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Evolution of a passive particle in a one-dimensional diffusive environment*

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Abstract

We study the behavior of a tracer particle driven by a one-dimensional fluctuating potential, defined initially as a Brownian motion, and evolving in time according to the heat equation. We obtain two main results. First, in the short time limit, we show that the fluctuations of the particle become Gaussian and sub-diffusive, with dynamical exponent $3/4$. Second, in the long time limit, we show that the particle is trapped by the local minima of the potential and evolves diffusively i.e. with exponent $1/2$.

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1 Introduction

Random walks in dynamical random environments have attracted a lot of attention recently. When the time correlations of the environment decay fast, several homogenization results have been obtained, see [29][7] as well as references therein. These results establish the existence of an asymptotic velocity for the walker (law of large numbers) and normal fluctuations around the average displacement (invariance principle). In the opposite extreme regime, when the environment becomes static, a detailed understanding of the behavior of the walker is available in dimension $d = 1$, see e.g. [33] for a review.

Diffusive environments in dimension $d = 1$ constitute an intermediate case where memory effects are expected to become relevant, since correlations decay with time only as $t^{-1/2}$. Homogenization results are known when the walker drifts away ballistically, and escapes from the correlations of the environment [4][18][8][22][30]. Recently, among other results, a law of large numbers with zero limiting speed was derived in [19] for

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the position of a walker evolving on top of the symmetric simple exclusion process (SSEP), at half filling, i.e. for a particle density equal to 1/2, and for jump distributions on empty sites and occupied sites symmetric to one another. However, the question of the size of the fluctuations in such a case remains largely elusive, and several conflicting conjectures appear in the literature [9][16][26][5][21]. In a particular scaling limit, Gaussian anomalous fluctuations were shown to hold for a walker on the SSEP [23], see also the discussion below.

The aim of this paper is to advance our understanding on the fluctuations of a walker in an unbiased, one-dimensional, diffusive random environment, and to make clear that *different behaviors may be observed depending on the time scales that we look at*, providing insight for the resolution of the above-mentioned seemingly paradoxical conjectures. For this, we introduce a new model where the evolution of the walker can be described in a fair bit of detail: The walker $X = (X_t)_{t\geq 0}$ is driven by a fluctuating potential V in the overdamped regime:

$$\begin{cases} X_0 \;=\; 0, \\ \partial_t X_t \;=\; -\partial_x V(t, X_t) \end{cases} \tag{1.1}$$

where the potential V solves the heat equation

$$\begin{cases} V(0,x) \;=\; B(x), \\ \partial_t V(t,x) \;=\; \partial_{xx} V(t,x) \end{cases} \tag{1.2}$$

and where B is a Brownian motion on $\mathbb{R}$. We now provide some more detailed motivations for the study of this specific model:

Long time behavior in an unbiased diffusive environment — As said above, the SSEP is a popular diffusive environment found in the literature, and it is easy to impose symmetry conditions so that the evolution of the walker is unbiased. To strengthen the analogy with our model, let us denote the SSEP at time $t \geq 0$ and at point $x \in \mathbb{Z}$ by $-\nabla V(t,x)$. Remarkably, the heat equation determines the average evolution of the SSEP:

$$\partial_t \langle \nabla V(t,x) \rangle \;=\; \gamma \Delta \langle \nabla V(t,x) \rangle$$

where $\langle \cdot \rangle$ denotes the average over all possible evolutions for a given initial environment, where Δ denotes the discrete Laplacean on $\mathbb{Z}$, and where γ is the jump rate of the SSEP. To unveil the long time behavior of the walker, it is thus a natural first step to understand its evolution in the averaged environment $-\langle \nabla V(t,x)) \rangle$, a set-up that can be regarded as a kind of mean-field version of the original problem. It is easier to carry out this step in the continuum, as we do in this paper (notice that $-\partial_x V(1,x)$ with V defined by (1.2) is a continuous analogue of the initial condition $-\nabla V(1,x)$ at half filling for the SSEP).

Theorem 2.6 below reveals that the potential V defined in (1.2) imposes potential barriers that will trap the particle: It moves to the deepest local minimum of the potential that becomes reachable thanks to the evolution of the potential. The mechanism at play is thus the same as in Sinai's walk [31], where the walker moves close to deeper and deeper local minima of the potential thanks to its fluctuations. It is thus worth noticing that the observed diffusive behavior is not a consequence of some homogenization but results from the fact that the particle is dragged by a field that evolves itself on diffusive scales. This lack of homogenization is reflected in the fact that the limiting distribution in Theorem 2.6 is non-Gaussian.

In addition, we do conjecture that the limiting diffusive behavior that we exhibit in Theorem 2.6 for the averaged environment yields an approximately correct picture of the behavior in the non-averaged environment, in agreement with numerical observations discussed below.

Random walks in cooling random environment (RWCRE) — The environment V defined in (1.2) is not stationary, a feature shared with the *cooling* environments introduced recently in a series of papers [3, 1, 2]. A RWRCE is a random walk in a time dependent random environment that is piecewise constant and refreshed at deterministic times $(\tau(k))_{k\geq 0}$, with $\tau(k) \to \infty$ as $k \to \infty$, as well as $T_k := \tau(k) - \tau(k-1) \to \infty$ as $k \to \infty$ (for the environment to be properly cooling). Our model features long time correlations and the environment is never properly refreshed but, at a given time $t > 0$, the evolution of the walker up to a time $2t$ depends mainly of the environment in a box of size of order $t^{1/2}$ at most, that will itself be roughly refreshed in a time of order t. Hence, the evolution of the environment is indeed cooling down, with effective refreshing times growing exponentially.

One of the interesting properties of RWCRE is that the law of their fluctuations depends on how fast the sequence of increments $(T_k)_{k\geq 1}$ diverges to infinity. If this divergence is slow enough, the walker admits Gaussian fluctuations after proper renormalisation thanks to homogenization, see [3, Theorem 1.6] for example. If instead the cooling is very fast, i.e. if $(T_k)_{k\geq 1}$ goes quickly enough to infinity, the displacement of the walker in the last time window becomes comparable or larger to its full displacement, and no homogenization occurs. This leads to non-Gaussian fluctuations, see [2, Theorem 2] and examples below this theorem (in particular Example 5 relative to double exponential cooling).

A quick heuristic analysis indicates that our model falls in the case where no homogenization occurs, as our results confirm:

1. For small $t > 0$, there is no trapping and, since the velocity field is of order $t^{-1/4}$, the particle travels a distance of order $t^{3/4}$ in a time of order t, which is of the same order as its full displacement. The fact that we obtain a Gaussian limiting distribution in Theorem 2.2 below does not actually result from homogenization but from choosing a Gaussian white noise for the initial environment.
2. For large $t > 0$, the particle spends most of its time in the vicinity of the local minima of the potential V. As these are separated by a distance of order $t^{1/2}$ and need a time of order t to be destroyed, the particle will travel a distance of order $\sqrt{t}$ between the time t and $2t$, thus again a displacement of the same order as the full displacement. As stressed already, lack of homogenization is reflected in the non-Gaussian limiting distribution.

Short and long time behaviors in a rough environment — Let us introduce a rough environment where different behaviors for the walker can be observed, depending on the time scales under scrutiny. We believe that this model may capture several regimes studied in the literature through specific scaling limits and help understanding the seemingly paradoxical conjectures quoted above. Our model in (1.1)-(1.2) is again obtained from this one by averaging over fluctuations, while preserving its rich behavior.

Let us consider a random potential $\mathcal{V} = (\mathcal{V}(t,x))_{t\geq 0, x\in\mathbb{R}}$ fluctuating with time according to the stochastic heat equation:

$$\begin{cases} \mathcal{V}(0,x) \;=\; B(x), \\ \partial_t \mathcal{V}(t,x) \;=\; \partial_{xx}\mathcal{V}(t,x) + \sqrt{2}\xi(t,x) \end{cases} \tag{1.3}$$

where B is a Brownian motion on $\mathbb{R}$, and where $\xi(t,x)$ is a space-time white noise. We refer to [17] for a gentle introduction to the stochastic heat equation. The process $\mathcal{V}$ is stationary, i.e. $\mathcal{V}(t,\cdot)$ is distributed as $B(\cdot)$ at all times $t \geq 0$, and evolves diffusively in time, i.e. the landscape described by $\mathcal{V}(t,\cdot)$ in a box of size L is refreshed after a time of order L^2. The potential $\mathcal{V}$ solving (1.3) can be obtained as the scaling limit of the height function of diffusive particle processes on the lattice, such as the SSEP, see e.g. Chapter 11 in [24].

We would like to consider a process $X = (X_t)_{t\geq 0}$ solving

$$\begin{cases} X_0 \ = \ 0, \\ \partial_t X_t \ = \ -\partial_x \mathcal{V}(t, X_t). \end{cases} \tag{1.4}$$

Since $\mathcal{V}$ is rough, it is not at all clear that we can make sense of the evolution equation (1.4). Three natural questions arise:

1. Can we find a limiting procedure so that the problem (1.4) admits a unique solution X almost surely?
2. If yes, how does X behave on short time scales?
3. And how does it behave in the long time limit?

While there seems to be no rigorous result on (1.4), a lot more is known if some external random force is added. Let us consider the problem

$$\begin{cases} \tilde{X}_0 \ = \ 0, \\ \mathrm{d}\tilde{X}_t \ = \ -\partial_x \tilde{\mathcal{V}}(t, \tilde{X}_t)\mathrm{d}t + \mathrm{d}\tilde{B}_t, \end{cases}$$

where $\tilde{\mathcal{V}}$ is some rough random field and where $\tilde{B}$ is a Brownian motion independent of $\tilde{\mathcal{V}}$. First, if $\tilde{\mathcal{V}}$ is independent of time, existence and uniqueness of $\tilde{X}$ is guaranteed for various potentials, in particular for $\tilde{\mathcal{V}}(t,x) = B(x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, see [10][20][6][14][15]. Moreover, in this case, the long time behavior of $\tilde{X}$ is analogous to that of Sinai's random walk [31]. When $\tilde{\mathcal{V}}$ depends explicitly on time, existence and uniqueness of $\tilde{X}$ has been shown in [12], provided that $\tilde{\mathcal{V}}$ is α-Hölder continuous with $\alpha > 1/3$ and can be endowed with a proper rough path structure. Theorem 22 in [12] does not apply as such to the field $\mathcal{V}$ defined in (1.3), because the initial condition would need to be α_b-Hölder continuous with $\alpha_b > 1/2$, but this is likely to be only a purely technical issue. In addition, the results in [12] would also guarantee that the drift term $(\tilde{X}_{t+h} - \tilde{X}_t) - (\tilde{B}_{t+h} - \tilde{B}_t)$ has a magnitude of order at most h^γ for h small enough, where γ is nearly equal to $3/4$. See also [11] and [27] for generalizations and further results.

Let us now come back to (1.4). As far as we know, this problem was first addressed in [9] by means of a heuristic fixed point argument. First, the authors conclude that if a process X solves (1.4), it fluctuates sub-diffusively on short time scales: $X_{t+\Delta t} - X_t$ is typically of order $(\Delta t)^{3/4}$ for small Δt. Second, the fluctuations of X become (almost) diffusive on long time scales: X_t is of order $(t \ln t)^{1/2}$ as t grows large.

The validity of these claims was analyzed in [21], by means of numerical simulations and theoretical arguments but, to the best of our knowledge, no rigorous proof has been provided so far. The conclusion of [21] confirms the findings of [9], though the existence of a logarithmic correction in the long time behavior could not be ascertain. Moreover, the analysis in [21] allows to view the process X as the limit of well defined processes. We defer to the Supplementary Material the few steps needed to recast the analysis developed in [21] into the present framework.

The occurence of two distinct behaviors, on short and long time scales, can be attributed to the two following mechanisms. On short time scales, if the velocity field $u = -\partial_x \mathcal{V}$ evolves much faster than the particle X, we can use the approximation

$$X_{t+\Delta t} \ \simeq \ X_t + \int_0^{\Delta t} u(t+s, X_t)\mathrm{d}s, \tag{1.5}$$

that we expect to become exact in the limit $\Delta t \to 0$. Assuming moreover that the fast fluctuations of u in the time interval $[t, t+\Delta t]$ are uncorrelated from X_t, we may further expect that the increments become stationary in the limit $\Delta t \to 0$, and we approximate $\int_0^{\Delta t} u(t+s, X_t)\mathrm{d}s$ by $\int_0^{\Delta t} u(s,0)\mathrm{d}s$. Therefore, since u is Gaussian and $\mathrm{E}(u(s,0)u(s',0)) = (4\pi|s-s'|)^{-1/2}$ for all $s, s' \geq 0$, we arrive at

$$\frac{X_{t+\Delta t} - X_t}{(\Delta t)^{3/4}} \quad \to \quad \mathcal{N}(0, D) \tag{1.6}$$

in law as $\Delta t \to 0$, with $D = 4/3\sqrt{\pi}$, and this result is consistent with the assumption that u evolves much faster than X. It is not very surprising that the exponent $3/4$ in (1.6) is the same as the exponent found in [12] for the drift term discussed above, since both can be obtained at a heuristic level through a similar reasoning. However, we do not expect (1.6) to be valid in the large time limit $\Delta t \to +\infty$, because $\mathcal{V}$ imposes potential barriers that will trap the particle, as already discussed above.

Let us now make the connection with the model (1.1)-(1.2) studied in this paper. In this simpler model, the two mechanisms described above can be clearly exhibited, and the intuitive reasonings made rigorous. In particular, we will make clear how sub-diffusive behavior on an initial short time scale, with dynamical exponenent $3/4$, can co-exist with a diffusive behavior on long time scales. Since the potential V evolves according to the heat equation, it becomes more and more regular as times evolves, and the sub-diffusive behavior (1.6) only persists at $t = 0$. On the other hand, trapping effects become more pronounced as the time grows large, leading to the eventual diffusive behavior of X.

Finally, let us mention that the recent mathematical result in [23] provides a partial and indirect support to the conjecture (1.6). Indeed, the authors of [23] study a random walk $W^n = (W^n_t)_{0\leq t\leq T}$, jumping on $\mathbb{Z}$ at a rate proportional to n, on top of the SSEP with a diffusion constant proportional to n^2. In the limit $n \to \infty$ and in the absence of drift, they derive that $W^n_t/\sqrt{n}$ converges to a sum of two Gaussian processes, with standard deviation at time t proportional to $t^{1/2}$ and $t^{3/4}$ respectively. As we explain in the Supplementary Material, once properly rescaled, the processes W^n converge to the putative process X solving (1.4), but only on a time domain that shrinks to 0 as $n \to \infty$.

Organization of this paper — In Section 2, we define properly the model studied in this paper, and we state our two main results. The first one, Theorem 2.2, deals with the short time behavior of the passive particle, and is shown in Section 4. The second one, Theorem 2.6, deals with its long time behavior, and is shown in Section 6. Some informations on the behavior of the environment are collected in Section 3, and some intermediate results on the behavior of the zeros of the velocity field are gathered in Section 5.

2 Definitions and results

We consider a one dimensional Brownian motion $B = (B(x))_{x\in\mathbb{R}}$ and we define the random potential $V = (V(t,x))_{t\geq 0, x\in\mathbb{R}}$ by

$$\begin{cases} V(0,x) \;=\; B(x) \quad \text{for all } x \in \mathbb{R} \\ \partial_t V(t,x) \;=\; \partial_{xx} V(t,x) \quad \text{for} \quad t > 0,\; x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Almost surely, the potential V is well defined and analytic as a function of $t > 0$ and $x \in \mathbb{R}$. Indeed, let $D = \{t \in \mathbb{C} : \Re(t) > 0\} \times \mathbb{C}$ and let us define the heat kernel as the complex function on D such that

$$(t,x) \;\mapsto\; P_t(x) \;=\; \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}. \tag{2.2}$$

The heat kernel is analytic as a function of the variables t and x, for $(t,x) \in D$. Moreover, almost surely, there exists $C > 0$ such that $|B(x)| \leq C(|x|+1)$, see e.g. [13]. Therefore, we can define a function V on D by

$$V(t,x) \;=\; \int_{\mathbb{R}} P_t(x-y)B(y)\mathrm{d}y,$$

it is analytic as a function of the variables t and x, for $(t,x) \in D$, and it solves (2.1) for $t > 0$ and $x \in \mathbb{R}$.

Let $u = -\partial_x V$ be a velocity field. For all $t > 0$ and $x \in \mathbb{R}$, the representation

$$u(t,x) \;=\; -\int_{\mathbb{R}} \partial_x P_t(x-y)B(y)\mathrm{d}y \;=\; -\int_{\mathbb{R}} P_t(x-y)\mathrm{d}B(y) \tag{2.3}$$

holds. We now introduce the process $X = (X_t)_{t\geq 0}$ that will be our main object of study, see also Fig. 1. We will prove the following proposition in Section 4:

Proposition 2.1. *There exists a unique process $X = (X_t)_{t\geq 0}$ satisfying almost surely*

$$\begin{cases} X_0 \;=\; 0 \\ \partial_t X_t \;=\; u(t,X_t) \quad \text{for} \quad t > 0, \end{cases} \tag{2.4}$$

continuous on $\mathbb{R}_+$ and smooth on $\mathbb{R}_+^*$.

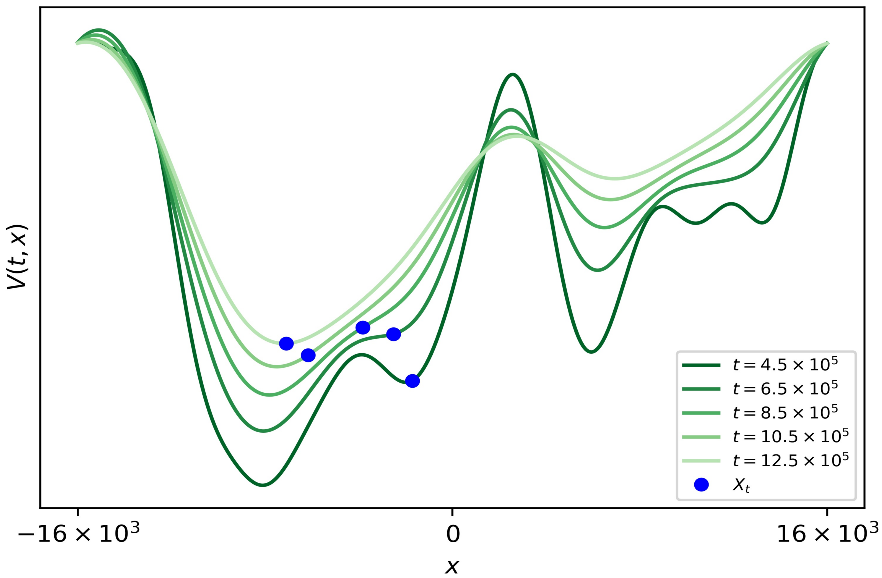

Figure 1: The process X and the potential V at different times. In the long run, the particle X sticks most of the time to a local minimum of the potential V, as made precise in Theorem 2.6. For the numerical simulation used to generate this plot, we have assumed periodic boundary conditions and we have taken the initial condition $X_{t_0} = 0$ with $t_0 = 4 \times 10^5$.

We want to show two results on the behavior of X. The first one characterizes its short time behavior:

Theorem 2.2. *When the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ is endowed with the topology of uniform convergence on compact sets, the following convergence holds:*

$$\left(\frac{X_{\theta T}}{T^{3/4}}\right)_{\theta\geq 0} \xrightarrow[T\to 0]{(law)} \left(\int_0^{\theta} u(s,0)\mathrm{d}s\right)_{\theta\geq 0}. \tag{2.5}$$

We need to introduce some preliminary material to formulate our second result, dealing with the long time behavior of X. Some of the objects are illustrated on Fig. 2. Given $t > 0$, let us define the set of zeros of the field u at time t:

$$\mathcal{Z}_t \;=\; \{x \in \mathbb{R} : u(t,x) = 0\}\,.$$

Let us distinguish attractive, or stable zeros, from repulsive, or unstable ones:

$$\begin{cases} \mathcal{Z}_t^{\mathrm{s}} \;=\; \{x \in \mathbb{R} : u(t,x) = 0, \partial_x u(t,x) < 0\}\,, \\ \mathcal{Z}_t^{\mathrm{u}} \;=\; \{x \in \mathbb{R} : u(t,x) = 0, \partial_x u(t,x) > 0\}. \end{cases}$$

We may also observe zeros that are neither stable nor unstable, say neutral:

$$\mathcal{Z}_t^{\mathrm{n}} \;=\; \{x \in \mathbb{R} : u(t,x) = 0, \partial_x u(t,x) = 0\}.$$

The next lemma allows to “trace back” a zero at time t up to time 0:

Lemma 2.3. *Almost surely, for all $t > 0$ and all $x \in \mathcal{Z}_t^{\mathrm{s}} \cup \mathcal{Z}_t^{\mathrm{u}}$, there exists a unique continuous function $r_{(t,x)} : [0,t] \to \mathbb{R}$ such that for all $0 < s \le t$,*

$$u(s, r_{(t,x)}(s)) = 0$$

and actually $r_{(t,x)}(s) \in \mathcal{Z}_s^{\mathrm{s}}$ if $x \in \mathcal{Z}_s^{\mathrm{s}}$ and $r_{(t,x)}(s) \in \mathcal{Z}_s^{\mathrm{u}}$ if $x \in \mathcal{Z}_s^{\mathrm{u}}$ (and thus in particular $\partial_x u(s, r_{(t,x)}(s)) \neq 0$). The function $r_{(t,x)}$ is smooth on $]0,t[$ and for all $0 < s \le t$,

$$\partial_s r_{(t,x)}(s) \;=\; -\frac{\partial_s u(s, r_{(t,x)}(s))}{\partial_x u(s, r_{(t,x)}(s))} \;=\; -\frac{\partial_{xx} u(s, r_{(t,x)}(s))}{\partial_x u(s, r_{(t,x)}(s))}. \tag{2.6}$$

Once properly rescaled, the long time behavior of X is described by the limiting process $Z = (Z_t)_{t\ge 0}$ introduced in the following proposition:

Proposition 2.4. *There exist unique processes $L = (L_t)_{t\ge 0}$ and $R = (R_t)_{t\ge 0}$ such that $L_0 = R_0 = 0$ and, almost surely, for all $t > 0$,*

$$\begin{cases} L_t \;=\; \max\{x \in \mathcal{Z}_t^{\mathrm{s}} \cup \mathcal{Z}_t^{\mathrm{u}} : r_{(t,x)}(0) < 0\}, \\ R_t \;=\; \min\{x \in \mathcal{Z}_t^{\mathrm{s}} \cup \mathcal{Z}_t^{\mathrm{u}} : r_{(t,x)}(0) > 0\}. \end{cases} \tag{2.7}$$

Moreover, almost surely, for all $t > 0$, one and only one of the following events occurs

$$\big(L_t \in \mathcal{Z}_t^{\mathrm{s}} \quad \text{and} \quad R_t \in \mathcal{Z}_t^{\mathrm{u}}\big) \qquad \text{or} \qquad \big(L_t \in \mathcal{Z}_t^{\mathrm{u}} \quad \text{and} \quad R_t \in \mathcal{Z}_t^{\mathrm{s}}\big). \tag{2.8}$$

We can thus define a process $Z = (Z_t)_{t\ge 0}$ by $Z_0 = 0$ and

$$\begin{cases} Z_t \;=\; L_t \quad \text{if} \quad L_t \in \mathcal{Z}_t^{\mathrm{s}} \\ Z_t \;=\; R_t \quad \text{if} \quad R_t \in \mathcal{Z}_t^{\mathrm{s}} \end{cases}$$

for $t > 0$. The following properties of Z hold:

1. *Almost surely, Z is càdlàg.*
2. *Almost surely, Z is discontinuous at some time $t > 0$ if and only if $Z_{t-} \in \mathcal{Z}_t^{\mathrm{n}}$.*
3. *Almost surely, for any compact interval $I \subset \mathbb{R}_+^*$, the number of discontinuities of $(Z_t)_{t\in I}$ is finite and $(Z_t)_{t\in I}$ is smooth away from the jumps.*
4. *$(Z_t)_{t\ge 0} = (T^{-\frac{1}{2}} Z_{Tt})_{t\ge 0}$ in law for all $T > 0$.*
5. *The variable Z_1 has a bounded density and there exists $c > 0$ such that, for all $z \ge 0$,*

$$c\,\mathrm{e}^{-z/c} \;\le\; \mathrm{P}(|Z_1| \ge z) \;\le\; \frac{1}{c}\,\mathrm{e}^{-cz}.$$

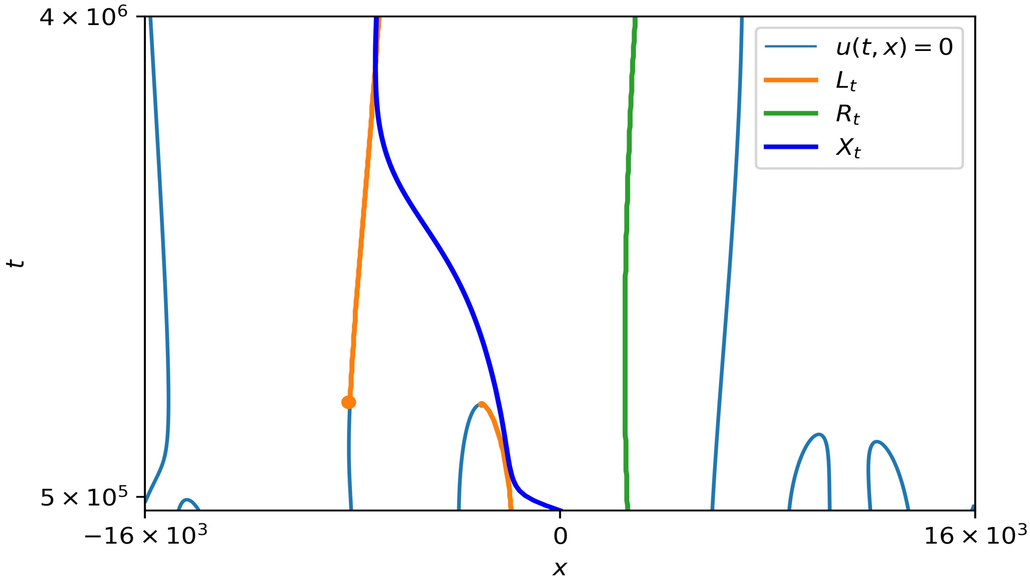

Figure 2: The processes L, R and X, as well as the zeros of the velocity field u. The realization of the environment is the same as on Fig. 1.

Remark 2.5. By a straightforward adaptation of the proof, items 1 to 5 above can be shown to hold as well with L or R in place of Z.

We now come to our result on the long time behavior of X:

Theorem 2.6. *When the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}$ is endowed with the Skorokhod's $\mathcal{M}_1$ topology for the convergence on compact sets, the following convergence holds:*

$$\frac{1}{T^{1/2}} \left(X_{\theta T} - Z_{\theta T}\right)_{\theta \geq 0} \xrightarrow[T \to +\infty]{(probability)} 0 \tag{2.9}$$

and thus in particular, by the scaling relation in item 4. in Proposition 2.4,

$$\left(\frac{X_{\theta T}}{T^{1/2}}\right)_{\theta \geq 0} \xrightarrow[T \to +\infty]{(law)} (Z_\theta)_{\theta \geq 0}. \tag{2.10}$$

Remark 2.7. Since the process $(X_{\theta T})_{\theta \geq 0}$ is continuous and the process $(Z_{\theta T})_{\theta \geq 0}$ has jumps, it is not possible to obtain the convergence in the Skorokhod's $\mathcal{J}$ topology. Let us remind the definition of the $\mathcal{M}_1$ topology, see [32]. Let $D([0,1], \mathbb{R})$ be the space of real càdlàg functions on $[0,1]$. For $f \in D([0,1], \mathbb{R})$, the completed graph of f is defined as

$$\mathcal{G}(f) \;=\; \{(t,x) \in [0,1] \times \mathbb{R} : x \in [f(t-) \wedge f(t), f(t-) \vee f(t)]\}$$

with $f(t-) = \lim_{s \to t, s<t} f(s)$. We define an order on $G(f)$ as follows

$$(t,x) \leq (s,y) \quad \Leftrightarrow \quad t < s \quad \text{or} \quad t = s \text{ and } |x - f(t-)| \leq |y - f(s-)|.$$

A parametrization of $\mathcal{G}(f)$ is defined to be a continuous map $\varphi : [0,1] \to \mathcal{G}(f)$ such that $\varphi(0) = (0, f(0))$, $\varphi(1) = (1, f(1))$ and φ is non-decreasing for the above order. The set of parametrizations of $\mathcal{G}(f)$ is denoted $\Pi(f)$. The $\mathcal{M}_1$ distance between two elements of $D([0,1], \mathbb{R})$ is defined as

$$d_{\mathcal{M}_1}(f,g) \;=\; \inf_{\varphi \in \Pi(f), \psi \in \Pi(g)} \left\{ \sup_{0 \leq \tau \leq 1} |\varphi(\tau) - \psi(\tau)|_\infty \right\}$$

where $|(t,x)|_\infty = \max\{|t|,|x|\}$. The definition is completely analogous on any other compact interval of $\mathbb{R}$.

Remark 2.8. Our two theorems could be generalized in several directions. First, it would be interesting to consider the same set-up in dimension $d > 1$, where the velocity field needs no longer to derive from a potential V; rather we could still consider that $(u(0,x))_{x\in\mathbb{R}^d}$ is a spatial white noise. In this case, the velocity at time t will be typically of order $t^{-d/4}$, and increasing the dimension will have rather opposite effects in the short and long times regimes. When $t \to 0$, the particle will move faster than in $d = 1$ and it is actually not clear that $(X_t)_{t\in I}$ will be well-defined on a compact interval I containing 0. For large times instead, the particle will move slower. So slowly that, from $d \geq 3$, we may actually expect that it will stay bounded and reach a finite limit as $t \to \infty$. This is consistent with the general belief that trapping effects cease to be relevant for $d \geq 3$.

Second, it would be possible to consider an initial velocity filed that is not a white noise, and reach presumably similar conclusions as long as $(u(0,x))_{x\in\mathbb{R}}$ satisfies some basic requirements, such as having zero average and being short-range correlated. Our proof relies however on the environment being Gaussian, and dropping this hypothesis would introduce technical difficulties. Last but not least, it would be highly interesting to reintroduce at least some fluctuations in the evolution of the velocity field, a task that would certainly require new ideas. We leave all these questions to further investigations.

3 Description of the environment

We establish here several features of the environment u, that will be used throughout this text. We first show the scaling property (3.1) below that will, among other things, play a key role in establishing Theorem 2.6. Second, we construct a grid of space-time points such that u keeps the same sign on some time interval around each of these points, see Proposition 3.1 below as well as the subsequent constructions. This grid allows to derive a priori bounds on the processes X, L, R and Z, that depend only on the sign of the velocity field u. Third, we obtain estimates on the supremum of u and its derivatives, see Lemma 3.4 below. These estimates will be mainly needed in the proof of Proposition 2.1.

Scaling property. For any $\alpha > 0$,

$$(u(t,x))_{t,x} \stackrel{(law)}{=} (\alpha^{1/4}u(\alpha t, \alpha^{1/2}x))_{t,x}. \tag{3.1}$$

Indeed, both fields are Gaussian, centred and have the same covariance and, from the representation of the field u in (2.3), we compute

$$\mathrm{E}(u(t,x)u(s,y)) = \int_{\mathbb{R}} P_t(x-z)P_s(z-y)\mathrm{d}z = P_{t+s}(x-y), \tag{3.2}$$

and $\alpha^{1/2}P_{\alpha(t+s)}(\alpha^{1/2}(x-y)) = P_{t+s}(x-y)$ from (2.2).

Sign of the field u. Given $\ell > 0$, let us define the event

$$D(\ell) = \{\exists y : |y| \leq \ell/2 \text{ and } \forall s \in [1/2,1], u(s,y) > 0\}. \tag{3.3}$$

Let us denote the complement of an event A by $\overline{A}$. The following proposition provides a control on the probability of $\overline{D(\ell)}$:

Proposition 3.1. *There exists $C > 0$ such that, for all $\ell > 0$,*

$$\mathrm{P}\left(\overline{D(\ell)}\right) \leq \frac{1}{C}\mathrm{e}^{-C\ell}.$$

Remark 3.2. By symmetry of u, by translation invariance and by the scaling property (3.1), we deduce from the above proposition that there exists $C>0$ such that, for any $x\in\mathbb{R}$ and any $t>0$,

$$\mathrm{P}\left(\overline{\{\exists y:|y-x|\le \ell\sqrt{t}/2 \text{ and } \forall s\in[t/2,t], \pm u(s,y)>0\}}\right) \;\le\; \frac{1}{C}\mathrm{e}^{-C\ell}.$$

Let us first provide some roadmap for the proof of Proposition 3.1. It is rather straightforward to see that, for any $\ell>0$, the event $D(\ell)$ has positive probability, cf. the proof of (3.11) below. The difficulty in establishing that $P(\overline{D(\ell)})$ decays exponentially with ℓ stems from the fact that the environments u at two different places never become fully independent, even though correlations decay faster than exponentially with the distance. To cope with this, we introduce a field $\tilde{u}$, that differs from u by the fact that the integral over $\mathbb{R}$ in (2.3) is replaced by an integral over a finite box of length $\ell_\star$, cf. (3.5) below. Therefore, the environments $\tilde{u}$ become truly independent over distances larger than $\ell_\star$. In addition, provided $\ell_\star$ is taken large enough, we can find an explicit condition on the initial environment so that $\tilde{u}$ is a good approximation of u, cf. (3.6) below. Finally and crucially, among $2N+1$ consecutive boxes of size $\ell_\star$, at least N of them will be such that $\tilde{u}$ evaluated in the middle of these boxes will be a good approximation of u evaluated at the same points, with a probability that goes exponentially fast to 1 as N grows large, cf. (3.7) below. The conclusion is obtained by combining this exponential bound with the exponential decay stemming from the independence of the variables $\tilde{u}$ at distant locations.

Proof of Proposition 3.1. We divide the proof into several steps.

1. Given a compact interval $I\subset\mathbb{R}$ and some $\alpha>0$,

$$\mathrm{P}\left(\sup\left\{\int_I \varphi(x)\mathrm{d}B(x)\;:\;\varphi\in\mathcal{C}^1(I), \|\varphi\|_{\mathcal{C}^1}\le 1\right\}>\alpha\right)\;\le\;\frac{1+2|I|^{3/2}}{\alpha} \tag{3.4}$$

with $\|\varphi\|_{\mathcal{C}^1}=\max_{x\in I}\{|\varphi(x)|+|\varphi'(x)|\}$.

Indeed, let $I=[a,b]$ be some compact interval, and let $\varphi\in\mathcal{C}^1(I)$. An integration by parts yields

$$\int_I\varphi(x)\mathrm{d}B(x)\;=\;\varphi(b)(B(b)-B(a))-\int_I\varphi'(x)(B(x)-B(a))\mathrm{d}x.$$

Hence,

$$\sup_{\varphi:\|\varphi\|_{\mathcal{C}^1}\le 1}\int_I\varphi(x)\mathrm{d}B(x)\;\le\;|B(b)-B(a)|+\int_I|B(x)-B(a)|\mathrm{d}x$$

and, by Markov inequality, for any $\alpha>0$,

$$\begin{aligned}\mathrm{P}\left(\sup_{\varphi:\|\varphi\|_{\mathcal{C}^1}\le 1}\int_I\varphi(x)\mathrm{d}B(x)\;>\;\alpha\right)\;&\le\;\frac{1}{\alpha}\mathrm{E}\left(|B(b)-B(a)|+\int_I|B(x)-B(a)|\mathrm{d}x\right)\\ &\le\;\frac{1+2(b-a)^{3/2}}{\alpha}.\end{aligned}$$

2. We introduce some definitions and notations. Let $\ell_\star\ge 1$. For $k\in\mathbb{Z}$, we define the points $x_k=k\ell_\star$ and the intervals

$$I_k\;=\;[x_k-\ell_\star/2,x_k+\ell_\star/2]\;,$$

as well as the variables

$$h_k\;=\;\frac{1}{\ell_\star^{3/2}}\sup\left\{\int_{I_k}\varphi(x)\mathrm{d}B(x)\;:\;\varphi\in\mathcal{C}^1(I_k),\|\varphi\|_{\mathcal{C}^1}\le 1\right\}.$$

Let also $n_0 \in \mathbb{N}^*$ and let us define the variables

$$H_k \;=\; 0 \vee \left\lfloor \frac{\ln h_k}{n_0} \right\rfloor .$$

We observe that, by (3.4), $\mathrm{P}(H_k = 0)$ goes to 1 as n_0 goes to infinity, uniformly in $\ell_\star$.

3. Given $k \in \mathbb{Z}$ and $t > 0$, let us define

$$\tilde{u}(t, x_k) \;=\; -\int_{I_k} P_t(x_k - y)\mathrm{d}B(y). \tag{3.5}$$

We want to control the difference between $\tilde{u}(t,x_k)$ and $u(t,x_k)$. For this, let us introduce the variables

$$\eta_k \;=\; \begin{cases} 1 & \text{if } \; H_j \le |k-j| \quad \forall j \in \mathbb{Z}, \\ 0 & \text{otherwise} \end{cases}$$

We claim that, given n_0, for all $\ell_\star$ large enough, and for all k such that $\eta_k = 1$,

$$\sup_{1/2 \le t \le 1} |u(t,x_k) - \tilde{u}(t,x_k)| < 1. \tag{3.6}$$

Let us show (3.6). By translation invariance, it suffices to consider the case $k = 0$. For all $t > 0$,

$$u(t,0) - \tilde{u}(t,0) \;=\; -\sum_{j \in \mathbb{Z}\setminus\{0\}} \int_{I_j} P_t(y)\mathrm{d}B(y).$$

Since $\eta_0 = 1$, it holds that $h_j \le \mathrm{e}^{|j|n_0}$ for all $j \in \mathbb{Z}\setminus\{0\}$. Moreover, there exist $C, c > 0$ such that $\| - P_t(\cdot)\|_{\mathcal{C}^1(I_j)} \le C\mathrm{e}^{-c|j|^2\ell_\star^2}$ for all $j \in \mathbb{Z}$, so that we finally obtain

$$\sup_{1/2 \le t \le 1} |u(t,0) - \tilde{u}(t,0)| \;\le\; \sum_{j \in \mathbb{Z}\setminus\{0\}} C\ell_\star^{3/2}\mathrm{e}^{-c|j|^2\ell_\star^2}\mathrm{e}^{|j|n_0},$$

and this becomes smaller than 1 for $\ell_\star$ large enough.

4. For all n_0 large enough, for all $\ell_\star \ge 1$, and for all $N \in \mathbb{N}$,

$$\mathrm{P}(|\eta^N|_1 < N) \;\le\; \mathrm{e}^{-N}, \tag{3.7}$$

where $\eta^N = (\eta_{-N}, \dots, \eta_N)$ and $|\eta^N|_1 = \sum_{k=-N}^{N} \eta_k$.

Indeed, on the event

$$E_N \;=\; \{H_j \le |\,|j| - N\,| \; \text{ for all } |j| > 2N\},$$

it holds that

$$\{k \in \mathbb{Z} : |k| \le N, \eta_k = 0\} \;\subset\; \bigcup_{-2N \le j \le 2N} \{j - (H_j - 1), \cdots, j + (H_j - 1)\}$$

with the convention $\{a, \cdots, b\} = \emptyset$ if $b < a$. Therefore

$$|\{k \in \mathbb{Z} : |k| \le N, \eta_k = 0\}| \;\le\; \sum_{-2N \le j \le 2N} [(2H_j - 1) \vee 0] \;\le\; \sum_{-2N \le j \le 2N} 2H_j.$$

We thus obtain

$$\mathrm{P}(|\eta^N|_1 < N) \;\le\; \mathrm{P}(\sum_{-2N \le j \le 2N} 2H_j > N+1) + \mathrm{P}(\overline{E_N}). \tag{3.8}$$

For the second term, since by (3.4), for any $n \in \mathbb{N}$, $\mathrm{P}(H_j \geq n) \leq 3\mathrm{e}^{-n_0 n}$ it holds that for n_0 large enough and all $N \geq 1$,

$$\mathrm{P}(\overline{E_N}) \;\leq\; \sum_{|j|>2N} \mathrm{P}(H_j > ||j|-N|) \;\leq\; \frac{1}{2}\mathrm{e}^{-N}. \tag{3.9}$$

For the first one, as the variables $(H_j)_{j\in\mathbb{Z}}$ are i.i.d., we obtain

$$\mathrm{P}(\sum_{-2N\leq j\leq 2N} 2H_j > N+1) \;\leq\; \mathrm{e}^{-2(N+1)}\mathrm{E}\left(\mathrm{e}^{4H_0}\right)^{4N+1}. \tag{3.10}$$

Finally, using once again that for any $n \in \mathbb{N}$, $\mathrm{P}(H_j \geq n) \leq 3\mathrm{e}^{-n_0 n}$, one can choose n_0 large enough so that for all $N \geq 1$, the last term in (3.10) is smaller that $\mathrm{e}^{-N}/2$. Inserting (3.9) and (3.10) into (3.8) yields the result.

5. There exists $p > 0$ such that, for all $\ell_\star \geq 1$ and for all $k \in \mathbb{Z}$,

$$\mathrm{P}\left(\inf_{1/2\leq t\leq 1} \tilde{u}(t, x_k) \;>\; 1\right) \;=\; p. \tag{3.11}$$

Again, to show this, it suffices to consider the case $k = 0$. We observe that, almost surely, $\inf_{1/2\leq t\leq 1} \tilde{u}(t, 0)$ is a continuous function of $\ell_\star$, converging to $\inf_{1/2\leq t\leq 1} u(t, 0)$ as $\ell_\star \to \infty$. Therefore, it is enough to establish the result for any fixed $\ell_\star \geq 1$ and for u instead of $\tilde{u}$. This last case can be handled with exactly the same proof, and we let $\ell_\star \geq 1$.

We first prove that for any $0 < a < b$, it holds that

$$\begin{aligned}&\mathrm{P}\left(\inf_{t\in[a,b]} \tilde{u}(t,0) \;\geq\; 1/2\right)\\ &\qquad \geq\; \mathrm{P}\left(\inf_{t\in[a,(a+b)/2]} \tilde{u}(t,0) \;\geq\; 1/2\right)\mathrm{P}\left(\inf_{t\in[(a+b)/2,b]} \tilde{u}(t,0) \;\geq\; 1/2\right).\end{aligned} \tag{3.12}$$

Indeed for all $n \geq 1$, the random vector $(u(t,0))_{t\in[a,b]\cap\mathbb{Z}/2^n}$ is gaussian and its coordinates are positively correlated as can be seen from the equivalent of (3.2) for $\tilde{u}$ so that, using [28],

$$\begin{aligned}&\mathrm{P}\left(\inf_{t\in[a,b]\cap\mathbb{Z}/2^n} \tilde{u}(t,0) \;\geq\; 1/2\right)\\ &\qquad \geq\; \mathrm{P}\left(\inf_{t\in[a,(a+b)/2]\cap\mathbb{Z}/2^n} \tilde{u}(t,0) \;\geq\; 1/2\right)\mathrm{P}\left(\inf_{t\in[(a+b)/2,b]\cap\mathbb{Z}/2^n} \tilde{u}(t,0) \;\geq\; 1/2\right).\end{aligned}$$

From this, we deduce (3.12) using that u is continuous.

Let us now assume that $\mathrm{P}\left(\inf_{1/2\leq t\leq 1} \tilde{u}(t,0) \geq 1/2\right) = 0$. From (3.12) we find a sequence of nested closed intervals $(I_n)_{n\geq 1}$ with $I_1 = [1/2, 1]$ and $|I_n| = 2^{-n}$ such that, for all $n \geq 1$, $\mathrm{P}\left(\inf_{t\in I_n} \tilde{u}(t,0) \geq 1/2\right) = 0$. Let $t_0 \in \cap_{n\geq 1} I_n$. Since $\tilde{u}(\cdot, 0)$ is continuous almost surely in t_0,

$$\{\tilde{u}(t_0,0) \geq 1/2\} \overset{a.s.}{=} \bigcup_{n\geq 1}\{\tilde{u}(t,0) \geq 1/2, t \in I_n\},$$

and thus $\mathrm{P}(\tilde{u}(t_0,0) \geq 1/2) = 0$. This is a contradiction.

6. We start now the proof of the proposition itself. Let p be the constant featuring in (3.11), let n_0 be large enough so that, for all $\ell_\star \geq 1$, $\mathrm{P}(H_0) \geq 1-p/2$ and such that (3.7) holds, and finally let $\ell_\star$ be large enough so that (3.6) holds.

Let $N \in \mathbb{N}^*$, and let $\ell = (2N+1)\ell_\star$. Since the events $D(\ell)$ are increasing with ℓ, it suffices to show the proposition for ℓ of this type. We start with

$$\begin{aligned} \mathrm{P}\left(\overline{D(\ell)}\right) \;&\leq\; \mathrm{P}\left(\inf_{1/2\leq t\leq 1} u(t,x) \;\leq\; 0,\ \forall x \in [-\ell/2, \ell/2]\right) \\ &\leq\; \mathrm{P}\left(\bigcap_{-N\leq i\leq N} \left\{\inf_{1/2\leq t\leq 1} u(t,x_i) \;\leq\; 0\right\}\right). \end{aligned} \tag{3.13}$$

Let us denote by A the event featuring in the right hand side of this last expression. From (3.7), we obtain

$$\mathrm{P}(A) \;\leq \sum_{\substack{\sigma\in\{0,1\}^{2N+1},\\ |\sigma|_1\geq N}} \mathrm{P}(A \mid \eta^N = \sigma)\mathrm{P}(\eta^N = \sigma) \;+\; \mathrm{e}^{-N}. \tag{3.14}$$

Moreover from (3.6), we deduce that for all $i \in \mathbb{Z}$,

$$\left\{\inf_{1/2\leq t\leq 1} u(t,x_i) \;\leq\; 0\right\} \cap \{\eta_i = 1\} \;\subset\; \left\{\inf_{1/2\leq t\leq 1} \tilde{u}(t,x_i) \;\leq\; 1\right\}.$$

Given σ such that $|\sigma|_1 \geq N$, we define $-N \leq i_1 < \cdots < i_N \leq N$ to be the N distinct smallest indexes such that for all $1 \leq j \leq N$, $\sigma_{i_j} = 1$. We denote by J the complementary set of the $(i_k)_{1\leq k\leq N}$ in $\mathbb{Z}$. The event $\{\eta^N = \sigma\}$ can be written as

$$\left(\bigcap_{1\leq k\leq N} \{H_{i_k} = 0\}\right) \cap \{(H_j)_{j\in J} \in B\}$$

with B some suitable event in $\mathbb{N}^J$. Therefore, as the $(H_j)_{j\in\mathbb{Z}}$ are i.i.d., we obtain

$$\mathrm{P}(A|\eta^N = \sigma) \;\leq\; \prod_{k=1}^{N} \mathrm{P}\left(\inf_{1/2\leq t\leq 1} \tilde{u}(t,x_{i_k}) \;\leq\; 1 \middle| H_{i_k} = 0\right) \;\leq\; \left(\frac{1-p}{1-p/2}\right)^N,$$

and the proof follows by inserting this bound into (3.14). □

In the following we make use of Proposition 3.1 to give a property of the environment that we will use repeatedly till the end of the article. Let $K \geq 1$ be a constant that will be fixed below. Given $k \geq 0$ and $\alpha \geq 1$, we define a finite family of space-time boxes covering $B(k,\alpha) = [0, 2^k] \times \left[-K\alpha\sqrt{2^k}, +K\alpha\sqrt{2^k}\right]$ in the following way: For all $n \geq 0$, we define

$$t_n(k) = 2^{k-n} \quad \text{and} \quad \ell_n(k,\alpha) = (\alpha + n^2)\sqrt{t_n(k)},$$

and also the space intervals

$$I_{n,j}(k,\alpha) = [j\ell_n(k,\alpha), (j+1)\ell_n(k,\alpha)], \quad j \in \mathbb{Z}.$$

We denote by $J_n(k,\alpha)$ the set of j such that $I_{n,j}(k,\alpha)$ intersects $[-K\alpha\sqrt{2^k}, K\alpha\sqrt{2^k}]$, so that $B(k,\alpha)$ is covered by the family of boxes $[t_{n+1}(k), t_n(k)] \times I_{n,j}(k,\alpha)$, for $n \geq 0$ and $j \in J_n(k,\alpha)$.

From now on we fix K large enough so that for all $\alpha \geq 1$ and $k \geq 0$,

$$\sum_{n\geq 0} 2\ell_n(k,\alpha) \;\leq\; \frac{K}{3}\alpha\sqrt{t_0(k)}. \tag{3.15}$$

The reason for defining K in this way will become clear later. For all $n \geq 0$ and $j \in J_n(k,\alpha)$, we consider the event

$$E_{n,j}(k,\alpha) \;=\; \{\exists y_1, y_2 \in I_{n,j}(k,\alpha) :\; \forall s \in [t_{n+1}(k), t_n(k)], u(s,y_1) > 0 \text{ and } u(s,y_2) < 0\}.$$

and we define

$$G(k,\alpha) \;=\; \bigcap_{n\geq 0, j\in J_n(k,\alpha)} E_{n,j}(k,\alpha). \tag{3.16}$$

We note that, uniformly in α and k,

$$\lfloor (2K)\ 2^{n/2}/(1+n^2) \rfloor \;\leq\; |J_n(k,\alpha)| \;\leq\; (2K)\ 2^{n/2}.$$

Hence, by Remark 3.2,

$$\mathrm{P}\left(\overline{G(k,\alpha)}\right) \;\leq\; \sum_{n\geq 0} (2K) 2^{n/2} \frac{1}{C} \mathrm{e}^{-C(\alpha+n^2)} \tag{3.17}$$

and we deduce that

$$\text{almost surely for all } k \geq 1 \text{ there exists } \alpha_k \geq 1 \text{ so that } G(k,\alpha_k) \text{ occurs.} \tag{3.18}$$

Actually Borel Cantelli lemma implies even that for all $k \geq 1$ almost surely $G(k,\alpha)$ occurs for α large enough. Property (3.18) will be useful in many of the following proofs. A first consequence is the following

Remark 3.3. An easy consequence of (3.18) is that almost surely, for any $i \in \mathbb{Z}$, the set $P_i = \{y \in \mathbb{R} : \forall s \in [2^{i-1}, 2^i], u(s,y) > 0\}$ is (infinite and) not bounded. Indeed almost surely, for all $k \geq i$ there are at least $|J_{k-i}(k,\alpha_k)|$ points in P_i separated by a distance at least $\sqrt{2^i}$. As $|J_{k-i}(k,\alpha_k)|$ goes to infinity when $k \to \infty$, this yields the result. The same result holds of course for the set $N_i = \{y \in \mathbb{R} : \forall s \in [2^{i-1}, 2^i], u(s,y) < 0\}$.

Expected size of u and its derivatives. We prove here some quantitative estimates on the field u and its derivatives.

Lemma 3.4. *For any $\delta > 0$, there exists $C > 0$ such that*

$$\mathrm{E}\left(\sup\left\{t^{\frac{1}{4}+\delta} u(t,x) \;:\; t \in]0,1], x \in [-1,1]\right\}\right) \;\leq\; C, \tag{3.19}$$

$$\mathrm{E}\left(\sup\left\{t^{\frac{3}{4}+\delta} \partial_x u(t,x) \;:\; t \in]0,1], x \in [-1,1]\right\}\right) \;\leq\; C, \tag{3.20}$$

$$\mathrm{E}\left(\sup\left\{t^{\frac{5}{4}+\delta} \partial_t u(t,x) \;:\; t \in]0,1], x \in [-1,1]\right\}\right) \;\leq\; C. \tag{3.21}$$

As the field $u \overset{(law)}{=} -u$, similar estimates hold for the infimum instead of the supremum.

Remark 3.5. We stress that the result is false if $\delta = 0$ as these suprema are infinite almost surely in this case. Indeed, let us consider for example the field u. First, for all $t > 0$ and $x \in \mathbb{R}$, the variables $t^{1/4} u(t,x)$ are identically distributed joint Gaussian variables. Second, given any $n \geq 1$ and points $x_1, \ldots, x_n$ with $-1 \leq x_1 < \cdots < x_n \leq 1$, the Gaussian vector $(t^{1/2} u(t,x_1), \ldots, t^{1/2} u(t,x_n))$ becomes uncorrelated as $t \to 0$. From this, one concludes that $\sup_{t\in]0,1],x\in[-1,1]} t^{1/4} u(t,x) = +\infty$ almost surely.

Proof. Before starting the proof, we remind that we have already obtained the expression $\mathrm{E}(u(t,x)u(s,y)) = P_{t+s}(x-y)$ in (3.2). Analogously, we derive

$$\partial_x u(t,x) = -\partial_x \int_{\mathbb{R}} P_t(x-y) \mathrm{d}B_y = -\int_{\mathbb{R}} P_t'(x-y) \mathrm{d}B_y$$

and thus

$$\begin{aligned}\mathrm{E}(\partial_x u(t,x)\partial_x u(s,y)) \;&=\; \int_{\mathbb{R}} P_t'(x-z)P_s'(y-z)\mathrm{d}z \;=\; \partial_x\partial_y P_{t+s}(x-y)\\ &=\; -\partial_{xx}P_{t+s}(x-y), \end{aligned} \tag{3.22}$$

as well as

$$\mathrm{E}(\partial_t u(t,x)\partial_t u(s,y)) = \partial_x^4 P_{t+s}(x-y).$$

The Lemma follows from Dudley's theorem, see e.g. [25]. Let us show (3.19). We define a metric d on $]0,1]\times[-1,1]$ by

$$d((t,x),(s,y)) \;=\; \left(\mathrm{E}\left(t^{\frac14+\delta}u(t,x) - s^{\frac14+\delta}u(s,y)\right)^2\right)^{\frac12}. \tag{3.23}$$

Given $\eta>0$, let $N(\eta)$ be the minimal number of balls of radius η for the metric d needed to cover $]0,1]\times[-1,1]$. Dudley's theorem asserts that

$$\mathrm{E}\left(\sup\left\{t^{\frac14+\delta}u(t,x)\,:\,t\in]0,1],x\in[-1,1]\right\}\right) \;\le\; 24\int_0^\infty (\ln N(\eta))^{\frac12}\mathrm{d}\eta. \tag{3.24}$$

Given $\eta>0$, let us derive a bound on $N(\eta)$. From (3.2) and (3.23), we compute

$$d((t,x),(s,y))^2 \;=\; (4\pi)^{-\frac12}\left(t^{2\delta}+s^{2\delta}-2s^{\frac14+\delta}t^{\frac14+\delta}\left(\frac{s+t}{2}\right)^{-\frac12}\mathrm{e}^{-\frac{(x-y)^2}{2(t+s)}}\right). \tag{3.25}$$

Hence the bounds

$$d((t,x),(s,y)) \;\le\; (4\pi)^{-\frac14}(t^{2\delta}+s^{2\delta})^{\frac12} \;\le\; (4\pi)^{-\frac14}\sqrt{2}. \tag{3.26}$$

Therefore $N(\eta)=1$ as soon as $\eta\ge(4\pi)^{-\frac14}\sqrt2$. Let us thus assume that $0<\eta<(4\pi)^{-\frac14}\sqrt2$. It follows from (3.26) that the set

$$B_0 \;=\; \left]0,c\,\eta^{\frac1\delta}\right]\times[-1,1] \quad\text{with}\quad c \;=\; \pi^{\frac{1}{4\delta}}$$

is contained in a single ball of radius η. On $(]0,1]\times[-1,1])\backslash B_0$, we will show that there exists $C>0$ such that

$$d((t,x),(s,y)) \;\le\; C\eta^{-\frac1\delta}\left(|t-s|+|x-y|\right). \tag{3.27}$$

This implies that there exists $C>0$ such that $N(\eta)\le C\eta^{-2\left(1+\frac1\delta\right)}$, hence that the integral in (3.24) converges (to a value that depends on δ).

Let us show (3.27). By the triangle inequality, it holds that

$$d((t,x),(s,y)) \;\le\; d((t,x),(t,y)) + d((t,y),(s,y)).$$

First, from (3.25),

$$d((t,x),(t,y))^2 \;=\; \pi^{-\frac12}t^{2\delta}\left(1-\mathrm{e}^{-\frac{(x-y)^2}{4t}}\right) \;\le\; C\frac{(x-y)^2}{t} \;\le\; C\eta^{-\frac1\delta}(x-y)^2$$

where we have used the bounds $t\le1$ and $1-\mathrm{e}^{-z}\le z$ for all $z\ge0$ to obtain the first inequality, and $t>c\eta^{\frac1\delta}$ to get the second one. Next, from (3.25) again,

$$\begin{aligned} d((t,y),(s,y))^2 \;&=\; (4\pi)^{-\frac12}t^{2\delta}\left(1+\left(1+\frac{s-t}{t}\right)^{2\delta}-2\left(1+\frac{s-t}{t}\right)^{\frac14+\delta}\left(1+\frac{s-t}{2t}\right)^{-\frac12}\right)\\ &=:\; t^{2\delta}\varphi\left(\frac{s-t}{t}\right) \;\le\; C\left(\frac{s-t}{t}\right)^2 \;\le\; C\eta^{-\frac2\delta}(t-s)^2 \end{aligned}$$

where the first inequality follows from the fact that $\varphi(0) = \varphi'(0) = 0$, and where the second one is obtained thanks to the bound $t > c\eta^{\frac{1}{\delta}}$.

The proof of (3.20) is analogous and we only outline the main steps. This time,

$$d((t,x),(s,y))^2 \;=\; (16\pi)^{-\frac{1}{2}} \left(t^{2\delta} + s^{2\delta} - 2s^{\frac{3}{4}+\delta} t^{\frac{3}{4}+\delta} \left(\frac{s+t}{2}\right)^{-\frac{3}{2}} \left(1 - \frac{(x-y)^2}{t+s}\right) \mathrm{e}^{-\frac{(x-y)^2}{2(t+s)}} \right).$$

Since the function $z \mapsto (1-z^2)\mathrm{e}^{-z^2/2}$ is bounded, we obtain a bound analogous to (3.26):

$$d((t,x),(s,y)) \;\le\; C(t^{2\delta} + s^{2\delta})^{\frac{1}{2}} \;\le\; C.$$

Hence, again, it is enough to show (3.27) for $t > c\eta^{\frac{1}{\delta}}$ for some $c > 0$, and the rest of the proof uses only completely similar computations.

The proof of (3.21) is analogous. □

4 Proof of Proposition 2.1 and Theorem 2.2

Proof of Proposition 2.1. Let us first show that, almost surely, there exists $\varepsilon > 0$ so that $(X_t)_{0\le t\le\varepsilon}$ is defined as the fixed point of the map Φ from $\mathcal{C}([0,\varepsilon],[-1,1])$ to itself such that

$$\Phi(f) : t \mapsto \int_0^t u(s, f_s)\mathrm{d}s$$

for any $f \in \mathcal{C}([0,\varepsilon],[-1,1])$. First, if we choose ε small enough, Φ is well defined. Indeed, thanks to Lemma 3.4, the time integral is a.s. convergent and moreover, taking for example $\delta = 1/10$, we find $C > 0$ such that, for all $f \in \mathcal{C}([0,\varepsilon],[-1,1])$,

$$\|\Phi(f)\|_\infty \;\le\; C \int_0^\varepsilon \frac{\mathrm{d}s}{s^{1/4+\delta}} \;\le\; 1$$

if ε is chosen small enough.

Next, Φ is contracting if ε is small enough. Indeed, there exists $C > 0$ such that, for all $f, g \in \mathcal{C}([0,\varepsilon],[-1,1])$,

$$\begin{aligned} \|\Phi(f) - \Phi(g)\|_\infty \;&\le\; \int_0^\varepsilon |u(s,f_s) - u(s,g_s)|\mathrm{d}s \\ &\le\; \|f-g\|_\infty \int_0^\varepsilon \sup_{x\in[-1,1]} |\partial_x u(s,x)|\mathrm{d}s \;\le\; C\|f-g\|_\infty \int_0^\varepsilon \frac{\mathrm{d}s}{s^{3/4+\delta}} \\ &\le\; \frac{1}{2}\|f-g\|_\infty \end{aligned}$$

if ε is chosen small enough.

It is thus almost surely possible to define $(X_t)_{0\le t\le\varepsilon}$ as the unique fixed point of Φ. Clearly this process is continuous and satisfies (2.4) for $0 \le t \le \varepsilon$. Let us show that this process can be extended on $\mathbb{R}_+$.

We define $(X_t)_{t\in I}$ as the maximal solution for the Cauchy problem $\partial_t X_t = u(t, X_t)$ with the condition that, at time $t = \varepsilon$, X_t coincides with X_ε found above. We already know that $\inf I = 0$ so that it remains to prove that $\sup I = +\infty$. If $t_\star = \sup I < \infty$ then, $(X_t)_{0<t<t_\star}$ explodes before time $t_\star$, i.e. $\lim_{t\to t_\star^-} |X_t| = +\infty$. This is impossible thanks to Remark 3.3. Indeed, let i be the integer such that $2^{i-1} < t_\star \le 2^i$ and choose $x \in P_i$ and $y \in N_i$ so that $x < X_{2^{i-1}} < y$. This implies that $X_s \in]x,y[$ for all $2^{i-1} \le s < t_\star$ and this is a contradiction. We notice that the fact that $I = \mathbb{R}_+^*$ also establishes that X is smooth on $\mathbb{R}_+^*$, since the field u is smooth on $\mathbb{R}_+^* \times \mathbb{R}$.

Let us finally prove the uniqueness of the process X. If X is continuous on $\mathbb{R}_+$ satisfying (2.4), then $s \mapsto u(s, X_s)$ is in $L^1_{loc}(\mathbb{R}_+)$ as the argument above shows and thus

$X_t = \int_0^t u(s, X_s)\mathrm{d}s$ for all $t \geq 0$. Hence, $(X_t)_{0 \leq t \leq \varepsilon}$ is the unique fixed point of the map Φ defined above. For larger times, uniqueness follows from the uniqueness of regular Cauchy problems. □

Proof of Theorem 2.2. We fix some $\theta_0 > 0$. For $0 < \theta \leq \theta_0$ and $T > 0$, we decompose

$$\frac{X_{\theta T}}{T^{3/4}} \;=\; \frac{1}{T^{3/4}} \int_0^{\theta T} u(s,0)\mathrm{d}s \;+\; \frac{1}{T^{3/4}} \int_0^{\theta T} \big(u(s,X_s) - u(s,0)\big)\mathrm{d}s.$$

Thanks to the scaling relation (3.1),

$$\left(\frac{1}{T^{3/4}} \int_0^{\theta T} u(s,0)\mathrm{d}s \right)_{0 \leq \theta \leq \theta_0} \;=\; \left(\int_0^{\theta} u(s,0)\mathrm{d}s \right)_{0 \leq \theta \leq \theta_0} \quad \text{in law.}$$

Hence, it is enough to show that almost surely

$$\sup_{0 < \theta \leq \theta_0} \left| \frac{1}{T^{3/4}} \int_0^{\theta T} u(s,X_s) - u(s,0)\ \mathrm{d}s \right| \;\to\; 0 \quad \text{as} \quad T \to 0.$$

Let $\delta = 1/10$. Thanks to Lemma 3.4, almost surely, there exists a constant $C > 0$ so that for all $0 < t \leq 1$ and all $x \in [-1,1]$, we have the bounds $|t^{\frac{1}{4}+\delta} u(t,x)| < C$ and $|t^{\frac{3}{4}+\delta} \partial_x u(t,x)| < C$. From now on, by continuity, we take T small enough so that $\theta_0 T \leq 1$ and $\sup_{s \leq \theta_0 T} |X_s| \leq 1$. It thus holds that for $0 \leq t \leq \theta_0 T$,

$$|X_t| \;\leq\; \Big|\int_0^t u(s,X_s)\ \mathrm{d}s\Big| \;\leq\; \int_0^t \sup_{x \in [-1,1]} |u(s,x)|\ \mathrm{d}s \;\leq\; C \int_0^t \frac{\mathrm{d}s}{s^{1/4+\delta}} \;\leq\; 2Ct^{3/4-\delta}.$$

Next, for all $0 \leq \theta \leq \theta_0$,

$$\begin{aligned} \left| \frac{1}{T^{3/4}} \int_0^{\theta T} u(s,X_s) - u(s,0)\mathrm{d}s \right| \;&\leq\; \frac{1}{T^{3/4}} \int_0^{\theta T} |u(s,X_s) - u(s,0)|\mathrm{d}s \\ &\leq\; \frac{1}{T^{3/4}} \int_0^{\theta T} \sup_{x \in [-1,1]} |\partial_x u(s,x)||X_s|\mathrm{d}s \\ &\leq\; \frac{2C^2}{T^{3/4}} \int_0^{\theta T} \frac{1}{s^{3/4+\delta}} s^{3/4-\delta}\mathrm{d}s \;\leq\; 4C^2 \frac{(\theta_0 T)^{1-2\delta}}{T^{3/4}} \end{aligned}$$

and the last bound (uniform on $0 \leq \theta \leq \theta_0$) converges to 0 as $T \to 0$. □

5 Proof of Lemma 2.3 and Proposition 2.4

We start with a Lemma, that guarantees that the zeros of u are almost surely never degenerate, i.e. either $\partial_x u$ or $\partial_t u$ is non-zero whenever u vanishes. This will enable us to invoke the implicit function theorem in several places. Moreover, we show also that there are only countably many isolated points where $\partial_x u$ vanishes, corresponding to the tops of the blue curves on Fig. 2. This is the key ingredient to show item 3 in Proposition 2.4.

Lemma 5.1. *The field u satisfies*

1. $\mathrm{P}\big(\exists (t,x) \in \mathbb{R}_+^* \times \mathbb{R} : u(t,x) = \partial_t u(t,x) = \partial_x u(t,x) = 0\big) \;=\; 0.$
2. *Almost surely, on any compact set $K \subset \mathbb{R}_+^* \times \mathbb{R}$, the set of points where $u(t,x) = \partial_x u(t,x) = 0$ is finite.*

Remark 5.2. As our proof shows, the first item holds actually for any smooth field u such that $(u, \partial_t u, \partial_x u)$ has a locally bounded density around 0.

Proof. For the first item, let us consider the field φ defined by

$$\varphi(t,x) \;=\; \big(u(t,x), \partial_t u(t,x), \partial_x u(t,x)\big), \quad t>0, x\in\mathbb{R}. \tag{5.1}$$

From the scaling relation (3.1), we obtain also

$$\big(\varphi(t,x)\big)_{t,x} \overset{(law)}{=} \big(\alpha^{1/4}u(\alpha t,\alpha^{1/2}x), \alpha^{5/4}\partial_t u(\alpha t,\alpha^{1/2}x), \alpha^{3/4}\partial_x u(\alpha t,\alpha^{1/2}x)\big)_{t,x}. \tag{5.2}$$

By sigma additivity, together with the scaling relation (5.2) and the space translation invariance we find that it suffices to show that

$$\mathrm{P}\big(\exists (t,x)\in[1,2]\times[-1,1] : \varphi(t,x)=0\big) \;=\; 0,$$

To prove this, let us first show that there exists some $C>0$ such that, for any $(t,x)\in[1,2]\times[-1,1]$ and for any $\varepsilon>0$,

$$\mathrm{P}(|\varphi(t,x)|<\varepsilon) \;<\; C\varepsilon^3 \tag{5.3}$$

where $|\cdot|$ denotes the Euclidean norm. First, since $t\le 2$, by the scaling relation (5.2) and translation invariance, for all $(t,x)\in[1,2]\times[-1,1]$

$$\mathrm{P}(|\varphi(t,x)|<\varepsilon) \le \mathrm{P}(|\varphi(2,0)|<\varepsilon).$$

Therefore, to show (5.3), since $\varphi(2,0)$ is Gaussian, it suffices to show that its covariance is non-degenerate, i.e. invertible. Since

$$\mathrm{E}(\varphi_i\varphi_j) \;=\; \int_{\mathbb{R}} \partial^i P_2(z)\partial^j P_2(z)\mathrm{d}z$$

for $1\le i,j\le 3$, with the notation $(\partial^1,\partial^2,\partial^3)=(1,\partial_t,\partial_x)$, and since $P_2(\cdot)$, $\partial_t P_2(\cdot)$ and $\partial_x P_2(\cdot)$ are linearly independent as elements of $L^2(\mathbb{R})$, the covariance is indeed non-degenerate.

Next, because φ is smooth, it holds that

$$\begin{aligned}\{\exists (t,x)\in[1,2]\times[-1,1] : \varphi(t,x)=0\} \;=&\\ \bigcup_{N\in\mathbb{N}^*} &\{\exists (t,x)\in[1,2]\times[-1,1] : \varphi(t,x)=0 \text{ and } \|\varphi'\|_\infty<N\}\end{aligned} \tag{5.4}$$

with $\|f\|_\infty = \max_{(t,x)\in[1,2]\times[-1,1]}\|f(t,x)\|$ for any continuous function f on $[1,2]\times[-1,1]$ with values in linear maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, and where $\|\cdot\|$ is the operator norm when both spaces are endowed with the euclidian norms. Thus it suffices to show that, for any $N\in\mathbb{N}^*$, the probability of the corresponding set in the union in the right hand side of (5.4) is zero. Let $N\in\mathbb{N}^*$, let $\varepsilon>0$ and let us define the points

$$(t_i,x_j) \;=\; (1+i\varepsilon, j\varepsilon), \quad i\in\mathbb{N},\ i<1/\varepsilon, \quad j\in\mathbb{Z},\ |j|<1/\varepsilon.$$

The number of such points is bounded by $2/\varepsilon^2$. Now, under the condition $\|\varphi'\|_\infty<N$, if $\varphi(t,x)=0$ for some $(t,x)\in[1,2]\times[-1,1]$, then $|\varphi(t_i,x_j)|\le\sqrt{2}\varepsilon N$ for one of the points (t_i,x_j) at least. Hence, using (5.3), we obtain

$$\begin{aligned}\mathrm{P}\left(\exists (t,x)\in[1,2]\times[-1,1] : \varphi(t,x)=0 \text{ and } \|\varphi'\|_\infty<N\right) \;&\le\; \sum_{i,j}\mathrm{P}\left(|\varphi(t_i,x_j)|\le\sqrt{2}\varepsilon N\right)\\ &\le\; \frac{2C}{\varepsilon^2}\left(\sqrt{2}\varepsilon N\right)^3.\end{aligned}$$

Since ε may be taken arbitrarily small for given $N \in \mathbb{N}^*$, the left hand side vanishes for any $N \in \mathbb{N}^*$.

We turn to the second item. As K is compact it is enough to prove that the set of $(s, y) \in K$, so that $u(s, y) = \partial_x u(s, y) = 0$, has only isolated points. Using item 1, we may assume that almost surely for all points of this set $\partial_t u(s, y) \neq 0$. We consider one of these points, and using the implicit function theorem, we know that there exists a real function S defined in a neighborhood of y so that the set of zeros of the field u coincides with the graph of this function on a neighborhood of (s, y). The function S satisfies

$$\begin{aligned} S'(z) &= -\frac{\partial_x u(S(z), z)}{\partial_t u(S(z), z)}, \\ S''(z) &= -1 + S'(z)\left(\frac{\partial_x u(S(z), z)\partial_{tt} u(S(z), z)}{(\partial_t u(S(z), z))^2} - 2\frac{\partial_{tx} u(S(z), z)}{\partial_t u(S(z), z)}\right) \end{aligned} \tag{5.5}$$

for z in a neighborhood of y. Therefore, in a neighborhood of y, $\partial_x u(S(z), z) = 0$ if and only if $S'(z) = 0$ and, since $S'(y) = 0$, we have $S''(z) < 0$, and thus also $S'(z) \neq 0$ for $z \neq y$. □

We have now all ingredients for the

Proof of Lemma 2.3. By Lemma 5.1, one may assume that almost surely on every point (s, y) such that $u(s, y) = 0$, either $\partial_t u(s, y) \neq 0$ or $\partial_x u(s, y) \neq 0$. The main observation is that, if (s, y) is such that $u(s, y) = 0$ and $\partial_x u(s, y) = 0$, then there exists $\varepsilon > 0$ such that

$$u(s', y') \neq 0 \text{ for all } (s', y') \in]s, s+\varepsilon[\times]y-\varepsilon, y+\varepsilon[. \tag{5.6}$$

Indeed, as $\partial_t u(s, y) = \partial_{xx} u(s, y) \neq 0$, we may assume that $\partial_t u(s, y) > 0$ (the other case being analogous). By continuity, there exists $\varepsilon > 0$ such that $\partial_t u(s', y') > 0$ for all $(s', y') \in]s, s+\varepsilon[\times]y-\varepsilon, y+\varepsilon[$, and $u(s, y') \geq 0$ for all $y' \in]y-\varepsilon, y+\varepsilon[$. Therefore $u(s', y') > u(s, y') \geq 0$ or all $(s', y') \in]s, s+\varepsilon[\times]y-\varepsilon, y+\varepsilon[$, which shows the claim. Let now $t > 0$ and $x \in \mathcal{Z}_t^{\mathrm{s}} \cup \mathcal{Z}_t^{\mathrm{u}}$. We consider the set

$$\mathcal{F} = \left\{ \begin{array}{l} 0 \leq s_0 < t : \text{there exists a function } r_{(t,x)} :]s_0, t] \to \mathbb{R}, \text{ and} \\ \text{a neighbourhood } \mathcal{V} \text{ of } (t,x) \text{ containing the graph of } r_{(t,x)}, \\ \text{such that for all } (s,y) \in \mathcal{V},\ u(s,y) = 0 \Leftrightarrow y = r_{(t,x)}(s) \end{array} \right\}.$$

Since $\partial_x u(t, x) \neq 0$, the implicit function theorem guarantees that $\mathcal{F} \neq \emptyset$. Let

$$s_{\min} = \inf \mathcal{F}$$

and let us prove by contradiction that $s_{\min} = 0$. Assume that $s_{\min} > 0$.

Remind the definitions of $G(k, \alpha)$ in (3.16), α_k in (3.17) and also property (3.18). We choose k large enough so that $t_0(k) = 2^k \geq t$ and $|x| \leq \frac{K}{3}\alpha_k\sqrt{t_0(k)}$. Since the graph of $r_{(t,x)}$ does not intersect

$$\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} :\ u(t,x) > 0\} \cup \{(t,x) \in \mathbb{R}^+ \times \mathbb{R} :\ u(t,x) < 0\}, \tag{5.7}$$

we obtain that for all $i \geq 0$, $|r_{(t,x)}(t_i(k)) - r_{(t,x)}(t_{i+1}(k))| \leq 2\ell_i(k)$. Hence, by definition of K in (3.15), for all $s_{\min} \leq s \leq t$, $|r_{(t,x)}(s)| \leq \frac{2K}{3}\alpha_k\sqrt{t_0(k)}$, and the set $\{(s, r_x(s)) : s_{\min} < s \leq t\}$ is bounded. By compactness, there exists $y \in \mathbb{R}$ such that $(s_{\min}, y)$ lies in its closure. Therefore, by continuity, $u(s_{\min}, y) = 0$ and, by the implicit function theorem again, $\partial_x u(s_{\min}, y) = 0$ as otherwise one should have $\inf \mathcal{F} < s_{\min}$. We reach a contradiction with (5.6).

Let us next show that $r_{(t,x)}$ is continuous in 0. This is a consequence of the fact that $r_{(t,x)}$ satisfies Cauchy property as s goes to 0. Indeed, with the same argument as above, for all j large enough and $0 < s, s' < t_j(k)$,

$$|r_{(t,x)}(s) - r_{(t,x)}(s')| \;\leq\; \sum_{n\geq j} 2\ell_n(k),$$

and this last sum goes to zero as $j \to +\infty$.

Finally, that stable zeros remain stable, and unstable ones remain unstable, follows from the fact that $\partial_x u(s, r_{(t,x)}(s)) \neq 0$ for all $s \in]0,t]$, as the above argument shows. The expression (2.6) follows from the implicit function theorem. □

Proof of Proposition 2.4: existence of the processes L and R satisfying (2.7) *and* (2.8). Let us first show that there exist processes L and R satisfying (2.7) almost surely for any $t > 0$. To fix the ideas, let us deal with L. As, almost surely, $u(t,\cdot)$ is analytic for any $t > 0$, $\mathcal{Z}_t$ has no accumulation points, and it is enough to prove that the set $\{x \in \mathcal{Z}_t^{\rm s} \cup \mathcal{Z}_t^{\rm u} : r_{(t,x)}(0) < 0\}$ is non-empty and bounded above. Let us fix $t > 0$ and show that, almost surely, for any $t' \in]0,t]$, $\{x \in \mathcal{Z}_{t'}^{\rm s} \cup \mathcal{Z}_{t'}^{\rm u} : r_{(t',x)}(0) < 0\}$ is non-empty and bounded above.

We choose k large enough so that $2^k \geq t$. Using Remark 3.3, almost surely, $u(t',\cdot)$ changes sign infinitely often on $]-\infty, -K\alpha_k\sqrt{2^k}[$. As u is continuous, each interval where u changes sign intersects $\mathcal{Z}_{t'}$. One can actually say more as, from the first item in Lemma 5.1, we may assume that $\partial_{xx}u = \partial_t u \neq 0$ whenever $u = \partial_x u = 0$ and thus, for all $y \in \mathcal{Z}_{t'}^{\rm n}$, the function $u(t,\cdot)$ vanishes but does not change sign in a neighborhood of y. From this one deduces that each interval where u changes sign intersects $\mathcal{Z}_{t'}^{\rm s} \cup \mathcal{Z}_{t'}^{\rm u}$ (we will use repeatedly this argument in the following).

Thus there exists $x \in]-\infty, -K\alpha_k\sqrt{2^k}[\cap (\mathcal{Z}_{t'}^{\rm s} \cup \mathcal{Z}_{t'}^{\rm u})$. Arguing as in the proof of Lemma 2.3, we obtain that for all $0 < s \leq t'$, $r_{t',x}(s) \leq -\frac{2K}{3}\alpha_k\sqrt{2^k}$ and in particular $r_{t',x}(0) < 0$. This implies that $\{x \in \mathcal{Z}_{t'}^{\rm s} \cup \mathcal{Z}_{t'}^{\rm u} : r_{(t',x)}(0) < 0\}$ is non-empty. Moreover it is also bounded above as, with the same argument, for $x \in \mathcal{Z}_{t'}^{\rm s} \cup \mathcal{Z}_{t'}^{\rm u}$ such that $x \geq K\alpha_k\sqrt{2^k}$, it holds that $r_{t',x}(s) \geq \frac{2K}{3}\alpha_k\sqrt{2^k} > 0$.

Second, let us show (2.8). For this, let us first prove that the probability of the event

$$\mathcal{W} \;=\; \{\exists (t,x) \in \mathbb{R}_+^* \times \mathbb{R} : r_{(t,x)}(0) = 0\} \tag{5.8}$$

vanishes. Let us decompose this event as

$$\mathcal{W} \;=\; \bigcup_{t>0} \mathcal{W}_t(0) \quad \text{with} \quad \mathcal{W}_t(y) \;=\; \{\exists x \in \mathbb{R} : r_{(t,x)}(0) = y\},\; y \in \mathbb{R}.$$

Since, by Lemma 2.3, the events $\mathcal{W}_t(0)$ increase as t decreases, it is enough to show that $\mathrm{P}(\mathcal{W}_t(0)) = 0$ for any $t > 0$. Let $t > 0$. As argued above, the set $\mathcal{Z}_t^{\rm s} \cup \mathcal{Z}_t^{\rm u}$ is almost surely unbounded above and below, and countable. Let us denote its elements by $(z_k)_{k\in\mathbb{Z}}$, with $z_k < z_{k+1}$ for all $k \in \mathbb{Z}$ and $z_0 = \min(\mathcal{Z}_t^{\rm s} \cup \mathcal{Z}_t^{\rm u}) \cap \mathbb{R}_+$. Therefore, given $y \in \mathbb{R}$, it holds that $\mathrm{P}(\mathcal{W}_t(y)) > 0$ if and only if $\mathrm{P}(r_{(t,z_k)}(0) = y) > 0$ for some $k \in \mathbb{Z}$. Since the atoms of a random variable are at most countable, the set of $y \in \mathbb{R}$ such that $\mathrm{P}(\mathcal{W}_t(y)) > 0$ is at most countable. As $\mathrm{P}(\mathcal{W}_t(y))$ is constant in $y \in \mathbb{R}$ by translation invariance, we deduce that $\mathrm{P}(\mathcal{W}_t(y)) = 0$ for all $y \in \mathbb{R}$.

On $\mathcal{W}^c$, let us assume by contradiction that there exists some $t > 0$ such that $L_t, R_t \in \mathcal{Z}_t^{\rm s}$ (one rules out analogously the case $L_t, R_t \in \mathcal{Z}_t^{\rm u}$). Since $u(t,x) < 0$ for $x > L_t$ in a neighborhood of L_t and since $u(t,x) > 0$ for $x < R_t$ in a neighborhood of R_t, we find that there exists $x \in Z_t^{\rm u} \cap]L_t, R_t[$. By (2.7), we would have $r_{(t,x)}(0) = 0$, but this is impossible if $\mathcal{W}^c$ is realized. □

Proof of item 1 in Proposition 2.4. First, let us show that Z is continuous in $t=0$. This follows from the fact that L and R are continuous in $t=0$. To fix the ideas, let us show this for L. We actually prove a bit more: For all $\varepsilon>0$, almost surely if $t>0$ is small enough, $|L_t|\le t^{1/2-\varepsilon}$. Let $\varepsilon>0$. As $r_{t,L_t}(0)<0$, arguing as in the proof of Lemma 2.3, we obtain that there exists $C>0$ such that, for all $0<t\le 1$,

$$L_t \;\le \sum_{n\ge\lfloor -\log_2 t\rfloor} 2\ell_n(0,\alpha_0) \;=\; \sum_{n\ge\lfloor -\log_2 t\rfloor} 2(\alpha_0+n^2)\sqrt{2^{-n}} \;\le\; C(\log_2 t)^2 t^{1/2}.$$

Thus, for $t>0$ small enough, the upper bound $L_t\le t^{1/2-\varepsilon}$ holds. Moreover, the above bound implies that for $t>0$ small enough,

$$t^{1/2-\varepsilon}-\sum_{n\ge\lfloor -\log_2 t\rfloor} 2\ell_n(0,\alpha_0) \;\ge\; 2\ell_{\lfloor -\log_2 t\rfloor}(0,\alpha_0).$$

Therefore, the function $u(t,\cdot)$ changes sign in $[-t^{1/2-\varepsilon},-\sum_{n\ge\lfloor -\log_2 t\rfloor}2\ell_n(0,\alpha_0)[$ so that this interval intersects $\mathcal{Z}_t$. Using the same argument as in the proof of Lemma 2.3, one can even say that this interval intersects $\mathcal{Z}_t^{\mathrm{s}}\cup\mathcal{Z}_t^{\mathrm{u}}$ and we consider some x in this intersection. Using once again the same argument, almost surely, $r_{t,x}(0)\le x+\sum_{n\ge\lfloor -\log_2 t\rfloor}2\ell_n(0,\alpha_0)<0$ and this implies that $L_t\ge x\ge -t^{1/2-\varepsilon}$.

Second, let $t>0$ and let us prove that Z is càdlàg at t. Because $L_t,R_t\in\mathcal{Z}_t^{\mathrm{s}}\cup\mathcal{Z}_t^{\mathrm{u}}$, the implicit function theorem implies that there exist $\varepsilon>0$ as well as $x_L,x_R\in\mathcal{Z}_{t+\varepsilon}^{\mathrm{s}}\cup\mathcal{Z}_{t+\varepsilon}^{\mathrm{u}}$ so that

$$L_t \;=\; r_{(t+\varepsilon,x_L)}(t),\qquad R_t \;=\; r_{(t+\varepsilon,x_R)}(t).$$

By definition of L_t and R_t, it holds that $]L_t,R_t[\cap(\mathcal{Z}_t^{\mathrm{s}}\cup\mathcal{Z}_t^{\mathrm{u}})=\emptyset$ so that the only zeros that could be in $]L_t,R_t[$ are neutral, and there is only a finite number of them since $\mathcal{Z}_t$ has no accumulation point. We call them z_i, $i=1,\cdots,n$ (of course n can be 0 and, even if we did not need to prove it for our purposes, we believe that n is at most 1). We claim that, for $\varepsilon>0$ small enough, there is no zero of u in the domain

$$\{(s,x)\in\mathbb{R}_+\times\mathbb{R}:\ t<s\le t+\varepsilon,\ x\in]r_{(t+\varepsilon,x_L)}(s),r_{(t+\varepsilon,x_R)}(s)[\}$$

Indeed, otherwise, as u is continuous there would be a sequence of zeros in this set converging to some z_i, and this is impossible due to (5.6), or to L_t or R_t and this is also impossible thanks to the implicit function theorem as both points are in $\mathcal{Z}_t^{\mathrm{s}}\cup\mathcal{Z}_t^{\mathrm{u}}$. This implies that $Z_s=r_{(t+\varepsilon,x_L)}(s)$ for all $s\in[t,t+\varepsilon]$ or $Z_s=r_{(t+\varepsilon,x_R)}(s)$ for all $s\in[t,t+\varepsilon]$, and this proves thus that Z is right continuous at t.

If $n\ge 1$ and if $i\in\{1,\dots,n\}$, using Lemma 5.1, almost surely $\partial_t u(t,z_i)\ne 0$ and from the implicit function theorem there exists a function S^i defined in a neighborhood of z_i such that the set of zeros of the field u in a neighborhood of (t,z_i) coincides with the graph of S^i. This argument is similar to the one used to define S in the proof of the second item of Lemma 5.1 so that S^i also satisfies (5.5) and $(S^i)''<0$ on a neighborhood of z_i. This implies that S^i defines two bijections: one from a left neighborhood of z_i into $]t-\varepsilon,t[$ (for a small enough $\varepsilon>0$) and another one from a right neighborhood of z_i into $]t-\varepsilon,t[$. Considering the inverse bijections, we define two continuous functions x_1^i and x_2^i, such that for all $t-\varepsilon\le s<t$, $x_1^i(s)$ and $x_2^i(s)$ are in $\mathcal{Z}_s^{\mathrm{s}}\cup\mathcal{Z}_s^{\mathrm{u}}$, $x_1^i(s)<z_i<x_2^i(s)$ and the graphs of x_1^i and x_2^i coincide with the graph of S^i in a neighborhood of (t,z_i).

Finally, we find that for all $t-\varepsilon\le s<t$,

$$\begin{aligned}&[r_{(t+\varepsilon,x_L)}(s),r_{(t+\varepsilon,x_R)}(s)]\cap(\mathcal{Z}_s^{\mathrm{s}}\cup\mathcal{Z}_s^{\mathrm{u}})\\ &\qquad=\;\{x_j^i(s),\ i=1,\dots,n;\ j=1,2\}\cup\{r_{(t+\varepsilon,x_L)}(s),r_{(t+\varepsilon,x_R)}(s)\}\end{aligned}$$

with the convention that the first set in the union in the right hand side is empty if $n = 0$. This implies that Z coincides on $[t-\varepsilon, t[$ with one of these $(n+2)$ functions and thus that it is left continuous at t. □

Proof of item 2 in Proposition 2.4. Suppose first that $Z_{t^-} \in \mathcal{Z}_t^{\mathrm{n}}$. Using (5.6), there exists $\varepsilon > 0$ so that there is no zero in $]t, t+\varepsilon[\times]Z_{t^-} - \varepsilon, Z_{t^-} + \varepsilon[$. This implies that Z is discontinuous at t. Suppose next that $Z_{t^-} \in \mathcal{Z}_t^{\mathrm{s}} \cup \mathcal{Z}_t^{\mathrm{u}}$, and thus by continuity that $Z_{t^-} \in \mathcal{Z}_t^{\mathrm{s}}$. Without loss of generality, we may assume that $r_{(t,Z_{t^-})}(0) < 0$. First, if $z \in \mathcal{Z}_t^{\mathrm{s}}$ satisfies $z < Z_{t^-}$, then $r_{(t,z)}(0) < r_{(t,Z_{t^-})} < 0$ and thus $z \neq Z_t$. Second, if $z \in \mathcal{Z}_t^{\mathrm{s}}$ satisfies $z > Z_{t^-}$, then there exists $z_0 \in]Z_{t^-}, z[\cap \mathcal{Z}_t^{\mathrm{u}}$ and, by continuity, there exists $s \in]0, t[$ such that $Z_s < r_{(t,z_0)}(s)$. Therefore, $r_{(t,z_0)}(0) > 0$ and since $r_{(t,z)}(0) > r_{(t,z_0)}(0)$, this implies also $z \neq Z_t$. We conclude that $Z_t = Z_{t^-}$. □

Proof of item 3 in Proposition 2.4. This follows from the second item in Lemma 5.1 and the fact that Z is discontinuous at t if and only if $Z_{t-} \in \mathcal{Z}_t^{\mathrm{n}}$. □

Proof of item 4 in Proposition 2.4. In this proof, it is convenient to write Z and r as functions of the environment. We fix $T > 0$ and define $(\tilde{Z}_\theta^T)_{\theta \geq 0} = (T^{-1/2} Z_{\theta T})_{\theta \geq 0}$. Given an environment u, we also define $(u_T(t,x))_{t,x} = (T^{1/4} u(Tt, T^{1/2}x))_{t,x}$. Our goal is to prove that $\tilde{Z}^T(u) = Z(u_T)$. Hence, since u and u_T have the same law, this will imply our claim. Let $\theta > 0$ and observe that

1. a real x belongs to $\mathcal{Z}_\theta(u_T)$ if and only if $T^{1/2}x \in \mathcal{Z}_{\theta T}(u)$,
2. in this case both zeros are of the same type and, if moreover x is not neutral, then for all $0 \leq s \leq \theta$
$$r_{(\theta,x)}(u_T)(s) \;=\; T^{-1/2} r_{(\theta T, T^{1/2}x)}(u)(sT).$$
   To prove this last point we observe that the function $\phi : s \to T^{-1/2} r_{(\theta T, T^{1/2}x)}(u)(sT)$ is continuous, satisfies $\phi(\theta) = x$ and $u_T(s, \phi(s)) = 0$ for all $0 < s \leq \theta$. This is enough to conclude as, by definition, $r_{(\theta,x)}(u_T)$ is the only function to have these properties.

By definition of the process Z, these two points imply that $\tilde{Z}^T(u) = Z(u_T)$. □

Proof of item 5 in Proposition 2.4. Let us first show that there exists $c > 0$ so that $\mathrm{P}(|Z_1| \geq z) \geq c\mathrm{e}^{-z/c}$ for all $z \geq 0$. Given $z \geq 0$, the bound

$$\mathrm{P}(|Z_1| \geq z) \;\geq\; \mathrm{P}(u(1,x) > 0,\ \forall x \in [-z, z])$$

holds. Since u is continuous almost surely, for any $x \in \mathbb{R}$,

$$\{u(1,x) > 1\} \;=\; \bigcup_{a>0} \{u(1,y) > 1,\ \forall y \in [x-a, x+a]\}.$$

Therefore, since $\mathrm{P}(u(1,x) > 0) > 0$, there exists $a > 0$ such that,

$$c \;:=\; \mathrm{P}(u(1,y) > 1,\ \forall y \in [x-a, x+a]) \;> 0.$$

For $z > 0$, using that $u(1,\cdot)$ is continuous we obtain

$$\begin{aligned}\mathrm{P}(u(1,x) > 0,\ \forall x \in [-z,z]) &\geq \mathrm{P}(u(1,x) \geq 1,\ \forall x \in [-z,z]) \\ &= \lim_n \mathrm{P}(u(1,x) \geq 1,\ \forall x \in [-z,z] \cap \mathbb{Z}/2^n).\end{aligned} \tag{5.9}$$

For all $n \geq 1$, the random vector $(u(1,x))_{x \in [-z,z] \cap \mathbb{Z}/2^n}$ is gaussian and its coordinates are positively correlated from (3.2) so that, using [28], and assuming $z > a$,

$$\mathrm{P}(u(1,x) \geq 1,\ \forall x \in [-z,z] \cap \mathbb{Z}/2^n) \geq \mathrm{P}(u(1,x) \geq 1,\ \forall x \in [-a,a] \cap \mathbb{Z}/2^n)^{\lceil z/a \rceil}. \tag{5.10}$$

We finally obtain, using again that $u(1,\cdot)$ is continuous, that

$$\begin{aligned} \mathrm{P}(u(1,x)>0,\ \forall x\in[-z,z]) &\geq \lim_n \mathrm{P}(u(1,x)\geq 1,\ \forall x\in[-a,a]\cap\mathbb{Z}/2^n)^{\lceil z/a\rceil} \\ &= \mathrm{P}(u(1,x)\geq 1,\ \forall x\in[-a,a])^{\lceil z/a\rceil} \\ &= e^{-\ln(1/c)\lceil z/a\rceil}. \end{aligned} \tag{5.11}$$

Second, let us show that there exists $c>0$ so that $\mathrm{P}(|Z_1|\geq z)\leq \mathrm{e}^{-cz}/c$. We first remind that, from (3.17), there exists $C>0$ such that for all $\alpha\geq 1$,

$$\mathrm{P}(\overline{G(0,\alpha)})\leq \frac{1}{C}\mathrm{e}^{-C\alpha}. \tag{5.12}$$

On $G(0,\alpha)$ the function $u(1,\cdot)$ changes sign on $]\frac{2}{3}K\alpha, K\alpha[$ so that, using the same argument as in the proof of Lemma 2.3, this interval intersects $\mathcal{Z}_1^{\mathrm{u}}\cup\mathcal{Z}_1^{\mathrm{s}}$. We consider a point x in this intersection. On $G(0,\alpha)$, $r_{1,x}(0)\geq\frac{1}{3}K\alpha>0$. With the same argument there exists $y\in]-K\alpha,-\frac{2}{3}K\alpha[\cap(\mathcal{Z}_1^{\mathrm{u}}\cup\mathcal{Z}_1^{\mathrm{s}})$ such that $r_{1,y}(0)\leq-\frac{1}{3}K\alpha<0$. This implies that $G(0,\alpha)\subset\{|Z_1|\leq\frac{2}{3}K\alpha\}$ and, together with (5.12), concludes the proof of this point.

Let us finally show that Z_1 has a bounded density. For this, it is enough to show that the cumulative distribution function of Z_1 is Lipschitz. Let us thus show that there exists $C>0$ such that, for any $\varepsilon>0$ and for any $x\in\mathbb{R}$,

$$\mathrm{P}(Z_1\in[x,x+\varepsilon])\ \leq\ C\varepsilon.$$

We start with the bound

$$\mathrm{P}(Z_1\in[x,x+\varepsilon])\ \leq\ \mathrm{P}([x,x+\varepsilon]\cap\mathcal{Z}_1\neq\emptyset)\ =\ \mathrm{P}([0,\varepsilon]\cap\mathcal{Z}_1\neq\emptyset).$$

In the sequel, to simplify writings, let us write $u(x)$ for $u(1,x)$ for any $x\in\mathbb{R}$. By a second order Taylor expansion, there exists a function $\theta:[0,\varepsilon]\to[0,\varepsilon]$ such that, for all $x\in[0,\varepsilon]$,

$$u(x)=u(0)+\partial_x u(0)x+\frac{\partial_{xx}u(\theta(x))x^2}{2}. \tag{5.13}$$

Let $\delta>0$ to be fixed later and let us decompose $\mathrm{P}([0,\varepsilon]\cap\mathcal{Z}_1\neq\emptyset)$ according to the following alternative:

$$\begin{aligned} \mathrm{P}([0,\varepsilon]\cap\mathcal{Z}_1\neq\emptyset)\ &\leq \mathrm{P}(\exists y\in[0,\varepsilon]:|u(0)+\partial_x u(0)y|\leq\delta) \\ &\quad+\ \mathrm{P}(\exists x\in[0,\varepsilon]:u(x)=0\quad\text{and}\quad\forall y\in[0,\varepsilon]:|u(0)+\partial_x u(0)y|>\delta). \end{aligned} \tag{5.14}$$

To get a bound on the first term, we notice that $u(0)$ and $\partial_x u(0)$ are independent Gaussian variables, and one finds that there exists $C>0$ such that, for any $\delta\in]0,\varepsilon]$ and any $\varepsilon>0$,

$$\mathrm{P}(\exists y\in[0,\varepsilon]:|u(0)+\partial_x u(0)y|\leq\delta)\ \leq\ C\varepsilon. \tag{5.15}$$

To get a bound on the second term, we use the expansion (5.13):

$$\begin{aligned} &\mathrm{P}(\exists x\in[0,\varepsilon]:u(x)=0\quad\text{and}\quad\forall y\in[0,\varepsilon]:|u(0)+\partial_x u(0)y|>\delta) \\ &=\ \mathrm{P}\left(\exists x\in[0,\varepsilon]:\frac{\partial_{xx}u(\theta(x))x^2}{2}=-(u(0)+\partial_x u(0)x)\quad\text{and}\quad\inf_{y\in[0,\varepsilon]}|u(0)+\partial_x u(0)y|>\delta\right) \\ &\leq\ \mathrm{P}(\exists x\in[0,\varepsilon]:|\partial_{xx}u(\theta(x))|x^2\geq 2\delta) \\ &\leq\ \mathrm{P}\left(\sup_{x\in[0,\varepsilon]}|\partial_{xx}u(x)|\geq\frac{2\delta}{\varepsilon^2}\right)\ \leq\ \frac{\varepsilon^2}{2\delta}\mathrm{E}\left(\sup_{x\in[0,\varepsilon]}\partial_{xx}u(x)\right)\ \leq\ \frac{C\varepsilon^2}{2\delta} \end{aligned}$$

where the last bound follows from Lemma 3.4. Therefore, taking $\delta=\varepsilon$, we obtain the claim by inserting this last bound together with (5.15) into (5.14). □

6 Proof of Theorem 2.6

Given $T > 0$, let us define the processes $(\tilde{Z}^T_\theta)_{\theta\geq 0} = (T^{-1/2} Z_{\theta T})_{\theta \geq 0}$ as well as $(\tilde{X}^T_\theta)_{\theta\geq 0} = (T^{-1/2} X_{\theta T})_{\theta\geq 0}$. Let us also define $(Y^T_\theta)_{\theta\geq 0}$ by $Y^T_0 = 0$ and, for $\theta > 0$,

$$\frac{\mathrm{d}Y^T_\theta}{\mathrm{d}\theta} \;=\; T^{1/4} u(\theta, Y^T_\theta).$$

Note that this definition makes sense, as can be shown exactly with the same arguments as in the proof of Proposition 2.1.

Let us first show that

$$\big(\tilde{Z}^T_\theta, \tilde{X}^T_\theta\big)_{\theta\geq 0} \;=\; \big(Z_\theta, Y^T_\theta\big)_{\theta\geq 0} \quad \text{in law.} \tag{6.1}$$

For this, it is convenient to explicitly write the couple of processes as a function of the environment. Given an environment u, let $(u_T(t,x))_{t\geq 0, x\in\mathbb{R}} = (T^{1/4} u(Tt, T^{1/2}x))_{t\geq 0, x\in\mathbb{R}}$ and let us show that

$$\big(\tilde{Z}^T_\theta, \tilde{X}^T_\theta\big)(u) \;=\; \big(Z_\theta, Y^T_\theta\big)(u_T)$$

for any $\theta \geq 0$. As u and u_T have the same law by the scaling relation (3.1), this will imply (6.1). The relation $\tilde{Z}^T_\theta(u) = Z_\theta(u_T)$ has already been shown in the proof of item 4 in Proposition 2.4. To show $\tilde{X}^T_\theta(u) = Y^T_\theta(u_T)$, we notice that $\tilde{X}^T_0 = 0$ and that for all $\theta > 0$,

$$\frac{\mathrm{d}\tilde{X}^T_\theta}{\mathrm{d}\theta} \;=\; T^{1/4} u_T(\theta, \tilde{X}^T_\theta)$$

and the claim follows from the fact that these relations characterize the process $(Y^T_\theta)_{\theta\geq 0}(u_T)$.

To prove Theorem 2.6, it is thus enough to prove that, almost surely, Y^T converges to Z in the $\mathcal{M}_1$ topology on compact sets as $T \to \infty$. Indeed, this implies that $Y^T - Z$ converges to 0 in probability as $T \to \infty$ and, thanks to (6.1), this implies that $(T^{-1/2}(X_{\theta T} - Z_{\theta T}))_{\theta\geq 0}$ converges to 0 in probability as $T \to \infty$. For notational convenience, we will show that $(Y^T_t)_{t\in[0,1]}$ converges to $(Z_t)_{t\in[0,1]}$, but our proof still holds for $[0,1]$ replaced by any compact interval.

We use characterization (v) of [32] for the convergence in the $\mathcal{M}_1$ topology. We first introduce some notations needed to state it. Given $a, b, c \in \mathbb{R}$, let

$$\|a - [b,c]\| \;=\; \min_{\tau\in[0,1]} |a - (\tau b + (1-\tau)c)|.$$

For $\delta > 0$ and $f, g \in D([0,1], \mathbb{R})$, let

$$v(f,g,t,\delta) = \sup\{|f(t_1) - g(t_2)|,\ 0 \vee (t-\delta) \leq t_1, t_2 \leq 1 \wedge (t+\delta)\}$$

and

$$w_s(f,t,\delta) = \sup\{\|f(t_2) - [f(t_1), f(t_3)]\|,\ 0 \vee (t-\delta) \leq t_1 < t_2 < t_3 \leq 1 \wedge (t+\delta)\}.$$

The characterization is the following: $f^T \to f$ converges to f as $T \to \infty$ for the $\mathcal{M}_1$ topology on $\mathcal{D}([0,1])$ if and only if

1. $f^T(1)$ converges to $f(1)$.
2. For all $0 \leq t \leq 1$ that is not a discontinuity point of f

$$\lim_{\delta\to 0} \lim_{T\to+\infty} v(f^T, f, t, \delta) \;=\; 0.$$

3. For all $0 \leq t \leq 1$ that is a discontinuity point of f

$$\lim_{\delta\to 0} \lim_{T\to+\infty} w_s(f^T, t, \delta) \;=\; 0.$$

The first point is actually a consequence of the second one as, for all $t \geq 0$ (and in particular for $t = 1$), almost surely, Z is continuous at t. Indeed we first observe, from item 4 in Proposition 2.4, that $s \to \mathrm{P}(Z_{s^-} \neq Z_s)$ is constant on $\mathbb{R}_+^*$. Then

$$\int \mathrm{P}(Z_{s^-} \neq Z_s)\, ds \;=\; \mathrm{E}\left(\int 1_{Z_{s^-} \neq Z_s}\, ds\right) \;=\; 0,$$

as almost surely the discontinuity points of Z are countable. This implies that $\mathrm{P}(Z_{s^-} \neq Z_s) = 0$ for all $s > 0$.

We start with the proof of the second item in the above characterization:

Lemma 6.1. *Almost surely, for all* $t_0 \in [0,1]$ *such that* Z *is continuous in* t_0*,*

$$\lim_{\delta \to 0} \lim_{T \to +\infty} v(Y^T, Z, t_0, \delta) = 0.$$

Proof. We first consider the case $t_0 = 0$. Almost surely, for all $T > 0$ and all $0 < t \leq 1$ small enough,

$$|Y_t^T| \leq \sum_{n \geq \lfloor -\ln_2 t \rfloor} 2\ell_n(0, \alpha_0) = \sum_{n \geq \lfloor -\ln_2 t \rfloor} 2(\alpha_0 + n^2)\sqrt{2^{-n}}.$$

This implies that, for any $\varepsilon > 0$, $t > 0$ small enough and for any $T > 0$,

$$|Y_t^T| \;\leq\; t^{\frac{1}{2} - \varepsilon}. \tag{6.2}$$

As Z is right continuous at 0 with limit 0 this gives the result in the case $t_0 = 0$.

We consider $0 < t_0 \leq 1$ so that Z is continuous at t_0 and fix some $\varepsilon > 0$. To fix ideas, and as the other case is analogous, let us assume that $Z_{t_0} = L_{t_0}$. Using Proposition 2.4, there exists $\delta > 0$ so that Z is continuous on $[t_0 - 2\delta, t_0 + \delta]$ so that, if $\delta > 0$ has been chosen small enough,

$$v(Z, Z, t_0, \delta) \;<\; \varepsilon,$$

and we only have to show that for $\delta > 0$ small enough and all T larger than some $T_0(\delta)$,

$$\sup\{|Z_s - Y_s^T|,\; t_0 - \delta \leq s \leq t_0 + \delta\} \;<\; \varepsilon. \tag{6.3}$$

We use the notations $t_i = t_0 + i\delta$, $i \in \{-2, -1, 0, 1\}$. We stress that for all $t \in [t_{-2}, t_1]$, $Z_t = r_{(t_1, L_{t_1})}(t)$ as, for $x \in \mathcal{Z}_{t_1}^{\mathrm{s}}$, $r_{(t_1,x)}$ is the only continuous function so that $r_{(t_1,x)}(t_1) = x$ and $u(s, r_{(t_1,x)}(s)) = 0$ for $0 < s \leq t_1$. Using Lemma 2.4, it is also possible to choose $\delta > 0$ small enough so that the only neutral zeros of u in the domain

$$\{(s,x) \in \mathbb{R}_+ \times \mathbb{R} :\; t_{-2} \leq s \leq t_1,\; x \in [r_{(t_1, L_{t_1})}(s), r_{(t_1, R_{t_1})}(s)]\} \tag{6.4}$$

lies in $\mathcal{Z}_{t_0}$. Using Remark 2.5, this choice for δ implies that R is continuous on $[t_{-2}, t_{-1}]$ and, with the same argument as above, that for all $t \in [t_{-2}, t_{-1}]$, $R_t = r_{(t_{-1}, R_{t_{-1}})}(t)$. Note however that it is not necessarily the case that $R_t = r_{(t_1, R_{t_1})}(t)$, as R could jump at time t_0.

Before going to the proof of (6.3) itself, let us first prove the following intermediate result: For all $t > 0$ so that $Z_t = L_t$, and if $\varepsilon > 0$ has been chosen small enough, there exists $T_0 > 0$ so that for all $T \geq T_0$:

$$r_{(t, L_t)}(s) - \varepsilon \;\leq\; Y_s^T \;\leq\; r_{(t, R_t)}(s) - \varepsilon \quad \text{for all } s \in [0, t]. \tag{6.5}$$

By the definition of L and R in Proposition 2.4, it holds that $r_{(t, L_t)}(0) < 0$ and $r_{(t, R_t)}(0) > 0$. Since the functions $r_{(t, L_t)}$ and $r_{(t, R_t)}$ are continuous, and since Y^T satisfies the bound (6.2), we conclude that there exists $\tau \in]0, t]$ so that (6.5) holds for $s \in [0, \tau]$.

Let us now assume that $s \in [\tau, t]$ and show the lower bound on Y^T in (6.5) (the proof of the upper bound is analogous). By Lemma 2.3, the function $s \to \partial_x u(s, r_{(t,L_t)}(s))$ is continuous and strictly negative on $[\tau, t]$ so that by compactness, there exists $c > 0$ such that

$$\partial_x u(s, r_{(t,L_t)}(s)) \;\leq\; -c \quad \text{for all} \quad s \in [\tau, t].$$

For $s \in [\tau, t]$, let

$$\tilde{r}_{(t,L_t)}(s) \;=\; r_{(t,L_t)}(s) - \varepsilon.$$

A second order expansion yields

$$u(s, \tilde{r}_{(t,L_t)}(s)) \;=\; -\varepsilon \partial_x u(s, r_{(t,L_t)}(s)) + \frac{\varepsilon^2}{2} \partial_{xx} u(s, y_\varepsilon(s))$$

with $y_\varepsilon(s) \in [\tilde{r}_{(t,L_t)}(t), r_{(t,L_t)}(s)]$. By continuity of $\partial_{xx} u$ and compactness, there exists $K \geq 0$ such that

$$u(s, \tilde{r}_{(t,L_t)}(s)) \;\geq\; c\varepsilon - K\varepsilon^2 \;\geq\; \frac{c\varepsilon}{2} \tag{6.6}$$

for all $s \in [\tau, t]$, provided $\varepsilon > 0$ was taken small enough. Suppose now that the lower bound in (6.5) is not satisfied so that there exists $s \in [\tau, t]$ such that $Y_s = \tilde{r}_{(t,L_t)}(s)$ and $\partial_s Y_s \leq \partial_s \tilde{r}_{(t,L_t)}(s) = \partial_s r_{(t,L_t)}(s)$ i.e. explicitly

$$T^{\frac{1}{4}} u(s, \tilde{r}_{(t,L_t)}(s)) \;\leq\; -\frac{\partial_{xx} u(r_{(t,L_t)}(s))}{\partial_x u(r_{(t,L_t)}(s))}. \tag{6.7}$$

Since the right hand side is uniformly bounded in $s \in [\tau, t]$, the lower bound (6.6) leads to a contradiction for T large enough. This concludes the proof of (6.5).

Let us now derive the result (6.3) from (6.5). We choose T_0 large enough so that (6.5) holds both for time t_1 and t_{-2}. It remains to show that for T large enough, $Y_t^T \leq r_{(t_1,L_{t_1})}(t) + \varepsilon$ for all $t \in [t_{-1}, t_1]$. For this, we first show that there exists $t_\star \in [t_{-2}, t_{-1}]$ such that $Y_{t_\star}^T \leq r_{(t_1,L_{t_1})}(t_\star) + \varepsilon$. By the definition of L and R, and since the set $\mathcal{W}$ defined in (5.8) has probability 0, for all $t > 0$, it holds that $]L_t \cap R_t[\cap(\mathcal{Z}_t^{\mathrm{s}} \cup \mathcal{Z}_t^{\mathrm{u}}) = \emptyset$. Hence, the choice of δ made before (6.4) implies that almost surely,

$$\{(s,x) \in \mathbb{R}_+ \times \mathbb{R} : \; t_{-2} \leq s \leq t_{-1}, \; x \in]L_s, R_s[, u(s,x) = 0\} \;=\; \emptyset.$$

As $Z_t = L_t$ for all $t \in [t_{-2}, t_{-1}]$, we obtain that $u(t,x) < 0$ for all $(t,x) \in \bigcup_{t_{-2} \leq s \leq t_{-1}}]L_s, R_s[$ and thus, by compactness, there exists $c > 0$ such that $u(t,x) < -c$ for all (t,x) such that $L_t + \varepsilon \leq x \leq R_t - \varepsilon$ with $t \in [t_0 - 2\delta, t_0 - \delta]$. Assume by contradiction that $Y_t > L_t + \varepsilon$ for all $t \in [t_{-2}, t_{-1}]$. Then, since we know that $Y_t \leq R_t - \varepsilon$, we conclude that

$$Y_t \leq Y_{t_{-2}} - cT^{1/4}(t - t_{-2}).$$

For T large enough, this yields a contradiction. Second, once we know that $Y_{t_\star} \leq r_{(t_1,L_{t_1})}(t_\star) + \varepsilon$, we may proceed as in the proof of (6.5) and show that, for T large enough, $Y_t \leq r_{(t_1,L_t)}(t) + \varepsilon$ for all $t \in [t_\star, t_1]$. □

Next, we turn to the proof of the third item in the above characterization:

Lemma 6.2. *Almost surely, for all $t_0 \in [0,1]$ such that t_0 is a jump point of Z,*

$$\lim_{\delta \to 0} \lim_{T \to +\infty} w_s(Y^T, t_0, \delta) = 0.$$

Proof. We first describe how the environment looks like around a fixed jump point $t_0 \in]0,1[$ of Z. In the following we will always suppose that $\delta > 0$ is small enough so that t_0 is the only jump of Z on $[t_0 - \delta, t_0 + \delta]$. As the three other cases are similar, we may also assume that $Z_s = L_s$ for all $t_0 - \delta \leq s < t_0$ and $Z_s = R_s$ for all $t_0 \leq s \leq t_0 + \delta$. We also consider δ small enough so that R is continuous on $[t_0, t_0 + \delta]$. Using Remark 2.5, this implies that

$$\{(s,x) \in \mathbb{R}_+ \times \mathbb{R} : \; t_0 < s \leq t_0 + \delta, \; x \in]L_s, R_s[; u(s,x) = 0\} \; = \; \emptyset. \tag{6.8}$$

We first focus on the behaviour of the environment just before the jump and prove that, for all $\varepsilon > 0$ (small enough), there exists $\delta > 0$ so that for all $t_0 - \delta \leq t < t_0$,

$$\begin{aligned} &u(t, Z_{t_0-} - \varepsilon) > 0, \\ &u(t, Z_{t_0-}) < 0, \\ &Z_{t_0-\delta} \in]Z_{t_0-} - \varepsilon, Z_{t_0-}[. \end{aligned} \tag{6.9}$$

We next describe the environment just after the jump at time t_0: For all $\varepsilon > 0$ (small enough) there exists $\delta > 0$ so that

$$\begin{aligned} &L_t < Z_{t_0-} - \varepsilon \quad \text{for all } t_0 \leq t \leq t_0 + \delta, \\ &\sup\{|Z_s - Z_t|, \; t_0 \leq s, t \leq t_0 + \delta\} \leq \varepsilon. \end{aligned} \tag{6.10}$$

We delay the proof of these two points and first assume that (6.9) and (6.10) hold for some $\varepsilon > 0$ and $\delta > 0$. We prove that it implies that, for T large enough,

$$\begin{aligned} &Z_{t_0-} - \varepsilon \leq Y_t^T \leq Z_{t_0-} \quad \text{for all } t_0 - \delta \leq t \leq t_0, \\ &Y^T \text{ is increasing on } [t_0, h] \text{ where } h = \inf\{t \geq t_0, \; Y_t^T \geq Z_t - \varepsilon\} \wedge (t_0 + \delta), \\ &Y_t^T \in [Z_t - \varepsilon, Z_t + \varepsilon] \quad \text{for all } h < t \leq t_0 + \delta. \end{aligned} \tag{6.11}$$

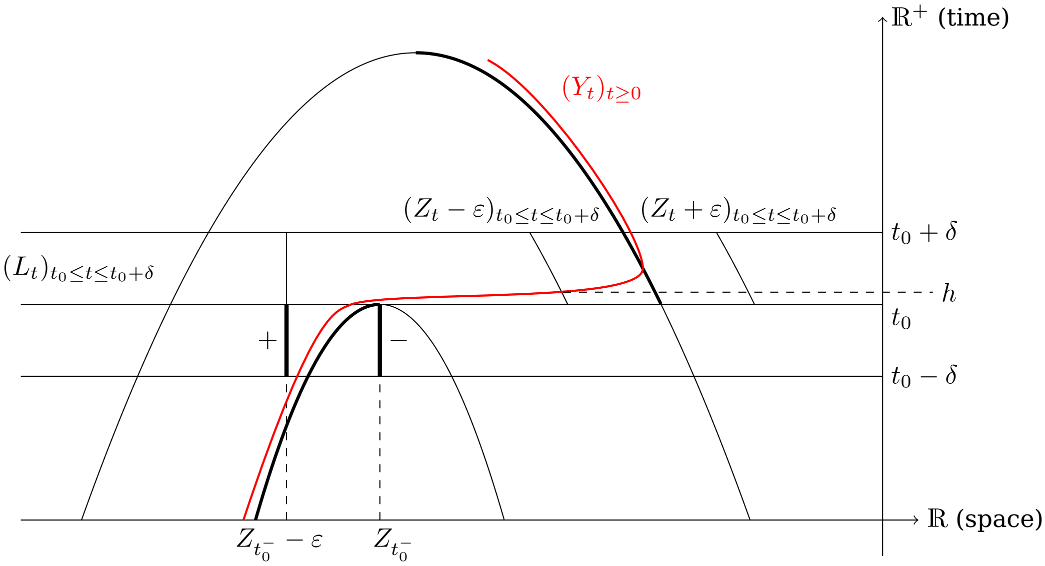

Figure 3: The environment and the process Y as in Equations (6.9),(6.10) and (6.11). The thick line represents the process Z and the red one the process Y, both of them near the time $t = t_0$.

Indeed, for the first point of (6.11), as $t_0 - \delta$ is not a jump point of Z and $Z_{t_0-\delta} \in]Z_{t_0-} - \varepsilon, Z_{t_0-}[$, Lemma 6.1 ensures that for T large enough $Y^T_{t_0-\delta}$ lies also in $]Z_{t_0-} - \varepsilon, Z_{t_0-}[$ and both barriers defined in (6.9) ensures that Y^T stays in this interval till t_0. For the second point of (6.11), from (6.8) and (6.10) we deduce that $u > 0$ in the domain $\{(t,x), t_0 \le t \le t_0+\delta; Z_{t_0-} - \varepsilon \le x \le R_t - \varepsilon\}$ and as $Y^T_{t_0} \in]Z_{t_0-} - \varepsilon, Z_{t_0-}[$ this implies that Y^T is increasing on $[t_0, h]$. The proof of the last point in (6.11) follows with the argument that has been used to prove (6.5).

One can check that conditions in (6.11) together with the second point in (6.10) implies that $w_s(Y^T, t_0, \delta) < 2\varepsilon$ and that concludes the proof. It remains to prove (6.9) and (6.10).

For (6.10), as $L_{t_0} < Z_{t_0-} - \varepsilon$ (if ε is small enough), continuity of L ensures that is still true for $t \in [t_0, t_0+\delta]$ if δ is taken small enough, and this yields the first point of (6.10). The second one follows from uniform continuity.

We turn to (6.9). As $Z_{t_0-} \in \mathcal{Z}^{\mathrm{n}}_t$, arguing as in the proof of the second item of Lemma 5.1, there exists a function S defined in a neighborhood of $Z_{t_0^-}$ so that, in a neighborhood B of $(t_0, Z_{t_0^-})$, the zeros of u coincide with the graph of S. Moreover S satisfies

$$S(x) - t_0 = -\frac{1}{2}(x - Z_{t_0-})^2 + \mathcal{O}(|x - Z_{t_0-}|^3) \tag{6.12}$$

as $x \to Z_{t_0-}$. As we assumed that $Z_s = L_s$ for all $t_0 - \delta \le s < t_0$ and $Z_s = R_s$ for all $t_0 \le s \le t_0 + \delta$, it holds that, for all $(t,x) \in B$, $u(t,x) > 0$ if $t > S(x)$ and has opposite sign if $t < S(x)$. Using (6.12), we deduce that for $\varepsilon > 0$ small enough there exists $\delta > 0$ so that for all $t_0 - \delta \le t \le t_0$, $u(t, Z_{t_0-} - \varepsilon) > 0$ and $u(t, Z_{t_0-}) < 0$. Moreover for $\delta > 0$ small enough

$$\{(t, Z_t) : t_0 - \delta \le t < t_0\} \;=\; \{(S(x), x) : Z_{t_0-\delta} \le x < Z_{t_0-}\},$$

so that from the continuity of Z and (6.12) we obtain that for $\delta > 0$ small enough $Z_{t_0-\delta} \in]Z_{t_0-} - \varepsilon, Z_{t_0-}[$. □

Supplementary material

We provide here the needed details to understand the implications of two earlier works, [21] and [23], for the understanding of the process X evolving in a rough potential, as described in the introduction, see (1.3) and (1.4). We can try to construct a process X solving (1.4) in three steps: First, we replace the velocity field u by a regularized field u^ℓ, varying smoothly in space on some length scale $\ell > 0$; second, we define the associated process X^ℓ; and third, we obtain X as the limit of the processes X^ℓ when the regularization is removed, i.e. for $\ell \to 0$. Concretely, for $\ell > 0$, let

$$u_\ell(t,\cdot) \;=\; P_{\ell^2} \star u(t,\cdot), \tag{6.13}$$

where the heat kernel P is defined in (2.2) and where $u = -\partial_x \mathcal{V}$ with $\mathcal{V}$ solving (1.3). Let then X^ℓ be the solution of the Cauchy problem (1.4) with u^ℓ instead of u, i.e. $X^\ell_0 = 0$ and

$$\partial_t X^\ell_t \;=\; u_\ell(t, X^\ell_t). \tag{6.14}$$

Let us first consider the analysis performed in [21]: We recall the main results found there, and we explain the connection with the above problem. Let $\lambda > 0$. In [21], the process S^λ satisfying $S^\lambda_0 = 0$ and solving

$$\partial_t S^\lambda_t \;=\; \lambda u_1(t, S^\lambda_t), \qquad t \ge 0, \tag{6.15}$$

is studied numerically for various values of $\lambda > 0$. The upshot is that, in the limit $\lambda \to 0$, and as far as numerical simulations can be reliably performed,

$$\mathrm{E}((S^\lambda_t)^2) \;\sim\; \lambda^2 t^{3/2} \quad \text{for} \quad 0 \le t \le \lambda^{-4} \qquad \text{and} \qquad \mathrm{E}((S^\lambda_t)^2) \;\sim\; t \quad \text{for} \quad t \ge \lambda^{-4}, \tag{6.16}$$

up to possible logarithmic corrections for $t \geq \lambda^{-4}$.

For $\ell > 0$, we can now define a process $\tilde{X}^\ell$ that will have the same law as X^ℓ solving (6.14): For all $t \geq 0$,

$$\tilde{X}^\ell_t \;=\; \ell S^{\ell^{1/2}}_{t/\ell^2}. \tag{6.17}$$

Indeed, since S^ℓ solves (6.15), the process $\tilde{X}^\ell$ solves

$$\partial_t \tilde{X}^\ell_t \;=\; \ell^{-1/2} u_1(\ell^{-2}t, \ell^{-1}\tilde{X}^\ell_t). \tag{6.18}$$

With the regularization (6.13), the scaling relation

$$\big(u_1(t,x)\big)_{t\geq 0, x\in\mathbb{R}} \;=\; \big(\ell^{1/2}u_\ell(\ell^2 t, \ell x)\big)_{t\geq 0, x\in\mathbb{R}} \tag{6.19}$$

holds in law for all $\ell > 0$, as can be checked by computing the covariance of both fields. Therefore, we conclude from (6.18) that $\tilde{X}^\ell = X^\ell$ in law. At this point, using (6.17) and the equality $\tilde{X}^\ell = X^\ell$ in law, we may reformulate (6.16) as:

$$\mathrm{E}((X^\ell_t)^2) \;\sim\; t^{3/2} \quad \text{for} \quad 0 \leq t \leq 1 \qquad \text{and} \qquad \mathrm{E}((X^\ell_t)^2) \;\sim\; t \quad \text{for} \quad t \geq 1.$$

Since these estimates do not depend on ℓ, they make the case for the existence of a limit process X solving (1.4).

Let us next move to the result in [23] quoted in the introduction. We have already described in the main text the convergence of the processes $(W^n)_{n\geq 1}$ studied in [23]. Here, to make our point, let us define a sequence of processes $U^n = (U^n_t)_{0\leq t\leq T}$ that can reasonably be expected to behave as the processes W^n, and for which the connection with (1.4) can be made very easily through a scaling argument. For $n \in \mathbb{N}^*$, let U^n be a real valued process satisfying $U^n_0 = 0$ and solving

$$\partial_t U^n_t \;=\; n u_1(n^2 t, U^n_t) \quad \text{for} \quad 0 \leq t \leq T. \tag{6.20}$$

For large values of t, and in the large n limit, we may expect that U^n and W^n behave in a similar way. In particular, we expect the scaling $\mathrm{E}(U^n(t)^2) \sim n t^{3/2}$ to hold in this regime.

Again, for $\ell > 0$, let us define a process $\hat{X}^\ell = (\hat{X}^\ell_t)_{0\leq t\leq \ell T}$ that will turn out to have the same law as X^ℓ for $0 \leq t \leq \ell T$:

$$\hat{X}^\ell_t \;=\; \ell U^{\ell^{-1/2}}_{t/\ell},$$

where we have assumed that ℓ is such that $\ell^{-1/2}$ is an integer. Indeed, from (6.20), we deduce that $\hat{X}^\ell_t$ solves

$$\partial_t \hat{X}^\ell_t \;=\; \ell^{-1/2} u_1(\ell^{-2}t, \ell^{-1}\tilde{X}^\ell_t) \quad \text{for} \quad 0 \leq t \leq \ell T$$

and, by the scaling relation (6.19), we deduce that $X^\ell = \hat{X}^\ell$ in law, for $t \in [0, \ell T]$. We observe also that $\mathrm{E}((\tilde{X}^\ell(t))^2) \sim t^{3/2}$ on this time interval. This brings thus some support to the validity of (1.6), but the time interval $[0, \ell T]$ shrinks to 0 as $\ell \to \infty$, and $\hat{X}^\ell$ should thus be controlled on longer time scales to reach a firm conclusion.

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