

On two-dimensional extensions of Bougerol’s identity in law*

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Abstract

Let $B = \{B_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion and denote by $A_t, t \geq 0$, the quadratic variation of semimartingale $e^{B_t}, t \geq 0$. The celebrated Bougerol’s identity in law (1983) asserts that, if $\beta = \{\beta_t\}_{t \geq 0}$ is another Brownian motion independent of B , then β_{A_t} has the same law as $\sinh B_t$ for every fixed $t > 0$. Bertoin, Dufresne and Yor (2013) obtained a two-dimensional extension of the identity involving, as the second coordinates, the local times of B and β at level zero. In this paper, we present a generalization of their extension in a situation that the levels of those local times are not restricted to zero. Our argument provides a short elementary proof of the original extension and sheds new light on that subtle identity.

Keywords: Brownian motion; exponential functional; Bougerol’s identity; local time.

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1 Introduction

This paper concerns a two-dimensional extension of Bougerol’s identity in law obtained by Bertoin, Dufresne and Yor [5]. Let $B = \{B_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion and set $A_t, t \geq 0$, by the quadratic variation of semimartingale $e^{B_t}, t \geq 0$, namely that of the stochastic integral $\int_0^t e^{B_s} dB_s, t \geq 0$:

$$A_t := \int_0^t e^{2B_s} ds.$$

This exponential additive functional of Brownian motion appears in a number of areas of probability theory such as the pricing of Asian options in mathematical finance, and is known for its close relationship with planar Brownian motion (or two-dimensional Bessel process); see the detailed surveys [13, 14] by Matsumoto and Yor. Let $\beta = \{\beta_t\}_{t \geq 0}$ be another one-dimensional standard Brownian motion which we assume to be independent of B throughout the paper. The celebrated Bougerol’s identity in law ([7]) then asserts that, for every fixed $t > 0$,

$$\beta_{A_t} \stackrel{(d)}{=} \sinh B_t. \tag{1.1}$$

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When $t > 0$, the exponential functional A_t admits a density with closed-form expression

$$\exp\left(\frac{\pi^2}{8t}\right) \mathbb{E}\left[\frac{\cosh B_t}{\sqrt{2\pi v^3}} \exp\left(-\frac{\cosh^2 B_t}{2v}\right) \cos\left(\frac{\pi}{2t} B_t\right)\right], \quad v > 0; \tag{1.2}$$

see [2, Equation (1.5)]. Because the expression involves an oscillatory integral, sometimes it is not easy to analyze; for instance, as $t \downarrow 0$, the prefactor in (1.2) grows rapidly while, in view of the Riemann–Lebesgue lemma, the expectation part converges to 0, which causes difficulties in numerical evaluation for a small value of t . Nonetheless, Bougerol's identity (1.1) makes the law of A_t fairly tractable; as an illustration, we readily see that its Mellin transform is expressed as

$$\mathbb{E}\left[(A_t)^{\nu-1}\right] = \frac{\sqrt{\pi}}{2^{\nu-1}\Gamma(\nu-1/2)} \mathbb{E}[|\sinh B_t|^{2\nu-2}], \quad \nu > 1/2$$

(see, e.g., [21, Equation (4.g)]), thanks to the identity. Here $\Gamma(\cdot)$ is the gamma function. For a more detailed account of Bougerol's identity, as well as for recent progress in its study such as extensions to other processes, see the survey [20] by Vakeroudis. We also refer to [10] for multidimensional extensions other than the ones dealt with in the present paper.

Let $L = \{L_t^x\}_{t \geq 0, x \in \mathbb{R}}$ and $\lambda = \{\lambda_t^x\}_{t \geq 0, x \in \mathbb{R}}$ be (the bicontinuous versions of) the local time processes of B and β , respectively. In [5], the following two-dimensional extension of Bougerol's identity (1.1) is shown:

Theorem 1.1 ([5, Theorem 1.1]). *For every fixed $t > 0$, it holds that*

$$(\beta_{A_t}, e^{-B_t} \lambda_{A_t}^0) \stackrel{(d)}{=} (e^{-B_t} \beta_{A_t}, \lambda_{A_t}^0) \stackrel{(d)}{=} (\sinh B_t, \sinh L_t^0).$$

We also refer the reader to [20, Theorem 4.1]. By virtue of the scaling property of Brownian motion, the first identity in law is an immediate consequence of the independence of B and β and the identity in law

$$(e^{B_t}, A_t) \stackrel{(d)}{=} (e^{-B_t}, e^{-2B_t} A_t), \tag{1.3}$$

which follows from the time-reversal: $\{B_{t-s} - B_t\}_{0 \leq s \leq t} \stackrel{(d)}{=} \{B_s\}_{0 \leq s \leq t}$. Roughly speaking, their proof of Theorem 1.1 is divided into three steps: first, observe that the validity of the claimed identity is reduced to showing

$$(e^{-B_t} |\beta_{A_t}|, \lambda_{A_t}^0) \stackrel{(d)}{=} (\sinh |B_t|, \sinh L_t^0); \tag{1.4}$$

then, for each $p > 0$, replace t by an exponential random variable with parameter p that is assumed to be independent of B and β ; and then, with this replacement, show that the joint Mellin transforms of both sides of (1.4) agree for any $p > 0$. While the original proof of Theorem 1.1 was rather involved (in fact, in the course of the proof, an auxiliary standard exponential random variable independent of the other elements, is also considered to make the computation progress), we have noticed in [15] that the validity of the theorem is explained via the following two standard facts: given $t > 0$, for every fixed $x \in \mathbb{R}$,

$$e^{B_t} \sinh x + \beta_{A_t} \stackrel{(d)}{=} \sinh(x + B_t), \tag{1.5}$$

and, for all $a \in \mathbb{R}$ and $b > 0$,

$$\mathbb{P}(B_t \leq a, L_t^0 \geq b) = \begin{cases} \mathbb{P}(b + B_t \leq 0) + \mathbb{P}(-a \leq b + B_t \leq 0) & (a \geq 0), \\ \mathbb{P}(b + B_t \leq a) & (a < 0). \end{cases} \tag{1.6}$$

To see how these two facts explain Theorem 1.1, for each $x \in \mathbb{R}$ and $y > 0$, consider the probability

$$\mathbb{P}(\beta_{A_t} \leq x, e^{-B_t} \lambda_{A_t}^0 \geq y).$$

In what follows, we denote by $\sinh^{-1} x$, $x \in \mathbb{R}$, the inverse function of the hyperbolic sine function. Then, in the case $x < 0$, by (1.6) for $a < 0$ and thanks to the independence of B and β , the above probability agrees with $\mathbb{P}(e^{B_t} y + \beta_{A_t} \leq x)$, which, by (1.5), is equal to

$$\begin{aligned} \mathbb{P}(\sinh(\sinh^{-1} y + B_t) \leq x) &= \mathbb{P}(\sinh^{-1} y + B_t \leq \sinh^{-1} x) \\ &= \mathbb{P}(B_t \leq \sinh^{-1} x, L_t^0 \geq \sinh^{-1} y) \\ &= \mathbb{P}(\sinh B_t \leq x, \sinh L_t^0 \geq y), \end{aligned}$$

where, for the second equality, we used (1.6) with $a = \sinh^{-1} x < 0$. The case $x \geq 0$ is treated in the same way by using (1.6) for $a \geq 0$. We remark that the second identity in Theorem 1.1 may also be proven by the same argument as above without appealing to (1.3).

Identity (1.5) was originally due to Alili and Gruet [2, Proposition 4]; for the sake of the completeness of the paper, we give a proof of (1.5) in Appendix, by means of stochastic differential equations (SDEs). Fact (1.6) is a consequence of the well-known formula for the joint density of B_t and L_t^0 :

$$\mathbb{P}(B_t \in da, L_t^0 \in db) = \frac{|a| + b}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(|a| + b)^2}{2t}\right\} dadb, \quad a \in \mathbb{R}, b > 0$$

(see, e.g., [12, Problem 6.3.4]), which may easily be deduced from the reflection principle of Brownian motion by virtue of Lévy's theorem for Brownian local time.

In this paper, we develop further the aforementioned idea to obtain the following generalization of Theorem 1.1 that extends (1.5):

Theorem 1.2. *For every fixed $t > 0$ and $x \in \mathbb{R}$, we have*

$$\begin{aligned} &\left(e^{B_t} \sinh x + \beta_{A_t}, e^{-B_t} \lambda_{A_t}^{-e^{B_t} \sinh x}\right) \\ &\stackrel{(d)}{=} \left(e^{-B_t} \sinh x + e^{-B_t} \beta_{A_t}, \lambda_{A_t}^{-\sinh x}\right) \\ &\stackrel{(d)}{=} \left(\sinh(x + B_t), \sinh(|x| + L_t^{-x}) - \sinh|x|\right). \end{aligned} \tag{1.7}$$

To our knowledge, the identity of the second coordinates, namely

$$\lambda_{A_t}^{\sinh x} \stackrel{(d)}{=} \sinh(|x| + L_t^x) - \sinh|x| \tag{1.8}$$

for every $t > 0$ and $x \in \mathbb{R}$, seems to be new in its own right. As will be seen in Remark 2.1, identity (1.8) may be explained by the original Bougerol's identity (1.1) and by the fact that

$$\{L_t^x\}_{t \geq 0} \stackrel{(d)}{=} \left\{ \left(\max_{0 \leq s \leq t} B_s - |x| \right)^+ \right\}_{t \geq 0}, \tag{1.9}$$

which follows from Lévy's theorem for Brownian local time and the strong Markov property of Brownian motion. Here, for every $a \in \mathbb{R}$, we denote $a^+ = \max\{a, 0\}$.

As it is closely related to the subject of the paper, another profound identity in law due to Dufresne [9] in the study of exponential functionals of Brownian motion, would be worth mentioning. Let μ be a positive real number and consider the additive functional

$$\int_0^t e^{2(B_s - \mu s)} ds, \quad t \geq 0.$$

Then its limit as $t \rightarrow \infty$ is known [9, Proposition 4.4.4(b)] to have the same law as $1/(2\gamma_\mu)$, where γ_μ is a gamma random variable with parameter μ :

$$\mathbb{P}(\gamma_\mu \in dx) = \frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-x} dx, \quad x > 0.$$

The above fact, known as Dufresne's identity in law, has connections with random polymer models, as well as generalizations to matrix-valued processes; see [18, 16, 19, 17, 3]. It would be of interest to explore Bougerol's identity in law (1.1) in those directions. For a matrix extension of the identity, see the recent paper [4] by Assiotis. We also refer to [11] for a process-level extension in terms of anticipative transforms of Brownian motion.

The rest of the paper is organized as follows: in the next section, preparing a lemma extending (1.6), we prove Theorem 1.2, and in Appendix, we give a proof of (1.5).

2 Proof of Theorem 1.2

In the sequel, we fix $t > 0$. We begin with the following lemma.

Lemma 2.1. *For every $x \in \mathbb{R}$, it holds that, for all $a \in \mathbb{R}$ and $b > 0$,*

$$\begin{aligned} & \mathbb{P}(B_t \leq a, L_t^x \geq b) \\ &= \begin{cases} \mathbb{P}(b + |x| + B_t \leq 0) + \mathbb{P}(x - a \leq b + |x| + B_t \leq 0) & (a \geq x), \\ \mathbb{P}(b + |x| + B_t \leq a - x) & (a < x). \end{cases} \end{aligned} \tag{2.1}$$

The above relation is obtained by the formula:

$$\begin{aligned} & \mathbb{P}(B_t \in da, L_t^x \in db) \\ &= \frac{|a - x| + b + |x|}{\sqrt{2\pi t^3}} \exp \left\{ -\frac{(|a - x| + b + |x|)^2}{2t} \right\} dadb, \quad a \in \mathbb{R}, b > 0 \end{aligned}$$

(see, e.g., [6, p.161, Formula 1.1.3.8]); one may also conclude (2.1) from (1.6) by the strong Markov property of Brownian motion. For ease of presentation, we slightly modify relation (2.1) in such a way that, given $x \in \mathbb{R}$, for all $a \in \mathbb{R}$ and $b > 0$,

$$\begin{aligned} & \mathbb{P}(x + B_t \leq a, L_t^{-x} \geq b) \\ &= \begin{cases} \mathbb{P}(b + |x| + B_t \leq 0) + \mathbb{P}(-a \leq b + |x| + B_t \leq 0) & (a \geq 0), \\ \mathbb{P}(b + |x| + B_t \leq a) & (a < 0), \end{cases} \end{aligned} \tag{2.2}$$

with which we prove Theorem 1.2.

Proof of Theorem 1.2. In view of (1.3) and the scaling property of Brownian motion, it suffices to prove the identity between the first line and the third line of the claimed identity (1.7). To this end, we will show that

$$\begin{aligned} & \mathbb{P} \left(e^{B_t \sinh x + \beta_{A_t}} \leq y, e^{-B_t} \lambda_{A_t}^{-e^{B_t} \sinh x} \geq z \right) \\ &= \mathbb{P}(\sinh(x + B_t) \leq y, \sinh(|x| + L_t^{-x}) - \sinh|x| \geq z) \end{aligned} \tag{2.3}$$

for any $y \in \mathbb{R}$ and $z > 0$; once this is done, then we also have, for any $y \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{P} \left(e^{B_t \sinh x + \beta_{A_t}} \leq y, e^{-B_t} \lambda_{A_t}^{-e^{B_t} \sinh x} = 0 \right) \\ &= \mathbb{P}(\sinh(x + B_t) \leq y, \sinh(|x| + L_t^{-x}) - \sinh|x| = 0) \end{aligned}$$

thanks to (1.5), and the proof is completed.

In the case $y < 0$, the left-hand side of (2.3) is rewritten, by the latter relation in (2.2) and by the independence of B and β , as

$$\begin{aligned} & \mathbb{P}\left(e^{B_t} \sinh x + \beta_{A_t} \leq y, \lambda_{A_t}^{-e^{B_t} \sinh x} \geq e^{B_t} z\right) \\ &= \mathbb{P}\left(e^{B_t} z + e^{B_t} \sinh |x| + \beta_{A_t} \leq y\right) \\ &= \mathbb{P}\left(\sinh\left(\sinh^{-1}(z + \sinh |x|) + B_t\right) \leq y\right), \end{aligned} \tag{2.4}$$

where the last line is due to (1.5). On the other hand, the right-hand side of (2.3) is rewritten as

$$\begin{aligned} & \mathbb{P}\left(x + B_t \leq \sinh^{-1} y, L_t^{-x} \geq \sinh^{-1}(z + \sinh |x|) - |x|\right) \\ &= \mathbb{P}\left(\sinh^{-1}(z + \sinh |x|) - |x| + |x| + B_t \leq \sinh^{-1} y\right), \end{aligned}$$

which agrees with (2.4). Here we used the latter relation in (2.2) for the equality.

The case $y \geq 0$ is treated similarly. In this case, by the former relation in (2.2), one sees that the left-hand side of (2.3) is rewritten as

$$\begin{aligned} & \mathbb{P}\left(e^{B_t}(z + \sinh |x|) + \beta_{A_t} \leq 0\right) + \mathbb{P}\left(-y \leq e^{B_t}(z + \sinh |x|) + \beta_{A_t} \leq 0\right) \\ &= \mathbb{P}\left(\sinh\left(\sinh^{-1}(z + \sinh |x|) + B_t\right) \leq 0\right) \\ &+ \mathbb{P}\left(-y \leq \sinh\left(\sinh^{-1}(z + \sinh |x|) + B_t\right) \leq 0\right). \end{aligned} \tag{2.5}$$

On the other hand, the right-hand side of (2.3) is rewritten as

$$\begin{aligned} & \mathbb{P}\left(x + B_t \leq \sinh^{-1} y, L_t^{-x} \geq \sinh^{-1}(z + \sinh |x|) - |x|\right) \\ &= \mathbb{P}\left(\sinh^{-1}(z + \sinh |x|) + B_t \leq 0\right) \\ &+ \mathbb{P}\left(-\sinh^{-1} y \leq \sinh^{-1}(z + \sinh |x|) + B_t \leq 0\right), \end{aligned}$$

which agrees with (2.5). Here we used the former relation in (2.2) for the equality. Therefore we have obtained (2.3) for all $y \in \mathbb{R}$ and $z > 0$ and the theorem is proven. \square

Remark 2.1. In view of (1.9), the left-hand side of (1.8) is identical in law with

$$\left(\max_{0 \leq s \leq A_t} \beta_s - \sinh |x|\right)^+$$

owing to the independence of B and β , which, by the reflection principle of Brownian motion, has the same law as

$$\left(|\beta_{A_t}| - \sinh |x|\right)^+,$$

and hence we have the identity in law

$$\lambda_{A_t}^{\sinh x} \stackrel{(d)}{=} \left(\sinh |B_t| - \sinh |x|\right)^+ \tag{2.6}$$

thanks to the original Bougerol's identity (1.1). On the other hand, by (1.9) and the reflection principle of Brownian motion again, the right-hand side of (1.8) is identical in law with

$$\begin{aligned} & \sinh \left\{ |x| + \left(\max_{0 \leq s \leq t} B_s - |x|\right)^+ \right\} - \sinh |x| \\ & \stackrel{(d)}{=} \sinh \left\{ |x| + \left(|B_t| - |x|\right)^+ \right\} - \sinh |x|, \end{aligned}$$

which coincides with the right-hand side of (2.6).

Appendix

This appendix is devoted to the proof of (1.5). We follow an inventive argument by Alili, Dufresne and Yor [1], which is also reproduced in [8, Appendix].

For every fixed $x \in \mathbb{R}$, define the process $X^x = \{X_t^x\}_{t \geq 0}$ by

$$X_t^x := e^{-Bt} \sinh x + e^{-Bt} \int_0^t e^{Bs} dW_s,$$

with $W = \{W_t\}_{t \geq 0}$ a one-dimensional standard Brownian motion independent of B . By Itô's formula, we see that X^x satisfies the SDE

$$dX_t^x = \sqrt{1 + (X_t^x)^2} d\gamma_t^x + \frac{1}{2} X_t^x dt, \quad X_0^x = \sinh x,$$

where the process $\gamma^x = \{\gamma_t^x\}_{t \geq 0}$ is defined by

$$\gamma_t^x := \int_0^t \frac{-X_s^x dB_s + dW_s}{\sqrt{1 + (X_s^x)^2}}.$$

The process γ^x is a continuous local martingale with quadratic variation $\langle \gamma^x \rangle_t = t$, $t \geq 0$, hence it is a Brownian motion. Since the coefficients of the above SDE are Lipschitz continuous, there exists a unique strong solution, and it is easily checked that the solution is given by

$$X_t^x = \sinh(x + \gamma_t^x), \quad t \geq 0. \tag{A.1}$$

On the other hand, because of the multidimensional time-change theorem (see, e.g., [12, Theorem 3.4.13]), we may find a one-dimensional standard Brownian motion $\beta = \{\beta_t\}_{t \geq 0}$ independent of B , such that, a.s.,

$$X_t^x = e^{-Bt} \sinh x + e^{-Bt} \beta_{A_t}, \quad t \geq 0.$$

Therefore, for every fixed $t > 0$, by the scaling property of Brownian motion and the independence of B and β ,

$$\begin{aligned} X_t^x &\stackrel{(d)}{=} e^{-Bt} \sinh x + \beta_{e^{-2Bt}A_t} \\ &\stackrel{(d)}{=} e^{Bt} \sinh x + \beta_{A_t}, \end{aligned} \tag{A.2}$$

where the second line follows from (1.3). Comparing (A.2) and (A.1) entails (1.5) because γ^x is a standard Brownian motion.

Another proof of identity (1.5) may be found in [10], in which the validity of the identity is reduced to a property of generalized inverse Gaussian distributions with particular set of parameters, a reduction which also allows one to replace time t by any stopping time of B of a certain class.

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