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Corrigendum to: The contact process on periodic trees*

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Abstract

In [1] we considered periodic trees in which the number of children in successive generations is $(n, a_1, \dots, a_k)$ with $\max_i a_i \le Cn^{1-\delta}$ and $(\log a_i)/\log n \to b_i$ as $n \to \infty$. Our proof contained an error. In this note we correct the proof. The theorem has changed: the critical value for local survival is asymptotically $\sqrt{\bar{c}_k(\log n)/n}$ where $l_k = \max\{i : 0 \le i \le k, a_i \ne 1\}$ and $\bar{c}_k = \min\{k+1-l_k-b_{l_k}, (k-b)/2\}$, where $b = \lim_{n\to\infty} \log(a_1 a_2 \cdots a_k)/\log n$.

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It has been pointed out to us that in our paper [1] published in Electronic Communications in Probability volume 23, paper no. 24 the proof of Lemma 3.1 is not correct. The problem is that on the periodic $(n, a_1, \dots, a_k)$ rooted at ρ, the k-neighborhood of a site with degree n on level $k+1$ is not isomorphic to the k-neighborhood of the root. For instance, when $a_k > 1$, the k-neighborhood of a site on level $k+1$ can contain other sites with degree n. Not only is the proof not correct but the result changes from the previous paper.

Old Theorem 1.2. *Consider the $(n, a_1, a_2, \dots, a_k)$ periodic tree where k is a fixed integer, and $\max_i a_i \le Cn^{1-\delta}$ for some $C, \delta > 0$. Suppose $b = \lim_{n\to\infty} \log(a_1 a_2 \cdots a_k)/\log n$. As $n \to \infty$ the critical value*

$$\lambda_2 \sim \sqrt{c_k \log n/n}$$

where $c_k = (k-b)/2$.

New Theorem 1.2. *Consider the $(n, a_1, a_2, \dots, a_k)$ periodic tree where k is a fixed integer, Let $b_i = \lim_{n\to\infty}(\log a_i)/(\log n)$ and $b = \sum_{i=1}^k b_i$. Suppose $\max_i b_i \le 1-\delta$ for some $\delta > 0$. Let $a_0 = n$ and define $l_k = \max\{i : 0 \le i \le k, a_i \ne 1\}$. As $n \to \infty$ the critical value*

$$\lambda_2 \sim \sqrt{\bar{c}_k \log n/n}$$

where $\bar{c}_k = \min\{k+1-l_k-b_{l_k}, (k-b)/2\}$.

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Old Theorem 1.3. *Under the assumptions of Theorem 1.2. Let* $\gamma_k = (k+1)/2 - (b+1)$.

(i) *If* $\gamma_k > 0$ *then as* $n \to \infty$ *the critical value* $\lambda_1 \sim \sqrt{\gamma_k \log n / n}$.

(ii) *If* $\gamma_k < 0$ *then* $(\log \lambda_1)/\log n \to -(b+1)/(k+1)$.

New Theorem 1.3. *Under the assumptions of Theorem 1.2. Let* $\gamma_k = (k+1)/2 - (b+1)$.

(i) *If* $\gamma_k > 0$ *then as* $n \to \infty$ *the critical value*

$$\lambda_1 \sim \sqrt{\frac{\bar{\gamma}_k \log n}{n}} \qquad \text{where } \bar{\gamma}_k = \min\{\gamma_k, k+1-l_k-b_{l_k}\}.$$

(ii) *If* $\gamma_k < 0$ *then* $(\log \lambda_1)/\log n \to -(b+1)/(k+1)$.

Notations. We will introduce a function $\ell : (n, a_1, \dots, a_k) \to \mathbb{Z}$ that assigns a level to every vertex in the tree. Let $\ell(\rho) = 0$. Now each vertex can be assigned a level according to their position relative to the root ρ. Specifically, for each x, $\ell(y) = \ell(x) - 1$ for exactly one neighbor y of x, and $\ell(y) = \ell(x) + 1$ for the other $d(x)$ neighbors of x, where $d(x)$ represents the number of children of x's. Let $L_i = \{y : \ell(y) = i\}$ denote level i. Let $y_{k+1} \in L_{k+1}$ be a vertex on level $k+1$. We will denote by $y_i \in L_i$ the vertex lying on the path between ρ and y_{k+1}.

1 Proof of Theorem 1.2

1.1 Proof of the new lower bound

Lemma 1.1. *Let* $\bar{c}_k = \min\{k+1-l_k-b_{l_k}, (k-b)/2\}$ *and* $\epsilon > 0$. *When* n *is sufficiently large,*

$$\lambda_2 \geq \sqrt{\frac{\bar{c}_k \log n}{(1+\epsilon)n}}.$$

Proof. Let $S(y_{k+1})$ be the maximum connected subgraph that contains y_{k+1} and vertices with degree smaller than n, see Figure 1.1. Suppose $\lambda = \sqrt{(c \log n)/n}$. The proof of Theorem 1.4 in [1] yields that for any $\eta > 0$, starting with y_{k+1} initially occupied the survival time on $S(y_{k+1})$ is upper bounded by $B_n := C_0 (\log n) n^{c(1+\eta)}$ for some $C_0 > 0$ when n is sufficiently large. Starting from y_{k+1} initially occupied, we run the contact process on $\cup_{i=1}^{2k+1} L_i$ and allow particles to be born at ρ and L_{2k+2}. Meanwhile we freeze any particle when it is born at ρ or L_{2k+2}. To upper bound the number of frozen particles at any $x \in \{\rho\} \cup L_{2k+2}$ during this time, we will use the following comparison process.

Let $S(y)$ denote the subgraph rooted at $y \in L_{k+1}$ that is isomorphic to $S(y_{k+1})$. Since $\cup_{i=1}^{2k+1} L_i = \cup_{y \in L_{k+1}} S(y)$, the contact process on $\cup_{i=1}^{2k+1} L_i$ can be upper bounded by a collection of independent contact processes on $S(y)$'s. To do that we first run the contact process $\zeta_t^{(0)}$ on $S(y_{k+1})$ starting with y_{k+1} initially occupied and allow particles to be born (and frozen simultaneously) at L_{k+1}, L_{2k+2} and ρ. For the j-th frozen particle at some $y \in L_{k+1}$, we start a new independent contact process $\zeta_t^{(j)}$ on $S(y)$ and freeze any particles born at L_{k+1}, L_{2k+2} and ρ. Again, each particle frozen at L_{k+1} is assigned an index according to its birth time. We continue this process until there is no more particle frozen at L_{k+1}. This comparison process dominates the original contact process on $\cup_{i=1}^{2k+1} L_i$ so it suffices to give an upper bound on the number of particles frozen at $x \in \{\rho\} \cup L_{2k+2}$ in the comparison process.

We begin by estimating the number of particles born at L_{k+1} during the process $\zeta_t^{(0)}$ on $S(y_{k+1})$. For $y \in L_{k+1}$, let $\Gamma(y_{k+1}, y)$ denote the path between y_{k+1} and y, and $d(y_{k+1}, y)$ denote the length of $\Gamma(y_{k+1}, y)$. If a particle is born at y, then there is a path leading from y_{k+1} to y. If this path has length $d(y_{k+1},y)+2m$, then there are $d(y_{k+1},y)+m$ steps towards y and m steps towards y_{k+1}. Let $(z_0,z_1,\dots,z_{d(y_{k+1},y)+2m})$ denote a path from y_{k+1} to y with $z_0=y_{k+1}$ and $z_{d(y_{k+1},y)+2m}=y$. To produce a particle at ρ, we need a birth from z_j to z_{j+1} to occur before the particle at z_j dies for all $j=0,\dots d(y_{k+1},y)+2m-1$. So the expected number of particles produced at y by this path is

$$(\lambda/(1+\lambda))^{d(y_{k+1},y)+2m}\le\lambda^{d(y_{k+1},y)+2m}.$$

If we let $d=Cn^{1-\delta}$ so that every vertex in $S(y_{k+1})$ has degree bounded by d, then the expected number of particles $N_{y_{k+1},y}$ that reaches y has

$$\begin{aligned}
EN_{y_{k+1},y}&\le\sum_{m=0}^{\infty}\binom{d(y_{k+1},y)+2m}{m}\lambda^{d(y_{k+1},y)+2m}d^m\\
&\le\lambda^{d(y_{k+1},y)}(1+\sum_{m=1}^{\infty}2^{d(y_{k+1},y)+2m}\lambda^{2m}d^m)\\
&=\lambda^{d(y_{k+1},y)}(1+2^{d(y_{k+1},y)}\sum_{m=1}^{\infty}(4\lambda^2d)^m)\le(1+\eta)\lambda^{d(y_{k+1},y)}.
\end{aligned}\tag{1.1}$$

Note this is exactly the calcualtion in (2.1) of [1]. Let $B(S(y_{k+1}),y)$ be the total number of particles frozen at y in $\zeta_t^{(0)}$. The expected survival time on $S(y_{k+1})$ is bounded by B_n when n is sufficiently large. If during this whole time y_{k+1} is occupied and pushing particles towards y, then the same calculation as in (3.1) in [1] shows

$$EB(S(y_{k+1}),y)\le B_n(1+\eta)\lambda^{d(y_{k+1},y)}.$$

Summing over all $y\in L_{k+1}$ according to their relative distance to y_{k+1} gives the expected number of particles frozen at L_{k+1} during the contact process $\zeta_t^{(0)}$ on $S(y_{k+1})$:

$$\begin{aligned}
&\sum_{y\in L_{k+1}}EB(S(y_{k+1}),y)\\
&\le B_n\cdot(1+\eta)\left(\lambda^{2(k-l_k+1)}a_{l_k}+\lambda^{2(k-l_k+2)}a_{l_k-1}a_{l_k}+\cdots+\lambda^{2k}a_1\cdots a_{l_k}\right)\\
&=(1+\eta)B_n\lambda^{2(k-l_k+1)}a_{l_k}\left(1+\lambda^2a_{l_k-1}+(\lambda^2a_{l_k-1})(\lambda^2a_{l_k-2})+\cdots+\prod_{i=1}^{a_{l_k-1}}\lambda^2a_i\right)\\
&\le 2B_n\lambda^{2(k-l_k+1)}a_{l_k}\le n^{c(1+2\eta)}n^{-(k-l_k+1)}n^{b_{l_k}}=n^{c(1+2\eta)-(k+1-l_k-b_{l_k})}.
\end{aligned}\tag{1.2}$$

where the last line follows from the fact that $\lambda^2a_i<1/2$ for $1\le i\le k$ when n is large. Since $c<k+1-l_k-b_{l_k}$, we can choose η sufficiently small and n sufficiently large so that (1.2) $<1/2$. Let R be the total rounds of independent contact processes run on the subgraphs isomorphic to $S(y_{k+1})$. Given that the expected number of frozen particles at L_{k+1} produced in $\zeta_t^{(0)}$ is less than $1/2$,

$$ER\le\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i=2.$$

In the process $\zeta_t^{(0)}$ on $S(y_{k+1})$, a calculation similar to (3.1) in [1] shows the expected number of particles frozen at the root is $\le B_n(1+\eta)\lambda^{k+1}$. Therefore, during the comparison process the expected number of particles frozen at the root at most

$$ER\cdot B_n(1+\eta)\lambda^{k+1}\le\lambda^{k+1}n^{c(1+2\eta)}.\tag{1.3}$$

The same calculation shows that for each vertex in L_{2k+2} the expected number of particles frozen there is also $\le \lambda^{k+1} n^{c(1+2\eta)}$. Thus we have established that starting with y_{k+1} occupied, during the contact process on $\cup_{i=1}^{2k+1} L_i$, the expected number of particles frozen at any $x \in \{\rho\} \cup L_{2k+2}$ is upper bounded by $\lambda^{k+1} n^{c(1+2\eta)}$.

Now we can use the original argument in the proof of Lemma 3.1 in [1] to show when we also have $c < (k-b)/2$ there is no strong survival. That is, for any $\epsilon > 0$, if $c = \bar{c}_k/(1+\epsilon) < \bar{c}_k$ and $\lambda = \sqrt{c(\log n)/n}$, the process does not survive strongly when n is large. □

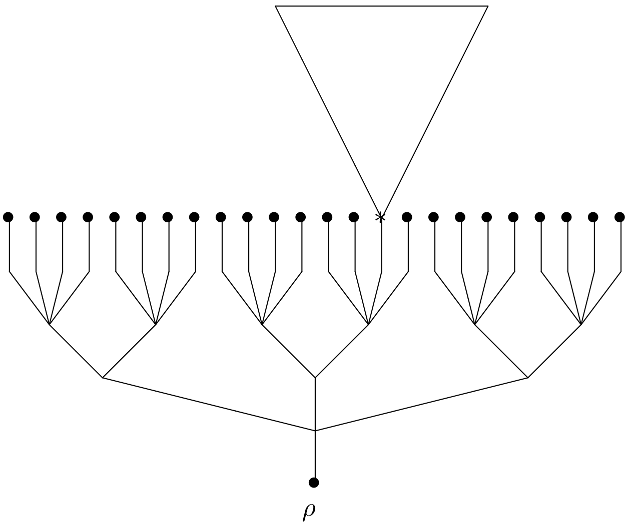

Figure 1: A picture of $S(y_{k+1})$ in the $(n,3,2,4,1)$ periodic tree (not containing the black dots). The $*$ marks y_{k+1}. The triangle above it is a copy of the tree of height k rooted at ρ (not containing the children of a_4).

1.2 Proof of the new upper bound

Lemma 1.2. *Let $\bar{c}_k = \min\{k - l_k + 1 - b_{l_k}, \frac{k-b}{2}\}$ and $\epsilon > 0$. When n is sufficiently large,*

$$\lambda_2 \le \sqrt{\frac{\bar{c}_k \log n}{(1-\epsilon)n}}.$$

The proof of the upper bound on λ_2 in [1] is not wrong. However since the lower upper bound is smaller we have to improve the upper bound too. There are two parts to the proof. Step 2: pushing the particles back to the root, which is based on a second moment calculation and an application of the Cauchy-Schwarz inequality, is almost the same as before. However, the first step changes considerably. We present Step 1 in detail and give a sketch of Step 2 for completeness.

Step 1: Pushing the particles out to distance $(k+1)m$. We need to estimate the probability of successfully pushing a particle from $y_{k+1} \in L_{k+1}$ to ρ and to L_{2k+2}. When $l_k = 0$, i.e., when the tree looks like $(n,1,\dots,1)$, our proof in [1] is correct and hence we consider only the case $l_k \ge 1$.

Let T_{l_k} be a subtree rooted at y_{l_k} with degree sequence $(a_{l_k}, 1, \dots, 1, n)$ (or (a_{l_k}, n) when $l_k = k$). Note that T_{l_k} contains the vertex y_{k+1}. In [1], the term “ignition” of a star graph refers to the event that starting from only the central vertex of a degree n star graph occupied, the number of occupied leaves increase to $L = (1-4\delta)\lambda n$ by time $n^c/4$, see Section 5 in [1].

For each $y \in L_{k+1}$, define the event

$$\begin{aligned} G_*(y) = &\{\text{the star at } y \text{ will become ignited by time } n^c/4\} \\ &\cap \{\text{there will be at least } \eta L \text{ occupied leaves during } I = [n^c/4, 3n^c/4]\} \\ &\cap \{\text{the center } y \text{ is occupied for at least one unit of time before } n^c/4\}. \end{aligned}$$

If $G_*(y)$ occurs for some $y \in L_{k+1}$, we will say the star at y is *fully ignited.*

Lemma 1.3. *Starting with $y \in L_{k+1}$ initially occupied, $P(G_*(y)) \to 1$ as $n \to \infty$.*

Proof. The first two events in $G_*(y)$ occurs with high probability according to Lemma 4.2 and Lemma 5.1 in [1]. For the last event, we first observe that by Lemma 5.1, $P(T_L < T_{0,0}, T_L < n^c/8) \to 1$ as $n \to \infty$, where $T_{0,0}$ is the extinction time on the star and T_L is the first time when there are L occupied leaves. By Lemma 4.2, with high probability there are always at least ηL occupied leaves during time $I' = [n^c/8, n^c/4]$. Then the proof of Lemma 6.1 in [1] shows that with high probability there is no interval of length $\geq t_* \equiv 2/(1-4\delta)\eta$ in I' during which the center y_{k+1} is always vacant. We can divide I' into $\geq \lfloor \frac{n^c}{8(t_*+2)} \rfloor$ intervals. In each interval the center y_{k+1} will become occupied before time t_* and try to stay occupied for one unit of time with success probability e^{-1}. Since y_{k+1} can try $O(n^c)$ many times, there is at least one success with high probability. □

For $1 \leq i \leq k+1$, let

$$t_i = \lambda^{-\delta} \cdot \lambda^{-(1+\delta)(k+1-i)}. \tag{1.4}$$

Recall that we use $y_1, \dots, y_k$ to denote the vertices lying on the path between ρ and y_{k+1} where the index i represents the corresponding level. Define, for $1 \leq i \leq k+1$,

$$G_i = \{\text{there is no interval of length} \geq t_i \text{ in } I \text{ during which } y_i \text{ is vacant}\}.$$

Lemma 1.4. *Suppose $G_*(y_{k+1})$ occurs. $P(G^c_{l_k}) \leq C\exp(-n^\epsilon)$ for some $C > 0$, $\epsilon > 0$.*

Proof. We use an induction on the index i from $k+1$ to l_k. When $i = k+1$, $t_{k+1} = \lambda^{-\delta}$. Using large deviations for a rate 1 Poisson process, the probability that there are more than n^c arrivals in the interval I of length $n^c/2$ is $\leq \exp(-\gamma n^c)$ for some $\gamma > 0$. On event $G_*(y_{k+1})$, the number of occupied leaves of y_{k+1} is always $\geq \eta L$. Once y_{k+1} is vacant, the time R_{k+1} needed for y_{k+1} to be occupied again satisfies

$$P(R_{k+1} > t_{k+1}) \leq \exp(-\lambda\eta L t_{k+1}) = \exp(-O(n^{\delta/2})).$$

Hence

$$P(G^c_{k+1}) \leq \exp(-\gamma n^c) + n^c P(R_{k+1} > t_{k+1}) \leq C_{k+1}\exp(-n^\epsilon)$$

for some $\epsilon < \delta/2$ and some constant C_{k+1}.

Suppose for $i = l+1$ the conclusion holds. We want to prove the result for $i = l$. Notice that $t_l = \lambda^{-(1+\delta)} t_{l+1}$. Suppose G_{l+1} occurs. Then an interval of length t_l can be partitioned into at least $\lambda^{-(1+\delta)}/2$ intervals of length $t_{l+1}+1$, within which y_{l+1} will become occupied at some time $s < t_{l+1}$ and try to infect y_l within the next one unit of time. The probability of success on one trial is $\geq e^{-1}(1-e^{-\lambda})$, so the probability of failing during the interval of length t_l is at most

$$(1 - e^{-1}(1-e^{-\lambda}))^{\lambda^{-(1+\delta)}/2} \leq \exp(-C\lambda^{-\delta})$$

since $1-e^{-\lambda} \geq \lambda/2$ when $\lambda > 0$ is small. That is, once y_l is vacant, the time R_l needed for y_l to be occupied again satisfies

$$P(R_l > t_l) \leq \exp(-C\lambda^{-\delta}).$$

Again using large deviations for a rate 1 Poisson process, the probability that there are more than n^c arrivals in the interval I of length $n^c/2$ is $\leq \exp(-\gamma n^c)$. Therefore, when G_{l+1} occurs the probability that G_l fails to occur is

$$P(G_l^c \cap G_{l+1}) \leq \exp(-\gamma n^c) + n^c P(R_l > t_l) \leq \exp(-\gamma n^c) + n^c \exp(-C\lambda^{-\delta}).$$

By assumption $P(G_{l+1}^c) \leq C_{l+1}\exp(-n^\epsilon)$ for some $C_{l+1} > 0$. It follows that

$$\begin{aligned} P(G_l^c) &\leq P(G_l^c \cap G_{l+1}) + P(G_{l+1}^c) \\ &\leq \exp(-\gamma n^c) + n^c \exp(-C\lambda^{-\delta}) + C_{l+1}\exp(-n^\epsilon) \leq C_l \exp(-n^\epsilon) \end{aligned}$$

for some constant $C_l > 0$. The induction is thus completed. □

Lemma 1.5. *Consider the contact process ξ_t on T_{l_k} starting with y_{k+1} initially occupied. Let $\sigma := |\{t : \xi_t(y) = 1$ for some $y \in L_{k+1}\}|$. Then there exists $\epsilon > 0$, so that*

$$P(\sigma > \exp(n^\epsilon/2)) \to 1 \quad \text{as } n \to \infty.$$

Proof. On event $G_*(y_{k+1}) \cap G_{l_k}$, we will partition the time interval $I = [n^c/4, 3n^c/4]$ into

$$M_\delta = \left\lfloor \frac{n^c}{2\lambda^{-\delta}(t_{l_k}+k)} \right\rfloor$$

subintervals $\{I_i : 1 \leq i \leq M_\delta\}$ of length $\lambda^{-\delta}(t_{l_k}+k)$ and write $I_i = [s_i, r_i)$. We further partition I_i into subintervals $\{I_{i,j} : 1 \leq j \leq \lfloor \lambda^{-\delta} \rfloor\}$ of equal length $t_{l_k}+k$. On the event G_{l_k} in each $I_{i,j}$ the vertex y_{l_k} becomes occupied before time t_{l_k} and has an independent probability e^{-1} of being occupied for at least one unit of time. Hence

$$P(y_{l_k} \text{ stays occupied for some } [s, s+1] \subseteq I_i \text{ where } s \leq r_i - k) \geq 1 - (1-e^{-1})^{\lfloor \lambda^{-\delta} \rfloor}.$$

Define event $A = \cap_{i=1}^{M_\delta}\{y_{l_k}$ stays occupied for some $[s, s+1] \subseteq I_i$ where $s \leq r_i - k\}$. It follows that

$$\begin{aligned} P(A | G_{l_k} \cap G_*(y_{k+1})) &\geq 1 - M_\delta \cdot (1-e^{-1})^{\lfloor \lambda^{-\delta} \rfloor} \\ &\geq 1 - \left\lfloor \frac{n^c}{2\lambda^{-\delta}(t_{l_k}+k)} \right\rfloor \cdot \exp(-e^{-1}\lfloor \lambda^{-\delta} \rfloor) \geq 1 - C\exp(-n^\epsilon) \end{aligned} \tag{1.5}$$

for some $\epsilon < \delta/2$.

On the event A, in each I_i there exists an interval of length 1 when y_{l_k} stays occupied. In this interval y_{l_k} can try to ignite the stars in $D = \{y \in L_{k+1} :\ y \text{ is a descendant of } y_{l_k}\}$ independently. For any $y \in D$, the probability that the star at y will be fully ignited is at least

$$(1-e^{-\lambda})(e^{-1}(1-e^{-\lambda}))^{k-l_k} P(G_*(y)) \geq C\lambda^{k+1-l_k}, \tag{1.6}$$

where we have $P(G_*(y)) \to 1$ as $n \to \infty$ by Lemma 1.3. So for each I_i ($1 \leq i \leq M_\delta$) we can start a trial as following. For each $y \in D$ we flip a coin with success probability $C\lambda^{k+1-l_k}$. If there is a success then we say the trial succeeds and stop. Otherwise we say the trial fails and wait for the next trial. The probability that all the trials fail is less than

$$\begin{aligned} &\prod_{i=1}^{M_\delta}(1-C\lambda^{k+1-l_k})^{a_{l_k}} = (1-C\lambda^{k+1-l_k})^{a_{l_k}M_\delta} \leq \exp(-C\lambda^{k+1-l_k}a_{l_k}M_\delta) \\ &\leq \exp(-Cn^c\lambda^{2\delta+(2+\delta)(k+1-l_k)}a_{l_k}) \equiv q. \end{aligned}$$

Therefore, on the event $A \cap G_{l_k} \cap G_*(y_{k+1})$, the star at y_{k+1} can get another star in D to be fully ignited with probability at least $1-q$. That is to say, the probability that a fully ignited star at y_{k+1} fails to get another star in D to be fully ignited is at most

$$q \cdot P(A|G_*(y_{k+1}) \cap G_{l_k})P(G_{l_k}|G_*(y_{k+1})) + P(A^c|G_*(y_{k+1}) \cap G_{l_k}) + P(G_{l_k}^c|G_*(y_{k+1})) \quad (1.7)$$

Recalling the definition of b_i we see that if $c > k - l_k + 1 - b_{l_k}$ we can choose δ sufficiently small so that $q \le \exp(-n^\epsilon)$ for some $\epsilon > 0$. Together with Lemma 1.4 and (1.5), we have $(1.7) \le C\exp(-n^\epsilon) \equiv q_1$.

We will flip coins with probability $1-q_1$ to get a head. The process will continue as long as we are getting heads and will stop once we get a tail. Let X be the number of flips before we get a tail. Then

$$P(X \ge \exp(n^\epsilon/2)) \ge P(X \ge \sqrt{1/q_1}) = (1-q_1)^{\sqrt{1/q_1}} \to 1 \quad \text{as } n \to \infty.$$

Each time a star at $y \in L_{k+1}$ becomes fully ignited, the central vertex is occupied for at least one unit of time before it starts to ignite other stars. Recalling that $\sigma := |\{t : \xi_t(y) = 1 \text{ for some } y \in L_{k+1}\}|$ we have

$$P(\sigma \ge \exp(n^\epsilon/2)|G_*(y_{k+1})) \ge P(X \ge \exp(n^\epsilon/2)) \to 1. \quad (1.8)$$

Combining (1.8) with Lemma 1.3 completes the proof. □

The set $\{t : \xi_t(y) = 1 \text{ for some } y \in L_{k+1}\}$ consists of disjoint intervals. We will partition this set into time blocks, each of total length $k+1$. At the start of each time block, choose one vertex $y \in L_{k+1}$ such that y is occupied at the time. The probability of successfully pushing a particle to the root within time $k+1$ is at least

$$(e^{-1}(1-e^{-\lambda}))^{k+1} \ge C'\lambda^{k+1}.$$

In every time block of σ we can try the push independently. Hence the probability of igniting the root is

$$\ge P(\sigma > \exp(n^\epsilon/2)) \cdot \left(1 - (1 - C'\lambda^{k+1})^{\exp(n^\epsilon/2)/(k+1)}\right) \cdot P(G_*(\rho)) \to 1 \quad \text{as } n \to \infty.$$

Notice that this estimate is also true for pushing a particle from level $k+1$ to level $2k+2$. This is to say, if $c > k - l_k + 1 - b_{l_k}$, an open site can make an adjacent site open with arbitrarily large probability in the comparison oriented percolation.

Step 2: Bringing a particle back to the root. Let $N = n \cdot (a_1 \cdots a_k)$ and let $\mathbb{T}_N$ denote the N-regular tree. We will compare the contact process on $(n, a_1, \dots, a_k)$ with a 2-dependent oriented percolation in $\mathbb{T}_N \times \mathbb{Z}_+$, where $\mathbb{Z}_+$ represents time. There exists a one to one map $f : (n, a_1, \dots, a_k) \to \mathbb{T}_N$ that takes each vertex in $L_{m(k+1)}$ for $m \in \mathbb{N}$ to a vertex on level m in $\mathbb{T}_N$. By the discussion in Step 1, the pushing events from y_{k+1} to ρ could involve another star $y \in L_{k+1}$. The graph distance between $f(y_{k+1})$ and $f(y)$ is 2 in $\mathbb{T}_N$. That is why the oriented percolation is 2-dependent.

Let p denote the probability of successfully pushing a particle to a neighbor on $\mathbb{T}_N$. By Step 1 we see that p can be arbitrarily close to 1 if $c > k - l_k + 1 - b_{l_k}$. Since $\mathbb{Z} \times \mathbb{Z}_+$ is embedded in $\mathbb{T}_N \times \mathbb{Z}_+$ and the 2-dependent oriented percolation on $\mathbb{Z} \times \mathbb{Z}_+$ has critical probability smaller than 1, if p is sufficiently large then a particle returns to the root at arbitrarily large times in the oriented percolation on $\mathbb{T}_N \times \mathbb{Z}_+$. Hence when $c > k - l_k + 1 - b_{l_k}$ the contact process has strong survival.

The original proof in [1] shows that if $c > \frac{k-b}{2}$ there is also strong survival. Therefore,

$$\lambda_2 \le \sqrt{\frac{c_k \log n}{(1-\epsilon)n}} \quad \text{where } c_k = \min\{k - l_k + 1 - b_{l_k}, \frac{k-b}{2}\}.$$

2 Proof of Theorem 1.3

Case 1: $\gamma_k > 0$**.** Let $\alpha_1 = \min\{\gamma_k, k+1-l_k-b_{l_k}\} - \epsilon$ and $\lambda = \sqrt{\alpha(\log n)/n}$. The same calculation that yields (1.3) shows that starting from y_{k+1} initially occupied, the total number of frozen particles on $\{\rho\} \cup L_{2k+2}$ is

$$\leq (1+n^{1+b})\lambda^{k+1}n^{\alpha(1+2\eta)}.$$

From this we see that if η is small then the expected number of particles that escape from $\cup_{i=1}^{2k+1} L_i$ is < 1 and comparing with a branching process implies that the process dies out.

Turning to the upper bound, let $\alpha_2 = \min\{\gamma_k, k+1-l_k-b_{l_k}\}+\epsilon$ and $\lambda = \sqrt{\alpha_2(\log n)/n}$. If $\alpha_2 = k+1-l_k-b_{l_k}+\epsilon$, then the same argument as in Step 2 in Section 1.2 implies survival of the process. If $\alpha_2 = \gamma_k + \epsilon$, then the original proof can be applied to prove survival of the process.

Case 2: $\gamma_k < 0$**.** Suppose $\lambda = n^{-\beta}$ with $\beta > 1/2$. In this case, Theorem 1.4 in [1] implies that the contact process survives for $O(\log n)$ on the graph $S(y_{k+1})$. Using (1.2) and (1.3) again, the expected number of particles that escape from $\cup_{i=1}^{2k+1} L_i$ is

$$\leq C(\log n)(1+\eta)\lambda^{k+1}(1+n^{b+1}) = C'(\log n)n^{b+1-\beta(k+1)}$$

for some positive constant C, C'. If $\beta > \frac{b+1}{k+1}$ the above is < 1 when n is large. Comparing with a branching process implies the the process dies out.

The proof of the lower bound is the same as before and thus omitted here.

References

[1] Huang, X., & Durrett, R. (2020). The contact process on periodic trees. *Electronic. Comm. in Prob. 25 (2020), paper 24.* MR4089731