Electronic Journal of Probability

Electron. J. Probab. **28** (2023), article no. 22, 1–84.
ISSN: `1083-6489` `https://doi.org/10.1214/22-EJP880`

General criteria for the study of quasi-stationarity*

Nicolas Champagnat[†] Denis Villemonais[†]

Abstract

For Markov processes with absorption, we provide general criteria ensuring the existence and the exponential non-uniform convergence in weighted total variation norm to a quasi-stationary distribution. We also characterize a subset of its domain of attraction by an integrability condition, prove the existence of a right eigenvector for the semigroup of the process and the existence and exponential ergodicity of the Q-process. These results are applied to one-dimensional and multi-dimensional diffusion processes, to pure jump continuous time processes, to reducible processes with several communication classes, to perturbed dynamical systems and discrete time processes evolving in discrete state spaces.

Keywords: Markov processes with absorption; quasi-stationary distribution; Q-process; mixing property; diffusion processes; birth and death processes; reducible processes; perturbed dynamical systems; Galton-Watson processes.
MSC2020 subject classifications: Primary 37A25; 60B10; 60F99; 60J05; 60J10; 60J25; 60J27, Secondary 60J60; 60J75; 60J80; 93E03.
Submitted to EJP on November 23, 2018, final version accepted on November 8, 2022.
Supersedes `arXiv:1712.08092v3`.
Supersedes `HAL:hal-01672979v2`.

Contents

---

*This work was partially funded by the Chair “Modélisation Mathématique et Biodiversité” of VEOLIA-Ecole Polytechnique-MNHN-F.X. N.C. was partially funded by the European Union (ERC, SINGER, 101054787). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

[†] Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France E-mail: `Nicolas.Champagnat@inria.fr`,`Denis.Villemonais@univ-lorraine.fr`

1 Introduction

Let $(X_t, t \in I)$ be a Markov process in $E \cup \{\partial\}$ where E is a measurable space and $\partial \notin E$, with set of time indices I which might be $\mathbb{R}_+$ or $\frac{1}{k}\mathbb{Z}_+$ for some $k \in \mathbb{N} := \{1, 2, \ldots\}$, where $\mathbb{Z}_+ := \{0, 1, \ldots\}$. For all $x \in E \cup \{\partial\}$, we denote as usual by $\mathbb{P}_x$ the law of X given $X_0 = x$ and for any probability measure μ on $E \cup \{\partial\}$, we define $\mathbb{P}_\mu = \int_{E\cup\{\partial\}} \mathbb{P}_x \, \mu(dx)$. We also denote by $\mathbb{E}_x$ and $\mathbb{E}_\mu$ the associated expectations. We assume that ∂ is absorbing, which means that $X_t = \partial$ for all $t \geq \tau_\partial$, $\mathbb{P}_x$-almost surely, where

$$\tau_\partial = \inf\{t \in I, \, X_t = \partial\}.$$

Our goal is to study the existence of *quasi-limiting distributions* on E for the process X, i.e. probability measures ν such that

$$\lim_{t\in I, \, t\to+\infty} \mathbb{P}_\mu(X_t \in A \mid t < \tau_\partial) = \nu(A)$$

for some probability measure μ on E and for all $A \subset E$ measurable. Such a measure ν is a *quasi-stationary distribution* for X, i.e. a probability measure such that $\mathbb{P}_\nu(X_t \in \cdot \mid t < \tau_\partial) = \nu(\cdot)$ for all $t \in I$. We refer the reader to [34, 83, 104] for general introductions to quasi-stationary distributions. In particular, it is well-known that there exists a constant $\lambda_0 \geq 0$, called the decay parameter of the quasi-stationary distribution ν, such that $\mathbb{P}_\nu(t < \tau_\partial) = e^{-\lambda_0 t}$ for all $t \in I$ (for discrete time processes, i.e. $I = \mathbb{Z}_+$, the term refers to $\theta_0 = e^{-\lambda_0}$).

More precisely, our first goal is to give general criteria involving Lyapunov-type functions $\varphi_1 \geq 1$ and $\varphi_2 \leq 1$ ensuring the existence of a quasi-stationary distribution ν_{QSD} such that

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV(\varphi_1)} \leq C\alpha^t \frac{\mu(\varphi_1)}{\mu(\varphi_2)}, \quad \forall t \in I, \tag{1.1}$$

for some constants $C \in (0, +\infty)$ and $\alpha \in (0, 1)$ and for all probability measure μ on E such that $\mu(\varphi_1) < +\infty$ and $\mu(\varphi_2) > 0$, where $\mu(\varphi) := \int_E \varphi(x) \, \mu(dx)$ and, for all probability measures μ_1 and μ_2,

$$\|\mu_1 - \mu_2\|_{TV(\varphi_1)} = \sup_{f:E\to\mathbb{R} \text{ measurable s.t. } |f|\leq\varphi_1} |\mu_1(f) - \mu_2(f)|.$$

When φ_1 is bounded, we recover convergence for the usual total variation distance $\|\cdot\|_{TV(1)}$ since the norms $\|\cdot\|_{TV(1)}$ and $\|\cdot\|_{TV(\varphi_1)}$ are equivalent. The measure ν_{QSD} in (1.1) is the only quasi-stationary distribution ν such that $\nu(\varphi_1) < +\infty$ and $\nu(\varphi_2) > 0$.

Our second goal is to show how our criteria can be applied to a wide range of Markov processes, including several classes of processes for which even the existence of a quasi-stationary distribution was not known, such as diffusions in irregular domains or perturbed dynamical systems in unbounded domains.

General criteria ensuring that the convergence in (1.1) holds uniformly with respect to the initial distribution μ have been studied in [10, 20]. In this case, ν_{QSD} is the quasi-limiting distribution of any initial distributions. However, these results do not apply to processes admitting several quasi-stationary distributions, which is known to happen in a variety of specific cases, even for processes irreducible in E (including branching processes [95, 2, 73, 76], one-dimensional birth and death processes [99, 47, 46, 108] and one-dimensional diffusion processes [75, 81]). In addition, as for non-absorbed processes, uniform convergence with respect to the initial distribution only happens for processes that come back quickly in compact sets [85, 20] or are killed fast [106]. The present paper provides general criteria generalizing those of [20] to cases of non-uniform convergence.

Given a quasi-stationary distribution ν, its domain of attraction is defined as the set of probability measures μ on E such that $\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial)$ converges in total variation norm to ν. In the case where the domain of attraction of ν contains all Dirac masses, ν is called the *Yaglom limit*, or the *minimal quasi-stationary distribution*. In all the models admitting several quasi-stationary distributions cited above, it has been proved that the minimal quasi-stationary distribution exists. The convergence (1.1) implies in addition that the domain of attraction of the Yaglom limit ν_{QSD} actually contains all measures μ such that $\mu(\varphi_1) < \infty$ and $\mu(\varphi_2) > 0$.

We provide in Section 2 criteria ensuring (1.1) for all $t \in \mathbb{Z}_+$. We also obtain several consequences, including a large subset of the domain of attraction of ν_{QSD} and the geometric uniform convergence of $x \mapsto e^{\lambda_0 n}\mathbb{P}_x(n < \tau_\partial)/\varphi_1(x)$ as $n \to +\infty$ to η/φ_1, where η is a function which satisfies $\mathbb{E}_x(\eta(X_n)\mathbb{1}_{n<\tau_\partial}) = e^{-\lambda_0 n}\eta(x)$ for all $n \in \mathbb{Z}_+$ and $x \in E$. We also obtain the existence of the process $(X_n, n \in \mathbb{Z}_+)$ conditioned to never be absorbed (the so-called Q-process) and its geometric ergodicity. Links between ergodicity of the Q-processes and quasi-limiting properties were already studied in various context (see for instance [1, 53, 86, 98, 49, 89]). All these results are proved in Sections 9 and 10.

The criterion developed in Section 2 assumes that $(X_n, n \in \mathbb{Z}_+)$ is aperiodic but of course applies to 1-periodic processes $(X_t, t \in I)$. Under additional aperiodicity assumptions, we show in Section 3 how the previous results extend to general time indices $t \in I$ and provide practical versions of our criteria for continuous-time processes. We also provide alternative conditions allowing to check our criteria, that are easier to check in some cases. We also show that the known criteria for uniform convergence in (1.1) obtained in [20] can be recovered using this new approach. These results are proved in Section 11.

These results allow us to put in a unified framework a large body of works on quasi-stationary distributions as illustrated by the rest of the paper, which is devoted to the application of our abstract criteria. We start in Section 4 with diffusion processes in $\mathbb{R}^d$, $d \geq 1$, absorbed at the boundary of a domain D. Our analysis provides for example the following general result.

Theorem 1.1. *Assume that $E = D$ is a bounded connected open subset of $\mathbb{R}^d$ and that $(X_t, t \in \mathbb{R}_+)$ is solution to*

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$

until its first exit time τ_∂ *from* D*, where* B *is a* r*-dimensional Brownian motion and* $b:\mathbb{R}^d\to\mathbb{R}^d$ *and* $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times r}$ *are Hölder functions, such that* σ *is uniformly elliptic. Then, the process* X *has a unique quasi-stationary distribution* ν_{QSD} *which satisfies*

$$\|\mathbb{P}_\mu(X_t\in\cdot\mid t<\tau_\partial)-\nu_{QSD}\|_{TV}\leq\frac{C}{\mu(\eta)}\,\alpha^t,\ \forall t\in[0,+\infty),$$

for some constants $C<+\infty$ *and* $\alpha\in(0,1)$*, where the function* η *is* $\mathcal{C}^2(D)$ *and satisfies*

$$\sum_{i=1}^d b_i(x)\frac{\partial\eta}{\partial x_i}(x)+\frac{1}{2}\sum_{i,j=1}^d\sum_{k=1}^r\sigma_{ik}(x)\sigma_{jk}(x)\frac{\partial^2\eta}{\partial x_i\partial x_j}(x)=-\lambda_0\eta(x),\quad\forall x\in D$$

and

$$\eta(x)=\lim_{t\to+\infty}e^{\lambda_0 t}\mathbb{P}_x(t<\tau_\partial),\quad\forall x\in D,$$

where the convergence is uniform in D*.*

We emphasize that one of the main contributions of this result with respect to the existing literature (see for example [90, 53, 15, 70, 43, 17, 26]) is that it applies to any bounded domain D without any regularity assumption, with possible applications to recent Monte-Carlo methods (see [92, 109]). Theorem 1.1 is in fact obtained in Section 4 as a particular case of a criterion for unbounded domains and coefficients b and σ only locally Hölder and locally uniformly elliptic in D. We also consider the case of diffusions with killing in Section 4.4. All these results are proved in Section 12.

Absorbed one-dimensional diffusions with or without killing have received a lot of attention (see for instance [78, 33, 75, 81, 96, 14, 74, 71, 61, 87, 23, 22]). We consider these models in Section 4.5. Our main contributions with respect to the literature are the characterization of a larger subset of the domain of attraction of the minimal quasi-stationary distribution, weaker regularity of the drift and diffusion coefficients and explicit general bounds on φ_1 and λ_0 allowing practical verification of our assumptions. Our criteria also provide alternative approaches to other classes of processes in continuous time and space, as those studied for example in [32, 6] using a spectral approach based on Tychonov's fixed point theorem, in [62, 49, 56, 13, 7] based on compactness or quasi-compactness properties, and in [80] for branching Markov processes using Lyapunov conditions on the conditioned semigroup.

The case of continuous-time Markov processes in discrete state spaces is considered in Section 5 with application to multitype birth and death processes absorbed at the exit of any connected $E\subset\mathbb{Z}_+^d$ (in the sense of the nearest neighbors structure of $\mathbb{Z}_+^d$). Note that the quasi-stationary behavior of finite state space processes [39] and of one-dimensional birth and death processes [67, 54, 16, 68, 99, 100] has been extensively studied using spectral methods that do not generalize easily to the multi-dimensional countable state-space setting. The quasi-stationary behavior of multi-dimensional birth and death processes was studied in the case of uniform convergence in (1.1) in [21, 26, 30, 31].

All the previous examples assumed irreducibility of X in E. In Section 6, we show that our criteria also apply to reducible cases, as those considered in [88] (for Galton-Watson processes), [55] (for discrete processes), [19] (for Feller diffusions) and [18, 104] (in the finite case). We first give a general criterion in Subsection 6.1 and we study in details an example with a countable infinity of communication classes in Subsection 6.2.

In Section 7, we consider general models in discrete time and continuous space, first extending the criteria of [10, 17] in order to cover the case of Euler schemes for stochastic differential equations absorbed at the boundary of a domain (as defined in [79, 51]) and penalized semigroups (as in [41, 42]; note that all our results naturally extend to penalized homogeneous semigroups, provided the penalization rate is bounded from above, see [24, 25]). We then study in details the case of perturbed dynamical systems, as those considered for example in [9, 5, 62], where the quasi-stationary behavior was studied using the criterion of [10]. As an illustration of our method, let us mention the following original result.

Theorem 1.2. *Let $E = D$ be a measurable set of $\mathbb{R}^d$ with positive Lebesgue measure and let $\partial \notin D$. Assume that*

$$X_{n+1} = \begin{cases} f(X_n) + \xi_n & \text{if } X_n \neq \partial \text{ and } f(X_n) + \xi_n \in D, \\ \partial & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally bounded measurable function such that

$$|x| - |f(x)| \xrightarrow[|x|\to+\infty]{} +\infty$$

and $(\xi_n)_{n\in\mathbb{N}}$ is an i.i.d. non-degenerate Gaussian sequence in $\mathbb{R}^d$. Then (1.1) is satisfied for $\varphi_1(x) = e^{|x|}$ and a positive measurable function φ_2 on D.

Finally, we study in Section 8 the case of processes in discrete time and discrete space. This is the most studied situation in the literature since it covers both the Galton-Watson processes [112, 59, 64, 2] and the general discrete case [38, 95, 47, 48, 46, 45, 55, 82]. We first show in Subsection 8.1 that our results allow to recover the general criterion of [45], based on the theory of R-positive matrices. We then consider general population processes dominated by population-dependent multi-type Galton-Watson processes in Subsection 8.2. The case of population-dependent Galton-Watson processes with a single type was studied in [55] using quasi-compactness methods. We also obtain as a corollary several results on subcritical multi-type Galton-Watson processes. We do not recover the optimal $L \log L$ assumption on the offspring distribution [64, 60] for the existence of a minimal quasi-stationary distribution ν_{QSD} having finite first moment, but we obtain a stronger form of convergence in (1.1), a larger subset of its domain of attraction and stronger moments properties on ν_{QSD}.

2 Main results

Let $(X_t, t \in I)$ be a Markov process in $E \cup \{\partial\}$ where E is a measurable space and $\partial \notin E$, with set of time indices I which might be $\mathbb{Z}_+ = \{0, 1, \ldots\}$, $\mathbb{R}_+$ or $\frac{1}{k}\mathbb{Z}_+$ for some $k \in \mathbb{N} = \{1, 2, \ldots\}$. We define the absorption time τ_∂ as

$$\tau_\partial = \inf\{t \in I,\, X_t = \partial\}.$$

In this section, we study the sub-Markovian transition semigroup of X considered at integer times, $(P_n)_{n\in\mathbb{Z}_+}$, defined as

$$P_n f(x) = \mathbb{E}_x\left(f(X_n)\mathbb{1}_{n<\tau_\partial}\right),\ \forall n \in \mathbb{Z}_+,$$

for all bounded or nonnegative measurable function f on E and all $x \in E$. We also define as usual the left-action of P_n on measures as

$$\mu P_n f = \mathbb{E}_\mu\left(f(X_n)\mathbb{1}_{n<\tau_\partial}\right) = \int_E P_n f(x)\,\mu(\mathrm{d}x),$$

for all probability measure μ on E. We make the following assumption.

Assumption (E). There exist a positive integer n_1, positive real constants $\theta_1, \theta_2, c_1, c_2, c_3$, two functions $\varphi_1, \varphi_2 : E \to \mathbb{R}_+$ and a probability measure ν on a measurable subset $K \subset E$ such that

(E1) *(Local Dobrushin coefficient).* $\forall x \in K$,

$$\mathbb{P}_x(X_{n_1} \in \cdot) \geq c_1 \nu(\cdot \cap K).$$

(E2) *(Global Lyapunov criterion).* We have $\theta_1 < \theta_2$ and

$$\begin{aligned}
&\inf_{x\in E} \varphi_1(x) \geq 1,\ \sup_{x\in K} \varphi_1(x) < \infty \\
&\inf_{x\in K} \varphi_2(x) > 0,\ \sup_{x\in E} \varphi_2(x) \leq 1, \\
&P_1\varphi_1(x) \leq \theta_1\varphi_1(x) + c_2\mathbb{1}_K(x),\ \forall x \in E \\
&P_1\varphi_2(x) \geq \theta_2\varphi_2(x),\ \forall x \in E.
\end{aligned}$$

(E3) *(Local Harnack inequality).* We have

$$\sup_{n\in\mathbb{Z}_+} \frac{\sup_{y\in K} \mathbb{P}_y(n < \tau_\partial)}{\inf_{y\in K} \mathbb{P}_y(n < \tau_\partial)} \leq c_3$$

(E4) *(Aperiodicity).* For all $x \in K$, there exists $n_4(x)$ such that, for all $n \geq n_4(x)$,

$$\mathbb{P}_x(X_n \in K) > 0.$$

Note that it follows from (E2) that $\theta_2 \leq 1$ and thus $\theta_1 < 1$. We also emphasize that our assumptions neither require that $\tau_\partial < +\infty$ $\mathbb{P}_x$-a.s., nor that $\mathbb{P}_x(n < \tau_\partial) > 0$ for all $t \geq 0$ and $x \in E$. Several examples of Markov processes satisfying Assumption (E) are provided in Sections 4 to 8.

Assumption (E) is an extension of the ergodicity criteria developed in [84]. Indeed, if we assume that $\tau_\partial = \infty$ $\mathbb{P}_x$-almost surely for all $x \in E$, then Condition (E3) becomes void and one can take $\varphi_2 \equiv 1$ in (E2), so that $\theta_2 = \theta_0 = 1$. We recognize in (E1) the standard "small set" assumption of [84], in (E2) for φ_1 a standard Foster-Lyapunov criterion and in (E4) an aperiodicity condition. As such, it is well-known that alternative formulations of these conditions can be given. In the general case, we provide in Section 3.1 conditions ensuring the existence of Lyapunov functions satisfying (E2) in terms of exponential moment of hitting times for φ_1 and exponential decay of the probability to be in K for φ_2, and conditions ensuring (E1) and (E3) based on comparisons between transition probabilities. Similarly as for the ergodicity criteria developed in [84], we extend our criterion to the continuous-time setting in Section 3.2.

In the rest of this section, we state the main general consequences of Assumption (E). We start with the exponential contraction in total variation of the conditional marginal distributions of the process given non-absorption. Its proof is given in Section 9.

Theorem 2.1. *Assume that Condition (E) holds true. Then there exist a constant $C > 0$, a constant $\alpha \in (0, 1)$, a probability measure ν_{QSD} on E such that $\nu_{QSD}(K) > 0$ and such that*

$$\left\| \frac{\mu P_n}{\mu P_n \mathbb{1}_E} - \nu_{QSD} \right\|_{TV(\varphi_1)} \leq C\, \alpha^n \frac{\mu(\varphi_1)}{\mu(\varphi_2)}, \quad \forall n \geq 0, \tag{2.1}$$

for all probability measure μ on E such that $\mu(\varphi_1) < \infty$ and $\mu(\varphi_2) > 0$. In addition, ν_{QSD} is the unique quasi-stationary distribution satisfying $\nu_{QSD}(\varphi_2) > 0$ and $\nu_{QSD}(\varphi_1) < \infty$.

Remark 2.2. For all $p \geq 1$, Hölder's inequality entails

$$P_1(\varphi_1^{1/p}) \leq (\theta_1\varphi_1 + c_2\mathbb{1}_K)^{1/p} \leq \theta_1^{1/p}\varphi_1^{1/p} + c_2^{1/p}\mathbb{1}_K,$$

so that $(\varphi_1^{1/p}, \varphi_2)$ satisfies Assumption (E) for all $p < \log\theta_1/\log\theta_2$. Therefore, the exponential convergence (2.1) actually holds true for the norm $\|\cdot\|_{TV(\varphi_1^{1/p})}$ and measures μ such that $\mu(\varphi_1^{1/p}) < +\infty$ for some $p < \log\theta_1/\log\theta_2$.

In the following result, we show the existence of an eigenfunction η of P_1 for the eigenvalue θ_0, where $\theta_0 \in (0,1]$ is such that

$$\mathbb{P}_{\nu_{QSD}}(n < \tau_\partial) = \theta_0^n, \quad \forall n \in \mathbb{N}.$$

We recall that the existence of the decay parameter θ_0 is a classical general result for quasi-stationary distributions [83, 34]. The proof of the following result is initiated in Section 10.1 and concluded in Section 10.3. To state this result, we define for all positive function ψ on E the space $L^\infty(\psi)$ as the set of measurable real functions f on E such that $\|f\|_{L^\infty(\psi)} := \sup_{x\in E} f(x)/\psi(x) < \infty$. Note that $(L^\infty(\psi), \|\cdot\|_{L^\infty(\psi)})$ is a Banach space.

Theorem 2.3. *Assume that Condition (E) holds true. Then there exists a function* $\eta : E \to \mathbb{R}_+$ *such that*

$$\eta(x) = \lim_{n\to+\infty} \frac{\mathbb{P}_x(n < \tau_\partial)}{\mathbb{P}_{\nu_{QSD}}(n < \tau_\partial)} = \lim_{n\to+\infty} \theta_0^{-n} \mathbb{P}_x(n < \tau_\partial), \quad \forall x \in E, \tag{2.2}$$

where the convergence is geometric in $L^\infty(\varphi_1)$*. In addition, we have* $\inf_{y\in K} \eta(y) > 0$*,* $\nu_{QSD}(\eta) = 1$*,* $\eta \in L^\infty\left(\varphi_1^{\log(1/\theta_0)/\log(1/\theta_1)}\right)$*,*

$$P_1\eta = \theta_0\eta \quad \textit{and} \quad \theta_0 \geq \theta_2 > \theta_1.$$

Remark 2.4. In general, there is no simple relation between φ_2 and η, in particular φ_2 is not necessarily an element of $L^\infty(\eta)$. However, it is true that, for all $x \in E$, $P_k\varphi_2(x) > 0$ for some $k \geq 0$ if and only if $\eta(x) > 0$ (see Corollary 2.10 below).

Remark 2.5. Note that, when η is bounded, the last result implies that one can actually take $\varphi_2 = \eta/\|\eta\|_\infty$ in Condition (E2). Results with unbounded φ_2 or $1/\varphi_1$ can also be obtained by taking the φ_1-transform of $(P_n)_{n\in\mathbb{Z}_+}$ (see [4, 25]).

We consider now the Q-process and its ergodicity properties under Condition (E). In the next result, proved in Section 10.2, $\Omega = E^{\mathbb{Z}_+}$ is the canonical state space of Markov chains on E and $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ is the associated canonical filtration. We emphasize that the constant α may differ from the one in Theorem 2.1. In the following result, we define

$$E' := \{x \in E, \eta(x) > 0\}.$$

Theorem 2.6. *Condition (E) implies the following properties.*

(i) Existence of the Q-process. *There exists a family* $(\mathbb{Q}_x)_{x\in E'}$ *of probability measures on* Ω *defined by*

$$\lim_{n\to+\infty} \mathbb{P}_x(A \mid n < \tau_\partial) = \mathbb{Q}_x(A)$$

for all $x \in E'$*, for all* $\mathcal{F}_m$*-measurable set* A *and for all* $m \geq 0$*. The process* $(\Omega, (\mathcal{F}_n)_{n\in\mathbb{Z}_+}, (X_n)_{n\in\mathbb{Z}_+}, (\mathbb{Q}_x)_{x\in E'})$ *is an* E'*-valued homogeneous Markov chain.*

(ii) Semigroup. *The semigroup of the Markov process* X *under* $(\mathbb{Q}_x)_{x\in E'}$ *is given for all bounded measurable function* φ *on* E' *and* $n \geq 0$ *by*

$$\widetilde{P}_n\varphi(x) = \frac{\theta_0^{-n}}{\eta(x)} P_n(\eta\varphi)(x). \tag{2.3}$$

(iii) Exponential ergodicity. *The probability measure* β *on* E' *defined by*

$$\beta(\mathrm{d}x) = \eta(x)\nu_{QSD}(\mathrm{d}x).$$

is the unique invariant distribution of the Markov process X under $(\mathbb{Q}_x)_{x\in E'}$. Moreover, there exist constants $C>0$ and $\alpha\in(0,1)$ such that, for all initial distributions μ on E' such that $\mu(\varphi_1/\eta)<\infty$ and

$$\left\|\mu\widetilde{P}_n-\beta(h)\right\|_{TV(\varphi_1/\eta)}\leq C\alpha^n\,\mu\left(\varphi_1/\eta\right),\quad\forall n\geq 0,\tag{2.4}$$

where $\mathbb{Q}_\mu=\int_{E'}\mathbb{Q}_x\,\mu(dx)$. In addition, for all initial distributions μ on E',

$$\left\|\mu\widetilde{P}_n-\beta\right\|_{TV}\xrightarrow[n\to\infty]{}0.\tag{2.5}$$

We conclude this section with corollaries of the last theorem. The following result is proved in Section 10.3.

Corollary 2.7. *Assume that Condition (E) holds true. Then there exist constants $C>0$ and $\alpha\in(0,1)$ such that, for all probability measure μ on E such that $\mu(\varphi_1)<+\infty$,*

$$\left\|\theta_0^{-n}\mu P_n-\mu(\eta)\nu_{QSD}\right\|_{TV(\varphi_1)}\leq C\,\alpha^n\mu(\varphi_1).\tag{2.6}$$

Remark 2.8. The proof of Theorem 2.6 makes use of [57, 58], which allows to derive explicit expressions for the constants C and α (we refer the interested reader to Remark 10.1). In particular, using these estimates in the proof of Corollary 2.7 would also provide explicit constants in (2.6).

Remark 2.9. The formulation (2.6) for the convergence of the semigroup is natural in this setting, since a property of equivalence between (2.6) and Condition (E) is proved in [4, 25].

The last corollary has consequences on the attraction domain of ν_{QSD}.

Corollary 2.10. *Assume that Condition (E) holds true. Then*

$$E'=\{x\in E:\exists k\geq 0,\ P_k\varphi_2(x)>0\}$$

and the domain of attraction of ν_{QSD} for the total variation norm contains all probability measures on E such that $\mu(E')>0$ and $\mu(\varphi_1^{1/p})<+\infty$ for some $p<\log\theta_1/\log\theta_2$. If in addition φ_1 is bounded, then the domain of attraction of ν_{QSD} is the set of probability measures on E such that $\mu(E')>0$ and ν_{QSD} is the unique quasi-stationnary distribution giving positive mass to E'.

Convergence estimates can also be obtained for initial distributions on E' satisfying $\mu(\eta)<+\infty$ but not necessarily $\mu(\varphi_1)<+\infty$. The following result is proved in 10.5.

Corollary 2.11. *Assume that Condition (E) holds true. Then, for all probability measures μ on E' such that $\mu(\eta)<+\infty$,*

$$\left\|\theta_0^{-n}\mu P^n-\mu(\eta)\nu_{QSD}\right\|_{TV(\eta)}\xrightarrow[n\to+\infty]{}0.\tag{2.7}$$

In particular, if η is positive on E, then ν_{QSD} is the unique quasi-stationary distribution of X such that $\nu_{QSD}(\eta)<+\infty$. If in addition η is lower bounded away from 0 on E, then for all probability measures μ on E such that $\mu(\eta)<+\infty$, we have

$$\|\mathbb{P}_\mu(X_n\in\cdot\mid n<\tau_\partial)-\nu_{QSD}\|_{TV(\eta)}\xrightarrow[n\to+\infty]{}0.\tag{2.8}$$

In particular, the domain of attraction of ν_{QSD} contains all probability measures μ on E such that $\mu(\eta)<+\infty$.

3 Other formulations and particular cases of Assumption (E)

In this section, we provide general comments on Assumption (E). Alternative formulations of our assumptions and simple criteria are gathered in Subsection 3.1. Subsection 3.2 focuses on criteria adapted to continuous time processes and we consider the case of uniform convergence in Theorem 2.1 in Subsection 3.3.

3.1 General comments on the assumptions

We propose here alternative formulations of Condition (E2) and criteria ensuring (E1) and (E3) when (E2) and (E4) are satisfied, that may be easier to check in some practical situations. In particular, we make strong use of these results in Sections 7 and 8.

3.1.1 Construction of Lyapunov functions satisfying (E2)

In order to prove the existence of functions φ_1 and φ_2 in Condition (E2), one may use probabilistic properties of the Markov process X, as stated in the following lemmas, proved in Sections 11.1 and 11.2. The first lemma shows a way to construct φ_2.

Lemma 3.1. *Let K be a measurable subset of E. If there exists $\theta_2 > 0$ such that*

$$\inf_{x\in K} \theta_2^{-n}\mathbb{P}_x(X_n \in K) \xrightarrow[n\to+\infty]{} +\infty,$$

then the function $\varphi_2 : E \to [0,1]$ defined by $\varphi_2(x) = \frac{\theta_2^{-1}-1}{\theta_2^{-\ell}-1}\sum_{k=0}^{\ell-1}\theta_2^{-k}\mathbb{P}_x(X_k \in K)$, for any ℓ is such that $\theta_2^{-\ell}\inf_{x\in K}\mathbb{P}_x(X_\ell \in K) \geq 1$, verifies $\inf_K \varphi_2 > 0$ and $P_1\varphi_2(x) \geq \theta_2\,\varphi_2(x)$. Moreover, (E4) is satisfied.

The second lemma shows how φ_1 can be constructed. This is a well-known result in the case without absorption [84], which can provide easier ways to check (E2) in some situations. We define

$$T_K = \inf\{n \in \mathbb{Z}_+,\ X_n \in K\}. \tag{3.1}$$

Lemma 3.2. *Let K be a measurable subset of E. If there exists a constant $\theta_1 > 0$ such that*

$$\mathbb{E}_x\left(\theta_1^{-T_K\wedge\tau_\partial}\right) < +\infty\ \forall x \in E \text{ and } C := \sup_{y\in K}\mathbb{E}_y\left(\mathbb{E}_{X_1}\left(\theta_1^{-T_K\wedge\tau_\partial}\right)\mathbb{1}_{1<\tau_\partial}\right) < +\infty,$$

then the function $\varphi_1 : E \to [1,+\infty)$ defined by $\varphi_1(x) = \mathbb{E}_x\left(\theta_1^{-T_K\wedge\lceil\tau_\partial\rceil}\right)$ satisfies

$$\sup_K \varphi_1 < +\infty \quad \text{and} \quad P_1\varphi_1 \leq \theta_1\varphi_1 + \frac{C}{\theta_1}\mathbb{1}_K.$$

Conversely, if there exist two constants $C > 0$, $\theta_1 > 0$ and a function $\varphi_1 : E \to [1,+\infty)$ such that $\sup_K \varphi_1 < +\infty$ and $P_1\varphi_1 \leq \theta_1\varphi_1 + C\mathbb{1}_K$, then, for all $\theta > \theta_1$, there exists a constant C_θ such that

$$\mathbb{E}_x\left(\theta^{-T_K\wedge\tau_\partial}\right) \leq C_\theta\varphi_1(x)\ \forall x \in E \text{ and } \sup_{y\in K}\mathbb{E}_y\left(\mathbb{E}_{X_1}\left(\theta^{-T_K\wedge\tau_\partial}\right)\mathbb{1}_{1<\tau_\partial}\right) < +\infty.$$

Note that the hitting time T_K is defined from the process $(X_n)_{n\in\mathbb{Z}_+}$. When $I \neq \mathbb{Z}_+$, it might be easier to use criteria based on the hitting time τ_K defined from the full process $(X_n)_{n\in I}$. We refer the reader to Lemma 3.6 below for that.

3.1.2 Checking (E1) and (E3) from comparisons between transition probabilities

Condition (E3) is a form of Harnack inequality, and one can indeed use general versions of these inequalities to check (E3) and (E1) (for example, our results on diffusions given in Section 4 use this idea, cf. Section 12.2). We propose below another criterion, based on comparison techniques on transition probabilities, to check that Conditions (E1) and (E3) hold true when Conditions (E2) and (E4) are satisfied. This result is proved in Subsection 11.3.

Proposition 3.3. *Assume that Conditions (E2) and (E4) are satisfied and that there exist two constants* $C > 0$ *and* $n_0 \leq m_0 \in \mathbb{N}$ *such that*

$$\mathbb{P}_x(X_{n_0} \in \cdot \cap K) \leq C\,\mathbb{P}_y(X_{m_0} \in \cdot),\ \forall x \in E \text{ and } y \in K. \tag{3.2}$$

Then Condition (E) is satisfied.

3.1.3 Optimal value of θ_2 in (E2)

As many results of Section 2 make use of the function $\varphi_1^{1/p}$ with a parameter $p \in [1, \log\theta_1/\log\theta_2)$, it is important to characterize the largest possible value of θ_2. This result is proved in Section 11.4.

Lemma 3.4. *If Condition (E) is satisfied for some functions* φ_1 *and* φ_2 *with constants* θ_1 *and* θ_2*, then, for all* $\theta_2' \in (\theta_1, \theta_0)$ *it is also satisfied for* φ_1 *and some function* φ_2' *with constants* θ_1 *and* θ_2'*.*

3.2 On continuous time

In Section 2, we only considered the conditional behavior of the process X at integer times. In general, the results of Section 2 do not give information about the process at intermediate times. In this section, we derive a sufficient condition which is well suited for practical verification in the case of continuous time Markov processes or for aperiodic Markov processes, in particular because (F2) below is usually easier to check than (E2). We consider an absorbed Markov process $(X_t)_{t\in I}$ with time parameter in $I = \mathbb{Z}_+$ or $[0, +\infty)$.

Assumption (F). There exist positive real constants $\gamma_1, \gamma_2, c_1, c_2$ and c_3, $t_1, t_2 \in I$, a measurable function $\psi_1 : E \to [1, +\infty)$, and a probability measure ν on a measurable subset $L \subset E$ such that

(F0) *(A strong Markov property).* Defining

$$\tau_L := \inf\{t \in I : X_t \in L\}, \tag{3.3}$$

assume that for all $x \in E$, $X_{\tau_L} \in L$, $\mathbb{P}_x$-almost surely on the event $\{\tau_L < \infty\}$ and for all $t \in I$ and all measurable $f : E \cup \{\partial\} \to \mathbb{R}_+$,

$$\mathbb{E}_x\left[f(X_t)\mathbb{1}_{\tau_L \leq t < \tau_\partial}\right] = \mathbb{E}_x\left[\mathbb{1}_{\tau_L \leq t\wedge\tau_\partial}\mathbb{E}_{X_{\tau_L}}\left[f(X_{t-u})\mathbb{1}_{t-u<\tau_\partial}\right]\big|_{u=\tau_L}\right].$$

(F1) *(Local Dobrushin coefficient).* $\forall x \in L$,

$$\mathbb{P}_x(X_{t_1} \in \cdot) \geq c_1\nu(\cdot \cap L).$$

(F2) *(Global Lyapunov criterion).* We have $\gamma_1 < \gamma_2$ and

$$\begin{aligned}
&\mathbb{E}_x(\psi_1(X_{t_2})\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial}) \leq \gamma_1^{t_2}\psi_1(x),\ \forall x \in E\\
&\mathbb{E}_x(\psi_1(X_t)\mathbb{1}_{t<\tau_\partial}) \leq c_2,\ \forall x \in L,\ \forall t \in [0, t_2] \cap I,\\
&\gamma_2^{-t}\mathbb{P}_x(X_t \in L) \xrightarrow[t\to+\infty]{} +\infty,\ \forall x \in L.
\end{aligned}$$

(F3) *(Local Harnack inequality).* We have

$$\sup_{t\geq 0}\frac{\sup_{y\in L}\mathbb{P}_y(t<\tau_\partial)}{\inf_{y\in L}\mathbb{P}_y(t<\tau_\partial)}\leq c_3$$

Be careful that the definition of τ_L in (3.3) is different from that of T_L in (3.1). Note also that, in (F2), the Lyapunov function φ_2 has been replaced by an alternative condition similar to Lemma 3.1. Both are actually equivalent thanks to (F0) (see the beginning of Section 11.5.1).

The following result is proved in Section 11.5.

Theorem 3.5. *Under Assumption (F), $(X_t)_{t\in I}$ admits a quasi-stationary distribution ν_{QSD}, which is the unique one satisfying $\nu_{QSD}(\psi_1)<\infty$ and $\nu_{QSD}(L)>0$ for some $t\in I$. Moreover, there exist constants $\alpha\in(0,1)$ and $C>0$ such that, for all probability measures μ on E satisfying $\mu(\psi_1)<\infty$ and $\mu(\psi_2)>0$,*

$$\|\mathbb{P}_\mu(X_t\in\cdot\mid t<\tau_\partial)-\nu_{QSD}\|_{TV(\psi_1)}\leq C\,\alpha^t\,\frac{\mu(\psi_1)}{\mu(\psi_2)},\ \forall t\in I, \tag{3.4}$$

where $\psi_2(x)=\sum_{k=0}^{n_0}\gamma_2^{-kt_2}\mathbb{P}_x(X_{kt_2}\in L)$ for some $n_0\geq 1$ large enough. In addition, there exists a constant $\lambda_0\geq 0$ such that $\lambda_0\leq\log(1/\gamma_2)<\log(1/\gamma_1)$ and $\mathbb{P}_{\nu_{QSD}}(t<\tau_\partial)=e^{-\lambda_0 t}$ for all $t\geq 0$, and there exists a function η such that

$$\eta(x)=\lim_{t\to+\infty}e^{\lambda_0 t}\mathbb{P}_x(t<\tau_\partial),\quad\forall x\in E, \tag{3.5}$$

where the convergence is exponential in $L^\infty(\psi_1^{1/p})$ for all $p\in[1,\log(1/\gamma_1)/\lambda_0)$, and $P_t\eta(x)=e^{-\lambda_0 t}\eta(x)$ for all $x\in E$ and $t\in I$.

A key point that guided our formulation of Condition (F) is that, for continuous-time Markov processes, usual practical conditions for the existence of ψ_1 are provided by Foster-Lyapunov inequalities (cf. [84]). They involve the extended infinitesimal generator $\bar{\mathcal{L}}$ of the process X (see e.g. [84, 26]) and take the form

$$\bar{\mathcal{L}}\psi_1(x)\leq-\lambda_1\psi_1(x)+C\mathbb{1}_K(x),\quad\forall x\in E. \tag{3.6}$$

This inequality does not imply, in general, that (E2) holds true for $\varphi_1=\psi_1$. However, Equation (3.6) implies (formally, assuming one can apply Dynkin's formula) that $\mathbb{E}_x[\mathbb{1}_{1\leq\tau_L\wedge\tau_\partial}\psi_1(X_1)]\leq e^{-\lambda_1}\psi_1(x)$ and $\mathbb{E}_x[\psi_1(X_t)\mathbb{1}_{t<\tau_\partial}]\leq e^{Ct}\psi_1(x)$. Hence the first two lines of (F2) can be deduced from classical Foster Lyapunov criteria. This will be used for diffusion processes in Section 4 or in discrete state space in Section 5.

Alternatively, one can use controls on the exponential moments for the return times in L. The following result, similar to Lemma 3.2, is proved in Section 11.6.

Lemma 3.6. *Assume that there exist positive constants $\gamma_1>0$ and $t_2\in I$ such that*

$$\mathbb{E}_x\left(\gamma_1^{-\tau_L\wedge\tau_\partial}\right)<\infty,\ \forall x\in E\quad\text{and}\quad\sup_{x\in L}\mathbb{E}_x\left(\mathbb{E}_{X_{t_2}}\left(\gamma_1^{-\tau_L\wedge\tau_\partial}\right)\mathbb{1}_{t_2<\tau_\partial}\right)<+\infty,$$

then $\psi_1(x)=\mathbb{E}_x\left(\gamma_1^{-\tau_L\wedge\tau_\partial}\right)$ satisfies

$$\begin{aligned}&\mathbb{E}_x(\psi_1(X_{t_2})\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial})\leq\gamma_1^{t_2}\psi_1(x),\ \forall x\in E\\&\mathbb{E}_x(\psi_1(X_t)\mathbb{1}_{t<\tau_\partial})\leq c_2,\ \forall x\in L,\ \forall t\in[0,t_2]\cap I,\end{aligned}$$

for some constant $c_2>0$.

Remark 3.7. In the proof of Theorem 3.5, we will show that Assumption (F) implies that Assumption (E) is satisfied for the sub-Markovian semigroup $(P_n)_{n\geq 0}$ of the absorbed Markov process $(X_{nt_2})_{n\in\mathbb{Z}_+}$, with the functions $\varphi_1 = \psi_1$ and $\varphi_2 = \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-(n_0+1)t_2}-1}\psi_2$, any $\theta_1 \in (\gamma_1^{t_2}, \gamma_2^{t_2})$, $\theta_2 = \gamma_2^{t_2}$ and the set

$$K = \left\{y \in E,\ \mathbb{P}_y(\tau_L \leq t_2)/\psi_1(y) \geq (\theta_1 - \gamma_1^{t_2})/c_2\right\} \supset L.$$

In particular, all the consequences of (E) stated in Section 2 hold true. Moreover, it is also possible to obtain a continuous-time version of Theorem 2.6 about the Q-process by adapting the proof given in Section 10.2.

Remark 3.8. If $I = \mathbb{R}_+$, it follows from the fact that $P_t\eta = e^{-\lambda_0 t}\eta$ that, setting $\eta(\partial) = 0$, the function η defined on $E \cup \{\partial\}$ belongs to the domain of the infinitesimal generator $\mathcal{L}$ of the semigroup of the Markov process X on $E \cup \{\partial\}$, seen as acting on $L^\infty(\psi_1^{1/p})$ for $p \in [1, \log(1/\gamma_1)/\lambda_0)$, and $\mathcal{L}\eta = -\lambda_0\eta$.

3.3 The case of uniform exponential convergence

We now want to characterize the case of exponential convergence in total variation of the conditional distributions of (X_n) to ν_{QSD}, uniformly with respect to the initial distribution μ. This question was already studied in [20]. The next result, proved in Section 11.7, gives a necessary and sufficient condition based on Condition (E).

Proposition 3.9. *There exists constants C and $\alpha < 1$ such that, for all probability measure μ on E and all integer n,*

$$\|\mathbb{P}_\mu(X_n \in \cdot \mid n < \tau_\partial) - \nu_{QSD}\|_{TV} \leq C\alpha^n, \tag{3.7}$$

if and only if Condition (E) is satisfied with a bounded function φ_1 and there exists an integer $n_4' > 0$ such that

$$\underline{c} := \inf_{x\in E} \mathbb{P}_x(X_{n_4'} \in K \mid n_4' < \tau_\partial) > 0. \tag{3.8}$$

4 Application to diffusion processes

In this section, we apply the criteria (E) and (F) to diffusion processes absorbed at the boundary of a domain. We give a general criterion in Subsection 4.1 and apply it to uniformly elliptic diffusions in Subsection 4.2 and to an example with vanishing diffusion coefficient at the boundary of the domain in Subsection 4.3. Our criteria are extended to diffusions with killing in Subsection 4.4 and the particular case of one-dimensional diffusions is studied in Subsection 4.5.

4.1 A general criterion in any dimension

We consider a diffusion process X on a connected, open domain $D \subset \mathbb{R}^d$ for some $d \geq 1$, solution to the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \tag{4.1}$$

where B is a standard, r-dimensional Brownian motion and $b : D \to \mathbb{R}^d$ and $\sigma : D \to \mathbb{R}^{d\times r}$ are locally Hölder functions, such that σ is locally uniformly elliptic in D, i.e.

$$\forall K \subset D \text{ compact},\quad \inf_{x\in K}\inf_{s\in\mathbb{R}^d\setminus\{0\}} \frac{s^*\sigma(x)\sigma^*(x)s}{|s|^2} > 0,$$

where $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^d$. We assume that the process is immediately absorbed at some cemetery point $\partial \notin D$ at its first exit time of D, denoted τ_∂.

The existence and basic properties of this process need some care since the coefficients b and σ are only defined in the open set D without any assumption on the boundary of D, and so may not be possible to extend as continuous functions out of this set. Details are given in Subsection 12.1. For the moment, let us only observe that, for all $k \geq 1$, defining the compact set

$$K_k = \{x \in D : |x| \leq k \text{ and } d(x, D^c) \geq 1/k\},$$

a weak solution to (4.1) can be constructed up to the first exit time $\tau_{K_k^c}$ of K_k as defined in (3.3). The proper definition of the absorption time τ_∂ is

$$\tau_\partial = \sup_{k \geq 1} \tau_{K_k^c}. \tag{4.2}$$

We introduce the differential operator associated to the SDE (4.1), related to the infinitesimal generator of the process X: for all $f \in \mathcal{C}^2(D)$, we define for all $x \in D$

$$\mathcal{L}f(x) := \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{r} \sigma_{ik}(x)\sigma_{jk}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x). \tag{4.3}$$

We also define the constant

$$\lambda_0 := \inf \left\{ \lambda > 0, \text{ s.t. } \liminf_{t \to +\infty} e^{\lambda t}\, \mathbb{P}_x\left(X_t \in B\right) > 0 \right\} \tag{4.4}$$

for some $x \in D$ and some open ball B such that $\overline{B} \subset D$. It is standard to prove using Harnack inequalities (proved in our case in Section 12.2) that, under the previous assumptions, $\lambda_0 < +\infty$ and its value is independent of the choice of $x \in D$ and of the non-empty, open ball B such that $\overline{B} \subset D$.

The following result is proved in Sections 12.1 to 12.3.

Theorem 4.1. *Assume that there exist some constants $C > 0$, $\lambda_1 > \lambda_0$, a $\mathcal{C}^2(D)$ function $\varphi : D \to [1, +\infty)$ and a subset $D_0 \subset D$ closed in D such that $\sup_{x \in D_0} \varphi(x) < +\infty$ and*

$$\mathcal{L}\varphi(x) \leq -\lambda_1 \varphi(x) + C \mathbb{1}_{x \in D_0}, \ \forall x \in D. \tag{4.5}$$

Assume also that there exists a time $s_1 > 0$ such that

$$\sup_{x \in D_0} \mathbb{P}_x(s_1 < \tau_{K_k} \wedge \tau_\partial) \xrightarrow[k \to \infty]{} 0. \tag{4.6}$$

Then X admits a quasi-stationary distribution ν_{QSD} which satisfies $\nu_{QSD}(\varphi^{1/p}) < +\infty$ for all $p > 1$. Moreover, for all $p \in (1, \lambda_1/\lambda_0)$, there exist a constant $\alpha_p \in (0,1)$, a constant C_p and a function $\varphi_{2,p} : D \to (0, +\infty)$ uniformly bounded away from 0 on compact subsets of D such that, for all probability measures μ on E satisfying $\mu(\varphi^{1/p}) < \infty$,

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV(\varphi^{1/p})} \leq C_p \alpha_p^t \frac{\mu(\varphi^{1/p})}{\mu(\varphi_{2,p})}, \ \forall t \in [0, +\infty).$$

In particular, ν_{QSD} is the only quasi-stationary distribution of X which satisfies $\nu_{QSD}(\varphi^{1/p}) < +\infty$ for at least one value of $p \in (1, \lambda_1/\lambda_0)$.

Remark 4.2. Note that $\tau_{K_k} = 0$ $\mathbb{P}_x$-a.s. for all $x \in K_k$, thus

$$\sup_{x \in D_0} \mathbb{P}_x(s_1 < \tau_{K_k} \wedge \tau_\partial) = \sup_{x \in D_0 \setminus K_k} \mathbb{P}_x(s_1 < \tau_{K_k} \wedge \tau_\partial).$$

Hence Condition (4.6) requires the process to be absorbed or return in K_k fast starting in $D_0 \setminus K_k$.

Remark 4.3. We shall actually prove that, under the conditions of the previous theorem, Assumption (F) is satisfied with $L = K_k$ for some $k \geq 1$, and $\psi_1 = \varphi^{1/p}$, for any $p \in (1, \lambda_1/\lambda_0)$.

Remark 4.4. In general, the assumptions of Theorem 4.1 do not ensure the non-explosion of the Markov process X. In the case of an explosive Markov process, the definition of τ_∂ in (4.2) implies that, in the event of an explosion, the absorption time τ_∂ is defined as equal to the explosion time.

The last result has other consequences of interest, gathered in the next corollary, proved in Section 12.4.

Corollary 4.5. *Under the assumptions of Theorem 4.1, the infimum defining the constant* λ_0 *in* (4.4) *is actually a minimum and it satisfies* $\mathbb{P}_{\nu_{QSD}}(t < \tau_\partial) = e^{-\lambda_0 t}$ *for all* $t \geq 0$*. In addition, the function* η *of Theorem 3.5 satisfies* $P_t\eta = e^{-\lambda_0 t}\eta$ *for all* $t \geq 0$*. In particular,* η *belongs to the domain of the infinitesimal generator of the semigroup of the process* X *defined as acting on the Banach space* $L^\infty(\varphi_1)$*, and it is an eigenfunction for the eigenvalue* $-\lambda_0$*. In addition,* $\eta \in \mathcal{C}^2(D)$ *and* $\mathcal{L}\eta(x) = -\lambda_0\eta(x)$ *for all* $x \in D$*.*

4.2 Application to uniformly elliptic diffusion processes

We consider the case where σ can be extended to $\mathbb{R}^d$ as a locally uniformly elliptic matrix-valued function. In the following corollary, we give a general situation where (4.6) holds true. We emphasize that, contrary to previous results on existence of quasi-stationary distributions for diffusions in a domain (see e.g. [90, 53, 70, 43, 17]), no regularity on the boundary of D is required.

Corollary 4.6. *Let* D *be an open connected subset of* $\mathbb{R}^d$*,* $d \geq 1$*. Let* X *be solution to the SDE*

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,\ t < \tau_\partial, \tag{4.7}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ *and* $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ *are locally Hölder continuous in* $\mathbb{R}^d$ *and* σ *is locally uniformly elliptic on* $\mathbb{R}^d$*. Recall the definition* (4.4) *of* λ_0 *and assume that there exist constants* $C > 0$*,* $\lambda_1 > \lambda_0$*, a* $\mathcal{C}^2(D)$ *function* $\varphi : D \to [1, +\infty)$ *and a bounded subset* $D_0 \subset D$ *closed in* D *such that*

$$\mathcal{L}\varphi(x) \leq -\lambda_1\varphi(x) + C\mathbb{1}_{x\in D_0},\ \forall x \in D. \tag{4.8}$$

Then the process X *absorbed at the boundary of* D *(in the sense of* (4.2)*) satisfies the assumptions of Theorem 4.1.*

Note that we do not assume that $\varphi(x) \to +\infty$ when $|x| \to +\infty$, hence the process X may be explosive (see Remark 4.4).

Proof. Let us consider the diffusion process Y solution to (4.7) on $\mathbb{R}^d$. Due to our regularity assumptions on b and σ, this process is well-defined up to a possibly finite explosion time τ_{expl}. The Harnack inequality (12.6) applied to Y on the compact set $\overline{D}_0$ ensures the existence of constants $\delta > 0$ and N such that, for all $f : \mathbb{R}^d \to [0, 1]$, for all $x \in \overline{D}_0$ and all $y \in B(x, \delta)$,

$$\mathbb{E}_x[\mathbb{1}_{\delta+\delta^2<\tau_{\text{expl}}} f(Y_{\delta+\delta^2})] \leq N\mathbb{E}_y[\mathbb{1}_{\delta+2\delta^2<\tau_{\text{expl}}} f(Y_{\delta+2\delta^2})].$$

By compactness of $\overline{D}_0$, there exist a positive integer n and $y_1, \dots, y_n \in D_0$ such that $\overline{D}_0 \subset \bigcup_{i=1}^n B(y_i, \delta)$. Setting $s_1 = \delta + \delta^2$, we deduce that, for all $k \geq 1$ and all $x \in D_0$,

$$\mathbb{P}_x(Y_{s_1} \in D \setminus K_k) \leq N \max_{1\leq i\leq n} \mathbb{P}_{y_i}(Y_{s_1+\delta^2} \in D \setminus K_k) \xrightarrow[k\to+\infty]{} 0.$$

Hence (4.6) is satisfied. This and Theorem 4.1 end the proof of Corollary 4.6. ☐

We give three examples of application.

Example 4.7. Assume that D is bounded. Then, one can choose $D_0 = D$ and $\varphi = 1$ in Corollary 4.6. This implies that Assumption (F) is satisfied for $\psi_1 = \varphi^p$ bounded (see Remark 4.3), so that it follows from Theorem 3.5 that the convergence of $e^{\lambda_0 t}\mathbb{P}_X(t < \tau_\partial)$ to η is uniform and that η is bounded. Theorem 3.5 also implies that Assumption (E) is satisfied for some bounded φ_1 and φ_2. Since $P_1\eta = e^{-\lambda_0}\eta$ and $e^{-\lambda_0} \geq \theta_2 > \theta_1$, we deduce that (E) is still satisfied if φ_2 is replaced by $\eta/\|\eta\|_\infty$. Therefore,

$$\|\mathbb{P}_\mu(X_n \in \cdot \mid n < \tau_\partial) - \nu_{QSD}\|_{TV} \leq \frac{C}{\mu(\eta)}\,\alpha^n,\ \forall n \in \mathbb{N}.$$

The extension to any $t \in [0, +\infty)$ can be obtained using the same argument as in Section 11.5.2 replacing φ_2 and φ_2' with η. This implies Theorem 1.1 of the introduction.

Example 4.8. Assume that $D \subset \mathbb{R}^d_+$ is open connected and that

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$

in D, where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ are locally Hölder continuous in $\mathbb{R}^d$, σ is locally uniformly elliptic on $\mathbb{R}^d$ and

$$\frac{\langle b(x), 1\rangle}{\langle x, 1\rangle} \xrightarrow[|x|\to+\infty]{} -\infty,$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean product in $\mathbb{R}^d$ and $|\cdot|$ is the associated norm. Then (4.8) is satisfied for $\varphi(x) = 1 + x_1 + \ldots + x_d$ and hence the process X absorbed at the boundary of D satisfies the assumptions of Theorem 4.1.

Example 4.9. Assume that $D \subset \mathbb{R}^d$ is open connected and that

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t$$

in D, where $b : \mathbb{R}^d \to \mathbb{R}^d$ is locally Hölder continuous in $\mathbb{R}^d$ and

$$\limsup_{|x|\to+\infty} \frac{\langle b(x), x\rangle}{|x|} < -\frac{3}{2}\sqrt{\lambda_0}, \tag{4.9}$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean product in $\mathbb{R}^d$ and λ_0 is defined in (4.4). Then the process X absorbed at the boundary of D satisfies the assumptions of Theorem 4.1.

Indeed, let us check that (4.8) is satisfied for $\varphi(x) = \exp(\sqrt{\lambda_0}|x|)$. One has, for all $x \neq 0$,

$$\begin{aligned}\mathcal{L}\varphi(x) &= \sum_{i=1}^d \frac{e^{\sqrt{\lambda_0}|x|}}{2}\left(\frac{\sqrt{\lambda_0}}{|x|} - \frac{\sqrt{\lambda_0}x_i^2}{|x|^3} + \frac{\lambda_0 x_i^2}{|x|^2}\right) + \sum_{i=1}^d e^{\sqrt{\lambda_0}|x|}\,\frac{\sqrt{\lambda_0}b_i(x)\,x_i}{|x|}\\ &\leq \sqrt{\lambda_0}\varphi(x)\left(\frac{d-1}{2|x|} + \frac{\sqrt{\lambda_0}}{2} + \frac{\langle b(x), x\rangle}{|x|}\right)\\ &\leq -(\lambda_0 + \varepsilon)\varphi(x)\end{aligned}$$

for some $\varepsilon > 0$ and for all x such that $|x|$ is large enough. This implies (4.8).

To apply this criterion, it is necessary to obtain a priori bounds on λ_0. We will give some ideas about how to do so for one-dimensional diffusions in Section 4.5. In general, one can also use of course that (4.9) is implied by

$$\lim_{|x|\to+\infty} \frac{\langle b(x), x\rangle}{|x|} = -\infty.$$

4.3 Non-uniformly elliptic diffusions: the Feller diffusion with competition

We provide an example where the diffusion matrix σ cannot be extended out of D as a locally uniformly elliptic matrix. This example deals with Feller diffusions with competition and is motivated by models of population dynamics with d species in interaction, where absorption corresponds to the extinction of one of the populations [15, 26].

Assume that $D = (0,\infty)^d$ and

$$\mathrm{d}X^i_t = \sqrt{\gamma_i X^i_t}\,\mathrm{d}B^i_t + X^i_t b_i(X_t)\,\mathrm{d}t,$$

where $\gamma_i > 0$ for all $1 \leq i \leq d$, $B^1,\dots,B^d$ are independent standard Brownian motions and b_i are locally Hölder in $(0,\infty)^d$ and locally bounded in $\mathbb{R}^d_+$.

Proposition 4.10. *Assume that there exist constants $c_0, c_1 > 0$ such that*

$$\sum_{i=1}^d \frac{x_i b_i(x)}{\gamma_i} \leq c_0 - c_1|x|, \quad \forall x \in (0,\infty)^d.$$

Then the process X absorbed at the boundary of D satisfies the assumptions of Theorem 4.1.

Compared to the existing literature on multi-dimensional Feller diffusions [15, 26], the main novelty of this result is that it covers cases where the process does not come down from infinity, e.g. $b_i(x) = r_i - \sum_{j=1}^d c_{ij}\frac{x_j}{1+x_j}$, for some positive constants r_i and c_{ij} such that $r_i < c_{ii}$ for all $1 \leq i \leq d$, and where b does not derive from a potential (see for instance [15], based on a spectral theoretic approach). While our results on existence and convergence to quasi-stationary distributions are more general than those of [15], we do not recover finer results on the spectrum of the process, such as its discreteness.

Proof. Our aim is to prove that the assumptions of Theorem 4.1 hold true with $\varphi(x) = \exp(c(x_1/\gamma_1 + \ldots + x_n/\gamma_n))$, where $c = c_1\,\min_i \gamma_i/\sqrt{d}$.

We have, for all $x \in D$,

$$\mathcal{L}\varphi(x) = \sum_{i=1}^d \left(\frac{x_i c^2}{2\gamma_i} + \frac{c x_i b_i(x)}{\gamma_i}\right)\varphi(x) \leq \left(c_0 c - \frac{c_1 c|x|}{2}\right)\varphi(x).$$

Choosing $\lambda_1 = \lambda_0 + 1$ and $D_0 = \{x \in D,\ \text{s.t. } |x| \leq (2c_0 + 2\lambda_1/c)/c_1\}$, one deduces that (4.5) holds true with $C = c_0 c\,\max_{D_0}\varphi$.

Let us now prove that

$$\mathbb{P}_x(1 < \tau_\partial) \xrightarrow[x\to\partial D, x\in D_0]{} 0, \tag{4.10}$$

which implies that (4.6) holds true with $s_1 = 1$. Fix $\varepsilon > 0$ and define the set $F = \left\{x \in \mathbb{R}^d_+,\ \text{s.t. } \varphi(x) \geq e^C \sup_{y\in D_0}\varphi(y)/\varepsilon\right\}$. Using Itô's formula (see the proof of (12.9) in Section 12.3 for details), we deduce from (4.5) that, For all $x \in D_0$,

$$\mathbb{P}_x(\tau_F \leq 1)\,e^C \sup_{y\in D_0}\varphi(y)/\varepsilon \leq \mathbb{E}_x\left(\varphi(X_{\tau_F\wedge 1})\mathbb{1}_{\tau_F\wedge 1 < \tau_\partial}\right) \leq e^C\varphi(x),$$

so that $\mathbb{P}_x(\tau_F \leq 1) \leq \varepsilon$ for all $x \in D_0$. Since F^c is bounded, we have

$$\beta := \sup_{x\in F^c, i\in\{1,\dots,d\}} |b_i(x)| < +\infty.$$

Let $(Z_t)_{t\in[0,+\infty)} := (Z^1_t,\dots,Z^d_t)_{t\in[0,+\infty)}$ be the solution of the system of SDEs

$$\mathrm{d}Z^i_t = \sqrt{\gamma_i Z^i_t}\,\mathrm{d}B^i_t + Z^i_t\beta\,\mathrm{d}t,\ Z^i_0 = X^i_0 \in (0,+\infty),$$

with absorption at the boundary of D. Note that the components of Z are independent one dimensional diffusion processes such that 0 is reachable and hence that

$$\mathbb{P}_x\left(\forall t\in[0,1],\,\forall i\in\{1,\dots,d\},\ Z^i_t>0\right)\xrightarrow[x\to\partial D]{}0.$$

Standard comparison arguments show that $X^i_t\leq Z^i_t$ for all $t<\tau_\partial\wedge\tau_F\wedge 1$ and all $i\in\{1,\dots,d\}$, so that

$$\mathbb{P}_x\left(\forall t\in[0,1],\,\forall i\in\{1,\dots,d\},\ X^i_t>0\text{ and }1<\tau_F\right)\xrightarrow[x\to\partial D]{}0.$$

But $\mathbb{P}_x(1<\tau_F)\geq 1-\varepsilon$, so that

$$\limsup_{x\to\partial D}\,\mathbb{P}_x\left(\forall t\in[0,1],\,\forall i\in\{1,\dots,d\},\ X^i_t>0\right)\leq\varepsilon.$$

Since this is true for all $\varepsilon>0$ and since $\{\forall t\in[0,1],\,\forall i\in\{1,\dots,d\},\ X^i_t>0\}=\{1<\tau_\partial\}$, we deduce that (4.10) holds true, which concludes the proof or Proposition 4.10. ☐

4.4 Diffusion processes with killing

This section is devoted to the study of diffusion processes with killing. More precisely, we consider as above a diffusion process X on a connected, open domain $D\subset\mathbb{R}^d$ for some $d\geq 1$, solution to the SDE

$$\mathrm{d}X_t=b(X_t)\mathrm{d}t+\sigma(X_t)\mathrm{d}B_t \tag{4.11}$$

absorbed in ∂ at its first exit time τ_{exit} of D, as defined in (4.2), with the same assumptions as in Section 4.1. We also assume that the process is subject to an additional measurable killing rate $\kappa:D\to\mathbb{R}_+$ which is locally bounded: there exists an independent exponential random variable ξ with parameter 1 such that the process is instantaneously sent to the cemetery point $\partial\notin D$ at time

$$\tau_\partial=\tau_{\text{exit}}\wedge\inf\left\{t\geq 0,\ \int_0^t\kappa(X_s)\,\mathrm{d}s\ >\xi\right\}.$$

Since κ is assumed to be locally bounded, one easily checks that λ_0 in (4.4) is finite, and that it does not depend on $x\in D$ or on the open ball B such that $\overline{B}\subset D$.

The following result is an extension to the multi-dimensional setting of [71, Theorem 4.3].

Theorem 4.11. *Assume that there exist a subset $D_0\subsetneq D$ closed in D such that*

$$\inf_{x\in D\setminus D_0}\kappa(x)>\lambda_0, \tag{4.12}$$

and a time $s_1>0$ such that

$$\sup_{x\in D_0}\mathbb{P}_x(s_1<\tau_\partial\wedge\tau_{K_k})\xrightarrow[k\to+\infty]{}0. \tag{4.13}$$

Then the process X absorbed at time τ_∂ admits a unique quasi-stationary distribution ν_{QSD} and there exist a positive function φ_2 on D (uniformly bounded away from 0 on compact subsets of D) and a positive constant C such that

$$\left\|\mathbb{P}_\mu(X_t\in\cdot\mid t<\tau_\partial)-\nu_{QSD}\right\|_{TV}\leq\frac{C}{\mu(\varphi_2)}\,\alpha^t,\ \forall t\in[0,+\infty)$$

for all probability measures μ on E.

Remark 4.12. Let us make some comments on the assumptions of the above result.

1. If the process without killing rate satisfies (4.13), then the process with killing rate also satisfies this property. Hence the analysis provided in Section 4.2 can also be used to check the assumptions of the above theorem.
2. If $\inf_{x\in D\setminus K_k}\kappa(x)\to+\infty$ when $k\to+\infty$, then the assumptions of Theorem 4.7 are trivially satisfied.
3. In order to reach the conclusion of Theorem 4.1 in the setting of killed diffusion, it is also possible to use a Lyapunov type criterion: the assumption (4.5) can be simply replaced by the assumption that there exist $\lambda>\lambda_0$ and $C>0$ such that
$$\mathcal{L}\varphi(x)-\kappa(x)\varphi(x)\leq-\lambda\varphi(x)+C\mathbb{1}_{x\in D_0}.$$
Note that (4.12) of course implies the last inequality for $\varphi\equiv 1$. This extension follows from a simple adaptation of the arguments of Theorem 4.1 observing that
$$\mathbb{E}_x\left[f(X_t)\mathbb{1}_{t<\tau_\partial}\right]=\mathbb{E}_x\left[f(X^D_t)\mathbb{1}_{t<\tau_{\text{exit}}}\exp\left(-\int_0^t\kappa(X^D_s)\mathrm{d}s\right)\right],$$
where the process X^D is the process solution to (4.11) without killing, absorbed at its first exit time of D, at time τ_{exit}.
4. If in addition the killing rate κ is locally Hölder in D, we can apply [50, Cor. 3.1] as in Section 12.4 to prove that η is $\mathcal{C}^2(D)$ and $\mathcal{L}\eta(x)-\kappa(x)\eta(x)=-\lambda_0\eta(x)$ for all $x\in D$.

Proof. The proof follows the same lines as the proof of Theorem 4.1 in Section 12. We emphasize that the construction of the process in Section 12.1 is still valid. The same is true for the Harnack inequalities of Section 12.2 since they are based on Krylov's and Safonov's general result [72] which is obtained for diffusion processes with a bounded and measurable killing rate. The rest of the proof is exactly the same, replacing $\varphi_1=\varphi$ by $\varphi_1=1$. □

4.5 The case of one-dimensional diffusions

In this section, we consider the case of one-dimensional diffusion processes. Here, the Hölder regularity of the coefficients is not needed. Let X be the solution in $D=(\alpha,\beta)$, where $-\infty\leq\alpha<\beta\leq+\infty$, to the SDE
$$\mathrm{d}X_t=\sigma(X_t)\,\mathrm{d}B_t+b(X_t)\,\mathrm{d}t,\quad X_0\in D,$$
where $\sigma:D\to(0,+\infty)$ and $b:D\to\mathbb{R}$ are measurable functions such that $(1+|b|)/\sigma^2$ is locally integrable on D. We assume that the process is sent to a cemetery point ∂ when it reaches the boundary of D and that it is subject to an additional killing rate $\kappa:D\to\mathbb{R}_+$ which is measurable and locally integrable w.r.t. Lebesgue's measure. This assumption implies that the killed process is regular in the sense that, for all $x,y\in D$, $\mathbb{P}_x(\tau_{\{y\}}<\infty)>0$.

We define λ_0 as in (4.4). The fact that λ_0 does not depend on x nor B is a consequence of the regularity of the process.

Let $\delta:D\to\mathbb{R}_+$ and $s:D\to\mathbb{R}$ be defined by
$$\delta(x)=\exp\left(-2\int_{\alpha_0}^x\frac{b(u)}{\sigma(u)^2}\,\mathrm{d}u\right)\quad\text{and}\quad s(x)=\int_{\alpha_0}^x\delta(u)\,\mathrm{d}u,$$
for some arbitrary $\alpha_0\in D$. We recall that s is the scale function of X (unique up to an affine transformation), meaning that $s(X_t)$ is a local martingale. We also recall that the

boundary α (and similarly for β) is said to be reachable (for the process without killing) if $s(\alpha_+) > -\infty$ and

$$\int_\alpha^+ \frac{s(x)-s(\alpha_+)}{\sigma(x)^2\delta(x)}\,\mathrm{d}x < +\infty.$$

Theorem 4.13. *Assume that one among the following conditions* (i)*,* (ii) *or* (iii) *holds true:*

(i) *α and β are reachable boundaries;*

(ii) *α is reachable and there exist $\lambda_1 > \lambda_0$, a $\mathcal{C}^2(D)$ function $\varphi : D \to [1,+\infty)$ and $x_1 \in D$ such that, for all $x \geq x_1$,*

$$\frac{\sigma(x)^2}{2}\varphi''(x) + b(x)\varphi'(x) - \kappa(x)\varphi(x) \leq -\lambda_1\varphi(x); \tag{4.14}$$

(iii) *there exist $\lambda_1 > \lambda_0$, a $\mathcal{C}^2(D)$ function $\varphi : D \to [1,+\infty)$ and $x_0 < x_1 \in D$ such that* (4.14) *holds true for all $x \in (\alpha, x_0) \cup (x_1, \beta)$.*

Then the conclusions of Theorem 4.1 hold true.

Remark 4.14. We shall not detail the proof of this result since it is very close to the proof of Theorem 4.1 given in Section 12. We only explain the places that need to be modified. First, weak existence, weak uniqueness and the strong Markov property are well-known under the assumptions that $\sigma > 0$ and $(1+|b|)/\sigma^2 \in L^1_{\mathrm{loc}}(D)$ (weak existence and uniqueness in law are proved up to an explosion time in [66, Thm. 5.5.15], so we can construct a unique weak solution and prove the strong Markov property as in Section 12.1). Second, in order to construct an appropriate function φ on D, we choose $D_0 = (\alpha, x_1]$ in case (ii) and $D_0 = [x_0, x_1]$ in case (iii) and we can extend φ on D_0 as a bounded $\mathcal{C}^2(D)$ function. In case (i), we can take $\varphi \equiv 1$ and $D_0 = D$. Third, (4.6) follows from the fact that the boundaries α and β are reachable in case (i) and α is reachable in case (ii), since

$$\sup_{x\in(\alpha,\alpha+1/k]} \mathbb{P}_x(s_1 < \tau_\partial) \leq \mathbb{P}_{\alpha+1/k}(s_1 < \tau_{\{\alpha\}}) \xrightarrow[k\to+\infty]{} 0.$$

In case (iii), the limit is trivial since $D_0 \subset K_k$ for k large enough. Finally, all the arguments using Harnack's inequality can be replaced by arguments using the regularity of the process and standard coupling arguments for one-dimensional diffusions (see [23, 22]).

In order to apply this result in practice, one needs to find computable estimates for λ_0 and candidates for φ. One may for instance use the bounds for the first eigenvalue of the (Dirichlet) infinitesimal generator of $(X_t, t \geq 0)$ obtained in a L^2 (symmetric) setting using Rayleigh-Ritz formula in [91, 110, 111], as observed in [71]. We propose here two different upper bounds for λ_0 which follow from the characterization (4.4) of the eigenvalue λ_0 and Dynkin's formula.

Proposition 4.15. *For all $\alpha < \mathfrak{a} < \mathfrak{b} < \beta$, we have*

$$\lambda_0 \leq \sup_{x\in[\mathfrak{a},\mathfrak{b}]} \left\{ \frac{1}{2}\left(\frac{\pi\sigma(x)}{\int_{\mathfrak{a}}^{\mathfrak{b}} \exp\left(-2\int_x^y \frac{b(z)}{\sigma^2(z)}\,\mathrm{d}z\right)\mathrm{d}y} \right)^2 + \kappa(x) \right\}.$$

If $x \mapsto b(x)/\sigma(x)^2$ is $\mathcal{C}^1([\mathfrak{a},\mathfrak{b}])$, then

$$\lambda_0 \leq \sup_{x\in[\mathfrak{a},\mathfrak{b}]} \frac{\pi^2\sigma(x)^2}{2(\mathfrak{b}-\mathfrak{a})^2} + \sigma(x)^2\left(\frac{b}{2\sigma^2}\right)'(x) + \frac{b(x)^2}{2\sigma(x)^2} + \kappa(x).$$

Proof. For the proof of the first inequality, set

$$f(x)=\sin\left(\pi\frac{s(x)-s(\mathfrak{a})}{s(\mathfrak{b})-s(\mathfrak{a})}\right).$$

Then, for all $x\in(\mathfrak{a},\mathfrak{b})$,

$$\begin{aligned}\frac{\sigma(x)^2}{2}f''(x)+b(x)f'(x)-\kappa(x)f(x)&=-\left(\frac{\pi^2\sigma(x)^2\delta(x)^2}{2(s(\mathfrak{b})-s(\mathfrak{a}))^2}+\kappa(x)\right)f(x)\\&=-\left(\frac{\pi^2\sigma(x)^2}{2\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\exp\left(-2\int_x^y\frac{b(z)}{\sigma^2(z)}\,\mathrm{d}z\right)\,\mathrm{d}y\right)^2}+\kappa(x)\right)f(x)\\&\geq -Cf(x),\end{aligned}$$

where

$$C:=\sup_{x\in[\mathfrak{a},\mathfrak{b}]}\left\{\frac{1}{2}\left(\frac{\pi\sigma(x)}{\int_{\mathfrak{a}}^{\mathfrak{b}}\exp\left(-2\int_x^y\frac{b(z)}{\sigma^2(z)}\,\mathrm{d}z\right)\,\mathrm{d}y}\right)^2+\kappa(x)\right\}.$$

Since f is C^2 and bounded, we deduce from Itô's formula that, for all $x\in(\mathfrak{a},\mathfrak{b})$,

$$\mathbb{E}_x(f(X_t)\mathbb{1}_{t<\tau_{\{\mathfrak{a},\mathfrak{b}\}}})\geq e^{-Ct}f(x).$$

Now, using the fact that $0<f(x)\leq 1$ for all $x\in(\mathfrak{a},\mathfrak{b})$, we deduce that

$$\mathbb{P}_x(X_t\in(\mathfrak{a},\mathfrak{b}))\geq e^{-Ct}f(x),\ \forall x\in D.$$

As a consequence, the definition of λ_0 entails $\lambda_0\leq C$.

The proof of the second inequality is the same, using instead the function

$$f(x):=\exp\left(-\int_{\mathfrak{c}}^x\frac{b(u)}{\sigma(u)^2}\,\mathrm{d}u\right)\sin\left(\pi\frac{x-\mathfrak{a}}{\mathfrak{b}-\mathfrak{a}}\right)$$

for some $\mathfrak{c}\in(\mathfrak{a},\mathfrak{b})$. □

The next result provides two candidates for φ. Its proof is a straightforward computation.

Proposition 4.16. *Let $\varphi:(0,+\infty)$ be any $\mathcal{C}^2(D)$ function such that, for some constants $\alpha_-<\alpha_0<\alpha_+\in D$,*

$$\varphi(x)=\begin{cases}\sqrt{s(x)} & \text{if } x\geq\alpha_+,\\ \sqrt{-s(x)} & \text{if } x\leq\alpha_-.\end{cases}\tag{4.15}$$

Then, for all $x\in(\alpha,\alpha_-]\cup[\alpha_+,\beta)$

$$\frac{\sigma(x)^2}{2}\varphi''(x)+b(x)\varphi'(x)-\kappa(x)\varphi(x)\leq-\left(\frac{\sigma(x)^2\delta(x)^2}{8s(x)^2}+\kappa(x)\right)\varphi(x).$$

If $x\mapsto b(x)/\sigma(x)^2$ is $C^1(D)$, then

$$\varphi(x)=\exp\left(-\int_{\alpha_0}^x\frac{b(u)}{\sigma^2(u)}\,\mathrm{d}u\right)\tag{4.16}$$

satisfies

$$\frac{\sigma(x)^2}{2}\varphi''(x)+b(x)\varphi'(x)-\kappa(x)\varphi(x)=-\left(\frac{b^2(x)}{2\sigma^2(x)}+\frac{\sigma^2(x)}{2}\left(\frac{b}{\sigma^2}\right)'(x)+\kappa(x)\right)\varphi(x).$$

Remark 4.17. The first function φ is always uniformly lower bounded on $(\alpha,\alpha_-]\cup[\alpha_+,\beta)$ by $\min\{\sqrt{s(\alpha_+)},\sqrt{-s(\alpha_-)}\}$. To ensure that the second one is also uniformly lower bounded, one needs further assumptions on the behavior of b/σ^2 close to α and β.

The above results can be used as follows. In the case where α is reachable and $b\equiv 0$, Condition (ii) of Theorem 4.13 holds true if

$$\liminf_{x\to\beta-}\frac{\sigma^2(x)}{8(x-\alpha)^2}+\kappa(x)>\lambda_0,$$

choosing $\alpha_0=\alpha$ and using the function φ of (4.15). Similarly, in the case where α is reachable, $\sigma\equiv 1$ and b is C^1, condition (ii) of Theorem 4.13 holds true if

$$\liminf_{x\to\beta-}\frac{b^2(x)}{2}+\frac{b'(x)}{2}+\kappa(x)>\lambda_0,$$

using the function φ of (4.16).

We give below more precise examples.

Example 4.18. Assume that $D=(0,+\infty)$, κ is locally bounded and that X is solution to the SDE in D

$$\mathrm{d}X_t=\sqrt{X_t}\mathrm{d}B_t-X_t\mathrm{d}t.$$

Then 0 is reachable for X and, since

$$\frac{\sigma(x)^2\delta(x)^2}{8s(x)^2}\xrightarrow[x\to+\infty]{}+\infty,$$

we deduce from Proposition 4.16 and Theorem 4.13 that X admits a quasi-stationary distribution ν_{QSD} and, for all $p\geq 1$, there exist positive constants C_p,γ_p and a positive function $\varphi_{2,p}$ on $(0,+\infty)$ such that

$$\|\mathbb{P}_\mu(X_t\in\cdot\mid t<\tau_\partial)-\nu_{QSD}\|_{TV(\exp(\cdot/p))}\leq C_p\frac{\int_{(0,+\infty)}\exp(x/p)\,\mu(\mathrm{d}x)}{\mu(\varphi_{2,p})}\,e^{-\gamma_p t},$$

for all probability measure μ on D. In particular, one deduces that the domain of attraction ν_{QSD} contains any initial distribution μ admitting a finite exponential moment. Note that, in the case where $\kappa\equiv 0$, the process X is a continuous state branching process (Feller diffusion), for which quasi-stationarity was already studied (see [73] and the references therein).

Example 4.19. Assume that $(\alpha,\beta)=\mathbb{R}$, that $b\equiv 0$ and σ is bounded measurable on $\mathbb{R}$. Assume also that the absorption of X is due to the killing rate $\kappa(x)=\kappa_0\left(1-\frac{1}{1+|x|}\right)$ for some constant $\kappa_0>0$. We deduce from the first inequality of Proposition 4.15 (taking $\mathfrak{b}>0$ and $\mathfrak{a}=-\mathfrak{b}$) that

$$\lambda_0\leq\frac{\pi^2\|\sigma\|_\infty^2}{8\mathfrak{b}^2}+\kappa_0\left(1-\frac{1}{1+\mathfrak{b}}\right)\leq\kappa_0\left(1-\frac{1}{1+2\mathfrak{b}}\right)$$

for $\mathfrak{b}$ large enough. Moreover, choosing $\varphi=1$ and $x_0=-3\mathfrak{b}$, $x_1=3\mathfrak{b}$, one deduces that, for all $x\notin[-x_1,x_1]$,

$$\frac{\sigma(x)^2}{2}\varphi''(x)-\kappa(x)\varphi(x)\leq-\kappa_0\left(1-\frac{1}{1+3\mathfrak{b}}\right)\varphi(x).$$

Hence Theorem 4.13 implies that there exists a unique quasi-stationary distribution ν_{QSD} for X and that it attracts all probability measures μ on D.

Example 4.20. We consider the case $(\alpha,\beta)=(0,+\infty)$, $\sigma(x)=1$, $b(x)=x\sin x$, and $\kappa(x)=\kappa_0\left(1-\frac{1}{1+x}\right)$ for some constant $\kappa_0>\pi^2+3$. This corresponds to a SDE $\mathrm{d}X_t=\mathrm{d}B_t+\nabla U(X_t)\mathrm{d}t$ where the potential $U(x)=\sin x-x\cos x$ has infinitely many wells with arbitrarily large depths, meaning that the process X without killing has a tendency to be "trapped" away from zero for large initial conditions. Nevertheless, thanks to the killing, we are able to prove convergence to a unique quasi-stationary distribution. Indeed, using the second inequality of Proposition 4.15, we have

$$\lambda_0\leq\sup_{x\in(0,1)}\frac{\pi^2}{2}+\frac{\sin x+x\cos x+x^2\sin^2 x}{2}+\kappa_0\left(1-\frac{1}{1+x}\right)\leq\frac{\pi^2}{2}+\frac{3}{2}+\kappa_0/2.$$

Moreover, 0 is a reachable boundary for X and, taking $\varphi=1$, one has, for all $x_1>0$ and all $x>x_1$,

$$\frac{\sigma(x)^2}{2}\varphi''(x)+b(x)\varphi'(x)-\kappa(x)\varphi(x)\leq-\kappa_0\left(1-\frac{1}{1+x_1}\right)\varphi(x)$$

Hence, since we assumed that $\kappa_0>\pi^2+3$, one deduces that there exists a unique quasi-stationary distribution ν_{QSD} for X and that it attracts all probability measures μ on D.

Remark 4.21. The case of general one-dimensional diffusion processes [65] can be handled using our framework, although using the infinitesimal generator is more tricky [63]. However, in the case of a regular diffusion process on $(0,+\infty)$ such that 0 is a reachable boundary and such that $+\infty$ is entrance, one easily shows (see for instance [23]) that, for all $\lambda>0$, there exists $y>0$ such that

$$\sup_{x\in(0,+\infty)}\mathbb{E}_x\left(e^{\lambda\tau_{[0,y]}}\right)<+\infty.$$

Hence, using the same proof as in Theorem 4.1 and using Lemma 3.6, we deduce that there exists a unique quasi-stationary distribution ν_{QSD} for X and that it satisfies

$$\|\mathbb{P}_\mu(X_t\in\cdot\mid t<\tau_\partial)-\nu_{QSD}\|_{TV}\leq\frac{1}{\mu(\varphi_2)}\,\alpha^t,\ \forall t\in[0,+\infty)$$

for some positive function φ_2 and some $\alpha<1$. Whether the convergence to ν_{QSD} holds uniformly with respect to the initial distribution (as in Proposition 3.9) without further assumptions remains an open problem. It has been shown to be true for a wide range of cases in [23, 22].

5 Application to processes in discrete state space and continuous time

Let X be a non-explosive[1] Markov process in a countable state space $E\cup\{\partial\}$ absorbed in ∂, with jump rate $q_{x,y}$ from x to $y\neq x$ such that $\sum_{y\in E\cup\{\partial\}\setminus\{x\}}q_{x,y}<\infty$ for all $x\in E$. The extended generator $\mathcal{L}$ acts on nonnegative real functions f on $E\cup\{\partial\}$ such that $\sum_{y\in E\cup\{\partial\}}q_{x,y}f(y)<\infty$ for all $x\in E$ as

$$\mathcal{L}f(x)=\sum_{y\neq x\in E\cup\{\partial\}}q_{x,y}(f(y)-f(x)),\quad\forall x\in E,\quad Lf(\partial)=0.\tag{5.1}$$

---

[1] One could actually consider the case of explosive Markov processes as in Section 4 (see Remark 4.4), with τ_∂ defined as the infimum between the first hitting time of ∂ and the explosion time.

Theorem 5.1. *Assume that there exists a finite subset* D_0 *of* E *such that* $\mathbb{P}_x(X_1 = y) > 0$ *for all* $x, y \in D_0$*, so that the constant*

$$\lambda_0 := \inf\left\{\lambda > 0,\ \textit{s.t. } \liminf_{t\to+\infty} e^{\lambda t}\,\mathbb{P}_x\left(X_t = x\right) > 0\right\}$$

is finite and independent of $x \in D_0$*. If in addition there exist constants* $C > 0$*,* $\lambda_1 > \lambda_0$*, a function* $\varphi : E \cup \{\partial\} \to \mathbb{R}_+$ *such that* $\varphi_{|E} \geq 1$*,* $\varphi(\partial) = 0$*,* $\sum_{y\in E\setminus\{x\}} q_{x,y}\varphi(y) < \infty$ *for all* $x \in E$ *and such that*

$$\mathcal{L}\varphi(x) \leq -\lambda_1\varphi(x) + C\mathbb{1}_{x\in D_0},\ \forall x \in E, \tag{5.2}$$

then Assumption (F) is satisfied with $L = D_0$*,* $\gamma_1 = e^{-\lambda_1}$*, any* $\gamma_2 \in (e^{-\lambda_1}, e^{-\lambda_0})$ *and* $\psi_1 = \varphi_{|E}$*. In addition,* $\mathbb{P}_{\nu_{QSD}}(t < \tau_\partial) = e^{-\lambda_0 t}$ *for all* $t \geq 0$*, the function* η *of Theorem 2.3 satisfies* $P_t\eta = e^{-\lambda_0 t}\eta$ *for all* $t \geq 0$ *and* $\sum_{y\in E\setminus\{x\}} q_{x,y}\eta(y) < \infty$ *and* $\mathcal{L}\eta(x) = -\lambda_0\eta(x)$ *for all* $x \in E$*.*

Remark 5.2. If in addition to the assumptions of Theorem 5.1 we assume that $\lambda_1 > \sup_{x\in E} q(x,\partial)$, it is possible to adapt the proof of Theorem 3.5 given in Section 11.5 to prove that the conclusion of Theorem 3.5 holds true with $\psi_2 \equiv 1$. Therefore, we obtain the improved convergence, for all $h \in L^\infty(\varphi)$,

$$|\mathbb{E}_\mu(h(X_t) \mid t < \tau_\partial) - \nu_{QSD}(h)| \leq C\,\mu(\varphi)\,\alpha^t\,\|h/\varphi\|_\infty,\ \forall t \geq 0,$$

instead of (3.4). If moreover φ is bounded over E, the convergence is uniform and there exists a unique quasi-stationary distribution.

Before turning to the proof of Theorem 5.1, we give an example of application.

Example 5.3. Assume that X is a birth and death process with killing on $E = \mathbb{N}$ and $\partial = 0$. This means that there exist non-negative numbers $(b_x)_{x\in\mathbb{N}}$, $(d_x)_{x\in\mathbb{N}}$, $(\kappa_x)_{x\in\mathbb{N}}$ such that $b_x > 0$ for all $x \geq 1$, $d_x > 0$ for all $x \geq 2$, and $d_1 = 0$, and such that, for all $x \in E$,

$$q_{x,y} = \begin{cases} b_x & \text{if } y = x+1, \\ d_x & \text{if } y = x-1, \\ \kappa_x & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We set

$$S := \sum_{k\geq 1} \frac{1}{d_k\alpha_k} \sum_{l\geq k} \alpha_l, \tag{5.3}$$

with $\alpha_k = \left(\prod_{i=1}^{k-1} b_i\right) / \left(\prod_{i=1}^{k} d_i\right)$. Recent advances on existence of quasi-stationary distribution of birth and death processes with killing were obtained in [35, 100, 101], see also the nice survey [104].

In this setting, Theorems 5.1 and 3.5 translate as follows: if there exists a function $\varphi : \mathbb{Z}_+ \to [1, +\infty)$ such that $\varphi(0) = 0$ and

$$\lambda_0 < \liminf_{x\to+\infty} -\frac{b_x(\varphi(x+1) - \varphi(x)) + d_x(\varphi(x-1) - \varphi(x))}{\varphi(x)} + \kappa_x, \tag{5.4}$$

then there exists a unique quasi-stationary distribution ν_{QSD} such that $\nu_{QSD}(\varphi) < +\infty$ which attracts exponentially fast all initial distributions integrating φ. To check (5.4), one may use in practice the fact that $\lambda_0 \leq \inf_{x\in\mathbb{N}} b_x + d_x + \kappa_x$, or adapt the ideas of Section 4.5 to birth and death processes, or use the finer upper bounds for λ_0 proved in [105]. We consider now three situations where the criterion (5.4) improves known results in the literature.

First, if $S < +\infty$, [35, Theorem 6.6] proves that there exists a unique quasi-stationary distribution for X assuming that $(\kappa_x)_{x\in\mathbb{N}}$ has finite support. We extend this result to any killing rates $(\kappa_x)_{x\in\mathbb{N}}$ and also prove that the unique quasi-stationary distribution attracts all initial distributions exponentially fast. We can indeed check that (5.4) is satisfied by a bounded function φ defined as follows: fix $\lambda_1 > \lambda_0$ and choose $x_0 \in \mathbb{N}$ large enough such that (see for instance [20, Equation (4.7)])

$$\sup_{x\in\mathbb{N}} \mathbb{E}_x(e^{\lambda_1 \tau_{D_0}\wedge\tau_\partial}) < +\infty.$$

where $D_0 = \{1,\dots,x_0\}$. Then we define $\varphi(0) = 0$ and

$$\varphi(x) = \mathbb{E}_x\left(e^{\lambda_1 \tau_{D_0}\wedge\tau_\partial}\right), \quad \forall x \in \mathbb{N}.$$

Using Markov's property at the first time of jump, one checks that (since $b_x + d_x \to +\infty$ when $x \to +\infty$, we assume w.l.o.g. that $b_x + d_x + \kappa_x > \lambda_1$ for all $x \geq x_0 + 1$),

$$\begin{aligned}\varphi(x) = \frac{d_x}{b_x + d_x + \kappa_x - \lambda_1}\varphi(x-1) + \frac{b_x}{b_x + d_x + \kappa_x - \lambda_1}\varphi(x+1)\\ + \frac{\kappa_x}{b_x + d_x + \kappa_x - \lambda_1},\ \forall x \geq x_0 + 1.\end{aligned}$$

Hence $\lambda_1 = -[b_x(\varphi(x+1) - \varphi(x)) + d_x(\varphi(x-1) - \varphi(x))]/\varphi(x) + \kappa_x$ for $x \geq x_0 + 1$ and (5.4) is satisfied.

Second, if $S = +\infty$ and if $\lambda_0 < \liminf_{x\to+\infty} \kappa_x$, it was proved in [101, Theorem 4.3] that there exists a quasi-stationary distribution. The criterion (5.4) improves this result since it implies that the quasi-stationary distribution is unique and that it attract all initial distributions exponentially fast. Indeed, (5.4) is clearly satisfied for $\varphi \equiv 1$.

Last, we can also extend [101, Theorem 4.3] to processes that do not necessarily admit a unique quasi-stationary distribution, and in particular that do not come down from infinity. For example, assuming that, for some $\varepsilon > 0$,

$$\liminf_{x\to+\infty}\ \kappa_x + \frac{\varepsilon}{1+\varepsilon}d_x - \varepsilon b_x > \lambda_0,$$

Condition (5.4) is satisfied for $\varphi(x) = (1+\varepsilon)^x$.

Note that, because of Corollary 2.7, our criteria imply the λ_0-positive recurrence of the process X (cf. e.g. [104, Eq. (26)]). Therefore, it can only apply to such situations. For results on birth and death processes with killing which are not λ_0-positive recurrent, we refer the reader to [101, Theorem 4.2].

Example 5.4. We consider general multitype birth and death processes in continuous time, taking values in a connected (in the sense of the nearest neighbors structure of $\mathbb{Z}^d$) subset E of $\mathbb{Z}_+^d$ for some $d \geq 1$, with transition rates

$$q_{x,y} = \begin{cases} b_i(x) & \text{if } y = x + e_i, \\ d_i(x) & \text{if } y = x - e_i, \\ 0 & \text{otherwise,} \end{cases}$$

with $e_i = (0,\dots,0,1,0,\dots,0)$ where the nonzero coordinate is the i-th one and with the convention that the process is sent instantaneously to ∂ when it jumps to a point $y \notin E$ according to the previous rates. To ensure irreducibility, it is sufficient (although not optimal) to assume that $b_i(x) > 0$ and $d_i(x) > 0$ for all $1 \leq i \leq d$ and $x \in E$.

We show below that Theorem 5.1 applies either under the assumption that

$$\frac{1}{1+|x|}\sum_{i=1}^d (d_i(x)-b_i(x)) \xrightarrow[x\in E,\ |x|\to+\infty]{} +\infty. \tag{5.5}$$

or that there exists $\delta>1$ such that

$$\sum_{i=1}^d (d_i(x)-\delta\, b_i(x)) \xrightarrow[x\in E,\ |x|\to+\infty]{} +\infty. \tag{5.6}$$

This improves the general criteria obtained in [26] since this reference assumes (among other assumptions) that $E=\mathbb{Z}_+^d$ and that $\sum_{i=1}^d (d_i(x)-b_i(x))\geq |x|^{1+\eta}$ for some $\eta>0$ and $|x|$ large enough.

Let us first show that (5.5) implies that the assumptions of Theorem 5.1 are satisfied. In order to do so, we define $\varphi(x)=|x|+1=x_1+\ldots+x_d+1$ and $\varphi(\partial)=0$ and obtain

$$\begin{aligned}\mathcal{L}\varphi(x)&=\sum_{i=1}^d (b_i(x)-d_i(x))-\sum_{i=1}^d \left(b_i(x)\mathbb{1}_{x+e_i\notin E}\varphi(x+e_i)+d_i(x)\mathbb{1}_{x-e_i\notin E}\varphi(x-e_i)\right)\\ &\leq -\varphi(x)\frac{\sum_{i=1}^d (d_i(x)-b_i(x))}{|x|+1}\end{aligned}$$

The proof is concluded by setting $D_0=\left\{x\in E,\ \text{s.t. } \frac{\sum_{i=1}^d (d_i(x)-b_i(x))}{|x|+1}\leq \lambda_0+1\right\}$.

Let us now show that (5.6) implies that the assumptions of Theorem 5.1 are satisfied. Setting $\varphi(x)=\exp\langle a,x\rangle$ for a given $a\in(0,\infty)^d$ and $\varphi(\partial)=0$, we obtain

$$\mathcal{L}\varphi(x)\leq -\varphi(x)\left(\sum_{i=1}^d (1-e^{-a_i})d_i(x)+(1-e^{a_i})b_i(x)\right).$$

Choosing $a=(\varepsilon,\ldots,\varepsilon)$ with ε small enough, we have

$$\liminf_{x\in E,\ |x|\to+\infty}\sum_{i=1}^d (1-e^{-a_i})d_i(x)+(1-e^{a_i})b_i(x)=+\infty.$$

Taking $D_0=\left\{x\in E,\ \text{s.t. } \sum_{i=1}^d (1-e^{-a_i})d_i(x)+(1-e^{a_i})b_i(x)\leq\lambda_0+1\right\}$ allows us to conclude the proof.

Proof of Theorem 5.1. The fact that λ_0 is independent of x is classical for irreducible processes (cf. e.g. [69]). We set $L=D_0$. Since X is a non-explosive pure jump continuous time process, it satisfies the strong Markov property and the entrance times τ_L and τ_∂ are stopping times. This entails (F0).

For all $x,y\in L$, we have

$$\mathbb{P}_x(X_2\in\cdot)\geq \inf_{u,v\in L}\mathbb{P}_u(X_1=v)\,\mathbb{P}_y(X_1\in\cdot),$$

where $\inf_{u,v\in L}\mathbb{P}_u(X_1=v)>0$ by assumption, which implies (F1) and (F3).

We set $\psi_1=\varphi$. For all $0\leq s\leq 1$, using (5.2) and Dynkin's formula, one has that for all $x\in L$

$$\mathbb{E}_x\left(\psi_1(X_s)\mathbb{1}_{s<\tau_\partial}\right)\leq e^{Cs}\sup_{y\in L}\psi_1(y).$$

Similarly, setting $\gamma_1 = e^{-\lambda_1}$, for all $x \in E \setminus L$,

$$\mathbb{E}_x\left(\psi_1(X_1)\mathbb{1}_{1<\tau_L\wedge\tau_\partial}\right) \leq e^{-\lambda_1}\psi_1(x) = \gamma_1\psi_1(x).$$

Choosing any $\gamma_2 \in (\gamma_1, e^{-\lambda_0})$, one obtains that (F2) is satisfied and the first part of Theorem 5.1 is proved.

The inequality $\sum_{y\in E\setminus\{x\}} q_{x,y}\eta(y) < \infty$ for all $x \in E$ follows from the fact that $\eta \in L^\infty(\psi_1)$ and the fact that $P_t\eta(x) = e^{-\lambda_0 t}\eta(x)$ was proved in Theorem 3.5. It then follows from Markov's property and the last equality that $(e^{\lambda_0 t}\eta(X_t), t \geq 0)$ is a martingale for the canonical filtration associated to X, with the convention that $\eta(\partial) = 0$. Now, it is standard to represent the Markov process X as a solution to a stochastic differential equation driven by a Poisson point process: assume that the elements of the finite or countable set E are labeled by distinct positive integers, that $\partial = 0$ and, for all $x, i \in \mathbb{Z}_+$, let $\kappa_i(x) = q_{x,0} + q_{x,1} + \ldots + q_{x,i}$ with the convention that $q_{x,x} = 0$ and $q_{x,i} = 0$ for all x or $i \notin E \cup \{\partial\}$ and set $q(x) = \sum_{i\in\mathbb{Z}_+} q_{x,i} < \infty$. Given a Poisson point measure $N(\mathrm{d}s, \mathrm{d}\theta)$ on $\mathbb{R}_+^2$ with intensity the Lebesgue measure on $\mathbb{R}_+^2$, the process X solution

$$X_t = X_0 + \int_0^t \int_0^{q(X_{s-})} \sum_{i=0}^{\infty} \mathbb{1}_{\theta\in[\kappa_{i+1}(X_{s-}),\kappa_i(X_{s-}))}(i - X_{s-})N(\mathrm{d}s, \mathrm{d}\theta)$$

is well-defined for all time $t \geq 0$ almost surely and is a Markov process with matrix of jump rates $(q_{i,j})_{i,j\in\mathbb{Z}_+}$. Introducing the compensated Poisson measure $\widetilde{N}(\mathrm{d}s, \mathrm{d}\theta) = N(\mathrm{d}s, \mathrm{d}\theta) - \mathrm{d}s\,\mathrm{d}\theta$, it follows from basic stochastic calculus for jump processes (cf. e.g. [93]) that

$$\begin{aligned} e^{\lambda_0 t}\eta(X_t) = X_0 &+ \int_0^t \int_0^{q(X_{s-})} e^{\lambda_0 s} \sum_{i=0}^{\infty} \mathbb{1}_{\theta\in[\kappa_{i+1}(X_{s-}),\kappa_i(X_{s-}))}(\eta(i) - \eta(X_{s-}))\widetilde{N}(\mathrm{d}s, \mathrm{d}\theta) \\ &+ \int_0^t e^{\lambda_0 s}\left(\sum_{i=0}^{\infty} q_{X_s,i}(\eta(i) - \eta(X_s)) + \lambda_0\eta(X_s)\right)\mathrm{d}s. \end{aligned}$$

Since $e^{\lambda_0 t}\eta(X_t)$ is a $\mathbb{P}_x$-martingale, the Doob-Meyer decomposition theorem entails that

$$\int_0^t e^{\lambda_0 s}\left(\sum_{i=0}^{\infty} q_{X_s,i}(\eta(i) - \eta(X_s)) + \lambda_0\eta(X_s)\right)\mathrm{d}s = 0$$

$\mathbb{P}_x$-almost surely for all $t \geq 0$ and all $x \in E$. Hence, if there exists $y \in E$ such that $\mathcal{L}\eta(y) \neq -\lambda_0\eta(y)$, by irreducibility, there exists an event with positive probability under $\mathbb{P}_x$ such that the previous integral is non-constant. We obtain a contradiction and hence $\mathcal{L}\eta(x) = -\lambda_0\eta(x)$ for all $x \in E$. □

6 On reducible examples

The criteria and examples studied in the last two sections assume that the process X is irreducible in E. However, the abstract results of Section 2 do not require the state space to be irreducible. Our goal in this section is to explain that our criteria are also well-suited to cases of reducible absorbed Markov processes, in the sense that the state space E can be partitioned in a finite or countable family of communication classes. The study of quasi-stationary behavior for such processes has been up to now restricted to the case of finite state spaces or to particular classes of models [77, 38, 88, 55, 19, 102, 103, 18, 104, 8]. Our criteria provide new practical tools to tackle this problem, further exploited in [28].

In Subsection 6.1, we consider a general setting with three successive sets. In Subsection 6.2, we consider a birth and death process with a countable infinity of communication classes.

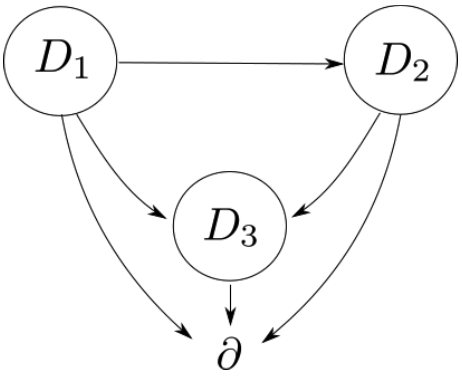

Figure 1: Transition graph displaying the relation between the sets D_1, D_2, D_3 and ∂.

6.1 Three successive sets

In this section, we consider a discrete time Markov process $(X_n, n \in \mathbb{Z}_+)$ evolving in a measurable set $E \cup \{\partial\}$ with absorption at $\partial \notin E$. We assume that the transition probabilities of X satisfy the structure displayed in Figure 1: one can find a partition $\{D_1, D_2, D_3\}$ of E such that the process starting from D_1 can access $D_1 \cup D_2 \cup D_3 \cup \{\partial\}$, the process starting from D_2 can only access $D_2 \cup D_3 \cup \{\partial\}$, and the process starting from D_3 can only access $D_3 \cup \{\partial\}$. More formally, we assume that $\mathbb{P}_x(T_{D_3} \wedge \tau_\partial < T_{D_1}) = 1$ for all $x \in D_2$ and that $\mathbb{P}_x(\tau_\partial < T_{D_1 \cup D_2}) = 1$ for all $x \in D_3$, where we recall that, for any measurable set $A \subset E$, $T_A = \inf\{n \in \mathbb{Z}_+,\ X_n \in A\}$.

Our aim is to provide sufficient conditions ensuring that X satisfies Assumption (E). In order to do so, we assume that Assumption (E) is satisfied by the process X before exiting D_2. This corresponds to the following assumption.

Assumption (H1). The absorbed Markov process Y evolving in $D_2 \cup \{\partial\}$, defined by

$$Y_n = \begin{cases} X_n & \text{if } n < T_{D_1 \cup D_3 \cup \{\partial\}}, \\ \partial & \text{if } n \geq T_{D_1 \cup D_3 \cup \{\partial\}}, \end{cases}$$

satisfies Assumption (E). In what follows, we denote the objects related to Y with a superscript Y, for instance, the constants of Assumption (E) for Y are denoted by $\theta_1^Y > 0$, $\theta_2^Y > 0$.

We also assume that the exit times from D_1 and D_3 for the process X admit exponential moments of sufficiently high order, as stated by the following assumption.

Assumption (H2). There exists a positive constant $\gamma < \theta_0^Y$ such that, for all $x \in D_1$,

$$\mathbb{E}_x\left(\gamma^{-T_{D_2}} \varphi_1^Y\left(X_{T_{D_2}}\right) \mathbb{1}_{T_{D_2} < T_{D_3} \wedge \tau_\partial}\right) < +\infty, \quad \mathbb{E}_x\left(\gamma^{-T_{D_3} \wedge \tau_\partial} \mathbb{1}_{T_{D_3} \wedge \tau_\partial < T_{D_2}}\right) < +\infty,$$

and such that

$$\sup_{x \in D_3} \mathbb{E}_x\left(\gamma^{-\tau_\partial}\right) < +\infty.$$

We are now able to state the main result of this section.

Theorem 6.1. *Under Assumptions (H1) and (H2), the process X satisfies Assumption (E) with $K = K^Y$,*

$$\varphi_1(x) = \mathbb{E}_x\left(\gamma^{-T_K \wedge \tau_\partial}\right) \quad \textit{and} \quad \varphi_2(x) \geq c \mathbb{1}_{x \in K},\ \forall x \in E, \tag{6.1}$$

for some constant $c > 0$. In particular, it admits a unique quasi-stationary distribution ν_{QSD} such that $\nu_{QSD}(\varphi_1) < \infty$ and $\nu_{QSD}(\varphi_2) > 0$. Moreover, there exist two constants

$C > 0$ *and* $\alpha \in (0,1)$ *such that, for all probability measure* μ *on* E *such that* $\mu(\varphi_1) < \infty$ *and* $\mu(\varphi_2) > 0$,

$$\|\mathbb{P}_\mu(X_n \in \cdot \mid n < \tau_\partial) - \nu_{QSD}\|_{TV(\varphi_1)} \leq C\alpha^n \, \frac{\mu(\varphi_1)}{\mu(\varphi_2)}. \tag{6.2}$$

Finally, $\theta_0 = \theta_0^Y$, $\nu_{QSD}(D_1) = 0$ *and the function* η *of Theorem 2.3 vanishes on* D_3*.*

Before turning to the proof of this result, let us make some remarks.

Remark 6.2.
1. The fact that there are three different sets D_1, D_2 and D_3 in the decomposition of E is not restrictive on the number of communication classes. Indeed, the three sets can contain several communication classes.
2. A similar result can be obtained for continuous time processes, based on Assumption (F) instead of (E), with the additional technical assumption that the strong Markov property can be applied at the exit times of D_1 and D_2.
3. We emphasize that, besides the exponential moment assumption, there is no additional requirement on the behavior of the Markov process in D_1 and D_3. In these sets, the process might be for instance periodic or deterministic and could satisfy that $\mathbb{P}_x(n < \tau_\partial) = 0$ for some $n \in \mathbb{N}$.
4. One easily checks from the proof that the function φ_1 in (6.1) is bounded (up to a multiplicative positive constant) from above by

   $$\mathbb{E}_x\left(\gamma^{-T_{D_2}}\varphi_1^Y\left(X_{T_{D_2}}\right)\mathbb{1}_{T_{D_2} < T_{D_3}\wedge\tau_\partial}\right) + \mathbb{E}_x\left(\gamma^{-T_{D_3}\wedge\tau_\partial}\mathbb{1}_{T_{D_3}\wedge\tau_\partial < T_{D_2}}\right)$$

   on D_1, by φ_1^Y on D_2 and by a constant on D_3.
5. In particular, if φ_1^Y is uniformly bounded and if the first statement in Assumption (H2) is replaced by

   $$\sup_{x \in D_1} \mathbb{E}_x\left(\gamma^{-T_{D_2 \cup D_3}\wedge\tau_\partial}\right) < +\infty,$$

   then one can also choose a bounded function φ_1 in Assumption (E) for X.

Remark 6.3. In general, processes on reducible state spaces may not satisfy Assumption (E). For example the convergence in (6.2) may not be exponential, or quasi-stationary distributions may not be unique, even if the process X restricted to D_1, D_2 or D_3 satisfy condition (E). We refer the reader to [28] for a more general discussion on quasi-stationary distributions and quasi-limiting behavior for general processes on reducible state spaces.

Proof of Theorem 6.1. Let us prove that Assumption (E) is satisfied by the process X. Note that, because of Lemma 3.4, one can assume without loss of generality that $\gamma < \theta_2^Y$.

Step 1. Assumption (E1).

We set $K = K^Y$, $n_1 = n_1^Y$, $c_1 = c_1^Y$ and $\nu = \nu^Y$. Assumption (E1) for X is an immediate consequence of Assumption (E1) for Y.

Step 2. Assumption (E2).

We set $\theta_2 = \theta_2^Y$ and

$$\varphi_2(x) = \begin{cases} \varphi_2^Y(x) & \text{if } x \in D_2 \\ 0 & \text{if } x \in D_1 \cup D_3. \end{cases}$$

Then the second and fourth lines of Assumption (E) for X are direct consequences of the same lines of Assumption (E) for Y.

Without loss of generality, we assume (increasing γ if necessary, which does not change the fact that Assumptions (H2) is true) that $\gamma \in (\theta_1^Y, \theta_2^Y)$. We define

$$\varphi_1(x) = \mathbb{E}_x\left(\gamma^{-T_K \wedge \tau_\partial}\right), \quad \forall x \in E \cup \{\partial\}.$$

Let us first check that φ_1 is finite on E. For all $x \in D_3$, using that $\mathbb{P}_x(\tau_\partial < T_{D_1 \cup D_2}) = 1$ and that $K \subset D_2$, one deduces that

$$\varphi_1(x) = \mathbb{E}_x\left(\gamma^{-\tau_\partial}\right) \leq A := \sup_{x \in D_3} \mathbb{E}_x\left(\gamma^{-\tau_\partial}\right) < +\infty.$$

For all $x \in D_2$, using the strong Markov property and inequality (9.7) for the process Y, one deduces that

$$\begin{aligned}
\varphi_1(x) &= \mathbb{E}_x\left(\gamma^{-T_K \wedge T_{D_2^c}} \mathbb{1}_{T_K < T_{D_2^c}}\right) + \mathbb{E}_x\left(\gamma^{-\tau_\partial} \mathbb{1}_{T_{D_2^c} < T_K}\right) \\
&= \mathbb{E}_x\left(\gamma^{-T_K \wedge T_{D_2^c}} \mathbb{1}_{T_K < T_{D_2^c}}\right) + \mathbb{E}_x\left(\gamma^{-T_K \wedge T_{D_2^c}} \mathbb{1}_{T_{D_2^c} < T_K} \mathbb{E}_{X_{T_{D_2^c}}}\left(\gamma^{-\tau_\partial}\right)\right) \\
&\leq A \mathbb{E}_x\left(\gamma^{-T_K \wedge T_{D_2^c}}\right) \leq \frac{A}{1 - \theta_1^Y / \gamma} \varphi_1^Y(x).
\end{aligned} \tag{6.3}$$

For all $x \in D_1$, one has, using the Markov property and the above inequalities,

$$\begin{aligned}
&\mathbb{E}_x\left(\gamma^{-T_K \wedge \tau_\partial}\right) = \mathbb{E}_x\left(\gamma^{-T_{D_2 \cup D_3} \wedge \tau_\partial} \varphi_1(X_{T_{D_2 \cup D_3} \wedge \tau_\partial})\right) \\
&\leq \frac{A}{1 - \theta_1^Y / \gamma}\left[\mathbb{E}_x\left(\gamma^{-T_{D_2}} \varphi_1^Y\left(X_{T_{D_2}}\right) \mathbb{1}_{T_{D_2} < T_{D_3} \wedge \tau_\partial}\right) + \mathbb{E}_x\left(\gamma^{-T_{D_3} \wedge \tau_\partial} \mathbb{1}_{T_{D_3} \wedge \tau_\partial < T_{D_2}}\right)\right],
\end{aligned} \tag{6.4}$$

which is finite by Assumption (H2).

The definition of φ_1 immediately implies that $\inf_E \varphi_1 \geq 1$ and, since φ_1^Y is uniformly bounded over $K \subset D_2$, (6.3) implies that $\sup_K \varphi_1 < +\infty$. Hence the first line of Assumption (E2) is satisfied. Moreover, for all $x \in K$,

$$\begin{aligned}
P_1 \varphi_1(x) &= \mathbb{E}_x\left(\mathbb{1}_{X_1 \in D_2} \mathbb{E}_{X_1}\left(\gamma^{-T_K \wedge \tau_\partial}\right)\right) + \mathbb{E}_x\left(\mathbb{1}_{X_1 \in D_3} \mathbb{E}_{X_1}\left(\gamma^{-\tau_\partial}\right)\right) \\
&\leq \mathbb{E}_x\left(\mathbb{1}_{X_1 \in D_2} \frac{A}{1 - \theta_1^Y / \gamma} \varphi_1^Y(X_1)\right) + A \\
&= \frac{A}{1 - \theta_1^Y / \gamma} P_1^Y \varphi_1^Y(x) + A \leq \frac{A}{1 - \theta_1^Y / \gamma}\left(\theta_1^Y \sup_K \varphi_1^Y + c_2^Y\right) + A.
\end{aligned}$$

Hence, the third line of (E2) for X with $\theta_1 = \gamma$ follows from Lemma 3.2.

Step 3. Assumption (E3).

For all $x \in K$, we have, for all $n \geq 1$,

$$\mathbb{P}_x(n < \tau_\partial) \leq \mathbb{P}_x(n < \tau_\partial \wedge T_{D_3}) + \mathbb{P}_x(T_{D_3} \leq n < \tau_\partial). \tag{6.5}$$

On the one hand, by Lemma 9.9, there exists a constant $C > 0$ such that

$$\mathbb{P}_x(n < \tau_\partial \wedge T_{D_3}) \leq \frac{C \varphi_1^Y(x)}{1 - \theta_1^Y / \theta_2^Y} \inf_{y \in K} \mathbb{P}_y(n < T_{D_2^c}) \leq \frac{C \sup_K \varphi_1^Y}{1 - \theta_1^Y / \theta_2^Y} \inf_{y \in K} \mathbb{P}_y(n < T_{D_2^c}).$$

On the other hand, using Markov's property and Markov's inequality,

$$\begin{aligned}
\mathbb{P}_x(T_{D_3} \leq n < \tau_\partial) &= \mathbb{E}_x\left(\mathbb{1}_{T_{D_3} \leq n} \mathbb{P}_{X_{T_{D_3}}}(n - u < \tau_\partial)\Big|_{u = T_{D_3}}\right) \\
&\leq \mathbb{E}_x\left(\mathbb{1}_{T_{D_3} \leq n} \varphi_1(X_{T_{D_3}}) \gamma^{n - T_{D_3}}\right) \leq A \mathbb{E}_x\left(\mathbb{1}_{T_{D_2^c} \leq n} \gamma^{n - T_{D_2^c}}\right),
\end{aligned}$$

since $\{T_{D_3} \leq n\} \subset \{T_{D_2^c} = T_{D_3}\}$. Now, using Theorem 2.3 and the fact that η^Y is uniformly bounded from above and away from 0 on K, we deduce that there exist constants $C, C' > 0$ such that

$$\begin{aligned}\mathbb{E}_x\left(\mathbb{1}_{T_{D_2^c}\leq n}\gamma^{n-T_{D_2^c}}\right) &= \sum_{k=1}^{n}\mathbb{P}_x(T_{D_2^c}=k)\,\gamma^{n-k} \leq \sum_{k=1}^{n}\mathbb{P}_x(T_{D_2^c}>k-1)\,\gamma^{n-k}\\ &\leq C\,\sum_{k=1}^{n}(\theta_0^Y)^{k-1}\,\gamma^{n-k} \leq C\,(\theta_0^Y)^{n-1}\frac{1}{1-\gamma/\theta_0^Y}\\ &\leq C\,C'\,\frac{(\theta_0^Y)^{-1}}{1-\gamma/\theta_0^Y}\,\inf_{y\in K}\mathbb{P}_y(n<T_{D_2^c}).\end{aligned}$$

Finally, we obtain from (6.5) that there exists a constant $C'' > 0$ such that, for all $x \in K$,

$$\mathbb{P}_x(n<\tau_\partial) \leq C''\inf_{y\in K}\mathbb{P}_y(n<T_{D_2}^c) \leq C''\inf_{y\in K}\mathbb{P}_y(n<\tau_\partial). \tag{6.6}$$

This concludes Step 3.

Step 4. Conclusion.

Assumption (E4) for the process X is an immediate consequence of Assumption (E4) for the process Y, and hence we have checked that X satisfies Assumption (E). The convergence result of Theorem 6.1 is exactly the convergence result obtained in Theorem 2.1.

Note that (6.6) entails that, for any $x \in K$,

$$\begin{aligned}\limsup_{n\to+\infty}(\theta_0^Y)^{-n}\mathbb{P}_x(n<T_{D_2^c}) &\leq \limsup_{n\to+\infty}(\theta_0^Y)^{-n}\mathbb{P}_x(n<\tau_\partial)\\ &\leq C''\limsup_{n\to+\infty}(\theta_0^Y)^{-n}\mathbb{P}_x(n<T_{D_2^c})\end{aligned}$$

and that Theorem 2.3 applied to Y entails

$$\limsup_{n\to+\infty}(\theta_0^Y)^{-n}\mathbb{P}_x(n<T_{D_2^c}) = \eta^Y(x) < +\infty.$$

Since it follows from Theorem 2.3 applied to X that $\lim_{n\to+\infty}\theta_0^{-n}\mathbb{P}_x(n<\tau_\partial) > 0$, we deduce that $\theta_0 = \theta_0^Y$.

Finally, for all $x \in K$, the structure of the transition graph of X implies that

$$0 = \mathbb{P}_x(X_n \in D_1 \mid n < \tau_\partial) \xrightarrow[n\to+\infty]{} \nu_{QSD}(D_1),$$

so that $\nu_{QSD}(D_1) = 0$. Moreover, for all $x \in D_3$, Markov's inequality and Assumption (H2) yield the inequality $\mathbb{P}_x(n<\tau_\partial) \leq A\,\gamma^n$, for all $x \in K$ and all $n \geq 1$. Since $\theta_0 = \theta_0^Y > \gamma$ by assumption, we deduce that, for all $x \in K$, $\lim_{n\to+\infty}\theta_0^{-n}\mathbb{P}_x(n<\tau_\partial) = 0$, which means that $\eta(x) = 0$.

This concludes the proof of Theorem 6.1. □

6.2 Countably many communication classes

In this section, we study a particular case of a continuous time càdlàg Markov process $(X_t)_{t\in[0,+\infty)}$ with a countable infinity of communication classes and we show that the process admits a quasi-stationary distribution.

More precisely, we assume that X evolves in the state space $\mathbb{N}\times\mathbb{Z}_+$ and, denoting $N_t \in \mathbb{N}$ and $Y_t \in \mathbb{Z}_+$ the two components of X_t for all $t \in [0,+\infty)$, that there exist three positive functions $b, d, f : \mathbb{N} \to (0,+\infty)$ such that

- N is a Poisson process with intensity 1,
- Y is a process such that, at time t,

$$Y \text{ jumps from } Y_t \text{ to } y \in \mathbb{Z}_+ \text{ with rate } \begin{cases} f(N_t)\, b(Y_t) & \text{if } y = Y_t + 1 \text{ and } Y_t \geq 1, \\ f(N_t)\, d(Y_t) & \text{if } y = Y_t - 1 \text{ and } Y_t \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\mathbb{N} \times \{0\}$ is absorbing for X and we are interested in the quasi-stationary behavior of X conditioned to not hit this set. Note that, in this case, each set $\{n\} \times \mathbb{N}$ is a communication class.

Remark 6.4. This process can be used to model the survival of an individual (for example a bacterium) whose metabolic efficiency (for example its ability to consume resources) changes with time, due to aging [96]. Here Y is the vitality of the individual, who dies when its vitality hits 0, $f(N)$ is the metabolic rate of the individual, which may for example decrease in the early life of the individual up to age n_0 and then accelerates progressively.

This can also model the accumulation of deleterious mutations in a population under the assumption that mutations do not overlap, i.e. that when a mutant succeeds to invade the population (either because they are advantaged or due to genetic drift for deleterious mutations), other types of mutants disappear rapidly. Here Y represents the size of the population and N the number of mutations. It is typical to assume that the first n_0 mutations that invade are advantageous (which corresponds to adaptation), and afterwards that deleterious mutations start to accumulate, hence accelerating the extinction of the species (extinction vortex [37, 36]).

In both cases, it is relevant to assume that f is decreasing on $\{1, 2, \ldots, n_0\}$ and increasing on $\{n_0, n_0 + 1, \ldots\}$.

We assume that $(d(y) - b(y))/y \to +\infty$ when $y \to +\infty$ or that there exists $\delta > 1$ such that $d(y) - \delta\, b(y) \to +\infty$. Hence the birth and death process Z evolving in $\mathbb{N}$, with birth rates $(b(z))_{z\in\mathbb{N}}$ and death rates $(d(z))_{z\in\mathbb{N}}$, satisfies Assumption (F) by Theorem 5.1 (see Example 5.4). In particular, there exist an eigenvalue $\lambda_0^Z > 0$ and eigenfunction $\eta^Z : \mathbb{N} \to (0, +\infty)$ such that, for all $z \in \mathbb{N}$, $\mathcal{L}^Z \eta^Z = -\lambda_0^Z \eta^Z$, where the operator $\mathcal{L}^Z$ is defined as the operator $\mathcal{L}$ in (5.1).

Theorem 6.5. *Assume also that there exists a unique* $n_0 \in \mathbb{N}$ *such that* $f(n_0) = \min_{n\in\mathbb{N}} f(n)$ *and that* $\liminf_{n\to+\infty} f(n) > f(n_0) + \frac{1}{\lambda_0^Z}$. *Then the process* X *satisfies Assumption (F) and admits a quasi-stationary distribution* ν_{QSD} *whose domain of attraction contains all Dirac measures* $\delta_{n,y}$*, with* $n \leq n_0$ *and* $y \in \mathbb{N}$.

Of course, all the consequences of Theorem 3.5 also apply here, taking the functions ψ_1 and ψ_2 as described in the proof.

Proof. The proof maks use of the special structure of the process Y, which can be constructes as

$$Y_t = Z_{\int_0^t f(N_s)\mathrm{d}s}, \quad \forall t \geq 0.$$

In general, we shall denote the objects related to Z with a superscript Z, for example ψ_1^Z is the functions involved in (F2) and L^Z is the set involved in (F) for Z. We can assume without loss of generality as in Theorem 5.1 that $L^Z = D_0^Z$, i.e.

$$\mathcal{L}^Z \psi_1^Z \leq -\lambda_1^Z \psi_1^Z + \bar{C}\mathbb{1}_{L^Z} \tag{6.7}$$

with $\psi_1^Z(0) = 0$ and $\lambda_1^Z > \lambda_0^Z$.

Our goal is to apply Theorem 5.1 to the process $X = (N, Y)$. We define the finite set $D_0 = \{n_0\} \times L^Z$, so that $\mathbb{P}_{(n_0,x)}(X_1 = (n_0, y)) > 0$ for all (n_0, x) and (n_0, y) in D_0, and check that $\lambda_0 \le f(n_0)\lambda_0^Z + 1$. Indeed, for all $y \in L^Z$,

$$\begin{aligned} e^{t(f(n_0)\lambda_0^Z+1)}\mathbb{P}_{(n_0,y)}((N_t, Y_t) = (n_0, y)) &\ge e^{tf(n_0)\lambda_0^Z}\mathbb{P}_y^Z(Z_{f(n_0)t} = y) \\ &\xrightarrow[t\to+\infty]{} \eta^Z(y)\nu^Z_{QSD}(\{y\}) > 0. \end{aligned}$$

We fix λ_1 such that

$$f(n_0)\lambda_0^Z + 1 < \lambda_1 < \left(\lambda_0^Z \inf_{n\neq n_0} f(n) + 1\right) \wedge \left(\lambda_0^Z \liminf_{n\to+\infty} f(n)\right) \wedge \left(\lambda_1^Z f(n_0) + 1\right)$$

and we choose

- $n_1 > n_0$ such that, for all $n \ge n_1$, $\lambda_1 < \lambda_0^Z f(n)$;
- $c > 0$ small enough so that $\psi_1^Z(x) \ge c\eta^Z(x)$ for all $x \ge 1$ (such a constant exists thanks to Theorem 2.3);
- $a > 0$ large enough so that $\lambda_1 < \lambda_1^Z f(n_0) + 1 - e^{-a}$;
- $\varepsilon > 0$ small enough so that $\lambda_1 < (\lambda_0^Z - \varepsilon)\inf_{n\neq n_0} f(n) + 1$;
- $b > a$ large enough so that $\lambda_1 < (\lambda_0^Z - \varepsilon)\inf_{n\neq n_0} f(n) + 1 - e^{-b}$ and $\bar{C}e^{a-b} < \varepsilon \inf_{y\in L^Z} \eta^Z(y)$, where the constant $\bar{C}$ is the one of (6.7).

We can now define

$$\psi_1(n, y) = \begin{cases} \psi_1^Z(y) & \text{if } n = n_0, \\ e^{a(n_0-n)}\psi_1^Z(y) + e^{b(n_0-n)}\eta^Z(y) & \text{if } n < n_0, \\ ce^{-a(n-n_0)}\eta^Z(y) & \text{if } n_0 < n < n_1, \\ ce^{-a(n_1-n_0)}\eta^Z(y) & \text{if } n_1 \le n. \end{cases}$$

In the case where $n < n_0$, it follows from (6.7) that

$$\begin{aligned} \mathcal{L}\psi_1(n, y) \le &- \left(\lambda_1^Z f(n) + 1 - e^{-a}\right) e^{a(n_0-n)}\psi_1^Z(y) \\ &- \left(\lambda_0^Z f(n) + 1 - e^{-b}\mathbb{1}_{n<n_0-1}\right) e^{b(n_0-n)}\eta^Z(y) \\ &+ \frac{\bar{C}}{\inf_{z\in L^Z} \eta^Z(z)} f(n)e^{a(n_0-n)}\eta^Z(y) \\ \le &- \lambda_1 e^{a(n_0-n)}\psi_1^Z(y) - \left[(\lambda_0^Z - \varepsilon)f(n) + 1 - e^{-b}\mathbb{1}_{n<n_0-1}\right] e^{b(n_0-n)}\eta^Z(y) \\ &+ \varepsilon f(n)e^{a(n_0-n)}\left(e^{b-a} - e^{(b-a)(n_0-n)}\right)\eta^Z(y) \\ \le &- \lambda_1\psi_1(n, y). \end{aligned}$$

When $n = n_0$, we have

$$\begin{aligned} \mathcal{L}\psi_1(n_0, y) &\le -\lambda_1^Z f(n_0)\psi_1^Z(y) + \bar{C}\mathbb{1}_{L^Z}(y)f(n_0) + ce^{-a}\eta^Z(y) - \psi_1^Z(y) \\ &\le -\lambda_1\psi_1(n_0, y) + \bar{C}f(n_0)\mathbb{1}_{D_0}(n_0, y). \end{aligned}$$

When $n_0 < n < n_1$, we have

$$\begin{aligned} \mathcal{L}\psi_1(n, y) &\le -\lambda_0^Z f(n)\, c\, e^{-a(n-n_0)}\eta^Z(y) + c\, e^{-a(n-n_0)}\eta^Z(y)\left(e^{-a} - 1\right) \\ &\le -\lambda_1\psi_1(n, y). \end{aligned}$$

When $n_1 \le n$, we have

$$\mathcal{L}\psi_1(n, y) \le -\lambda_0^Z f(n)\eta^Z(y) \le -\lambda_1\psi_1(n, y).$$

Finally we have proved that $\mathcal{L}\psi_1(n,y) \leq -\lambda_1\psi_1(n,y) + \bar{C}f(n)\mathbb{1}_{D_0}(n,y)$, where $\lambda_1 > \lambda_0$. Now, note that, since Z is a birth-death process, basic comparison arguments imply that $\eta^Z(k) \geq \eta^Z(1) > 0$ for all $k \geq 1$. Therefore, the function ψ_1 is uniformly lower bounded, so that it satisfies the assumptions of Theorem 5.1 up to a multiplicative constant.

Hence, Theorem 5.1 allows us to conclude the proof. The fact that all Dirac masses $\delta_{(n,y)}$ with $n \leq n_0$ belong to the domain of attraction follows from Corollary 2.10. □

7 Application to processes in continuous state space and discrete time

Discrete time Markov models in continuous state space and with absorption naturally arise in many applications. Examples of such processes are given by perturbed dynamical systems, cf. e.g. [44, 9, 5, 62], or piecewise deterministic Markov processes when one looks at the process at jump times only (see e.g. [3]). We provide in Section 7.1 a general criterion applying to such processes with arbitrarily close to 1, state-dependent killing probability, and we give applications to Euler schemes for diffusions absorbed at the boundary of a domain. In Section 7.2, we consider perturbed dynamical systems in finite dimension. We first consider the case of unbounded domains with unbounded perturbation. Subsection 7.2.1 assumes that the perturbation has bounded density with respect to Lebesgue's measure and Subsection 7.2.2 provides examples with perturbations with unbounded density. Finally, the case of bounded perturbations is studied in Subsection 7.2.3. Theorem 1.2 of the Introduction is obtained as an application of the results of Section 7.2.1.

7.1 Two sided estimates for processes with killing

Let $(Y_n, n \in \mathbb{Z}_+)$ be a Markov process evolving on a measurable state space $E \cup \{\partial\}$ with transition kernel $(Q(y,\cdot)_{y\in E\cup\{\partial\}})$ such that $\partial \notin E$ is absorbing (i.e. $Q(\partial,\{\partial\}) = 1$) satisfying a two-sided estimate (see for instance [10, 40, 17]), which means that there exist a probability measure ζ on E, a positive function $g : E \to (0,+\infty)$ and a constant $C > 1$ such that, for all $y \in E$ and all measurable sets $A \subset E$,

$$g(y)\zeta(A) \leq Q(y,A) \leq Cg(y)\zeta(A). \tag{7.1}$$

Condition (7.1) is known to be satisfied for various models (see e.g. [9] or the references in [17]). It is also well known (see [10, 17]) that this implies that Y admits a unique quasi-stationary distribution ν^Y_{QSD} for which the convergence in (2.1) holds true for the total variation distance with geometric speed uniform with respect to the initial distribution μ on E. Our aim is to generalize this result to processes obtained from Y with additional killing (or penalization).

More precisely, let $p : E \times E \to (0,1]$ be measurable and consider the Markov process X evolving in $E \cup \{\partial\}$ with transition kernel $P(x,\cdot)_{x\in E\cup\{\partial\}}$ defined by

$$P(x,\mathrm{d}y) = \begin{cases} p(x,y)Q(x,\mathrm{d}y) + (1-p(x,y))\delta_\partial(\mathrm{d}y) & \text{if } x \in E \\ \delta_\partial(\mathrm{d}y) & \text{if } x = \partial. \end{cases}$$

Observe that Condition (7.1) may not be satisfied by the kernel P in cases where $\inf_{x,y\in E} p(x,y) = 0$. We also emphasize that the kernel P generates a penalized semi-group of $(Y_n)_{n\in\mathbb{Z}_+}$, in the sense that, for any function $f : E \to \mathbb{R}_+$, all $x \in E$ and all $n \geq 1$, one has

$$\mathbb{E}_x\left(f(X_n)\mathbb{1}_{n<\tau_\partial}\right) = \mathbb{E}_x\left(p(x,Y_1)\cdots p(Y_{n-1},Y_n)\, f(Y_n)\mathbb{1}_{n<\tau^Y_\partial}\right),$$

where τ_∂, resp. τ^Y_∂, is the absorption time for X, resp. Y, in ∂.

Theorem 7.1. *Assume that there exists an increasing sequence* $(L_k)_{k\geq 1}$ *of measurable subsets of* E *such that* $E = \cup_{k=1}^{+\infty} L_k$ *and* $\inf_{x,y\in L_k} p(x,y) > 0$ *for all* $k \geq 1$. *Then* X *satisfies Assumption (E) with* $\varphi_1 = 1$ *and* φ_2 *positive on* E. *In particular,* X *admits a unique quasi-stationary distribution whose domain of attraction contains all probability measures on* E.

Example 7.2. Typical examples of discrete-time Markov processes in continuous state space are given by Euler schemes for stochastic differential equations. We consider the SDE $\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}B_t$ in $\mathbb{R}^d$, with b and σ bounded measurable on $\mathbb{R}^d$ and σ uniformly elliptic on $\mathbb{R}^d$. Its standard Euler scheme with time-step δ is the Markov chain $(X_n, n \geq 0)$ defined as

$$X_{n+1} = b(X_n)\delta + \sqrt{\delta}\sigma(X_n)G_n, \tag{7.2}$$

where $(G_n, n \geq 0)$ is an i.i.d. sequence of $\mathcal{N}(0, \mathrm{Id})$ Gaussian variables in $\mathbb{R}^d$. In the case of a SDE absorbed at its first exit time of a bounded open connected domain $D \subset \mathbb{R}^d$, the “naive” Euler scheme, constructed as above with the additional rule that X_n is immediately sent to ∂ when $X_n \notin D$, is not good in terms of weak error. Indeed, when X_n is close to the boundary of D and X_{n+1} remains in D, the path of the SDE Y in the time interval $[n\delta, (n+1)\delta]$ might have exited D. In this case, it is more efficient to construct the Brownian path that links 0 to G_n on the time interval $[n\delta, (n+1)\delta]$ as a Brownian bridge $(\tilde{G}_t, t \in [n\delta, (n+1)\delta])$ such that $\tilde{G}_{n\delta} = 0$ and $\tilde{G}_{(n+1)\delta} = G_n$, so that one can approximate the path of the diffusion on this time interval as

$$\tilde{X}_t = b(X_n)(t - n\delta) + \sqrt{\delta}\sigma(X_n)\tilde{G}_t, \quad \forall t \in [n\delta, (n+1)\delta],$$

and approximate the absorption event as $\{\exists t \in [n\delta, (n+1)\delta] : \tilde{X}_t \notin D\}$. The corresponding Euler scheme is thus obtained as the Markov chain X as defined in (7.2) with the penalization $p(X_n, X_{n+1}) = \mathbb{P}(\exists t \in [n\delta, (n+1)\delta] : \tilde{X}_t \notin D)$. For a detailed presentation and study of this kind of modified Euler schemes, we refer the reader to [79, 51, 52, 11].

Using Theorem 7.1, we obtain the existence and convergence to a unique quasi-stationary distribution for this modified Euler scheme. Indeed, (7.1) is satisfied for the naive Euler scheme with ζ equal to the restriction of Lebesgue’s measure to D and a constant function g, thanks to the boundedness of the domain D, the uniform ellipticity of σ and the boundedness of b and σ. In addition, it follows from the connectedness of the domain D, the uniform ellipticity of σ and the boundedness of b and σ that $\inf_{x,y\in K} p(x,y) > 0$ for any compact subset K of D.

Proof of Theorem 7.1. For all $k \geq 1$, we define the set $K_k = \{x \in L_k \text{ s.t. } g(x) \geq 1/k\}$. Let k_0 be large enough so that $\zeta(K_{k_0}) > 0$. Then one has, for all $k \geq k_0$, all $x \in K_k$ and all measurable set $A \subset E$,

$$\mathbb{P}_x(X_1 \in A \cap K_{k_0}) \geq g(x) \int_{A\cap K_{k_0}} p(x,y)\,\zeta(\mathrm{d}y) \geq \frac{\zeta(K_{k_0}) \inf_{u,v\in L_k} p(u,v)}{k}\,\nu(A \cap K_{k_0}), \tag{7.3}$$

where ν is the probability measure on K_{k_0} defined by $\nu(A) = \zeta(A)/\zeta(K_{k_0})$. We fix $k \geq k_0$ large enough so that $C/k < \frac{\zeta(K_{k_0}) \inf_{u,v\in L_{k_0}} p(u,v)}{k_0}$, where the constant C is the one of (7.1), and set $K = K_k$.

Let us now check that Condition (E) is satisfied with the above choices of K and ν (extended by 0 to $K_k \setminus K_{k_0}$), and with $\theta_1 = C/k$ and $\theta_1 < \theta_2 < \frac{\zeta(K_{k_0}) \inf_{u,v\in L_{k_0}} p(u,v)}{k_0}$.

Setting $\varphi_1 = 1$, one has

$$\begin{aligned} &P_1\varphi_1(x) \leq 1,\ \forall x \in K,\\ &P_1\varphi_1(x) \leq C\,g(x) \leq \theta_1 = \theta_1\varphi_1(x),\ \forall x \in E \setminus K, \end{aligned}$$

so that the first and third lines of Condition (E2) are satisfied. Using Markov's property, one deduces from (7.3) that $\theta_2^{-n}\inf_{x\in K}\mathbb{P}_x(X_n\in K_{k_0})\to+\infty$ when $n\to+\infty$. Hence Lemma 3.1 implies that the second and fourth lines of Condition (E2) are satisfied. It also implies that Condition (E4) is satisfied. Note also that the function φ_2 provided by Lemma 3.1 is positive on E since g is positive in (7.1).

Moreover, for all $x\in E$, all $y\in K$ and all measurable set $A\subset E$,

$$\begin{aligned}\mathbb{P}_x(X_1\in A\cap K)\leq Cg(x)\zeta(A\cap K)&\leq\frac{Cg(x)kg(y)}{\inf_{K\times K}p}\int_{A\cap K}p(y,z)\,\zeta(\mathrm{d}z)\\&\leq\frac{C\|g\|_\infty k}{\inf_{K\times K}p}\,\mathbb{P}_y(X_1\in A\cap K).\end{aligned}$$

We deduce from Proposition 3.3 with $n_0=m_0=1$ that Conditions (E1) and (E3) are satisfied, which concludes the proof of Theorem 7.1. ☐

7.2 Perturbed dynamical systems

We consider the following perturbed dynamical system

$$X_{n+1}=f(X_n)+\xi_n,$$

where $f:\mathbb{R}^d\to\mathbb{R}^d$ is a measurable function and $(\xi_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence in $\mathbb{R}^d$. We assume that the process evolves in a measurable set D of $\mathbb{R}^d$ with positive Lebesgue measure, meaning that it is immediately sent to $\partial\notin\mathbb{R}^d$ as soon as $X_n\notin D$. We shall consider two situations below, where the random variables ξ_n are unbounded or almost surely bounded. In the unbounded case, different methods must be used depending on whether ξ_n has a bounded density with respect to Lebesgue's measure or not.

The same arguments would also work if $X_{n+1}=f(X_n)+\xi_n(X_n)$, where the sequence of random maps $(x\mapsto\xi_n(x))_{n\geq 0}$ are i.i.d. We leave the appropriate extensions of our assumptions and arguments to the reader.

7.2.1 The case of unbounded perturbation with bounded density

We consider here the case where the random variables ξ_n have support $\mathbb{R}^d$.

Proposition 7.3. *Assume that f is locally bounded, that the law of ξ_n has a bounded density $g(x)$ with respect to Lebesgue's measure such that*

$$\inf_{|x|\leq R}g(x)>0,\quad\forall R>0,$$

and that there exists a locally bounded function $\varphi:\mathbb{R}^d\to[1,+\infty)$ such that $x\mapsto\mathbb{E}(\varphi(x+\xi_1))$ is locally bounded on $\mathbb{R}^d$ and such that

$$\limsup_{|x|\to+\infty,\,x\in D}\frac{\mathbb{E}(\varphi(f(x)+\xi_1))}{\varphi(x)}=0.\tag{7.4}$$

Then Condition (E) is satisfied with $\varphi_1=\varphi$ and φ_2 positive on D.

Note that, if D is bounded, the last result is already a consequence of the classical criterion based on (7.1). Before proving this result, let us give three applications.

Example 7.4. If there exists $\alpha>0$ such that $\mathbb{E}e^{\alpha|\xi_1|}<+\infty$ and if $|x|-|f(x)|\to+\infty$ when $|x|\to+\infty$, then Proposition 7.3 applies. Indeed, choosing $\varphi(x)=\exp(\alpha|x|)$, we have

$$\frac{\mathbb{E}\varphi(|f(x)+\xi_1|)}{\varphi(x)}\leq e^{\alpha(|f(x)|-|x|)}\mathbb{E}e^{\alpha|\xi_1|}\xrightarrow[|x|\to+\infty]{}0.$$

For instance, this covers the case of Gaussian perturbations, as stated in Theorem 1.2 in the introduction.

Example 7.5. If there exists $p > 0$ such that $\mathbb{E}(\xi_1^p) < +\infty$ and if $|f(x)| = o(|x|)$ when $|x| \to +\infty$, then Proposition 7.3 applies. Indeed, choosing $\varphi(x) = (1+|x|)^p$, we have

$$\frac{\mathbb{E}\varphi(|f(x)+\xi_1|)}{\varphi(x)} \leq \frac{(1+|f(x)|)^p}{(1+|x|)^p}\mathbb{E}[(1+|\xi_1|)^p] \xrightarrow[|x|\to+\infty]{} 0.$$

Example 7.6. If $\mathbb{E}\log(1+|\xi_1|) < \infty$ and $|f(x)| \leq C|x|^{\varepsilon(x)}$ for some $C > 0$ and some $\varepsilon(x) \to 0$ when $|x| \to +\infty$, then Proposition 7.3 applies. Indeed, choosing $\varphi(x) = \log(e+|x|)$, we have

$$\frac{\mathbb{E}\varphi(|f(x)+\xi_1|)}{\varphi(x)} \leq \frac{\log(e+C)+\varepsilon(x)\log(e+|x|)}{\log(e+|x|)} + \frac{\mathbb{E}\log(1+|\xi_1|)}{\log(e+|x|)}.$$

Proof of Proposition 7.3. We first prove Conditions (E2) and (E4) and conclude the proof with Proposition 3.3.

Step 1. Conditions (E2) and (E4) are satisfied.

Let $K_1 \subset D$ be a bounded measurable set with positive Lebesgue measure. Then, for all $x \in K_1$, denoting by λ_d the Lebesgue measure on $\mathbb{R}^d$,

$$\mathbb{P}_x(X_1 \in K_1) = \mathbb{P}(f(x)+\xi_1 \in K_1) \geq \lambda_d(K_1) \inf_{u \in K_1 + B(0,\sup_{K_1}|f|)} g(u) > 0.$$

Fix $\theta_2 \in (0, \lambda_d(K_1)\inf_{u\in K_1+B(0,\sup_{K_1}|f|)} g(u))$, we deduce that, for all $x \in K_1$,

$$\theta_2^{-n} \inf_{x\in K_1} \mathbb{P}_x(X_n \in K_1) \geq \theta_2^{-n} \inf_{x \in K_1} \mathbb{P}_x(X_1 \in K_1, \dots, X_n \in K_1) \xrightarrow[n\to+\infty]{} +\infty.$$

Fix $0 < \theta_1 < \theta_2$, and, using (7.4), consider a bounded subset $K \subset D$ containing K_1 and such that, for all $x \in D \setminus K$, $P_1\varphi(x) \leq \theta_1\varphi(x)$. Since K is bounded, one has

$$\inf_{x\in K} \mathbb{P}_x(X_1 \in K_1) \geq \lambda_d(K_1) \inf_{u\in K_1+B(0,\sup_K |f|)} g(u) > 0,$$

so that

$$\theta_2^{-n} \inf_{x\in K} \mathbb{P}_x(X_n \in K) \geq \theta_2^{-n}\lambda_d(K_1) \inf_{u \in K_1+B(0,\sup_K|f|)} g(u) \inf_{x\in K_1} \mathbb{P}_x(X_{n-1} \in K_1)$$

and thus $\theta_2^{-n}\inf_{x\in K}\mathbb{P}_x(X_n \in K)$ converges to $+\infty$ when $n \to +\infty$. Lemma 3.1 then entail that Condition (E4) is satisfied and that there exists a function $\varphi_2 : D \to [0,1]$ such that $P_1\varphi_2(x) \geq \theta_2\varphi_2(x)$ for all $x \in D$ and such that $\inf_K \varphi_2 > 0$. In addition, for all $x \in D$, $\mathbb{P}_x(X_1 \in K) \geq \lambda_d(K)\inf_{u\in K-f(x)} g(u) > 0$, so that $P_1\mathbb{1}_K(x) > 0$. Hence, the function φ_2 of Lemma 3.1 also satisfies that $\varphi_2(x) > 0$ for all $x \in E$.

Setting $\varphi_1 = \varphi$, we deduce that Conditions (E2) and (E4) are satisfied for the set K.

Step 2. Comparison of transition probabilities.

Let us prove that Proposition 3.3 applies with $n_0 = m_0 = 1$. For all $x \in D$, we have

$$\mathbb{P}_x(X_1 \in \cdot \cap K) \leq \sup_{u\in\mathbb{R}^d} g(u)\,\lambda_d(\cdot \cap K).$$

Moreover, for all $y \in K$,

$$\begin{aligned}\mathbb{P}_y(X_1 \in \cdot) &\geq \mathbb{P}(f(y)+\xi_1 \in \cdot \cap K)\\ &\geq \inf_{u\in K+B(0,\sup_K|f|)} g(u)\,\lambda_d(\cdot\cap K).\end{aligned}$$

Hence, for all $x \in E$ and all $y \in K$,

$$\mathbb{P}_x(X_1 \in \cdot \cap K) \leq \frac{\sup_{\mathbb{R}^d} g}{\inf_{K+B(0,\sup_K|f|)} g}\,\mathbb{P}_y(X_1 \in \cdot).$$

We deduce from Step 1 and Proposition 3.3 that Condition (E) is satisfied with the functions φ_1 and φ_2, which concludes the proof. □

7.2.2 An example with unbounded perturbation and singular density

The last result made strong use of the boundedness of g. Actually, our criteria also apply to perturbations with singular density. We consider here the following example: assume that $f(x) = Ax + B$, where A is an invertible $d \times d$ matrix and $B \in \mathbb{R}^d$, and that there exists $a > 0$ such that the density g of ξ_n satisfies for some constant C_g

$$g(x) \leq C_g \left(\frac{1}{|x|^{d-a}} \vee 1 \right) \quad \forall x \in \mathbb{R}^d. \tag{7.5}$$

We have the following result.

Proposition 7.7. *Let $\| \cdot \|$ be a norm on $\mathbb{R}^d$ and assume that*

$$\sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{\|Ax\|}{\|x\|} < 1. \tag{7.6}$$

Assume also that $\mathbb{E} e^{\alpha |\xi_1|} < \infty$ for some $\alpha > 0$ and that

$$\inf_{|x| \leq R} g(x) > 0, \quad \forall R > 0.$$

Then Condition (E) is satisfied with $\varphi_1 = \varphi$ and φ_2 positive on D.

The proof of Proposition 7.3 made use of Proposition 3.3 with $n_0 = m_0 = 1$. The proof of Proposition 7.7 requires to apply Proposition 3.3 with $n_0 \geq 2$.

Proof. The first step of the proof of Proposition 7.3 remains valid taking $\varphi(x) = e^{\alpha \|x\|}$ for $\alpha > 0$ small enough and using (7.6) and the equivalence of the norms $| \cdot |$ and $\| \cdot \|$ (the computation is similar to the one of Example 7.4). So we only have to prove that (3.2) is satisfy and apply Proposition 3.3.

We define $n_0 = \lceil d/a \rceil$ and we assume without loss of generality (reducing slightly a if needed) that $n_0 a > d$. We observe that

$$X_{n_0} = A^{n_0} x + A^{n_0 - 1}(B + \xi_1) + \cdots + B + \xi_{n_0}.$$

Using (7.5) and the fact that $\sup_{x \neq 0} \frac{|Ax|}{|x|} \leq C^2_{\|\cdot\|}$ where the constant $C_{\|\cdot\|}$ is such that $C^{-1}_{\|\cdot\|} | \cdot | \leq \| \cdot \| \leq C_{\|\cdot\|} | \cdot |$, the density g_2 of $A\xi_1 + \xi_2$ satisfies

$$\begin{aligned} g_2(x) &= \frac{1}{|\mathrm{det} A|} \int_{\mathbb{R}^d} g(x-y) g(A^{-1}y) \mathrm{d}y \\ &\leq \frac{C_g^2}{|\mathrm{det} A|} \int_{\{y : |A^{-1}y| \leq 1\} \cap B(x,1)} \frac{1}{|x-y|^{d-a}} \frac{1}{|A^{-1}y|^{d-a}} \mathrm{d}y + C_g \left(1 + \frac{1}{|\mathrm{det} A|} \right) \\ &\leq \frac{C_g^2 C_{\|\cdot\|}^{2(d-a)}}{|\mathrm{det} A|} \int_{B(0, C^2_{\|\cdot\|})} \frac{1}{|x-y|^{d-a}} \frac{1}{|y|^{d-a}} \mathrm{d}y + C_g \left(1 + \frac{1}{|\mathrm{det} A|} \right) \\ &= \frac{C_g^2 C_{\|\cdot\|}^{2(d-a)}}{|\mathrm{det} A|} \frac{1}{|x|^{d-2a}} \int_{B(0, C^2_{\|\cdot\|}/|x|)} \frac{1}{\left| \frac{x}{|x|} - u \right|^{d-a}} \frac{1}{|u|^{d-a}} \mathrm{d}u + C_g \left(1 + \frac{1}{|\mathrm{det} A|} \right), \end{aligned} \tag{7.7}$$

where we made the change of variable $u = y/|x|$.

If $2a > d$ (i.e. if $n_0 = 2$), we can bound the integral in the right-hand side as follows:

$$\begin{aligned} \int_{B\left(0, \frac{C^2_{\|\cdot\|}}{|x|}\right)} \frac{1}{\left| \frac{x}{|x|} - u \right|^{d-a}} \frac{1}{|u|^{d-a}} \mathrm{d}u &\leq C + 2^d \int_{B\left(0, \frac{C^2_{\|\cdot\|}}{|x|}\right) \setminus B(0,2)} \frac{1}{|u|^{2d-2a}} \mathrm{d}u \\ &\leq C + \frac{C}{2a-d} \frac{1}{|x|^{2a-d}}, \end{aligned}$$

where the constant C may change from line to line. Therefore, g_2 is bounded if $2a > d$.

Otherwise, if $2a < d$, the integral in the right-hand side of (7.7) can be bounded by the same integral over $\mathbb{R}^d$ and thus it is uniformly bounded with respect to x, so g_2 is bounded by $C(1 \vee 1/|x|^{d-2a})$. In this case, we can proceed similarly to bound the density g_3 of $A^2\xi_1 + A\xi_2 + \xi_3$, and prove by induction that the density g_{n_0} of $A^{n_0-1}\xi_1 + \cdots + \xi_{n_0}$ is bounded.

We deduce that

$$\mathbb{P}_x(X_{n_0} \in \cdot \cap K) \leq \sup_{u \in \mathbb{R}^d} g_{n_0}(u)\, \lambda_d(\cdot \cap K).$$

The end of the proof is the same as for Proposition 7.3, using Proposition 3.3 with $m_0 = n_0$. ☐

7.2.3 Two examples with bounded perturbation

The case where ξ_1 is a bounded random variable is more involved. To avoid complications, we will focus on the case where ξ_n is a uniform random variable on the unit ball $B(0,1)$ of $\mathbb{R}^d$. Extensions to different distributions are possible.

We start with the simpler case of bounded domain D and contracting dynamical system f.

Proposition 7.8. *Assume that D is a bounded, connected open set of $\mathbb{R}^d$, that f is continuous and satisfies $|f(x) - x| < 1$ for all $x \in D$. Then Condition (E) is satisfied.*

Proof. Again, the proof makes use of the criterion of Proposition 3.3.

Step 1. Construction and properties of the sets K_ε, $\varepsilon > 0$.

For all $\varepsilon > 0$, let K'_ε be the connected component of $\{x \in D : d(x, \partial D) \geq 2\varepsilon\}$ with larger Lebesgue measure and let

$$K_\varepsilon := \bigcup_{x \in K'_\varepsilon} \overline{B(x, \varepsilon)},$$

which is a also a connected compact subset of D with distance to D^c larger than ε. For all $\delta > 0$ and all $x, y \in K_\varepsilon$, we call a sequence $(x_0, x_1, \ldots, x_n) \in K_\varepsilon^{n+1}$ for some $n \in \mathbb{N}$ a δ-path linking x to y in K_ε if $x_0 = x$, $x_n = y$ and $|x_k - x_{k-1}| < \delta$ for all $1 \leq k \leq n$. By construction, the set K_ε satisfies that, for all $\delta > 0$ and all $x, y \in K_\varepsilon$, there exists a δ-path linking x to y in K_ε. In addition, since K_ε is compact, there exists an integer $n_{\varepsilon,\delta}$ depending only on ε and δ such that, for all $x, y \in K_\varepsilon$, there exists a δ-path in K_ε linking x to y with length less than $n_{\varepsilon,\delta}$. For all $x \in K_\varepsilon$ and all $k \in \{1, \ldots, n_{\varepsilon,\delta}\}$ let us define

$$K^{(k)}_{\varepsilon,\delta}(x) = \Big\{y \in \mathbb{R}^d : \exists x_1, \ldots, x_{k-1} \in K_\varepsilon,\ |x_\ell - x_{\ell-1}| < \delta \text{ for all } 1 \leq \ell \leq k \\ \text{with } x_0 = x \text{ and } x_k = y\Big\}.$$

Note that in general, $K^{(k)}_{\varepsilon,\delta}$ is not included in K_ε, but it is included in D if $\delta < \varepsilon$. It follows from above that $K^{(n_{\varepsilon,\delta})}_{\varepsilon,\delta}(x) \supset K_\varepsilon$ for all $x \in K_\varepsilon$.

Let us also prove that $\cup_{\varepsilon>0} K_\varepsilon = D$. Let $(x_n)_{n\geq 1}$ be a dense sequence in D and for all $n \geq 1$, let $r_n = d(x_n, \partial D)/2$. Since $D = \cup_{n\geq 1} B(x_n, r_n)$, there exists $n_0 \geq 1$ such that $\cup_{1\leq n\leq n_0} B(x_n, r_n)$ has Lebesgue measure larger than $\lambda_d(D)/2$. Since D is connected, there exists a continuous path in D linking x_i to x_j for all $1 \leq i, j \leq n_0$. Since the distance between this path and ∂D is positive, there exists $\varepsilon > 0$ small enough such that all the points $x_1, \ldots, x_{n_0}$ belong to the same connected component of $\{x \in D : d(x, \partial D) \geq 2\varepsilon\}$. We can assume without loss of generality that $\varepsilon < r_n/2$ for all $1 \leq n \leq n_0$, so that this connected component actually contains $\cup_{1\leq n\leq n_0} B(x_n, r_n)$ and hence has the largest Lebesgue measure among all the connected components of $\{x \in D : d(x, \partial D) \geq 2\varepsilon\}$. In particular, K_ε contains $B(x_1, r_1)$ for all ε small enough. Now, given any $x \in D$, there exists a path linking x to x_1 in D. Since the distance between this path and ∂D is positive, x belongs to K_ε for all $\varepsilon > 0$ small enough. Hence, we have proved that $\cup_{\varepsilon>0} K_\varepsilon = D$ and that the family $(K_\varepsilon)_{\varepsilon>0}$ is non-increasing with respect to $\varepsilon > 0$ when ε is small enough.

Step 2. Proof of Condition (3.2) *of Proposition 3.3.*

For all $\varepsilon > 0$, since f is continuous,

$$\delta_\varepsilon := \left(1 - \sup_{x\in K_\varepsilon} |f(x) - x|\right) \wedge \varepsilon > 0.$$

Hence, for all $x \in K_\varepsilon$,

$$\mathbb{P}_x(X_1 \in \cdot \cap B(x, \delta_\varepsilon)) \geq c_d \lambda_d(\cdot \cap B(x, \delta_\varepsilon)), \tag{7.8}$$

for a positive constant c_d only depending on the dimension of the space. In other words, for all $x \in K_\varepsilon$,

$$\mathbb{P}_x(X_1 \in \cdot) \geq c_d \mathbb{P}(x + U \in \cdot)$$

where U is a uniform random variable on $B(0, \delta_\varepsilon)$. Hence, defining the Markov chain $Y_n = Y_0 + U_1 + \ldots + U_n$ where U_i are i.i.d. uniform random variable on $B(0, \delta_\varepsilon)$, we deduce that

$$\mathbb{P}_x(X_k \in \cdot) \geq c_d^k \mathbb{P}_x(Y_1, \ldots, Y_{k-1} \in K_\varepsilon \text{ and } Y_k \in \cdot), \quad \forall x \in K_\varepsilon,\ \forall k \in \mathbb{N}. \tag{7.9}$$

In view of Step 1, the following Lemma 7.9 about the process Y implies that there exists a constant $c' > 0$ such that

$$\mathbb{P}_x(X_{n_{\varepsilon,\delta_\varepsilon/3}} \in \cdot) \geq c' \lambda_d(\cdot \cap K_\varepsilon), \quad \forall x \in K_\varepsilon. \tag{7.10}$$

Since the law of X_1 is dominated by the Lebesgue measure independently of X_0, we have proved that, for all $\varepsilon > 0$, (3.2) is satisfied for $K = K_\varepsilon$, $n_0 = 1$ and $m_0 = n_{\varepsilon,\delta_\varepsilon/3}$. This concludes Step 2 of the proof.

Lemma 7.9. *For all* $1 \leq k \leq n_{\varepsilon,\delta_\varepsilon/3}$*, there exists a constant* $c'_k > 0$ *such that, for all* $x \in K_\varepsilon$*,*

$$\mathbb{P}_x(Y_1, \ldots, Y_{k-1} \in K_\varepsilon \textit{ and } Y_k \in \cdot) \geq c'_k \lambda_d\left(\cdot \cap K^{(k)}_{\varepsilon,\delta_\varepsilon/3}(x)\right), \tag{7.11}$$

where λ_d *is Lebesgue's measure on* $\mathbb{R}^d$*.*

Step 3. Proof of (E2) and (E4).

Fix $\varepsilon_0 > 0$ such that K_{ε_0} is non-empty and $(K_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ is non-increasing. It follows from the definition of K_ε that $\inf_{x\in K_{\varepsilon_0}} \lambda_d(K_{\varepsilon_0} \cap B(x, \delta_{\varepsilon_0})) > 0$. Fixing

$$\theta_2 < 4 \wedge \left\{c_d \inf_{x\in K_{\varepsilon_0}} \lambda_d(K_{\varepsilon_0} \cap B(x, \delta_{\varepsilon_0}))\right\},$$

we deduce from (7.8) that

$$\lim_{n\to+\infty} \theta_2^{-n} \inf_{x\in K_{\varepsilon_0}} \mathbb{P}_x(X_n \in K_{\varepsilon_0}) = +\infty. \tag{7.12}$$

Since the law of X_1 is dominated by the Lebesgue measure and $D = \cup_{0<\varepsilon\leq\varepsilon_0} K_\varepsilon$, there exists $\varepsilon_1 \in (0, \varepsilon_0]$ small enough such that

$$\sup_{x\in D} \mathbb{P}_x(X_1 \in D \setminus K_{\varepsilon_1}) \leq \theta_2/4.$$

Hence, the function

$$\varphi_1 : x \in D \mapsto \begin{cases} 1 & \text{if } x \in K_\varepsilon, \\ 4/\theta_2 & \text{if } x \in D \setminus K_{\varepsilon_1}, \end{cases}$$

satisfies $P_1\varphi_1(x) \leq 2 \leq (\theta_2/2)\varphi_1(x)$ for all $x \in D \setminus K_{\varepsilon_1}$. Hence the first and third lines of Condition (E2) are satisfied with $\theta_1 = \theta_2/2$ and $K = K_{\varepsilon_1}$.

We also deduce from (7.10), (7.12), the fact that $K_{\varepsilon_0} \subset K_{\varepsilon_1}$ and Markov's property that

$$\lim_{n\to+\infty} \theta_2^{-n} \inf_{x\in K_{\varepsilon_1}} \mathbb{P}_x(X_n \in K_{\varepsilon_1}) = +\infty.$$

Hence, it follows from Lemma 3.1 that (E4) is satisfied with $K = K_{\varepsilon_1}$ and that there exists a function φ_2 satisfying the conditions of (E2) with θ_2 defined above and $K = K_{\varepsilon_1}$.

Therefore, the result follows from Step 2 and Proposition 3.3 with $K = K_{\varepsilon_1}$, $n_0 = 1$ and $m_0 = n_{\varepsilon_1,\delta_{\varepsilon_1}/3}$. □

Proof of Lemma 7.9. We prove this result by induction over k. Since $Y_1 = x + U_1$ is uniform in $B(x, \delta_\varepsilon)$, the case $k = 1$ is clear since $K^{(1)}_{\varepsilon,\delta_\varepsilon/3} = B(x, \delta_\varepsilon/3) \subset B(x, \delta_\varepsilon)$.

So assume that (7.11) is satisfied for some $1 \leq k \leq n_{\varepsilon,\delta_\varepsilon/3} - 1$ and let us prove it for $k+1$. Let $A \subset \mathbb{R}^d$ be measurable. Using (7.11) for k and the fact that Y_{k+1} is uniform in $B(Y_k, \delta_\varepsilon)$ conditionally on Y_k, we have

$$\begin{aligned}
&\mathbb{P}_x(Y_1, \dots, Y_k \in K_\varepsilon,\ Y_{k+1} \in A) \\
&\qquad \geq \mathbb{P}_x\left(Y_1, \dots, Y_{k-1} \in K_\varepsilon,\ Y_k \in K^{(k)}_{\varepsilon,\delta_\varepsilon/3}(x) \cap K_\varepsilon,\ Y_{k+1} \in A \cap B(Y_k, \delta_\varepsilon)\right) \\
&\qquad \geq \frac{c'_k}{\lambda_d(B(0,\delta_\varepsilon))} \int_{K^{(k)}_{\varepsilon,\delta_\varepsilon/3}(x)\cap K_\varepsilon} \mathrm{d}y \int_{A\cap B(y,\delta_\varepsilon)} \mathrm{d}z \\
&\qquad = \frac{c'_k}{\lambda_d(B(0,\delta_\varepsilon))} \int_A \lambda_d\left\{K^{(k)}_{\varepsilon,\delta_\varepsilon/3}(x) \cap K_\varepsilon \cap B(z, \delta_\varepsilon)\right\} \mathrm{d}z \\
&\qquad \geq \frac{c'_k}{\lambda_d(B(0,\delta_\varepsilon))} \int_{A\cap K^{(k+1)}_{\varepsilon,\delta_\varepsilon/3}(x)} \lambda_d\left\{K^{(k)}_{\varepsilon,\delta_\varepsilon/3}(x) \cap K_\varepsilon \cap B(z, \delta_\varepsilon)\right\} \mathrm{d}z,
\end{aligned}$$

where the third equality follows from Fubini's theorem.

Now, for all $z \in K^{(k+1)}_{\varepsilon,\delta_\varepsilon/3}(x)$, there exists a path $x_0 = x, x_1, \dots, x_k \in K_\varepsilon$ such that $|x_\ell - x_{\ell-1}| < \delta_\varepsilon/3$ for all $1 \leq \ell \leq k$ and $|x_k - z| < \delta_\varepsilon/3$. By definition of K_ε, there exists $y \in K_\varepsilon$ such that $x_{k-1} \in B(y, \varepsilon) \subset K_\varepsilon$. Let y' be the unique point such that $|y' - x_{k-1}| = \delta_\varepsilon/6$ of the half-line with initial point x_{k-1} and containing y. Then $B(y', \delta_\varepsilon/6) \subset K_\varepsilon$. Since $|x_k - z| < \delta_\varepsilon/3$ and $|x_{k-1} - x_k| < \delta_\varepsilon/3$, we also have $B(y', \delta_\varepsilon/6) \subset B(z, \delta_\varepsilon)$. In addition, for all $y'' \in B(y', \delta_\varepsilon/6)$, the path $x_0 = x, x_1, \dots, x_{k-1}, y''$ lies in K_ε and has distance between consecutive point smaller than $\delta_\varepsilon/3$. Therefore, $B(y', \delta_\varepsilon/6) \subset K^{(k)}_{\varepsilon,\delta_\varepsilon/3}(x)$. We conclude that, for all $z \in K^{(k+1)}_{\varepsilon,\delta_\varepsilon/3}(x)$,

$$\lambda_d\left\{K^{(k)}_{\varepsilon,\delta_\varepsilon/3}(x) \cap K_\varepsilon \cap B(z, \delta_\varepsilon)\right\} \geq \lambda_d(B(0, \delta_\varepsilon/6)).$$

Hence

$$\mathbb{P}_x(Y_1, \dots, Y_k \in K_\varepsilon,\ Y_{k+1} \in A) \geq c'_{k+1}\lambda_d\left(A \cap K^{(k+1)}_{\varepsilon,\delta_\varepsilon/3}(x)\right)$$

for a positive constant c'_{k+1}. □

The general case of dynamical systems with bounded perturbations raises several additional difficulties. We illustrate two of them with the next example in dimension 1. We consider the Markov process in $D = (0, +\infty)$ defined as

$$X_0 \in (0, +\infty), \quad X_{n+1} = \alpha X_n - \frac{1}{1 + X_n} + \xi_n, \quad \forall n \geq 0$$

where $\alpha \in (0,1)$ and ξ_n are i.i.d. with uniform distribution on $[-1,1]$ and the process is immediately sent to the cemetery point ∂ when it leaves D. The first difficulty comes from the fact that

$$\mathbb{P}_x(X_1 > 0) = \left[1 - \left(\frac{1}{1+x} - \alpha x\right)\right] \vee 0 \xrightarrow[x\to 0+]{} 0,$$

which means that the probability of immediate absorption converges to 1 when x approaches the boundary of D. The second difficulty comes from the fact that $|f(x)-x|$ is unbounded on D (in contrast with Proposition 7.8). This example is covered by the following general result.

Proposition 7.10. *Assume that $X_{n+1} = f(X_n) + \xi_n$ with $D = (0,+\infty)$, ξ_n i.i.d. uniform on $[-1,1]$, f continuous and there exists $x^* \in D$ such that*

$$(0,x^*) = \{x \in D : |f(x)-x| < 1\} \quad \textit{and} \quad [x^*,+\infty) = \{x \in D : f(x)+1 \le x\}.$$

Then Condition (E) is satisfied.

Proof. Fix $K_0 \subset (0,x^*)$ a closed interval with non-empty interior. As in the proof of Proposition 7.8, using in particular (7.9) and (7.11), there exists $n_0 \ge 1$ and $c_0 > 0$ such that, for all $x \in K_0$,

$$\mathbb{P}_x(X_{n_0} \in \cdot) \ge c_0 \lambda_1(\cdot \cap K_0).$$

Hence there exists a constant $\theta_2 \in (0,1)$ such that

$$\theta_2^{-n} \inf_{x\in K_0} \mathbb{P}_x(X_n \in K_0) \xrightarrow[n\to+\infty]{} +\infty. \tag{7.13}$$

Fix now $\theta_1 < \theta_2$ and $K \subset (0,x^*)$ a closed interval such that $K_0 \subset K$ and

$$\lambda_1\{(0,x^*) \setminus K\} \le \frac{\theta_1}{M},$$

where

$$M := \frac{2(1+e^{(x^*+2)/\theta_1})}{\theta_1}.$$

As above, there exists $n_1 \ge 1$ and $c_1 > 0$ such that, for all $x \in K$,

$$\mathbb{P}_x(X_{n_1} \in \cdot) \ge c_1 \lambda_1(\cdot \cap K).$$

In particular, $\inf_{x\in K} \mathbb{P}_x(X_{n_1} \in K_0) > 0$, so that, using Markov property and (7.13), we deduce that

$$\theta_2^{-n} \inf_{x\in K} \mathbb{P}_x(X_n \in K) \xrightarrow[n\to+\infty]{} +\infty.$$

Using Lemma 3.1, we deduce that there exists a function φ_2 satisfying the conditions of (E2) and that (E4) is satisfied. For all $x \in D$, let

$$\varphi_1(x) = \begin{cases} 1 & \text{if } x \in K, \\ M & \text{if } x \in (0,x^*) \setminus K, \\ e^{x/\theta_1} & \text{if } x \ge x^*. \end{cases}$$

For $x \ge x^*$, using the fact that the density of X_1 on D with respect to Lebesgue measure is bounded by $\frac{1}{2}\mathbb{1}_D$ for all value of X_0, we have

$$\begin{aligned} P_1\varphi_1(x) &\le \mathbb{E}_x(e^{X_1/\theta_1}\mathbb{1}_{X_1 \ge x^*}) + \mathbb{P}_x(X_1 \in K) + M\mathbb{P}_x(X_1 \in (0,x^*) \setminus K) \\ &\le \mathbb{E}_x(e^{X_1/\theta_1}) + \frac{M}{2}\lambda_1\{(0,x^*) \setminus K\} \\ &\le \varphi_1(x) e^{(f(x)-x)/\theta_1} \mathbb{E}_x e^{\xi_1/\theta_1} + \frac{\theta_1}{2} \\ &\le \varphi_1(x) e^{-\theta_1^{-1}} \frac{e^{\theta_1^{-1}} - e^{-\theta_1^{-1}}}{2\theta_1^{-1}} + \frac{\theta_1}{2}\varphi_1(x) \le \theta_1 \varphi_1(x). \end{aligned}$$

For $x \in (0, x^*) \setminus K$, since $f(x) + \xi_1 \leq x + 2 \leq x^* + 2$,

$$\begin{aligned} P_1\varphi_1(x) &\leq \mathbb{P}_x(X_1 \in K) + e^{(x^*+2)/\theta_1}\mathbb{P}_x(X_1 \geq x^*) + M\mathbb{P}_x(X_1 \in (0, x^*) \setminus K) \\ &\leq 1 + e^{(x^*+2)/\theta_1} + \frac{M}{2}\lambda_1\{(0, x^*) \setminus K\} \\ &\leq M\left(\frac{\theta_1}{2} + \frac{\theta_1}{2M}\right) \leq \theta_1\varphi_1(x). \end{aligned}$$

Since $P_1\varphi_1(x)$ is clearly bounded for $x \leq x^*$, we have proved (E2).

To conclude, it remains to observe that (3.2) can be deduced for $n_0 = 1$ and m_0 large enough exactly as in the proof of Proposition 7.8. Hence the result follows from Proposition 3.3. □

8 Irreducible processes in discrete state space and discrete time

The theory of R-positive matrices is a powerful tool to study absorbed Markov processes in discrete time and space [45]. The goal of Section 8.1 is to show that our criteria allow to recover the results on convergence to quasi-stationarity of this theory. We then study in Section 8.2 a class of discrete Markov chains in discrete time to which criteria based on R-positive matrices do not apply easily.

8.1 R-positive matrices

We consider a Markov chain $(X_n, n \in \mathbb{Z}_+)$ in a countable state space $E \cup \{\partial\}$ with $\partial \notin E$ an absorbing point and with irreducible transition probabilities in E, i.e. such that for all $x, y \in E$, there exists $n = n(x, y) \geq 1$ such that $\mathbb{P}_x(X_n = y) > 0$. In this case, the most general criterion for existence and convergence to a quasi-stationary distribution is provided in [45]. In this paper, the authors obtain a convergence result similar to the one of Theorem 2.1 restricted to Dirac initial distributions, and the pointwise convergence to η as in Theorem 2.3, using the theory of R-positive matrices. In this section, we show how our criterion allows to recover these results, providing in addition the several refinements of Section 2 (including the characterization of a non-trivial subset of the domain of attraction, the convergence of (2.1) for unbounded functions f and a stronger convergence to η).

We denote by P the transition matrix of the chain $(X_n, n \in \mathbb{Z}_+)$ and we assume that the absorption time τ_∂ is almost surely finite. Without loss of generality, we will assume that the process is aperiodic, meaning that $\mathbb{P}_x(X_n = y) > 0$ for all $x, y \in E$ provided n is large enough; the extension to general periodic processes is routine, as observed in [45] (see also [27] on this topic in our general setting).

Proposition 8.1. *The assumptions of [45, Theorem 1] imply Assumption (E).*

Proof. Since E is finite or countable and because of the irreducibility assumption, it is known [107] that the limit

$$\frac{1}{R} := \lim_{n\to+\infty} \mathbb{P}_x(X_n = y)^{1/n} \tag{8.1}$$

exists with $1 \leq R < \infty$, and is independent of $x, y \in E$. Using [45, Lemma 1], the assumptions of [45, Theorem 1] can be stated as follows: there exist a non-empty set $K \subset E$ and $x_0 \in K$ such that

(a) there exist $\varepsilon_0 > 0$ and a constant C_1 such that, for all $x \in K$ and all $n \geq 0$,

$$\mathbb{P}_x(n < \sigma_K \wedge \tau_\partial) \leq C_1(R + \varepsilon_0)^{-n},$$

where σ_K is the first return time in K

$$\sigma_K := \inf\{n \geq 1, X_n \in K\}.$$

(b) there exists a constant C_2 such that, for all $x \in K$ and $n \geq 0$,

$$\mathbb{P}_x(n < \tau_\partial) \leq C_2 \mathbb{P}_{x_0}(n < \tau_\partial);$$

(c) there exist $n_0 \geq 0$ and a constant $C_3 > 0$ such that, for all $x \in K$,

$$\mathbb{P}_x(T_{\{x_0\}} \leq n_0) \geq C_3,$$

where we recall that $T_L := \inf\{n \in \mathbb{Z}_+ : X_n \in L\}$ for all $L \subset E$.

Let us first prove (E1). By aperiodicity and irreducibility, there exists $m_1 \geq 1$ such that, for all $n \geq m_1$, $\mathbb{P}_{x_0}(X_n = x_0) > 0$. Combining this with (c), the Markov Property entails that, for all $x \in K$,

$$\mathbb{P}_x(X_{n_0+m_1} = x_0) \geq C_3 \min_{m_1 \leq k \leq n_0+m_1} \mathbb{P}_{x_0}(X_k = x_0).$$

This is (E1) with $\nu = \delta_{x_0}$ and $n_1 = n_0 + m_1$.

We now prove (E2) and (E4). Condition (a) implies that

$$\left(R + \frac{\varepsilon_0}{2}\right) \sup_{y \in K} \mathbb{E}_y \left[\mathbb{1}_{1<\tau_\partial} \mathbb{E}_{X_1}\left(\left(R + \frac{\varepsilon_0}{2}\right)^{T_K \wedge \tau_\partial}\right)\right] = \sup_{y \in K} \mathbb{E}_y \left[\left(R + \frac{\varepsilon_0}{2}\right)^{\sigma_K \wedge \tau_\partial}\right] < \infty.$$

For all $x \in E \setminus K$, the irreducibility assumption implies that there exist $y \in K$ and $n = n(x,y) \geq 1$ such that $\mathbb{P}_y(X_n = x \text{ and } n < \sigma_K) > 0$. By Markov's property,

$$\mathbb{E}_y \left[\left(R + \frac{\varepsilon_0}{2}\right)^{\sigma_K \wedge \tau_\partial}\right] \geq \mathbb{P}_y(X_n = x \text{ and } n < \sigma_K) \mathbb{E}_x \left[\left(R + \frac{\varepsilon_0}{2}\right)^{\sigma_K \wedge \tau_\partial}\right].$$

Since $\sigma_K = T_K$ almost surely under $\mathbb{P}_x$ for $x \in E \setminus K$, Lemma 3.2 provides a function φ_1 satisfying the conditions of (E2), with $\theta_1 := (R + \frac{\varepsilon_0}{2})^{-1}$. According to [45, (1.16)], which holds true under their assumption by [45, Theorem 1], and setting $\theta_2 = \left(R + \frac{\varepsilon_0}{3}\right)^{-1}$, one has

$$\lim_{n \to +\infty} \theta_2^{-n} \mathbb{P}_{x_0}(X_n = x_0) = +\infty.$$

Using Markov's property, Condition (c) immediately entails that

$$\lim_{n \to +\infty} \theta_2^{-n} \inf_{x \in K} \mathbb{P}_x(X_n \in K) = +\infty.$$

Using Lemma 3.1, we deduce that there exists a function $\varphi_2 : E \to [0,1]$ satisfying the conditions of (E2) and that (E4) holds true. This concludes the proof of (E2) and (E4).

To conclude, Conditions (b) and (E1) imply, for all $n \geq 0$,

$$\inf_{y \in K} \mathbb{P}_y(n < \tau_\partial) \geq \inf_{y \in K} \mathbb{P}_y(n + n_1 < \tau_\partial) \geq c_1 \mathbb{P}_{x_0}(n < \tau_\partial) \geq \frac{c_1}{C_2} \sup_{y \in K} \mathbb{P}_y(n < \tau_\partial).$$

This proves (E3) and concludes the proof of Proposition 8.1. ☐

Remark 8.2. One can actually prove that, in the particular case of a discrete state space E and aperiodic and irreducible transition probability on E, Assumption (E) is equivalent to the Conditions (a), (b) and (c) of [45]. Besides the additional properties provided in Section 2, one of our main contribution in this particular setting is to provide a more tractable criterion. Indeed, the use of Lyapunov type functions has the advantage to be quite flexible.

8.2 Application to the extinction of biological populations dominated by Galton-Watson processes

In this section, we show how our criteria can be applied to general population processes dominated by population-dependent Galton-Watson processes. In particular, we refine existing results for the classical multi-type Galton-Watson process.

More precisely, we consider an aperiodic and irreducible Markov population process $(Z_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}_+^d = E\cup\{\partial\}$ absorbed at $\partial=0$ such that, for all $n\geq 0$,

$$\|Z_{n+1}\| \leq \sum_{i=1}^{|Z_n|} \xi_{i,n}^{(Z_n)}, \tag{8.2}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^d$ and $|z|=z_1+\ldots+z_d$ for all $z\in\mathbb{Z}_+^d$ and, for all $n\geq 0$, the nonnegative random variables $\xi_{1,n}^{(Z_n)},\ldots,\xi_{|Z_n|,n}^{(Z_n)}$ are assumed independent (but not necessarily identically distributed) given Z_n and the families $(\xi_{i,n}^{(z)}, z\in\mathbb{Z}_+^d, 1\leq i\leq |z|)$ are i.i.d. for $n\in\mathbb{Z}_+$.

We assume that

$$\mathbb{E}\left(\sum_{i=1}^{|z|}\xi_{i,n}^{(z)}\right)\leq m\|z\|,\quad \forall z\in\mathbb{Z}_+^d \text{ such that } |z|\geq n_0, \tag{8.3}$$

for some $m<1$ and $n_0\in\mathbb{N}$. This means that the population size has a tendency to decrease (in mean) when it is too large. This also implies that $\tau_\partial<\infty$ a.s.

In the following theorem, $R>0$ is the limiting value defined in (8.1).

Theorem 8.3. *Assume that* $(Z_n, n\in\mathbb{Z}_+)$ *is aperiodic irreducible, that it satisfies the assumptions* (8.2) *and* (8.3) *and that, for some* $q_0>\frac{\log R}{\log(1/m)}\vee 1$,

$$\sup_{z\in\mathbb{Z}_+^d,\ 1\leq i\leq |z|}\mathbb{E}[(\xi_{i,1}^{(z)})^{q_0}]<\infty,$$

Then Condition (E) holds true with $\varphi_1(x)=\|x\|^q$*, for all* $q\in\left(\frac{\log R}{\log(1/m)}\vee 1, q_0\right]$.

Remark 8.4. This result easily applies if $\sup_{z\in\mathbb{Z}_+^d,\ 1\leq i\leq |z|}\mathbb{E}[(\xi_{i,n}^{(z)})^q]<\infty$ for all $q>0$. In other cases, we need an upper bound for $R>0$ to check the assumptions of Theorem 8.3. For instance, one may use the fact that $R\leq 1/\sup_{z\in\mathbb{Z}_+^d}\mathbb{P}_z(Z_1=z)$. One may also use Lyapunov techniques, in the same spirit as in Section 4.15 for diffusion processes.

Remark 8.5. A particular case of application of the above theorem is when Z is obtained from a Galton-Watson multi-type process (see below for a more precise definition) with additional population-dependent death rates. For example, one can assume that additional death events may affect a fraction of the population, modelling global death events. In this case, compared to the Galton-Watson case, the independence between the progeny of individuals breaks down. Another situation covered by the above result is the case where the domain of absorption of Z is a larger set than 0, for example the process may be absorbed when it reaches one edge of $\mathbb{Z}_+^d$ (i.e. when one type disappears). Another typical application of Theorem 8.3 is the case of population-dependent Galton-Watson processes, i.e. of processes such that, given Z_n, Z_{n+1} is the sum of $|Z_n|$ i.i.d. random variables whose law may depend on Z_n. In this situation, Theorem 8.3 and its consequences stated in Section 2 generalize the results of [55] to the multi-type situation and provides finer results on the domain of attraction of the minimal quasi-stationary distribution. The reducible cases considered in [55] can also be recovered using the criterion of Theorem 6.1 in Section 6.1 or the criteria of [28]. Of course, the above cases may be combined.

Let us now consider the case of multi-type Galton-Watson processes. A Markov process $(Z_n, n \in \mathbb{Z}_+)$ evolving in $\mathbb{Z}_+^d = E \cup \{\partial\}$ absorbed at $\partial = 0$ is called a Galton-Watson process with d types if, for all $n \geq 0$ and all $i \in \{1, \dots, d\}$,

$$Z_{n+1}^i = \sum_{k=1}^d \sum_{\ell=1}^{Z_n^k} \zeta_{k,i}^{(n,\ell)}, \tag{8.4}$$

where the random variables $(\zeta_{k,1}^{(n,\ell)}, \dots, \zeta_{k,d}^{(n,\ell)})_{n,\ell,k}$ in $\mathbb{Z}_+$ are assumed independent and such that, for all $k \in \{1, \dots, d\}$, $(\zeta_{k,1}^{(n,\ell)}, \dots, \zeta_{k,d}^{(n,\ell)})_{n,\ell}$ is an i.i.d. family. We define the matrix $M = (M_{k,i})_{1 \leq k,i \leq d}$ of mean offspring as

$$M_{k,i} = \mathbb{E}(\zeta_{k,i}^{(n,\ell)}), \quad \forall k, i \in \{1, \dots, d\},$$

and assume that $M_{k,i} < +\infty$ and that there exists $n \geq 1$ such that $[M^n]_{k,i} > 0$ for all $k, i \in \{1, \dots, d\}$.

Using the classical formalism of [59], we consider a positive right eigenvector v of the matrix M of mean offspring and we denote by $\rho(M)$ its spectral radius. The sub-critical case corresponds to $\rho(M) < 1$. It is well-known [64] (see also [60, 2]) that this implies the existence of a quasi-stationary distribution whose domain of attraction contains all Dirac measures (a so-called *Yaglom limit* or *minimal quasi-stationary distribution*). The authors also prove that $\nu_{QSD}(|\cdot|) < \infty$ if and only if $\mathbb{E}[|Z_1| \log(|Z_1|) \mid Z_0 = (1, \dots, 1)] < \infty$. While the following result makes the stronger assumption that $\mathbb{E}[|Z_1|^{q_0} \mid Z_0 = (1, \dots, 1)] < \infty$ for some $q_0 > 1$, we obtain the finer results of Section 2, including a stronger form of convergence (in total variation norm with exponential speed), a non-trivial subset of the domain of attraction of the minimal quasi-stationary distribution and stronger moment properties for this quasi-stationary distribution.

Corollary 8.6. *If $(Z_n, n \geq 0)$ is a d-type irreducible, aperiodic sub-critical Galton-Watson process, and if, for some $q_0 > 1$,*

$$\mathbb{E}[|Z_1|^{q_0} \mid Z_0 = (1, \dots, 1)] < \infty,$$

then Condition (E) holds true with $\varphi_1(z) = |z|^q$ for any $q \in (1, q_0]$. In particular, the domain of attraction of ν_{QSD} contains all the probability measures such that $\mu(|\cdot|^q) < \infty$ for some $q > 1$.

This corollary easily derives from Theorem 8.3. Indeed, setting $\|z\| = \langle v, z \rangle$ and $\xi_{i,n}^{(Z_n)} = \sum_{j=1}^d v_j \zeta_{k,j}^{(n,\ell)}$ (assuming that i is the $\ell - th$ individual of type k in the population), one obtains

$$\|Z_{n+1}\| = \sum_{i=1}^{|Z_n|} \xi_{i,n}^{(Z_n)}$$

and

$$\mathbb{E}\left(\sum_{i=1}^{|Z_n|} \xi_{i,n}^{(Z_n)} \,\middle|\, Z_n = z \right) = \sum_{k=1}^d \sum_{\ell=1}^{z_k} \sum_{j=1}^d v_j \mathbb{E}\left(\zeta_{k,j}^{(n,\ell)}\right) = \rho(M)\|z\|,$$

for all $z \in \mathbb{Z}_+^d$. Since, in the case of multi-type Galton-Watson process, one has $R = 1/\rho(M)$ (see for instance Theorems 2 and 3 of [64]), Theorem 8.3 applies with $m = \rho(M)$.

To prove Theorem 8.3, we use the following lemma.

Lemma 8.7. *For all $q \in \left(\frac{\log R}{\log(1/m)} \vee 1, q_0\right]$, there exists a constant C_q such that, for all $z \in \mathbb{Z}_+^d$,*

$$\mathbb{E}\left[\left(\sum_{i=1}^{|z|} \xi_{i,n}^{(z)} - \mathbb{E}(\xi_{i,n}^{(z)})\right)^q\right] \leq C_q |z|^{1 \vee (q/2)}.$$

Proof. If $q \in (1,2]$, this is exactly Lemma 1 of [29]. If $q \geq 2$, Burkholder's inequality [12] implies that there exists a constant c_q such that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^{|z|}\xi_{i,n}^{(z)}-\mathbb{E}(\xi_{i,n}^{(z)})\right)^q\right] &\leq c_q\mathbb{E}\left[\left(\sum_{i=1}^{|z|}\left\{\xi_{i,n}^{(z)}-\mathbb{E}(\xi_{i,n}^{(z)})\right\}^2\right)^{q/2}\right]\\
&= c_q|z|^{q/2}\mathbb{E}\left[\left(\frac{1}{|z|}\sum_{i=1}^{|z|}\left\{\xi_{i,n}^{(z)}-\mathbb{E}(\xi_{i,n}^{(z)})\right\}^2\right)^{q/2}\right]\\
&\leq c_q|z|^{q/2}\mathbb{E}\left[\frac{1}{|z|}\sum_{i=1}^{|z|}\left|\xi_{i,n}^{(z)}-\mathbb{E}(\xi_{i,n}^{(z)})\right|^q\right]\\
&\leq c_q|z|^{q/2}\mathbb{E}\left[\frac{1}{|z|}\sum_{i=1}^{|z|}\left|\xi_{i,n}^{(z)}\right|^q+\mathbb{E}(\xi_{i,n}^{(z)})^q\right]\\
&\leq 2c_q|z|^{q/2}\sup_{z\in\mathbb{Z}_+^d,\ 1\leq i\leq |z|}\mathbb{E}[(\xi_{i,n}^{(z)})^q],
\end{aligned}$$

where we used Jensen's inequality in the third line, that the r.v. $\xi_{i,n}^{(z)}$ are nonnegative in the fourth line and Hölder's inequality in the last inequality. □

Proof of Theorem 8.3. We introduce an increasing sequence $(K_k, k\geq 0)$ of finite subsets of $\mathbb{Z}_+^d\setminus\{\partial\}$, where K_k is the smallest set containing $\{z\in\mathbb{Z}_+^d : 1\leq |z|\leq k\}$ such that the process Z restricted to K_k is irreducible and aperiodic. The existence of this set follows from the irreducibility assumption and the fact that $\mathbb{Z}_+^d$ is countable. We shall choose $K=K_k$ for an appropriate value of $k\geq 0$.

Fix $q\in\left(\frac{\log R}{\log(1/m)}\vee 1, q_0\right]$, $\theta_1\in(m^q,1/R)$, $\theta_2\in(\theta_1,1/R)$ and $\varphi_1(z)=\|z\|^q$. Using Minkowski's inequality in the first inequality, Lemma 8.7 in the third line and the equivalence between norms on $\mathbb{R}_+^d$,

$$\begin{aligned}
P_1\varphi_1(z)=\mathbb{E}\left(\left|\sum_{i=1}^{|z|}\xi_{i,n}^{(z)}\right|^q\right) &\leq \left[\mathbb{E}\left(\left|\sum_{i=1}^{|z|}\xi_{i,n}^{(z)}-\mathbb{E}(\xi_{i,n}^{(z)})\right|^q\right)^{1/q}+\sum_{i=1}^{|z|}\mathbb{E}(\xi_{i,n}^{(z)})\right]^q\\
&\leq\left[\left(C_q|z|^{1\vee(q/2)}\right)^{1/q}+m\|z\|\right]^q\\
&= m^q\|z\|^q\left(1+C'_q|z|^{1/(q\wedge 2)-1}\right)^q\\
&\leq m^q\|z\|^q+C''_q|z|^{q-1+1/(q\wedge 2)},
\end{aligned}\tag{8.5}$$

for constants C'_q and C''_q only depending on q and m. Since $q-1+1/(q\wedge 2)<q$, there exists $k_1\geq 0$ such that, for all $z\notin K_{k_1}$,

$$P_1\varphi_1(z)\leq\theta_1\varphi_1(z).\tag{8.6}$$

We also deduce that, for all $z\in K_{k_1}$,

$$P_1\varphi_1(z)\leq\max_{x\in K_{k_1}} m^q\|x\|^q+C''_q|x|^{q-1+1/(q\wedge 2)}<+\infty.$$

Setting $K=K_{k_1}$, we deduce that the first and third lines of Condition (E2) are satisfied.

By definition of R, we have $\lim_{n\to\infty}\theta_2^{-n}\inf_{z\in K}\mathbb{P}_z(X_n\in K)=+\infty$ and hence, using Lemma 3.1, there exists a function $\varphi_2 : E\to[0,1]$ such that the second and fourth lines of Condition (E2) are satisfied. It also implies that Condition (E4) holds true.

Since the process is irreducible and aperiodic and K is finite, (3.2) is clearly satisfied for $n_0 = 1$ and m_0 large enough, so that Theorem 8.3 follows from Proposition 3.3. □

9 Proof of Theorem 2.1

In all the proof, the constants C are all positive and finite and may change from line to line. We first assume from Subsections 9.1 to 9.6 that for all $n \geq 0$ and all $x \in E$, $\mathbb{P}_x(n < \tau_\partial) > 0$. The general case will be handled in Subsection 9.8.

9.1 Main steps of the proof

The proof is based on a careful study of the semigroup of the process conditioned to not be absorbed before time T. In this section, we give the main ideas and steps of the proof of (2.1) for the total variation norm $\|\cdot\|_{TV} := \|\cdot\|_{TV(1)}$ in place of $\|\cdot\|_{TV(\varphi_1)}$, and leave the details for the following subsections, where preliminary results and the following Propositions 9.1, 9.2, 9.3 and Lemma 9.4 are proved. The general case of the $\|\cdot\|_{TV(\varphi_1)}$ norm is handled in Subsection 9.7.

For any $T \in \mathbb{Z}_+$, we consider the law of the process X conditioned to not be absorbed before time T. We introduce the linear operators $(S^T_{m,n})_{0\leq m\leq n\leq T}$ defined by

$$S^T_{m,n} f(x) = \mathbb{E}(f(X_n) \mid X_m = x,\ T < \tau_\partial) = \frac{P_{n-m}\left(f P_{T-n}\mathbb{1}_E\right)(x)}{P_{T-m}\mathbb{1}_E(x)}.$$

It is well-known that $(S^T_{m,n})_{0\leq m\leq n\leq T}$ forms a time-inhomogeneous semigroup (i.e. $S^T_{m,n}S^T_{n,p} = S^T_{m,p}$ for all $m \leq n \leq p \leq T$) and that the process $(X_n, 0 \leq n \leq T)$ under $\mathbb{P}^{S^T_{0,\cdot}}_x$ is a (time-inhomogeneous) Markov process, where we denote by $\mathbb{P}^{S^T_{0,\cdot}}_x$ the law of the process $(X_n, 0 \leq n \leq T)$ conditionally on $T < \tau_\partial$ and $X_0 = x$.

Fix $\theta \in (\theta_1/\theta_2, 1)$. For any $T \geq 0$, we set, for $x \in E$,

$$\psi_T(x) = \mathbb{E}_x(\theta^{-T_K\wedge T} \mid T < \tau_\partial) = \mathbb{E}^{S^T_{0,\cdot}}_x\left(\theta^{-T_K\wedge T}\right),$$

where

$$T_K := \inf\{n \in \mathbb{Z}_+ : X_n \in K\}$$

is the first hitting time of K by the process $(X_n, n \in \mathbb{Z}_+)$. Be careful that T_K is not the first hitting time of K by the full process $(X_t, t \in I)$, unless $I = \mathbb{Z}_+$.

The following proposition provides a Lyapunov-type property for the inhomogeneous semigroup S.

Proposition 9.1. *There exists a constant* $\bar{C} > 0$ *such that, for all* $0 \leq m < T$ *and* $1 \leq k \leq T - m$,

$$S^T_{m,m+k}\psi_{T-(m+k)}(x) \leq \theta^k \psi_{T-m}(x) + \bar{C}, \quad \forall x \in E. \tag{9.1}$$

The next proposition provides a Dobrushin coefficient-type property for the inhomogeneous semigroup S.

Proposition 9.2. *There exists a constant* $\alpha_0 \in (0,1)$ *such that, for all* $R > 0$*, there exists* $k_R \geq 1$ *such that, for all* $T \geq k_R$ *and all* $x, y \in E$ *such that* $\psi_T(x) + \psi_T(y) \leq R$*, we have*

$$\left\|\delta_x S^T_{0,k_R} - \delta_y S^T_{0,k_R}\right\|_{TV} \leq 2(1-\alpha_0).$$

The following property is a consequence of the two previous ones.

Proposition 9.3. *There exist constants* $n_0 \geq 1$, $C > 0$ *and* $\alpha \in (0,1)$ *such that,* $\forall n \geq 1$ *and all* $x, y \in E$,

$$\left\| \delta_x S^{n_0 n}_{0,n_0 n} - \delta_y S^{n_0 n}_{0,n_0 n} \right\|_{TV} \leq C\alpha^n (2 + \psi_{n_0 n}(x) + \psi_{n_0 n}(y)).$$

Let us now deduce (2.1) with the total variation norm in place of $\| \cdot \|_{TV(\varphi_1)}$, from the last proposition. We have, for all $x, y \in E$,

$$\begin{aligned} &\left\| \delta_x P_{nn_0} - \delta_x P_{nn_0} \mathbb{1}_E \, \delta_y S^{n_0 n}_{0,n_0 n} \right\|_{TV} \\ &\qquad \leq C\alpha^n \left(2\delta_x P_{nn_0} \mathbb{1}_E + \mathbb{E}_x \left(\theta^{-T_K \wedge nn_0} \mathbb{1}_{nn_0 < \tau_\partial} \right) + \psi_{n_0 n}(y) \delta_x P_{nn_0} \mathbb{1}_E \right). \end{aligned}$$

Hence, for any probability measure μ on E, integrating the above inequality over $\mu(\mathrm{d}x)$ leads to

$$\begin{aligned} &\left\| \mu P_{nn_0} - \mu P_{nn_0} \mathbb{1}_E \, \delta_y S^{n_0 n}_{0,n_0 n} \right\|_{TV} \\ &\qquad \leq C\alpha^n \left(2\mu P_{nn_0} \mathbb{1}_E + \mathbb{E}_\mu \left(\theta^{-T_K \wedge nn_0} \mathbb{1}_{nn_0 < \tau_\partial} \right) + \psi_{n_0 n}(y) \mu P_{nn_0} \mathbb{1}_E \right). \end{aligned}$$

We make use of the following lemma.

Lemma 9.4. *For all* $\theta \in (\theta_1/\theta_2, 1)$*, there exists a constant* C *such that, for all* $0 \leq m \leq T$ *and all probability measure* μ *over* E *such that* $\mu(\varphi_2) > 0$,

$$\mathbb{E}_\mu \left(\theta^{-T_K \wedge T} \mathbb{1}_{T < \tau_\partial} \right) \leq C \, \frac{\mu(\varphi_1)}{\mu(\varphi_2)} \mathbb{P}_\mu \left(T < \tau_\partial \right).$$

This implies that, for all μ such that $\mu(\varphi_2) > 0$,

$$\begin{aligned} &\left\| \mu P_{nn_0} - \delta_y S^{n_0 n}_{0,n_0 n} \mu P_{nn_0} \mathbb{1}_E \right\|_{TV} \\ &\qquad \leq C\alpha^n \left(2\mu P_{nn_0} \mathbb{1}_E + \frac{\mu(\varphi_1)}{\mu(\varphi_2)} \mu P_{nn_0} \mathbb{1}_E + \psi_{n_0 n}(y) \mu P_{nn_0} \mathbb{1}_E \right). \end{aligned}$$

Hence

$$\left\| \frac{\mu P_{nn_0}}{\mu P_{nn_0} \mathbb{1}_E} - \delta_y S^{n_0 n}_{0,n_0 n} \right\|_{TV} \leq C\alpha^n \left(2 + \frac{\mu(\varphi_1)}{\mu(\varphi_2)} + \psi_{n_0 n}(y) \right).$$

Using the same procedure w.r.t. y, we deduce that, for any probability measures μ_1 and μ_2 on E such that $\mu_1(\varphi_2) > 0$ and $\mu_2(\varphi_2) > 0$,

$$\left\| \frac{\mu_1 P_{nn_0}}{\mu_1 P_{nn_0} \mathbb{1}_E} - \frac{\mu_2 P_{nn_0}}{\mu_2 P_{nn_0} \mathbb{1}_E} \right\|_{TV} \leq C\alpha^n \left(\frac{\mu_1(\varphi_1)}{\mu_1(\varphi_2)} + \frac{\mu_2(\varphi_1)}{\mu_2(\varphi_2)} \right),$$

where we used the fact that $\mu(\varphi_1)/\mu(\varphi_2) \geq 1$ for all probability measure μ on E such that $\mu(\varphi_2) > 0$.

Because of Lemma 9.6 below, we deduce that, for some constant $D_1 > 0$ and for all $0 \leq k < n_0$,

$$\begin{aligned} \left\| \frac{\mu_1 P_{nn_0+k}}{\mu_1 P_{nn_0+k} \mathbb{1}_E} - \frac{\mu_2 P_{nn_0+k}}{\mu_2 P_{nn_0+k} \mathbb{1}_E} \right\|_{TV} &\leq C\alpha^n \left(\frac{\mu_1 P_k \varphi_1}{\mu_1 P_k \varphi_2} + \frac{\mu_2 P_k \varphi_1}{\mu_2 P_k \varphi_2} \right) \\ &\leq C\alpha^n \left(\frac{\mu_1(\varphi_1)}{\mu_1(\varphi_2)} \vee D_1 + \frac{\mu_2(\varphi_1)}{\mu_2(\varphi_2)} \vee D_1 \right). \end{aligned}$$

Therefore, up to a change in the constant C and replacing α by α^{1/n_0}, we deduce that, for all probability measures μ_1 and μ_2 on E such that $\mu_1(\varphi_2) > 0$ and $\mu_2(\varphi_2) > 0$ and for all $n \geq 0$,

$$\left\| \frac{\mu_1 P_n}{\mu_1 P_n \mathbb{1}_E} - \frac{\mu_2 P_n}{\mu_2 P_n \mathbb{1}_E} \right\|_{TV} \leq C\alpha^n \left(\frac{\mu_1(\varphi_1)}{\mu_1(\varphi_2)} + \frac{\mu_2(\varphi_1)}{\mu_2(\varphi_2)} \right). \tag{9.2}$$

Fix $x_0 \in K$. We set $\mu_1 = \delta_{x_0}$ and $\mu_2 = \frac{\mu_1 P_1}{\mu_1 P_1 \mathbb{1}_E}$ in (9.2). Since $\frac{\mu_1\varphi_1}{\mu_1\varphi_2} < \infty$ and because of Lemma 9.6 below, we have $\frac{\mu_2\varphi_1}{\mu_2\varphi_2} < \infty$. We deduce that, for some constant $C > 0$,

$$\left\| \frac{\delta_{x_0} P_{n+1}}{\delta_{x_0} P_{n+1} \mathbb{1}_E} - \frac{\delta_{x_0} P_n}{\delta_{x_0} P_n \mathbb{1}_E} \right\|_{TV} \leq C\alpha^n,$$

and hence, using the completeness of the space of probability measures on E for the total variation norm, we deduce that there exists a quasi-limiting measure ν_{QSD} (which is hence a quasi-stationary distribution) such that

$$\left\| \frac{\delta_{x_0} P_n}{\delta_{x_0} P_n \mathbb{1}_E} - \nu_{QSD} \right\|_{TV} \leq \frac{2C}{1-\alpha}\alpha^n.$$

In particular, it follows from Lemma 9.8 below that $\nu_{QSD}(K) > 0$ and hence that $\nu_{QSD}(\varphi_2) > 0$. Since Lemma 9.6 implies that $\frac{P_n\varphi_1(x_0)}{P_n\mathbb{1}_E(x_0)}$ is uniformly bounded in $n \geq 0$, we deduce that $\nu_{QSD}(\varphi_1 \wedge M)$ is bounded uniformly in $M > 0$ and hence $\nu_{QSD}(\varphi_1) < \infty$.

Using (9.2) again (up to another change of the constant C), we obtain that, for all probability measure μ on E such that $\frac{\mu(\varphi_1)}{\mu(\varphi_2)} < \infty$,

$$\left\| \frac{\mu P_n}{\mu P_n \mathbb{1}_E} - \nu_{QSD} \right\|_{TV} \leq C\alpha^n \frac{\mu(\varphi_1)}{\mu(\varphi_2)}.$$

This also entails that there exists a unique quasi-stationary distribution such that $\nu_{QSD}(\varphi_1)/\nu_{QSD}(\varphi_2) < \infty$.

This ends the proof of (2.1) for the total variation norm. The general case with the norm $\|\cdot\|_{TV(\varphi_1)}$ is proved in Subsection 9.7.

9.2 Preliminary results

We start by proving two basic inequalities which are direct consequences of (E2).

Lemma 9.5. *For all $x \in E \setminus K$ and all $n \geq 0$,*

$$\mathbb{P}_x(n < T_K \wedge \tau_\partial) \leq \mathbb{E}_x[\varphi_1(X_n)\mathbb{1}_{n<T_K\wedge\tau_\partial}] \leq \theta_1^n \varphi_1(x).$$

For all $x \in E$ and $n \geq 0$,

$$\mathbb{P}_x(n < \tau_\partial) \geq \mathbb{E}_x[\varphi_2(X_n)\mathbb{1}_{n<\tau_\partial}] \geq \theta_2^n \varphi_2(x).$$

Proof of Lemma 9.5. These two properties follow easily by induction from (E2). For example, the first one makes use of the following relation: for all $n \geq 1$ and $x \in E$,

$$\mathbb{E}_x[\varphi_1(X_n)\mathbb{1}_{n<T_K\wedge\tau_\partial}] = \mathbb{1}_{x\in E\setminus K}\, P_1 \left[\mathbb{E}_\cdot \left(\varphi_1(X_{n-1})\mathbb{1}_{n-1<T_K\wedge\tau_\partial}\right)\right](x).$$

This and (E2) entail the property at time $n = 1$ and, by induction, at any time $n \geq 1$. ☐

The next lemma states that the expectation of $\varphi_1(X_n)$ is controlled by the expectation of $\varphi_2(X_n)$ uniformly in time.

Lemma 9.6. *For all $\theta \in (\theta_1/\theta_2, 1]$, there exists a finite constant $D_\theta > 0$ such that, for all probability measure μ on E such that $\mu(\varphi_1)/\mu(\varphi_2) < \infty$, for all $T \in \mathbb{Z}_+$ and all $x \in E$,*

$$\frac{\mu P_T \varphi_1}{\mu P_T \varphi_2} \leq \left(\theta^T \frac{\mu(\varphi_1)}{\mu(\varphi_2)}\right) \vee D_\theta. \tag{9.3}$$

Proof of Lemma 9.6. It follows from (E2) that

$$\mu P_{T+1}\varphi_1 \leq \theta_1 \mu P_T \varphi_1 + C\mu P_T \mathbb{1}_K$$

and

$$\mu P_{T+1}\varphi_2 \geq \theta_2 \mu P_T \varphi_2.$$

Hence

$$\begin{aligned}\frac{\mu P_{T+1}\varphi_1}{\mu P_{T+1}\varphi_2} &\leq \frac{\theta_1 \mu P_T \varphi_1 + C\mu P_T \mathbb{1}_K(x)}{\theta_2 \mu P_T \varphi_2} \\ &\leq \frac{\theta_1}{\theta_2}\,\frac{\mu P_T \varphi_1}{\mu P_T \varphi_2} + \frac{C}{\theta_2 \inf_{y\in K}\varphi_2(y)}.\end{aligned}$$

Since $\theta_1/\theta_2 < \theta$, these arithmetico-geometric inequalities entail (9.3). □

We now give an irreducibility inequality.

Lemma 9.7. *For all* $C \geq 1$*, there exists a time* $n_5(C) \in \mathbb{N}$ *such that*

$$a_5(C) := \inf_{\mu\in\mathcal{M}_1(E)\text{ s.t. }\mu(\varphi_1)\leq C\mu(\varphi_2)} \mathbb{P}_\mu(X_{n_5(C)} \in K) > 0. \tag{9.4}$$

Proof of Lemma 9.7. It follows from (E4) that there exists a time $n_\nu \in \mathbb{N}$ such that, for all $n \geq n_\nu$, $\mathbb{P}_\nu(X_n \in K) > 0$, and, using (E1), that for all $n \geq n_\nu + n_1$,

$$\inf_{x\in K} \mathbb{P}_x(X_n \in K) \geq c_1 \mathbb{P}_\nu(X_{n-n_1} \in K) > 0.$$

Let $C \geq 1$ and μ be such that $\mu(\varphi_1) \leq C\mu(\varphi_2)$. It follows from Lemma 9.5 that, for all $n \geq 1$,

$$\mathbb{P}_\mu(T_K \wedge \tau_\partial > n) \leq \mathbb{E}_\mu\left[\varphi_1(X_n)\mathbb{1}_{T_K\wedge\tau_\partial > n}\right] \leq \theta_1^n \mu(\varphi_1) \leq C\theta_1^n \mu(\varphi_2).$$

and

$$\mathbb{P}_\mu(n < \tau_\partial) \geq \mathbb{E}_\mu[\varphi_2(X_n)] \geq \theta_2^n \mu(\varphi_2).$$

Therefore,

$$\mathbb{P}_\mu(T_K \leq n < \tau_\partial) \geq (\theta_2^n - C\theta_1^n)\,\mu(\varphi_2).$$

Choosing $n(C) = \lceil 2C/\log(\theta_2/\theta_1)\rceil$, we deduce that

$$\mathbb{P}_\mu(T_K \leq n(C) < \tau_\partial) \geq \frac{\theta_2^{n(C)}}{2}\mu(\varphi_2) \geq \frac{\theta_2^{n(C)}}{2C}.$$

Therefore,

$$\begin{aligned}\mathbb{P}_\mu(X_{n(C)+n_\nu+n_1} \in K) &\geq \mathbb{E}_\mu\left[\mathbb{1}_{T_K\leq n(C)}\mathbb{P}_{X_{T_K}}(X_{n(C)+n_\nu+n_1-k} \in K)\big|_{k=T_K}\right] \\ &\geq \min_{n_\nu+n_1\leq k\leq n_\nu+n_1+n(C)} \inf_{x\in K} \mathbb{P}_x(X_k \in K)\,\frac{\theta_2^{n(C)}}{2C}.\end{aligned}$$

Hence we have proved Lemma 9.7 with $n_5(C) = n_\nu + n_1 + n(C)$. □

The next lemma shows that conditional distributions with initial conditions in K give to K a mass uniformly bounded from below.

Lemma 9.8. *There exists a time* $n_6 \in \mathbb{N}$ *such that*

$$\inf_{T\geq n_6} \inf_{x\in K} \mathbb{P}_x(X_T \in K \mid T < \tau_\partial) > 0.$$

Proof of Lemma 9.8. Since φ_1/φ_2 is bounded over K, we deduce from Lemma 9.6 that, setting $C := D_1 + \sup_{x\in K}\frac{\varphi_1(x)}{\varphi_2(x)}$, we have for all $x\in K$ and all $T\geq n_5(C)$,

$$\frac{P_{T-n_5(C)}\varphi_1(x)}{P_{T-n_5(C)}\varphi_2(x)} \leq C. \tag{9.5}$$

Using Lemma 9.7 applied to $\mu = \frac{\delta_x P_{T-n_5(C)}}{\delta_x P_{T-n_5(C)}\mathbb{1}_E}$, we deduce that, for all $x\in K$ and $T\geq n_5(C)$,

$$\mathbb{P}_x(X_T\in K\mid T<\tau_\partial) = \frac{\mu P_{n_5(C)}\mathbb{1}_K}{\mu P_{n_5(C)}\mathbb{1}_E} \geq \mu P_{n_5(C)}\mathbb{1}_K \geq a_5(C). \qquad \square$$

The next lemma shows that survival probabilities are controlled by the function φ_1.

Lemma 9.9. *There exists a constant $C>0$ such that, for all $p\in[1,\log\theta_1/\log\theta_2)$, $x\in E$ and $n\geq 1$,*

$$\mathbb{P}_x(n<\tau_\partial) \leq C\,\frac{\varphi_1(x)^{1/p}}{1-\theta_1^{1/p}/\theta_2}\inf_{y\in K}\mathbb{P}_y(n<\tau_\partial). \tag{9.6}$$

Proof of Lemma 9.9. It follows from Lemma 9.5 that, for all $p\geq 1$, $x\in E\setminus K$ and $n\geq 1$,

$$\mathbb{P}_x(n<T_K\wedge\tau_\partial) \leq \theta_1^{n/p}\varphi_1(x)^{1/p}. \tag{9.7}$$

Note that this inequality is trivial for $x\in K$. In particular, for $p\geq 1$ such that $\theta_1^{1/p}<\theta_2$, for all $x\in K$,

$$\mathbb{E}_x(\theta_2^{-T_K\wedge\tau_\partial}) \leq \frac{\varphi_1(x)^{1/p}}{1-\theta_1^{1/p}/\theta_2}. \tag{9.8}$$

Fix $p\in[1,\log\theta_1/\log\theta_2)$. Using (9.7), the second inequality of Lemma 9.5 and (E3), we have for all $x\in E$

$$\begin{aligned}
\mathbb{P}_x(n<\tau_\partial) &= \mathbb{P}_x(n<T_K\wedge\tau_\partial) + \mathbb{P}_x(T_K\wedge\tau_\partial\leq n<\tau_\partial)\\
&\leq \theta_2^n\varphi_1(x)^{1/p} + \sum_{k=0}^n \mathbb{P}_x(T_K\wedge\tau_\partial=k)\sup_{y\in K}\mathbb{P}_y(n-k<\tau_\partial)\\
&\leq \frac{\inf_{z\in K}\mathbb{P}_z(n<\tau_\partial)}{\inf_{z\in K}\varphi_2(z)}\varphi_1(x)^{1/p} + c_3\sum_{k=0}^n \mathbb{P}_x(T_K\wedge\tau_\partial=k)\inf_{y\in K}\mathbb{P}_y(n-k<\tau_\partial)\\
&\leq C\inf_{z\in K}\mathbb{P}_z(n<\tau_\partial)\,\varphi_1(x)^{1/p} + C\inf_{z\in K}\mathbb{P}_z(n<\tau_\partial)\sum_{k=0}^n \mathbb{P}_x(T_K\wedge\tau_\partial=k)\theta_2^{-k},
\end{aligned} \tag{9.9}$$

where we used the fact that, for some constant $C>0$, for all $n\geq k\geq 0$ and all $z\in K$,

$$\mathbb{P}_z(n<\tau_\partial) \geq C\theta_2^k\inf_{y\in K}\mathbb{P}_y(n-k<\tau_\partial). \tag{9.10}$$

This is proved using the three following equations. For all $n\geq k\geq n_6$ and all $z\in K$, by Lemmata 9.8 and 9.5,

$$\begin{aligned}
\mathbb{P}_z(n<\tau_\partial) &\geq \mathbb{P}_z(X_k\in K\mid k<\tau_\partial)\mathbb{P}_z(k<\tau_\partial)\inf_{y\in K}\mathbb{P}_y(n-k<\tau_\partial)\\
&\geq C\theta_2^k\varphi_2(z)\inf_{y\in K}\mathbb{P}_y(n-k<\tau_\partial)\\
&\geq C\theta_2^k\inf_{y\in K}\mathbb{P}_y(n-k<\tau_\partial),
\end{aligned}$$

since $\inf_{z\in K}\varphi_2(z)>0$. Also, for all $n\geq n_6\geq k$, using the last inequality,

$$\begin{aligned}\mathbb{P}_z(n<\tau_\partial)&\geq C\theta_2^{n_6}\inf_{y\in K}\mathbb{P}_y(n-n_6<\tau_\partial)\\&\geq C\theta_2^{n_6}\inf_{y\in K}\mathbb{P}_y(n-k<\tau_\partial)\\&\geq (C\theta_2^{n_6})\,\theta_2^k\inf_{y\in K}\mathbb{P}_y(n-k<\tau_\partial).\end{aligned}$$

Finally, for all $k\leq n<n_6$,

$$\mathbb{P}_z(n<\tau_\partial)\geq\mathbb{P}_z(n_6<\tau_\partial)\geq C\theta_2^{n_6}\geq(C\theta_2^{n_6})\,\theta_2^k\inf_{y\in K}\mathbb{P}_y(n-k<\tau_\partial),$$

so (9.10) is proved.

Combining (9.8) and (9.9) ends the proof of Lemma 9.9. □

9.3 Proof of Proposition 9.1

Markov's property implies that, for all $x\in E\setminus K$ and $T,m\geq 1$,

$$S^T_{0,1}\psi_{T-1}(x)=S^{T+m}_{m,m+1}\psi_{T-1}(x)=\theta\psi_T(x). \tag{9.11}$$

Indeed,

$$\begin{aligned}\theta\psi_T(x)&=\frac{\mathbb{E}_x(\theta^{1-T_K\wedge T}\mathbb{1}_{T<\tau_\partial})}{\mathbb{P}_x(T<\tau_\partial)}\\&=\frac{\mathbb{E}_x\left[\mathbb{1}_{1<\tau_\partial}\mathbb{E}_{X_1}(\theta^{-T_K\wedge(T-1)}\mid T-1<\tau_\partial)\mathbb{P}_{X_1}(T-1<\tau_\partial)\right]}{\mathbb{P}_x(T<\tau_\partial)}=S^T_{0,1}\psi_{T-1}(x).\end{aligned}$$

Similarly, for all $x\in$K,

$$S^T_{0,1}\psi_{T-1}(x)=S^{T+m}_{m,m+1}\psi_{T-1}(x)=\theta\mathbb{E}_x^{S^T_{0,\cdot}}(\theta^{-\sigma_K\wedge T}), \tag{9.12}$$

where

$$\sigma_K:=\min\{n\geq 1,\ X_n\in K\}$$

is the first return time in K. Setting

$$C:=\sup_{T\geq 0}\sup_{x\in K}\mathbb{E}_x^{S^T_{0,\cdot}}(\theta^{-\sigma_K\wedge T}),$$

which is finite (see Lemma 9.10), we can apply recursively (9.11) and (9.12) to obtain

$$\begin{aligned}S^T_{m,m+k}\psi_{T-(m+k)}&=S^T_{m,m+k-1}\left(\mathbb{1}_{E\setminus K}S^T_{m+k-1,m+k}(\psi_{T-(m+k)})\right)\\&\quad+S^T_{m,m+k-1}\left(\mathbb{1}_K S^T_{m+k-1,m+k}(\psi_{T-(m+k)})\right)\\&\leq\theta S^T_{m,m+k-1}\psi_{T-(m+k-1)}+C\theta\\&\leq\ldots\leq\theta^k\psi_{T-m}(x)+C\sum_{\ell=1}^k\theta^\ell.\end{aligned}$$

Hence Proposition 9.1 follows from the next lemma.

Lemma 9.10. *For all $\theta\in(\theta_1/\theta_2,1)$,*

$$\sup_{T\geq 0}\sup_{x\in K}\mathbb{E}_x^{S^T_{0,\cdot}}(\theta^{-\sigma_K\wedge T})<\infty.$$

Proof of Lemma 9.10. Fix $x \in K$. On the one hand, by Lemma 9.9 (with $p = 1$), we have for any $1 \le n < T$,

$$\begin{aligned}\mathbb{P}_x(n < \sigma_K \text{ and } T < \tau_\partial) &= \mathbb{E}_x(\mathbb{1}_{n<\sigma_K \wedge \tau_\partial} \mathbb{P}_{X_n}(T - n < \tau_\partial)) \\ &\le C \inf_{y \in K} \mathbb{P}_y(T - n < \tau_\partial) \mathbb{E}_x(\mathbb{1}_{n<\sigma_K \wedge \tau_\partial} \varphi_1(X_n)).\end{aligned}$$

Using (E2) and Markov's property as in the proof of Lemma 9.5, we deduce

$$\mathbb{P}_x(n < \sigma_K \text{ and } T < \tau_\partial) \le C \inf_{y \in K} \mathbb{P}_y(T - n < \tau_\partial) \theta_1^{n-1} P_1 \varphi_1(x) \tag{9.13}$$

$$\le C \inf_{y \in K} \mathbb{P}_y(T - n < \tau_\partial) \theta_1^n. \tag{9.14}$$

On the other hand, Lemma 9.8 implies the existence of a constant $C > 0$ such that, for all $x \in K$ and all $n \ge n_6$,

$$\mathbb{P}_x(X_n \in K) \ge C \mathbb{P}_x(n < \tau_\partial).$$

We deduce from Markov's property and Lemma 9.5 that, for all $T \ge n \ge n_6$,

$$\begin{aligned}\mathbb{P}_x(T < \tau_\partial) &\ge \mathbb{P}_x(X_n \in K) \inf_{y \in K} \mathbb{P}_y(T - n < \tau_\partial) \\ &\ge C \mathbb{P}_x(n < \tau_\partial) \inf_{y \in K} \mathbb{P}_y(T - n < \tau_\partial) \\ &\ge C \theta_2^n \inf_{y \in K} \mathbb{P}_y(T - n < \tau_\partial).\end{aligned}$$

Combining this with (9.13), we finally deduce that there exists a constant $C > 0$ such that, for all $x \in K$ and all $T \ge n \ge n_6$,

$$\mathbb{P}_x(n < \sigma_K \mid T < \tau_\partial) \le C \left(\frac{\theta_1}{\theta_2} \right)^n. \tag{9.15}$$

The extension to any $T \ge n$ is trivial, so the conclusion follows. □

9.4 Proof of Proposition 9.2

We start by stating a lemma proved at the end of this subsection.

Lemma 9.11. *For all $x \in K$ and $n_1 + n_6 \le n \le T$,*

$$\mathbb{P}_x(X_n \in \cdot \mid T < \tau_\partial) \ge c_1' \nu, \tag{9.16}$$

where the measure ν and the integer n_1 are the one of Condition (E1), the integer n_6 is from Lemma 9.7 and $c_1' > 0$ is independent of x, n and T.

Fix $\theta \in (\theta_1/\theta_2, 1)$ and set $k_R = \lceil \log(2R)/\log(1/\theta) \rceil + n_1 + n_6$ and fix $T \ge k_R$. For all $x \in E$ such that $\psi_T(x) \le R$, Markov's inequality implies that

$$\mathbb{P}_x(T_K > k_R - n_1 - n_6 \mid T < \tau_\partial) = \mathbb{P}_x^{S_{0,\cdot}^T}(T_K > k_R - n_1 - n_6) \le \frac{R}{\theta^{-k_R + n_1 + n_6}} \le \frac{1}{2}.$$

It follows from Lemma 9.11 that, for all measurable $A \subset E$,

$$\begin{aligned}\mathbb{P}_x^{S_{0,\cdot}^T}(X_{k_R} \in A) &\ge \frac{\mathbb{E}_x \left[\sum_{k=1}^{k_R - n_1 - n_6} \mathbb{1}_{T_K = k} \mathbb{P}_{X_k}(X_{k_R - k} \in A,\, T - k < \tau_\partial) \right]}{\mathbb{P}_x(T < \tau_\partial)} \\ &\ge c_1' \nu(A) \frac{\mathbb{E}_x \left[\sum_{k=1}^{k_R - n_1 - n_6} \mathbb{1}_{T_K = k} \mathbb{P}_{X_k}(T - k < \tau_\partial) \right]}{\mathbb{P}_x(T < \tau_\partial)} \\ &= c_1' \nu(A) \mathbb{P}_x(T_K \le k_R - n_1 - n_6 \mid T < \tau_\partial) \\ &\ge \frac{1}{2} c_1' \nu(A).\end{aligned}$$

This concludes the proof of Proposition 9.2 with $\alpha_0 = c_1'/2$.

Proof of Lemma 9.11. For all measurable set $A \subset K$, we deduce from Markov's property that, for all $x \in K$ and all $T \geq n \geq n_1 + n_6$,

$$\begin{aligned}\mathbb{P}_x(X_n \in A,\, T < \tau_\partial) &\geq \mathbb{E}_x\left[\mathbb{1}_{X_{n-n_1}\in K}\,\mathbb{E}_{X_{n-n_1}}\left(\mathbb{1}_{X_{n_1}\in A}\mathbb{P}_{X_{n_1}}(T-n<\tau_\partial)\right)\right]\\ &\geq \mathbb{E}_x\left[\mathbb{1}_{X_{n-n_1}\in K}\,\mathbb{P}_{X_{n-n_1}}(X_{n_1}\in A)\right]\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial)\\ &\geq c_1\nu(A)\mathbb{P}_x\left(X_{n-n_1}\in K\right)\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial), \qquad (9.17)\end{aligned}$$

where we used (E1). Now, using Lemma 9.9, we deduce that there exists a constant $c > 0$ such that

$$\begin{aligned}\mathbb{P}_x(T<\tau_\partial) \leq \mathbb{P}_x(T-n_1<\tau_\partial) &= \mathbb{E}_x\left(\mathbb{1}_{n-n_1<\tau_\partial}\mathbb{P}_{X_{n-n_1}}(T-n<\tau_\partial)\right)\\ &\leq c\mathbb{E}_x\left(\mathbb{1}_{n-n_1<\tau_\partial}\varphi_1(X_{n-n_1})\right)\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial).\end{aligned}$$

Since $\varphi_1(x)/\varphi_2(x)$ is uniformly bounded over $x \in K$, Lemma 9.6 implies that there exists a constant $c' > 0$ such that, for all $x \in K$,

$$\mathbb{E}_x\left[\mathbb{1}_{n-n_1<\tau_\partial}\varphi_1(X_{n-n_1})\right] \leq c'\mathbb{E}_x\left[\mathbb{1}_{n-n_1<\tau_\partial}\varphi_2(X_{n-n_1})\right] \leq c'\mathbb{P}_x\left(n-n_1<\tau_\partial\right).$$

But $n - n_1 \geq n_6$, hence Lemma 9.8 entails that there exists a constant $c'' > 0$ such that, for all $x \in K$,

$$\mathbb{P}_x\left(n-n_1<\tau_\partial\right) \leq c''\mathbb{P}_x(X_{n-n_1}\in K).$$

Hence we obtain

$$\mathbb{P}_x(T<\tau_\partial) \leq cc'c''\,\mathbb{P}_x\left(X_{n-n_1}\in K\right)\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial).$$

Combining this with (9.17), we obtain

$$\mathbb{P}_x(X_n\in A \mid T<\tau_\partial) \geq \frac{c_1}{cc'c''}\,\nu(A).$$

This ends the proof of Lemma 9.11. □

9.5 Proof of Proposition 9.3

We transpose the ideas of [57] (see also [58]) to the time-inhomogeneous setting. We fix the constants $R = 4\bar{C}/(1-\theta)$ and $\beta = \alpha_0/2\bar{C}$, where $\bar{C}$ is the constant of Proposition 9.1. For all $T \geq 0$ and all $\varphi : E \to \mathbb{R}$, we set

$$|||\varphi|||_T = \sup_{x,y\in E}\frac{|\varphi(x)-\varphi(y)|}{2+\beta\psi_T(x)+\beta\psi_T(y)}.$$

Fix n and $T \geq 0$ such that $(n+1)k_R \leq T$ and let φ be such that $|||\varphi|||_{T-(n+1)k_R} \leq 1$. Then, replacing φ by $\varphi + c$ for some appropriate constant c, one has $|\varphi| \leq 1 + \beta\psi_{T-(n+1)k_R}$ (see Lemma 3.8 p.14 in [57]).

If $\psi_{T-nk_R}(x) + \psi_{T-nk_R}(y) > R$, then, using Proposition 9.1,

$$\begin{aligned}&\left|S^T_{nk_R,(n+1)k_R}\varphi(x) - S^T_{nk_R,(n+1)k_R}\varphi(y)\right|\\ &\qquad\leq 2+\theta\beta\psi_{T-nk_R}(x)+\theta\beta\psi_{T-nk_R}(y)+2\beta\bar{C}\\ &\qquad\leq 2+(\theta+(1-\theta)/2)\left(\beta\psi_{T-nk_R}(x)+\beta\psi_{T-nk_R}(y)\right)\\ &\qquad\qquad-(R\beta)(1-\theta)/2+2\beta\bar{C}\\ &\qquad\leq(1-\alpha_1)(2+\beta\psi_{T-nk_R}(x)+\beta\psi_{T-nk_R}(x)),\end{aligned}$$

where $\alpha_1 \in (0,1)$ is such that $2+(\theta+(1-\theta)/2)\,y \leq (1-\alpha_1)(2+y)$ for all $y \geq \beta R$.

If $\psi_{T-nk_R}(x)+\psi_{T-nk_R}(y) \leq R$, then, considering

$$\varphi = \varphi' + \varphi'',$$

with $|\varphi'| \leq 1$ and $|\varphi''| \leq \beta\psi_{T-(n+1)k_R}$, Propositions 9.1 and 9.2 entail

$$\begin{aligned}\Big|S^T_{nk_R,(n+1)k_R}\varphi(x) - S^T_{nk_R,(n+1)k_R}\varphi(y)\Big| \\ \leq 2(1-\alpha_0) + \beta\theta\psi_{T-nk_R}(x) + \beta\theta\psi_{T-nk_R}(y) + 2\beta\bar{C}.\end{aligned}$$

Our choice $\beta = \alpha_0/2\bar{C}$ implies that

$$\Big|S^T_{nk_R,(n+1)k_R}\varphi(x) - S^T_{nk_R,(n+1)k_R}\varphi(y)\Big| \leq (1-\alpha_2)(2+\beta\psi_{T-nk_R}(x)+\beta\psi_{T-nk_R}(y)).$$

for the constant $\alpha_2 = \frac{\alpha_0}{2} \wedge (1-\theta) > 0$.

Hence, we obtained

$$\Big|\Big|\Big|S^T_{nk_R,(n+1)k_R}\varphi\Big|\Big|\Big|_{T-nk_R} \leq (1-\alpha_1\wedge\alpha_2)|||\varphi|||_{T-(n+1)k_R},$$

which implies by iteration that

$$\Big|\Big|\Big|S^{nk_R}_{0,nk_R}\varphi\Big|\Big|\Big|_{nk_R} \leq (1-\alpha_1\wedge\alpha_2)^n|||\varphi|||_0 \leq (1-\alpha_1\wedge\alpha_2)^n\|\varphi\|_\infty/(1+\beta).$$

This concludes the proof of Proposition 9.3.

9.6 Proof of Lemma 9.4

This lemma in a generalization of Lemma 9.10. Its proof is based on similar computations. We give the details for sake of completeness.

For all probability measure μ on E, for any $0 \leq n < T$, using Lemma 9.9 for the second inequality and Lemma 9.5 for the third inequality, we have

$$\begin{aligned}\mathbb{P}_\mu(n < T_K \text{ and } T < \tau_\partial) &\leq \mathbb{E}_\mu(\mathbb{1}_{n<T_K}\mathbb{P}_{X_n}(T-n<\tau_\partial)) \\ &\leq C\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial)\mathbb{E}_\mu(\mathbb{1}_{n<T_K}\varphi_1(X_n)) \\ &\leq C\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial)\theta_1^n\mu(\varphi_1). \qquad (9.18)\end{aligned}$$

For all integer $n \geq n_\mu$, where

$$n_\mu := \left\lceil n_5(D_\theta) + \frac{\log\frac{\mu(\varphi_1)}{D_\theta\mu(\varphi_2)}}{\log(1/\theta)}\right\rceil,$$

it follows from Lemma 9.6 that

$$\frac{\mu P_{n-n_5(D_\theta)}\varphi_1}{\mu P_{n-n_5(D_\theta)}\varphi_2} \leq D_\theta \vee \left(\theta^{n-n_5(D_\theta)}\frac{\mu(\varphi_1)}{\mu(\varphi_2)}\right) \leq D_\theta$$

and from Lemma 9.7 that

$$\frac{\mu P_n\mathbb{1}_K}{\mu P_n\mathbb{1}_E} \geq a_5(D_\theta) > 0.$$

Therefore, we obtain from the Markov property and Lemma 9.5 that

$$\begin{aligned}\mathbb{P}_\mu(T<\tau_\partial)&\geq \mathbb{P}_\mu(X_n\in K)\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial)\\&\geq a_5(D_\theta)\mathbb{P}_\mu(n<\tau_\partial)\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial)\\&\geq a_5(D_\theta)\theta_2^n\mu(\varphi_2)\inf_{y\in K}\mathbb{P}_y(T-n<\tau_\partial).\end{aligned}$$

Combining this with (9.18), we obtain that, for all $n\geq n_\mu$,

$$\mathbb{P}_\mu(n<T_K\text{ and }T<\tau_\partial)\leq\frac{C}{a_5(D_\theta)}\left(\frac{\theta_1}{\theta_2}\right)^n\frac{\mu(\varphi_1)}{\mu(\varphi_2)}\mathbb{P}_\mu(T<\tau_\partial).$$

Hence

$$\mathbb{E}_\mu\left(\theta^{-T_K\wedge T}\mathbb{1}_{T_K\geq n_\mu,\ T<\tau_\partial}\right)\leq C\frac{\mu(\varphi_1)}{\mu(\varphi_2)}\mathbb{P}_\mu\left(T<\tau_\partial\right).$$

We deduce that

$$\mathbb{E}_\mu\left(\theta^{-T_K\wedge T}\mathbb{1}_{T<\tau_\partial}\right)\leq\left(C\frac{\mu(\varphi_1)}{\mu(\varphi_2)}+\theta^{-n_\mu}\right)\mathbb{P}_\mu\left(T<\tau_\partial\right).$$

Since $\theta^{-n_\mu}\leq\frac{\theta^{-(n_5(D_\theta)+1)}\mu(\varphi_1)}{D_\theta\mu(\varphi_2)}$, we have proved Lemma 9.4.

9.7 Conclusion of the proof of (2.1) for the norm $\|\cdot\|_{TV(\varphi_1)}$

For all $n\geq 1$, we introduce the linear operator on $L^\infty(\varphi_1)$, defined for all $h\in L^\infty(\varphi_1)$ as

$$R_nh(x)=\mathbb{E}_x\left(h(X_n)\mathbb{1}_{T_K\leq n<\tau_\partial}\right),\quad\forall x\in E. \tag{9.19}$$

Note that this operator is well-defined since $|R_nh(x)|\leq\|h\|_{L^\infty(\varphi_1)}P_n\varphi_1(x)<\infty$. We first give some properties of R_n, which can be seen as a bounded approximation of P_n in $L^\infty(\varphi_1)$.

Lemma 9.12. *We have*

$$\bar{R}:=\sup_{n\geq 1}\sup_{x\in E}R_n\varphi_1(x)<\infty,$$

and for all $n\geq 1$ *and* $x\in E$*,*

$$0\leq P_n\varphi_1(x)-R_n\varphi_1(x)\leq\theta_1^n\varphi_1(x).$$

Proof. Using Markov's property,

$$\begin{aligned}R_n\varphi_1(x)&=\sum_{k\leq n}\mathbb{E}_x[\mathbb{1}_{T_K=k}P_{n-k}\varphi_1(X_k)]\\&\leq\sup_{y\in K,\ k\geq 0}P_k\varphi_1(y)\,\mathbb{P}_x(T_K\leq n)\\&\leq\sup_{y\in K,\ k\geq 0}\frac{P_k\varphi_1(y)}{P_k\varphi_2(y)}\leq D_1\vee\sup_{y\in K}\frac{\varphi_1(y)}{\varphi_2(y)}<+\infty\end{aligned}$$

by Lemma 9.6. This proves the first inequality. For the second one, we observe that for all $x\in E$,

$$P_n\varphi_1(x)-R_n\varphi_1(x)=\mathbb{E}_x(\varphi_1(X_n)\mathbb{1}_{n<T_K})\leq\theta_1^n\varphi_1(x)$$

by Lemma 9.5. □

We fix $1 \leq k \leq n$, h such that $|h| \leq \varphi_1$ and μ such that $\mu(\varphi_1)/\mu(\varphi_2) \leq D_\theta$, where $\theta = \frac{1+\theta_1/\theta_2}{2}$ and D_θ is from Lemma 9.6. The inequality (2.1) with $\|\cdot\|_{TV}$ in place of $\|\cdot\|_{TV(\varphi_1)}$ and Lemma 9.12 entail

$$\left|\frac{\mu P_{n-k} R_k h}{\mu P_{n-k}\mathbb{1}_E} - \nu_{QSD}(R_k h)\right| \leq C\alpha^{n-k}\frac{\mu(\varphi_1)}{\mu(\varphi_2)}\sup_{x\in E}|R_k h(x)| \leq C D_\theta \bar{R}\alpha^{n-k}.$$

The second inequality of Lemma 9.12 implies

$$|\nu_{QSD}[(P_k - R_k)h]| \leq \theta_1^k \nu_{QSD}(\varphi_1)$$

and, by Lemma 9.6,

$$\frac{\mu P_{n-k}(P_k - R_k)h}{\mu P_{n-k}\mathbb{1}_E} \leq \theta_1^k \frac{\mu P_{n-k}\varphi_1}{\mu P_{n-k}\varphi_2} \leq \theta_1^k \left(D_\theta \vee \frac{\mu(\varphi_1)}{\mu(\varphi_2)}\right) = \theta_1^k D_\theta.$$

Combining the last three inequalities and recalling that $\nu_{QSD} P_k h = \theta_0^k \nu_{QSD}(h)$, we obtain that, for some constant $C > 0$,

$$\left|\frac{\mu P_n h}{\theta_0^k \mu P_{n-k}\mathbb{1}_E} - \nu_{QSD}(h)\right| \leq C\left(\alpha^{n-k}\theta_0^{-k} + (\theta_1/\theta_0)^k\right).$$

Applying the last inequality to $h = \mathbb{1}_E$, we obtain

$$\left|\frac{1}{\theta_0^k \mu P_{n-k}\mathbb{1}_E} - \frac{1}{\mu P_n \mathbb{1}_E}\right| \leq \frac{C(\alpha^{n-k}\theta_0^{-k} + (\theta_1/\theta_0)^k)}{\mu P_n \mathbb{1}_E}$$

so that, using Lemma 9.6,

$$\left|\frac{\mu P_n h}{\theta_0^k \mu P_{n-k}\mathbb{1}_E} - \frac{\mu P_n h}{\mu P_n \mathbb{1}_E}\right| \leq C(\alpha^{n-k}\theta_0^{-k} + (\theta_1/\theta_0)^k)\,\frac{\mu P_n \varphi_1}{\mu P_n \mathbb{1}_E} \leq C D_\theta(\alpha^{n-k}\theta_0^{-k} + (\theta_1/\theta_0)^k).$$

Hence, for some $\bar{\alpha} < 1$, for all $n \geq 0$,

$$\left|\frac{\mu P_n h}{\mu P_n \mathbb{1}_E} - \nu_{QSD}(h)\right| \leq C(\alpha^{n-k}\theta_0^{-k} + (\theta_1/\theta_0)^k) \leq C\bar{\alpha}^n.$$

Finally, if $\mu(\varphi_1)/\mu(\varphi_2) > D_\theta$, then let $T = \left\lceil \frac{\ln(D_\theta \mu(\varphi_1)/\mu(\varphi_2))}{-\ln\theta} \right\rceil$, so that $\mu P_T \varphi_1/\mu P_T \varphi_2 \leq D_\theta$ according to Lemma 9.6. We deduce from the previous inequality applied to $\mu P_T/\mu P_T \mathbb{1}_E$ that, for all $n \geq 0$,

$$\left|\frac{\mu P_{T+n} h}{\mu P_{T+n}\mathbb{1}_E} - \nu_{QSD}(h)\right| \leq C\bar{\alpha}^n \leq C\bar{\alpha}^n \theta^{T-1}\frac{\mu(\varphi_1)}{\mu(\varphi_2)}$$

while, using again Lemma 9.6, we obtain, for all $n \in \{0, T-1\}$,

$$\left|\frac{\mu P_n h}{\mu P_n \mathbb{1}_E} - \nu_{QSD}(h)\right| \leq D_\theta \vee \left(\theta^n \frac{\mu(\varphi_1)}{\mu(\varphi_2)}\right) + \nu_{QSD}(\varphi_1) \leq C\theta^n \frac{\mu(\varphi_1)}{\mu(\varphi_2)}.$$

The last two inequalities conclude the proof of (2.1) with $\alpha = \bar{\alpha} \vee \theta$ and hence of Theorem 2.1.

9.8 The case where $\mathbb{P}_x(n < \tau_\partial) = 0$ for some $x \in E$ and $n \geq 1$

In this section, we assume that X satisfies assumption (E), but we do not assume anymore that $\mathbb{P}_x(n < \tau_\partial) > 0$ for all $x \in E$ and all $n \geq 1$. We introduce $\bar{E} = \{x \in E,\ \mathbb{P}_x(n < \tau_\partial) > 0\ \forall n \geq 0\}$ and $\bar{\bar{E}} = E \setminus \bar{E}$. One immediately deduces from (E2) that, for all $x \in \bar{\bar{E}}$ and all $n \geq 0$, $\varphi_2(x) = 0$ and $\mathbb{P}_x(X_n \in K) = 0$, and hence that $\delta_x P_n \varphi_1 \leq \theta_1^n \varphi_1(x)$

by Lemma 9.5. In addition, one easily checks that the semi-group P restricted to $\bar{E}\cup\{\partial\}$ still satisfies assumption (E), and in particular (2.1) applies.

Let μ be a probability measure on E such that $\mu(\varphi_2)>0$ and $\mu(\varphi_1)<+\infty$. Then, for all $n\geq 0$ and all $|h|\leq\varphi_1$,

$$\begin{aligned}|\mu P_n h-\nu_{QSD}(h)\mu P_n\mathbb{1}_E| &\leq \left|\mu P_n h-\mu_{|\bar{E}}P_n h\right|+\left|\mu_{|\bar{E}}P_n h-\nu_{QSD}(h)\mu_{|\bar{E}}P_n\mathbb{1}_E\right|\\ &\quad+\nu_{QSD}(\varphi_1)\left|\mu_{|\bar{E}}P_n\mathbb{1}_E-\mu P_n\mathbb{1}_E\right|.\end{aligned}$$

Each term can be bounded as follows:

$$\begin{aligned}&\left|\mu P_n h-\mu_{|\bar{E}}P_n h\right|\leq\mu_{|\bar{\bar{E}}}P_n\varphi_1\leq\theta_1^n\mu\varphi_1,\\ &\left|\mu_{|\bar{E}}P_n h-\nu_{QSD}(h)\mu_{|\bar{E}}P_n\mathbb{1}_E\right|\leq C\alpha^n\frac{\mu_{|\bar{E}}(\varphi_1)}{\mu_{|\bar{E}}(\varphi_2)}\mu_{|\bar{E}}P_n\mathbb{1}_E\leq C\alpha^n\frac{\mu(\varphi_1)}{\mu(\varphi_2)}\mu P_n\mathbb{1}_E,\\ &\nu_{QSD}(\varphi_1)\left|\mu_{|\bar{E}}P_n\mathbb{1}_E-\mu P_n\mathbb{1}_E\right|\leq\nu_{QSD}(\varphi_1)\mu_{|\bar{\bar{E}}}P_n\varphi_1\leq\nu_{QSD}(\varphi_1)\theta_1^n\mu\varphi_1.\end{aligned}$$

Since $\mu P_n\mathbb{1}_E\geq\theta_2^n\mu(\varphi_2)$, we deduce that

$$\left|\frac{\mu P_n h}{\mu P_n\mathbb{1}_E}-\nu_{QSD}(h)\right|\leq\left((\theta_1/\theta_2)^n+\nu_{QSD}(\varphi_1)(\theta_1/\theta_2)^n+C\alpha^n\right)\frac{\mu(\varphi_1)}{\mu(\varphi_2)}.$$

This concludes the proof of (2.1) in the general case.

10 Proof of the other results of Section 2

The previous section ensures the existence of a quasi-stationary distribution ν_{QSD} such that $\nu_{QSD}(\varphi_1)<+\infty$ and $\nu_{QSD}(K)>0$. Denoting by θ_0 its associated decay parameter, we observe that $\theta_2\leq\theta_0$, since Lemma 9.5 entails that, for all $n\geq 1$,

$$\theta_0^n=\mathbb{P}_{\nu_{QSD}}(n<\tau_\partial)\geq\nu_{QSD}(K)\inf_{y\in K}\mathbb{P}_y(n<\tau_\partial)\geq\nu_{QSD}(K)\theta_2^n\inf_{y\in K}\varphi_2(y).$$

We begin to prove Theorem 2.3 in Section 10.1, except for the exponential convergence in $L^\infty(\varphi_1)$. We then prove Theorem 2.6 in Section 10.2. In Section 10.3, we conclude the proof of Theorem 2.3 and prove Corollary 2.7. We prove Corollary 2.11 in Subsection 10.5.

10.1 Proof of the existence of the eigenfunction η

In this section, we show that the limit (2.2) is well defined pointwise, $\nu_{QSD}(\eta)=1$, $P_1\eta=\theta_0\eta$, η is lower bounded away from 0 on K and $\eta\in L^\infty(\varphi_1^{\log(1/\theta_0)/\log(1/\theta_1)})$.

For all $n\geq 0$ and $x\in E\cup\{\partial\}$, let us denote

$$\eta_n(x)=\theta_0^{-n}\mathbb{P}_x(n<\tau_\partial)=\frac{\mathbb{P}_x(n<\tau_\partial)}{\mathbb{P}_{\nu_{QSD}}(n<\tau_\partial)}.$$

By Lemma 9.9, for all $x\in E$,

$$\begin{aligned}\eta_n(x)&\leq C\theta_0^{-n}\inf_{y\in K}\mathbb{P}_y(n<\tau_\partial)\varphi_1(x)\\ &\leq\frac{C}{\nu_{QSD}(K)}\theta_0^{-n}\mathbb{P}_{\nu_{QSD}}(n<\tau_\partial)\varphi_1(x)=\frac{C\varphi_1(x)}{\nu_{QSD}(K)}.\end{aligned}\tag{10.1}$$

This implies that the sequence $(\eta_n)_{n\geq 0}$ is uniformly bounded in $L^\infty(\varphi_1)$.

For all probability measure μ on E and for all $n,m\geq 0$, by Markov's property,

$$\mu(\eta_{n+m})=\mu(\eta_n)\mathbb{E}_\mu\left[\theta_0^{-m}\mathbb{P}_{X_n}(m<\tau_\partial)\mid n<\tau_\partial\right].$$

Hence, by Theorem 2.1, for all μ such that $\mu(\varphi_2) > 0$ and $\mu(\varphi_1) < +\infty$,

$$\begin{aligned}|\mu(\eta_{n+m}) - \mu(\eta_n)| &= \mu(\eta_n)\,|\mathbb{E}_\mu(\eta_m(X_n) \mid n < \tau_\partial) - 1| \\ &= \mu(\eta_n)\,|\mathbb{E}_\mu(\eta_m(X_n) \mid n < \tau_\partial) - \nu_{QSD}(\eta_m)| \\ &\leq C\mu(\varphi_1)\alpha^n \frac{\mu(\varphi_1)}{\mu(\varphi_2)}.\end{aligned}$$

For any $x \in E$, applying this result to $\mu = (\delta_x + \nu_{QSD})/2$, we deduce that

$$|\eta_{n+m}(x) - \eta_n(x)| \leq C\varphi_1(x)^2\alpha^n.$$

This shows that $(\eta_n(x))_{n\geq 0}$ is a Cauchy sequence and hence that, for all $x \in E$,

$$\eta(x) = \lim_{n\to+\infty} \theta_0^{-n}\mathbb{P}_x(n < \tau_\partial)$$

and, by (10.1), that $\eta \in L^\infty(\varphi_1)$.

Then, since η_n is bounded in $L^\infty(\varphi_1)$, we deduce by dominated convergence that $\nu_{QSD}(\eta) = 1$ and that, for all $x \in E$,

$$\delta_x P_1 \eta = \lim_{n\to+\infty} \delta_x P_1 \eta_n = \lim_{n\to+\infty} \theta_0 \eta_{n+1}(x) = \theta_0 \eta(x). \tag{10.2}$$

The fact that η is lower bounded away from 0 on K is an immediate consequence of Lemma 9.9 (integrating (9.6) with respect to $\nu_{QSD}(\mathrm{d}x)$) and the fact that $\nu_{QSD}(\varphi_1) < +\infty$.

It only remains to prove that $\eta \in L^\infty\left(\varphi_1^{\log\theta_0/\log\theta_1}\right)$. To prove this, we use the operator R_n introduced in (9.19). By Lemma 9.12 and using the fact that $\eta \in L^\infty(\varphi_1)$, for all $x \in E$,

$$\begin{aligned}\eta(x) = \theta_0^{-n} P_n \eta(x) &\leq C\theta_0^{-n}\left[R_n\varphi_1(x) + (P_n - R_n)\varphi_1(x))\right] \\ &\leq C\bar{R}\theta_0^{-n} + C\left(\frac{\theta_1}{\theta_0}\right)^n \varphi_1(x).\end{aligned}$$

Applying this inequality for $n = \lfloor -\log\varphi_1(x)/\log\theta_1 \rfloor$, we deduce

$$\eta(x) \leq C\exp\left(\frac{\log\varphi_1(x)}{\log\theta_1}\,\log\theta_0\right) \leq C\varphi_1(x)^{\log\theta_0/\log\theta_1},$$

which concludes the proof.

10.2 Proof of Theorem 2.6

We start with Point (i). We introduce $\Gamma_n = \mathbb{1}_{n<\tau_\partial}$ and define for all $x \in E'$ and $n \geq 0$ the probability measure

$$Q_n^{\Gamma,x} = \frac{\Gamma_n}{\mathbb{E}_x(\Gamma_n)}\mathbb{P}_x,$$

so that the Q-process exists if and only if $Q_n^{\Gamma,x}$ admits a proper limit when $n \to \infty$. For all $0 \leq k \leq n$, we have by the Markov property

$$\frac{\mathbb{E}_x(\Gamma_n \mid \mathcal{F}_k)}{\mathbb{E}_x(\Gamma_n)} = \frac{\mathbb{1}_{k<\tau_\partial}\mathbb{P}_{X_k}(n-k < \tau_\partial)}{\mathbb{P}_x(n < \tau_\partial)}.$$

By the pointwise convergence in (2.2) (proved in Subsection 10.1), this converges almost surely as $n \to +\infty$ to

$$M_k := \mathbb{1}_{k<\tau_\partial}\theta_0^{-k}\frac{\eta(X_k)}{\eta(x)} = \theta_0^{-k}\frac{\eta(X_k)}{\eta(x)},$$

and $\mathbb{E}_x(M_k)=\theta_0^{-k}\frac{P_k\eta(x)}{\eta(x)}=1$. These two properties allow to apply the penalization's theorem of Roynette, Vallois and Yor [94, Theorem 2.1], which implies that M is a martingale under $\mathbb{P}_x$ and that $Q_n^{\Gamma,x}(A)$ converges to $\mathbb{E}_x\left(M_k\mathbb{1}_A\right)$ for all $A\in\mathcal{F}_k$ when $n\to\infty$. This means that $\mathbb{Q}_x$ is well defined and

$$\left.\frac{\mathrm{d}\mathbb{Q}_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_k}=M_k.$$

Note that the fact that $\eta(x)=0$ for all $x\in E\setminus E'$ implies that $(X_n,n\geq 0)$ is E'-valued $\mathbb{Q}_x$-almost surely for all $x\in E'$. The fact that X is Markov under $(\mathbb{Q}_x)_{x\in E'}$ and Point (ii) can be easily deduced from the last formula (see e.g. [20, Section 6.1]).

It remains to prove Point (iii). We define the function $\psi=\varphi_1/\eta\times\|\eta\|_{L^\infty(\varphi_1)}$ on E'. Note that, since $\eta\in L^\infty(\varphi_1)$, ψ is uniformly lower bounded. Moreover, for all $x\in E'$,

$$\widetilde{P}_1\psi(x)=\frac{\theta_0^{-1}\|\eta\|_{L^\infty(\varphi_1)}}{\eta(x)}P_1\varphi_1(x)\leq\frac{\theta_1}{\theta_0}\psi(x)+\frac{c_2\|\eta\|_{L^\infty(\varphi_1)}}{\theta_0\eta(x)}\mathbb{1}_K(x)\leq\widetilde{\theta}\psi(x)+\widetilde{c},$$

where $\widetilde{\theta}=\theta_1/\theta_0$ and

$$\widetilde{c}=\frac{c_2\|\eta\|_{L^\infty(\varphi_1)}}{\theta_0\inf_K\eta}.$$

Hence, for all $x\in E$ and all $n\geq 1$,

$$\widetilde{P}_n\psi(x)\leq\widetilde{\theta}\widetilde{P}_{n-1}\psi(x)+\widetilde{c}\leq\ldots\leq\widetilde{\theta}^n\psi(x)+\frac{\widetilde{c}}{1-\widetilde{\theta}}.\tag{10.3}$$

Using Lemma 9.5, we have that, for all $x\in E'$,

$$\mathbb{Q}_x(T_K>n)=\mathbb{E}_x\left(\theta_0^{-n}\frac{\varphi_1(X_n)}{\eta(X_n)}\mathbb{1}_{T_K>n}\mathbb{1}_{X_n\in E'}\right)\leq\widetilde{\theta}^n\psi(x)\tag{10.4}$$

Now, choosing m_K large enough so that $\sup_{x\in K}\widetilde{\theta}^{m_K}[\sup_K\psi+\tilde{c}/(1-\tilde{\theta})]\leq 1/2$, we deduce that, for all $x\in K$ and all $n_0\geq 0$,

$$\mathbb{Q}_x(\exists n\in\{n_0,\ldots,n_0+m_K\},\ X_n\in K)\geq 1-\widetilde{\theta}^{m_K}\widetilde{P}_{n_0}\psi(x)\geq 1/2.\tag{10.5}$$

Now, let $n_K\geq 1$ be such that $\inf_{x\in K}\mathbb{P}_x(X_n\in K)>0$ for all $n\geq n_K$ (such a n_K exists by (E1) and (E4), see the proof of Lemma 9.7) and let

$$a:=\inf_{n\in\{n_K,\ldots,n_K+m_K\}}\inf_{x\in K}\mathbb{P}_x(X_n\in K)>0.$$

so that, for all $x\in K$, all $n\in\{n_K,\ldots,n_K+m_K\}$ and all $A\subset E$ measurable,

$$\begin{aligned}\mathbb{Q}_x(X_{n+n_1}\in A)\geq\mathbb{Q}_x(X_n\in K,X_{n+n_1}\in A)&=\frac{\theta_0^{-n-n_1}}{\eta(x)}\mathbb{E}_x(\mathbb{1}_{X_n\in K}\mathbb{E}_{X_n}(\eta(X_{n_1})\mathbb{1}_{X_{n_1}\in A}))\\&\geq\frac{\theta_0^{-n-n_1}}{\eta(x)}ac_1\nu(\eta\mathbb{1}_A)=\frac{\theta_0^{-n-n_1}\nu(\eta)}{\eta(x)}ac_1\nu_\eta(A)\geq\frac{ac_1}{c_3}\nu_\eta(A),\end{aligned}$$

where we used (E1) and (E3) and defined $\nu_\eta(\mathrm{d}x):=\frac{\eta(x)\nu(\mathrm{d}x)}{\nu(\eta)}$. We deduce from the last inequality and (10.5) that, for all $n_0\geq 0$,

$$\mathbb{Q}_x(X_{n_0+n_1+n_K+m_K}\in\cdot)\geq\sum_{n=n_0}^{n_0+m_K}\mathbb{Q}_x\left[\mathbb{1}_{X_n\in K}\mathbb{Q}_{X_n}(X_{n_0+n_1+n_K+m_K-n}\in\cdot)\right]\geq\frac{ac_1}{2c_3}\nu_\eta.$$

Hence, for all $n\geq n_K+m_K+n_1$,

$$\mathbb{Q}_x(X_n\in\cdot)\geq\frac{ac_1}{2c_3}\nu_\eta,\quad\forall x\in K.$$

For all $x \in E'$, setting $k_x = \lceil \frac{\log(2\psi(x))}{-\log \widetilde{\theta}} \rceil$, it follows from (10.4) that $\mathbb{Q}_x(T_K \leq k_x) \geq \frac{1}{2}$, and hence

$$\mathbb{Q}_x(X_{k_x+n_K+m_K+n_1} \in \cdot) \geq \frac{ac_1}{4c_3}\nu_\eta.$$

In particular, for all $R > 0$, setting $k_R = \lceil \frac{\log(2R)}{-\log \widetilde{\theta}} \rceil + n_K + m_k + n_1$, we have, for all $x, y \in E'$ such that $\psi(x) + \psi(y) < R$,

$$\|\delta_x P_{k_R} - \delta_y P_{k_R}\|_{TV} \leq 1 - \frac{ac_1}{4c_3}. \tag{10.6}$$

By [57, Thm 3.9], together with (10.3), the last assertion implies that there exist constants $C > 0$ and $\widetilde{\alpha}_1 \in (0,1)$ such that, for all real function h on E' such that $|\!|\!| h |\!|\!| < \infty$,

$$\left|\!\left|\!\left| \widetilde{P}_n h \right|\!\right|\!\right| \leq C \widetilde{\alpha}_1^n |\!|\!| h |\!|\!|, \tag{10.7}$$

where

$$|\!|\!| h |\!|\!| = \sup_{x,y \in E'} \frac{|h(x) - h(y)|}{2 + \psi(x) + \psi(y)}.$$

This implies (2.4). In particular, for all $x \in E'$,

$$\|\delta_x \widetilde{P}_n - \beta\|_{TV} \xrightarrow[n \to +\infty]{} 0.$$

Hence, (2.5) is a consequence of Lebesgue's dominated convergence theorem. This ends the proof of Theorem 2.6.

Remark 10.1. As noted in [57, Remark 3.10], it is possible to obtain explicit constants $\widetilde{C}$ and $\widetilde{\alpha}_1$ in (10.7) from the parameters in (10.3) and (10.6) (note that a slight modification of the proof of Lemma 9.9 entails that one can actually take $\widetilde{c} = \frac{1/\theta_1}{1-\theta_1/\theta_0} \leq \frac{1/\theta_1}{1-\theta_1/\theta_2}$). More precisely, setting $\alpha = \frac{ac_1}{4c_3}$ and $K = \widetilde{c}/(1-\widetilde{\theta})$ and $\gamma = \widetilde{\theta}$, then taking any $\alpha_0 \in (0, \alpha)$ and $R > \frac{2K}{1-\gamma}$, and setting $b = \frac{2\alpha_0}{\gamma R + 2K}$,

$$\alpha_R = (1 - \alpha + \alpha_0) \vee \frac{2 + b\gamma R + b\gamma K}{2 + bR} \in (0,1),$$

and $C_R = \frac{2/b + 1 + K + K/(1-\gamma)}{\alpha_R}$, we obtain that, for all $f \in L^\infty(\varphi_1/\eta)$,

$$\left| \widetilde{P}_n f - \beta(f) \right| \leq C_R \alpha_R^{n/k_R} \|f\|_{L^\infty(\varphi_1/\eta)}.$$

10.3 Proof of Corollary 2.7 and end of the proof of Theorem 2.3

Let $|g| \leq \varphi_1$ and set $h = g/\eta$. Then (2.4) entails that, for all $x \in E'$,

$$\left| \theta_0^{-n} \mathbb{E}_x(g(X_n) \mathbb{1}_{X_n \in E'}) - \eta(x) \nu_{QSD}(g \mathbb{1}_{E'}) \right| \leq C\, \bar{\alpha}^n \varphi_1(x).$$

In what follows, we set $\nu' = \nu_{QSD}(\cdot \cap E')$ and, for all $k \geq 1$,

$$g_k(x) = \mathbb{1}_{x \in E'} \mathbb{E}_x \left(\mathbb{1}_{X_1 \notin E'} g(X_k) \mathbb{1}_{k < \tau_\partial} \right).$$

Note that, defining $E'' := E \setminus E'$, $E'' \cup \{\partial\}$ is an absorbing set. Since $K \subset E'$, we thus have

$$g_k(x) \leq \mathbb{1}_{x \in E'}\, \mathbb{E}_x \left(\mathbb{1}_{k < T_K \wedge \tau_\partial} \varphi_1(X_k) \right) \leq \theta_1^k \varphi_1(x).$$

We also define the measure ν'' on E'' by

$$\nu'' = \sum_{\ell \geq 1} \theta_0^{-\ell} \mathbb{E}_{\nu'} \left(\mathbb{1}_{X_1 \notin E'} \mathbb{1}_{X_\ell \in \cdot} \right) = \sum_{\ell \geq 1} \theta_0^{-\ell} \nu'(g_\ell).$$

Hence we have for all $n \geq 1$ and all $x \in E'$

$$\begin{aligned}\left|\theta_0^{-n}\mathbb{E}_x(g(X_n)\mathbb{1}_{X_n\in E''}) - \eta(x)\nu''(g)\right| &\leq \sum_{\ell=1}^{n}\left|\theta_0^{-n}\mathbb{E}_x(g_\ell(X_{n-\ell})) - \theta_0^{-\ell}\eta(x)\nu'(g_\ell)\right| \\ &\leq \sum_{\ell=1}^{n}\theta_0^{-\ell}\left|\theta_0^{-(n-\ell)}\mathbb{E}_x(g_\ell(X_{n-\ell})) - \eta(x)\nu'(g_\ell)\right| \\ &\leq \sum_{\ell=1}^{n}\theta_0^{-\ell}\varphi_1(x)C\bar{\alpha}^{n-\ell}\|g_\ell\|_{L^\infty(\varphi_1)} \\ &\leq C\varphi_1(x)\sum_{\ell=1}^{n}\left(\frac{\theta_1}{\theta_0}\right)^\ell\bar{\alpha}^{n-\ell}.\end{aligned}$$

We thus proved that, setting $\nu_0 = \nu' + \nu''$, and up to a change in the constants C and $\bar{\alpha}$, for all $x \in E'$,

$$\left|\theta_0^{-n}\mathbb{E}_x(g(X_n)\mathbb{1}_{n<\tau_\partial}) - \eta(x)\nu_0(g)\right| \leq C\,\bar{\alpha}^n\varphi_1(x).$$

Now, by Lemma 9.5, for all $x \in''$, $|\mathbb{E}_x g(X_n)\mathbb{1}_{n<\tau_\partial}| \leq \theta_1^n\varphi_1(x)$. This, we have proved that, up to a change in $\bar{\alpha}$, for all $x \in E$,

$$\left|\theta_0^{-n}\mathbb{E}_x(g(X_n)) - \eta(x)\nu_0(g)\right| \leq C\,\bar{\alpha}^n\varphi_1(x). \tag{10.8}$$

Integrating with respect to ν_{QSD} shows that $\nu_0 = \nu_{QSD}$. Thus, we have proved (2.6).

To conclude, we can now end the proof of Theorem 2.3. Indeed, taking $g \equiv 1$ immediately entails that the convergence (2.2) is geometric in $L^\infty(\varphi_1)$.

10.4 Proof of Corollary 2.10

If $\eta(x) > 0$, it follows from Corollary 2.7 and the fact that $\nu_{QSD}(\varphi_2) > 0$ that there exists $k \geq 0$ such that $\delta_x P_k\varphi_2 > 0$. Hence $E' \subset \{x \in E : \exists k \geq 0,\ P_k\varphi_2(x) > 0\}$. Conversely, if $P_k\varphi_2(x) > 0$, we apply Theorem 2.1 to $\mu = \frac{\delta_x P_k}{\delta_x P_k \mathbb{1}_E}$. Since $\nu_{QSD}(\eta) > 0$, there exists $n \geq 0$ such that $0 < \frac{\mu P_n \eta}{\mu P_n \mathbb{1}_E} = \frac{\delta_x P_{n+k}\eta}{\delta_x P_{n+k}\mathbb{1}_E} = \frac{\theta_0^{n+k}}{\delta_x P_{n+k}\mathbb{1}_E}\eta(x)$. Hence we have proved that $E' = \{x \in E : \exists k \geq 0,\ P_k\varphi_2(x) > 0\}$.

The fact that any μ such that $\mu(E') > 0$ and $\mu(\varphi_1^{1/p}) < +\infty$ for some $p < \log\theta_1/\log\theta_2$ belongs to the domain of attraction of ν_{QSD} follows from Remark 2.2 and Corollary 2.7.

In the case where φ_1 is bounded, the domain of atttraction contains all measures μ such that $\mu(E') > 0$. If $\mu(\eta) = 0$, then $\mu P_k\eta = 0$ for all $k \geq 0$, which means that μP_k gives no mass to E'. Hence the convergence of conditional distributions to ν_{QSD} cannot hold true. The uniqueness of the quasi-stationary distribution follows immediately.

10.5 Proof of Corollary 2.11

Applying (2.5) with $\mu(\eta\cdot)/\mu(\eta)$ instead of μ and recalling that $\mu(E \setminus E') = 0$, we obtain

$$\sup_{f:E'\to\mathbb{R},\,\|f\|_\infty\leq 1}\left|\theta_0^{-n}\frac{\mu P_n(\eta f)}{\mu(\eta)} - \beta(f)\right| \xrightarrow[n\to+\infty]{} 0.$$

This entails the convergence result (2.7).

Assume from now on that η is positive on E. Let ν be a quasi-stationary distribution on E such that $\nu(\eta) < +\infty$ and denote by $\bar{\theta}_0 \in (0, 1]$ the associated decay parameter, such that $\mathbb{P}_\nu(X_n \in \cdot) = \bar{\theta}_0^n\nu$ for all $n \geq 0$. Then, according to (2.7), we have, for all $g \in L^\infty(\eta)$,

$$\left|\theta_0^{-n}\bar{\theta}_0^n\nu(g) - \nu(\eta)\nu_{QSD}(g)\right| \xrightarrow[n\to+\infty]{} 0.$$

This entails that $\bar{\theta}_0 = \theta_0$ and that ν is proportional to ν_{QSD}. Since they both are probability measures, we deduce that $\nu = \nu_{QSD}$, which concludes the proof of the second claim of Corollary 2.11.

Finally, assuming that η is lower bounded away from 0 on E, we deduce from (2.7) with $g \equiv 1$ that, for all probability measure μ on E such that $\mu(\eta) < +\infty$,

$$\theta_0^{-n} \mu P^n \mathbb{1}_E \xrightarrow[n \to +\infty]{} \mu(\eta) > 0.$$

This and (2.7) imply that

$$\sup_{g : E \to \mathbb{R},\, \|g\|_{L^\infty(\eta)} \leq 1} \left| \mathbb{E}_\mu(g(X_n) \mid n < \tau_\partial) - \nu_{QSD}(g) \right| \xrightarrow[n \to +\infty]{} 0,$$

hence (2.8) holds true and the proof of Corollary 2.11 is completed.

11 Proof of the results of Section 3

In this section are proved Lemma 3.1 in Subsection 11.1, Lemma 3.2 in Subsection 11.2, Proposition 3.3 in Subsection 11.3 and Lemma 3.4 in Subsection 11.4. Then we prove Theorem 3.5 in Subsection 11.5, Lemma 3.6 in Subsection 11.6 and Proposition 3.9 in Subsection 11.7.

11.1 Proof of Lemma 3.1

The function φ_2 defined in the statement satisfies, for all $x \in E$, $\varphi_2(x) \in [0,1]$ and, for all $x \in K$, $\varphi_2(x) \geq \frac{\theta_2^{-1} - 1}{\theta_2^{-\ell} - 1} > 0$. Moreover, we have, for all $x \in E$,

$$P_1 \varphi_2(x) = \theta_2 \varphi_2(x) - \frac{\theta_2^{-1} - 1}{\theta_2^{-\ell} - 1} \left(\theta_2 \mathbb{1}_K(x) - \theta_2^{-\ell+1} P_\ell \mathbb{1}_K(x) \right) \geq \theta_2 \varphi_2(x)$$

since ℓ is chosen such that $\theta_2^{-\ell} P_\ell \mathbb{1}_K(x) \geq \mathbb{1}_K(x)$ for all $x \in E$.

Our assumption also implies that there exists n_0 such that, for all $n \geq n_0$, $\theta_2^{-n} \inf_{x \in K} \mathbb{P}_x(X_n \in K) \geq 1$. Choosing $n_4(x) = n_0$ for all $x \in K$ entails (E4), which concludes the proof of Lemma 3.1.

11.2 Proof of Lemma 3.2

Assume that

$$\mathbb{E}_x \left(\theta_1^{-T_K \wedge \tau_\partial} \right) < +\infty \ \forall x \in E \text{ and } \sup_{y \in K} \mathbb{E}_y \left(\mathbb{E}_{X_1} \left(\theta_1^{-T_K \wedge \tau_\partial} \right) \mathbb{1}_{1 < \tau_\partial} \right) < +\infty$$

and set $\varphi_1(x) = \mathbb{E}_x \left(\theta_1^{-T_K \wedge \lceil \tau_\partial \rceil} \right)$ for all $x \in E$. Then, for all $x \in E \setminus K$, using Markov's property at time 1,

$$P_1 \varphi_1(x) = \mathbb{E}_x \left(\mathbb{E}_{X_1} \left(\theta_1^{-T_K \wedge \lceil \tau_\partial \rceil} \right) \mathbb{1}_{1 < \tau_\partial} \right) \leq \mathbb{E}_x \left(\theta_1^{-(T_K \wedge \lceil \tau_\partial \rceil - 1)} \right) = \theta_1 \varphi_1(x).$$

Moreover, for all $x \in K$, $P_1 \varphi_1(x) \leq \theta_1^{-1} \sup_{y \in K} \mathbb{E}_y \left(\mathbb{E}_{X_1} \left(\theta_1^{-T_K \wedge \tau_\partial} \right) \mathbb{1}_{1 < \tau_\partial} \right)$, and hence the first part of the lemma is proved.

Assume now that there exist two constants $C > 0$, $\theta_1 > 0$ and a function $\varphi_1 : E \to [1, +\infty)$ such that $\sup_K \varphi_1 < +\infty$ and $P_1 \varphi_1 \leq \theta_1 \varphi_1 + C \mathbb{1}_K$. Then, for all $n \geq 1$ and all $x \in E \setminus K$,

$$\mathbb{E}_x \left(\varphi_1(X_n) \mathbb{1}_{n < T_K \wedge \tau_\partial} \right) \leq \theta_1^n \varphi_1(x).$$

Thus we deduce that, for all $x \in E$,

$$\mathbb{P}_x\left(n < T_K \wedge \tau_\partial\right) \le \theta_1^n \varphi_1(x).$$

In particular, for all $\theta > \theta_1$ and all $x \in E$,

$$\mathbb{E}_x\left(\theta^{-T_K \wedge \tau_\partial}\right) \le \frac{1}{\theta - \theta_1}\,\varphi_1(x) < +\infty.$$

We also deduce that

$$\sup_{x \in K} \mathbb{E}_x\left(\mathbb{E}_{X_1}\left(\theta^{-T_K \wedge \tau_\partial}\right)\right) \le \frac{1}{\theta - \theta_1} \sup_{x \in K} P_1 \varphi_1(x) < +\infty.$$

This concludes the proof of Lemma 3.2.

11.3 Proof of Proposition 3.3

Condition (E4) implies that there exists $x_0 \in E$ such that $\mathbb{P}_{x_0}(X_{n_0} \in K) > 0$. We then deduce from our assumption (3.2) that Condition (E1) is satisfied with the probability measure ν on K defined by

$$\nu(\cdot) = \frac{\mathbb{P}_{x_0}(X_{n_0} \in \cdot \cap K)}{\mathbb{P}_{x_0}(X_{n_0} \in K)}$$

and the constants $c_1 = \mathbb{P}_{x_0}(X_{n_0} \in K)/C > 0$ and $n_1 = m_0$.

Let us now check Condition (E3) and the last part of Proposition 3.3. We define $T_K^{(n_0)} = \inf\{n \ge n_0 \text{ s.t. } X_n \in K\}$. Lemma 9.5 (which only makes use of Condition (E2)) implies that, for all $x \in E$, $\mathbb{P}_x(n < T_K \wedge \tau_\partial) \le \theta_1^n \varphi_1(x)$. Hence, for all $x \in E$ and all $n \ge n_0$,

$$\begin{aligned}
\mathbb{P}_x(n < \tau_\partial \wedge T_K^{(n_0)}) &= \mathbb{E}_x\left(\mathbb{1}_{n_0 < \tau_\partial} \mathbb{P}_{X_{n_0}}(n - n_0 < \tau_\partial \wedge T_K)\right) \\
&\le \theta_1^{n-n_0}\, \mathbb{E}_x\left(\mathbb{1}_{n_0 < \tau_\partial} \varphi_1(X_{n_0})\right) \\
&\le (\theta_1 + c_2)^{n_0} \theta_1^{n-n_0}\, \varphi_1(x).
\end{aligned}$$

Since $\varphi_1 \ge 1$, we also have $\mathbb{P}_x(n < \tau_\partial) \le C\theta_1^n \varphi_1(x)$ for all $n < n_0$. Hence we proved that, for all $x \in E$ and $n \ge 0$,

$$\mathbb{P}_x(n < \tau_\partial \wedge T_K^{(n_0)}) \le C\theta_1^n \varphi_1(x). \tag{11.1}$$

Therefore, for some constant $C > 0$,

$$\begin{aligned}
\mathbb{P}_x(n < \tau_\partial) &\le \mathbb{P}_x(n < \tau_\partial \wedge T_K^{(n_0)}) + \mathbb{P}_x(T_K^{(n_0)} \le n < \tau_\partial) \\
&\le C\,\varphi_1(x)\theta_1^n + \sum_{k=n_0}^{n} \mathbb{E}_x\left(\mathbb{1}_{T_K^{(n_0)} = k} \mathbb{P}_{X_k}(n - k < \tau_\partial)\right).
\end{aligned} \tag{11.2}$$

Now, for all $x \in E$, all $y \in K$ and all $k \in \{n_0, \dots, n\}$, (3.2) and (11.1) entail

$$\begin{aligned}
\mathbb{E}_x\left(\mathbb{1}_{T_K^{(n_0)} = k} \mathbb{P}_{X_k}(n - k < \tau_\partial)\right) &\le \mathbb{E}_x\left(\mathbb{1}_{k - n_0 < T_K^{(n_0)} \wedge \tau_\partial} \mathbb{E}_{X_{k-n_0}}\left(\mathbb{1}_{X_{n_0} \in K}\, \mathbb{P}_{X_{n_0}}(n - k < \tau_\partial)\right)\right) \\
&\le \mathbb{E}_x\left(\mathbb{1}_{k - n_0 < T_K^{(n_0)} \wedge \tau_\partial} C\, \mathbb{P}_y(n + m_0 - k < \tau_\partial)\right) \\
&\le \theta_1^{k-n_0} \varphi_1(x)\, C\, \mathbb{P}_y(n - k < \tau_\partial),
\end{aligned}$$

where the constant C may change from line to line. Using Lemma 9.8, which only makes use of (E1), (E2) and (E4), there exists $n_6 \in \mathbb{Z}_+$ such that, for all $y \in K$ and for all $n, k \in \mathbb{Z}_+$ such that $n - k \ge n_6$,

$$\begin{aligned}
\mathbb{P}_y(n < \tau_\partial) &\ge \mathbb{P}_y(X_{n-k} \in K) \inf_{z \in K} \mathbb{P}_z(k < \tau_\partial) \\
&\ge \mathbb{P}_y(n - k < \tau_\partial) \inf_{T \ge n_6} \inf_{z \in K} \mathbb{P}_z(X_T \in K \mid T < \tau_\partial) \inf_{z \in K} P_k \varphi_2(z) \\
&\ge C'' \theta_2^k\, \mathbb{P}_y(n - k < \tau_\partial),
\end{aligned}$$

where $C'' > 0$. Hence,

$$\mathbb{E}_x\left(\mathbb{1}_{T_K^{(n_0)}=k}\mathbb{P}_{X_k}(n-k<\tau_\partial)\right) \leq \varphi_1(x)\left(\frac{\theta_1}{\theta_2}\right)^k \frac{\theta_1^{-n_0}C}{C''}\,\mathbb{P}_y(n<\tau_\partial).$$

Now, we deduce from (11.2) and (11.1) that, for all $x \in E$ and all $y \in K$,

$$\begin{aligned}\mathbb{P}_x(n<\tau_\partial) &\leq C\,\varphi_1(x)\left[\theta_1^n + \mathbb{P}_y(n<\tau_\partial)\sum_{k=1}^{n-n_6}\left(\frac{\theta_1}{\theta_2}\right)^k\right] + \mathbb{P}_x(T_K^{(n_0)}\wedge\tau_\partial \geq n-n_6)\\ &\leq C\,\varphi_1(x)\left[\theta_1^n + \mathbb{P}_y(n<\tau_\partial) + \theta_1^{n-n_6}\right]\\ &\leq C\,\varphi_1(x)\,\mathbb{P}_y(n<\tau_\partial)\end{aligned}$$

since $\mathbb{P}_y(n<\tau_\partial) \geq \theta_2^n \inf_K \varphi_2$. This implies (E3) since $\sup_K \varphi_1 < \infty$.

11.4 Proof of Lemma 3.4

Combining Theorem 2.3 and the fact that $\inf_K \eta > 0$, we deduce that

$$\liminf_{n\to+\infty}\ \inf_{x\in K}\theta_0^{-n}\mathbb{P}_x(n<\tau_\partial) > 0.$$

Let $\theta_2' < \theta_0$. Using Lemma 9.8,

$$\lim_{n\to+\infty}(\theta_2')^{-n}\inf_{x\in K}\mathbb{P}_x(X_n\in K) = +\infty.$$

Hence the result follows from Lemma 3.1.

11.5 Proof of Theorem 3.5

We assume that Assumption (F) is satisfied. In Subsection 11.5.1, we prove that Assumption (E) holds true for the sub-Markovian semigroup $(P_n)_{n\geq 0}$ of the absorbed Markov process $(X_{nt_2}, n\in\mathbb{Z}_+)$. In Subsection 11.5.2, we prove the existence of a quasi-stationary distribution for $(X_t)_{t\in I}$ with the claimed properties and in Subsection 11.5.3, we prove the convergence of $e^{\lambda_0 t}\mathbb{P}_x(t<\tau_\partial)$ to $\eta(x)$ for $t\in I$, $t\to+\infty$.

11.5.1 Proof of (E)

We fix $\theta_1 \in (\gamma_1^{t_2}, \gamma_2^{t_2})$ and set $\theta_2 = \gamma_2^{t_2}$. Let us first remark that the last line of Condition (F2) implies that $\gamma_2^{-t}\mathbb{P}_\nu(X_t\in L)\to+\infty$ when $t\to+\infty$. Hence, using Condition (F1), we deduce that

$$\gamma_2^{-t}\inf_{x\in L}\mathbb{P}_x(X_t\in L)\xrightarrow[t\to+\infty]{}+\infty. \tag{11.3}$$

We consider a number $n_0\in\mathbb{N}^*$ large enough so that $\gamma_2^{-t}\inf_{x\in L}\mathbb{P}_x(X_t\in L)\geq 1\vee\frac{c_2}{\theta_1-\gamma_1^{t_2}}$, for all $t\geq(n_0-1)t_2$ and we set

$$\varphi_1 = \psi_1 \quad\text{and}\quad \varphi_2 = \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}\sum_{k=0}^{n_0-1}\gamma_2^{-kt_2}P_k\mathbb{1}_L.$$

Step 1. Proof of (E2), (E4) and (E1) for $(P_n)_{n\in\mathbb{Z}_+}$.

For all $x\in E\setminus L$, it follows from (F0) and the second line of (F2) that

$$\begin{aligned}P_1\psi_1(x) &= \mathbb{E}_x\left(\psi_1(X_{t_2})\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial}\right) + \mathbb{E}_x\left(\mathbb{1}_{\tau_L\leq t_2\wedge\tau_\partial}\mathbb{E}_{X_{\tau_L}}\left(\mathbb{1}_{t_2-s<\tau_\partial}\psi_1(X_{t_2-s})\right)\big|_{s=\tau_L}\right)\\ &\leq \gamma_1^{t_2}\psi_1(x) + \mathbb{P}_x(\tau_L\leq t_2)c_2.\end{aligned}$$

We define

$$K = \left\{y \in E,\ \mathbb{P}_y(\tau_L \le t_2)/\psi_1(y) \ge (\theta_1 - \gamma_1^{t_2})/c_2\right\}.$$

The second line of (F2) at time $t = 0$ and the fact that $\theta_1 - \gamma_1^{t_2} < 1$ imply that $L \subset K$. Moreover, we have, for all $x \notin K$,

$$P_1\psi_1(x) \le \theta_1\psi_1(x). \tag{11.4}$$

Hence, for all $x \in E$,

$$P_1\psi_1(x) \le \theta_1\psi_1(x) + c_2\mathbb{1}_K(x). \tag{11.5}$$

Note that it immediately follows from the definition of K that $\sup_{x\in K}\psi_1(x) < \infty$. In particular, the first and third lines of (E2) are proved.

Moreover, using the Markov property provided by (F0) and the definition of n_0, we deduce that, for all $t \ge n_0t_2$,

$$\inf_{x\in K}\gamma_2^{-t}\mathbb{P}_x(X_t \in L) \ge \inf_{x\in K}\mathbb{P}_x(\tau_L \le t_2)\inf_{s\in[0,t_2]}\inf_{y\in L}\gamma_2^{-t}\mathbb{P}_y\left(X_{t-s} \in L\right) \ge 1, \tag{11.6}$$

where we used the fact that, for all $x \in K$, $\mathbb{P}_x(\tau_L \le t_2) \ge \frac{\theta_1-\gamma_1^{t_2}}{c_2}$. In particular,

$$P_1\varphi_2 = \gamma_2^{t_2}\varphi_2 + \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}\left(\gamma_2^{-(n_0-1)t_2}P_{n_0}\mathbb{1}_L - \gamma_2^{t_2}\mathbb{1}_L\right) \ge \gamma_2^{t_2}\varphi_2 = \theta_2\varphi_2.$$

In addition, for all $x \in K$,

$$\varphi_2(x) \ge \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}\gamma_2^{-(n_0-1)t_2}\mathbb{P}_x(X_{n_0t_2} \in L) \ge \frac{1-\gamma_2^{t_2}}{\gamma_2^{-n_0t_2}-1}.$$

Hence (E2) is proved. Moreover, (11.6) also entails that (E4) holds true.

Fix $n_1 > n_0$ such that $n_1t_2 - t_1 \ge n_0t_2$. Condition (F1) and then (11.6) imply that, for all $x \in K$,

$$\mathbb{P}_x(X_{n_1t_2} \in \cdot \cap K) \ge \mathbb{P}_x(X_{n_1t_2-t_1} \in L)c_1\nu(\cdot \cap L) \ge \gamma_2^{n_1t_2-t_1}c_1\nu(\cdot \cap L).$$

Extending ν as a probability measure on K, we obtain (E1).

Step 3. Estimation of the survival probability.

Our goal here is to prove a version of Lemma 9.9, where (9.6) is replaced by

$$\mathbb{P}_x(nt_2 < \tau_\partial) \le C\frac{\varphi_1(x)}{1-\theta_1/\theta_2}\inf_{y\in L}\mathbb{P}_y(nt_2 < \tau_\partial), \quad \forall x \in E, \forall n \in \mathbb{N}. \tag{11.7}$$

Since the proof is similar, we only highlight the main differences. First, Lemma 9.8 only uses (E1), (E2) and (E4), so that there exist $n_6 \ge 1$ and $\zeta_1 > 0$ such that, for all $x \in K$ and all $n \ge n_6$,

$$\delta_xP_n\mathbb{1}_K \ge \zeta_1\delta_xP_n\mathbb{1}_E.$$

Hence, for all $x \in K$ and all $N \ge n_0 + n_6$, using (11.6),

$$\delta_xP_N\mathbb{1}_L \ge \gamma_2^{n_0t_2}\,\delta_xP_{N-n_0}\mathbb{1}_K \ge \zeta_1\gamma_2^{n_0t_2}\,\delta_xP_{N-n_0}\mathbb{1}_E \ge \zeta_1\gamma_2^{n_0t_2}\,\delta_xP_N\mathbb{1}_E.$$

Hence,

$$\inf_{N\ge n_0+n_6}\inf_{x\in K}\mathbb{P}_x(X_{Nt_2} \in L \mid Nt_2 < \tau_\partial) > 0. \tag{11.8}$$

Third, it follows from (F2) that, for all $x \in E \setminus L$,

$$\mathbb{P}_x(nt_2 < \tau_L \wedge \tau_\partial) \le \gamma_1^{nt_2}\psi_1(x) = \theta_1^n\varphi_1(x). \tag{11.9}$$

and from (E2) that, for all $x \in E$,

$$\mathbb{P}_x(nt_2 < \tau_\partial) \geq \gamma_2^{nt_2}\varphi_2(x). \tag{11.10}$$

Therefore, following the same lines as in (9.9) (replacing K with L), we deduce from (11.9) and (11.10) that, for all $x \in E$

$$\begin{aligned}\mathbb{P}_x(nt_2 < \tau_\partial) &\leq \theta_1^n\varphi_1(x) + c_3\int_0^{nt_2}\inf_{y\in L}\mathbb{P}_y\left(\left(n-\lceil s/t_2\rceil\right)t_2 < \tau_\partial\right)\mathbb{P}_x(\tau_L\wedge\tau_\partial\in \mathrm{d}s)\\ &\leq C\inf_{z\in L}\mathbb{P}_z(nt_2<\tau_\partial)\varphi_1(x) + \frac{c_3\gamma_2^{-t_2}}{c}\inf_{z\in L}\mathbb{P}_z(nt_2<\tau_\partial)\mathbb{E}_x\left(\gamma_2^{-\tau_L\wedge\tau_\partial}\right),\end{aligned}$$

which entails (11.7), where we used in the second inequality the fact that

$$\mathbb{P}_x(nt_2 < \tau_\partial) \geq c\gamma_2^{kt_2}\inf_{y\in L}\mathbb{P}_y\left((n-k)t_2 < \tau_\partial\right), \quad \forall x \in L,$$

which is deduced from (11.8) exactly as in Lemma 9.9.

Step 4. Proof of (E3).

Using (11.7) and the fact that $\sup_{x\in K}\varphi_1(x) < +\infty$, we deduce that there exists a constant $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\sup_{x\in K}\mathbb{P}_x(nt_2<\tau_\partial) \leq C\inf_{y\in L}\mathbb{P}_y(nt_2<\tau_\partial).$$

Moreover, using the Markov property at time n_0t_2 and (11.6), we have that, for all $t \geq 0$,

$$\inf_{x\in K}\mathbb{P}_x(t<\tau_\partial) \geq \inf_{x\in K}\mathbb{P}_x(t+n_0t_2<\tau_\partial) \geq \gamma_2^{n_0t_2}\inf_{y\in L}\mathbb{P}_y(t<\tau_\partial).$$

These inequalities imply (E3).

11.5.2 Existence of a quasi-stationary distribution for $(X_t)_{t\in I}$

Subsection 11.5.1 and Theorem 2.1 imply that there exists a probability measure ν_{QSD} on E such that

$$\mathbb{P}_{\nu_{QSD}}(X_{nt_2}\in\cdot\mid nt_2<\tau_\partial) = \nu_{QSD},\ \forall n\in\mathbb{Z}_+,$$

such that $\nu_{QSD}(\varphi_1) < \infty$ and $\nu_{QSD}(\varphi_2) > 0$, which is equivalent to $\nu_{QSD}(L) > 0$ because of the quasi-stationarity and the form of φ_2. For all $t \in [0, t_2]$, let us define the probability measure ν_t on E by

$$\nu_t = \mathbb{P}_{\nu_{QSD}}(X_t\in\cdot\mid t<\tau_\partial).$$

For all $n \in \mathbb{Z}_+$, we have, using the Markov property and the fact that ν_{QSD} is a quasi-stationary distribution for $(X_{nt_2})_{n\geq 0}$,

$$\mathbb{P}_{\nu_t}(X_{nt_2}\in\cdot\mid nt_2<\tau_\partial) = \mathbb{E}_{\nu_{QSD}}(\mathbb{P}_{X_{nt_2}}(X_t\in\cdot\mid t<\tau_\partial)\mid nt_2<\tau_\partial) = \mathbb{P}_{\nu_{QSD}}(X_t\in\cdot\mid t<\tau_\partial),$$

hence ν_t is a quasi-stationary distribution for $(P_n)_{n\geq 0}$. Moreover, the third line of (F2) and the quasi-stationarity of ν_t imply that $\nu_t(L)$ is positive.

Fix $\rho_1 \in (\theta_1^{1/t_2}, \gamma_2)$. It follows from (11.9) that there exists a constant $C > 0$ such that, for all $x \in E$,

$$\varphi_1'(x) := \mathbb{E}_x\left(\rho_1^{-\tau_L\wedge\tau_\partial}\right) \leq C\,\varphi_1(x).$$

We also have that, for all $x \in E \setminus L$,

$$\begin{aligned}\mathbb{E}_x\left(\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial}\varphi_1'(X_{t_2})\right) &= \rho_1^{t_2}\mathbb{E}_x\left(\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial}\rho_1^{-\tau_L\wedge\tau_\partial}\right)\\ &\leq \rho_1^{t_2}\varphi_1'(x) \qquad (11.11)\end{aligned}$$

and the inequality is trivial for $x \in L$. In addition, for all $t \in [0, t_2]$ and all $x \in L$, $\mathbb{E}_x\left(\varphi_1'(X_t)\mathbb{1}_{t<\tau_\partial}\right) \leq C\mathbb{E}_x\left(\psi_1(X_t)\mathbb{1}_{t<\tau_\partial}\right) \leq Cc_2$. Hence Condition (F) is satisfied replacing γ_1 with ρ_1 and ψ_1 with φ_1'. Therefore, we can apply Step 1 to prove that (E) is satisfied with φ_1' and φ_2' where

$$\varphi_2' = \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0't_2}-1}\sum_{k=0}^{n_0'-1}\gamma_2^{-kt_2}P_k\mathbb{1}_L$$

for an integer n_0' that can be chosen larger than n_0. We also deduce as in the beginning of Step 2 that ν_{QSD} is the unique quasi-stationary distribution of $(P_n)_{n\geq 0}$ such that $\nu_{QSD}(\varphi_1') < \infty$ and $\nu_{QSD}(L) > 0$.

Moreover, by Markov's property at time t we have for all $x \in E$ and $t \geq 0$,

$$\begin{aligned}\varphi_1'(x) &= \mathbb{E}_x\left[\mathbb{1}_{t<\tau_L\wedge\tau_\partial}\rho_1{}^{-\tau_L\wedge\tau_\partial}\right] + \mathbb{E}_x\left[\mathbb{1}_{t\geq\tau_L\wedge\tau_\partial}\rho_1{}^{-\tau_L\wedge\tau_\partial}\right]\\ &\leq \rho_1{}^{-t}\mathbb{E}_x\left[\mathbb{1}_{t<\tau_L\wedge\tau_\partial}\varphi_1'(X_t)\right] + \rho_1{}^{-t}\mathbb{P}_x(t\geq\tau_L\wedge\tau_\partial)\\ &\leq \rho_1{}^{-t}\left(\mathbb{E}_x[\mathbb{1}_{t<\tau_\partial}\varphi_1'(X_t)] + 1\right) \qquad (11.12)\end{aligned}$$

so that, for all $t \in [0, t_2]$,

$$\begin{aligned}\nu_t(\varphi_1') &\leq \rho_1{}^{-(t_2-t)}\left[\mathbb{E}_{\nu_{QSD}}\left(\mathbb{1}_{t_2<\tau_\partial}\varphi_1'(X_{t_2})\right)/\mathbb{P}_{\nu_{QSD}}(t<\tau_\partial) + 1\right]\\ &\leq \rho_1{}^{-(t_2-t)}\left[\mathbb{E}_{\nu_{QSD}}\left(\mathbb{1}_{t_2<\tau_\partial}\varphi_1'(X_{t_2})\right)/\mathbb{P}_{\nu_{QSD}}(t_2<\tau_\partial) + 1\right]\\ &= \rho_1{}^{-(t_2-t)}\left(\nu_{QSD}(\varphi_1') + 1\right) < \infty.\end{aligned}$$

Since we observed that $\nu_t(L) > 0$, we deduce that $\nu_t = \nu_{QSD}$ for all $t \in I \cap [0, t_2]$.

Using the Markov property, we deduce that $\nu_t = \nu_{QSD}$ for all $t \in I$ and hence that ν_{QSD} is a quasi-stationary distribution for $(X_t)_{t\in I}$. Since any quasi-stationary distribution for $(X_t)_{t\in I}$ is also a quasi-stationary distribution for $(P_n)_{n\geq 0}$, we deduce that ν_{QSD} is the unique quasi-stationary distribution for $(X_t)_{t\in I}$ such that $\nu_{QSD}(\varphi_1) < +\infty$ and $\nu_{QSD}(L) > 0$.

Let $t \geq t_2$ be fixed and define $k \in \mathbb{N}$ such that $0 \leq t - kt_2 < t_2$. It follows from the fact that $P_1\varphi_1' \leq \bar{C}\varphi_1'$ and from (11.12) that

$$\begin{aligned}\mathbb{E}_x[\mathbb{1}_{t<\tau_\partial}\varphi_1'(X_t)] &\leq \bar{C}^k\mathbb{E}_x\left[\mathbb{1}_{t-kt_2<\tau_\partial}\varphi_1'(X_{t-kt_2})\right]\\ &\leq \bar{C}^k\rho_1{}^{-(k+1)t_2+t}\mathbb{E}_x\left[\mathbb{1}_{t_2<\tau_\partial}\varphi_1'(X_{t_2}) + \mathbb{1}_{t-kt_2<\tau_\partial}\right]\\ &\leq C\bar{C}^k\rho_1{}^{-(k+1)t_2+t}\mathbb{E}_x\left[\mathbb{1}_{t_2<\tau_\partial}\varphi_1(X_{t_2}) + 1\right]\\ &\leq C\bar{C}^k\rho_1{}^{-(k+1)t_2+t}(\theta_1 + c_2 + 1)\varphi_1(x). \qquad (11.13)\end{aligned}$$

Note that a similar inequality may not hold true with φ_1' replaced by φ_1 under our assumptions. This explains why we need to introduce φ_1'.

Now, let μ be a probability measure such that $\mu(\varphi_1) < \infty$ and $\mu(\varphi_2) > 0$. Then, for all $t \geq n_0t_2$, it follows from (11.6) that, for all $k \geq 0$,

$$\mathbb{P}_\mu(X_{t+kt_2} \in L) \geq \mathbb{P}_\mu(X_{kt_2} \in L)\inf_{y\in L}\mathbb{P}_y(X_t \in L) \geq \gamma_2^t\mathbb{P}_\mu(X_{kt_2} \in L).$$

Therefore, for all $t \in [n_0t_2,(n_0+1)t_2]$,

$$\begin{aligned}\mathbb{E}_\mu(\varphi_2(X_t)) &= \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}\sum_{k=0}^{n_0-1}\gamma_2^{kt_2}\mathbb{P}_\mu(X_{t+kt_2}\in L)\\ &\geq \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}\gamma_2^{(n_0+1)t_2}\sum_{k=0}^{n_0-1}\gamma_2^{kt_2}\mathbb{P}_\mu(X_{kt_2}\in L) = \gamma_2^{(n_0+1)t_2}\mu(\varphi_2).\end{aligned}$$

This and inequality (11.13) imply that (using that $n_0' \geq n_0$), for all $t \in [n_0t_2,(n_0+1)t_2]$ and for a constant $C>0$ that may change from line to line,

$$\frac{\mu_t(\varphi_1')}{\mu_t(\varphi_2')} \leq C\frac{\mu_t(\varphi_1')}{\mu_t(\varphi_2)} \leq C\frac{\mu(\varphi_1)}{\mu(\varphi_2)},$$

where $\mu_t := \mathbb{P}_\mu(X_t\in\cdot \mid t<\tau_\partial)$. It then follows the fact that (E) is satisfied by $(P_n, n\geq 0)$ with the functions φ_1' and φ_2' that there exist constants $\alpha<1$ and $C>0$ such that, for all $t\in[n_0t_2,(n_0+1)t_2]$,

$$\left\|\frac{\mu_tP_n}{\mu_tP_n\mathbb{1}_E}-\nu_{QSD}\right\|_{TV} \leq C\alpha^n\,\frac{\mu(\varphi_1)}{\mu(\varphi_2)},$$

Using Markov property, we deduce that

$$\|\mathbb{P}_\mu(X_{nt_2+t}\in\cdot\mid nt_2+t<\tau_\partial)-\nu_{QSD}\|_{TV}\leq C\alpha^n\,\frac{\mu(\varphi_1)}{\mu(\varphi_2)}.$$

This ends the proof of (3.4).

11.5.3 Convergence to η

To finish the proof of Theorem 3.5, it remains to prove that the convergence (3.5) is exponential in $L^\infty(\psi_1^{1/p})$ and that $P_t\eta = e^{-\lambda_0 t}\eta$. Because of Remark 2.2, it is enough to prove this for $p=1$. Since we proved that (E) holds true for the semigroup $(P_n)_{n\geq 0}$ and for the functions φ_1' and φ_2', it follows from Theorem 2.3 that there exist constants $\lambda_0\in[0,\log(1/\gamma_2)]$, $\alpha\in(0,1)$ and $C>0$ such that, for all $y\in E$,

$$\left|e^{\lambda_0 nt_2}\mathbb{P}_y(nt_2<\tau_\partial)-\eta(y)\right|\leq C\alpha^n\varphi_1'(y).$$

For any $t\in[t_2,2t_2]$, integrating this inequality with respect to $\mathbb{P}_x(X_t\in dy; t<\tau_\partial)$, we deduce from (11.13) that

$$\left|e^{\lambda_0 nt_2}\mathbb{P}_x(nt_2+t<\tau_\partial)-\mathbb{E}_x(\eta(X_t)\mathbb{1}_{t<\tau_\partial})\right|\leq C\alpha^n\varphi_1(x)$$

for a constant C independent of $t\in[t_2,2t_2]$. Setting $\eta_t(x)=\mathbb{E}_x\left[e^{\lambda_0 t}\eta(X_t)\mathbb{1}_{t<\tau_\partial}\right]$, we obtain for all $t\in[t_2,2t_2]$

$$\left|e^{\lambda_0(nt_2+t)}\mathbb{P}_x(nt_2+t<\tau_\partial)-\eta_t(x)\right|\leq Ce^{2\lambda_0t_2}\alpha^n\varphi_1(x).$$

Proceeding as in (10.2), we deduce, letting $n\to+\infty$, that $P_1\eta_t = e^{-\lambda_0 t_2}\eta_t$. It then follows from Corollary 2.7 that $\eta_t(x)=\eta(x)\nu_{QSD}(\eta_t)$ for all $x\in E$. Since we proved above that ν_{QSD} is a quasi-stationary distribution with decay parameter λ_0, by definition of η_t, $\nu_{QSD}(\eta_t)=1$ and thus $P_t\eta=e^{-\lambda_0 t}\eta$. This ends the proof of Theorem 3.5.

11.6 Proof of Lemma 3.6

Proceeding as in (11.11) and (11.12), we have that, for all $x \in E$ and $t \in I$,

$$\mathbb{E}_x\left(\psi_1(X_{t_2})\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial}\right) \leq \gamma_1^{t_2}\psi_1(x) \quad \text{and} \quad \psi_1(x) \leq \gamma_1^{-t}\left(\mathbb{E}_x\left[\mathbb{1}_{t<\tau_\partial}\psi_1(X_t)\right]+1\right).$$

Therefore, for all $t \leq t_2$ and all $x \in L$,

$$\begin{aligned}\mathbb{E}_x\left[\mathbb{1}_{t<\tau_\partial}\psi_1(X_t)\right] &\leq \gamma_1^{-(t_2-t)}\mathbb{E}_x\left\{\left[\mathbb{E}_{X_t}\left(\mathbb{1}_{t_2-t<\tau_\partial}\psi_1(X_{t_2-t})\right)+1\right]\mathbb{1}_{t<\tau_\partial}\right\}\\ &\leq \gamma_1^{-(t_2-t)}\left[\mathbb{E}_x\left(\mathbb{1}_{t_2<\tau_\partial}\psi_1(X_{t_2})\right)+1\right]\\ &\leq c_2 := \gamma_1^{-t_2}\left[\sup_{y\in L}\mathbb{E}_y\left(\mathbb{1}_{t_2<\tau_\partial}\psi_1(X_{t_2})\right)+1\right].\end{aligned}$$

This concludes the proof of Lemma 3.6.

11.7 Proof of Proposition 3.9

Let us first assume that (E) is satisfied with φ_1 bounded and (3.8) and prove that (3.7) holds true. Theorem 2.1 and Remark 2.2 entail that, for all $n \geq n_4'$,

$$\begin{aligned}\left\|\frac{\mu P_n}{\mu P_n\mathbb{1}_E}-\nu_{QSD}\right\|_{TV} &\leq \alpha^{n-n_4'}\frac{\|\varphi_1\|_\infty}{\inf_{x\in K}\varphi_2(x)}\frac{\mu P_{n_4'}\mathbb{1}_E}{\mu P_{n_4'}\mathbb{1}_K}\\ &\leq \alpha^{n-n_4'}\frac{\|\varphi_1\|_\infty}{\underline{c}\inf_{x\in K}\varphi_2(x)}.\end{aligned}$$

Hence the convergence is uniform.

Let us now assume that (3.7) holds true. It was proved in [20] that this is equivalent to the following condition.

Condition (A). There exist positive constants c_1, c_2, a positive integer k_0 and a probability measure ν on E such that

(A1) *(Conditional Dobrushin coefficient)* For all $x \in E$,

$$\mathbb{P}_x(X_{k_0} \in \cdot \mid k_0 < \tau_\partial) \geq c_1\nu.$$

(A2) *(Global Harnack inequality)* We have

$$\sup_{k\in\mathbb{Z}_+}\frac{\sup_{y\in E}\mathbb{P}_y(k<\tau_\partial)}{\mathbb{P}_\nu(k<\tau_\partial)} \leq c_2.$$

Several consequences of Condition (A) were deduced in [20], among which the fact that the convergence (2.2) in Theorem 2.3 holds true with respect to the L^∞ norm on E with $\eta(x) > 0$ for all $x \in E$. In particular, η is bounded, $P_1\eta = \theta_0\eta$ and there exists a constant C' such that, for all $n \geq 0$,

$$\sup_{x\in E}\mathbb{P}_x(n<\tau_\partial) \leq C'\theta_0^n. \tag{11.14}$$

We fix $\varepsilon \in (0, 1/(4C'))$. Since η is positive on E, there exists $\delta > 0$ such that the set $K := \{x \in E : \eta(x) \geq \delta\}$ satisfies $\nu_{QSD}(K) \geq 1-\varepsilon$ and $\nu(K) > 0$. Setting $\varphi_2 = \eta/\|\eta\|_\infty$, the part of (E2) dealing about φ_2 is satisfied with $\theta_2 = \theta_0$. Since the convergence in Theorem 2.3 holds true with respect to the L^∞ norm, we deduce from the choice of K that there exists $k \geq k_0$ such that

$$c := \inf_{x\in K}\mathbb{P}_x(k_0<\tau_\partial) \geq \inf_{x\in K}\mathbb{P}_x(k<\tau_\partial) > 0.$$

It follows from (A1) and (A2) that, for all $n \geq 0$,

$$\inf_{x\in K} \mathbb{P}_x(n < \tau_\partial) \geq \inf_{x\in K} \mathbb{P}_x(n + k_0 < \tau_\partial) \geq c_1 c\, \mathbb{P}_\nu(n < \tau_\partial) \geq \frac{c_1 c}{c_2} \sup_{y\in E} \mathbb{P}_y(n < \tau_\partial).$$

This implies (E3) and that $\inf_{x\in K} \mathbb{P}_x(k_0 < \tau_\partial) > 0$. Hence, (E1) follows from (A1) with the probability measure $\frac{\nu(\cdot\cap K)}{\nu(K)}$. Moreover, for any n large enough to have $C\alpha^n \leq 1/2$ where the constants C and α are those of (3.7), we have $\mathbb{P}_x(X_n \in K \mid t < \tau_\partial) \geq \nu_{QSD}(K) - C\alpha^n \geq 1/2 - \varepsilon > 0$ and hence (E4) is satisfied. The last computation also entails (3.8) with $n'_4 = n$.

It remains to construct a function φ_1 satisfying (E2) with $\theta_1 < \theta_0$. For all $x \in E$,

$$\mathbb{P}_x(X_n \in E \setminus K \mid n < \tau_\partial) \leq \nu_{QSD}(E \setminus K) + C\alpha^n \leq \varepsilon + C\alpha^n.$$

Using (11.14), we deduce that

$$\mathbb{P}_x(X_n \in E \setminus K) \leq C'(\varepsilon + C\alpha^n)\theta_0^n,$$

so that there exists n_0 large enough such that

$$\mathbb{P}_x(n_0 < T_K \wedge \tau_\partial) \leq \frac{1}{3}\theta_0^{n_0} = \left(\frac{\theta_0}{3^{1/n_0}}\right)^{n_0}.$$

From this follows that, for all $k \in \mathbb{N}$ and all $x \in E$,

$$\mathbb{P}_x(kn_0 < T_K \wedge \tau_\partial) \leq \left(\frac{\theta_0}{3^{1/n_0}}\right)^{kn_0}.$$

In particular, for $\theta_1 := \theta_0/2^{1/n_0}$,

$$\varphi_1(x) := \mathbb{E}_x\left(\theta_1^{-T_K\wedge\lceil\tau_\partial\rceil}\right), \quad \forall x \in E,$$

is a bounded function on E and Lemma 3.2 implies that, for all $x \in E$,

$$P_1\varphi_1(x) \leq \theta_1\varphi_1(x) + \|\varphi_1\|_\infty \mathbb{1}_K(x).$$

Since $\theta_1 < \theta_0$, (E2) is proved.

12 Proof of the results of Section 4.1

In order to prove Theorem 4.1, we check Condition (F). The goal of Subsection 12.1 is to give the construction of the process X and to check (F0) with $L = K_k$ for any $k \geq 1$. In Subsection 12.2, we explain how (F1) and (F3) can be deduced from general Harnack inequalities. Finally, Subsection 12.3 completes the proof of Theorem 4.1. The proof of Corollary 4.5 is then given in Subsection 12.4.

12.1 Construction of the diffusion process X and Markov property

The goal of this section is to construct a weak solution X to the SDE (4.1) with absorption out of D, and prove that it is Markov and satisfies a strong Markov property at appropriate stopping times, enough to entail Condition (F0) for $L = K_k$ for any $k \geq 1$. We introduce the natural path space for the process X as

$$\mathcal{D} := \Big\{ w : \mathbb{R}_+ \to D \cup \{\partial\} : \ \forall k \geq 1,\ w \text{ is continuous on } [0, \tau_k(w)] \\ \text{and } w(t) = \partial,\ \forall t \geq \sup_{k\geq 1} \tau_k(w) \Big\},$$

where $\tau_k(w) := \inf\{t \geq 0 : w_t \in D \setminus K_k\}$. Note that $\mathcal{D}$ contains functions which are not càdlàg since they may not have a left limit at $\tau_\partial-$ and, indeed, it is easy to construct examples where X is not càdlàg $\mathbb{P}$-a.s.[2] Note also that this definition means that we are looking for a process X such that

$$\tau_\partial := \sup_{k\geq 1} \tau_{D\setminus K_k},$$

which is the natural definition of τ_∂ when the left limit of X at time τ_∂ does not exist.

We endow the path space $\mathcal{D}$ with its natural filtration

$$\mathcal{F}_t = \sigma(w_s, s \leq t) = \bigvee_{n\geq 1, 0\leq t_1<t_2<\ldots<t_n\leq t} \sigma(w_{t_1}, w_{t_2}, \ldots, w_{t_n})$$

and we follow the usual method which consists in constructing for all $x \in D$ a probability measure $\mathbb{P}_x$ on $\mathcal{D}$ and a stochastic process $(B_t, t \geq 0)$ on $\mathcal{D} \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}^r)$, such that B is a standard r-dimensional Brownian motion under $\mathbb{P}_x \otimes \mathbb{W}^r$, where $\mathbb{W}^r$ is the r-dimensional Wiener measure and such that $w_0 = x$ $\mathbb{P}_x \otimes \mathbb{W}^r$-almost surely and the canonical process $(w_t, t \geq 0)$ solves the SDE (4.1) for this Brownian motion B on the time interval $[0, \sup_k \tau_k(w))$ [3].

For this construction, we use the fact that b and σ can be extended out of K_k to $\mathbb{R}^d$ as globally Hölder and bounded functions b_k and σ_k and such that σ_k is uniformly elliptic on $\mathbb{R}^d$. Hence (see e.g. [66, Rk. 5.4.30]) the martingale problem is well-posed for the SDE

$$\mathrm{d}X^k_t = b_k(X^k_t)\mathrm{d}t + \sigma_k(X^k_t)\mathrm{d}B_t.$$

Let us denote by $\mathbb{P}^k_x$ the solution to this martingale problem for the initial condition $x \in \mathbb{R}^d$. This is a probability measure on $\mathcal{C} := \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, equipped with its canonical filtration $(\mathcal{G}_t)_{t\geq 0}$.

For all $k \geq 1$, we define $\tau'_k(w) = \inf\{t \geq 0, w_t \notin \mathrm{int}(K_k)\}$, where $\mathrm{int}(K_k)$ is the interior of K_k. Since the paths $w \in \mathcal{D}$ or $\mathcal{C}$ are continuous at time τ'_k and $\mathbb{R}^d \setminus \mathrm{int}(K_k)$ is closed, it is standard to prove that τ'_k is a stopping time for the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ on $\mathcal{D}$ and for the canonical filtration $(\mathcal{G}_t)_{t\geq 0}$ on $\mathcal{C}$. We define as usual the stopped σ-fields $\mathcal{F}_{\tau'_k}$ and $\mathcal{G}_{\tau'_k}$, and we define for all $x \in \mathrm{int}(K_k)$ the restriction of $\mathbb{P}_x$ to $\mathcal{F}_{\tau'_k}$ as the restriction of $\mathbb{P}^k_x$ to $\mathcal{G}_{\tau'_k}$, where we can identify the events of the two filtrations since they both concern continuous parts of the paths. This construction is consistent for k and $k+1$ (meaning that if $x \in K_k$, they give the same probability to events of $\mathcal{F}_{\tau_k}$) by uniqueness of the solutions $\mathbb{P}^k_x$ and $\mathbb{P}^{k+1}_x$ to the above martingale problems. Hence there exists a unique extension $\mathbb{P}_x$ of the above measures to $\bigvee_{k\geq 1} \mathcal{F}_{\tau'_k}$. Note that, because of the specific structure of the path space $\mathcal{D}$, we have

$$\bigvee_{k\geq 1} \mathcal{F}_{\tau'_k} = \mathcal{F}_\infty. \tag{12.1}$$

---

[2] For example, one may consider D the open disc of radius 1 centered at 0 in $\mathbb{R}^2$, $\sigma = \mathrm{Id}$ and $b(x) = (-x_2\beta(|x|), x_1\beta(|x|))$ where $x = (x_1, x_2) \in D$. Decomposing the process in polar coordinates $(R_t, \theta_t) := (|X_t|, \arctan(X^{(1)}_t/X^{(2)}_t))$, the radius R_t is a 2-dimensional Bessel process, and X_t is sent to ∂ when R_t hits 1 (in a.s. finite time). The angle θ_t is solution to $\mathrm{d}\theta_t = R_t^{-1}\mathrm{d}W_t - \beta(R_t)\mathrm{d}t$ before τ_∂, for some Brownian motion W. Hence, if $\beta(r)$ converges sufficiently fast to $+\infty$ when $r \to 1$, θ_t a.s. converges to $-\infty$ when $t \to \tau_\partial-$, so X does not admit a left limit at time τ_∂.

[3] Since $\sigma(x)$ is non-degenerate for all $x \in D$, the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^r)$ equipped with the Wiener measure $\mathbb{W}^r$ is only used to construct the Brownian path B_t after time $\sup_k \tau_k(w)$ and could be omitted for our purpose since we only need to construct the process B up to time $\sup_k \tau_k(w)$.

To check this, it suffices to observe that, for all $t \geq 0$ and all measurable $A \subset D \cup \{\partial\}$,

$$\begin{aligned}\{w_t \in A\} &= \{t < \tau_\partial,\ w_t \in A \cap D\} \cup \{\tau_\partial \leq t,\ \partial \in A\} \\ &= \left(\bigcup_{k\geq 1} \{t < \tau'_k,\ w_t \in A \cap D\}\right) \cup \left(\bigcap_{k \geq 1} \{\tau'_k \leq t, \partial \in A\}\right),\end{aligned} \tag{12.2}$$

hence $\{w_t \in A\} \in \bigvee_{k\geq 1} \mathcal{F}_{\tau'_k}$, and, proceeding similarly, the same property holds for events of the form $\{w_{t_1} \in A_1, \dots, w_{t_n} \in A_n\}$.

We recall (see [66, Section 5.4]) that $(\mathbb{P}^k_x)_{x\in\mathbb{R}^d}$ forms a strong Markov family on the canonical space $\mathcal{C}$. Our goal is now to prove that the family of probability measures $(\mathbb{P}_x)_{x\in D\cup\{\partial\}}$, where $\mathbb{P}_\partial$ is defined as the Dirac measure on the constant path equal to ∂, forms a Markov kernel of probability measures, for which the strong Markov property applies at well-chosen stopping times.

We first need to prove that $(\mathbb{P}_x)_{x\in D}$ defines a kernel of probability measures, i.e. that $x \mapsto \mathbb{P}_x(\Gamma)$ is measurable for all events Γ of $\mathcal{F}_\infty$. We prove it for an event of the form $\{w_t \in A\}$, the extension to events of the form $\{w_{t_1} \in A_1, \dots, w_{t_n} \in A_n\}$, and hence to all events of $\mathcal{F}_\infty$, being easy. This follows from (12.2):

$$\begin{aligned}\mathbb{P}_x(w_t \in A) &= \lim_{k\to+\infty} \mathbb{P}_x(t < \tau'_k,\ w_t \in A \cap D) + \mathbb{1}_{\partial\in A} \lim_{k\to+\infty} \mathbb{P}_x(\tau'_k \leq t) \\ &= \lim_{k\to+\infty} \mathbb{P}^{k+1}_x(t < \tau'_k,\ w_t \in A \cap D) + \mathbb{1}_{\partial\in A} \lim_{k\to+\infty} \mathbb{P}^{k+1}_x(\tau'_k \leq t).\end{aligned}$$

Since all the probabilities in the right-hand side are measurable functions of x, so is $x \mapsto \mathbb{P}_x(w_t \in A)$.

Now, let us prove that $(X_t, t \geq 0)$ is Markov. It is well-known that this is implied by the following property: for all $n \geq 1$ and $0 \leq t_1 \leq \ldots \leq t_{n+1}$ and $A_1, \dots, A_{n+1}$ measurable subsets of $D \cup \{\partial\}$,

$$\mathbb{P}_x(w_{t_1} \in A_1, \dots, w_{t_{n+1}} \in A_{n+1}) = \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A_1,\dots,w_{t_n}\in A_n} \mathbb{P}_{w_{t_n}}(w_{t_{n+1}-t_n} \in A_{n+1})\right].$$

We prove this property only for $n = 1$. It is easy to extend the proof to all values of $n \geq 1$. We have

$$\begin{aligned}\mathbb{P}_x(w_{t_1} \in A_1, w_{t_2} \in A_2) &= \mathbb{P}_x(w_{t_1} \in A_1, w_{t_2} \in A_2, \tau_\partial > t_2) \\ &\quad + \mathbb{P}_x(w_{t_1} \in A_1, t_1 < \tau_\partial \leq t_2)\mathbb{1}_{\partial\in A_2} + \mathbb{P}_x(\tau_\partial \leq t_1)\mathbb{1}_{\partial\in A_1\cap A_2}.\end{aligned}$$

Now, using that $(\mathbb{P}^k_x)_{x\in\mathbb{R}^d}$ is a Markov family for all $k \geq 1$,

$$\begin{aligned}&\mathbb{P}_x(w_{t_1} \in A_1, w_{t_2} \in A_2, \tau_\partial > t_2) \\ &\qquad = \lim_{k\to\infty} \mathbb{P}_x(w_{t_1} \in A_1, w_{t_2} \in A_2, \tau_k > t_2) \\ &\qquad = \lim_{k\to\infty} \mathbb{P}^k_x(w_{t_1} \in A_1, w_{t_2} \in A_2, \tau_k > t_2) \\ &\qquad = \lim_{k\to\infty} \mathbb{E}^k_x\left[\mathbb{1}_{w_{t_1}\in A_1, t_1<\tau_k} \mathbb{P}^k_{w_{t_1}}(w_{t_2-t_1} \in A_2, \tau_k > t_2 - t_1)\right] \\ &\qquad = \lim_{k\to\infty} \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A_1, t_1<\tau_k} \mathbb{P}_{w_{t_1}}(w_{t_2-t_1} \in A_2, \tau_k > t_2 - t_1)\right] \\ &\qquad = \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A_1, t_1<\tau_\partial} \mathbb{P}_{w_{t_1}}(w_{t_2-t_1} \in A_2, \tau_\partial > t_2 - t_1)\right]\end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{P}_x(w_{t_1} \in A_1, t_1 < \tau_\partial \leq t_2)\mathbb{1}_{\partial\in A_2} &= \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A_1, t_1<\tau_\partial} \mathbb{P}_{w_{t_1}}(\tau_\partial \leq t_2 - t_1)\right]\mathbb{1}_{\partial\in A_2} \\ &= \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A_1, t_1<\tau_\partial} \mathbb{P}_{w_{t_1}}(\tau_\partial \leq t_2 - t_1, w_{t_2-t_1} \in A_2)\right].\end{aligned}$$

Since

$$\mathbb{P}_x(\tau_\partial \leq t_1)\mathbb{1}_{\partial\in A_1\cap A_2} = \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A_1,\tau_\partial\leq t_1}\mathbb{P}_{w_{t_1}}(w_{t_2-t_1}\in A_2)\right],$$

we have proved that $\mathbb{P}_x(w_{t_1}\in A_1, w_{t_2}\in A_2) = \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A_1}\mathbb{P}_{w_{t_1}}(w_{t_2-t_1}\in A_2)\right]$. This ends the proof of the Markov property.

To conclude this subsection, let us prove that the strong Markov property holds for all stopping times τ_F where $F\subset D$ is closed in D. Note that τ_F is indeed a stopping time for the filtration $\mathcal{F}_t$ since $\tau_F = \sup_k \tau_F\wedge\tau'_k = \sup_k \tau_{(F\cup D^c)\cup\mathrm{int}(K_k)^c}$, where the complement is understood in $\mathbb{R}^d$, $(F\cup D^c)\cup\mathrm{int}(K_k)^c$ is a closed subset of $\mathbb{R}^d$ and all $w\in\mathcal{D}$ is continuous at time $\tau_{(F\cup D^c)\cup\mathrm{int}(K_k)^c}$. Let $x\in D$, $t_1,t_2,s\geq 0$ and $A,B\subset D$ be measurable sets. We proceed as above: first, observe that

$$\begin{aligned}
&\{w_{t_1}\in A,\ t_1<\tau_F\leq t_2,\ w_{\tau_F+s}\in B\}\\
&\qquad\qquad = \bigcup_{\ell\geq 1}\{w_{t_1}\in A,\ t_1<\tau_F\leq t_2,\ w_{\tau_F+s}\in B,\ w_r\in K_\ell\ \forall r\in[0,\tau_F+s]\}\\
&\qquad\qquad = \bigcup_{\ell\geq 1}\{w_{t_1}\in A,\ t_1<\tau_F\wedge\tau'_\ell\leq t_2,\ w_{\tau_F\wedge\tau'_\ell+s}\in B,\ \tau'_\ell>\tau_F+s\}.
\end{aligned}$$

Since $\tau_F\wedge\tau'_\ell$ is a $\mathcal{G}_t$-stopping time on $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$ and using the strong Markov property under $\mathbb{P}^\ell$, we deduce that

$$\begin{aligned}
&\mathbb{P}_x(w_{t_1}\in A,\ t_1<\tau_F\leq t_2,\ w_{\tau_F+s}\in B)\\
&\qquad = \lim_{\ell\to+\infty}\mathbb{P}^\ell_x(w_{t_1}\in A,\ t_1<\tau_F\wedge\tau'_\ell\leq t_2,\ w_{\tau_F\wedge\tau'_\ell+s}\in B,\ \tau'_\ell>\tau_F+s)\\
&\qquad = \lim_{\ell\to+\infty}\mathbb{E}^\ell_x\left[\mathbb{1}_{w_{t_1}\in A,\ t_1<\tau_F\wedge\tau'_\ell\leq t_2}\mathbb{P}^\ell_{w_{\tau_F\wedge\tau'_\ell}}(w_s\in B,\ s<\tau'_\ell)\right]\\
&\qquad = \lim_{\ell\to+\infty}\mathbb{E}^\ell_x\left[\mathbb{1}_{w_{t_1}\in A,\ t_1<\tau_F\leq\tau'_\ell\wedge t_2}\mathbb{P}^\ell_{w_{\tau_F}}(w_s\in B,\ s<\tau'_\ell)\right]\\
&\qquad = \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A,\ t_1<\tau_F\leq\tau_\partial\wedge t_2}\mathbb{P}_{w_{\tau_F}}(w_s\in B,\ s<\tau_\partial)\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{P}_x(w_{t_1}\in A,\ t_1<\tau_F\leq t_2, w_{\tau_F+s}=\partial)\\
&\qquad\qquad = \lim_{\ell\to+\infty}\mathbb{P}^\ell_x(w_{t_1}\in A,\ t_1<\tau_F\leq t_2\wedge\tau'_\ell,\ \tau'_\ell\leq\tau_F+s)\\
&\qquad\qquad = \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A,\ t_1<\tau_F\leq t_2\wedge\tau_\partial}\mathbb{P}_{w_{\tau_F}}(w_s=\partial)\right]
\end{aligned}$$

and thus

$$\mathbb{P}_x(w_{t_1}\in A,\ t_1<\tau_F\leq t_2,\ w_{\tau_F+s}\in B) = \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A,\ t_1<\tau_F\leq t_2\wedge\tau_\partial}\mathbb{P}_{w_{\tau_F}}(w_s\in B)\right]$$

for all $A,B\subset D\cup\{\partial\}$ measurable. The previous computation extends without difficulty to prove

$$\begin{aligned}
&\mathbb{P}_x\left(w_{t_1}\in A_1,\dots,w_{t_n}\in A_n,\ t_n<\tau_F\leq t_{n+1},\ w_{\tau_F+s_1}\in B_1,\dots,w_{\tau_F+s_m}\in B_m\right)\\
&\qquad = \mathbb{E}_x\left[\mathbb{1}_{w_{t_1}\in A_1,\dots,w_{t_n}\in A_n,\ t_n<\tau_F\leq t_{n+1}}\mathbb{P}_{w_{\tau_F}}(w_{s_1}\in B_1,\dots,w_{s_m}\in B_m)\right]
\end{aligned}\tag{12.3}$$

for all $n,m\geq 1$, $0\leq t_1\leq\ldots\leq t_{n+1}$, $0\leq s_1\leq\ldots\leq s_m$ and $A_1,\dots,A_n,B_1,\dots,B_m\subset D\cup\{\partial\}$ measurable. This implies the strong Markov property at time τ_F, in the sense that, for all $k\geq 1$, all $x\in E$ and all $\Gamma\in\mathcal{F}_\infty$,

$$\mathbb{P}_x\left(w^{\tau_F}\in\Gamma\mid\mathcal{H}_{\tau_F}\right) = \mathbb{P}_{w_{\tau_F}}(\Gamma),\quad \mathbb{P}_x\text{-almost surely,}$$

where $w^{\tau_F} = (w_{\tau_F+s}, s \geq 0)$ and

$$\mathcal{H}_{\tau_F} = \sigma\Big(\{w_{t_1} \in A_1, \dots, w_{t_n} \in A_n, t_n < \tau_F \leq t_{n+1}\}, n \in \mathbb{N}, \\ 0 \leq t_1 \leq \dots \leq t_{n+1},\ A_1, \dots, A_n \in D \text{ measurable}\Big).$$

This form of strong Markov property at time τ_F is enough for our purpose, since it entails (F0) for $L = K_k$ for all $k \geq 1$.

12.2 Harnack inequalities

Our goal here is to check Conditions (F1) and (F3) for the diffusion process constructed above. We will make use of general Harnack inequalities of Krylov and Safonov [72].

Proposition 12.1. *There exist a probability measure ν on D and a constant $t_\nu > 0$ such that, for all $k \geq 1$, there exists a constant $b_k > 0$ such that*

$$\mathbb{P}_x(X_{t_\nu} \in \cdot) \geq b_k \nu(\cdot),\ \forall x \in K_k. \tag{12.4}$$

Moreover, for all $k \geq 1$ such that K_k is non-empty,

$$\inf_{t \geq 0} \frac{\inf_{x \in K_k} \mathbb{P}_x(t < \tau_\partial)}{\sup_{x \in K_k} \mathbb{P}_x(t < \tau_\partial)} > 0. \tag{12.5}$$

Proof. Consider a bounded measurable function $f : D \to \mathbb{R}_+$ with $\|f\|_\infty \leq 1$ and define the application $u : (t, x) \in \mathbb{R}_+ \times E \mapsto \mathbb{E}_x[\mathbb{1}_{t<\tau_\partial} f(X_t)]$. It is proved in [26] using [72] that, for all $k \geq 1$, there exist two constants $N_k > 0$ and $\delta_k > 0$, which do not depend on f (provided $\|f\|_\infty \leq 1$), such that

$$u(\delta_k^2, x) \leq N_k u(2\delta_k^2, y), \text{ for all } x, y \in K_k \text{ such that } |x - y| \leq \delta_k/2. \tag{12.6}$$

Note that the proof given in [26] makes use of the following strong Markov property: for all open ball B such that $B \subset K_k$ for some $k \geq 1$, all $x \in B$, $t \geq 0$ and all measurable $f : D \cup \{\partial\} \to \mathbb{R}_+$,

$$\mathbb{E}_x\left[f(X_t)\mathbb{1}_{\tau_{D\setminus B} \leq t < \tau_\partial}\right] = \mathbb{E}_x\left[\mathbb{1}_{\tau_{D\setminus B} \leq t} \mathbb{E}_{X_{\tau_{D\setminus B}}}\left[f(X_{t-u})\mathbb{1}_{t-u<\tau_\partial}\right]\Big|_{u=\tau_{D\setminus B}}\right].$$

This property follows from (12.3).

Step 1: Proof of (12.4)

Fix $x_1 \in D$ and $k_1 \geq 1$ such that $x_1 \in \text{int}(K_{k_1})$. Let ν denote the conditional law $\mathbb{P}_{x_1}(X_{\delta_{k_1}^2} \in \cdot \mid \delta_{k_1}^2 < \tau_\partial)$. Then, for all measurable $A \subset D \cup \{\partial\}$, Harnack's inequality (12.6) with $f = \mathbb{1}_A$ entails that, for all $x \in D$ such that $|x - x_1| < \frac{\delta_{k_1}}{2} \wedge d(x_1, D \setminus K_{k_1})$,

$$\mathbb{P}_x(2\delta_{k_1}^2 \in A) \geq \frac{\mathbb{P}_{x_1}(\delta_{k_1}^2 < \tau_\partial)}{N_{k_1}} \nu(A).$$

Since the diffusion is locally elliptic and D is connected, for all $k \geq 1$, there exists a constant $d_k > 0$ such that

$$\inf_{x \in K_k} \mathbb{P}_x(X_1 \in B(x_1, (\delta_{k_1}/2) \wedge d(x_1, D \setminus K_{k_1})) \geq d_k.$$

This and Markov's property entail that, for all $x \in K_k$,

$$\mathbb{P}_x(X_{1+2\delta_{k_1}^2} \in \cdot) \geq d_k \frac{\mathbb{P}_{x_1}(\delta_{k_1}^2 < \tau_\partial)}{N_{k_1}} \nu.$$

This implies the first part of Proposition 12.1.

Step 2: Proof of (12.5)

Fix $k \geq 1$ such that K_k is non-empty and consider $\ell > k$ such that K_k is included in one connected component of $\operatorname{int}(K_\ell)$. For all $t \geq 2\delta_\ell^2$, the inequality (12.6) applied to $f(x) = \mathbb{P}_x(t - 2\delta_\ell^2 < \tau_\partial)$ and the Markov property entail that

$$\mathbb{P}_x(t - \delta_\ell^2 < \tau_\partial) \leq N_\ell \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in K_\ell \text{ such that } |x-y| \leq \delta_\ell/2.$$

Since $s \mapsto \mathbb{P}_x(s < \tau_\partial)$ is non-increasing, we deduce that

$$\mathbb{P}_x(t < \tau_\partial) \leq N_\ell \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in K_\ell \text{ such that } |x-y| \leq \delta_\ell/2.$$

Since K_k has a finite diameter and is included in a connected component of K_ℓ, we deduce that there exists N'_k equal to some power of N_ℓ such that, for all $t \geq 2\delta_\ell^2$,

$$\mathbb{P}_x(t < \tau_\partial) \leq N'_k \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in K_k.$$

Now, for $t \leq 2\delta_\ell^2$, we simply use the fact that, for all $x \in K_k$, $\mathbb{P}_x(2\delta_\ell^2 < \tau_\partial) \geq \mathbb{P}_x(2\delta_\ell^2 < \tau_B)$ where $B = (x, 1/2k)$ and hence $x \mapsto \mathbb{P}_x(2\delta_\ell^2 < \tau_\partial)$ is uniformly bounded from below on K_k by a constant $1/N''_k > 0$. In particular,

$$\mathbb{P}_x(t < \tau_\partial) \leq 1 \leq N''_k \mathbb{P}_y(2\delta_\ell^2 < \tau_\partial) \leq N''_k \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in K_k.$$

This concludes the proof of Proposition 12.1. □

12.3 Proof of Theorem 4.1

Our aim is to prove that Condition (F) holds true with $L = K_k$ for some $k \geq 1$ large enough. We have already proved (F0), (F1) and (F3) with $L = K_k$ for any $k \geq 1$. Hence we only have to check (F2). Fix $\rho_1 \in (\lambda_0, \lambda_1)$, $\rho_2 \in (\lambda_0, \rho_1)$ and $p \in (1, \lambda_1/\rho_1)$ and define

$$\psi_1(x) = \varphi(x)^{1/p}, \ \forall x \in D. \tag{12.7}$$

Fix $\rho'_1 \in (\rho_1, \lambda_1/p)$ and

$$t_2 \geq \frac{2s_1(C+\lambda_1)}{\lambda_1 - p\rho'_1} \vee \frac{\log 2}{\rho'_1 - \rho_1},$$

where the constant C comes from (4.5). Set $L = K_{k_0}$ with k_0 large enough so that $\nu(K_{k_0}) > 0$ and, using (4.6),

$$\mathbb{P}_x(s_1 < \tau_{K_{k_0}} \wedge \tau_\partial) \leq e^{-(\rho'_1 + C/p)t_2}$$

for all $x \in D_0$.

From the definition of λ_0 and applying the same argument as in Step 2 of the proof of Proposition 12.1 with $f(x) = \mathbb{P}_x(X_{t-2\delta_\ell^2} \in L)$ with ℓ large enough to have K_{k_0} included in one connected component of K_ℓ, we deduce that

$$\liminf_{t \to +\infty} e^{\rho_2 t} \inf_{x \in L} \mathbb{P}_x(X_t \in L) = +\infty,$$

and hence the last line of (F2) is proved with $\gamma_2 = e^{-\rho_2}$.

Let us now check that the first line of Assumption (F2) holds true for all $x \in D_0$ and then for all $x \in D \setminus D_0$. For all $x \in D_0$, we have $\psi_1(x) \leq \sup_{x \in D_0} \varphi^{1/p}(x) < +\infty$, and hence, for all $t \in [s_1, t_2]$, using Hölder's inequality and the definition of k_0,

$$\begin{aligned} \mathbb{E}_x\left(\psi_1(X_t)\mathbb{1}_{t < \tau_L \wedge \tau_\partial}\right) &\leq \mathbb{E}_x\left(\mathbb{1}_{t<\tau_\partial}\varphi(X_t)\right)^{1/p} \mathbb{P}_x(t < \tau_L \wedge \tau_\partial)^{\frac{p-1}{p}} \\ &\leq \varphi(x)^{1/p} e^{Ct_2/p} \mathbb{P}_x(s_1 < \tau_L \wedge \tau_\partial)^{\frac{p-1}{p}} \\ &\leq e^{-\rho'_1 t_2} \leq e^{-\rho_1 t_2}\psi_1(x). \end{aligned} \tag{12.8}$$

To prove (12.8), we used the fact that $\mathcal{L}\varphi \leq C \leq C\varphi$ and Itô's formula to obtain $P_t\varphi \leq e^{Ct}\varphi$. Since this argument is used repeatedly in the sequel, we give it in details for sake of completeness. It follows from Itô's formula that, for all $k \geq 1$, $\mathbb{P}_x$-almost surely,

$$\begin{aligned} e^{-C\left(t\wedge\tau_{K_k^c}\right)}\varphi\left(X_{t\wedge\tau_{K_k^c}}\right) = \varphi(x) &+ \int_0^t \mathbb{1}_{s\leq\tau_{K_k^c}} e^{-Cs}\left(\mathcal{L}\varphi(X_s) - C\varphi(X_s)\right)\mathrm{d}s \\ &+ \int_0^t \mathbb{1}_{s\leq\tau_{K_k^c}} e^{-Cs}\nabla\varphi(X_s)^*\sigma(X_s)\mathrm{d}B_s. \end{aligned}$$

Since $\nabla\varphi(x)$ and $\sigma(x)$ are uniformly bounded on K_k, the last term has zero expectation, and thus

$$\mathbb{E}_x\left[e^{-C\left(t\wedge\tau_{K_k^c}\right)}\varphi\left(X_{t\wedge\tau_{K_k^c}}\right)\right] \leq \varphi(x).$$

Letting $k \to +\infty$, we deduce form Fatou's lemma that

$$\mathbb{E}_x\left[e^{-Ct}\mathbb{1}_{t<\tau_\partial}\varphi(X_t)\right] \leq \varphi(x) \tag{12.9}$$

as claimed.

This proves the second line of (F2) for all $x \in D_0$ and $\gamma_1 = e^{-\rho_1}$.

Now, for all $x \in D \setminus D_0$, since D_0 is closed in D, it follows from the strong Markov property (12.3) at time τ_{D_0} that

$$\begin{aligned} \mathbb{E}_x\left(\psi_1(X_{t_2})\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial}\right) &= \mathbb{E}_x\left(\mathbb{1}_{t_2-s_1<\tau_L\wedge\tau_\partial\wedge\tau_{D_0}}\mathbb{E}_{X_{t_2-s_1}}\left(\psi_1(X_{s_1})\mathbb{1}_{s_1<\tau_L\wedge\tau_\partial}\right)\right) \\ &+ \mathbb{E}_x\left(\mathbb{1}_{\tau_{D_0}\leq t_2-s_1}\mathbb{E}_{X_{\tau_{D_0}}}\left(\psi_1(X_{t_2-u})\mathbb{1}_{t_2-u<\tau_\partial\wedge\tau_L}\right)\big|_{u=\tau_{D_0}}\right). \end{aligned} \tag{12.10}$$

Using Hölder's inequality and (12.9), we deduce that, for all $y \in D$,

$$\mathbb{E}_y\left(\psi_1(X_{s_1})\mathbb{1}_{s_1<\tau_L\wedge\tau_\partial}\right) \leq \mathbb{E}_y\left(\varphi(X_{s_1})\mathbb{1}_{s_1<\tau_\partial}\right)^{1/p} \leq e^{\frac{s_1C}{p}}\varphi(y)^{1/p} = e^{\frac{s_1C}{p}}\psi_1(y).$$

Hence, the first term in the right-hand side of (12.10) satisfies

$$\mathbb{E}_x\left(\mathbb{1}_{t_2-s_1<\tau_L\wedge\tau_\partial\wedge\tau_{D_0}}\mathbb{E}_{X_{t_2-s_1}}\left(\psi_1(X_{s_1})\mathbb{1}_{s_1<\tau_L\wedge\tau_\partial}\right)\right) \leq e^{\frac{s_1C}{p}}\mathbb{E}_x\left(\mathbb{1}_{t_2-s_1<\tau_L\wedge\tau_\partial\wedge\tau_{D_0}}\psi_1(X_{t_2-s_1})\right).$$

As a consequence, using again Hölder's inequality and applying as above Itô's formula using that $\mathcal{L}\varphi(x) \leq -\lambda_1\varphi(x)$ for all $x \notin D_0$, one has

$$\begin{aligned} \mathbb{E}_x\left(\mathbb{1}_{t_2-s_1<\tau_L\wedge\tau_\partial\wedge\tau_{D_0}}\mathbb{E}_{X_{t_2-s_1}}\left(\psi_1(X_{s_1})\mathbb{1}_{s_1<\tau_L\wedge\tau_\partial}\right)\right) &\leq e^{-\lambda_1\frac{t_2-s_1}{p}}e^{\frac{s_1C}{p}}\varphi(x)^{1/p} \\ &\leq e^{-t_2\frac{\rho_1'+\lambda_1/p}{2}}\psi_1(x), \end{aligned}$$

where we used in the last inequality that $t_2 \geq \frac{2s_1(C+\lambda_1)}{\lambda_1 - p\rho_1'}$. Moreover, using (12.8), we obtain that the second term in the right-hand side of (12.10) satisfies

$$\begin{aligned} &\mathbb{E}_x\left(\mathbb{1}_{\tau_{D_0}\leq t_2-s_1}\mathbb{E}_{X_{\tau_{D_0}}}\left(\psi_1(X_{t_2-u})\mathbb{1}_{t_2-u<\tau_\partial\wedge\tau_L}\right)\big|_{u=\tau_{D_0}}\right) \\ &\qquad\qquad \leq e^{-\rho_1't_2}\mathbb{P}_x(\tau_{D_0} \leq t_2 - s_1) \leq e^{-\rho_1't_2}\psi_1(x). \end{aligned}$$

We finally deduce from (12.10) that, for all $x \in D \setminus D_0$,

$$\mathbb{E}_x\left(\psi_1(X_{t_2})\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial}\right) \leq 2e^{-\rho_1't_2}\psi_1(x) \leq e^{-\rho_1t_2}\psi_1(x),$$

where we used that $t_2 \geq \log 2/(\rho_1' - \rho_1)$. This concludes the proof that the first line of (F2) holds true with $\gamma_1 = e^{-\rho_1}$.

Since φ is locally bounded, $\sup_L \varphi < \infty$, and hence, using again (12.9), we deduce that, for all $t \geq 0$,

$$\sup_{x\in L} \mathbb{E}_x(\psi_1(X_t)\mathbb{1}_{t<\tau_\partial}) \leq \sup_{x\in L} \mathbb{E}_x(\varphi(X_t)\mathbb{1}_{t<\tau_\partial}) \leq e^{Ct} \sup_{x\in L} \varphi(x) < \infty,$$

which implies the scond line of Assumption (F2).

In addition, because of the local uniform ellipticity of the diffusion X, for all $n_0 \geq 1$, $\psi_2 := \sum_{k=0}^{n_0} P_k \mathbb{1}_L$ is uniformly bounded away from zero on all compact subsets of D. This and Theorem 3.5 concludes the proof of Theorem 4.1.

12.4 Proof of Corollary 4.5

Using Theorem 3.5, there exists λ_0' such that, for all $x \in D$,

$$\eta(x) = \lim_{t\to+\infty} e^{\lambda_0' t}\mathbb{P}_x(t < \tau_\partial).$$

We choose in the definition of λ_0 a ball B such that $\nu_{QSD}(B) > 0$ (recall that λ_0 is independent of the choice of B). Given $x \in D$ such that $\eta(x) > 0$,

$$\lim_{t\to+\infty} e^{\lambda_0' t}\mathbb{P}_x(X_t \in B) = \eta(x)\nu_{QSD}(B) \in (0, +\infty).$$

Hence, $\lambda_0 = \lambda_0'$ and the infimum in the definition of λ_0 is a minimum. The facts that $\mathbb{P}_{\nu_{QSD}}(t < \tau_\partial) = e^{-\lambda_0 t}$ and $P_t\eta = e^{-\lambda_0 t}\eta$ are then direct consequences of Theorem 3.5.

Let us now prove that η is $\mathcal{C}^2$. First, it follows from [97, Theorem 7.2.4] that $x \mapsto e^{\lambda_0 t}\mathbb{P}_x(t < \tau_\partial)$ is continuous for all $t \geq 0$ (see e.g. [26] for a detailed proof). Hence the uniform convergence in Theorem 2.3 implies that η is continuous on D.

Now, let B be any non-empty open ball such that $\overline{B} \subset D$. We consider the following initial-boundary value problem (in the terminology of [50]) associated to the differential operator $\mathcal{L}$ defined in (4.3)

$$\begin{cases} \partial_t u(t,x) - \mathcal{L}u(t,x) - \lambda_0 u(t,x) = 0 & \text{for all } (t,x) \in (0,T] \times B, \\ u(0,x) = \eta(x) & \text{for all } x \in B, \\ u(t,x) = \eta(x) & \text{for all } (t,x) \in (0,T] \times \partial B. \end{cases}$$

Since the coefficients of $\mathcal{L}$ are Hölder and uniformly elliptic in $\overline{B}$ and since η is continuous, we can apply Corollary 1 of Chapter 3 of [50] to obtain the existence and uniqueness of a solution u to the above problem, continuous on $[0,T] \times \overline{B}$ and $\mathcal{C}^{1,2}((0,T] \times B)$. Now, we can apply Itô's formula to $e^{\lambda_0 s}u(T-s, X_s)$: for all $s < \tau_{B^c} \wedge T$ and all $x \in B$, $\mathbb{P}_x$-almost surely,

$$\begin{aligned} e^{\lambda_0 s}u(T-s, X_s) = u(T,x) &+ \int_0^s e^{\lambda_0 r}\left(-\frac{\partial u}{\partial t} + \mathcal{L}u + \lambda_0 u\right)(T-r, X_r)\,\mathrm{d}r \\ &+ \int_0^s e^{\lambda_0 r}\nabla u(T-r, X_r)\sigma(X_r)\,\mathrm{d}B_r. \end{aligned}$$

Since u is bounded and continuous on $[0,T] \times \overline{B}$ and $\nabla u(t,x)$ is locally bounded in $(0,T] \times B$, it follows from standard localization arguments that

$$\begin{aligned} u(T,x) &= \mathbb{E}_x\left[e^{\lambda_0(T\wedge\tau_{B^c})}u(T - (T\wedge\tau_{B^c}), X_{T\wedge\tau_{B^c}})\right] \\ &= \mathbb{E}_x\left[e^{\lambda_0(T\wedge\tau_{B^c})}\eta(X_{T\wedge\tau_{B^c}})\right]. \end{aligned}$$

Now, the Markov property and the fact that $P_t\eta = e^{-\lambda_0 t}\eta$ entail that $e^{\lambda_0 t}\eta(X_t)$ is a martingale on $(\mathcal{D}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$, hence

$$\eta(x) = \mathbb{E}_x\left[e^{\lambda_0(T\wedge\tau_{B^c})}\eta(X_{T\wedge\tau_{B^c}})\right] = u(T,x).$$

Therefore, $\eta \in \mathcal{C}^2(D)$ and $\mathcal{L}\eta(x) = -\lambda_0\eta(x)$ for all $x \in D$.

References

[1] E. Arjas, E. Nummelin, and R. L. Tweedie. Semi-Markov processes on a general state space: α-theory and quasistationarity. *J. Austral. Math. Soc. Ser. A*, 30(2):187–200, 1980/81. MR0607930

[2] K. B. Athreya and P. E. Ney. *Branching processes*. Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196. MR0373040

[3] R. Azaïs, J.-B. Bardet, A. Génadot, N. Krell, and P.-A. Zitt. Piecewise deterministic Markov process—recent results. In *Journées MAS 2012*, volume 44 of *ESAIM Proc.*, pages 276–290. EDP Sci., Les Ulis, 2014. MR3178622

[4] V. Bansaye, B. Cloez, P. Gabriel, and A. Marguet. A non-conservative Harris' ergodic theorem. *arXiv e-prints*, page arXiv:1903.03946, Mar 2019. MR4498558

[5] M. Baudel and N. Berglund. Spectral theory for random poincaré maps. *SIAM J. Appl. Math.*, 49(6):4319–4375, 2017. MR3719020

[6] M. Benaïm, N. Champagnat, W. Oçafrain, and D. Villemonais. Degenerate processes killed at the boundary of a domain. *arXiv preprint arXiv:2103.08534*, 2021.

[7] M. Benaïm, N. Champagnat, W. Oçafrain, and D. Villemonais. Quasi-compactness criterion for strong Feller kernels with an application to quasi-stationary distributions. *HAL preprint hal-03640205*, Apr. 2022.

[8] M. Benaïm, B. Cloez, and F. Panloup. Stochastic approximation of quasi-stationary distributions on compact spaces and applications. *Ann. Appl. Probab.*, 28(4):2370–2416, 2018. MR3843832

[9] N. Berglund and D. Landon. Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh-Nagumo model. *Nonlinearity*, 25(8):2303–2335, 2012. MR2946187

[10] G. Birkhoff. Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.*, 85:219–227, 1957. MR0087058

[11] F. M. Buchmann. Simulation of stopped diffusions. *J. Comput. Phys.*, 202(2):446–462, 2005. MR2145388

[12] D. L. Burkholder. Martingale transforms. *Ann. Math. Statist.*, 37:1494–1504, 1966. MR0208647

[13] M. M. Castro, J. S. W. Lamb, G. O. Méndez, and M. Rasmussen. Existence and uniqueness of quasi-stationary and quasi-ergodic measures for absorbing markov chains: a banach lattice approach. *arXiv preprint arXiv:2111.13791*, 2021.

[14] P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, 37(5):1926–1969, 2009. MR2561437

[15] P. Cattiaux and S. Méléard. Competitive or weak cooperative stochastic Lotka-Volterra systems conditioned to non-extinction. *J. Math. Biol.*, 60(6):797–829, 2010. MR2606515

[16] J. A. Cavender. Quasi-stationary distributions of birth-and-death processes. *Adv. Appl. Probab.*, 10(3):570–586, 1978. MR0501388

[17] N. Champagnat, K. A. Coulibaly-Pasquier, and D. Villemonais. Criteria for exponential convergence to quasi-stationary distributions and applications to multi-dimensional diffusions. *Séminaire de Probabilités XLIX*, pages 165–182, 2018. MR3837104

[18] N. Champagnat, P. Diaconis, and L. Miclo. On Dirichlet eigenvectors for neutral two-dimensional Markov chains. *Electron. J. Probab.*, 17:no. 63, 41, 2012. MR2968670

[19] N. Champagnat and S. Rœlly. Limit theorems for conditioned multitype Dawson-Watanabe processes and Feller diffusions. *Electron. J. Probab.*, 13:no. 25, 777–810, 2008. MR2399296

[20] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. *Probab. Theory Rel.*, 164(1):243–283, 2016. MR3449390

[21] N. Champagnat and D. Villemonais. Population processes with unbounded extinction rate conditioned to non-extinction. *ArXiv e-prints*, Nov. 2016.

[22] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution for absorbed one-dimensional diffusions with killing. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(1):177–199, 2017. MR3622466

[23] N. Champagnat and D. Villemonais. Uniform convergence of conditional distributions for absorbed one-dimensional diffusions. *Adv. in Appl. Probab.*, 50(1):178–203, 2018. MR3781982

[24] N. Champagnat and D. Villemonais. Uniform convergence of penalized time-inhomogeneous Markov processes. *ESAIM Probab. Stat.*, Mar. 2018. To appear. MR3877329

[25] N. Champagnat and D. Villemonais. Practical criteria for R-positive recurrence of unbounded semigroups. *Electron. Commun. Probab.*, 25(6):1–11, 2020. MR4066299

[26] N. Champagnat and D. Villemonais. Lyapunov criteria for uniform convergence of conditional distributions of absorbed Markov processes. *Stoch. Proc. Appl.*, 135:51–74, 2021. MR4222402

[27] N. Champagnat and D. Villemonais. Quasi-limiting estimates for periodic absorbed Markov chains. *arXiv preprint*, 2022.

[28] N. Champagnat and D. Villemonais. Quasi-stationary distributions in reducible state spaces. *arXiv preprint arXiv:2201.10151*, 2022.

[29] S. D. Chatterji. An L^p-convergence theorem. *Ann. Math. Statist.*, 40:1068–1070, 1969. MR0251769

[30] J.-R. Chazottes, P. Collet, and S. Méléard. Sharp asymptotics for the quasi-stationary distribution of birth-and-death processes. *Probab. Theory Related Fields*, 164(1-2):285–332, 2016. MR3449391

[31] J.-R. Chazottes, P. Collet, and S. Méléard. On time scales and quasi-stationary distributions for multitype birth-and-death processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 55(4):2249–2294, 2019. MR4029154

[32] P. Collet, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions for structured birth and death processes with mutations. *Probab. Theory Related Fields*, 151:191–231, 2011. 10.1007/s00440-010-0297-4. MR2834717

[33] P. Collet, S. Martínez, and J. San Martín. Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption. *Ann. Probab.*, 23(3):1300–1314, 1995. MR1349173

[34] P. Collet, S. Martínez, and J. San Martín. *Quasi-stationary distributions*. Probability and its Applications (New York). Springer, Heidelberg, 2013. Markov chains, diffusions and dynamical systems. MR2986807

[35] P. Coolen-Schrijner and E. A. van Doorn. Quasi-stationary distributions for a class of discrete-time Markov chains. *Methodol. Comput. Appl. Probab.*, 8(4):449–465, 2006. MR2329282

[36] C. Coron. *Stochastic modeling and eco-evolution of a diploid population*. PhD thesis, Palaiseau, Ecole polytechnique, 2013.

[37] C. Coron, S. Méléard, E. Porcher, and A. Robert. Quantifying the mutational meltdown in diploid populations. *Am. Nat.*, 181(5):623–636, 2013.

[38] J. N. Darroch and E. Seneta. On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *J. Appl. Probab.*, 2:88–100, 1965. MR0179842

[39] J. N. Darroch and E. Seneta. On quasi-stationary distributions in absorbing continuous-time finite Markov chains. *J. Appl. Probab.*, 4:192–196, 1967. MR0212866

[40] E. B. Davies and B. Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.*, 59(2):335–395, 1984. MR0766493

[41] P. Del Moral. *Feynman-Kac formulae*. Probability and its Applications (New York). Springer-Verlag, New York, 2004. Genealogical and interacting particle systems with applications. MR2044973

[42] P. Del Moral. *Mean field simulation for Monte Carlo integration*, volume 126 of *Monographs on Statistics and Applied Probability*. CRC Press, Boca Raton, FL, 2013. MR3060209

[43] P. Del Moral and D. Villemonais. Exponential mixing properties for time inhomogeneous diffusion processes with killing. *Bernoulli*, 24(2):1010–1032, 2018. MR3706785

[44] M. Faure and S. J. Schreiber. Quasi-stationary distributions for randomly perturbed dynamical systems. *Ann. Appl. Probab.*, 24(2):553–598, 2014. MR3178491

[45] P. A. Ferrari, H. Kesten, and S. Martínez. R-positivity, quasi-stationary distributions and ratio limit theorems for a class of probabilistic automata. *Ann. Appl. Probab.*, 6(2):577–616, 1996. MR1398060

[46] P. A. Ferrari, H. Kesten, S. Martínez, and P. Picco. Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.*, 23(2):501–521, 1995. MR1334159

[47] P. A. Ferrari, S. Martínez, and P. Picco. Some properties of quasi-stationary distributions in the birth and death chains: a dynamical approach. In *Instabilities and nonequilibrium structures, III (Valparaíso, 1989)*, volume 64 of *Math. Appl.*, pages 177–187. Kluwer Acad. Publ., Dordrecht, 1991. MR1177850

[48] P. A. Ferrari, S. Martínez, and P. Picco. Existence of nontrivial quasi-stationary distributions in the birth-death chain. *Adv. Appl. Probab.*, 24(4):795–813, 1992. MR1188953

[49] G. Ferré, M. Rousset, and G. Stoltz. More on the long time stability of Feynman-Kac semigroups. *Stochastics and Partial Differential Equations: Analysis and Computations*, 9(3):630–673, 2021. MR4297235

[50] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. MR0181836

[51] E. Gobet. Weak approximation of killed diffusion using Euler schemes. *Stoch. Process. Appl.*, 87(2):167–197, 2000. MR1757112

[52] E. Gobet. Euler schemes and half-space approximation for the simulation of diffusion in a domain. *ESAIM Probab. Statist.*, 5:261–297, 2001. MR1889160

[53] G. L. Gong, M. P. Qian, and Z. X. Zhao. Killed diffusions and their conditioning. *Probab. Theory Related Fields*, 80(1):151–167, 1988. MR0970476

[54] P. Good. The limiting behavior of transient birth and death processes conditioned on survival. *J. Austral. Math. Soc.*, 8:716–722, 1968. MR0240879

[55] F. Gosselin. Asymptotic behavior of absorbing Markov chains conditional on nonabsorption for applications in conservation biology. *Ann. Appl. Probab.*, 11(1):261–284, 2001. MR1825466

[56] A. Guillin, B. Nectoux, and L. Wu. Quasi-stationary distribution for strongly Feller Markov processes by Lyapunov functions and applications to hypoelliptic Hamiltonian systems. *HAL preprint hal-03068461*, Dec. 2020.

[57] M. Hairer. Convergence of markov processes (lecture notes). www.hairer.org/notes/Convergence.pdf, 2010.

[58] M. Hairer and J. C. Mattingly. Yet another look at Harris' ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progr. Probab.*, pages 109–117. Birkhäuser/Springer Basel AG, Basel, 2011. MR2857021

[59] T. E. Harris. *The theory of branching processes*. Die Grundlehren der Mathematischen Wissenschaften, Bd. 119. Springer-Verlag, Berlin; Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963. MR0163361

[60] C. R. Heathcote, E. Seneta, and D. Vere-Jones. A refinement of two theorems in the theory of branching processes. *Teor. Verojatnost. i Primenen.*, 12:341–346, 1967. MR0217889

[61] A. Hening and M. Kolb. Quasistationary distributions for one-dimensional diffusions with singular boundary points. *Stoch. Process. Appl.*, 129(5):1659–1696, 2019. MR3944780

[62] G. Hinrichs, M. Kolb, and V. Wachtel. Persistence of one-dimensional AR(1)-sequences. *J. Theoret. Probab.*, 33(1):65–102, 2020. MR4064294

[63] K. Itô and H. P. McKean, Jr. *Diffusion processes and their sample paths*, volume 125 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1974. Second printing, corrected. MR0345224

[64] A. Joffe and F. Spitzer. On multitype branching processes with $\rho \leq 1$. *J. Math. Anal. Appl.*, 19:409–430, 1967. MR0212895

[65] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002. MR1876169

[66] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. MR1121940

[67] S. Karlin and J. McGregor. The classification of birth and death processes. *Trans. Amer. Math. Soc.*, 86:366–400, 1957. MR0094854

[68] M. Kijima and E. Seneta. Some results for quasi-stationary distributions of birth-death processes. *J. Appl. Probab.*, 28(3):503–511, 1991. MR1123824

[69] J. F. C. Kingman. The exponential decay of Markov transition probabilities. *Proc. London Math. Soc. (3)*, 13:337–358, 1963. MR0152014

[70] R. Knobloch and L. Partzsch. Uniform conditional ergodicity and intrinsic ultracontractivity. *Potential Anal.*, 33:107–136, 2010. MR2658978

[71] M. Kolb and D. Steinsaltz. Quasilimiting behavior for one-dimensional diffusions with killing. *Ann. Probab.*, 40(1):162–212, 2012. MR2917771

[72] N. V. Krylov and M. V. Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1):161–175, 239, 1980. MR0563790

[73] A. Lambert. Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. *Electron. J. Probab.*, 12:no. 14, 420–446, 2007. MR2299923

[74] J. Littin C. Uniqueness of quasistationary distributions and discrete spectra when ∞ is an entrance boundary and 0 is singular. *J. Appl. Probab.*, 49(3):719–730, 2012. MR3012095

[75] M. Lladser and J. San Martín. Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process. *J. Appl. Probab.*, 37(2):511–520, 2000. MR1781008

[76] P. Maillard. The λ-invariant measures of subcritical Bienaymé-Galton-Watson processes. *Bernoulli*, 24(1):297–315, 2018. MR3706758

[77] P. Mandl. Sur le comportement asymptotique des probabilités dans les ensembles des états d'une chaîne de Markov homogène. *Časopis Pěst. Mat.*, 84:140–149, 1959. MR0108854

[78] P. Mandl. Spectral theory of semi-groups connected with diffusion processes and its application. *Czechoslovak Math. J.*, 11 (86):558–569, 1961. MR0137143

[79] R. Mannella. Absorbing boundaries and optimal stopping in a stochastic differential equation. *Phys. Lett. A*, 254(5):257–262, 1999. MR1687576

[80] A. Marguet. A law of large numbers for branching Markov processes by the ergodicity of ancestral lineages. *ESAIM Probab. Statist.*, 23:638–661, 2019. MR4011569

[81] S. Martínez and J. San Martín. Classification of killed one-dimensional diffusions. *Ann. Probab.*, 32(1A):530–552, 2004. MR2040791

[82] S. Martínez, J. San Martín, and D. Villemonais. Existence and uniqueness of a quasistationary distribution for Markov processes with fast return from infinity. *J. Appl. Probab.*, 51(3):756–768, 2014. MR3256225

[83] S. Méléard and D. Villemonais. Quasi-stationary distributions and population processes. *Probab. Surv.*, 9:340–410, 2012. MR2994898

[84] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes. III. Foster-Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.*, 25(3):518–548, 1993. MR1234295

[85] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Cambridge University Press, Cambridge, second edition, 2009. With a prologue by Peter W. Glynn. MR2509253

[86] Y. Miura. Ultracontractivity for Markov semigroups and quasi-stationary distributions. *Stoch. Anal. Appl.*, 32(4):591–601, 2014. MR3219695

[87] Y. Miura. Ultracontractivity for Markov semigroups and quasi-stationary distributions. *Stoch. Anal. Appl.*, 32(4):591–601, 2014. MR3219695

[88] Y. Ogura. Asymptotic behavior of multitype Galton-Watson processes. *J. Math. Kyoto Univ.*, 15(2):251–302, 1975. MR0383560

[89] W. Oçafrain. Convergence to quasi-stationarity through Poincaré inequalities and Bakry-Émery criteria. *Electron. J. Probab.*, 26:1–30, 2021. MR4278594

[90] R. G. Pinsky. On the convergence of diffusion processes conditioned to remain in a bounded region for large time to limiting positive recurrent diffusion processes. *Ann. Probab.*, 13(2):363–378, 1985. MR0781410

[91] R. G. Pinsky. Explicit and almost explicit spectral calculations for diffusion operators. *J. Funct. Anal.*, 256(10):3279–3312, 2009. MR2504526

[92] M. Pollock, P. Fearnhead, A. M. Johansen, and G. O. Roberts. The scalable langevin exact algorithm: Bayesian inference for big data. *arXiv preprint arXiv:1609.03436*, 2016.

[93] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability. MR2020294

[94] B. Roynette, P. Vallois, and M. Yor. Some penalisations of the Wiener measure. *Jpn. J. Math.*, 1(1):263–290, 2006. MR2261065

[95] E. Seneta and D. Vere-Jones. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probab.*, 3:403–434, 1966. MR0207047

[96] D. Steinsaltz and S. N. Evans. Markov mortality models: Implications of quasistationarity and varying initial conditions. *Theo. Pop. Bio.*, 65(65):319–337, 2004.

[97] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition. MR2190038

[98] M. Takeda. Existence and uniqueness of quasi-stationary distributions for symmetric markov processes with tightness property. *J. Theoret. Probab.*, Jan 2019. MR4020697

[99] E. A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. Appl. Probab.*, 23(4):683–700, 1991. MR1133722

[100] E. A. van Doorn. Conditions for the existence of quasi-stationary distributions for birth-death processes with killing. *Stoch. Process. Appl.*, 122(6):2400–2410, 2012. MR2922634

[101] E. A. van Doorn. Weighted sums of orthogonal polynomials related to birth-death processes with killing. *Adv. Dyn. Syst. Appl.*, 8(2):401–412, 2013. MR3162157

[102] E. A. van Doorn and P. K. Pollett. Survival in a quasi-death process. *Linear Algebra Appl.*, 429(4):776–791, 2008. MR2428129

[103] E. A. van Doorn and P. K. Pollett. Quasi-stationary distributions for reducible absorbing Markov chains in discrete time. *Markov Process. Related Fields*, 15(2):191–204, 2009. MR2538313

[104] E. A. van Doorn and P. K. Pollett. Quasi-stationary distributions for discrete-state models. *European J. Oper. Res.*, 230(1):1–14, 2013. MR3063313

[105] E. A. van Doorn and A. I. Zeifman. Extinction probability in a birth-death process with killing. *J. Appl. Probab.*, 42(1):185–198, 2005. MR2144903

[106] A. Velleret. Unique quasi-stationary distribution, with a possibly stabilizing extinction. *Stoch. Process. Appl.*, 148:98–138, 2022. MR4393344

[107] D. Vere-Jones. Ergodic properties of nonnegative matrices. I. *Pacific J. Math.*, 22:361–386, 1967. MR0214145

[108] D. Villemonais. Minimal quasi-stationary distribution approximation for a birth and death process. *Electron. J. Probab.*, 20:no. 30, 18, 2015. MR3325100

[109] A. Q. Wang, M. Kolb, G. O. Roberts, and D. Steinsaltz. Theoretical properties of quasi-stationary monte carlo methods. *Ann. Appl. Probab.*, 29(1):434–457, 2019. MR3910008

[110] J. Wang. First eigenvalue of one-dimensional diffusion processes. *Electron. Commun. Probab.*, 14:232–244, 2009. MR2507752

[111] J. Wang. Sharp bounds for the first eigenvalue of symmetric Markov processes and their applications. *Acta Math. Sin. (Engl. Ser.)*, 28(10):1995–2010, 2012. MR2966951

[112] A. M. Yaglom. Certain limit theorems of the theory of branching random processes. *Doklady Akad. Nauk SSSR (N.S.)*, 56:795–798, 1947. MR0022045