

# On the CLT for stationary Markov chains with trivial tail sigma field\*

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## Abstract

In this paper we consider stationary Markov chains with trivial two-sided tail sigma field, and prove that additive functionals satisfy the central limit theorem provided the variance of partial sums divided by  $n$  is bounded.

**Keywords:** Markov chains; central limit theorem; tail sigma field; additive functionals.

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## 1 Introduction

One of the most useful theorems for stationary sequences is the central limit theorem for partial sums  $S_n$  with the normalization  $\sqrt{n}$ . For several classes of additive functionals of stationary Markov chains, the size of the variance of partial sums determines the limiting distribution. For instance for additive functionals of reversible, stationary and ergodic Markov chains, with centered and square integrable variables, Kipnis and Varadhan [10] proved that if  $E(S_n^2)/n$  converges to a finite limit, then the CLT holds. On the other hand, for additive functionals of Harris recurrent and aperiodic Markov chains with centered and square integrable variables, Chen ([7], Theorem II. 3.1) proved that if  $S_n/\sqrt{n}$  is stochastically bounded, it satisfies the CLT.

These results suggest and motivate the study of limiting distribution for stationary Markov chains with additive functionals satisfying  $\sup_n E(S_n^2)/n < \infty$ . Recently, Peligrad [12] introduced a new idea, which involves conditioning with respect to both the past and the future of the process. By using this approach she proved that functions of a Markov chain which is stationary and totally ergodic (in the ergodic theoretical sense), and with  $\sup_n E(S_n^2)/n < \infty$ , satisfy the CLT, provided that a random centering is used. In this paper we show that the random centering is not needed if the two-sided tail sigma field of the Markov chain is trivial.

Our result simply states that if a stationary Markov chain has two-sided tail sigma field trivial, then any additive functional with finite second moment, centered at expectation and with  $\sup_n E(S_n^2)/n < \infty$  satisfies the central limit theorem (CLT). The interest of such a result consists in the fact that does not require fine computations of the rate of convergence of mixing coefficients.

Examples of stationary processes with trivial two-sided tail sigma field include absolutely regular Markov chains and interlaced mixing Markov chains. The definitions will

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be given in this paper. We also refer to Subsection 2.5 in Bradley [3] for a survey and Bradley [4] for the proofs of the results in that survey.

It should be noted that, for a stationary Markov chain, the condition  $\sup_n E(S_n^2)/n < \infty$  alone is not enough for the CLT (see for instance Bradley [2] or Cuny and Lin [8], Prop. 9.5(ii), among other examples). On the other hand, if the stationary sequence is not Markov, the condition  $\sup_n E(S_n^2)/n < \infty$  together with the two-sided tail sigma field is trivial, are not enough for the CLT. Indeed, Bradley [5] constructed a stationary sequence, such that any 5 variables are independent,  $\sup_n E(S_n^2)/n < \infty$ , the two-sided tail sigma field is trivial, but the CLT does not hold.

## 2 Results

We assume that  $(\xi_n)_{n \in \mathbb{Z}}$  is a strictly stationary Markov chain, defined on a complete probability space  $(\Omega, \mathcal{F}, P)$  with values in a Polish space  $(S, \mathcal{A})$ . Denote by  $\mathcal{F}_n = \sigma(\xi_k, k \leq n)$  and by  $\mathcal{F}^n = \sigma(\xi_k, k \geq n)$ , completed with the sets of measure 0 with respect to  $P$ . The invariant distribution on  $\mathcal{A}$  is denoted by  $\pi(A) = \mathbb{P}(\xi_0 \in A)$ . We shall construct the Markov chain in a canonical way on  $\Omega = S^{\mathbb{Z}}$  from a kernel  $P(x, A)$  and the marginal distribution  $\pi$  (see the section “A standard nice probability space”, pages 17-18 in Ch. 1, Bradley [4]).

We define the two-sided tail sigma field by

$$\mathcal{T}_d = \bigcap_{n \geq 1} (\mathcal{F}_{-n} \vee \mathcal{F}^n).$$

We say that  $\mathcal{T}_d$  is trivial if for any  $A \in \mathcal{T}_d$  we have  $P(A) = 0$  or 1. Note that this sigma field might be larger than the sigma algebra generated by the union of one-sided tail sigma fields defined as  $\mathcal{T}_l = \bigcap_{n \geq 1} (\mathcal{F}_{-n})$  and  $\mathcal{T}_r = \bigcap_{n \geq 1} (\mathcal{F}^n)$ . For simplicity, when we refer to the tail sigma field we shall always understand the two-sided one,  $\mathcal{T}_d$ .

Let  $\mathbb{L}_0^2(\pi)$  be the set of measurable functions on  $S$  such that  $\int f^2 d\pi < \infty$  and  $\int f d\pi = 0$ . For a function  $f \in \mathbb{L}_0^2(\pi)$  let

$$X_i = f(\xi_i), \quad S_n = \sum_{i=1}^n X_i. \tag{2.1}$$

We denote by  $\|X\|$  the norm in  $L^2(\Omega, \mathcal{F}, P)$  and  $\Rightarrow$  denotes convergence in distribution.

The goal of this note is to establish the following two results.

**Theorem 2.1.** *Let  $(X_n)_{n \in \mathbb{Z}}$  and  $(S_n)_{n \geq 1}$  be as defined in (2.1) with*

$$\sup_n \frac{E(S_n^2)}{n} < \infty. \tag{2.2}$$

*Assume  $(\xi_n)$  has trivial tail sigma field  $\mathcal{T}_d$ . Then, for some  $c \geq 0$ , the following limit exists*

$$\lim_{n \rightarrow \infty} \frac{E(|S_n|)}{\sqrt{n}} = \frac{c}{\sqrt{2\pi}} \geq 0,$$

and

$$\frac{S_n}{\sqrt{n}} \Rightarrow N(0, c^2) \text{ as } n \rightarrow \infty.$$

As a consequence of this result we obtain a necessary and sufficient condition for the CLT for additive functionals of Markov chains with trivial tail sigma field.

**Theorem 2.2.** *Let  $(X_n)_{n \in \mathbb{Z}}$  and  $(S_n)_{n \geq 1}$  be as defined in (2.1) and assume  $(\xi_n)$  has trivial tail sigma field  $\mathcal{T}_d$ . Then the following are equivalent:*

- (i)  $(S_n^2/n)_{n \geq 1}$  is uniformly integrable.
- (ii) There is  $c \geq 0$  such that

$$\lim_{n \rightarrow \infty} \frac{E(S_n^2)}{n} = c^2 \text{ and } \frac{S_n}{\sqrt{n}} \Rightarrow N(0, c^2) \text{ as } n \rightarrow \infty.$$

**2.1 Two classes of stationary Markov Chains with trivial  $\mathcal{T}_d$ .**

We shall give here two examples of Markov chains with  $\mathcal{T}_d$  trivial.

**Absolutely regular Markov chains**

For a stationary Markov chain  $\bar{\xi} = (\xi_k)_{k \in \mathbb{Z}}$  with values in a separable Banach space endowed with the Borel sigma algebra  $\mathcal{B}$ , the coefficient of absolute regularity is defined by (see Proposition 3.22 in Bradley, [4])

$$\beta_n = \beta_n(\xi) = \beta(\xi_0, \xi_n) = E \left( \sup_{A \in \mathcal{B}} |\mathbb{P}(\xi_n \in A | \xi_0) - \mathbb{P}(\xi_0 \in A)| \right),$$

where  $\mathcal{B}$  denotes the Borel sigma field.

Equivalently, (see Corollary 3.30 in Bradley [4])

$$\beta_n = \beta_n(\xi) = \beta(\xi_0, \xi_n) = \sup_{C \in \mathcal{B}^2} |P((\xi_0, \xi_n) \in C) - P((\xi_0, \xi_n^*) \in C)|,$$

where  $(\xi_0, \xi_n^*)$  are independent and identically distributed. This coefficient was introduced by Volkonskii and Rozanov [15] and was attributed there to Kolmogorov.

If  $\beta_n \rightarrow 0$ , the Markov chain is called absolutely regular and the tail sigma field  $\mathcal{T}_d$  is trivial (see Subsection 2.5 in Bradley [3]). It follows that both Theorem 2.1 and Theorem 2.2 hold.

Let us mention that there are numerous examples of stationary absolutely regular Markov chains. We know that a strictly stationary, countable state Markov chain is absolutely regular if and only if the chain is irreducible and aperiodic. Also, any strictly stationary Harris recurrent and aperiodic Markov chain is absolutely regular. For easy reference we refer to Section 3 in Bradley [3] survey paper and to the references mentioned there.

In general, the CLT for this class requires the knowledge of the rates of convergence to 0 of the  $(\beta_n)$  coefficients (see for instance Doukhan et al. [9] and Peligrad [12] for a discussion on the CLT under  $\beta_n \rightarrow 0$ ).

**Interlaced mixing Markov chains**

Another example where our results apply is the class of interlaced mixing Markov chains. Let  $\mathcal{A}, \mathcal{B}$  be two sub  $\sigma$ -algebras of  $\mathcal{F}$ . Define the maximal coefficient of correlation

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{f \in \mathbb{L}_0^2(\mathcal{A}), g \in \mathbb{L}_0^2(\mathcal{B})} \frac{|E(fg)|}{\|f\| \cdot \|g\|},$$

where  $\mathbb{L}_0^2(\mathcal{A})$  ( $\mathbb{L}_0^2(\mathcal{B})$ ) is the space of random variables that are  $\mathcal{A}$ -measurable (respectively  $\mathcal{B}$ -measurable), zero mean and square integrable. For a sequence of random variables,  $(\xi_k)_{k \in \mathbb{Z}}$ , we define

$$\rho_n^* = \sup \rho(\sigma(\xi_i, i \in S), \sigma(\xi_j, j \in T)),$$

where the supremum is taken over all pairs of disjoint sets,  $T$  and  $S$  of  $\mathbb{R}$  such that  $\min\{|t - s| : t \in T, s \in S\} \geq n$ . We call the sequence  $\rho^*$ -mixing if  $\rho_n^* \rightarrow 0$  as  $n \rightarrow \infty$ .

The  $\rho^*$ -mixing condition goes back to Stein [14] and to Rosenblatt [13]. It is well-known that  $\rho^*$ -mixing implies that the tail sigma field  $\mathcal{T}_d$  is trivial (see Subsection 2.5 in Bradley [3]). It follows that both Theorem 2.1 and Theorem 2.2 hold. Although these theorems are not new for this class, the results in this paper provide an unified approach for different classes of Markov chains and make possible the treatment of other classes of Markov chains. For further reaching results concerning  $\rho^*$ -mixing sequences see for

instance Theorem 11.18 in Bradley [4] and Corollary 9.16 in Merlevède, Peligrad and Utev [11].

These two classes, absolutely regular and interlaced mixing Markov chains, are of independent interest. There are known examples (see Example 7.16 in Bradley, [4]) of  $\rho^*$ -mixing sequences which are not absolutely regular. On the other hand there are known examples of absolutely regular Markov chains which are not  $\rho^*$ -mixing. An example of reversible, absolutely regular Markov chain, which is not  $\rho^*$ -mixing was constructed by Bradley [6]. Furthermore, if we consider, for instance,  $(\xi_n)_{n \in \mathbb{Z}}$ , and  $(\eta_n)_{n \in \mathbb{Z}}$ , two independent Markov chains, each with trivial two-sided tail sigma field, then the Markov chain  $(\xi_n, \eta_n)_{n \in \mathbb{Z}}$  is again stationary with trivial two-sided tail sigma field (see Theorem A072 page 456, Vol. 1, in [4]).

### 3 Proofs

The proofs of both theorems are based on the following result, which is Theorem 1 in Peligrad [12], combined with Lemma 3.2 below.

For reader's convenience, let us state first the main result in Peligrad [12]. It uses the notion of totally ergodic Markov chain. To explain it, let us consider the operator  $P$  induced by the kernel  $P(x, A)$  on bounded measurable functions on  $(S, \mathcal{A})$  defined by  $Pf(x) = \int_S f(y)P(x, dy)$ . We call  $(\xi_n)_{n \in \mathbb{Z}}$  totally ergodic if and only if the powers  $P^m$  are ergodic with respect to  $\pi$ , for all  $m \in \mathbb{N}$  (i.e.  $P^m f = f$  for  $f$  bounded on  $(S, \mathcal{A})$  implies  $f$  is constant  $\pi$ -a.s.).

**Theorem 3.1.** *Let  $(X_n)_{n \in \mathbb{Z}}$  and  $(S_n)_{n \geq 1}$  be as defined in (2.1),  $(\xi_n)$  is totally ergodic and (2.2) is satisfied. Then, the following limit exists*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|S_n - E(S_n | \xi_0, \xi_n)\|^2 = c^2 \tag{3.1}$$

and

$$\frac{S_n - E(S_n | \xi_0, \xi_n)}{\sqrt{n}} \Rightarrow N(0, c^2) \text{ as } n \rightarrow \infty.$$

Next lemma deals with the random centering in Theorem 3.1.

**Lemma 3.2.** *Let  $(\xi_n)_{n \in \mathbb{Z}}$  be a stationary sequence not necessarily Markov with trivial tail sigma field  $\mathcal{T}_d$ . Let  $(X_n)_n$  and  $(S_n)_n$  as defined in (2.1). Then*

$$E \left| E \left( \frac{S_n}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.2}$$

If in addition we assume that  $(S_n^2/n)_{n \geq 1}$  is uniformly integrable then

$$E \left( E^2 \left( \frac{S_n}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.3}$$

*Proof of Lemma 3.2.* To prove (3.2) it is clear that it is enough to prove that from any subsequence of indexes  $(n')$  convergent to infinity, we can extract one  $(n'')$ , also convergent to infinity, and such that (3.2) holds along  $(n'')$ . Obviously, condition (2.2) implies that  $(S_n/\sqrt{n})$  is tight. Denote by  $\bar{\xi} = (\xi_n)_{n \in \mathbb{Z}}$ . Consider the vector  $W_n = (S_n/\sqrt{n}, \bar{\xi})$  defined in a canonical way on  $R \times S^{\mathbb{Z}}$  with values in  $R \times S^{\mathbb{Z}}$ . Note that  $W_n$  is tight because  $(S_n/\sqrt{n})$  is tight and  $\bar{\xi}$  does not depend on  $n$ . Therefore, from any subsequence  $(n')$  we can extract one  $(n'')$  such that  $W_{n''}$  is convergent in distribution, say  $W_{n''} \Rightarrow W = (L, \bar{\xi}')$ , where  $\bar{\xi}'$  is distributed as  $\bar{\xi}$  and  $S_n/\sqrt{n} \Rightarrow L$ . Because  $R \times S^{\mathbb{Z}}$  is separable, by the Skorohod representation theorem, (see Theorem 6.7 in Billingsley [1]), we can expand the probability space to  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$  and construct on this expanded probability space,

vectors  $\tilde{W}_{n^n} = (\tilde{S}_{n^n}, \tilde{\xi}^{n^n})$  and  $\tilde{W} = (\tilde{L}, \tilde{\xi}')$  such that for each  $n^n$ ,  $\tilde{W}_{n^n}$  is distributed as  $W_{n^n}$ ,  $\tilde{W}$  is distributed as  $W$ , and  $\tilde{W}_{n^n} \rightarrow \tilde{W}$  a.s. Note that for each  $n^n$  we have  $\tilde{\xi}^{n^n} = \tilde{\xi}'$  a.s. and so  $(\tilde{S}_{n^n}, \tilde{\xi}^{n^n}) = (\tilde{S}_{n^n}, \tilde{\xi}')$  a.s. Denote  $\tilde{\mathcal{F}}_n = \sigma(\tilde{\xi}'_k, k \leq n)$  and  $\tilde{\mathcal{F}}^n = \sigma(\tilde{\xi}'_k, k \geq n)$ , completed with the sets of measure 0 and  $\tilde{\mathcal{T}}_d = \cap_{n \geq 1} (\tilde{\mathcal{F}}_{-n} \vee \tilde{\mathcal{F}}^n)$ . Note that the Skorohod representation (see page 71 in Billingsley [1]) starts with the construction of  $(\tilde{L}, \tilde{\xi}')$  in a canonical way on  $R \times S^Z$ , such that the marginals are distributed as  $L$  and  $\tilde{\xi}$ . But because  $\tilde{\xi}$  was also constructed in a canonical way, we have that the tail sigma field  $\tilde{\mathcal{T}}_d$  of  $\tilde{\xi}'$  is also trivial.

To simplify the notation let us re-denote the index  $n^n$  by  $n$ . Clearly,

$$\frac{\tilde{S}_n}{\sqrt{n}} \rightarrow \tilde{L} \text{ a.s. as } n \rightarrow \infty.$$

Now (2.2) implies that  $(\tilde{S}_n/\sqrt{n})$  is uniformly integrable, so we also have

$$\tilde{E} \left| \frac{\tilde{S}_n}{\sqrt{n}} - \tilde{L} \right| \rightarrow 0 \text{ as } n \rightarrow \infty, \tag{3.4}$$

and because  $E(X_1) = 0$ , by the convergence of moments in the weak laws (see Theorem 3.5 in Billingsley [1]), we have that

$$\tilde{E}(\tilde{L}) = 0. \tag{3.5}$$

By the Fatou lemma, we also have that  $\tilde{E}(\tilde{L}^2) < \infty$ .

By stationarity and the triangle inequality, note that for every  $m \in N, m \leq n$ ,

$$\begin{aligned} E \left| E \left( \frac{S_n}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right| &\leq E \left| E \left( \frac{S_n - S_m}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right| + \\ &E \left| E \left( \frac{S_n - S_m}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) - E \left( \frac{S_n}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right| \\ &\leq E \left| E \left( \frac{S_{n-m}}{\sqrt{n}} \middle| \mathcal{F}_{-m} \vee \mathcal{F}^{n-m} \right) \right| + \frac{E|S_m|}{\sqrt{n}}. \end{aligned} \tag{3.6}$$

Because  $W_n$ , and  $\tilde{W}_{n^n}$  have the same distribution,

$$E \left| E \left( \frac{S_{n-m}}{\sqrt{n}} \middle| \mathcal{F}_{-m} \vee \mathcal{F}^{n-m} \right) \right| = \tilde{E} \left| \tilde{E} \left( \frac{\tilde{S}_{n-m}}{\sqrt{n}} \middle| \tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^{n-m} \right) \right|. \tag{3.7}$$

Now we use the following inequality:

$$\begin{aligned} &\tilde{E} \left| \tilde{E} \left( \frac{\tilde{S}_{n-m}}{\sqrt{n}} \middle| \tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^{n-m} \right) \right| \\ &\leq \tilde{E} \left| \tilde{E} \left( \frac{\tilde{S}_{n-m}}{\sqrt{n}} - \tilde{L} \middle| \tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^{n-m} \right) \right| + \tilde{E}|\tilde{E}(\tilde{L} | \tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^{n-m})|. \end{aligned} \tag{3.8}$$

We treat now the first term in the right hand side of (3.8). By the properties of the conditional expectation

$$\tilde{E} \left| \tilde{E} \left( \frac{\tilde{S}_{n-m}}{\sqrt{n}} - \tilde{L} \middle| \tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^{n-m} \right) \right| \leq \tilde{E} \left| \frac{\tilde{S}_{n-m}}{\sqrt{n}} - \tilde{L} \right|. \tag{3.9}$$

Overall, starting from (3.6) combined to (3.7), (3.8) and (3.9), we obtain for  $n, m \in N, m \leq n$ ,

$$E \left| E \left( \frac{S_n}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right| \leq \tilde{E} \left| \frac{\tilde{S}_{n-m}}{\sqrt{n}} - \tilde{L} \right| + \tilde{E} \left| \tilde{E}(\tilde{L} \middle| \tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^{n-m}) \right| + \frac{E|S_m|}{\sqrt{n}}.$$

Therefore, for  $m \in N$  fixed, by letting  $n \rightarrow \infty$ , and by taking into account (3.4) and the fact that  $\tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^{n-m}$  is decreasing in  $n$

$$\limsup_n E \left| E \left( \frac{S_n}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right| \leq \tilde{E} \left| \tilde{E}(\tilde{L} \middle| \cap_{n \geq 1} (\tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^n)) \right|. \tag{3.10}$$

Now, by letting  $m \rightarrow \infty$ , and using the fact that  $\cap_{n \geq 1} (\tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^n)$  is decreasing in  $m$ , we obtain by (3.5) that

$$\begin{aligned} \limsup_n E \left| E \left( \frac{S_n}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right| &\leq \tilde{E} \left| \tilde{E}(\tilde{L} \middle| \cap_{m \geq 1} \cap_{n \geq 1} (\tilde{\mathcal{F}}_{-m} \vee \tilde{\mathcal{F}}^n)) \right| \\ &\leq \tilde{E} \left| \tilde{E}(\tilde{L} \middle| \tilde{\mathcal{T}}_d) \right| = \left| \tilde{E}(\tilde{L}) \right| = 0. \end{aligned} \tag{3.11} \quad \square$$

*Proof of Theorem 2.1.* First of all we mention that, by Proposition 2.12 in the Vol. 1 of Bradley [4], the sequence  $(\xi_k)_{k \in Z}$  is totally ergodic. Since we assumed (2.2), by Theorem 3.1

$$\frac{S_n - E(S_n | \xi_0, \xi_n)}{\sqrt{n}} \Rightarrow N(0, c^2) \text{ as } n \rightarrow \infty.$$

By Lemma 3.2, and by the Markov property

$$E \left| E \left( \frac{S_n}{\sqrt{n}} \middle| \xi_0, \xi_n \right) \right| = \left| E \left( \frac{S_n}{\sqrt{n}} \middle| \mathcal{F}_0 \vee \mathcal{F}^n \right) \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore

$$\frac{S_n}{\sqrt{n}} \Rightarrow N(0, c^2) \text{ as } n \rightarrow \infty.$$

Now, because  $E(S_n^2/n) < \infty$ , it follows that (see Theorem 3.5 in Billingsley [1])

$$\lim_{n \rightarrow \infty} E \frac{|S_n|}{\sqrt{n}} = E|N(0, c^2)| = \frac{c}{\sqrt{2\pi}}.$$

So,  $c$  can be identified as

$$c = \sqrt{2\pi} \lim_{n \rightarrow \infty} E \frac{|S_n|}{\sqrt{n}}. \quad \square$$

*Proof of Theorem 2.2.* Assume (i). Since  $(S_n^2/n)_{n \geq 1}$  is uniformly integrable, it follows that Theorem 2.1 holds. In addition, (see Theorem 3.5 in Billingsley [1])

$$\lim_{n \rightarrow \infty} \frac{E(S_n^2)}{n} = c^2.$$

Because by Theorem 3.1

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|S_n - E(S_n | \xi_0, \xi_n)\|^2 = c^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \|S_n\|^2 - \lim_{n \rightarrow \infty} \frac{1}{n} \|E(S_n | \xi_0, \xi_n)\|^2,$$

we also have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|E(S_n | \xi_0, \xi_n)\|^2 = 0.$$

and so (i) implies (ii). On the other hand, if (ii) holds, by the convergence of moments in the CLT (Theorem 3.6 in Billingsley [1]), we have that  $(S_n^2/n)_{n \geq 1}$  is uniformly integrable.  $\square$

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<sup>4</sup>BS: Bernoulli Society <http://www.bernoulli-society.org/>

<sup>5</sup>Project Euclid: <https://projecteuclid.org/>

<sup>6</sup>IMS Open Access Fund: <https://imstat.org/shop/donation/>