alpha g(x) \geq \mathbb{E}_x\big[e^{-\alpha \tau_y} R_\alpha g(\mathsf{X}_{\tau_y}) \mathbb{1}_{\{\tau_y < \infty\}}\big] = R_\alpha g(y) \mathbb{E}_x\big[e^{-\alpha \tau_y}\big]. \tag{2.8}$$

As $R_\alpha g \in C_0(J)$, this inequality implies (2.1). As a referee has pointed out, the inequality in (2.8) can also be deduced from the strong Markov property:

$$\begin{aligned} R_\alpha g(x) &\geq \mathbb{E}_x\Big[\int_{\tau_y}^\infty e^{-\alpha s} g(X_s) ds \mathbb{1}_{\{\tau_y < \infty\}}\Big] \\ &= \mathbb{E}_x\Big[\int_0^\infty e^{-\alpha(s+\tau_y)} \mathbb{E}_x\big[g(X_{s+\tau_y}) | \mathcal{F}_{\tau_y}\big] ds \mathbb{1}_{\{\tau_y < \infty\}}\Big] \\ &= \mathbb{E}_x\Big[\int_0^\infty e^{-\alpha(s+\tau_y)} \mathbb{E}_{\mathsf{X}_{\tau_y}}\big[g(X_s)\big] ds \mathbb{1}_{\{\tau_y < \infty\}}\Big] \\ &= R_\alpha g(y) \mathbb{E}_x\big[e^{-\alpha \tau_y}\big]. \end{aligned}$$

The argument based on the optional stopping theorem also works when X is a (not necessarily strong) Markov process.

Conversely, assume that (2.1) holds. By [23, Proposition III.2.4], X is an FD process if and only if $T_t(C_0(J)) \subset C_0(J)$ for all $t > 0$. As X is a Feller process, we only need to show that $T_t f$ vanishes at infinity for every $f \in C_0(J)$ and $t > 0$. Of course, for this property we can restrict our attention to open boundaries, cf. (1.2).

Denote the left boundary point of J by l and the right boundary point by r. Let g_1 be as in the proof of Lemma 2.2 and assume that l is open and finite. For $l < x < r$ a little calculus yields that

$$\begin{aligned} \infty > (x - l) g_1^+(x) + g_1(l) - g_1(x) &= \int_l^x \big(g_1^+(x) - g_1^+(z)\big) dz \\ &= \int_l^x \int_{(z,x]} 2\alpha g_1(u) \mathfrak{m}(du) dz \\ &\geq 2\alpha g_1(l) \int_l^x \mathfrak{m}((z,x]) dz. \end{aligned}$$

As l is open (i.e., $\mathfrak{u}(l) = \infty$, where $\mathfrak{u}$ is as in (1.1)), this inequality yields $g_1(l) = 0$. Similarly, $g_2(r) = 0$ holds in case r is open and finite. In summary, (2.1) holds for all open boundaries irrespective whether these are finite or infinite.

Take $f \in C_0(J)$ and $\alpha, \varepsilon > 0$. For $l < y < x < r$ we have

$$\mathbb{P}_x(\mathsf{X}_\alpha < y) \le \mathbb{P}_x(\tau_y < \alpha) \le e^{\alpha^2}\mathbb{E}_x\big[e^{-\alpha\tau_y}\big]. \tag{2.9}$$

Suppose that the right boundary r is open. Then, as $f \in C_0(J)$, there exists a $z \in J^\circ$ such that $|f(x)| \le \varepsilon$ for all $z \le x$. Now, taking (2.1) and (2.9) into account, we obtain

$$\begin{aligned}|T_\alpha f(x)| &\le \mathbb{E}_x\big[|f(\mathsf{X}_\alpha)|\mathbb{1}_{\{\mathsf{X}_\alpha \ge z\}}\big] + \mathbb{E}_x\big[|f(\mathsf{X}_\alpha)|\mathbb{1}_{\{\mathsf{X}_\alpha < z\}}\big] \\ &\le \varepsilon + \|f\|_\infty \mathbb{P}_x(\mathsf{X}_\alpha < z) \to \varepsilon \text{ as } x \to r.\end{aligned}$$

This implies that $T_\alpha f(x) \to 0$ as $x \to r$.

Similarly, when the left boundary l is open it follows that $T_\alpha f(x) \to 0$ as $x \to l$. We conclude that $T_\alpha f$ vanishes at infinity. The proof is complete. □

Proof of Lemma 2.1 (ii). By Lemma 2.2, (2.2) holds if and only if all infinite boundary points are natural. Thus, (2.2) holds for the diffusions X and $\mathsf{X}_{\cdot\wedge\zeta}$ simultaneously. Consequently, we can w.l.o.g. assume that $\mathsf{X} = \mathsf{X}_{\cdot\wedge\zeta}$.

Let l be the left boundary point of J and let r be the right boundary point. In case $-\infty < l < r < +\infty$ the process $\mathsf{X} = \mathsf{X}_{\cdot\wedge\zeta}$ is bounded and the claim of Lemma 2.1 (ii) is obvious. Below we distinguish between the cases where $-\infty < l < r = \infty$ and $-\infty = l < r = \infty$. The remaining case $-\infty = l < r < \infty$ is similar to the former.

If $\mathsf{X}_t \in L^1(\mathbb{P}_x)$ for all $t > 0$, then the Markov property yields that

$$\mathbb{P}_x\text{-a.s.} \quad \mathbb{E}_x\big[\mathsf{X}_t|\mathcal{F}_s\big] = \mathbb{E}_{\mathsf{X}_s}\big[\mathsf{X}_{t-s}\big], \quad 0 \le s < t.$$

Hence, as martingales always have constant expectation, we have the following:

Lemma 2.4. *X is a $\mathbb{P}_x$-martingale for all $x \in J^\circ$ if and only if $\mathsf{X}_t \in L^1(\mathbb{P}_x)$ and $\mathbb{E}_x[\mathsf{X}_t] = x$ for all $x \in J^\circ$ and $t > 0$.*

In the following we prove that the latter condition from Lemma 2.4 is equivalent to (2.2).

Case 1: $-\infty < l < r = \infty$. Fix $x \in J^\circ = (l, \infty)$ and $t > 0$. First of all, $\mathsf{X}_t \in L^1(\mathbb{P}_x)$ follows from Fatou's lemma as X is a local martingale which is bounded from below. For $x < y < r = \infty$ the stopped process $\mathsf{X}_{\cdot\wedge\tau_y}$ is $\mathbb{P}_x$-a.s. bounded and consequently, a $\mathbb{P}_x$-martingale. As $\mathsf{X}_t \in L^1(\mathbb{P}_x)$, the dominated convergence theorem yields that

$$\begin{aligned}\mathbb{E}_x\big[\mathsf{X}_t\big] &= \lim_{y\to\infty} \mathbb{E}_x\big[\mathsf{X}_t \mathbb{1}_{\{\tau_y > t\}}\big] \\ &= \lim_{y\to\infty} \big(\mathbb{E}_x\big[\mathsf{X}_{t\wedge\tau_y}\big] - \mathbb{E}_x\big[\mathsf{X}_{\tau_y}\mathbb{1}_{\{\tau_y \le t\}}\big]\big) \\ &= x - \lim_{y\to\infty} y\mathbb{P}_x(\tau_y \le t).\end{aligned}$$

Thus, by Lemma 2.4, X is a $\mathbb{P}_x$-martingale for all $x \in J^\circ$ if and only if

$$\lim_{y\to\infty} y\mathbb{P}_x(\tau_y \le t) = 0$$

for all $x \in J^\circ$ and $t > 0$. Taking this observation into consideration, the next lemma completes the proof of Lemma 2.1 (ii) for the current case.

Lemma 2.5. *Let $x \in J^\circ$. Then, $\lim_{y\to\infty} y\mathbb{P}_x(\tau_y \le t) = 0$ for all $t > 0$ if and only if $\lim_{y\to\infty} y\mathbb{E}_x[e^{-\alpha\tau_y}] = 0$ for all $\alpha > 0$.*

Proof. Take $\alpha > 0$. Fubini's theorem yields that

$$\int_0^\infty e^{-\alpha t}\mathbb{P}_x(\tau_y \le t)dt = \iint_u^\infty e^{-\alpha t} dt\, \mathbb{P}_x(\tau_y \in du) = \tfrac{1}{\alpha}\mathbb{E}_x\big[e^{-\alpha\tau_y}\big].$$

Furthermore, for every $y \geq x$ we have

$$(y-l)\mathbb{P}_x(\tau_y \leq t) = \mathbb{E}_x\big[(\mathsf{X}_{t\wedge\tau_y} - l)\mathbb{1}_{\{\tau_y \leq t\}}\big] \leq \mathbb{E}_x\big[\mathsf{X}_{t\wedge\tau_y}\big] - l = x - l,$$

which implies $|y|\mathbb{P}_x(\tau_y \leq t) \leq x - l + |l|$. Thus, if $\lim_{y\to\infty} y\mathbb{P}_x(\tau_y \leq t) = 0$ for all $t > 0$, then the dominated convergence theorem yields

$$\lim_{y\to\infty} y\mathbb{E}_y\big[e^{-\alpha\tau_y}\big] = \lim_{y\to\infty} \int_0^\infty \alpha e^{-\alpha t} y\mathbb{P}_x(\tau_y \leq t)dt = 0.$$

This is the *only if* implication.

Conversely, if $\lim_{y\to\infty} y\mathbb{E}_x[e^{-\alpha\tau_y}] = 0$, then

$$\lim_{y\to\infty} y\mathbb{P}_x(\tau_y \leq \alpha) \leq e^{\alpha^2} \lim_{y\to\infty} y\mathbb{E}_x\big[e^{-\alpha\tau_y}\big] = 0. \tag{2.10}$$

This gives the *if* implication. The proof is complete. □

Case 2: $-\infty = l < r = \infty$. We start with a version of [19, Lemma 1]:

Lemma 2.6. *For all $t > 0$ and $x \in J = \mathbb{R}$ we have $\mathsf{X}_t \in L^1(\mathbb{P}_x)$.*

For completeness, we provide a proof for Lemma 2.6 at the end of this section. Suppose now that (2.2) holds and take $x \in \mathbb{R}$. As in (2.10), we obtain

$$\lim_{y\to\infty} y\mathbb{P}_x(\tau_y \leq t) = \lim_{y\to\infty} y\mathbb{P}_x(\tau_{-y} \leq t) = 0, \quad t > 0.$$

Now, by virtue of Lemma 2.6, the dominated convergence theorem yields that

$$\begin{aligned}\mathbb{E}_x\big[\mathsf{X}_t\big] &= \lim_{y\to\infty} \mathbb{E}_x\big[\mathsf{X}_t\mathbb{1}_{\{\tau_y\wedge\tau_{-y} > t\}}\big] \\ &= \lim_{y\to\infty}\big(\mathbb{E}_x\big[\mathsf{X}_{t\wedge\tau_y\wedge\tau_{y-}}\big] - \mathbb{E}_x\big[\mathsf{X}_{\tau_y\wedge\tau_{-y}}\mathbb{1}_{\{\tau_y\wedge\tau_{-y}\leq t\}}\big]\big) \\ &= x - \lim_{y\to\infty}\big(y\mathbb{P}_x(\tau_y \leq t, \tau_y < \tau_{-y}) - y\mathbb{P}_x(\tau_{-y} \leq t, \tau_{-y} < \tau_y)\big) \\ &= x\end{aligned}$$

for all $t > 0$. Hence, the process X is a $\mathbb{P}_x$-martingale by Lemma 2.4.

Conversely, assume that X is a $\mathbb{P}_x$-martingale for all $x \in \mathbb{R}$ and take $a \in \mathbb{R}$. By the optional stopping theorem, the stopped process $\mathsf{X}_{\cdot\wedge\tau_a}$ is a $\mathbb{P}_x$-martingale for all $x \in \mathbb{R}$. For suitable initial values, $\mathsf{X}_{\cdot\wedge\tau_a}$ is a diffusion with state space $[a,\infty)$ (or with state space $(-\infty, a]$). Notice that $\mathsf{X}_{\cdot\wedge\tau_a}$ has the same boundary behavior at ∞ (or at $-\infty$) as the unstopped process X, see [15, Section 3.9, pp. 102 – 105]. Now, the previous case and Lemma 2.2 yield that ∞ and $-\infty$ are natural. Hence, again by Lemma 2.2, (2.2) holds and the proof is complete. □

Proof of Lemma 2.6. We use a suitable Lyapunov function. Such a function was also used in the proof of [19, Lemma 1], but it was not given explicitly. Let $-\infty < a < 0 < b < \infty$ and let $g\colon \mathbb{R} \to [0,1]$ be a continuous function such that $g \equiv 0$ off $[a,b]$ and $g > 0$ on $[a,b]$. Furthermore, define

$$f(x) \triangleq \begin{cases} \displaystyle\int_0^x \int_{(0,y]} 2g(z)\mathfrak{m}(dz)dy, & \text{for } x \geq 0, \\ \displaystyle\int_x^0 \int_{(y,0]} 2g(z)\mathfrak{m}(dz)dy, & \text{for } x \leq 0. \end{cases}$$

Notice that $\frac{1}{2}\frac{d}{d\mathfrak{m}}\frac{d^+ f}{dx} = g$, $\lim_{x\to\infty} f(x)/x > 0$ and $\lim_{x\to-\infty} f(x)/(-x) > 0$.

Take $y > (-a) \vee b$. As $\frac{1}{2}\frac{d}{d\mathfrak{m}}\frac{d^+}{dx}$ is the generator of the stopped diffusion $\mathsf{X}_{\cdot\wedge\tau_y\wedge\tau_{-y}}$ and f is in its domain (see [13, Section 2.7]), Dynkin's formula ([13, Lemma 48, p. 119]) yields

$$\mathbb{E}_x\big[f(\mathsf{X}_{t\wedge\tau_y\wedge\tau_{-y}})\big] = f(x) + \mathbb{E}_x\Big[\int_0^{t\wedge\tau_y\wedge\tau_{-y}} g(\mathsf{X}_s)ds\Big] \le f(x) + t\|g\|_\infty.$$

Finally, letting $y \to \infty$ and using Fatou's lemma yields that $f(\mathsf{X}_t) \in L^1(\mathbb{P}_x)$. As $\lim_{x\to\infty} f(x)/x > 0$ and $\lim_{x\to-\infty} f(x)/(-x) > 0$, this implies $\mathsf{X}_t \in L^1(\mathbb{P}_x)$ and the proof is complete. □

References

[1] R. Azencott. Behavior of diffusion semi-groups at infinity. *Bulletin de la Société Mathématique de France*, 102:193–240, 1974. MR0356254

[2] L. Breiman. *Probability*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 1992. MR1163370

[3] E. Bayraktar and H. Xing. On the uniqueness of classical solutions of Cauchy problems. *Proceedings of the American Mathematical Society*, 138(6):2061–2064, 2010. MR2596042

[4] U. Çetin. Diffusion transformations, Black–Scholes equation and optimal stopping. *The Annals of Applied Probability*, 28(5):3102–3151, 2018. MR3847983

[5] U. Çetin and K. Larsen. Uniqueness in Cauchy problems for diffusive real-valued strict local martingales. *Transactions of the American Mathematical Society Series B*, 10:381–406, 2023. MR4556218

[6] K. L. Chung and J. B. Walsh. *Markov processes, Brownian motion, and time symmetry*. Springer Science+Business Media, 2nd edition, 2005. MR2152573

[7] P. Clément and C. A. Timmermans. On C_0-semigroups generated by differential operators satisfying Ventcel's boundary conditions. *Indagationes Mathematicae (Proceedings)*, 89(4):379–387, 1986. MR0869754

[8] D. Criens. Lyapunov criteria for the Feller–Dynkin property of martingale problems. *Stochastic Processes and their Applications*, 130(5):2693–2736, 2020. MR4080726

[9] F. Delbaen and H. Shirakawa. No arbitrage condition for positive diffusion price processes. *Asia-Pacific Financial Markets*, 9(3):159–168, 2002.

[10] A. Eberle. *Uniqueness and Non-Uniqueness of Semigroups Generated by Singular Diffusion Operators*. Springer Berlin Heidelberg, 1999. MR1734956

[11] S. N. Ethier and T. G. Kurtz. *Markov Processes Characterization and Convergence*. Wiley, 2005. MR0838085

[12] W. Feller. The parabolic differential equations and the associated semi-group of transformations. *Annals of Mathematics, Second Series*, 55(3):468–519, 1952. MR0047886

[13] D. Freedman. *Brownian Motion and Diffusion*. Springer New York Heidelberg Berlin, 1983. MR0686607

[14] H. Hulley. *Strict local martingales in continuous financial market models*. Ph.D. Thesis, University of Sydney, Sydney, Australia, 2009.

[15] K. Itô and H. P. McKean, Jr.. *Diffusion Processes and their Sample Paths*. Springer Berlin Heidelberg, 1974. MR0345224

[16] O. Kallenberg. *Foundations of Modern Probability*. Springer Nature Switzerland, 3rd edition, 2021. MR4226142

[17] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer New York, 2nd edition, 1991. MR1121940

[18] F. B. Knight. *Essentials of Brownian Motion and Diffusion*. American Mathematical Society, 1981. MR0613983

[19] S. Kotani. On a condition that one-dimensional diffusion processes are martingales. In *In Memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, 149–156. Springer, 2006. MR2276894

[20] P. Mandl. *Analytic Treatment of One-dimensional Markov Processes*. Springer Berlin Heidelberg New York, 1968. MR0247667

[21] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer New York, 1983. MR0710486

[22] S. C. Port and C. J. Stone. *Brownian Motion and Classical Potential Theory*. Academic Press, 1978. MR0492329

[23] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer, Berlin Heidelberg, 3rd edition, 1999. MR1725357

[24] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes, and Martingales. Volume 1 - Foundations*. John Wiley & Sons, 2nd edition, 1994. MR0531031

[25] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes, and Martingales. Volume 2 - Itô Calculus*. Cambridge University Press, 2nd edition, 2000. MR1780932

[26] M. Urusov and M. Zervos. Necessary and sufficient conditions for the r-excessive local martingales to be martingales. *Electronic Communications in Probability*, 22(10):1–6, 2017. MR3607805

[27] R. Vyborny and R. Nester. L'Hôpital's rule, a counterexample. *Elemente der Mathematik*, 44:116–121, 1989. MR1006321

[28] L. Wu and Y. Zhang. A new topological approach to the L^∞-uniqueness of operators and the L^1-uniqueness of Fokker–Planck equations. *Journal of Functional Analysis*, 241:557–610, 2006. MR2271930

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