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On non-extinction in a Fleming-Viot-type particle model with Bessel drift

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Abstract

Motivated by the work [6] of Mariusz Bieniek, Krzysztof Burdzy and Soumik Pal we study a Fleming-Viot-type particle system consisting of independently moving particles each driven by generalized Bessel processes on the positive real line. Upon hitting the boundary $\{0\}$ this particle is killed and an uniformly chosen different one branches into two particles. Using the symmetry of the model and the self similarity property of Bessel processes, we obtain a criterion to decide whether the particles converge to the origin at a finite time. This addresses open problem 1.4 in [6]. Specifically, inspired by [6, Open Problem 1.5], we investigate the case of three moving particles and refine the general result of [6, Theorem 1.1(ii)] extending the regime of drift parameters, where convergence does not occur – even to values, where it does occur when considering the case of only two particles.

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1 Introduction

In [9], the authors analyzed the following particle system which was introduced earlier in [10]: Consider a fixed open connected subset D of the Euclidean space as domain and a system of particles starting at some deterministic point. As long as no particle hits the boundary ∂D of the domain they move independently according to Brownian motion. If a particle does hit the boundary, it jumps to the position of some independently uniformly randomly chosen other one keeping the number of particles constant. Then, they again move independently according to Brownian motion, and so on. In [9] different limit theorems are proven, one of them shows, that the empirical distribution at a fixed finite time converges to the law of Brownian motion conditioned to stay inside the domain as the number of particles goes to infinity. These results have been later improved and generalized to processes other than Brownian motion (e.g. [18],

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[1], [2], [21]) and also refined limit theorems such as theorems of central limit type have been obtained. Interesting recent work include: [13], [24], [14], [12], [22], [4] and [26].

It turned out, that proving the non-extinction of the particle system, i.e. proving that with probability one only finitely many jumps occur in finite time, is much more subtle than initially thought. Under suitable regularity assumptions this property was proved in [5, Theorem 5.4] and [20, Theorem 1] as well as in [28, Theorem 2.1]. In [6] the authors demonstrated that extinction of the particle system can actually occur. In this paper the authors in particular proved, that in the case of $D = (0, \infty)$ and two particles driven by Bessel processes there is a phase transition. In fact in [6, Theorem 1.1(i)] it is shown, that the two particle process goes extinct if and only if the parameter ν of the Bessel process (in the parametrization (2.1)) is negative. Furthermore the authors are able to prove the non-extinction of a N-particle system driven by Bessel processes, if the parameter ν is greater or equal to $2/N$.

In this work we are interested in the case of $N \geq 3$ particles. We give sufficient conditions ensuring the extinction and non-extinctions of the particle system in terms of an integral test, which involves a probability measure, which seems difficult to calculate explicitly. Still we demonstrate that the criterion can be applied efficiently to establish non-extinction of the particle system. In particular, we in some sense give an affirmative answer to a converse of [6, Open Problem 1.5] in Theorem 5.4: There exists a Fleming-Viot-type process with extinction almost surely, for the 2-particle system, but non-extinction with probability one for the 3-particle system. This illustrates that generally speaking adding another particle to the system potentially does cause non-extinction.

The rest of the paper is organized as follows: Section 2 is used to formalize the problem and to rigorously formulate the notion of the model's inherent symmetry. Brownian Scaling for generalized Bessel processes is put to use in order for Corollary 2.7 to entail an alternative description of the law in question. In the subsequent section a few calculations of some density functions are carried out to be used later on. In Section 4, aspects of the theory of Markov chains in uncountable state spaces is utilized to see, that the underlying Hidden Markov Model behaves well implying the criterion 4.5 as result. As an application, in the succeeding section the case of three particles is considered and the result 5.4 is achieved.

Finally, in the closing sections we briefly discuss two possible directions of related further research and append some formulas used from external sources and prove two integral formulas we use in Section 3.

2 Notation and basic properties

This section follows the notion, that we only need to know the particles' positions at jumping times and how long it took for the next jump to occur. Without loss of generality the positive positions of the particles not jumping may be indexed in ascending order. Next, by a scaling property of generalized Bessel processes, we may transform the problem to polar coordinates and see that the next position only depends on the angles of the old positions. The dependency structure is expressible as Hidden Markov Model and entails us to give an alternative expression for the extinction probability. This will imply an abstract criterion in Section 4.

2.1 Problem formulation

Let us start by giving a more formal description of the problem under investigation: We consider a system $(X_t)_{t\geq 0} = (X_t^1, \dots, X_t^N)_{t\geq 0}$ of $N \in \mathbb{N}, N \geq 2$ particles starting in $X_0 = x_0 \in (0, \infty)^N$. We will generally use the superindex to distinguish components and denote the time in the subindex. $\mathbb{P}_{x_0}(\cdot)$ and $\mathbb{E}_{x_0}[\cdot]$ represent probabilities and expectations regarding events and functionals of X_t starting in x_0. As long as no particle reaches 0, they move independently according to the generalized Bessel processes with parametrization

$$dX_t^j = dB_t^j + \frac{(\nu-1)/2}{X_t^j}\,dt, \quad j = 1, \dots, N. \tag{2.1}$$

Here, B_t^j are independent Brownian motions. The stopping time

$$\tau_1 := \min_{j=1}^{N} \inf\{t > 0 : \lim_{s\uparrow t} X_s^j = 0\}$$

denotes the first time any of the particles would hit 0. Note, that $\mathbb{P}_{x_0}(\tau_1 < \infty) = 1$ if and only if $\nu < 2$ and in that case there is some unique j with $\tau_1 = \inf\{t > 0 : \lim_{s\uparrow t} X_s^j = 0\}$ almost surely. This particle with superindex j will be set independently and uniformly to the position of one of the other $N-1$ particles at time τ_1. Therefore, the system stays in the state space $(0,\infty)^N$ and the jump is implemented in a fashion, the paths of all particles being càdlàg. After the jump the particles again move as independent generalized Bessel processes until time τ_2 where $\tau_n := \tau_{n-1} + \tau \circ \theta_{\tau_{n-1}}$ for $n \geq 2$ and θ_s denotes the time shift $\theta_s((X_t)_{t\geq 0}) := (X_{s+t})_{t\geq 0}$. The mechanism is repeated inductively; the system (X_t) is a Markovian process in continuous time $0 \leq t < \lim_{n\to\infty} \tau_n$ and state space $(0,\infty)^N$ with càdlàg paths. Since there is no natural way to define the process for time points $t \geq \lim_{n\to\infty} \tau_n =: \tau_\infty$ the question arises if that limit in fact diverges (i.e. no extinction occurs) almost surely. Without loss of generality we may assume $\nu < 2$.

2.2 Symmetry of the model

Let $X_{t-} := \lim_{s\uparrow t} X_s$ denote the limit from the left. To simplify notation we formally define

$$X_{0-} := x_{0-} \in H := \dot{\bigcup_{j=1}^{N}} (0,\infty)^{j-1} \times \{0\} \times (0,\infty)^{N-j} \tag{2.2}$$

meaning that with $x_{0-}^j = 0$ for all $l \in \{1, \dots, N\} \setminus \{j\}$ it holds

$$\mathbb{P}_{x_{0-}}\left(X_0 = (x_{0-}^1, \dots, x_{0-}^{j-1}, x_{0-}^l, x_{0-}^{j+1}, \dots, x_{0-}^N)\right) = 1/(N-1).$$

Upon setting $\tau_0 := 0$ we may consider the jump time chain $(\mathring{X}_n)_{n\in\mathbb{N}_0} := (X_{\tau_n-})_{n\in\mathbb{N}_0}$ as discrete time Markov chain on state space H.

For $x \in H$ and a permutation $\pi \in S_N$ on $\{1, \dots, N\}$ let us introduce the notation $x^\pi := (x^{\pi(1)}, \dots, x^{\pi(N)})$ and for subsets $A \subseteq H$ let us define $A^\pi := \{a^\pi : a \in A\}$. Observe, that for starting values $x_0 \in H$, permutations $\pi \in S_N$, measurable sets $A \in \mathcal{B}(H)$ and time indices $n \in \mathbb{N}_0$:

$$\mathbb{P}_{x_0}\left(\mathring{X}_n \in A\right) = \mathbb{P}_{x_0^\pi}\left(\mathring{X}_n^{\pi^{-1}} \in A\right) = \mathbb{P}_{x_0^\pi}\left(\mathring{X}_n \in A^\pi\right).$$

For $x, y \in H$ let the equivalence relation $x \leftrightsquigarrow y$ hold, if and only if $y = x^\pi$ for some $\pi \in S_N$. Let $[x] := \{x^\pi : \pi \in S_n\}$ the equivalence class of $x \in H$. For subsets $A \subseteq H$ let $A/\leftrightsquigarrow := \{[a] : a \in A\}$ the corresponding set of equivalence classes. The set $\mathcal{B}(H)/\leftrightsquigarrow := \{A/\leftrightsquigarrow : A \in \mathcal{B}(H)\}$ is a σ-field. Note, that $A/\leftrightsquigarrow = B/\leftrightsquigarrow$ implies $\bigcup_{\pi\in S_N} A^\pi = \bigcup_{\pi\in S_N} B^\pi$ and that $[x] = [y]$ implies $\kappa(x, \bigcup_{\pi\in S_N} A^\pi) = \kappa(y, \bigcup_{\pi\in S_N} A^\pi)$ where

$$\kappa : H \times (\mathcal{B}(H))^{\otimes\mathbb{N}_0} \to [0,1], \quad \kappa(x, A) = \mathbb{P}_x\left((\mathring{X}_n)_{n\in\mathbb{N}_0} \in A\right)$$

denotes the distribution of the jumping time chain $(\mathring{X}_n)_{n\in\mathbb{N}_0}$.

We can therefore define the Markov chain $(\mathring{Z}_n)_{n\in\mathbb{N}_0} = (\mathring{X}_n/\leftrightsquigarrow)_{n\in\mathbb{N}_0}$ on state space $H/\leftrightsquigarrow$ with probability measures $(Q_x)_{x\in H/\leftrightsquigarrow}$ via

$$Q_{[x_0]}\left(\mathring{Z}_n \in A/\leftrightsquigarrow\right) := \mathbb{P}_{x_0}\left(\mathring{X}_n \in \bigcup_{\pi\in S_N} A^\pi\right).$$

For $x \in H$ let $x^\downarrow$ the uniquely determined representative of $[x]$ with

$$x^1 \geq x^2 \geq \ldots \geq x^N = 0$$

and for $A \subseteq H$ let $A^\downarrow := \{a^\downarrow : a \in A\}$. Using this notation it holds

$$Q_{[x_0]}\left(\mathring{Z}_n \in A/\leftrightsquigarrow\right) = \mathbb{P}_{x_0}\left(\mathring{X}_n \in \bigcup_{\pi\in S_N} A^\pi\right) = \mathbb{P}_{x_0^\downarrow}\left(\mathring{X}_n^\downarrow \in A^\downarrow\right).$$

We might as well consider the Markov chain $(\mathring{X}_n^\downarrow)_{n\in\mathbb{N}_0}$ on $\{x \in (0,\infty)^{N-1} \times \{0\} : x^1 \geq x^2 \geq \ldots x^N = 0\}$. This suggests to neglect the redundant 0 in the last N-th component. Finally, we may define

$$\begin{aligned}
&Q_{[z_0]}(Z_n \in A/\leftrightsquigarrow) := Q_{[(z_0^1,\ldots,z_0^{N-1},0)]}(\mathring{Z}_n \in \{[(a^1,\ldots,a^{N-1},0)] : a \in A\})\\
&= \mathbb{P}_{(z_0^1,\ldots,z_0^{N-1},0)}(\mathring{X}_n \in \{(a^1,\ldots,a^{N-1},0)^\pi : a \in A, \pi \in S_N\})\\
&= \frac{1}{N-1}\sum_{l=1}^{N-1} \mathbb{P}_{(z_0^1,\ldots,z_0^{N-1},z_0^l)}(X_{\tau_n-} \in \{(a^1,\ldots,a^{N-1},0)^\pi : a \in A, \pi \in S_N\}).
\end{aligned}$$

This leads to

Definition 2.1. *Let* $(Y_n)_{n\in\mathbb{N}_0} := ((X_{\tau_n-}^\downarrow)^1,\ldots,(X_{\tau_n-}^\downarrow)^{N-1})_{n\in\mathbb{N}_0}$ *denote the Markov chain on the state space* $(0,\infty)^{N-1\downarrow}$.

2.3 Self-similarity of generalized Bessel processes

In what follows we substantially want to exploit the scaling property of generalized Bessel processes. Let $\|x\| := \left(\sum_j \left(x^j\right)^2\right)^{1/2}$ denote the Euclidean norm. Let again $H := \dot\bigcup_{j=1}^N (0,\infty)^{j-1} \times \{0\} \times (0,\infty)^{N-j}$ as in (2.2) denote the subspace of $[0,\infty)^N$ where exactly one component equals 0 and let $m : H \to \{1,\ldots,N\}$, $m(x) := \arg\{j : x^j = 0\}$ the mapping picking that component. Moreover, for $x \in H$ and $l \in \{1,\ldots,N\}\setminus\{m(x)\}$ define

$$y(x,l) := (x^1,\ldots,x^{m(x)-1},x^l,x^{m(x)+1},\ldots,x^N) \in (0,\infty)^N.$$

We now describe given $N \in \mathbb{N}$, $N \geq 2$, a "normed" Fleming-Viot N-particle process $(\overline{X}_t)_{t\geq 0}$ to the parameter $\nu < 2$ on the state space $(0,\infty)^N$. In a nutshell, ahead each jumping time, we normalize the $(N-1)$ positive particles by dividing by their norm $\|\overline{X}_{\overline{\tau}_n-}\|$. Just after the jump, their norm will then be strictly larger than 1.

Definition 2.2. *Let* $(V_n)_{n\in\mathbb{N}_0}$ *a family of independent random variables uniformly distributed on* $\{1,\ldots,N-1\}$ *independently from anything else. For* $n \in \mathbb{N}_0$ *let* $\overline{\tau}_n := \tau_n((\overline{X}))$, *i.e.*

$$\overline{\tau}_0 = 0, \qquad \overline{\tau}_n = \min_{j=1}^N \inf\{t > \overline{\tau}_{n-1} : \overline{X}_{s-}^j = 0\} \text{ for } n \in \mathbb{N}.$$

The components $(\overline{X}_t^j)$, $j = 1,\ldots,N$ *move during time points* $t \in (\overline{\tau}_{n-1},\overline{\tau}_n)$ *independently from* (V_n) *and independently from each other according to a Bessel process starting in*

$\overline{X}^j_{\overline{\tau}_{n-1}}$ *and global parameter* $\nu < 2$ *in the sense of* (2.1)*. Denoting* $b_n : \{1, \ldots, N-1\} \to \{1, \ldots, N\} \setminus \{m(\overline{X}_{\overline{\tau}_n -})\}$ *for the unique order preserving bijection we set*

$$\overline{X}_{\overline{\tau}_n} := y\left(\frac{\overline{X}_{\overline{\tau}_n -}}{\|\overline{X}_{\overline{\tau}_n -}\|}, b_n(V_n)\right), \quad n \in \mathbb{N}_0.$$

In order to facilitate an alternative problem formulation we must relate the normed Fleming-Viot process $\overline{X}_t$ with the original process X_t, i.e. we must be able to properly scale it back so we have no loss of information:

Definition 2.3. *We define for* $n \in \mathbb{N}_0$ *the backscaled (series of) jumping times*

$$\tau_n^{bs} := \sum_{k=1}^{n} \prod_{j=1}^{k} \|\overline{X}_{\overline{\tau}_{j-1}-}\|^2 \cdot (\overline{\tau}_k - \overline{\tau}_{k-1})$$

and the backscaled process

$$X_t^{bs} := \sum_{n=1}^{\infty} \mathbb{1}_{\left[\tau_{n-1}^{bs}, \tau_n^{bs}\right)}(t) \left(\prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\overline{\tau}_{n-1} + (t - \tau_{n-1}^{bs}) / \prod_{j=1}^{n} \|\overline{X}_{\overline{\tau}_{j-1}-}\|^2}\right).$$

Proposition 2.4 (cf. the alternative construction of the Fleming-Viot process in the beginning of the proof of [6, Theorem 1.1 (i)].)**.** *Let* $X_{0-} := x_{0-} \in H$ *arbitrary and set* $\overline{X}_{0-} := x_{0-}$*. Then the processes* (X_t) *and* (X_t^{bs}) *are identically distributed.*

Proof. We will use the following scaling invariance: If $c > 0$ and X_t a generalized Bessel process stopped at the origin starting in $x_0 > 0$ and Y_t an independent generalized Bessel process stopped at the origin starting in x_0/c, the processes $(\mathbb{1}_{[0, \inf\{s : X_s = 0\})}(t)\, X_t)$ and $(\mathbb{1}_{[0, c^2 \cdot \inf\{s : Y_s = 0\})}(t)\, c \cdot Y_{t/c^2})$ are identically distributed.

To this end, let us inductively show for $M \in \mathbb{N}$:

$$\sum_{n=1}^{M} \mathbb{1}_{[\tau_{n-1}^{\mathrm{bs}}, \tau_n^{\mathrm{bs}})} X^{\mathrm{bs}} \overset{d}{=} \sum_{n=1}^{M} \mathbb{1}_{[\tau_{n-1}, \tau_n)} X.$$

Start of induction ($M = 1$*):* Firstly, by assumption

$$X_0^{\mathrm{bs}} = \|x_{0-}\| \cdot y\left(x_{0-}/\|x_{0-}\|, b_0(V_0)\right) = y(x_{0-}, b_0(V_0)) \overset{d}{=} X_0.$$

By the scaling property

$$\mathbb{1}_{[\tau_0^{\mathrm{bs}}, \tau_1^{\mathrm{bs}})} X^{\mathrm{bs}} = \mathbb{1}_{[0, \|x_{0-}\|^2 \overline{\tau}_1)} \|x_{0-}\| \cdot \overline{X}_{\bullet / \|x_{0-}\|^2} \overset{d}{=} \mathbb{1}_{[\tau_0, \tau_1)} X.$$

Inductive step ($M \to M+1$*):* By induction hypothesis, $\tau_M^{\mathrm{bs}} \overset{d}{=} \tau_M$ and $X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-} \overset{d}{=} X_{\tau_M -}$. Hence

$$\begin{aligned} X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}} &= \prod_{j=1}^{M+1} \|\overline{X}_{\overline{\tau}_{j-1}-}\| \cdot \overline{X}_{\overline{\tau}_M} = \|X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}\| \cdot y\left(\overline{X}_{\overline{\tau}_n -}/\|\overline{X}_{\overline{\tau}_n -}\|, b_n(V_n)\right) \\ &= \|X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}\| \cdot y\left(X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}/\|X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}\|, b_n(V_n)\right) = y\left(X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}, b_M(U_M)\right) \overset{d}{=} X_{\tau_M} \end{aligned}$$

and again by scaling

$$\mathbb{1}_{[\tau_M^{\mathrm{bs}}, \tau_{M+1}^{\mathrm{bs}})} X^{\mathrm{bs}} = \mathbb{1}_{[\tau_M^{\mathrm{bs}}, \tau_{M+1}^{\mathrm{bs}})} \|X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}\| \cdot \overline{X}_{\overline{\tau}_M + (\bullet - \tau_M^{\mathrm{bs}}) / \|X^{\mathrm{bs}}_{\tau_M^{\mathrm{bs}}-}\|^2} \overset{d}{=} \mathbb{1}_{[\tau_M, \tau_{M+1})} X. \qquad \square$$

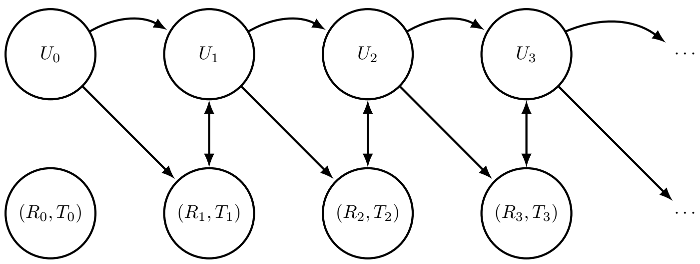
Remark 2.5. Definition 2.2 and Definition 2.3 may be regarded to as generalization of the construction given in the beginning of the proof of [6, Theoream 1.1 (i)] where the scaling invariance of generalized Bessel processes was used as well. Here, for $N > 2$ the sequence of random variables $(\overline{X}^{\downarrow}_{\overline{\tau}_n-}/\|\overline{X}^{\downarrow}_{\overline{\tau}_n-}\|)_{n\in\mathbb{N}_0}$ is not deterministic.

We are now ready to consider the normed process written with descending ordering and introduce some new letters for sequences of random variables to be used later on.

Definition 2.6. *Let us change to polar coordinates upon defining*

$$\begin{aligned}(\overline{Y}_n, T_n)_{n\in\mathbb{N}_0} &:= (((\overline{X}^{\downarrow}_{\overline{\tau}_n-})^1, \dots, (\overline{X}^{\downarrow}_{\overline{\tau}_n-})^{N-1}), \overline{\tau}_n - \overline{\tau}_{(n-1)\vee 0})_{n\in\mathbb{N}_0}\\ &\simeq \left(\frac{\overline{Y}_n}{\|\overline{Y}_n\|}, \|\overline{Y}_n\|, T_n\right)_{n\in\mathbb{N}_0} =: (U_n, R_n, T_n)_{n\in\mathbb{N}_0}.\end{aligned}$$

The following picture illustrates the dependency structure implied by Proposition 2.4: Given the direction U_n of the particles immediately ahead a jump, there is a Markovian transition kernel to the direction U_{n+1} immediately ahead the next jump, the factor R_{n+1} by which the magnitude will change and the time T_{n+1} it will take. The marginals of $(U_{n+1}, R_{n+1}, T_{n+1})$ given U_n are not independent.



In other words: $(M_n)_{n\in\mathbb{N}_0} := ((U_n, U_{n+1}), (R_{n+1}, T_{n+1}))_{n\in\mathbb{N}_0}$ is a hidden Markov model (HMM) with hidden chain (U_n, U_{n+1}) and observed chain (R_{n+1}, T_{n+1}) on the product space $\mathsf{S}^2 \times (0,\infty)^2$ where

$$\mathsf{S} := \{x = (x^1, \dots, x^{N-1}) \in (0,\infty)^{N-1\downarrow} : \|x\| = 1\}. \tag{2.3}$$

Corollary 2.7. *Under the assumptions of Proposition 2.4 it holds*

$$\tau_\infty \overset{d}{=} \|x_{0-}\|^2 \sum_{k=1}^{\infty} \prod_{j=1}^{k-1} \|\overline{X}_{\overline{\tau}_j-}\|^2 \cdot (\overline{\tau}_k - \overline{\tau}_{k-1}) = R_0^2 \sum_{k=1}^{\infty} \prod_{j=1}^{k-1} R_j^2 \cdot T_k.$$

3 Densities

In this section we calculate several density functions for later usage. Some of the technical aspects are outsourced to the appendix.

Lemma 3.1. *The Markov chain (Y_n) from Definition 2.1 admits a density function $h_{y_0}(y)$ of the form*

$$h_{y_0}(y) = \frac{2}{N-1} \sum_{j,k=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(k)\le\pi(j)}} \int_0^\infty g_{y_0^k}(t)\, f_t(y_0^j, y^{\pi(k)}) \prod_{\substack{s=1\\ s\neq k}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, dt,$$

where

$$g_x(t) = \mathbb{P}_x(\inf\{t > 0 : X_t^1 = 0\} \in dt) \tag{3.1}$$

is the density function of the hitting time of 0 *of a generalized Bessel process and*

$$f_t(x,y) = \mathbb{P}_x(X_t^1 \in dy, \min\{X_s : s \leq t\} > 0) \tag{3.2}$$

is the transition density of a generalized Bessel process stopped in the origin starting at $x > 0$ *and moving to* $y > 0$ *at time* $t > 0$.

Proof. Let $y_0 \in (0,\infty)^{N-1\downarrow}$ and $A \in \mathcal{B}\left((0,\infty)^{N-1\downarrow}\right)$ arbitrary. By independence the transition density of $((X_t^{\downarrow})^1, \ldots, (X_t^{\downarrow})^{N-1})$ up to the first jump is given by

$$\begin{aligned}
&\mathbb{P}_{y_0}(Y_1 \in A) = \frac{1}{N-1}\sum_{j=1}^{N-1} \mathbb{P}_{(y_0^1,\ldots,y_0^{N-1},y_0^j)}(X_{\tau-} \in \{(a^1,\ldots,a^{N-1},0)^\pi : a \in A, \pi \in S_N\}) \\
&= \frac{1}{N-1}\sum_{j=1}^{N-1}\Bigg(2\int_0^\infty \int_{\bigcup_{\pi\in S_{N-1}} A^\pi} \prod_{s=1}^{N-1} f_t(y_0^s, y^s)\, dy\, g_{y_0^j}(t)\, dt \\
&\qquad + \sum_{\substack{k=1\\k\neq j}}^{N-1} \int_0^\infty \int_{\bigcup_{\pi\in S_{N-1}} A^\pi} f_t(y_0^j, y^k) \prod_{\substack{s=1\\s\neq k}}^{N-1} f_t(y_0^s, y^s)\, dy\, g_{y_0^k}(t)\, dt\Bigg). \\
&= \int_A \frac{2}{N-1}\sum_{j=1}^{N-1}\Bigg(\sum_{\pi\in S_{N-1}} \int_0^\infty \prod_{s=1}^{N-1} f_t(y_0^s, y^{\pi(s)})\, g_{y_0^j}(t)\, dt \\
&\qquad + \sum_{\substack{k=1\\k\neq j}}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(k)<\pi(j)}} \int_0^\infty f_t(y_0^j, y^{\pi(k)}) \prod_{\substack{s=1\\s\neq k}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, g_{y_0^k}(t)\, dt\Bigg)\, dy \\
&= \int_A \frac{2}{N-1}\sum_{j,k=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(k)\leq\pi(j)}} \int_0^\infty f_t(y_0^j, y^{\pi(k)}) \prod_{\substack{s=1\\s\neq k}}^{N-1} f_t(y_0^s, y^{\pi(s)})\, g_{y_0^k}(t)\, dt\, dy.
\end{aligned}$$

In the second equation we have first considered both cases where the dying particle starts from the doubly occupied position. In the third equation we have used that for $\pi_1 \neq \pi_2 \in S_{N-1}$ it holds

$$A^{\pi_1} \cap A^{\pi_2} \subseteq (0,\infty)^{N-1\downarrow^{\pi_1}} \cap (0,\infty)^{N-1\downarrow^{\pi_2}} \subseteq \bigcup_{\substack{j,k=1\\ j\neq k}}^{N-1} \{a \in (0,\infty)^{N-1} : a^j = a^k\}$$

which is a Lebesgue null set. □

Remark 3.2. Accounting for the time length and using polar coordinates the Markov transition from (U_n) to $(U_{n+1}, R_{n+1}, T_{n+1})$ as in Definition 2.6 is thereby given by the density

$$\widetilde{h}_{u_0}(u,r,t) = \frac{2r^{N-2}}{N-1}\sum_{l,k=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(k)\leq\pi(l)}} g_{u_0^k}(t)\, f_t(u_0^l, r\cdot u^{\pi(k)}) \prod_{\substack{s=1\\s\neq k}}^{N-1} f_t(u_0^s, r\cdot u^{\pi(s)})$$

for $u_0, u \in \mathsf{S}$ as in Definition 2.3 and $r, t \in (0,\infty)$ with the factor r^{N-2} accounting for the functional determinant of the coordinate transformation.

Definition 3.3. *Let* $0 < w \coloneqq 1 - \nu/2$ *an alternative representation of the drift parameter in the generalized Bessel processes.*

Lemma 3.4 (cf. [27, Theorem 8 (ii)].). *Using the parameterization with $w > 0$ the density function $h_{y_0}(y)$ in Lemma 3.1 may be computed to*

$$
\begin{aligned}
&h_{y_0}(y) \\
&= \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s\right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s\right)}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s!\right]} \prod_{s=1}^{N-1} (y^s)^{2k_s} \times \right. \\
&\quad \left. \times \sum_{i=1}^{N-1} \frac{\left(y_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(y_0^{\pi(s)}\right)^{2k_s} \right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1} \left[\left(y_0^s\right)^2 + \left(y^s\right)^2 \right] \right)^{(w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s}} \right].
\end{aligned}
$$

Proof. According to [7, Appenix 1.21] for $w > 0$ the transition density of a generalized Bessel process stopped in the origin starting at $x > 0$ and moving to $y > 0$ at time $t > 0$ is given by

$$
\begin{aligned}
&\mathbb{P}_x(X_t^1 \in dy, \min\{X_s^1 : s \le t\} > 0) \\
&= f_t(x,y) := \frac{x^{2w} y}{2^w t^{w+1}} \exp\left(-\frac{x^2 + y^2}{2t}\right) \sum_{k=0}^{\infty} \frac{\left(\frac{xy}{2t}\right)^{2k}}{k!\,\Gamma(k+1+w)}
\end{aligned}
\tag{3.3}
$$

and according to [25, Proposition 2.9] or to [19, Expression (15)] which is true for $\nu < 0$ also, the density of the hitting time of 0 is given by

$$
\mathbb{P}_x(\inf\{t > 0 : X_t^1 = 0\} \in dt) = g_x(t) := \frac{x^{2w}}{2^w t^{w+1} \Gamma(w)} \exp\left(-\frac{x^2}{2t}\right). \tag{3.4}
$$

Using monotone convergence theorem we generally rearrange expressions of the form $\int_0^\infty \prod_{j=1}^{N-1} f_t\left(x^j, y^j\right) g_{x^N}(t)\, dt$ with all variables positive and apply Lemma A.1:

$$
\begin{aligned}
&\int_0^\infty \prod_{j=1}^{N-1} f_t\left(x^j, y^j\right) g_{x^N}(t)\, dt \\
&= \frac{\left(\prod_{j=1}^{N} x^j\right)^{2w} \prod_{j=1}^{N-1} y^j}{2^{Nw}\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\left(\prod_{j=1}^{N-1} \frac{\left(\frac{x^j y^j}{2}\right)^{2k_j}}{k_j! \Gamma(k_j + w + 1)} \right) \times \right. \\
&\qquad \left. \times \int_0^\infty t^{-N(w+1) - 2\sum_{j=1}^{N-1} k_j} \exp\left(- \frac{\sum_{j=1}^{N} \left(x^j\right)^2 + \sum_{j=1}^{N-1} \left(y^j\right)^2}{2t} \right) dt \right] \\
&= \frac{2^{N-1} \left(\prod_{j=1}^{N} x^j\right)^{2w} \prod_{j=1}^{N-1} y^j}{\Gamma(w)} \times \\
&\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left(-1 + N(w+1) + 2\sum_{j=1}^{N-1} k_j\right)}{\prod_{j=1}^{N-1} \Gamma(k_j + w + 1) k_j!} \frac{\prod_{j=1}^{N-1} \left(x^j y^j\right)^{2k_j}}{\left(\sum_{j=1}^{N} (x^j)^2 + \sum_{j=1}^{N-1} (y^j)^2\right)^{-1 + N(w+1) + 2\sum_{j=1}^{N-1} k_j}} \right].
\end{aligned}
$$

In view of Lemma 3.1 this implies for the density function

$$
h_{y_0}(y) = \frac{2}{N-1} \sum_{i,j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \le \pi(j)}} \int_0^\infty f_t(y_0^i, y^{\pi(j)}) \prod_{\substack{s=1 \\ s \neq j}}^{N-1} f_t(y_0^s y^{\pi(s)})\, g_{y_0^j}(t)\, dt
$$

$$
\begin{aligned}
&= \frac{2}{N-1} \sum_{i,j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \leq \pi(j)}} \frac{2^{N-1} \left(y_0^i \prod_{s=1}^{N-1} y_0^s \right)^{2w} \prod_{s=1}^{N-1} y^{\pi(s)}}{\Gamma(w)} \times \\
&\quad \times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s \right)}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s! \right]} \times \right. \\
&\quad \left. \times \frac{\left(y_0^i y^{\pi(j)} \right)^{2k_j} \prod_{\substack{s=1 \\ s \neq j}}^{N-1} \left(y_0^s y^{\pi(s)} \right)^{2k_s}}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1} \left[\left(y_0^s\right)^2 + \left(y^{\pi(s)}\right)^2 \right] \right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \right] \\
&= \frac{2^N \left(\prod_{s=1}^{N-1} y_0^s \right)^{2w} \prod_{s=1}^{N-1} y^s}{(N-1)\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1} k_s \right)}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s! \right]} \prod_{s=1}^{N-1} \left(y^s\right)^{2k_s} \times \right. \\
&\quad \left. \times \sum_{i=1}^{N-1} \frac{\left(y_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \leq \pi(j)}} \left[\left(y_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(y_0^{\pi(s)} \right)^{2k_s} \right]}{\left(\left(y_0^i\right)^2 + \sum_{s=1}^{N-1} \left[\left(y_0^s\right)^2 + \left(y^s\right)^2 \right] \right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}} \right]. \qquad \square
\end{aligned}
$$

Definition 3.5. *Let $\sigma \coloneqq \sigma^{N-2}$ the $(N-2)$-dimensional Riemannian measure on the sphere as $(N-2)$-dimensional manifold in $\mathbb{R}^{N-1}$.*

Lemma 3.6. *The density function of the chain $(U_n)_{n \in \mathbb{N}_0}$ from Definition 2.6 with respect to the Riemannian measure σ may be expressed as*

$$
\begin{aligned}
&p(u_0, u) \coloneqq \mathbb{P}_{u_0}(U_1 \in du)/d\sigma(u_0) = \int_0^\infty \int_0^\infty \widetilde{h}_{u_0}(u, r, t)\, dt\, dr = \int_0^\infty r^{N-2} h_{u_0}(r \cdot u)\, dr \\
&= \frac{2^{N-1} \left(\prod_{s=1}^{N-1} u_0^s \right)^{2w} \prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times \\
&\quad \times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left(\sum_{s=1}^{N-1} k_s + Nw \right) \left(\sum_{s=1}^{N-1} k_s + N - 2 \right)!}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s! \right]} \prod_{s=1}^{N-1} \left(u^s\right)^{2k_s} \times \right. \\
&\quad \left. \times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \leq \pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)} \right)^{2k_s} \right]}{\left(1 + \left(u_0^i\right)^2 \right)^{\sum_{s=1}^{N-1} k_s + Nw}} \right].
\end{aligned}
$$

Proof. By applying Lemma A.5 in the Appendix we attain

$$
\begin{aligned}
\int_0^\infty r^{N-2} h_{u_0}(r \cdot u)\, dr &= \frac{2^N \left(\prod_{s=1}^{N-1} u_0^s \right)^{2w} \prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times \\
&\quad \times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left((w+1)N - 1 + 2\sum_{s=1}^{N-1} k_s \right)}{\prod_{s=1}^{N-1} \left[\Gamma(k_s + w + 1) k_s! \right]} \prod_{s=1}^{N-1} \left(u^s\right)^{2k_s} \times \right. \\
&\quad \times \sum_{i=1}^{N-1} \left[\left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(i) \leq \pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1 \\ s \neq \pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)} \right)^{2k_s} \right] \times \right. \\
&\quad \left. \left. \times \int_0^\infty \frac{r^{2N-3+2\sum_{s=1}^{N-1} k_s}}{\left(\left(u_0^i\right)^2 + 1 + r^2 \right)^{(w+1)N-1+2\sum_{s=1}^{N-1} k_s}}\, dr \right] \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^{N-1}\left(\prod_{s=1}^{N-1} u_0^s\right)^{2w}\prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times \\
&\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \Bigg[\frac{\Gamma\left(\sum_{s=1}^{N-1} k_s + Nw\right)\left(\sum_{s=1}^{N-1} k_s + N - 2\right)!}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \prod_{s=1}^{N-1} (u^s)^{2k_s} \times \\
&\qquad \times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}\left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}}\prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1}\left(u_0^{\pi(s)}\right)^{2k_s}\right]}{\left(1+\left(u_0^i\right)^2\right)^{\sum_{s=1}^{N-1} k_s + Nw}} \Bigg]. \qquad \square
\end{aligned}
$$

Remark 3.7. By construction, the index $i \in \{1,\dots,N-1\}$ indicates which particle replicates and from the position u_0^j one (out of at most two) particle dies. Particularly, the mappings

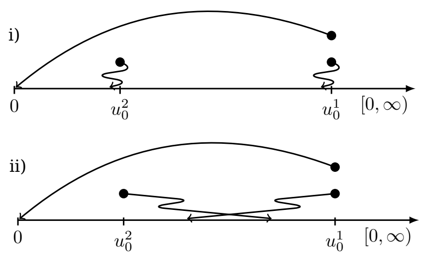
$$
\begin{aligned}
p^i : (u_0,u) \mapsto \mathbb{P}_{u_0}(U_1 \in d\sigma(u), \text{particle } i \text{ replicates})/d\sigma(u) &= \frac{2^{N-1}\left(\prod_{s=1}^{N-1} u_0^s\right)^{2w}\prod_{s=1}^{N-1} u^s}{(N-1)\Gamma(w)} \times \\
\times \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \Bigg[\frac{\Gamma\left(\sum_{s=1}^{N-1} k_s + Nw\right)\left(\sum_{s=1}^{N-1} k_s + N - 2\right)!}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]} \prod_{s=1}^{N-1} (u^s)^{2k_s} \times \\
\times \frac{\left(u_0^i\right)^{2w}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}\left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}}\prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1}\left(u_0^{\pi(s)}\right)^{2k_s}\right]}{\left(1+\left(u_0^i\right)^2\right)^{\sum_{s=1}^{N-1} k_s + Nw}} \Bigg]
\end{aligned}
$$

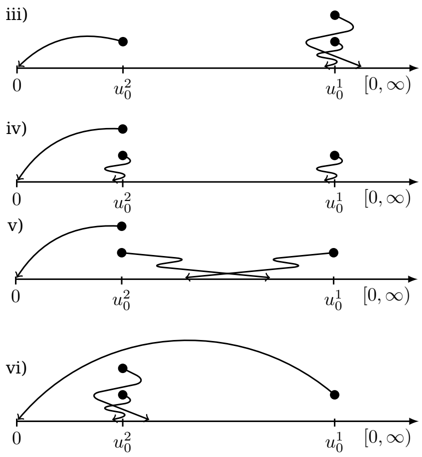
define transition densities to subkernels with total mass $1/(N-1)$.

Example 3.8. In the case of $N = 3$ the expression in Lemma 3.6 reads

$$
\begin{aligned}
p(u_0,u) &= \frac{2\left(u_0^1\right)^{2w}\left(u_0^1\right)^{2w} u^1 u^2}{(N-1)\Gamma(w)} \times \\
&\times \sum_{k_1,k_2=0}^{\infty} \Bigg[\frac{\Gamma\left(k_1+k_2+3w\right)\cdot(k_1+k_2+1)!}{\Gamma(k_1+w+1)\,\Gamma(k_2+w+1)} \frac{\left(u^1\right)^{2k_1}}{k_1!} \frac{\left(u^2\right)^{2k_2}}{k_2!} \times \\
&\qquad \times \Bigg(\frac{\left(u_0^1\right)^{2w}\left[\left(u_0^1\right)^{2k_1}\left(u_0^2\right)^{2k_2} \boldsymbol{+} \left(u_0^2\right)^{2k_1}\left(u_0^1\right)^{2k_2} \boldsymbol{+} \left(u_0^1\right)^{2k_1}\left(u_0^1\right)^{2k_2}\right]}{\left(1+\left(u_0^1\right)^2\right)^{k_1+k_2+3w}} \\
&\qquad\quad + \frac{\left(u_0^2\right)^{2w}\left[\left(u_0^1\right)^{2k_1}\left(u_0^2\right)^{2k_2} \boldsymbol{+} \left(u_0^2\right)^{2k_1}\left(u_0^1\right)^{2k_2} \boldsymbol{+} \left(u_0^2\right)^{2k_1}\left(u_0^2\right)^{2k_2}\right]}{\left(1+\left(u_0^2\right)^2\right)^{k_1+k_2+3w}} \Bigg) \Bigg].
\end{aligned}
$$

Those six summands with their plus sign bold faced correspond to the following cases:





The figures in the first line illustrate $i = j = 1$ where in the left one the surviving particles preserve their order and on the right they switch. The figure in the second line of the display shows the situation $i = 1, j = 2$. In the third line, the particle from position u_0^2 replicated ($i = 2$) and also one of the two particles starting in u_0^2 dies ($j = 2$). Finally, in last figure, $i = 2$, $j = 1$.

4 (Non-)extinction criterion

The main result of this section is Theorem 4.5 which may be seen as simplification of the original problem in question. It would be highly desirable to compute or approximate the invariant probability measure η in Definition 4.3 in order to state more precise results.

4.1 Markov chain analysis

We work in the framework of Markov chains in uncountable state space as laid out in [15, Chapter 5, 9-11, 15]. The basic proof idea for Proposition 4.1 is that in the critical regime of the state space in question, at least one component is small. But then by the model under consideration the time-continuous moving has only a small amount of time to emerge and the process is dominated by the jump mechanism. This allows at least for a positive probability $1/(N-1)$ to make one small particle large by jumping to the largest of the $(N-1)$ others, which is at least $1/\sqrt{N-1}$. Iterating this $N-2$ times ensures all particles to be sufficiently large even at a geometric rate.

Technically, the transition function p of the chain of directions U_n on S may be written in terms of Lauricella series (cf. Definition A.8 in the Appendix). Starting at $u_0 \in \mathsf{S}$ with u_0^{N-1} close to zero means the particles are scarcely given time to evolve and the discontinuous jump mechanism dominates. The density $p(u_0, \cdot)$ becoming singular corresponds to the arguments of the Lauricella series approaching the boundary of the domain of convergence.

Proposition 4.1. *The Markov chain (U_n) is irreducible with the $(N-2)$-dimensional spherical measure σ as maximal irreducibility measure, strongly aperiodic, positive Harris and uniformly geometrically ergodic. The unique invariant probability measure admits a density with respect to σ which is strictly positive σ– a.e.*

Proof. In view of Lemma 3.6 the density function $p : \mathsf{S}^2 \to (0, \infty)$ is a positive continuous mapping and therefore has a positive minimum on the compact set

$$C_0 := \left\{ u \in \mathsf{S} : u^{N-1} \geq \left(\frac{1}{2\sqrt{2}} \right)^{N-2} / \sqrt{N-1} \right\};$$

i.e.

$$\min_{u_0,u\in C_0} p(u_0,u) =: \delta > 0.$$

Consequently, C_0 is an $(1,\xi)$-small set for the kernel of the Markov chain (U_n) in the sense of [15, Definition 9.1.1] where

$$\xi : \mathcal{B}(\mathsf{S}) \to [0,\infty), \quad \xi(A) \coloneqq \delta\sigma(A\cap C_0).$$

Due to $\xi(C_0) = \delta\sigma(C_0) > 0$ we further see, that C_0 is strongly aperiodic in the sense of [15, Definition 9.1.2]. Moreover, any set $A \in \mathcal{B}(\mathsf{S})$ with $\sigma(A) > 0$ is accessible in the sense of [15, Definition 3.5.1] since for arbitrary $u_0 \in \mathsf{S}$ for the return time $\sigma_A \coloneqq \inf\{n \in \mathbb{N} : U_n \in A\}$

$$\mathbb{P}_{u_0}(\sigma_A < \infty) \geq \mathbb{P}_{u_0}(U_1 \in A) = \int_A p(u_0,u)\, d\sigma(u) > 0.$$

Particularly, the set C_0 is accessible and the kernel of (U_n) is seen to be irreducible by means of [15, Definition 9.2.1]. The argument also exhibits σ to be an irreducibility measure in the sense of [15, Definition 9.2.2]. Conversely, consider a set $A \in \mathcal{B}(\mathsf{S})$ with $\sigma(A) = 0$. Then,

$$\begin{aligned}\mathbb{P}_{u_0}(\sigma_A < \infty) &\leq \sum_{n=1}^{\infty} \mathbb{P}_{u_0}(U_n \in A) \leq \sum_{n=1}^{\infty} \sup_{u_0\in\mathsf{S}} \mathbb{P}_{u_0}(U_1 \in A)\\ &= \sum_{n=1}^{\infty} \sup_{u_0\in\mathsf{S}} \int_A p(u_0,u)\, d\sigma(u) = 0\end{aligned}$$

for $u_0 \in \mathsf{S}$. This shows that σ is a maximal irreducibility measure, that is, the set of accessible sets is given by $\{A \in \mathcal{B}(\mathsf{S}) : \sigma(A) > 0\}$. Following [15, Definition 9.3.5] not only the set C_0 but also the kernel of (U_n) is strongly aperiodic.

We now turn to Harris recurrence and positivity properties of (U_n). Observe, that for compact sets $K \subseteq \mathsf{S}$, it holds

$$\inf_{u_0\in K} \mathbb{P}_{u_0}(U_1 \in C_0) = \min_{u_0\in K} \int_{C_0} p(u_0,u)\, d\sigma(u) > 0, \tag{4.1}$$

since the mapping $\mathsf{S} \to (0,1], u_0 \mapsto \int_{C_0} p(u_0,u)\, d\sigma(u)$ is continuous and p is strictly positive. The space $K \coloneqq \mathsf{S}$ is not compact, but still we will show

$$\inf_{u_0\in S} \mathbb{P}_{u_0}(\cup_{n=1}^{N-2}\{U_n \in C_0\}) > 0.$$

For this to end, define given an index $j \in \{1,\dots,N-1\}$, a permutation $\pi \in S_{N-1}$ and $u_0 \in \mathsf{S}$ the symbol $\widetilde{u_0}_{\pi,j} \coloneqq \widetilde{u_0}_{\pi,j}(u_0)$ by

$$\widetilde{u_0}^s_{\pi,j} \coloneqq \begin{cases} u_0^{\pi(s)}, & s \neq \pi^{-1}(j),\\ u_0^1, & s = \pi^{-1}(j),\end{cases}$$

and additionally given $u \in \mathsf{S}$ define $x_{\pi,j} \coloneqq x_{\pi,j}(u_0,u)$ by

$$x^s_{\pi,j} \coloneqq \frac{(u^s)^2}{1+(u_0^1)^2} \cdot \left(\widetilde{u_0}^s_{\pi,j}\right)^2.$$

Further, for $k \in \{0,\dots,N-2\}$ we consider the sets

$$C_k \coloneqq \left\{u \in \mathsf{S} : u^{N-1-k} \geq \left(\frac{1}{2\sqrt{2}}\right)^{N-2-k} / \sqrt{N-1}\right\}.$$

We will now show, that for all $j \in \{1, \dots, N-1\}$, $\pi \in S_{N-1}$ and $k \in \{1, \dots, N-2\}$ it holds

$$\sup_{\substack{u_0 \in C_k \\ u \in \mathsf{S} \setminus C_{k-1}}} \sum_{s=1}^{N-1} \left(x^s_{\pi,j}\right)^{1/2} < 1. \tag{4.2}$$

Proof of (4.2)*:* Let $\langle x, y \rangle := \sum_{s=1}^{N-1} x^s \cdot y^s$ denote the scalar product on the Euclidean space $\mathbb{R}^{N-1}$. By the classical rearrangement inequality [23, Theorem 368]

$$\max_{j=1}^{N-1} \max_{\pi \in S_{N-1}} \sum_{s=1}^{N-1} \left(x^s_{\pi,j}\right)^{1/2} = \max_{j=1}^{N-1} \left\langle \frac{(u_0^1, u_0^1, \dots, u_0^{j-1}, u_0^{j+1}, \dots, u_0^{N-1})}{\sqrt{1 + (u_0^1)^2}}, u \right\rangle.$$

In the case $j \le N - 1 - k$ we find using $\langle x, y \rangle \le \|x\| \cdot \|y\|$

$$\sup_{u_0 \in C_k} \left\langle \frac{(u_0^1, u_0^1, \dots, u_0^{j-1}, u_0^{j+1}, \dots, u_0^{N-1})}{\sqrt{1 + (u_0^1)^2}}, u \right\rangle \le \sup_{u_0 \in C_k} \left(\frac{1 + (u_0^1)^2 - \left(u_0^j\right)^2}{1 + (u_0^1)^2} \right)^{1/2} < 1$$

and in the case $j \ge N - k$ we find using $\langle x, y \rangle = (\|x\|^2 + \|y\|^2 - \|x - y\|^2)/2$

$$\begin{aligned} &\sup_{\substack{u_0 \in C_k \\ u \in \mathsf{S} \setminus C_{k-1}}} \left\langle \frac{(u_0^1, u_0^1, \dots, u_0^{j-1}, u_0^{j+1}, \dots, u_0^{N-1})}{\sqrt{1 + (u_0^1)^2}}, u \right\rangle \\ &\le 1 - \frac{1}{2} \inf_{\substack{u_0 \in C_k \\ u \in \mathsf{S} \setminus C_{k-1}}} \left(\frac{u_0^{N-k-1}}{\sqrt{1 + (u_0^1)^2 - \left(u_0^j\right)^2}} - u^{N-k} \right)^2 < 1. \end{aligned}$$

This finishes the proof of (4.2). ■

Let further denote $(a)_n := \Gamma(a + n)/\Gamma(a)$ the Pochhammer symbol,

$$F_C^{(n)}(a, b, c_1, \dots, c_n, x_1, \dots, x_n) := \sum_{k_1, \dots, k_n = 0}^{\infty} \frac{(a)_{k_1 + \dots + k_n} (b)_{k_1 + \dots + k_n}}{(c_1)_{k_1} \dots (c_n)_{k_n} k_1! \cdots k_n!} x_1^{k_1} \cdots x_n^{k_n}$$

the C-type Lauricella hypergeometric series [17, (2.1.3)] convergent on $\{|x_1|^{1/2} + \dots + |x_n|^{1/2} < 1\}$ and the abbreviation

$$F(x) = F(x^1, \dots, x^{N-1}) := F_C^{(N-1)}(Nw, N - 1, w + 1, w + 1, \dots, w + 1; (x^s)_{s=1}^{N-1}).$$

With this notation by hand, (4.2) implies: For all $j \in \{1, \dots, N-1\}$, $\pi \in S_{N-1}$, $k \in \{1, \dots, N-2\}$ it holds

$$\sup_{\substack{u_0 \in C_k \\ u \in \mathsf{S} \setminus C_{k-1}}} F(x_{\pi,j}(u_0, u)) < \infty. \tag{4.3}$$

For $k \in \{1, \dots, N-2\}$ and $\delta > 0$ let us introduce the sets

$$C_k^\delta := \{u \in C_k : u^{N-1} < \delta\}.$$

With $c_{w,N}$ some constant depending on the Bessel parameter w and the number of particles N only, if follows by (4.3) with the notation from Remark 3.7:

$$\begin{aligned} \sup_{u_0 \in C_k^\delta} \int_{C_{k-1}} p^1(u_0, u)\, d\sigma(u) &\le \sigma(C_{k-1}) \sup_{\substack{u_0 \in C_k^\delta \\ u \in \mathsf{S} \setminus C_{k-1}}} p^1(u_0, u) \\ &\le \sigma(C_{k-1})\, c_{w,N}\, \delta^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi \in S_{N-1} \\ \pi(1) \le \pi(j)}} \sup_{\substack{u_0 \in C_k \\ u \in \mathsf{S} \setminus C_{k-1}}} F(x_{\pi,j}(u_0, u)) \xrightarrow[\delta \downarrow 0]{} 0. \end{aligned} \tag{4.4}$$

Provided by (4.4) let $\delta > 0$ such that

$$\sup_{u_0 \in C_k^\delta} \int_{\mathsf{S}\setminus C_{k-1}} p^1(u_0, u)\, d\sigma(u) < \frac{1}{N-1}/2.$$

If we then split

$$\inf_{u_0 \in C_k} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) = \inf_{u_0 \in C_k \setminus C_k^\delta} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) \wedge \inf_{u_0 \in C_k^\delta} P_{u_0}(U_1 \in C_{k-1}),$$

we see that the first infimum is positive since $C_k \setminus C_k^\delta$ is compact and we may apply (4.1). As for the second infimum, recalling Remark 3.7,

$$\begin{aligned}
&\inf_{u_0 \in C_k^\delta} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) \geq \inf_{u_0 \in C_k^\delta} \int_{C_{k-1}} p^1(u_0, u)\, d\sigma(u) \\
&= \frac{1}{N-1} - \sup_{u_0 \in C_k^\delta} \int_{\mathsf{S}\setminus C_{k-1}} p^1(u_0, u)\, d\sigma(u) \geq \frac{1}{N-1}/2 > 0.
\end{aligned}$$

Therefore, for $k \in \{1, \dots, N-2\}$ it holds

$$\inf_{u_0 \in C_k} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) > 0. \tag{4.5}$$

We are now ready to prove for $k \in \{1, \dots, N-2\}$:

$$\inf_{u_0 \in C_k} \mathbb{P}_{u_0}(\sigma_{C_0} \leq k) > 0. \tag{4.6}$$

Proof of (4.6)*:* Let us show the claim by induction over $k \in \{1, \dots, N-2\}$. The base case $k = 1$ is a consequence of Claim 4. Assuming the assertion for $k - 1 \in \{1, \dots, N-3\}$ implies

$$\begin{aligned}
&\inf_{u_0 \in C_k} \mathbb{P}_{u_0}(\sigma_{C_0} \leq k) \geq \inf_{u_0 \in C_k} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) \cdot \inf_{u_0 \in C_k} \mathbb{P}_{u_0}(\sigma_{C_0} \mid U_1 \in C_{k-1}) \\
&\geq \inf_{u_0 \in C_k} \mathbb{P}_{u_0}(U_1 \in C_{k-1}) \cdot \inf_{u_0 \in C_{k-1}} \mathbb{P}_{u_0}(\sigma_{C_0} \leq k-1),
\end{aligned}$$

where the first factor is positive by (4.5) and the second factor is positive by induction hypothesis. This finishes the proof of (4.6). ■

As a consequence of (4.6), i.e. specifying to the case of $k = N-2$,

$$\kappa \coloneqq \inf_{u_0 \in \mathsf{S}} \mathbb{P}_{u_0}(\sigma_{C_0} \leq N-2) > 0;$$

for each trial of $N-2$ consecutive transitions, there is at least probability $\kappa > 0$ to return to C_0 during this time period. In other words, for $G \sim \mathrm{Geo}(\kappa)$ geometrically distributed supported on $\mathbb{N}$ with success parameter κ, i.e. $\mathbb{P}(G = l) = \kappa(1-\kappa)^{l-1}$ for $l \in \mathbb{N}$, it holds

$$\sup_{u_0 \in \mathsf{S}} \mathbb{E}_{u_0}[\sigma_{C_0}] \leq \mathbb{E}[(N-2)\cdot G] = (N-2)/\kappa < \infty. \tag{4.7}$$

Following [15, Proposition 10.2.4], we deduce that (U_n) is Harris recurrent and in view of [15, Corollary 11.2.9] is further positive. Therefore, by [15, Theorem 9.2.15] the unique invariant probability measure is a maximal irreducibility measure. We infer, that it admits a positive density function. By [15, Lemma 9.4.8 (ii)], (4.7) shows that the state space S is petite. Since our kernel is irreducible and aperiodic, every petite set is small. Hence, also S is small and according to [15, Theorem 15.3.1] the kernel is uniformly geometrically ergodic in the sense of [15, Definition 15.2.1]. □

We use the formalism to transfer Proposition 4.1 to the enlarged state space applicable to the HMM.

Proposition 4.2. *The Markov kernel K of the HMM $M = ((U_n, U_{n+1}), (R_{n+1}, T_{n+1}))_{n\in\mathbb{N}_0}$ is irreducible with maximal irreducibility measure $\sigma\big|_{\mathcal{B}(\mathsf{S})}^{\otimes 2}\otimes\mathrm{Leb}\,\big|_{\mathcal{B}((0,\infty))}^{\otimes 2}$, aperiodic, positive Harris and uniformly geometrically ergodic. The invariant probability has positive density $\widetilde{h}_{u_1}(u_2, r_0, t_0)\, p_\eta(u_1)$ with $\widetilde{h}$ from Remark 3.2 and p_η the invariant density of (U_n).*

Proof. Let $\delta_a(x)$ denote the Dirac-Delta distribution and $d\sigma(u_1, u_2) \coloneqq d\sigma(u_2)\, d\sigma(u_1)$ for $u = (u_1, u_2) \in \mathsf{S}^2$. The Markov kernel Q of the hidden chain $(U_n, U_{n+1})_{n\in\mathbb{N}_0}$ on $(\mathsf{S}^2, \mathcal{B}(\mathsf{S}^2))$ is given by

$$Q(u, V) = \int_V \delta_{u_2}(v_1)\, p(u_2, v_2)\, d\sigma(v),$$

where $u \in \mathsf{S}^2$ and $V \in \mathcal{B}(\mathsf{S}^2)$. The transition is independent of u_1. Consider the kernel G from $(\mathsf{S}^2, \mathcal{B}(\mathsf{S}^2))$ to $((0,\infty)^2, \mathcal{B}((0,\infty)^2))$:

$$G((u_1, u_2), A) = \frac{\int_A h_{u_1}(u_2, r, t)\, d(r,t)}{\int_{(0,\infty)^2} h_{u_1}(u_2, r, t)\, d(r,t)} = \frac{\int_A h_{u_1}(u_2, r, t)\, d(r,t)}{p(u_1, u_2)}.$$

The Markov kernel of the HMM $M = ((U_n, U_{n+1}), (R_{n+1}, T_{n+1}))_{n\in\mathbb{N}_0}$ is given by

$$\begin{aligned} &K(((u_1, u_2), (r_0, t_0)), D) = \int_D G((v_1, v_2), d(r,t))\, Q((u_1, u_2), dv) \\ &= \int_D \frac{h_{v_1}(v_2, r, t)\, d(r,t)}{p(v_1, v_2)}\, \delta_{u_2}(v_1)\, p(u_2, v_2)\, d\sigma(v) = \int_D h_{v_1}(v_2, r, t)\, d(r,t)\, \delta_{u_2}(v_1)\, d\sigma(v) \end{aligned}$$

as in [11, Equation (2.14)] and is independent of u_1, r_0, t_0.

We have seen in the proof of Proposition 4.1 that the set

$$C_0 \coloneqq \left\{ u \in \mathsf{S} : u^{N-1} \geq \left(\frac{1}{2\sqrt{2}}\right)^{N-2} \Big/ \sqrt{N-1} \right\}$$

is small for (U_n) because $\delta \coloneqq \min_{u_0, u \in C_0} p(u_0, u) > 0$ is positive. The measure

$$\mu(D) \coloneqq \delta \int_{D\cap(C_0\times\mathsf{S}\times(0,\infty)^2)} h_{w_1}(w_2, \rho, s)\, d(\rho, s)\, d\sigma(w_1, w_2)$$

on $\mathcal{B}(\mathsf{S}^2 \times (0,\infty)^2)$ is nonzero as

$$\mu(C_0 \times \mathsf{S} \times (0,\infty)^2) = \delta \int_{C_0\times\mathsf{S}} p(w_1, w_2) d\sigma(w_1, w_2) = \delta \int_{C_0} d\sigma(w_1) = \delta\, \sigma(C_0) > 0$$

and the set $\mathsf{S} \times C_0 \times (0,\infty)^2$ is $(2, \mu)$-small with respect to K: For arbitrary $x = (u_1, u_2, r_0, t_0) \in \mathsf{S} \times C_0 \times (0,\infty)^2$ and $D \in \mathcal{B}(\mathsf{S}^2 \times (0,\infty)^2)$:

$$\begin{aligned} &K^2(x, D) \geq K^2(x, D \cap (C_0 \times \mathsf{S} \times (0,\infty)^2)) \\ &= \int_{D\cap(C_0\times\mathsf{S}\times(0,\infty)^2)} h_{w_1}(w_2, \rho, s)\, d(\rho, s)\ p(u_2, w_1)\, d\sigma(w_1, w_2) \geq \mu(D). \end{aligned}$$

Next, we show, that sets $D \in \mathcal{B}(\mathsf{S}^2 \times (0,\infty)^2)$ with $\int_D d(\rho, s)\, d\sigma(w_1, w_2) > 0$ are accessible: Let $x = (u_1, u_2, r_0, t_0) \in \mathsf{S}^2 \times (0,\infty)^2$ arbitrary. Then

$$\mathbb{P}_x(\sigma_D < \infty) \geq \mathbb{P}_x(\sigma_D \in \{1, 2\}) \geq K^2(x, D)$$

$$= \int_D h_{w_1}(w_2,\rho,s)\,d(\rho,s)\,p(u_2,w_1)\,d\sigma(w_1,w_2) > 0.$$

This shows ϕ is an irreducibility measure and particularly, the small set $\mathsf{S}\times C_0\times(0,\infty)^2$ is accessible since

$$\phi(\mathsf{S}\times C_0\times(0,\infty)^2) = \sigma(\mathsf{S})\cdot\sigma(C_0)\cdot\mathrm{Leb}((0,\infty))^2 = \infty > 0.$$

Therefore K is seen to be irreducible. Let us now consider a set $D\in\mathcal{B}(\mathsf{S}^2\times(0,\infty)^2)$ with $\phi(D)=0$. From Fubini's theorem it follows with $D_{v_1,r,t} := \{v_2\in\mathsf{S} : (v_1,v_2,r,t)\in D\}$

$$0 = \int_{(0,\infty)^2}\int_{\mathsf{S}} \sigma(D_{v_1,r,t})\,d\sigma(v_1)\,d(r,t).$$

There must exist $u_2\in\mathsf{S}, r_0, t_0\in(0,\infty)$ such that $\sigma(D_{u_2,r_0,t_0})=0$. Then for $x=(u_1,u_2,r_0,t_0)$ with $u_1\in\mathsf{S}$ arbitrary

$$\mathbb{P}_x(\tau_D<\infty) \le K(x,D) + \sum_{n=2}^{\infty}\sup_{x\in\mathsf{S}^2\times(0,\infty)^2} K^2(x,D).$$

Both summands vanish because for the first one

$$K(x,D) = \int_{(0,\infty)^2}\int_{D_{u_2,r,t}} h_{u_2}(v_2,r,t)\,d\sigma(v_2)\,d(r,t)$$

where the inner integral is equal to zero since $\sigma(D_{u_2,r,t})=0$ and for the second summand

$$\sup_{x\in\mathsf{S}^2\times(0,\infty)^2} K^2(x,D) = \sup_{x\in\mathsf{S}^2\times(0,\infty)^2}\int_D h_{w_1}(w_2,\rho,s)\,p(u_2,w_1)\,d(\rho,s)\,d\sigma(w_1,w_2) = 0$$

since $\phi(D)=0$. In summary, we have computed ϕ to be a maximal irreducibility measure for K. What is more, K is aperiodic since $\inf_{x\in\mathsf{S}\times C_0\times(0,\infty)^2} K(x,\mathsf{S}\times C_0\times(0,\infty)^2)>0$.

Writing

$$\sigma_D((M_n)) = \inf\{n\in\mathbb{N} : M_n\in D\}$$

for $D\in\mathcal{B}(\mathsf{S}^2\times(0,\infty)^2)$ to emphasize that σ_D is a return time for the process $(M_n)_{n\in\mathbb{N}_0}$ the random values $\sigma_{\mathsf{S}\times C_0\times(0,\infty)^2}((M_n))$ and $\sigma_{C_0}((U_n))$ relate in an easy way:

$$\mathbb{P}^{-1}_{(u_{0,1},u_{0,2},r_0,t_0)}\circ\sigma_{\mathsf{S}\times C_0\times(0,\infty)^2}((M_n)) = \mathbb{P}^{-1}_{u_{0,2}}\circ\sigma_{C_0}((U_n)).$$

This then enables us to use the same machinery as in the proof of Proposition 4.1 to deduce the HMM M to be positive Harris, uniformly geometrically ergodic with the invariant probability having positive density with respect to $\sigma|^{\otimes 2}_{\mathcal{B}(\mathsf{S})}\otimes\mathrm{Leb}\,|^{\otimes 2}_{\mathcal{B}((0,\infty))}$. Moreover, if η denotes the invariant probability measure of (U_n) and p_η its density such that $\eta(du)=p_\eta(u)\,d\sigma(u)$ for $u\in\mathsf{S}$, the invariant probability measure for K has density $h_{u_1}(u_2,r_0,t_0)\,p_\eta(u_1)$, since for $D\in\mathcal{B}(\mathsf{S}^2\times(0,\infty)^2)$

$$\begin{aligned}&\int_{\mathsf{S}^2\times(0,\infty)^2} K(((u_1,u_2),(r_0,t_0)),D)\,h_{u_1}(u_2,r_0,t_0)\,p_\eta(u_1)\,d(r_0,t_0)\,d\sigma(u_1,u_2)\\ &= \int_D h_{v_1}(v_2,r,t)\,p_\eta(v_1)\,d(r,t)\,d\sigma(v_1,v_2).\end{aligned}$$

□

Definition 4.3. *Let η denote the invariant probability of U_n on S and μ denote the invariant probability of M_n on $\mathsf{S}^2\times(0,\infty)^2$.*

4.2 Application of Birkhoff's ergodic theorem

Definition 4.4. *Let $\mathbb{P}_{u_0}$ denote the probability measure associated to the density $\widetilde{h}_{u_0}$ from Remark 3.2 and $\mathbb{E}_{u_0}$ the corresponding expectation. Let $\mathbb{E}_\eta[\cdot] = \int_{\mathsf{S}} \mathbb{E}_{u_0}[\cdot]\,\eta(du_0)$.*

First assume that $\ln R_1$ and $\max\{0, \ln T_1\}$ are elements of $L^1(\mu)$ which will be proved later on (cf. Lemma 4.6 and Lemma 4.8). Then the following Theorem holds.

Theorem 4.5 (for the second part cf. [8, Theorem 9.1.1].)**.** *If $\mathbb{E}_\eta[\ln R_1] > 0$ then $\tau_\infty = \infty$ a.s. and if $\mathbb{E}_\eta[\ln R_1] < 0$ then $\tau_\infty < \infty$ a.s.*

Proof. In view of Corollary 2.7 it is desirable to use Cauchy's root test on the series. By Proposition 4.1 combined with [15, Theorem 5.2.6], Lemma 4.6 and Lemma 4.8 below tell us, that we may apply Birkhoff's theorem for Markov chains [15, Theorem 5.2.9] on the the HMM M to deduce for μ-almost all $x = (u_1, u_2, r, t) \in \mathsf{S}^2 \times (0, \infty)^2$

$$\frac{1}{k} \sum_{j=1}^{k} \ln R_j \to \mathbb{E}_\mu[\ln R_1] = \mathbb{E}_\eta[\ln R_1]$$

$\mathbb{P}_x$-a.s. Similarly,

$$\frac{1}{k} \sum_{j=1}^{k} \ln^+ T_j \xrightarrow[k\to\infty]{\mathbb{P}_x-a.s.} \mathbb{E}_\mu[\ln^+ T_1] < \infty.$$

Thus, if $\mathbb{E}_\eta[\ln R_1] < 0$ then a.s.

$$\limsup_{k\to\infty} \left(T_k \prod_{j=1}^{k-1} R_j^2 \right)^{1/k} \le \exp\left(\limsup_{k\to\infty} \left(\frac{1}{k} \sum_{j=1}^{k} \ln^+ T_j - \frac{1}{k} \sum_{j=1}^{k-1} \ln^+ T_j + \frac{2}{k} \sum_{j=1}^{k-1} \ln R_j \right) \right) < 1,$$

which by Cauchy's root test implies $\tau_\infty < \infty$ a.s.

On the other hand, [15, Corollary 5.2.13] implies that for μ-almost all $x \in \mathsf{S}^2 \times (0,\infty)^2$, $\mathbb{P}_x$-almost surely, for $A := \{l \in \mathbb{N} : T_l \ge 1\}$ it holds $|A| = \infty$. In other words, there exists an increasing subsequence $(k_l)_{l\in\mathbb{N}}$ such that $\{k_l : l \in \mathbb{N}\} = A$ and we infer in the case $\mathbb{E}_\eta[\ln R_1] > 0$

$$\limsup_{k\to\infty} \left(T_k \prod_{j=1}^{k-1} R_j^2 \right)^{1/k} \ge \exp\left(\limsup_{l\to\infty} \left(\frac{2}{k_l} \sum_{j=1}^{k_l-1} \ln R_j \right) \right) > 1.$$

This finishes the proof. ☐

4.3 Integrability of the ergodic elements

We use some calculation techniques already used before to show some quantities under consideration are in $L^1(\mu)$. Together with Lemma 4.8 this can be seen as preparation in order to use Birkhoff's ergodic theorem in the proof of this section's main result Theorem 4.5.

Lemma 4.6. *The expectation $\mathbb{E}_\mu[|\ln R_1|] < \infty$ is finite and for arbitrary $u_0 \in \mathsf{S}$ the expectation $\mathbb{E}_{u_0}[|\ln R_1|] < \infty$ is finite.*

Proof. Define the interval $I := [1/2, 2\sqrt{2}]$ and $I^c := (0,\infty) \setminus I = (0, 1/2) \cup (2\sqrt{2}, \infty)$. Then

$$\begin{aligned} \mathbb{E}_\mu[|\ln R_1|] = &\int_{\mathsf{S}^2} \int_{1/2}^{2\sqrt{2}} |\ln r|\, r^{N-2}\, h_{u_0}(r\cdot u)\, dr\, p_\eta(u_0)\, d\sigma(u_0, u) \\ &+ \int_{\mathsf{S}^2} \int_{I^c} |\ln r|\, r^{N-2}\, h_{u_0}(r\cdot u)\, dr\, p_\eta(u_0)\, d\sigma(u_0, u). \end{aligned} \tag{4.8}$$

The first summand in Equation (4.8) is finite since

$$\int_{\mathsf{S}^2}\int_{1/2}^{2\sqrt{2}} |\ln r|\, r^{N-2}\, h_{u_0}(r\cdot u)\, dr\, p_\eta(u_0)\, d\sigma(u_0,u) \le \max_{1/2\le r\le 2\sqrt{2}} |\ln r| < \infty.$$

Let us turn to the second summand in Equation (4.8). For $u_0, u \in \mathsf{S}$, $r>0$ by Lemma 3.4,

$$\begin{aligned} h_{u_0}(r\cdot u) = {} & \frac{r^{N-1}2^N\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}\prod_{s=1}^{N-1}u^s}{(N-1)\Gamma(w)}\times \\ & \times \sum_{k_1,\dots,k_{N-1}=0}^{\infty}\Bigg[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]}\prod_{s=1}^{N-1}(r\cdot u^s)^{2k_s}\times \\ & \times\sum_{i=1}^{N-1}\frac{\left(u_0^i\right)^{2w}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}\left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}}\prod_{\substack{s=1\\ s\ne\pi^{-1}(j)}}^{N-1}\left(u_0^{\pi(s)}\right)^{2k_s}\right]}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\Bigg]. \end{aligned}$$

Given indices $i,j\in\{1,\dots,N-1\}$, a permutation $\pi\in S_{N-1}$ and $u_0\in\mathsf{S}$, we define the symbol $\widetilde{u_0}_{\pi,j,i} \coloneqq \widetilde{u_0}_{\pi,j,i}(u_0)$ by

$$\widetilde{u_0}^s_{\pi,j,i} \coloneqq \begin{cases} u_0^{\pi(s)}, & s\ne\pi^{-1}(j),\\ u_0^i, & s=\pi^{-1}(j),\end{cases}$$

and additionally given $r>0$ and $u\in\mathsf{S}$ define $x_{\pi,j,i,r} \coloneqq x_{\pi,j,i,r}(u_0,u)$ by

$$x^s_{\pi,j,i,r} \coloneqq \left(\frac{2r\,\widetilde{u_0}^s_{\pi,j,i}\,u^s}{\left(u_0^i\right)^2+1+r^2}\right)^2$$

and let

$$F\left((x^s)_{s=1}^{N-1}\right) \coloneqq F_C^{(N-1)}\left(\frac{(w+1)N-1}{2},\frac{(w+1)N}{2},w+1,\dots,w+1,(x^s)_{s=1}^{N-1}\right)$$

the corresponding Lauricella series to write using the Legendre duplication formula for the Gamma function

$$\begin{aligned} h_{u_0}(r\cdot u) = {} & \frac{r^{N-1}2^{(w+2)N-2}\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}\prod_{s=1}^{N-1}u^s\,\Gamma\left(\frac{(w+1)N-1}{2}\right)\Gamma\left(\frac{(w+1)N}{2}\right)}{\sqrt{\pi}\,(N-1)\,\Gamma(w)\,\Gamma(w+1)^{N-1}}\times \\ & \times\sum_{i=1}^{N-1}\frac{\left(u_0^i\right)^{2w}}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1}}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}F\left(x_{\pi,j,i,r}(u_0,u)\right). \end{aligned}$$

With some constant $C_{w,N}$ we bound the second summand in Equation (4.8) according to

$$\begin{aligned} & \int_{\mathsf{S}^2}\int_{I^c}|\ln r|\,r^{N-2}\,h_{u_0}(r\cdot u)\,dr\,p_\eta(u_0)\,d\sigma(u_0,u) \\ & \le C_{w,N}\int_{\mathsf{S}^2}\int_{I^c}\frac{|\ln r|\,r^{2N-3}}{(1+r^2)^{N-1+Nw}}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}F\left(x_{\pi,j,i,r}(u_0,u)\right)dr\,p_\eta(u_0)\,d\sigma(u_0,u). \end{aligned}$$

Note,

$$\sup\left\{\sum_{s=1}^{N-1}\left(x^s_{\pi,j,i,r}(u_0,u)\right)^{1/2} : u_0,u\in\mathsf{S},\pi\in S_{N-1},r\in I^c,i,j=1,\dots,N-1\right\}$$
$$\leq \sup\left\{\frac{2r\sqrt{1+\xi}}{\xi+1+r^2} : \xi\in[0,1],r\in I^c\right\}\leq \max\left\{\sup_{r\in I^c}\frac{2r}{1+r^2},\sup_{r\in I^c}\frac{2\sqrt{2}r}{2+r^2}\right\}<1.$$

Turning back to the integral, we finally achieve using the same argument used to deduce assertion (4.3) in the proof of Proposition 4.1

$$C_{w,N}\int_{\mathsf{S}^2}\int_{I^c}\frac{|\ln r|\, r^{2N-3}}{(1+r^2)^{N-1+Nw}}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} F\left(x_{\pi,j,i,r}(u_0,u)\right) dr\, p_\eta(u_0)\, d\sigma(u_0,u)$$
$$\leq \text{const}\cdot C_{w,N}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}}\int_{\mathsf{S}^2}\int_{I^c}\frac{|\ln r|\, r^{2N-3}}{(1+r^2)^{N-1+Nw}}\, dr\, p_\eta(u_0)\, d\sigma(u_0,u)<\infty.$$

This shows the finiteness of the second summand in Equation (4.8) also, and therefore finishes the proof of the first assertion $\mathbb{E}_\mu[|\ln R_1|]<\infty$. The second assertion $\mathbb{E}_{u_0}[|\ln R_1|]<\infty$ for $u_0\in\mathsf{S}$ arbitrary, follows along the lines. □

Definition 4.7. *Let us denote the mapping*

$$\ln^+ : (0,\infty)\to[0,\infty),\quad x\mapsto \ln^+(x) \coloneqq \max\{0,\ln x\}=\begin{cases}0, & x\leq 1,\\ \ln x, & x>1,\end{cases}$$

as positive part of the natural logarithm.

Lemma 4.8. *The expectation $\mathbb{E}_\mu[\ln^+ T_1]<\infty$ is finite.*

Proof. According to [6, Equation (2.1)], the hitting time of the origin of a Bessel process started at $x>0$ is distributed as $\frac{\sqrt{x}}{2G}$ where $G\sim\text{Gamma}(w)$ is Gamma-distributed, that is

$$\mathbb{P}(G\in dt)=\frac{1}{\Gamma(w)}t^{w-1}e^{-t}\, dt,\quad t>0.$$

Letting $(G,G_0,\dots,G_{N-1})$ independent and $\Gamma(w)$-distributed we bound

$$\mathbb{E}_\mu[\ln^+ T_1]\leq\int_{\mathsf{S}}\ln^+\left(\min\left\{\frac{\sqrt{u_0^1}}{2G_0},\frac{\sqrt{u_0^1}}{2G_1},\dots,\frac{\sqrt{u_0^{N-1}}}{2G_{N-1}}\right\}\right)\eta(du_0)\leq\mathbb{E}\left[\ln^+\frac{1}{2G}\right]$$
$$=\int_0^\infty\ln^+\left(\frac{1}{2t}\right)\frac{1}{\Gamma(w)}t^{w-1}e^{-t}\, dt=\frac{1}{2^w\Gamma(w)}\int_0^\infty s\cdot\exp\left(-s\cdot w-e^{-s}/2\right)\, ds<\infty.\quad □$$

4.4 Computation of the integrand $\ln R_1$ in Theorem 4.5

With the intention to use Theorem 4.5 more explicitly, let us give an explicit formula for $\mathbb{E}_{u_0}[\ln R_1]$. We will then rederive [6, Theorem 1.1 (ii)] and in Section 5 we will show that for $\nu\geq -0.03$ the particle system does almost surely not extinct in Theorem 5.4.

Lemma 4.9. *The integrand in Theorem 4.5 is given by*

$$\mathbb{E}_{u_0}[\ln R_1]=\frac{\sum_{s=1}^{N-1}\ln\left(1+(u_0^s)^2\right)}{2\,(N-1)}+$$

$$
\begin{aligned}
&+\frac{2\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}}{(N-1)\,(N-1)!\,\Gamma(w)}\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\left[\frac{\Gamma\left(wN+\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\Gamma(k_s+w+1)}\times\right.\\
&\times\sum_{i=1}^{N-1}\frac{\left(u_0^i\right)^{2w}}{\left(\left(u_0^i\right)^2+1\right)^{wN+\sum_{s=1}^{N-1}k_s}}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}\left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}}\prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1}\left(u_0^{\pi(s)}\right)^{2k_s}\right]\times\\
&\left.\times\frac{\psi\left(N-1+\sum_{s=1}^{N-1}k_s\right)-\psi\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2}\right].
\end{aligned}
$$

Proof. Applying Lemma 3.4 and Fubini's theorem results in

$$
\begin{aligned}
&\mathbb{E}_{u_0}[\ln R_1]\\
&=\frac{2^N\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}}{(N-1)\Gamma(w)}\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\left[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]}\int_{\mathsf{S}}\prod_{s=1}^{N-1}\left(u^s\right)^{2k_s+1}\,d\sigma(u)\times\right.\\
&\times\sum_{i=1}^{N-1}\left(u_0^i\right)^{2w}\sum_{j=1}^{N-1}\sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\le\pi(j)}}\left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}}\prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1}\left(u_0^{\pi(s)}\right)^{2k_s}\right]\times\\
&\left.\times\int_0^\infty\frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}\ln r}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\,dr\right].
\end{aligned}
\tag{4.9}
$$

By the constructed symmetry and [3, Equation (8)]

$$
\int_{\mathsf{S}}\prod_{s=1}^{N-1}\left(u^s\right)^{2k_s+1}\,d\sigma(u)=\frac{\prod_{s=1}^{N-1}(k_s!)}{2^{N-2}(N-1)!\,\left(N-2+\sum_{s=1}^{N-1}k_s\right)!}.
\tag{4.10}
$$

By Lemma A.7 in the Appendix

$$
\begin{aligned}
&\int_0^\infty\frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}\ln r}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\,dr\\
&=\frac{\ln\left(\left(u_0^i\right)^2+1\right)}{2}\int_0^\infty\frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\,dr+\\
&+\frac{\psi\left(N-1+\sum_{s=1}^{N-1}k_s\right)-\psi\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2}\times\\
&\times\frac{\Gamma\left(N-1+\sum_{s=1}^{N-1}k_s\right)\,\Gamma\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2\left(\left(u_0^i\right)^2+1\right)^{wN+\sum_{s=1}^{N-1}k_s}\,\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}.
\end{aligned}
\tag{4.11}
$$

Plugging (4.10) and (4.11) into (4.9), recalling Remark 3.7, this implies

$$
\frac{2^N\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}}{(N-1)\Gamma(w)}\sum_{k_1,\ldots,k_{N-1}=0}^{\infty}\left[\frac{\Gamma\left((w+1)N-1+2\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\left[\Gamma(k_s+w+1)k_s!\right]}\int_{\mathsf{S}}\prod_{s=1}^{N-1}\left(u^s\right)^{2k_s+1}\,d\sigma(u)\times\right.
$$

$$\times \sum_{i=1}^{N-1} \left(u_0^i\right)^{2w} \sum_{j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s}\right] \times$$
$$\times \int_0^\infty \frac{r^{2N-3+2\sum_{s=1}^{N-1}k_s}\ln r}{\left(\left(u_0^i\right)^2+1+r^2\right)^{(w+1)N-1+2\sum_{s=1}^{N-1}k_s}}\,dr\Bigg]$$
$$= \frac{\sum_{s=1}^{N-1}\ln\left(1+(u_0^s)^2\right)}{2\,(N-1)} + \frac{2\left(\prod_{s=1}^{N-1}u_0^s\right)^{2w}}{(N-1)\,(N-1)!\,\Gamma(w)} \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \left[\frac{\Gamma\left(wN+\sum_{s=1}^{N-1}k_s\right)}{\prod_{s=1}^{N-1}\Gamma(k_s+w+1)}\times\right.$$
$$\times \sum_{i=1}^{N-1} \frac{\left(u_0^i\right)^{2w}}{\left(\left(u_0^i\right)^2+1\right)^{wN+\sum_{s=1}^{N-1}k_s}} \sum_{j=1}^{N-1} \sum_{\substack{\pi\in S_{N-1}\\ \pi(i)\leq\pi(j)}} \left[\left(u_0^i\right)^{2k_{\pi^{-1}(j)}} \prod_{\substack{s=1\\ s\neq\pi^{-1}(j)}}^{N-1} \left(u_0^{\pi(s)}\right)^{2k_s}\right] \times$$
$$\left.\times \frac{\psi\left(N-1+\sum_{s=1}^{N-1}k_s\right)-\psi\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2}\right]. \qquad \square$$

As first application, we immediately obtain

Corollary 4.10. *For $\nu \geq 2/N$ the particle system does almost surely not extinct.*

Proof. In view of Theorem 4.5 it suffices to show $\mathbb{E}_{u_0}[\ln R_1] > 0$ uniformly for all $u_0 \in \mathsf{S}$. This is the case by Lemma 4.9: Since the digamma function restricted to $(0,\infty)$ is strictly monotonously increasing, for $\nu \geq 2/N \Leftrightarrow w \leq (N-1)/N$ the difference

$$\frac{\psi\left(N-1+\sum_{s=1}^{N-1}k_s\right)-\psi\left(wN+\sum_{s=1}^{N-1}k_s\right)}{2} \geq 0$$

is non negative and it holds

$$\inf_{u_0\in\mathsf{S}} \mathbb{E}_{u_0}[\ln R_1] \geq \inf_{u_0\in\mathsf{S}} \frac{\sum_{s=1}^{N-1}\ln\left(1+(u_0^s)^2\right)}{2\,(N-1)} = \frac{\ln 2}{2\,(N-1)} > 0. \qquad \square$$

Remark 4.11. Corollary 4.10 recovers [6, Theorem 1.1 (ii)]. There, the argumentation suffices if each particle is only reflected upon hitting 0 instead of performing an actual jump. The basic idea is to use the fact that for $Z = (Z^1,\dots,Z^N)$ consisting of $N-1$ independent Bessel processes of parameter ν, the process $\|Z\|$ is a Bessel process with parameter $N\cdot\nu$. Setting $\nu \coloneqq 2/N$ leading to a Bessel process with parameter $N\cdot 2/N = 2$ has the law of $\|(B^1,B^2)\|$ with two independent Brownian motions and thereby never hits 0. Observe, also by Lemma 4.9 for $\nu \coloneqq 2/N$ it holds with notation $X_{0-} \coloneqq u_0$

$$\mathbb{E}_{u_0}[\ln R_1] = \frac{1}{N-1}\sum_{s=1}^{N-1} \ln\|(u_0^1,\dots,u_0^s,u_0^s,\dots,u_0^{N-1})\| = \mathbb{E}_{u_0}[\ln\|X_0\|] > 0$$

as expected since by the explanation above $\ln\|X_t\| \overset{d}{=} \ln\|(B^1,B^2)\|$ for $\nu = 2/N$ is a local martingale.

If the particles actually do jump (and therefore interact) this generally may be taken into account by considering $\inf_{u_0\in\mathsf{S}}\mathbb{E}_{u_0}[\ln R_1]$ instead where the contributions of different p^i's might outweigh each other. This approach still neglects the ratios of the particles infinitesimally ahead the jumping times τ_n so we do not need to know the stationary distribution η any more explicit.

5 Three particles

In this section we fix $N \coloneqq 3$ and in the same fashion as in the proof of Corollary 4.10 we want to find regimes of parameter values ν with $\inf_{u_0 \in \mathbb{S}} \mathbb{E}_{u_0}[\ln R_1] > 0$. Then Criterion 4.5 implies non-extinction. For this purpose we may without loss of generality assume $w \geq 2/3$ in what follows. Lemma 4.9 specifies for $N = 3$ using the angle parametrization $\cos\varphi_0 = u_0^1$ and $\sin\varphi_0 = u_0^2$ to

$$\begin{aligned}
\mathbb{E}_{\varphi_0}[\ln R_1] = {} & \frac{\ln(1+\cos^2\varphi_0) + \ln(1+\sin^2\varphi_0)}{4} + \\
& + \frac{(\cos\varphi_0 \sin\varphi_0)^{2w}}{2\,\Gamma(w)} \sum_{k_1,k_2=0}^{\infty} \frac{\Gamma(3w+k_1+k_2)}{\Gamma(k_1+w+1)\Gamma(k_2+w+1)} \times \\
& \times \Bigg(\frac{\cos^{2w}(\varphi_0)}{(1+\cos^2\varphi_0)^{3w+k_1+k_2}} \cdot \frac{\psi(2+k_1+k_2) - \psi(3w+k_1+k_2)}{2} \cdot \\
& \qquad \cdot \left(2\cos^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0 + \cos^{2k_1}\varphi_0 \cos^{2k_2}\varphi_0\right) + \\
& \quad + \frac{\sin^{2w}(\varphi_0)}{\left(1+\sin^2\varphi_0\right)^{3w+k_1+k_2}} \cdot \frac{\psi(2+k_1+k_2) - \psi(3w+k_1+k_2)}{2} \cdot \\
& \qquad \cdot \left(2\cos^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0 + \sin^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0\right) \Bigg).
\end{aligned}$$

Let us recall Example 3.8; e.g. the summand with $2\cos^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0$ in it corresponds to the situation $i = j = 1$, the one with $\cos^{2k_1}\varphi_0 \cos^{2k_2}\varphi_0$ to $i = 1$ and $j = 2$.

Since the first term $\frac{\ln(1+\cos^2\varphi_0)+\ln(1+\sin^2\varphi_0)}{4} \geq 0$ is non-negative which corresponds to the particle system performing a jump, we can allow the remainder to be slightly negative accordingly which corresponds to the continuous drift to be more negative.

Lemma 5.1. *For $\nu \geq 0.2404$ the particle system does almost surely not extinct.*

Proof. Because the derivative of the digamma function, the trigamma function, is strictly decreasing when restricted to $(0,\infty)$, the difference $\psi'(3w+k) - \psi'(2+k)$ is negative for all $k \geq 0$. Thereby the function $k \mapsto \psi(3w+k) - \psi(2+k)$ is recognized to be strictly decreasing yielding the uniform bound

$$\psi(2+k_1+k_2) - \psi(3w+k_1+k_2) \geq \psi(2) - \psi(3w)$$

whence

$$\inf_{\varphi_0} \mathbb{E}_{\varphi_0}[\ln R] \geq \inf_{\varphi_0} \frac{\ln(2+\sin^2\varphi_0\cos^2\varphi_0)}{4} + \frac{\psi(2)-\psi(3w)}{2} \geq \frac{\ln 2}{4} + \frac{\psi(2)-\psi(3w)}{2}.$$

Here, in spirit of Remark 3.7, we have used that

$$\begin{aligned}
& \frac{(\cos\varphi_0 \sin\varphi_0)^{2w}}{2\,\Gamma(w)} \sum_{k_1,k_2=0}^{\infty} \frac{\Gamma(3w+k_1+k_2)}{\Gamma(k_1+w+1)\Gamma(k_2+w+1)} \times \\
& \times \left(\frac{\cos^{2w}(\varphi_0)}{(1+\cos^2\varphi_0)^{3w+k_1+k_2}} \cdot \left(2\cos^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0 + \cos^{2k_1}\varphi_0 \cos^{2k_2}\varphi_0\right) \right) \\
= {} & \frac{(\cos\varphi_0 \sin\varphi_0)^{2w}}{2\,\Gamma(w)} \sum_{k_1,k_2=0}^{\infty} \frac{\Gamma(3w+k_1+k_2)}{\Gamma(k_1+w+1)\Gamma(k_2+w+1)} \times \\
& \times \left(\frac{\sin^{2w}(\varphi_0)}{\left(1+\sin^2\varphi_0\right)^{3w+k_1+k_2}} \cdot \left(2\cos^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0 + \sin^{2k_1}\varphi_0 \sin^{2k_2}\varphi_0\right) \right) = \frac{1}{2}.
\end{aligned}$$

This shows the assertion for $w \leq 0.8798$, respectively $\nu = 2 \cdot (1-w) \geq 0.2404$. □

In order to achieve finer estimates we split the domain of summation $(k_1, k_2) \in \mathbb{N}_0^2$ into $\{(0,0)\}$ and $\mathbb{N}_0^2 \setminus \{(0,0)\}$. With the following definition we measure the contribution $B(w, \sin^2\varphi_0)$ induced by the term with $k_1 = k_2 = 0$.

Definition 5.2. *Let*

$$\begin{aligned} &B : (0,\infty) \times [0,1/2] \to [0,1]; \\ &B(w,\xi) \coloneqq \frac{3\,\Gamma(3w)}{2\,\Gamma(w)\,\Gamma(w+1)^2} \cdot (\xi(1-\xi))^w \cdot \left(\left(\frac{1-\xi}{(2-\xi)^3} \right)^w + \left(\frac{\xi}{(1+\xi)^3} \right)^w \right). \end{aligned}$$

Lemma 5.3. *For $\nu \geq 0$ the particle system does almost surely not extinct.*

Proof. Firstly according to Definition 5.2 and by using $\psi(2+k_1+k_2) - \psi(3w+k_1+k_2) \geq \psi(3) - \psi(3w+1)$ for $k_1 + k_2 \geq 1$ and for the equality the identity $\psi(x+1) - \psi(x) = 1/x$ for positive $x > 0$:

$$\begin{aligned} \mathbb{E}_{\varphi_0}[\ln R^2] = 2\,\mathbb{E}_{\varphi_0}[\ln R] \geq{} & \frac{\ln(2+\cos^2\varphi_0 \sin^2\varphi_0)}{2} + \\ & + B(w,\sin^2\varphi_0)\cdot(\psi(2)-\psi(3w)) + (1-B(w,\sin^2\varphi_0))\cdot(\psi(3)-\psi(3w+1)) \\ = \frac{\ln(2+\cos^2\varphi_0\sin^2\varphi_0)}{2} & + \psi(3) - \psi(3w+1) + B(w,\sin^2\varphi_0)\cdot\left(\frac{1}{3w} - \frac{1}{2}\right). \end{aligned} \tag{5.1}$$

By introducing the abbreviation $\xi \coloneqq \sin^2\varphi_0$ and by using the generalized Bernoulli's inequality $(1+x)^r \leq 1 + r \cdot x$ for $x > -1, 0 \leq r \leq 1$ it follows writing $\zeta \coloneqq (1-\xi)\cdot\xi = \cos^2\varphi_0 \sin^2\varphi_0$:

$$\begin{aligned} B(w,\xi)\cdot\left(\frac{1}{3w}-\frac{1}{2}\right) &= -(3w-2)\cdot\frac{\Gamma(3w)}{4\,\Gamma(w+1)^3}\left(\left(\frac{(1-\xi)^2\xi}{(2-\xi)^3}\right)^w + \left(\frac{(1-\xi)\xi^2}{(1+\xi)^3}\right)^w\right) \\ &\geq -(3w-2)\cdot\frac{\Gamma(3w)}{4\,\Gamma(w+1)^3}\left(2 - 2w + w\cdot\zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}\right). \end{aligned} \tag{5.2}$$

Whence

$$\begin{aligned} \mathbb{E}_{\varphi_0}[\ln R^2] \geq{} & \frac{\ln(2+\zeta)}{2} + \psi(3) - \psi(3w+1) - \\ & - (3w-2)\cdot\frac{\Gamma(3w)}{4\,\Gamma(w+1)^3}\left(2-2w+w\cdot\zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}\right) \eqqcolon h(\zeta). \end{aligned} \tag{5.3}$$

In the relevant domain $w \in [0.8798, 1], \zeta \in [0, 1/4]$ differentiating with respect to ζ computes to

$$\frac{d}{d\zeta}h(\zeta) = \frac{1}{2(2+\zeta)}\left(1 - (3w-2)\cdot w\cdot\frac{\Gamma(3w)}{\Gamma(w+1)^3}\left(\frac{-11(\zeta+2)^2 + 63(\zeta+2) - 81}{(\zeta+2)^3}\right)\right). \tag{5.4}$$

It holds

$$0 < (3w-2)\cdot w \leq 1 \tag{5.5}$$

and by

$$\frac{d}{dw}\frac{\Gamma(3w)}{\Gamma(w+1)^3} = \frac{3\,\Gamma(3w)}{\Gamma(w+1)^3}\cdot(\psi(3w) - \psi(w+1)) > 0$$

also

$$\frac{\Gamma(3w)}{\Gamma(w+1)^3} \leq \frac{\Gamma(3\cdot 1)}{\Gamma(1+1)^3} = 2. \tag{5.6}$$

Furthermore, due to

$$\frac{d}{d\zeta}\frac{-11(\zeta+2)^2+63(\zeta+2)-81}{(\zeta+2)^3} = \frac{11}{(\zeta+2)^4}\left(\frac{5}{11}-\zeta\right)(7-\zeta)>0$$

we get the estimate

$$\frac{-11(\zeta+2)^2+63(\zeta+2)-81}{(\zeta+2)^3} \leq \frac{-11\cdot(9/4)^2+63\cdot 9/4-81}{(9/4)^3} = 4/9. \tag{5.7}$$

Using (5.5), (5.6) and (5.7) in (5.4) we deduce

$$\frac{d}{d\zeta}h(\zeta) \geq \frac{1}{2(2+\zeta)}\left(1-2\cdot 4/9\right)>0$$

and therefore by recalling the bound given in (5.3)

$$\mathbb{E}_{\varphi_0}[\ln R^2] \geq h(0) = \ln(2)/2+\psi(3)-\psi(3w+1)-(3w-2)\cdot(1-w)\cdot\frac{\Gamma(3w)}{2\,\Gamma(w+1)^3}.$$

Because the gamma function $\Gamma(x)$ is strictly increasing for values larger than $x > 1.46163..$, the unique positive root of the digamma function, it holds $\Gamma(3w)/(2\Gamma(w+1)^3) \leq \Gamma(1.8798)^{-3}$ and we may further estimate:

$$\mathbb{E}_{\varphi_0}[\ln R^2] \geq \ln(2)/2+\psi(3)-\psi(3w+1)-(3w-2)\cdot(1-w)\,/\Gamma(1.8798)^3 =: \alpha(w).$$

Differentiating with respect to w leads to

$$\alpha'(w) = 3\left(\frac{2w-5/3}{\Gamma(1.8798)^3}-\psi'(3w+1)\right).$$

Differentiating once again

$$\alpha''(w) = 9\cdot\left(\frac{2}{3\,\Gamma(1.8798)^3}-\psi''(3w+1)\right)>0,$$

because $\psi''\mid_{(0,\infty)}$ is a strictly negative function.

Due to $\alpha'(0.8798)\approx -0.627691<0<0.296638\approx\alpha'(1)$ in the interval $w\in[0.8798,1]$ there is an unique global minimum of α in the interior $(0.8798,1)$. By $\alpha'(0.9611)\approx -0.000299177<0<0.000466669\approx\alpha'(0.9612)$ we can narrow down more and estimate

$$\begin{aligned}&\alpha(w)\geq \ln(2)/2+\psi(3)-\psi(3\cdot 0.9612)-(3\cdot 0.9612-2)\cdot(1-0.9611)/\Gamma(1.8798)^3\\ &\approx 0.00736878>0.\end{aligned}$$

We have now shown $\mathbb{E}_{\varphi_0}[\ln R_1^2]>0$ uniformly in $(w,\varphi_0)\in[0.8798,1]\times[0,\pi/4]$, which shows the assertion. □

In the case of $N=2$ particles there is a critical parameter value $\nu=0$ (cf. [5, Theorem 1.1 (i)]). This is not longer true for $N=3$ particles as the following main result shows. The assumption $\nu\geq 0$ in the preceding lemma was of technical nature only; we may replace Bernoulli's inequality adequately by the estimate $x^w+y^w\leq(x+y)^w$ valid for $w\geq 1$ and $x,y>0$.

Theorem 5.4. *For $\nu\geq -0.03$ the particle system does almost surely not extinct.*

Proof. We want to recycle a few computations from the proof of Lemma 5.3 and again write $\xi:=\sin^2\varphi_0$ and $\zeta:=\xi\cdot(1-\xi)$.

Firstly, again by (5.1) and (5.2)

$$\mathbb{E}_{\varphi_0}[\ln R_1^2] = \mathbb{E}_{\arcsin\sqrt{\xi}}[\ln R_1^2] \geq \frac{\ln(2+\cos^2\varphi_0\sin^2\varphi_0)}{2} + \psi(3) - \psi(3w+1) - \\ - (3w-2)\cdot\frac{\Gamma(3w)}{4\,\Gamma(w+1)^3}\left(\left(\frac{(1-\xi)^2\xi}{(2-\xi)^3}\right)^w + \left(\frac{(1-\xi)\xi^2}{(1+\xi)^3}\right)^w\right).$$

For $w \geq 1$ we may now apply the inequality $x^w + y^w \leq (x+y)^w$ for $x, y > 0$ and attain

$$\left(\frac{(1-\xi)^2\xi}{(2-\xi)^3}\right)^w + \left(\frac{(1-\xi)\xi^2}{(1+\xi)^3}\right)^w \leq \left(\zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}\right)^w.$$

Due to $\zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3} \leq \frac{1}{4}\cdot\frac{1+10/4}{(2+0)^3} = 7/64 < 1$ we further estimate

$$\left(\zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}\right)^w \leq \zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3} \leq w\cdot\zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}$$

and altogether

$$\mathbb{E}_{\varphi_0}[\ln R_1^2] \geq \frac{\ln(2+\zeta)}{2} + \psi(3) - \psi(3w+1) - \\ - (3w-2)\cdot\frac{\Gamma(3w)}{4\,\Gamma(w+1)^3}\cdot w\cdot\zeta\cdot\frac{1+10\zeta-2\zeta^2}{(2+\zeta)^3}.$$

This expression read with respect to ζ is the same as the already analyzed bound (5.3) up to some additive constant; we can directly transfer, that the minimum is attained in $\zeta = 0$:

$$\mathbb{E}_{\varphi_0}[\ln R_1^2] \geq \ln(2)/2 + \psi(3) - \psi(3w+1).$$

The unique root is located at $w \approx 1.01565264025354...$. □

6 Open problems

It is clear that the bound of Theorem 5.4 can not be sharp. With the same method of requiring $\ln R$ to be positive uniformly for all $\varphi_0 \in [0, \pi/4]$, numerical approximations suggest that the critical value is at $w \approx 1.20360229090196$ which corresponds to $\nu \approx -0.40720458180392$; there, the minimum is attained for $\varphi_0 = \pi/4$. The authors believe that in the case $N = 3$ there is $\nu_\star$ sufficiently small, such that the particle system extincts almost surely for all $\nu \leq \nu_\star$. For the criterion Theorem 4.5 to be useful in proving so, since the integrand $\ln R_1$ will be negative only in some regime of S near $(1, 0)$ and positive near $(1/\sqrt{2}, 1/\sqrt{2})$, it seems to be a reasonable strategy to partition $\mathsf{S} = \biguplus_j \mathsf{S}_j$ properly and to show bounds for $\ln R_1$ with $u_0 \in \mathsf{S}_j$ and for $\eta(\mathsf{S}_j)$. The later might be achieved even in considering only one-step transition probabilities.

Another concern addresses asymptotics for $N \to \infty$: The result Theorem 5.4 shows that adding particles potentially really enlarges the domain of parameter values where non-extinction occurs almost surely. It would be very interesting to know, whether for all $\nu \in \mathbb{R}$ there exists N_ν, such that for all $N \geq N_\nu$ the particle system does not extinct almost surely.

A Properties of hypergeometric functions

In this Appendix we collect some explicit properties of special functions which are used during the paper.

The Gamma function $\Gamma(x)$ for $x > 0$ may be represented by the integral $\Gamma(x) = \int_0^\infty e^{-t}\,t^{x-1}\,dt$. This implies the following integral formula:

Lemma A.1. *For $a > 0$, $b > 1$ it holds*

$$\int_0^\infty t^{-b}\, e^{-a/t}\, dt = \Gamma(b-1)/a^{b-1}.$$

Proof. Substituting $s \coloneqq a/t$, we derive

$$\int_0^\infty t^{-b}\, e^{-a/t}\, dt = \int_0^\infty (s/a)^b\, e^{-s}\, a/s^2\, ds = \Gamma(b-1)/a^{b-1}.$$ □

Definition A.2. *We denote by*

$$(x)_k \coloneqq \prod_{j=0}^{k-1}(x+j),\ k \in \mathbb{N}_0$$

the Pochhammer symbol (rising factorial).

Definition A.3. *The Gausian hypergeometric function is defined for $|z| < 1$ as*

$${}_2F_1(a,b;c;z) \coloneqq \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.$$

Lemma A.4. *The hypergeometric function obeys for $a, b > 0$ and $x < 1$ the identity*

$${}_2F_1(a,b;a;x) = (1-x)^{-b} \tag{A.1}$$

and according to [16, 2.12 (5)] has the following integral representation:

$$\begin{aligned} {}_2F_1(a,b;c;1-z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty s^{b-1}(1+s)^{a-c}(1+sz)^{-a}\, ds, \\ \textit{Re}\ c > \textit{Re}\ b > 0, |\arg z| < \pi. \end{aligned} \tag{A.2}$$

Lemma A.5. *For $a, c, \beta > 0, \gamma > \frac{1+\delta}{\beta} > 0$ it holds*

$$\int_0^\infty \frac{y^\delta}{(c + a\cdot y^\beta)^\gamma}\, dy = \beta^{-1}\, c^{-\gamma} \left(\frac{c}{a}\right)^{\frac{1+\delta}{\beta}} \cdot \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)}.$$

Proof. By (A.2) and (A.1)

$$\begin{aligned} \int_0^\infty \frac{y^\delta}{(c + a\cdot y^\beta)^\gamma}\, dy &= \frac{c^{-\gamma}}{\beta} \int_0^\infty \frac{z^{\frac{1+\delta-\beta}{\beta}}}{\left(1 + \frac{a}{c}\cdot z\right)^\gamma}\, dz \\ &= \frac{c^{-\gamma}}{\beta} \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)} \cdot {}_2F_1\left(\gamma, \frac{1+\delta}{\beta}; \gamma; 1 - \frac{a}{c}\right) \\ &= \frac{c^{-\gamma}}{\beta} \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)} \cdot \left(\frac{c}{a}\right)^{\frac{1+\delta}{\beta}}. \end{aligned}$$ □

Definition A.6. *Let $\psi = (\ln \circ \Gamma)' = \Gamma'/\Gamma$ denote the Digamma function.*

Lemma A.7. *For $a, c, \beta > 0, \gamma > \frac{1+\delta}{\beta} > 0$*

$$\begin{aligned} &\int_0^\infty \frac{y^\delta \cdot \ln y}{(c + a\cdot y^\beta)^\gamma}\, dy \\ &= \beta^{-2}\, c^{-\gamma} \left(\frac{c}{a}\right)^{\frac{1+\delta}{\beta}} \cdot \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)} \left(\ln\frac{c}{a} + \psi\left(\frac{1+\delta}{\beta}\right) - \psi\left(\gamma - \frac{1+\delta}{\beta}\right)\right) \\ &= \frac{\ln\frac{c}{a} + \psi\left(\frac{1+\delta}{\beta}\right) - \psi\left(\gamma - \frac{1+\delta}{\beta}\right)}{\beta} \int_0^\infty \frac{y^\delta}{(c + a\cdot y^\beta)^\gamma}\, dy. \end{aligned}$$

Proof. Recalling Lemma A.5,

$$\int_0^\infty \frac{y^\delta \cdot \ln y}{(c + a \cdot y^\beta)^\gamma}\, dy = \frac{\partial}{\partial \delta} \int_0^\infty \frac{y^\delta}{(c + a \cdot y^\beta)^\gamma}\, dy$$

$$= \beta^{-2}\, c^{-\gamma} \left(\frac{c}{a}\right)^{\frac{1+\delta}{\beta}} \cdot \frac{\Gamma\left(\frac{1+\delta}{\beta}\right)\Gamma\left(\gamma - \frac{1+\delta}{\beta}\right)}{\Gamma(\gamma)} \left(\ln \frac{c}{a} + \psi\left(\frac{1+\delta}{\beta}\right) - \psi\left(\gamma - \frac{1+\delta}{\beta}\right)\right). \quad \square$$

Definition A.8. *Let*

$$F_C^{(n)}(a, b, c_1, \ldots, c_n, x_1, \ldots, x_n) := \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{(a)_{k_1 + \ldots + k_n} (b)_{k_1 + \ldots + k_n}}{(c_1)_{k_1} \ldots (c_n)_{k_n} k_1! \cdots k_n!} x_1^{k_1} \cdots x_n^{k_n}$$

denote the C-type Lauricella hypergeometric series as can be found e.g. in [17, Equation (2.1.3)].

Proposition A.9. *The C-type Lauricella series converges absolutely on*

$$\{|x_1|^{1/2} + \ldots + |x_n|^{1/2} < 1\}.$$

Proof. A discussion with proof can be found in [17, Section 2.2] and is omitted here; cf. also [17, Section 2.9] for a general theory of convergence of multiple hypergeometric series. $\square$

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