

The Foata–Fuchs proof of Cayley’s formula, and its probabilistic uses*

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Abstract

We present a very simple bijective proof of Cayley’s formula due to Foata and Fuchs (1970). This bijection turns out to be very useful when seen through a probabilistic lens; we explain some of the ways in which it can be used to derive probabilistic identities, bounds, and growth procedures for random trees with given degrees, including random d -ary trees. We also introduce a partial order on the degree sequences of rooted trees, and conjecture that it induces a stochastic partial order on heights of random rooted trees with given degrees.

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1 Introduction

A *rooted tree* is a triple $t = (V, E, \rho)$, where (V, E) is a finite connected graph without cycles and where $\rho \in V$; we call ρ *the root of* t . By “tree” we always mean “rooted tree” unless otherwise specified. We write

$$\mathcal{T}_n = \{t = (V, E, \rho) : t \text{ is a tree with } V = [n]\}$$

for the set of trees with vertices labelled by $[n] := \{1, \dots, n\}$. *Cayley’s formula* (which to the best of current knowledge was first established by Borchardt [11]) provides a very simple formula for $|\mathcal{T}_n|$.

Theorem 1.1 (Cayley’s formula). $|\mathcal{T}_n| = n^{n-1}$.

There are numerous proofs of Cayley’s formula in the literature. The 1967 survey by Moon [34] presents ten such proofs, including the proof via so-called *Prüfer codes*, which

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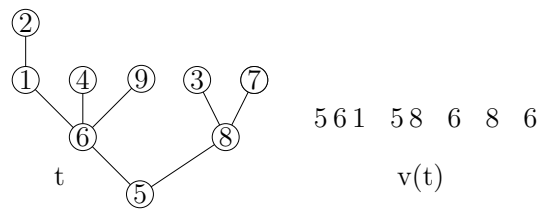


Figure 1: A tree t and the corresponding sequence $v(t)$.

is probably the one most frequently presented in undergraduate texts. More recent proofs include those discovered by Joyal [29, Section 2.2], which considers doubly-rooted trees; by Pitman [38], which analyzes a coalescent process for building trees; and by van der Hofstad [45, Section 3.6], which uses the connection between random plane trees and conditioned Poisson branching processes.

In this work, we present a proof of Cayley’s formula due to Foata and Fuchs [23], via a so-called *line-breaking* construction. We believe this is the simplest proof of Cayley’s formula yet discovered, but it is not well-known. (In fact, in the first version of this manuscript we believed it to be new; we thank Adrien Segovia for pointing out the work [23] to us.) The reason it has been overlooked may be because the focus of the paper [23] is on a bijection between the set of sequences $\{(x_1, \dots, x_n) : x_i \in [n], 1 \leq i \leq n\}$ and the set of functions $f : [n] \rightarrow [n]$. The Foata–Fuchs bijection map yields a bijection between sequences with $x_1 = x_2$ and \mathcal{T}_n : given such a sequence (x_1, \dots, x_n) , if $f : [n] \rightarrow [n]$ is the resulting function, then the corresponding tree in \mathcal{T}_n has root x_1 and edge set $\{(i, f(i)), i \in [n] \setminus \{x_1\}\}$. For an English-language presentation of the full Foata–Fuchs bijection, see [13].

We discuss related constructions for unrooted trees, rooted trees with marked vertices and rooted forests in Section 2. The bijection is extremely useful for the analysis of random combinatorial trees; Section 3 of this paper discusses some of its probabilistic implications. In that section, we also define an algorithm for growing random trees with given vertex degrees. When applied to regular trees, the algorithm yields a sequence $(T_m, m \geq 1)$, where for each $m \geq 1$, T_m is a uniformly random, leaf-labelled d -ary tree with m internal nodes, and T_{m+1} is generated from T_m by local regrafting. Our algorithm is somewhat similar in spirit to Rémy’s algorithm [41], which generates a sequence of uniform binary leaf-labelled trees.

Before giving the proof of Cayley’s formula, we introduce some terminology. Suppose t is a tree, S is a connected subset of its vertices, and x is a vertex. The *path from S to x* in t is the unique path in t which starts at a vertex of S , does not visit any other vertex of S , and ends at x . If P is a path, we may also write P to denote the set of vertices of the path. Finally, a *leaf* of t is a non-root vertex of t with degree 1.

Proof of Cayley’s formula. The term n^{n-1} counts sequences $v = (v_1, \dots, v_{n-1}) \in [n]^{n-1}$. We prove the theorem by introducing a bijection between \mathcal{T}_n and $[n]^{n-1}$.

Bijection

For a tree t on $[n]$ with root ρ :

- Let $\ell_1 < \ell_2 < \dots < \ell_k$ be the leaves of t .
- Let $S_0 = \{\rho\}$.
- Recursively, for $i = 1, \dots, k$, let P_i be the path in t from S_{i-1} to ℓ_i , and let $S_i = S_{i-1} \cup P_i$. Let P_i^* be P_i omitting its final point.
- Let $v(t)$ be the concatenation of $P_1^*, P_2^*, \dots, P_k^*$.

We claim that $v(t) \in [n]^{n-1}$. Indeed, by definition, for each i , $P_i^* \subset [n]$. Moreover, note that the edge sets of P_1, P_2, \dots, P_k form a partition of the edge set of t , and for each i , $|P_i^*|$ equals the number of edges in P_i . This implies that the length of $v(t)$ is equal to the number of edges in t , which equals $n - 1$.

To show that the above construction is a bijection, we describe its inverse. For a sequence $v = v_1, v_2, \dots, v_m$, we say that $j \in \{2, \dots, m\}$ is the location of a repeat of v if $v_j = v_i$ for some $i < j$.

Inverse of the bijection

Given a sequence $v = (v_1, v_2, \dots, v_{n-1}) \in [n]^{n-1}$:

- Let $j(0) = 1$, let $j(1) < j(2) < \dots < j(k - 1)$ be the locations of the repeats of the sequence v , and let $j(k) = n$.
- Let $\ell_1 < \ell_2 < \dots < \ell_k$ be the elements of $[n]$ not occurring in v .
- For $i = 1, \dots, k$, let P_i be the path $(v_{j(i-1)}, \dots, v_{j(i)-1}, \ell_i)$ with $j(i) - j(i - 1)$ edges.
- Let $t(v)$ be the graph with vertex set $[n]$, with root v_1 and with edge set given by the union of the edges of the paths P_1, P_2, \dots, P_k .

We claim that $t(v)$ is a tree with vertex set $[n]$. By construction, for $i = 2, \dots, k$, the set of vertices $\cup_{m < i} P_m$ and the path P_i intersect at a single point (namely $v_{j_{i-1}}$, the first point of P_i). Hence, by induction, for each i the union of the edges of the paths P_1, \dots, P_i is a tree, and in particular taking $i = k$ we have that $t(v)$ is a tree. Since for $i = 1, \dots, k$, the final point of P_i is ℓ_i , and since no element ℓ_i appears as the first point of any other path P_m , we have that the leaves of $t(v)$ are ℓ_1, \dots, ℓ_k .

We will argue that if $v \in [n]^{n-1}$, then $v(t(v)) = v$. Let $k, P_1, \dots, P_k, \ell_1, \dots, \ell_k$, and $v_1, v_{j_1}, \dots, v_{j_k}$ be defined as in the inverse of the bijection applied to v . By definition, v_1 is the root in $t(v)$. Note that P_1 is a path contained in $t(v)$ with endpoints v_1 and ℓ_1 . The fact that $t(v)$ is a tree implies that P_1 is the unique path from v_1 to ℓ_1 in $t(v)$. Similarly, for $1 < i \leq k$, P_i is a path contained in $t(v)$ with endpoints $v_{j_{i-1}}$ and ℓ_i , and with $P_i \cap \cup_{m < i} P_m = \{v_{j_{i-1}}\}$, so that P_i is the unique path in $t(v)$ from the union of P_1, \dots, P_{i-1} to ℓ_i . Hence $v(t(v))$ is the concatenation of P_1^*, \dots, P_k^* , giving $v(t(v)) = v$ as desired. \square

2 Extensions and related constructions

Trees with given degrees

The bijection has the property that for a given tree t , in the associated sequence v the number of times a given integer k appears is precisely the number of children of the vertex with label k in t .

The *type* of a tree t is the vector $n = (n_c, c \geq 0)$, where n_c is the number of vertices of t with exactly c children. Writing $n = \sum_{c \geq 0} n_c$ and

$$\mathcal{T}(n) = \{t \in \mathcal{T}_n : t \text{ has type } n\},$$

then

$$|\mathcal{T}(n)| = \binom{n}{n_c, c \geq 0} \cdot \frac{(n - 1)!}{\prod_{c \geq 0} (c!)^{n_c}}.$$

This identity (also found in [37, Theorem 1.5] and [43, Corollary 3.5]) follows from the specializations of the bijection to trees of a given type: the right-hand side counts the number of ways to first choose the labels of the vertices with c children for each $c \geq 0$, then choose a sequence of length $n - 1$ in which the label of each vertex with c children appears exactly c times.

Unrooted trees

A simple trick allows us to directly study unrooted trees with this bijection by considering vertex 1 as the root and removing the leading 1 of the corresponding coding sequence. This gives a proof of the more well-known form of Cayley’s formula which states that there are n^{n-2} unrooted trees of size n .

There is another way to adapt the bijection to encode unrooted trees of size n by sequences in $[n]^{n-2}$. It consists of letting P_1^* be the path between the first and second-lowest-labelled vertices of degree 1 (excluding both endpoints) and continuing as in the bijection for rooted trees.

Rooted trees with marked vertices

The bijection can also be modified to encode rooted trees with $r \geq 1$ distinguishable marks on the vertices, $(t, m_1, \dots, m_r) \in \mathcal{T}_n \times [n]^r$, by sequences in $[n]^{n+r-1}$. The modification consists of changing the definition of P_i^* in the recursive step slightly when constructing the sequence from the tree: for $i = 1, \dots, r$, P_i^* is the path from S_{i-1} to the i -th marked point (including the final point of this path), and for $i = 1, \dots, k$, P_{r+i}^* is the path from S_{r+i-1} to the i -th unmarked leaf ℓ_i (excluding its final point). Here the number of appearances of a vertex in the coding sequence is the sum of its number of children and its number of marks.

By taking the $r \rightarrow \infty$ limit of this bijection and applying it to an i.i.d. sequence of elements of $[n]$, one recovers the construction of the so-called p -trees from [15]. (Briefly, p -trees may be seen as random trees with fixed but *random* types, where the type distribution has a product structure.)

Rooted forests

The bijection above also extends to rooted forests. Given a set $S \subset [n]$, write \mathcal{F}_n^S for the set of forests F with vertex set $[n]$ and root set S . Setting $s = |S|$, we describe a bijection between \mathcal{F}_n^S and the set

$$\{v = (v_1, \dots, v_{n-s}) \in [n]^{n-s} : v_1 \in S\},$$

which has cardinality sn^{n-s-1} . The analogue of the “inverse bijection” is the easiest to describe, so we begin with that. To construct a sequence (v_1, \dots, v_{n-s}) from $F \in \mathcal{F}_n^S$, proceed just as in the above proof of Cayley’s formula, but start from the s -vertex forest F_0 containing only the root vertices S , and at each step append the vertex labels along the path to the smallest labelled leaf not already in the current forest (excluding the leaf itself).

Conversely, here is how to construct a forest in \mathcal{F}_n^S from a sequence $v = (v_1, \dots, v_{n-s})$ with $v_1 \in S$. Say that i is the location of a repeat if $i > 1$ and either $v_i \in S$ or there is $1 \leq j < i$ such that $v_i = v_j$. Denote the locations of repeats of v by $j(1), \dots, j(k-1)$ in increasing order, and let $j(k) = n - s + 1$. List the integers from $[n] \setminus S$ which do not appear in v as ℓ_1, \dots, ℓ_k in increasing order.

Form a graph $F_S = F_S(v)$ with vertices $[n]$, root set S , and edge set

$$\{v_i v_{i+1} : i \in [n - s - 1], i + 1 \notin \{j(1), \dots, j(k)\}\} \cup \{v_{j(i)-1} \ell_i, 1 \leq i \leq k\}.$$

Essentially the same argument as in the proof of Cayley’s formula shows that the connected components of F_S are trees and that there are s components of F_S , each containing exactly one vertex of S . Thus, rooting each component of F_S at its unique element of S turns F_S into an element of \mathcal{F}_n^S .

Since the construction is bijective, this yields the following extension of Cayley’s formula (which is in fact stated by both Borchardt [11] and Cayley [16]):

$$|\mathcal{F}_n^S| = |\{v = (v_1, \dots, v_{n-s}) \in [n]^{n-s} : v_1 \in S\}| = sn^{n-s-1}.$$

This extension of the bijection can also be specialized to count forests with fixed types.

Other coding sequences of trees

A variant of Prüfer codes for rooted trees, first described by Neville [35], gives another bijection between \mathcal{T}_n and $[n]^{n-1}$. It coincides with the original Prüfer code if one sees vertex n as the root and then removes the resulting trailing n . Deo and Micikevicius [19] found that reading the rooted Prüfer code in reverse leads to a simpler description of the bijection, which shares some features of the bijection presented in this paper. This bijection was independently discovered by Seo and Shin [42] under the name *Reverse-Prüfer codes* to study leader vertices in trees, and also by Fleiner [21], who remarked that it can be used to describe the law of the height of a uniformly sampled vertex in a random tree (see Proposition 3.2, below).

3 Random trees

The bijection above has numerous consequences for random trees. This section first explains how the bijection can be used to study typical and extreme distances in random trees, then discusses how it can be used to define growth procedures for random trees, somewhat in the spirit of Rémy’s algorithm [41] for growing uniformly random leaf-labelled binary trees.

For a finite set S , we will write $X \in_u S$ to mean that X is chosen uniformly at random from the set S .

Distances in random trees

For a tree $t = (V, E, \rho)$ and a vertex $v \in V$, write $|v| = |v|_t$ for the graph distance from ρ to v .

Proposition 3.1. Let $T \in_u \mathcal{T}_n$ and let L be a uniformly random leaf of T . Also, let $V = (V_i, i \geq 1)$ be a sequence of independent uniformly random elements of $[n]$ and let $I = \min(i \geq 2 : V_i \in \{V_1, \dots, V_{i-1}\})$ be the index of the first repeated element of V . Then $|L| + 1 \stackrel{d}{=} \min(I, n)$.

Proof. Write $V = (V_1, \dots, V_{n-1}) \in_u [n]^{n-1}$. Then $T = t(V) \in_u \mathcal{T}_n$. Moreover, recalling that the repeated entries of (V_1, \dots, V_{n-1}) are $j(1), \dots, j(k-1)$ and that $j(k) = n$, we have $\min(I, n) = j(1)$. The first leaf $\ell_1(T)$ is a child of $V_{j(1)-1}$, so

$$|\ell_1(T)| = |V_{j(1)-1}| + 1 = j(1) - 1 + 1 = \min(I, n) - 1.$$

But since T is a uniformly random tree, randomly permuting its leaf labels does not change its distribution, so $|\ell_1(T)|$ has the same distribution as $|L|$ for L a uniformly random leaf of T . \square

One can also use the version of the bijection for rooted trees with one marked vertex to show the following similar result, whose proof is left to the reader.

Proposition 3.2. Let $T \in_u \mathcal{T}_n$ and let $U \in_u [n]$ be independent of T . Then $|U| \stackrel{d}{=} I - 2$, where I is as in Proposition 3.1.

These propositions in particular imply that if D_n is the distance from the root to a uniformly random vertex (or leaf) in $T_n \in_u \mathcal{T}_n$, then $n^{-1/2}D_n \xrightarrow{d} R$, where R is Rayleigh distributed: $\mathbf{P}\{R \geq x\} = e^{-(x^2/2)}\mathbf{1}_{\{x \geq 0\}}$. It is not hard to build on these statements in order to precisely characterize the joint distribution of the lengths of the branches to the k smallest labelled leaves (using the bijection) or to k uniform vertices (using its extension to trees with k marked vertices). The latter recovers the asymptotic *line-breaking construction* of uniformly random trees, proposed by Aldous [5]. In his work, Aldous used a discrete version of the line-breaking construction based on the so-called *Aldous-Broder algorithm* [7, 12], a procedure for efficiently sampling a uniformly random spanning tree of a fixed connected graph. In the setting of the complete graph K_n , the Aldous-Broder algorithm is a close cousin of the sampling procedure obtained by applying our bijection to a sequence (U_1, \dots, U_{n-1}) of independent, uniformly random elements of $[n]$, though the two are not equivalent.

These distributional identities also make it possible to compare the distributions of typical distances in different trees. Define a partial order \prec on type sequences $n = (n_c, c \geq 0)$ with $\sum_{c \geq 0} n_c = n$ and $\sum_{c \geq 0} cn_c = n - 1$ by the following covering¹ relation: $m = (m_c, c \geq 0)$ covers $(n_c, c \geq 0)$ if there are positive integers a, b such that $m_a = n_a + 1, m_b = n_b + 1$ and $m_0 = n_0 - 1, m_{a+b} = n_{a+b} - 1$. (In words, to obtain m from n we replace one vertex with $a + b$ children by two vertices, one with a children and one with b children, and reduce the number of leaves accordingly; then $n \prec m$.)

Given types n, m related as in the previous paragraph, we may couple the constructions of random trees with types n and m as follows. Let $V_n = (V_1, \dots, V_{n-1})$ be a uniformly random sequence of elements of $[n]$ subject to the constraint that for each $c \geq 0$, the number of values from $[n]$ which occur precisely c times in the sequence is n_c . Conditionally on V_n , choose $X \in [n]$ uniformly at random from among those integers in $[n]$ which appear exactly $a + b$ times in V_n , and independently choose Y uniformly at random from among those integers in $[n]$ which do not appear in V_n . Choose a of the instances where X appears in V_n , uniformly at random, and replace each of them by the integer Y ; call the resulting sequence V_m . Then the trees T_n and T_m corresponding to V_n and V_m under the bijection are uniformly random elements of $\mathcal{T}(n)$ and $\mathcal{T}(m)$, respectively. Moreover, the index of the first repeated element of V_n is at most that of the first repeated element of V_m , so the distance from the root to the smallest labelled leaf in T_n is at most the corresponding distance in T_m .

It follows from this coupling that for any types n and m with $n \preceq m$, if $T_n \in_u \mathcal{T}(n)$ and $T_m \in_u \mathcal{T}(m)$, and L_n and L_m are uniformly random leaves of T_n and T_m , respectively, then $|L_n| = |L_n|_{T_n}$ is stochastically dominated by $|L_m| = |L_m|_{T_m}$, by which we mean that $\mathbf{P}\{|L_n| \leq t\} \geq \mathbf{P}\{|L_m| \leq t\}$ for all $t \in \mathbb{R}$; we denote this relation by $|L_n| \preceq_{st} |L_m|$. We conjecture that this stochastic relation also holds for the heights of the trees: writing $\text{ht}(t) := \max(|v|, v \text{ is a vertex of } t)$, then

$$\text{ht}(T_n) \preceq_{st} \text{ht}(T_m),$$

whenever $T_n \in_u \mathcal{T}(n)$ and $T_m \in_u \mathcal{T}(m)$ and $n \preceq m$.

The above coupling may also be extended to the construction of trees with a marked vertex, using the version of the bijection for rooted trees with a marked vertex presented in Section 2, and it follows that for T_n and T_m as in the preceding paragraph, if $U \in_u [n]$ then $|U|_{T_n} \preceq_{st} |U|_{T_m}$. This stochastic inequality may also be proven using Proposition 3.2.

Some of the most valuable probabilistic consequences of the bijection arise when studying distances in trees with given degrees, using the variant of the bijection that

¹For a partially ordered set (\mathcal{P}, \prec) , $y \in \mathcal{P}$ covers $x \in \mathcal{P}$ if $x \prec y$ and for all $z \in \mathcal{P}$, if $x \preceq z \preceq y$ then $z \in \{x, y\}$.

we introduce at the start of Section 2. In particular, in recent work, Blanc-Renaudie [9] uses the version of the bijection for trees with given degrees to prove convergence toward inhomogeneous continuum random trees for random combinatorial trees with a fixed type, in great generality, thereby resolving a conjecture from [6] on Lévy trees being a ‘mixture’ of inhomogeneous continuum random trees; he also proved exponential upper tail bounds for the heights of such random trees, which are tight up to constant factors in many cases. In other recent work, Addario-Berry and Donderwinkel [3] use the bijection in order to prove several conjectures from [1], [4], [28] and [33] regarding the asymptotic behaviour of the height of random trees and tree-like structures; the work [3] in particular shows that *all* random combinatorial trees with n nodes have typical height $O(n^{1/2})$, unless they are extremely “path-like”, possessing $n - o(n)$ nodes with exactly one child.

The bijection can also be adapted to study the structure of random graphs which are not trees (see [2, Chapter 1]), and we expect that it will prove useful in proving convergence results in such settings as well; a first example is found in [10].

Growth procedures for trees with a given degree sequence

In this section, we present a growth procedure for trees with a given degree sequence — and in particular for d -ary trees — which is built from a version of the inverse bijection presented in the proof of Theorem 1.1. To the best of our knowledge, this is the first growth procedure for uniformly random trees with a given degree sequence. Other growth procedures for uniform d -ary trees have appeared in the literature, but we believe that the procedure obtained by specializing our general method to d -ary trees is the simplest one known so far. We briefly describe the two other growth procedures for uniform d -ary trees that we are aware of at the end of this section.

Fix an abstract set $\{\ell_i, i \geq 1\}$ of leaf labels. Let $\mathbf{d} = (d_1, \dots, d_m)$ be a sequence of positive integers, and let $L_{\mathbf{d}} = 1 + \sum_{i=1}^m (d_i - 1) \geq 1$. Let $\mathcal{T}_{\mathbf{d}}$ denote the set of trees t with vertex set $[m] \cup \{\ell_1, \dots, \ell_{L_{\mathbf{d}}}\}$ such that for each $i \in [m]$, i has d_i children in t (and $\ell_1, \dots, \ell_{L_{\mathbf{d}}}$ are leaves of t). Also, write

$$\mathcal{S}_{\mathbf{d}} := \{(v_1, \dots, v_{m+L_{\mathbf{d}}-1}) : |\{k : v_k = i\}| = d_i \text{ for all } i \in [m]\} .$$

Then the following modification of the inverse bijection from Theorem 1.1 gives a bijection between $\mathcal{S}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}}$. Fix $\mathbf{v} = (v_1, \dots, v_{m+L_{\mathbf{d}}-1}) \in \mathcal{S}_{\mathbf{d}}$.

Bijection between $\mathcal{S}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}}$.

- Let $j(0) = 1$, let $j(1) < j(2) < \dots < j(L_{\mathbf{d}} - 1)$ be the locations of the repeats of the sequence \mathbf{v} , and let $j(L_{\mathbf{d}}) = m + L_{\mathbf{d}}$.
- For $i = 1, \dots, L_{\mathbf{d}}$, let P_i be the path $(v_{j(i-1)}, \dots, v_{j(i)-1}, \ell_i)$.
- Let $t(\mathbf{v}) \in \mathcal{T}_{\mathbf{d}}$ have root v_1 and edge set given by the union of the edges of the paths $P_1, P_2, \dots, P_{L_{\mathbf{d}}}$.

In the above bijection, the path P_i ends with the leaf ℓ_i . However, any other fixed ordering of $\ell_1, \dots, \ell_{L_{\mathbf{d}}}$ would also yield a bijective correspondence. In fact, this is even true if the choice of leaf ordering depends on the sequence \mathbf{v} , provided that the leaf-ordering rule still has the property that different sequences result in different trees. We exploit this flexibility below, in order to design a simple growth procedure.

Write \preceq for the total ordering of the vertices of $t(\mathbf{v})$ which is the order the vertices first appear along the paths $P_1, \dots, P_{L_{\mathbf{d}}}$.

For $d \in \mathbb{N}$ and $\mathbf{d}' := (d_1, \dots, d_m, d)$, note that we can construct an element $S' \in \mathcal{S}_{\mathbf{d}'}$ starting from an element $S \in \mathcal{S}_{\mathbf{d}}$ by inserting d repeats of the integer $m + 1$. (Later, we

also write $\mathbf{d}' = (d_1, \dots, d_m, d_{m+1})$, so $d_{m+1} = d$.) This corresponds to choosing $S \in \mathcal{S}_{\mathbf{d}}$ and a multiset of size d with elements in $[m + L_{\mathbf{d}}]$; one then constructs $S' \in \mathcal{S}_{\mathbf{d}'}$ by inserting the i 'th repeat of $m + 1$ immediately before the j 'th entry of S , where j is the i 'th element of the multiset (unless $j = m + L_{\mathbf{d}}$, in which case we insert the i 'th repeat of $m + 1$ at the end of S). Equivalently, we may construct an element of $\mathcal{S}_{\mathbf{d}'}$ starting from a tree $T \in \mathcal{T}_{\mathbf{d}}$ and a multiset of d vertices of T , where we use the convention that $i \in [m + L_{\mathbf{d}}]$ corresponds to the i 'th vertex in the total ordering \preceq of the vertices of T defined above. The following growth procedure provides a way to use this correspondence to sample a uniform random element of $\mathcal{T}_{\mathbf{d}'}$ given $T \in_u \mathcal{T}_{\mathbf{d}}$ and an independent uniform random multiset of size d of vertices of T . It yields the same shape of tree that one would obtain starting from T by first transforming the sequence S corresponding to T under the above bijection as just described, then applying the bijection to the resulting sequence. However, it yields a different labelling of the leaves.

Constructing a tree $T' \in \mathcal{T}_{\mathbf{d}'}$ from a tree $T \in \mathcal{T}_{\mathbf{d}}$ and a multiset of d vertices of T

Denote the multiset as $\{w_1, \dots, w_d\}$, where $w_1 \preceq w_2 \preceq \dots \preceq w_d$. Then T' is obtained from T by adding vertices $m + 1, \ell_{L_{\mathbf{d}}+1}, \dots, \ell_{L_{\mathbf{d}}+d-1}$ and modifying the edge-set as follows.

1. If w_1 was the root of T , connect $m + 1$ to w_1 and re-root the tree at $m + 1$. Otherwise, replace the edge vw_1 that connects w_1 to its parent v with two edges $v(m + 1)$ and $(m + 1)w_1$.
2. For $j = 2, \dots, d$,
 - (a) if $w_j = w_{j-1}$, add an edge from $m + 1$ to $\ell_{L_{\mathbf{d}}+j-1}$,
 - (b) if $w_j \neq w_{j-1}$, remove the edge from w_j to its parent v , then add edges from v to $\ell_{L_{\mathbf{d}}+j-1}$ and from $m + 1$ to w_j .

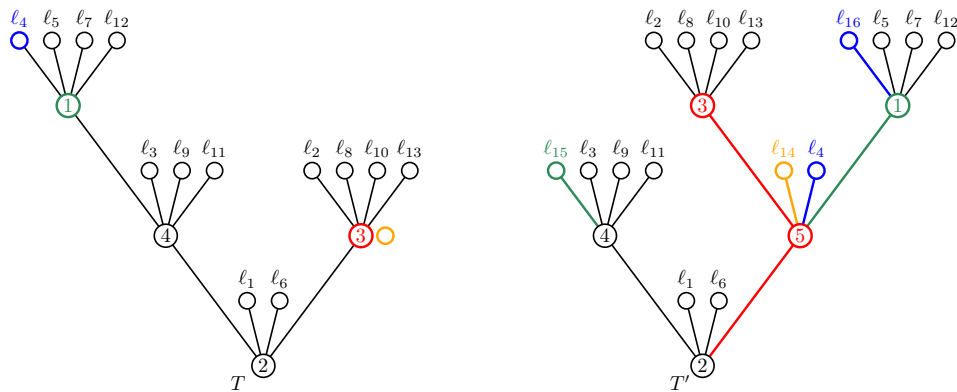


Figure 2: The growth procedure is used to add one internal vertex to the quaternary tree T . Here, $(w_1, w_2, w_3, w_4) = (3, 3, 1, \ell_4)$.

An example is provided in Figure 2. In the example, the coding sequence of T under the bijection presented above is $(2, 2, 3, 2, 4, 4, 1, 1, 2, 1, 3, 4, 3, 4, 1, 3)$. The total ordering \preceq of vertices of T is

$$2 \prec l_1 \prec 3 \prec l_2 \prec 4 \prec l_3 \prec 1 \prec l_4 \prec l_5 \prec l_6 \prec \dots \prec l_{13}.$$

The chosen multiset $(3, 3, 1, \ell_4)$ of vertices of T corresponds to the multiset $(3, 3, 7, 8)$, of

elements of $[17] = \{1, 2, \dots, 17\}$, and thence to transforming the coding sequence into

$$(2, 2, \mathbf{5}, \mathbf{5}, 3, 2, 4, 4, \mathbf{5}, 1, \mathbf{5}, 1, 2, 1, 3, 4, 3, 4, 1, 3).$$

The new sequence gives T' if we modify the bijection so that the new leaves $\ell_{14}, \ell_{15}, \ell_{16}$ are inserted when the repeats of 5 occur, and the other leaves are inserted according to their original ordering. In general, the growth procedure yields a modified bijection which reads as follows. With d' as above, fix $v' = (v'_1, \dots, v'_{m+L_{d'}}) \in \mathcal{S}_{d'}$, and note that $L_{d'} = L_d + d - 1$.

Modified bijection between $\mathcal{S}_{d'}$ and $\mathcal{T}_{d'}$.

- Let $j(0) = 1$, let $j(1) < j(2) < \dots < j(L_{d'} - 1)$ be the locations of the repeats of the sequence v' , and let $j(L_{d'}) = m + 1 + L_{d'}$.
- Reorder $\ell_1, \dots, \ell_{L_{d'}}$ as $\hat{\ell}_1, \dots, \hat{\ell}_{L_{d'}}$ as follows. For $1 \leq i \leq L_{d'} - 1$:
 - If $v'_{j(i)}$ is the $(k + 1)$ 'st appearance of $(m + 1)$ then let $\hat{\ell}_i = \ell_{L_{d'}+k}$.
 - If $v'_{j(i)}$ is the k 'th repeated entry which is not equal to $(m + 1)$ then let $\hat{\ell}_i = \ell_k$.

Let $\hat{\ell}_{L_{d'}} = \ell_{L_{d'}}$.

- For $i = 1, \dots, L_{d'}$, let P'_i be the path $(v_{j(i-1)}, \dots, v_{j(i)-1}, \hat{\ell}_i)$.
- Let $t'(v') \in \mathcal{T}_{d'}$ have root v'_1 and edge set given by the union of the edges of the paths $P'_1, P'_2, \dots, P'_{L_{d'}}$.

We now describe the inverse of the modified bijection, which takes as input a tree $t' \in \mathcal{T}_{d'}$ — i.e., a tree t' with vertex set $[m + 1] \cup \{\ell_1, \dots, \ell_{L_{d'}}\}$ such that for each $i \in [m + 1]$, i has d_i children, and $\ell_1, \dots, \ell_{L_{d'}}$ are leaves — and outputs a sequence $v' = (v'_1, \dots, v'_{m+L_{d'}}) \in \mathcal{S}_{d'}$. We leave it to the reader to verify that the two procedures are indeed inverses. This justifies the fact that the above growth procedure takes the uniform distribution on \mathcal{T}_d to the uniform distribution on $\mathcal{T}_{d'}$.

Inverse of the modified bijection

- Let $v'_{m+L_{d'}}$ be equal to the parent of $\ell_{L_{d'}}$ in t' . Say that $\ell_{L_{d'}}$ is used, and that $\ell_1, \dots, \ell_{L_{d'}-1}$ and $\ell_{L_{d'}+1}, \dots, \ell_{L_{d'}}$ are unused.
- For $j = m + L_{d'} - 1, \dots, 2, 1$:
 - If the number of occurrences of v'_{j+1} in $(v'_{j+1}, \dots, v'_{m+L_{d'}})$ is equal to the number of children of v'_{j+1} in t' , then let v'_j be equal to the parent of v'_{j+1} in t' .
 - Otherwise, define v'_j as follows.
 - * If $v'_{j+1} = m + 1$, let i^* be the maximum $L_{d'} + 1 \leq i \leq L_{d'}$ such that ℓ_i is unused.
 - * If $v'_{j+1} \neq m + 1$, let i^* be the maximum $1 \leq i \leq L_{d'} - 1$ such that ℓ_i is unused.

Let v'_j be equal to the parent of ℓ_{i^*} in t' . Say that ℓ_{i^*} is used.

We conclude the paper by justifying the assertion from Section 1 that the growth procedure yields an algorithm for sampling a sequence of random d -ary leaf-labelled trees. For this, let $\mathbf{d}^{(m)} = (d, d, \dots, d)$ have length m . Starting from the unique tree $T^{(1)} \in_u \mathcal{T}_{d(1)}$, for each $m \geq 1$ let $T^{(m+1)}$ be constructed from $T^{(m)}$ according to the above

procedure. Now let T_m be obtained from $T^{(m)}$ by unlabelling the non-leaf vertices; then T_m is a uniformly random leaf-labelled d -ary tree with m internal nodes, and $(T_m, m \geq 1)$ is the random sequence alluded to in the introduction. (In fact, given a multiset of size d of vertices of $T^{(m)}$, its ordering, as defined in the construction of $T^{(m+1)}$ from $T^{(m)}$, does not depend on the labels of the internal vertices. Therefore, it is straightforward to define the pushforward of the growth procedure directly on leaf-labelled trees.) Then, given T_{m+1} , the non-leaf vertex that was added to obtain T_{m+1} from T_m is a uniformly random non-leaf vertex of T_{m+1} . In the binary case, this distinguishes the procedure from Rémy’s algorithm, for which, given the shape of T_{m+1} , the non-leaf vertex that was added is instead the parent of a uniformly random leaf of T_{m+1} .

The growth procedure for planar d -ary trees described in [32] is most comparable to ours. Like our procedure, it is based on moving subtrees in the tree, but while we sample a uniform multiset of vertices, the author of [32] samples a uniform set of edges (possibly including ‘fake’ edges called *buds*) and an independently chosen uniform element in $[d]$ to define a step of the growth procedure. In the procedure in [32], the subtrees always get attached to a new vertex below the old root, while our attachment point is distributed as a uniformly random non-leaf vertex. The growth procedure in [32] is arguably more complex, both to describe and to verify, but a single growth step takes constant time, so the procedure yields a linear time algorithm for growing uniform d -ary trees. In our procedure, the part of a growth step that we do not see how to execute in constant time is determining the order under \preceq of the elements in the selected multiset.

In [31], the authors show the existence of a local growth procedure for planar d -ary trees, in which a random leaf is replaced by a new vertex with d children. However, the probabilities with which the leaves are chosen are implicitly defined, which makes the growth procedure more challenging to implement in practice.

Previous applications of the Foata–Fuchs bijection

This section provides a brief history of some of the previous applications of the Foata–Fuchs bijection that have appeared in the literature. Though it is not exhaustive, we believe it does include all references to the work of Foata and Fuchs [23] which in papers which either have a probabilistic focus or have a connection to the study of trees and forests.

- A number of papers [14, 25, 26, 26, 30, 39, 40, 44] mention the Foata–Fuchs bijection in connection with Abel–Cayley–Hurwitz identities and/or the *forest volume formula*,

$$\sum_{\{F: \text{roots}(F)=R\}} \prod_{v \in [n]} x_v^{|c_F(v)|} = \left(\sum_{r \in R} x_r \right) \left(\sum_{v \in [n]} x_v \right)^{n-|R|-1},$$

where the sum is over rooted forests F with vertex set $[n]$ and with a fixed set of roots $R \subset [n]$, and where $c_F(v)$ denotes the number of children of v in F . Most of these works do not use the Foata–Fuchs bijection directly, but rather build on a result of Françon [25], who showed how the Foata–Fuchs bijection can be used to prove Abel’s identities.

In [39, 40], Pitman discusses some probabilistic interpretations and applications of the forest volume formula, mostly in the context of coalescent processes, but also in connection with the model of random trees called p -trees, mentioned in Section 2.

- The paper [24] describes two bijections between acyclic mappings and parking functions; this paper does not use the Foata–Fuchs bijection, but cites it as providing a tree encoding different from the Prüfer code. (The study of *random parking*

functions, and parking functions on random trees, has seen a fairly recent surge of activity; see, e.g., [8, 17, 18, 20, 27, 36].)

- The paper [22] uses the Foata–Fuchs bijection to enumerate k -trees (labelled graphs that can be built starting from a clique of size k by repeatedly attaching new vertices such that at each step, the neighbourhood of the added vertex is a clique of size k).

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