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On SDEs with Lipschitz coefficients, driven by continuous, model-free martingales*

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Abstract

We prove the existence and uniqueness of solutions of SDEs with Lipschitz coefficients, driven by continuous, model-free martingales. The main tool in our reasoning is Picard's iterative procedure and a model-free version of the Burkholder-Davis-Gundy inequality for integrals driven by model-free, continuous martingales. We work with a new outer measure which assigns zero value exactly to those properties which are instantly blockable.

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1 Introduction

The main purpose of this paper is to prove the existence and uniqueness of solutions of differential equations driven by continuous, model-free martingales. Continuous, model-free martingales were introduced in a recent book by Glenn Schafer and Vladimir Vovk [9]. Roughly speaking, model-free martingales are processes representing evolution of values of a dynamic portfolio consisting of several financial assets; they are related to model-free price paths. *Typical*, model-free price paths represent evolution of prices of financial assets which do not allow to obtain infinite wealth during finite time by risking small amount and trading these assets. From pioneering works of Vovk [10], [11], [12], [13] it is well known that typical model-free price paths reveal many properties of local martingales. The case of continuous price paths is understood much better than the case of càdlàg paths.

However, even in the case of continuous, model-free price paths there are still many topics which need to be understood better. One of such topics is the existence and uniqueness of solutions of differential equations driven by such paths. The first results in this direction are proven in [1], even for Hilbert space-valued processes. In [1] the authors assume, similarly as we do here, that the coefficients of the differential equations are Lipschitz continuous, but they additionally assume some growth condition on the

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quadratic variation process of the coordinate process, see [1, Sect. 2, Remark 2.7]. Another related paper is [5], where existence and uniqueness result for one-dimensional differential equations, driven by typical paths, with non-Lipschitz continuous coefficients in the spirit of Yamada-Watanabe as well as an approximation result in the spirit of Doss-Sussmann were proven.

Our approach is different. First, the driving processes of our equations are more general processes – model-free, continuous martingales. Second, we work with the properties which hold with *instant enforcement*. Roughly speaking, they are such properties (subsets of $\Omega \times [0,+\infty)$) consisting of pairs of $\omega \in \Omega$ (which may be interpreted as an elementary event or outcome of reality) and $t \in [0,+\infty)$ (time) that a trader (skeptic) is able to become infinitely rich as soon as they cease to hold, see [9, Chapt. 14] and the next section. *Typical* paths are on the other hand the trajectories of the canonical process $S_t(\omega) = \omega(t)$ for such properties (subsets) of $\Omega \ni \omega$ that a trader (skeptic) is able to become infinitely rich when they do not hold, but it may take some time until a given time horizon $T \in (0,+\infty)$ that she/he becomes infinitely rich, see for example [6].

In this paper we consider the following differential equation (or rather integral equation) driven by model-free, continuous martingales $X^1, X^2, \ldots X^d \in \mathcal{M}$ (the family $\mathcal{M}$ is defined in subsection 2.1):

$$\mathbf{Y}_t(\omega) = \mathbf{Y}_0(\omega) + \int_0^t K(s, \mathbf{Y}(\omega), \omega)\, \mathrm{d}\mathbf{A}_s + \int_0^t F(s, \mathbf{Y}(\omega), \omega)\, \mathrm{d}\mathbf{X}_s(\omega), \tag{1.1}$$

where $\mathbf{A} : [0,+\infty) \times \Omega \to \mathbb{R}^d$ is a continuous, adapted, finite-variation process, $\mathbf{X}$ is a d-dimensional process with coordinates $X^1, X^2, \ldots X^d$, $\mathbf{X} = (X^1, X^2, \ldots X^d)$, and K, F : $[0,+\infty) \times \left([-\infty,+\infty]^d\right)^{[0,+\infty)} \times \Omega \to \mathbb{R}^{d\times d}$ (for formal reasons we allow $\mathbf{X}$ and $\mathbf{Y}$ to attain values from $[-\infty,+\infty]^d$) are non-anticipating, matrix-valued and Lipschitz in the sense that there exists $L \in (0,+\infty)$ such that for all $t \in [0,+\infty)$, $\mathbf{x}, \mathbf{y} : [0,+\infty) \to [-\infty,+\infty]^d$ and $\omega \in \Omega$

$$|K(t,\mathbf{x},\omega) - K(t,\mathbf{y},\omega)| + |F(t,\mathbf{x},\omega) - F(t,\mathbf{y},\omega)| \le L \sup_{s\in[0,t]} |\mathbf{x}(s) - \mathbf{y}(s)|, \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ with $n = d \times d$ on the left side of (1.2) and $n = d$ on the the right side of (1.2), for example: $|K(t,\mathbf{x},\omega) - K(t,\mathbf{y},\omega)| = \left(\sum_{i,j=1}^d \left(K^{i,j}(t,\mathbf{x},\omega) - K^{i,j}(t,\mathbf{y},\omega)\right)^2\right)^{1/2}$, with the convention that $+\infty - (+\infty) = 0$ and $-\infty - (-\infty) = 0$. The definition of non-anticipating functionals and formal statement of all assumptions is given in Sect. 3.

Equation (1.1) may be written as the system of integral equations: for $j = 1, 2, \ldots, d$,

$$Y_t^j(\omega) = Y_0^j(\omega) + \sum_{i=1}^d \int_0^t K^{i,j}(s, \mathbf{Y}(\omega), \omega)\, \mathrm{d}A_s^i + \sum_{i=1}^d \int_0^t F^{i,j}(s, \mathbf{Y}(\omega), \omega)\, \mathrm{d}X_s^i(\omega) \tag{1.3}$$

or, equivalently,

$$Y_t^j(\omega) = Y_0^j(\omega) + \int_0^t K^j(s, \mathbf{Y}(\omega), \omega)\, \mathrm{d}\mathbf{A}_s + \int_0^t F^j(s, \mathbf{Y}(\omega), \omega)\, \mathrm{d}\mathbf{X}_s(\omega),$$

where $\int_0^t K^j(s, \mathbf{Y}(\omega), \omega)\, \mathrm{d}\mathbf{A}_s = \sum_{i=1}^d \int_0^t K^{i,j}(s, \mathbf{Y}(\omega), \omega)\, \mathrm{d}A_s^i$, $K^j(s, \mathbf{Y}(\omega), \omega) = \left(K^{i,j}(s, \mathbf{Y}(\omega), \omega)\right)_{i=1,2,\ldots,d}$ and a similar notation is used for F. The integrals appearing in the first sum in (1.3) are understood as the standard Lebesgue-Stieltjes integrals, while integrals appearing in the second sum as model-free Itô integrals introduced in the next section.

Condition (1.2) is sufficient for our purpose. The same condition is used in [7, Chapt. IX, Sect. 2] but it differs from that used in [1].

This paper is organized as follows. In the next section we introduce necessary definitions, notations and tools (like the model-free BDG inequality). In the last section we apply these tools and Picard's iterative procedure (used in a similar way as in [2]) to prove the existence and uniqueness of the solution of (1.1).

2 Definitions, notation and auxiliary results

First we outline a general setting in which we will work and which follows closely [9, Chapt. 14] and [3]. $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of non-negative integers and $b, d \in \mathbb{N} \setminus \{0\}$. We will work with a martingale space which is a quintuple

$$\left(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, J = \{1, 2, \ldots, b\}, \{S^j, j \in J\}\right)$$

of the following objects: Ω is a space of possible outcomes of reality, $\mathcal{F}$ is a σ-field of the subsets of Ω which we call *events*, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration and $\{S^j, j \in J\} = \{S^1, S^2, \ldots S^b\}$ is a family of *basic continuous martingales*, that is for any $t \in [0, +\infty)$ and $j \in J$, S^j_t is a $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}))$-measurable *real variable* $S^j_t : \Omega \to \mathbb{R}$ such that for each $\omega \in \Omega$ the trajectory $[0, +\infty) \ni t \mapsto S^j_t(\omega)$ is continuous ($\mathcal{B}(\mathbb{R})$ denotes the σ-field of Borel subsets of $\mathbb{R}$).

The financial interpretation of the introduced objects is that J is a set of securities and S^j_t is the price of security j at time t, see also [9, Sect. 14.1]. In the case of one security ($J = \{1\}$), S^1 is often assumed to be the coordinate process $S^1_t(\omega) = \omega(t)$ defined on the space Ω of *all* continuous functions $\omega : [0, +\infty) \to \mathbb{R}$.

Throughout the paper the filtration $\mathbb{F}$ is fixed, moreover, we assume that $\mathcal{F}_0$ is trivial, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, thus all $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}))$-measurable variables S^j_0, $j \in J$, are deterministic. Moreover, we assume that the filtration $\mathbb{F}$ is such that all random times we define in this paper are indeed stopping times with respect to $\mathbb{F}$. This holds for example if we assume that for any $t \geq 0$ and any instantly blockable set $B \subseteq [0, +\infty) \times \Omega$ (blockable sets are defined later) the projection of $B \cap ([0, t] \times \Omega)$ onto Ω belongs to $\mathcal{F}_t$ (such assumption has a natural interpretation – that at the moment $t \geq 0$ we are able to say if there was any trading strategy making us infinitely rich (after investing a small positive amount at the moment 0); see also [3]).

2.1 Definitions and auxiliary results

A *real process* $X : [0, +\infty) \times \Omega \to \mathbb{R}$ is a collection of real variables $X_t : \Omega \to \mathbb{R}$, $t \in [0, +\infty)$, such that X_t is $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}))$-measurable, thus all processes which we consider are adapted to $\mathbb{F}$.

A d-*dimensional real process* $\mathbf{Y}$ is a d-tuple $\left(Y^1, Y^2, \ldots, Y^d\right)$ of real processes Y^1, Y^2, …, Y^d.

A *process* $Y : [0, +\infty) \times \Omega \to \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$, is a collection of *extended variables* $Y_t : \Omega \to [-\infty, +\infty]$, $t \in [0, +\infty)$, such that Y_t is $(\mathcal{F}_t, \mathcal{B}([-\infty, +\infty]))$-measurable (any set in $\mathcal{B}([-\infty, +\infty])$ is of the form A, $A \cup \{-\infty\}$, $A \cup \{+\infty\}$ or $A \cup \{-\infty, +\infty\}$, where $A \in \mathcal{B}(\mathbb{R})$).

A d-*dimensional process* $\mathbf{Y}$ is a d-tuple $\left(Y^1, Y^2, \ldots, Y^d\right)$ of processes $Y^1, Y^2, \ldots, Y^d$.

For *any* function $Y : [0, +\infty) \times \Omega \to [-\infty, +\infty]$ we define its supremum Y^* as

$$Y^*_t(\omega) := \sup_{0\leq s\leq t} |Y_t(\omega)|,$$

where we denote $Y_t(\omega) := Y(t, \omega)$. Y is *globally bounded* iff $|Y_t(\omega)| < +\infty$ for all $(t, \omega) \in [0, +\infty) \times \Omega$.

Throughout the whole paper we apply the following convention. A sequence of real numbers a_n, where $n = 0, 1, 2, \ldots$, is denoted by (a_n) or $(a_n)_n$ and a sequence of real numbers a^n, where $n = 0, 1, 2, \ldots$, is denoted by (a^n) or $(a^n)_n$ (without indication that n ranges over the set of nonnegative integers $\mathbb{N}$). A similar convention will be applied to infinite sequences of stopping times, variables etc.

A *simple trading strategy* is a triplet $G = (c, (\tau_n), (g_n))$ which consists of the initial capital $c \in \mathbb{R}$, a sequence of $\mathbb{F}$-stopping times (τ_n) and a sequence of $(\mathcal{F}_{\tau_n}, \mathcal{B}(\mathbb{R}))$-measurable real variables $g_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, such that $g_n(\omega) = 0$ whenever $\tau_n(\omega) = +\infty$. The stopping times are assumed to be non-decreasing (for all $n \in \mathbb{N}$ and *each* $\omega \in \Omega$, $\tau_{n+1}(\omega) \geq \tau_n(\omega)$) and such that $\tau_0 \equiv 0$, and for *each* $\omega \in \Omega$ the sequence $(\tau_n(\omega))$ is divergent to $+\infty$ or there exists some $n \in \mathbb{N}$ such that $\tau_n(\omega) = \tau_{n+1}(\omega) = \ldots \in [0, +\infty]$.

For a simple trading strategy $G = (c, (\tau_n), (g_n))$ and a real process $X : [0, +\infty) \times \Omega \to \mathbb{R}$ we define

$$(G \cdot X)_t(\omega) := c + \sum_{n=1}^{+\infty} g_{n-1}(\omega) \left(X_{\tau_n(\omega) \wedge t} - X_{\tau_{n-1}(\omega) \wedge t} \right). \tag{2.1}$$

(For $s, t \in [-\infty, +\infty]$ we define $s \wedge t = \min\{s, t\}$.) Let us note that by the assumptions about the sequence (τ_n), there is only finite number of non-zero summands in the sum $\sum_{n=1}^{+\infty} g_{n-1}(\omega) \left(X_{\tau_n(\omega) \wedge t} - X_{\tau_{n-1}(\omega) \wedge t} \right)$ appearing in the definition of $(G \cdot X)_t(\omega)$.

We define the *simple capital process* corresponding to the vector $\mathbf{G} = \left(G^j\right)_{j \in J}$ of simple trading strategies G^j, $j \in J$, as

$$(\mathbf{G} \cdot \mathbf{S})_t(\omega) := \sum_{j \in J} (G^j \cdot S^j)_t(\omega). \tag{2.2}$$

The simple capital process has a very natural interpretation – it is the capital accumulated till time t by the application of the simple trading strategy G^j to the asset whose price is equal to the basic martingale S^j, $j \in J$.

The class $\mathcal{C}$ of *nonnegative supermartingales* is defined as the smallest class with the following properties

- $\mathcal{C}$ contains all simple capital processes which are nonnegative;
- whenever $X \in \mathcal{C}$, Y is a simple capital process and $X + Y$ is nonnegative then $X + Y \in \mathcal{C}$;
- for any sequence (X^n) such that $X^n \in \mathcal{C}$ for $n \in \mathbb{N}$, we have that $X := \liminf_{n \to +\infty} X^n$ also belongs to $\mathcal{C}$.

The class $\mathcal{C}$ may also be described using transfinite induction on the countable ordinals α: $\mathcal{C} = \bigcup_\alpha \mathcal{C}^\alpha$, where

- $\mathcal{C}^0$ contains all simple capital processes which are nonnegative;
- for $\alpha > 0$, whenever $X \in \mathcal{C}^{<\alpha} := \bigcup_{\beta < \alpha} \mathcal{C}^\beta$, Y is a simple capital process and $X + Y$ is nonnegative then $X + Y \in \mathcal{C}^\alpha$;
- for $\alpha > 0$ and for any sequence (X^n) such that $X^n \in \mathcal{C}^{<\alpha}$ for $n \in \mathbb{N}$, we have that $X := \liminf_{n \to +\infty} X^n$ belongs to $\mathcal{C}^\alpha$.

A property $E \subseteq [0, +\infty) \times \Omega$ is *instantly enforceable*, or holds *with instant enforcement*, w.i.e. in short, if there exists a nonnegative supermartingale X such that $X_0 = 1$ and

$$(t, \omega) \notin E \Longrightarrow X_t(\omega) = +\infty.$$

Complements of instantly enforceable properties (sets) are called *instantly blockable*.

For $Y:[0,+\infty)\times\Omega\to[-\infty,+\infty]$ we define its *upper expectation* (or cost of super-hedging or super-replication) in the following way

$$\begin{aligned}\overline{\mathbb{E}}Y := \inf\{\lambda\in\mathbb{R} : \exists X\in\mathcal{C} \text{ such that } \forall(t,\omega)\in[0,+\infty)\times\Omega, X_0(\omega)\le\lambda \\ \text{and } X_t(\omega)\ge Y_t(\omega)\}\end{aligned}$$

and for $A\subseteq[0,+\infty)\times\Omega$ we define its *outer measure* as $\overline{\mathbb{P}}(A)=\overline{\mathbb{E}}\mathbf{1}_A$. We have the following result (see [3, Lemma 2.3]).

Proposition 2.1. *The set $B\subseteq[0,+\infty)\times\Omega$ is instantly blockable iff $\overline{\mathbb{P}}(B)=0$.*

Next to the class of nonnegative supermartingales, other important class of processes which we will work with is the family of *martingales*. The class of martingales $\mathcal{M}$ is defined as the smallest $\lim$-closed class of real (w.i.e.) processes such than it contains all simple capital processes. By the fact that $\mathcal{M}$ is $\lim$-closed we mean that whenever $X^n\in\mathcal{M}$, $n\in\mathbb{N}$, and X is a real (w.i.e.) process such that for any $(t,\omega)\in[0,+\infty)\times\Omega$,

$$\lim_{n\to+\infty}\sup_{s\in[0,t]}|X_s(\omega)-X_s^n(\omega)|=0 \quad w.i.e. \tag{2.3}$$

then also $X\in\mathcal{M}$. Condition (2.3) guarantees that the limit process X is real and continuous w.i.e., see [3, Fact 2.9].

For any $X\in\mathcal{M}$ there exists its *quadratic variation* process denoted by $[X]$ (see [3, Proposition 4.3]), which is non-decreasing, real and continuous w.i.e. (and one may take a version which is non-decreasing, real and continuous for all $\omega\in\Omega$), and for any $p\ge1$ the following BDG inequalities hold (see [3, Proposition 4.8]):

$$c_p\overline{\mathbb{E}}[X]^{p/2}\le\overline{\mathbb{E}}((X-X_0)^*)^p\le C_p\overline{\mathbb{E}}[X]^{p/2}.$$

In the case $p>1$ one may take $C_p=6^p(p-1)^{p-1}$ and $c_p=1/C_p$, while in the case $p=1$ one may take $C_p=6$ and $c_p=1/3$.

2.2 Stochastic integrals

Until now we have not defined integrals appearing in (1.3). Integrals with respect to model-free, typical paths as integrators were defined in several papers. The case of continuous, model-free typical paths as integrators was considered first in [6], see also [14]. In [14], integrals with more general, typical, model-free càdlàg integrators were also considered; see also [4]. Integrals with respect to model-free, continuous martingales were introduced in [8], [9], see also [3]. In [8], [9], [3], it was proven that the property that such integrals exist holds with instant enforcement.

Although the mentioned integrals may be useful, now we will define integrals suiting our needs better. Let $\mathbf{X}$ be a d-dimensional process with coordinates $X^1,X^2,\dots X^d$, ($\mathbf{X}=(X^1,X^2,\dots X^d)$), which are martingales. To do this, let us introduce the spaces $\mathcal{G}^0=\mathcal{G}^0_{\mathbf{X}}$ and $\mathcal{H}=\mathcal{H}_{\mathbf{X}}$ of (equivalence classes of) d-dimensional processes and processes respectively, equipped with the norms:

$$\|\mathbf{Y}\|^{\mathcal{G}}_{\infty,\mathbf{X},loc}:=\sum_{N=1}^{\infty}2^{-N}\overline{\mathbb{E}}|\mathbf{Y}|^*_{\cdot\wedge\sigma(\mathbf{X},N)},\ \|Z\|^{\mathcal{H}}_{\infty,\mathbf{X},loc}:=\sum_{N=1}^{\infty}2^{-N}N^{-2}\overline{\mathbb{E}}|Z|^*_{\cdot\wedge\sigma(\mathbf{X},N)}$$

where $\mathbf{Y}:[0,+\infty)\times\Omega\to[-\infty,+\infty]^d$, $|\mathbf{Y}|=\sqrt{\sum_{i=1}^d(Y^j)^2}$ and we define

$$\sigma(\mathbf{X},N):=\sigma\left([X^1],N\right)\wedge\sigma\left([X^2],N\right)\wedge\ldots\wedge\sigma\left(\left[X^d\right],N\right),$$

where for the martingale X,

$$\sigma\left([X],N\right)=\inf\{t\ge0:[X]_t\ge N\}.$$

To deal with the values $+\infty$ and $-\infty$ which may be attained by some components of (a representative of) some element $\mathbf{Y}$ of $\mathcal{G}^0$ we recall the convention that $+\infty-(+\infty)=0$ and $-\infty-(-\infty)=0$. Our definitions seemingly restrict the space of possible solutions $\mathbf{Y}$ of Eq. (1.1) (they can not grow to fast on the intervals $[0,\sigma(\mathbf{X},N)]$), but it is not the case thanks to the Definition 3.1 of a solution of (1.1).

Further, let $\mathcal{G}=\mathcal{G}_{\mathbf{X}}$ be a closure of the linear subspace of $\mathcal{G}^0$ spanned by càdlàg d-*dimensional step processes* of the form $\mathbf{G}=\left(G^i\right)_{i=1,2,\ldots,d}$, where

$$G_t^i(\omega):=\sum_{n=1}^{+\infty} g_{n-1}^i(\omega)\mathbf{1}_{\left[\tau_{n-1}^i(\omega),\tau_n^i(\omega)\right)}(t),$$

and $G^i=(0,(\tau_n^i),(g_n^i))$, $i=1,2,\ldots,d$, are simple trading strategies. For the càdlàg d-dimensional step process $\mathbf{G}$ we define the *simple integral process* as

$$\mathbf{G}\cdot\mathbf{X}=\sum_{i=1}^{d} G^i\cdot X^i,$$

where $G^i\cdot X^i$ are defined by (2.1). Instead of $G^i\cdot X^i$ we will also write $\int_0^{\cdot} G_s^i\,\mathrm{d}X_s^i$ and instead of $\left(G^i\cdot X^i\right)_t$ we will also write $\int_0^t G_s^i\,\mathrm{d}X_s^i$.

Remark 2.2. More appropriate notation (consistent with the Stieltjes integral) to denote simple capital processes defined in (2.2) and just defined simple integrals, would be $\mathbf{G}_-\cdot\mathbf{S}$ and $\mathbf{G}_-\cdot\mathbf{X}$ respectively. Similarly, more appropriate notation in (1.1) would be $\int_{(0,t]} K\left(s-,\mathbf{Y}\left(\omega\right),\omega\right)\,\mathrm{d}\mathbf{A}_s$ and $\int_{(0,t]} F\left(s-,\mathbf{Y}\left(\omega\right),\omega\right)\,\mathrm{d}\mathbf{X}_s\left(\omega\right)$, but we will not use it to be consistent with the notation used in [9] and [3].

In the sequel we will also use the fact that for a simple trading strategy G and a continuous martingale X, $[G\cdot X]=\int_0^{\cdot}\left(G_s\right)^2\,\mathrm{d}[X]_s$ w.i.e., see [3, Fact 5.1].

Proposition 2.3. *The spaces $\mathcal{G}$ and $\mathcal{H}$ are Banach spaces. Two processes $\mathbf{Y}^1$ and $\mathbf{Y}^2$ are representatives of the same classes in $\mathcal{G}$ iff $\mathbf{Y}^1=\mathbf{Y}^2$ w.i.e., which is equivalent with $\overline{\mathbb{E}}|\mathbf{Y}^1-\mathbf{Y}^2|=0$. A similar statement holds for processes in $\mathcal{H}$. For any $\mathbf{G}\in\mathcal{G}$ which is the limit of d-dimensional step processes $\mathbf{G}^n$ in $\mathcal{G}$, there exists the limit of $\mathbf{G}^n\cdot\mathbf{X}$ in $\mathcal{H}$ and we define $\mathbf{G}\cdot\mathbf{X}$ as this limit. Moreover, $\mathbf{G}\cdot\mathbf{X}$ has a representative in $\mathcal{H}$ which is a martingale, which implies that any representative of $\mathbf{G}\cdot\mathbf{X}$ in $\mathcal{H}$ is a martingale.*

Proof. The proof that $\|\cdot\|_{\infty,\mathbf{X},loc}^{\mathcal{G}}$ defines a norm and that two processes $\mathbf{Y}^1$ and $\mathbf{Y}^2$ are representatives of the same classes in $\mathcal{G}$ iff $\mathbf{Y}^1=\mathbf{Y}^2$ w.i.e., which is equivalent with $\overline{\mathbb{E}}|\mathbf{Y}^1-\mathbf{Y}^2|=0$, is omitted. To prove the completeness let (Y^n) be a Cauchy sequence with respect to the metric $d_{\infty,\mathbf{X},loc}^{\mathcal{G}}$ induced by the norm $\|\cdot\|_{\infty,\mathbf{X},loc}^{\mathcal{G}}$. Let (d_k) be any sequence of positive reals such that $\sum_{k=1}^{+\infty} d_k<+\infty$. There exists a subsequence $(\mathbf{Y}^{n_k})$ such that for $n\geq n_k$, $n,k=1,2,\ldots$ one has $d_{\infty,\mathbf{X},loc}^{\mathcal{G}}\left(\mathbf{Y}^n,\mathbf{Y}^{n_k}\right)\leq d_k$. Taking $\mathbf{Y}:=\liminf_{l\to+\infty}\mathbf{Y}^{n_l}$ ($\liminf$ is defined component-wise), for $n\geq n_k$ we get

$$d_{\infty,\mathbf{X},loc}^{\mathcal{G}}\left(\mathbf{Y}^n,\mathbf{Y}\right)\leq d_{\infty,\mathbf{X},loc}^{\mathcal{G}}\left(\mathbf{Y}^n,\mathbf{Y}^{n_k}\right)+\sum_{l=k}^{+\infty} d_{\infty,\mathbf{X},loc}^{\mathcal{G}}\left(\mathbf{Y}^{n_l},\mathbf{Y}^{n_{l+1}}\right)\leq d_k+\sum_{l=k}^{+\infty} d_l,$$

thus $\mathbf{Y}$ is the limit of the sequence $(\mathbf{Y}^n)$ (as a limit one can also take $\limsup_{l\to+\infty}\mathbf{Y}^{n_l}$). Similarly we prove the completeness of $\mathcal{H}$.

For two step processes $\mathbf{G}^m$ and $\mathbf{G}^n$, using the BDG inequality for $p=1$, we estimate

$$\begin{aligned}
&\overline{\mathbb{E}}|\mathbf{G}^m\cdot\mathbf{X}-\mathbf{G}^n\cdot\mathbf{X}|^*_{\cdot\wedge\sigma(\mathbf{X},N)}\le\sum_{i=1}^d\overline{\mathbb{E}}|G^{m,i}\cdot X^i-G^{n,i}\cdot X^i|^*_{\cdot\wedge\sigma(\mathbf{X},N)}\\
&\le C_1\sum_{i=1}^d\overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\sigma(\mathbf{X},N)}\left(G^{m,i}-G^{n,i}\right)^2_s\mathrm{d}[X^i]_s\right)^{1/2}\le C_1\sum_{i=1}^d\overline{\mathbb{E}}\left(\left(G^{m,i}-G^{n,i}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)}N^{1/2}\right)\\
&=C_1N^{1/2}\sum_{i=1}^d\overline{\mathbb{E}}\left(G^{m,i}-G^{n,i}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)}\le C_2N^{1/2}d\cdot\overline{\mathbb{E}}\left(\mathbf{G}^m-\mathbf{G}^n\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)}\\
&\le C_1N^{1/2}d2^N\|\mathbf{G}^m-\mathbf{G}^n\|^{\mathcal{G}}_{\infty,\mathbf{X},loc}.
\end{aligned}$$

From the last estimate it follows that $(\mathbf{G}^n\cdot\mathbf{X})$ is a Cauchy sequence in $\mathcal{H}$, since

$$\begin{aligned}
&\|\mathbf{G}^m\cdot\mathbf{X}-\mathbf{G}^n\cdot\mathbf{X}\|^{\mathcal{H}}_{\infty,\mathbf{X},loc}=\sum_{N=1}^\infty 2^{-N}N^{-2}\overline{\mathbb{E}}|\mathbf{G}^m\cdot\mathbf{X}-\mathbf{G}^n\cdot\mathbf{X}|^*_{\cdot\wedge\sigma(\mathbf{X},N)}\\
&\le\sum_{N=1}^\infty 2^{-N}N^{-2}C_1N^{1/2}d2^N\|\mathbf{G}^m-\mathbf{G}^n\|^{\mathcal{G}}_{\infty,\mathbf{X},loc}=\left(C_1d\sum_{N=1}^\infty N^{-3/2}\right)\|\mathbf{G}^m-\mathbf{G}^n\|^{\mathcal{G}}_{\infty,\mathbf{X},loc}.
\end{aligned}$$

$\mathbf{G}\cdot\mathbf{X}$ is a limit in $\mathcal{H}$ of $\mathbf{G}^n\cdot\mathbf{X}$, which are martingales. To prove that it has a representative in $\mathcal{H}$ which is a martingale let $(n_k)_k$ be any subsequence of the sequence of all natural numbers such that $M:=\sum_{k=1}^{+\infty}\|\mathbf{G}\cdot\mathbf{X}-\mathbf{G}^{n_k}\cdot\mathbf{X}\|^{\mathcal{H}}_{\infty,\mathbf{X},loc}<+\infty$ and let $B\subseteq[0,+\infty)\times\Omega$ be the set of (t,ω) where $\left(\mathbf{G}\cdot\mathbf{X}-\mathbf{G}^{n_k}\cdot\mathbf{X}\right)^*_t(\omega)\nrightarrow 0$. Let $(t,\omega)\in B$ and $N\in\mathbb{N}$ be such that $\sigma(\mathbf{X},N)(\omega)\ge t$. We have

$$\sum_{k=1}^{+\infty}\left(\mathbf{G}\cdot\mathbf{X}-\mathbf{G}^{n_k}\cdot\mathbf{X}\right)^*_{\sigma(\mathbf{X},N)(\omega)}(\omega)\ge\sum_{k=1}^{+\infty}\left(\mathbf{G}\cdot\mathbf{X}-\mathbf{G}^{n_k}\cdot\mathbf{X}\right)^*_t(\omega)=+\infty.$$

As a result, for any $\varepsilon>0$

$$\varepsilon\sum_{k=1}^{+\infty}\sum_{N=1}^{+\infty}2^{-N}N^{-2}\left(\mathbf{G}\cdot\mathbf{X}-\mathbf{G}^{n_k}\cdot\mathbf{X}\right)^*_{\sigma(\mathbf{X},N)(\omega)}(\omega)=+\infty.$$

On the other hand, since

$$\begin{aligned}
&\overline{\mathbb{E}}\sum_{k=1}^{+\infty}\sum_{N=1}^{+\infty}2^{-N}N^{-2}\left(\mathbf{G}\cdot\mathbf{X}-\mathbf{G}^{n_k}\cdot\mathbf{X}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)}\\
&\le\sum_{k=1}^{+\infty}\overline{\mathbb{E}}\sum_{N=1}^{+\infty}2^{-N}N^{-2}\left(\mathbf{G}\cdot\mathbf{X}-\mathbf{G}^{n_k}\cdot\mathbf{X}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)}\\
&=\sum_{k=1}^{+\infty}\|\mathbf{G}\cdot\mathbf{X}-\mathbf{G}^{n_k}\cdot\mathbf{X}\|^{\mathcal{H}}_{\infty,\mathbf{X},loc}=M<+\infty,
\end{aligned}$$

we know that there exist a non-negative supermartingale which starts from a capital no greater than εM and attains value $+\infty$ on B. Since ε is arbitrary positive real, B is instantly blockable, which implies that $\mathbf{G}\cdot\mathbf{X}$ is a martingale. □

3 Theorem on existence and uniqueness of the solutions of SDEs with Lipschitz coefficients, driven by continuous, model-free martingales

In this section we prove the existence and uniqueness of the solution of SDE (1.1). We will assume the following:

1. $A^j = A^{j,u} - A^{j,v}$, $j = 1, 2, \ldots, d$, and $|A|^j = A^{j,u} + A^{j,v}$, $|A| = \left(\sum_{j=1}^d \left(|A|^j\right)^2\right)^{1/2}$, where $A^{j,u}, A^{j,v} : [0, +\infty) \times \Omega \to \mathbb{R}$ are continuous, non-decreasing, adapted processes, starting from 0;

2. $K, F : [0, +\infty) \times \left([-\infty, +\infty]^d\right)^{[0,+\infty)} \times \Omega \to \mathbb{R}^{d\times d}$, and K and F are *non-anticipating*, by which we mean that

   (a) for any $t \in [0, +\infty)$, $\omega \in \Omega$ and two functions $\mathbf{x}, \mathbf{y} : [0, +\infty) \to [-\infty, +\infty]^d$, $K(t, \mathbf{x}, \omega) = K(t, \mathbf{y}, \omega)$ and $F(t, \mathbf{x}, \omega) = F(t, \mathbf{y}, \omega)$ whenever $\mathbf{x}(s) = \mathbf{y}(s)$ for all $s \in [0, t]$;

   (b) for any adapted càdlàg process $\mathbf{Y} : [0, +\infty) \times \Omega \to [-\infty, +\infty]^d$ the processes $K_t(\omega) := K(t, \mathbf{Y}(\omega), \omega)$, $F_t(\omega) := F(t, \mathbf{Y}(\omega), \omega)$ are adapted and càdlàg;

3. K and F satisfy condition (1.2).

For a process $\mathbf{Y} : [0, +\infty) \times \Omega \to \mathbb{R}^d$ instead of $K(s, \mathbf{Y}(\omega), \omega)$ we will often write $K(s, \mathbf{Y})$ and instead of $F(s, \mathbf{Y}(\omega), \omega)$ we will often write $F(s, \mathbf{Y})$.

Now let us define what we will mean by the solution of (1.1).

Definition 3.1. *Let $\mathbf{X}$ be a d-dimensional process with coordinates $X^1, X^2, \ldots X^d$, ($\mathbf{X} = (X^1, X^2, \ldots X^d)$), which are martingales. A solution of* (1.1) *is any d-dimensional process $\mathbf{Y}$ such that there exist a sequence of non-decreasing $\mathbb{F}$-stopping times (τ_n), which tend to $+\infty$ for all $\omega \in \Omega$ and such that $\mathbf{Y}_{\cdot\wedge\tau_n}$ ($\mathbf{Y}$ stopped at the time τ_n) is a representative of some element of $\mathcal{G}$ and the following equalities for $j = 1, 2, \ldots, d$, $n \in \mathbb{N}$ and any $t \in [0, +\infty)$ hold:*

$$Y^j_{t\wedge\tau_n} - Y^j_0 - \int_0^t K^j(s, \mathbf{Y}_{t\wedge\tau_n}) \mathbf{1}_{[0,\tau_n)}(s) \, \mathrm{d}\mathbf{A}_s = \int_0^t F^j(s, \mathbf{Y}_{t\wedge\tau_n}) \mathbf{1}_{[0,\tau_n)}(s) \, \mathrm{d}\mathbf{X}_s, \tag{3.1}$$

where the integral on the left hand side of (3.1) *is understood as the usual Lebesque-Stieltjes integral while the integral on the right hand side of* (3.1) *as the integral defined in Proposition 2.3 (by the Lipschitz assumption* (1.2)*, the d-dimensional processes $F^j(\cdot, \mathbf{Y}_{\cdot\wedge\tau_n})$, $j = 1, 2, \ldots, d$, belong to (are representatives of some elements of) $\mathcal{G}$ and the integrals $\int_0^t F^j(s, \mathbf{Y}_{t\wedge\tau_n}) \mathbf{1}_{[0,\tau_n)}(s) \, \mathrm{d}\mathbf{X}_s$ are well defined). The equality in* (3.1) *is understood as the fact that the process $Y^j_{t\wedge\tau_n} - Y^j_0 - \int_0^t K^j(s, \mathbf{Y}_{t\wedge\tau_n}) \mathbf{1}_{[0,\tau_n)}(s) \, \mathrm{d}\mathbf{A}_s$ is a representative of the same equivalence class in $\mathcal{H}$ which is on the right hand side of* (3.1).

We will use a model-free version of the BDG inequality for $p = 1$ and Picard's iterative procedure (used in a similar way as in [2]) to prove the following theorem.

Theorem 3.2. *Under the assumptions 1.-3. stated above, integral equation (1.1) has a solution in the sense of Definition 3.1 and this solution is unique in the sense that for any two solutions $\mathbf{G}$ and $\mathbf{H}$ we have $\overline{\mathbb{E}}(\mathbf{G} - \mathbf{H})^* = 0$, or, eqiuvalently, $\mathbf{G} = \mathbf{H}$ w.i.e.*

Remark 3.3. Theorem 3.2 implies the existence of a solution of (1.1) in the sense of Definition 3.1. Naturally, for many equations, like for example the one-dimensional Black-Scholes equation $Y_t = y_0 + \int_0^t Y_s \, \mathrm{d}A_s + \sigma \int_0^t Y_s \, \mathrm{d}X_s$ (x_0, σ - deterministic) we can write the solution explicitly $Y_t = y_0 \exp\left(A_t - \frac{1}{2}\sigma^2 [X]_t + \sigma(X_t - X_0)\right)$ and verify that it satisfies the Black-Scholes equation using the (model-free) Itô formula (see [13]). However, for more general equations we often have no explicit solutions and the existence of a solution is not obvious.

3.1 Proof of Theorem 3.2

3.1.1 Existence

Let us define $q = 1/(3L)$, $r = 1/(3C_1 d^2 L)$,

$$\vartheta_0 := \inf\{t \geq 0 : |A|_t \geq q\},\ \sigma_0 := \inf\left\{t \geq 0 : \max_{j=1,2,\ldots,d} [X^j]_t \geq r\right\},$$

$$\theta_0 = \vartheta_0 \wedge \sigma_0$$

and for any $\mathbf{G}$ in $\mathcal{G}$ define

$$\begin{aligned}\left(T^0\mathbf{G}\right)_t &= \mathbf{Y}_0 + \int_0^{t\wedge\theta_0} K\left(s, \mathbf{G}\right) \mathrm{d}\mathbf{A}_s + \int_0^{t\wedge\theta_0} F\left(s, \mathbf{G}\right) \mathrm{d}\mathbf{X}_s \\ &= \mathbf{Y}_0 + \int_0^t K\left(s, \mathbf{G}\right) \mathbf{1}_{[0,\theta_0)}(s)\, \mathrm{d}\mathbf{A}_s + \int_0^t F\left(s, \mathbf{G}\right) 1_{[0,\theta_0)}(s)\, \mathrm{d}\mathbf{X}_s.\end{aligned}$$

Let us fix $N \in \mathbb{N}$. By the inequality

$$\begin{aligned}&\left(\int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} F\left(s, \mathbf{G}\right) \mathrm{d}\mathbf{X}_s\right)^* \\ &\leq \sum_{i=1}^d \left(\sum_{j=1}^d \int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} \left\{F^{i,j}\left(s, \mathbf{G}\right) - F^{i,j}\left(s, \mathbf{H}\right)\right\} \mathrm{d}X_s^j\right)^* \\ &\leq \sum_{i,j=1}^d \left(\int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} \left\{F^{i,j}\left(s, \mathbf{G}\right) - F^{i,j}\left(s, \mathbf{H}\right)\right\} \mathrm{d}X_s^j\right)^*\end{aligned}$$

(which follows from the estimate $\sqrt{\sum_{i=1}^d a_i^2} \leq \sum_{i=1}^d |a_i|$), the subadditivity of $\overline{\mathbb{E}}$, the Lipschitz property and the BDG inequality, for any $\mathbf{G}, \mathbf{H} \in \mathcal{G}$ we estimate

$$\begin{aligned}&\overline{\mathbb{E}}\left(T^0\mathbf{G} - T^0\mathbf{H}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)} \\ &\leq \overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} \left|K\left(s, \mathbf{G}\right) - K\left(s, \mathbf{H}\right)\right| \mathrm{d}|A|_s\right)^* \\ &\quad + \sum_{i,j=1}^d \overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} \left\{F^{i,j}\left(s, \mathbf{G}\right) - F^{i,j}\left(s, \mathbf{H}\right)\right\} \mathrm{d}X_s^j\right)^* \\ &\leq \overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} L\left(\mathbf{G} - \mathbf{H}\right)_s^* \mathrm{d}|A|_s\right)^* \\ &\quad + C_1 \sum_{i,j=1}^d \overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} L^2\left(\left(\mathbf{G} - \mathbf{H}\right)_s^*\right)^2 \mathrm{d}[X^j]_s\right)^{1/2} \\ &\leq \overline{\mathbb{E}}\left(L\left(\mathbf{G} - \mathbf{H}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)} |A|_{\cdot\wedge\theta_0}\right) \\ &\quad + C_1 \sum_{i,j=1}^d \overline{\mathbb{E}}\left(L\left(\mathbf{G} - \mathbf{H}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)} \left([X^j]_{\cdot\wedge\theta_0}\right)^{1/2}\right) \\ &\leq \overline{\mathbb{E}}\left(L\left(\mathbf{G} - \mathbf{H}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)} \frac{1}{3L}\right) + C_1 \sum_{i,j=1}^d \overline{\mathbb{E}}\left(L\left(\mathbf{G} - \mathbf{H}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)} \frac{1}{3C_1 d^2 L}\right) \\ &= \frac{2}{3}\overline{\mathbb{E}}\left(\mathbf{G} - \mathbf{H}\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)}. \end{aligned} \tag{3.2}$$

We have also used the fact that for $\mathbf{G} \in \mathcal{G}$, $i, j = 1, 2, \ldots, d$, $G^i \cdot X^j$ is a martingale (which follows from Proposition 2.3) and that $\left[G^i \cdot X^j\right] = \int_0^\cdot \left(G^i_s\right)^2 \mathrm{d}[X^j]_s$ w.i.e., which follows from [3, Fact 5.1] by simple approximation arguments (see also [3, Fact 5.4]).

From (3.2) we get

$$\|T^0\mathbf{G} - T^0\mathbf{H}\|^{\mathcal{G}}_{\infty,\mathbf{X},loc} \le \frac{2}{3}\|\mathbf{G} - \mathbf{H}\|^{\mathcal{G}}_{\infty,\mathbf{X},loc}. \tag{3.3}$$

By (3.3), $\|T^0\mathbf{G}\|^{\mathcal{G}}_{\infty,\mathbf{X},loc} \le \frac{2}{3}\|\mathbf{G}\|^{\mathcal{G}}_{\infty,\mathbf{X},loc} < +\infty$. We will now show that $T^0\mathbf{G}$ is a limit of step processes in $\mathcal{G}$ from which it will follow that $T^0\mathbf{G} \in \mathcal{G}$ ($T^0\mathbf{G}$ is a representative of an element of $\mathcal{G}$). First, notice that $\mathbf{G} \in \mathcal{G}$ thus it is a limit (in $\mathcal{G}$) of a sequence of d-dimensional step processes $\mathbf{G}^n \in \mathcal{G}$, $n \in \mathbb{N}$. By (3.3),

$$\|T^0\mathbf{G}^n\|^{\mathcal{G}}_{\infty,\mathbf{X},loc} \le \frac{2}{3}\|\mathbf{G}^n\|^{\mathcal{G}}_{\infty,\mathbf{X},loc} \le \|\mathbf{G}\|^{\mathcal{G}}_{\infty,\mathbf{X},loc}$$

for sufficiently large n. For each step (thus càdlàg) process $\mathbf{G}^n$, $K\left(s, \mathbf{G}^n\right)$ and $F\left(s, \mathbf{G}^n\right)$ are again adapted càdlàg processes (assumption 2(b)) which may be uniformly approximated by step processes K^n and F^n with given accuracy $\varepsilon > 0$, respectively. For example if we define $\tau^{n,\varepsilon}_0 := 0$, $f^{n,\varepsilon}_0 := 0$ and for $m \in \mathbb{N} \setminus \{0\}$

$$\tau^{n,\varepsilon}_m := \inf\left\{t > \tau^{n,\varepsilon}_{m-1} : |F^n_t - f^{n,\varepsilon}_{m-1}| \ge \varepsilon\right\}, \quad f^{n,\varepsilon}_m = F^n_{\tau^{n,\varepsilon}_m},$$

then $F^{n,\varepsilon}_t := \sum_{m=1}^{+\infty} f^{n,\varepsilon}_{m-1}\mathbf{1}_{\left[\tau^{n,\varepsilon}_{m-1}, \tau^{n,\varepsilon}_m\right)}(t)$ approximates F^n uniformly on $[0, +\infty)$ with accuracy ε, that is $\sup_{t\in[0,+\infty)} |F^{n,\varepsilon}_t - F^n_t| \le \varepsilon$. The integrals $\int_0^{\cdot\wedge\theta_0} K^n \,\mathrm{d}\mathbf{A}_s$ and $\int_0^{\cdot\wedge\theta_0} F^n \,\mathrm{d}\mathbf{X}_s$ are continuous. We also have the estimate

$$\begin{aligned}
&\overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\theta_0} K\left(s, \mathbf{G}^n\right) \mathrm{d}\mathbf{A}_s - \int_0^{\cdot\wedge\theta_0} K^n \,\mathrm{d}\mathbf{A}_s\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)} \\
&\le \overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} \varepsilon \ \mathrm{d}|A|_s\right)^* \le \varepsilon q.
\end{aligned}$$

and by the BDG inequality we estimate

$$\begin{aligned}
&\overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\theta_0} F\left(s, \mathbf{G}^n\right) \mathrm{d}\mathbf{X}_s - \int_0^{\cdot\wedge\theta_0} F^n \,\mathrm{d}\mathbf{X}_s\right)^*_{\cdot\wedge\sigma(\mathbf{X},N)} \\
&\le C_1 \sum_{i,j=1}^{d} \overline{\mathbb{E}}\left(\int_0^{\cdot\wedge\theta_0\wedge\sigma(\mathbf{X},N)} \varepsilon^2 \,\mathrm{d}[X^j]_s\right)^{1/2} \le C_1 d^2 \varepsilon r.
\end{aligned}$$

From last two inequalities we infer that $\|T^0\mathbf{G}^n - \mathbf{Y}_0 - \int_0^{\cdot\wedge\theta_0} K^n \,\mathrm{d}\mathbf{A}_s - \int_0^{\cdot\wedge\theta_0} F^n \,\mathrm{d}\mathbf{X}_s\|^{\mathcal{G}}_{\infty,\mathbf{X},loc}$ may be as small as we please, thus $T^0\mathbf{G}^n$ may be approximated with arbitrary accuracy by continuous processes in $\mathcal{G}$, thus the same holds for $T^0\mathbf{G}$, thus $T^0\mathbf{G} \in \mathcal{G}$.

Now we know that T^0 may be viewed as a mapping $T^0 : \mathcal{G} \to \mathcal{G}$, which by (3.3) is a contraction. This contraction has a unique fixed point $\mathbf{Y}^0$ which for any $t \in [0, +\infty)$ satisfies

$$\mathbf{Y}^0_{t\wedge\theta_0} = \mathbf{Y}_0 + \int_0^{t\wedge\theta_0} K\left(s, \mathbf{Y}^0\right) \mathrm{d}\mathbf{A}_s + \int_0^{t\wedge\theta_0} F\left(s, \mathbf{Y}^0\right) \mathrm{d}\mathbf{X}_s,$$

Next, on the set $\{\omega \in \Omega : \theta_0(\omega) < +\infty\}$ we define

$$\vartheta_1 := \inf\{t \ge 0 : |A|_t - |A|_{\theta_0} \ge q\}, \ \sigma_1 := \inf\left\{t \ge 0 : \max_{j=1,2,\ldots,d}\left([X^j]_t - [X^j]_{\theta_0}\right) \ge r\right\},$$

otherwise we define $\theta_1 = +\infty$. Next we set

$$\theta_1 := \vartheta_1 \wedge \sigma_1,$$

and introduce the following operator T^1:,

$$\left(T^1\mathbf{G}\right)_t := \mathbf{Y}^0_{t\wedge\theta_0} + \int_{t\wedge\theta_0}^{t\wedge\theta_1} K\left(s,\mathbf{G}\right)\,\mathrm{d}\mathbf{A}_s + \int_{t\wedge\theta_0}^{t\wedge\theta_1} F\left(s,\mathbf{G}\right)\,\mathrm{d}\mathbf{X}_s.$$

Similarly as before, we prove that $T^1 : \mathcal{G} \to \mathcal{G}$, T^1 is a contraction and has a fixed point $\mathbf{Y}^1 \in \mathcal{G}$. Moreover, $\mathbf{Y}^0$ and $\mathbf{Y}^1$ agree on the interval $[0,\theta_0] \setminus \{+\infty\}$ and thus $\mathbf{Y}^1$ for any $t \in [0,+\infty)$ satisfies

$$\mathbf{Y}^1_{t\wedge\theta_1} = \mathbf{Y}_0 + \int_0^{t\wedge\theta_1} K\left(s,\mathbf{Y}^1\right)\,\mathrm{d}\mathbf{A}_s + \int_0^{t\wedge\theta_1} F\left(s,\mathbf{Y}^1\right)\,\mathrm{d}\mathbf{X}_s.$$

Similarly, having defined θ_n, $T^n : \mathcal{G} \to \mathcal{G}$, and its fixed point $\mathbf{Y}^n$, $n = 0,1,\ldots$ we define the stopping time θ_{n+1} and introduce the operator $T^{n+1} : \mathcal{G} \to \mathcal{G}$,

$$\left(T^{n+1}\mathbf{G}\right)_t := \mathbf{Y}^n_{t\wedge\theta_n} + \int_{t\wedge\theta_n}^{t\wedge\theta_{n+1}} K\left(s,\mathbf{G}\right)\,\mathrm{d}\mathbf{A}_s + \int_{t\wedge\theta_n}^{t\wedge\theta_{n+1}} F\left(s,\mathbf{G}\right)\,\mathrm{d}\mathbf{X}_s$$

and its fixed point $\mathbf{Y}^{n+1}$, which agrees with $\mathbf{Y}^n$ on the interval $[0,\theta_n] \setminus \{+\infty\}$.

Finally, setting

$$\mathbf{Y} := \lim_{n\to+\infty} \mathbf{Y}^n$$

we get that for any $t \in [0,+\infty)$ and $n \in \mathbb{N}$, $\mathbf{Y}$ satisfies

$$\mathbf{Y}_{t\wedge\theta_n} = \mathbf{Y}_0 + \int_0^{t\wedge\theta_n} K\left(s,\mathbf{Y}\right)\,\mathrm{d}\mathbf{A}_s + \int_0^{t\wedge\theta_n} F\left(s,\mathbf{Y}\right)\,\mathrm{d}\mathbf{X}_s.$$

Now we will prove that $\lim_{n\to+\infty}\theta_n(\omega) = +\infty$ for all $\omega \in \Omega$. Let us notice that for any $T > 0$ from the inequality $\theta_n(\omega) \le T$, $n \in \mathbb{N}$, it follows that $|A|_T(\omega) + [X]_T(\omega) \ge \min(q,r)\cdot n$. Since $|A|$ and $[X]$ are continuous for all $\omega \in \Omega$ (we choose such version of $[X]$), thus $|A|_T(\omega)$ and $[X]_T(\omega)$ are finite for all $\omega \in \Omega$ and

$$\begin{aligned}\left\{\omega \in \Omega : \lim_{n\to+\infty}\theta_n(\omega) < +\infty\right\} &= \bigcup_{N=1}^{+\infty}\left\{\omega \in \Omega : \lim_{n\to+\infty}\theta_n(\omega) \le N\right\}\\ &\subseteq \bigcup_{N=1}^{+\infty}\left(\bigcap_{n=1}^{+\infty}\left\{\omega \in \Omega : |A|_N(\omega) + [X]_N(\omega) \ge \min(q,r)n\right\}\right) = \emptyset.\end{aligned}$$

3.1.2 Uniqueness

In general, we can not guarantee that $\mathbf{Y} \in \mathcal{G}$, because we do not control the growth of the process $\mathbf{A}$. However, we have just proved that it is a solution in the sense of Definition 3.1. Now we will prove that any two such solutions must be equal w.i.e.

Let $\mathbf{G}$ and $\mathbf{H}$ be two solutions of (1.1) satisfying, together with sequences of stopping times (γ_n) and (η_n) respectively, conditions of Definition 3.1. Let us define $\tilde{\theta}_0 = \theta_0 \wedge \gamma_0 \wedge \eta_0$ and

$$\left(\tilde{T}^0\mathbf{G}\right)_t = \mathbf{Y}_0 + \int_0^{t\wedge\tilde{\theta}_0} K\left(s,\mathbf{G}\right)\,\mathrm{d}\mathbf{A}_s + \int_0^{t\wedge\tilde{\theta}_0} F\left(s,\mathbf{G}\right)\,\mathrm{d}\mathbf{X}_s.$$

Similarly as in (3.2) we prove that

$$\overline{\mathbb{E}}\left(\tilde{T}^0\mathbf{G}-\tilde{T}^0\mathbf{H}\right)^*_{\cdot\wedge\tilde{\theta}_0}\le\frac{2}{3}\overline{\mathbb{E}}\left(\mathbf{G}-\mathbf{H}\right)^*_{\cdot\wedge\tilde{\theta}_0}. \tag{3.4}$$

On the other hand, since $\mathbf{G}$ and $\mathbf{H}$ are solutions of (1.1), $\overline{\mathbb{E}}\left(\tilde{T}^0\mathbf{G}_{\cdot\wedge\tilde{\theta}_0}-\mathbf{G}_{\cdot\wedge\tilde{\theta}_0}\right)^*=0$, $\overline{\mathbb{E}}\left(\tilde{T}^0\mathbf{H}_{\cdot\wedge\tilde{\theta}_0}-\mathbf{H}_{\cdot\wedge\tilde{\theta}_0}\right)^*=0$. From this and (3.4) we get $\overline{\mathbb{E}}\left(\mathbf{G}_{\cdot\wedge\tilde{\theta}_0}-\mathbf{H}_{\cdot\wedge\tilde{\theta}_0}\right)^*=0$.

Next, defining $\tilde{\theta}_1=\theta_1\wedge\gamma_1\wedge\eta_1$,

$$\left(\tilde{T}^1\mathbf{G}\right)_t=\mathbf{G}_{t\wedge\tilde{\theta}_0}+\int_{t\wedge\tilde{\theta}_0}^{t\wedge\tilde{\theta}_1}K\left(s,\mathbf{G}\right)\,\mathrm{d}\mathbf{A}_s+\int_{t\wedge\tilde{\theta}_0}^{t\wedge\tilde{\theta}_1}F\left(s,\mathbf{G}\right)\,\mathrm{d}\mathbf{X}_s.$$

and reasoning similarly as before we get $\overline{\mathbb{E}}\left(\mathbf{G}_{\cdot\wedge\tilde{\theta}_1}-\mathbf{H}_{\cdot\wedge\tilde{\theta}_1}\right)^*=0$.

Similarly, for $\tilde{\theta}_n=\theta_n\wedge\gamma_n\wedge\eta_n$ we get $\overline{\mathbb{E}}\left(\mathbf{G}_{\cdot\wedge\tilde{\theta}_n}-\mathbf{H}_{\cdot\wedge\tilde{\theta}_n}\right)^*=0$. Since $\tilde{\theta}_n(\omega)\to+\infty$ as $n\to+\infty$ for all $\omega\in\Omega$, we have

$$\overline{\mathbb{E}}\left(\mathbf{G}-\mathbf{H}\right)^*\le\sum_{n=1}^{+\infty}\overline{\mathbb{E}}\left(\mathbf{G}_{\cdot\wedge\tilde{\theta}_n}-\mathbf{H}_{\cdot\wedge\tilde{\theta}_n}\right)^*=0.$$

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