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The Brown measure of the sum of a self-adjoint element and an elliptic element

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Abstract

We completely determine the Brown measure of the sum of a self-adjoint element and an elliptic element, which is the limiting eigenvalue distribution of the random matrix

$$Y_N + \sqrt{s - \frac{t}{2}} X_N + i\sqrt{\frac{t}{2}} X'_N$$

where Y_N is an $N \times N$ deterministic Hermitian matrix whose eigenvalue distribution converges as $N \to \infty$ and X_N and X'_N are independent Gaussian unitary ensembles. We also study various asymptotic behaviors of this Brown measure as the variance of the elliptic element approaches infinity.

Keywords: Brown measure; elliptic element; non-Hermitian random matrix.
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1 Introduction

1.1 The sum of a self-adjoint element and an elliptic element

An elliptic element is an element in a W^*-probability space of the form $z = x + iy$ where x and y are freely independent semicircular elements, possibly with different variances. By substracting the mean $\tau(z)$ if necessary, we only consider the case $\tau(z) = 0$ in this paper. The variance of such an element is given by

$$\tau(z^*z) = \tau(x^*x) + \tau(y^*y).$$

Once the variance of z is given, say s, there are several possibilities for the variances of x and y. We use the parameters $t = 2\tau(y^*y)$, and $\tau(x^*x) = s - \frac{t}{2}$. Under the parameters s, t, the elliptic element z then has the form

$$\tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$$

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where $\tilde{\sigma}_{s-\frac{t}{2}}$ and $\sigma_{\frac{t}{2}}$ are freely independent centered semicircular elements with variances $s-\frac{t}{2}$ and $\frac{t}{2}$ respectively in a certain W^*-probability space.

Suppose that y_0 is a bounded self-adjoint element in the W^*-probability space containing $\tilde{\sigma}_{s-\frac{t}{2}}$ and $\sigma_{\frac{t}{2}}$; suppose also that all the three elements are freely independent. In this paper, we compute the Brown measure of the element

$$y_0+\tilde{\sigma}_{s-\frac{t}{2}}+i\sigma_{\frac{t}{2}}.$$

We show that the Brown measure of $y_0+\tilde{\sigma}_{s-\frac{t}{2}}+i\sigma_{\frac{t}{2}}$ is a push-forward of the Brown measure of y_0+c_s where $c_s=\tilde{\sigma}_{\frac{s}{2}}+i\sigma_{\frac{s}{2}}$ is the Voiculescu's circular element. The Brown measure of y_0+c_s was computed and analyzed by Zhong and the author [25]. We also study the asymptotic behavior of the Brown measure of $y_0+\tilde{\sigma}_{s-\frac{t}{2}}+i\sigma_{\frac{t}{2}}$ as

1. $s,t\to\infty$ such that the ratio s/t remains as a constant $>\frac{1}{2}$;
2. $s\to\infty$ and t is kept fixed; and
3. $s,t\to\infty$ such that the ratio $s/t=\frac{1}{2}$.

If $s\geq t$, our results can be computed by the results of Zhong and the author [25] in which the Brown measure of x_0+c_t is computed, with $x_0=y_0+\tilde{\sigma}_{s-t}$, where c_t is a circular element, freely independent of x_0. If $s<t$, $y_0+\tilde{\sigma}_{s-\frac{t}{2}}+i\sigma_{\frac{t}{2}}$ is not a sum of a self-adjoint element and a circular element. We need a more general method.

We use the result in [21] to compute the Brown measure of $y_0+\tilde{\sigma}_{s-\frac{t}{2}}+i\sigma_{\frac{t}{2}}$ in terms of the Hermitian part $y_0+\tilde{\sigma}_{s-\frac{t}{2}}$ and t (the parameter of the semicircular element in the skew-Hermitian part). We combine this method with techniques in free probability to determine the Brown measure of $y_0+\tilde{\sigma}_{s-\frac{t}{2}}+i\sigma_{\frac{t}{2}}$ in terms of y_0, and s and t. The results in [21] used a PDE method introduced in the work of Driver, Hall and Kemp [12]; this method has been used in subsequent work by other authors [10, 21, 25]. See also the expository article [19] by Hall for an introduction to the PDE method.

Our results have direct connections to random matrix theory. If X_N and X_N' are independent Gaussian unitary emsembles (GUEs), and Y_N is a sequence of $N\times N$ self-adjoint deterministic matrices whose empirical eigenvalue distributions converge weakly to the law of y_0, then Y_N, X_N and X_N' are asymptotically free in the sense of Voiculescu [33]. If $s>\frac{t}{2}$, by [29, Theorem 6], the empirical eigenvalue distribution of the (almost surely non-normal) random matrix

$$Y_N+\sqrt{s-\frac{t}{2}}X_N+i\sqrt{\frac{t}{2}}X_N'$$

converges to the Brown measure of $y_0+\tilde{\sigma}_{s-\frac{t}{2}}+i\sigma_{\frac{t}{2}}$ as $N\to\infty$. The Brown measure of the case $s=\frac{t}{2}$ is studied in [21], and it is a special case of the results in this paper. In this $s=\frac{t}{2}$ special case, the random matrix model is not a sum of a random matrix and a Ginibre ensemble. We cannot apply [29] to conclude that the empirical eigenvalue distribution converges to the Brown measure; it is still an open problem to give a mathematical proof of the convergence. Nevertheless, numerical simulations in [21] suggest that the Brown measure of $y_0+i\sigma_{\frac{t}{2}}$ is indeed the limiting eigenvalue distribution of $Y_N+i\sqrt{t/2}X_N$, where Y_N and X_N are the same matrices as above.

The Brown measure computed in the case where $y_0=0$ is the elliptic law [8] (see also [15]); its name is due to the fact that its support is a region bounded by an ellipse centered at the origin. In the even more special case $s=t$, the Brown measure is called the circular law since its support is a disk centered at the origin. The circular law was first discovered by Ginibre [13] as a limiting eigenvalue distribution of a random matrix model with Gaussian entries, now commonly called the Ginibre ensemble, then by Girko [14] in the case when the entries come with more relaxed assumptions. The assumptions of random matrix models were then further relaxed, for example, by Bai [1], and Tao and Vu [31]. In the $s \neq t$ case, the elliptic law was first computed by Girko [15] as a limiting eigenvalue distribution of a certain random matrix model. The Brown measure, in the operator framework, was computed by Biane and Lehner [8] and various later work of others.

The Brown measure of operators of the form $X + iY$ where X and Y are freely independent has been analyzed at a nonrigorous level in the physics literature. Stephanov [30] used the case when X is Bernoulli distributed and Y is a GUE to provide a model of QCD. Janik et al. [26] identified the domain where the eigenvalues cluster in the large-N limit when X is an arbitrary self-adjoint random matrix and Y is a GUE. Jarosz and Nowak [27, 28] computed the limiting eigenvalue distribution for general self-adjoint X and Y. Belinschi et al. [3, 4] put the results in [27, 28] on a more rigorous basis; however, there have not been analytic results about the Brown measure of $X + iY$ obtained under this framework.

Since this article was posted on the arXiv, the results of this article have been extended by several papers. In [24], Theorem 1.2 is extended to the case when y_0 is an unbounded self-adjoint element. Zhong [35] computes the Brown measure of $y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$ for arbitrary bounded operator y_0. Hall and the author [20] compute the Brown measure of the multiplicative analogue of the operator considered in this paper.

1.2 Statements of results

Let y_0 be a bounded self-adjoint element, $\tilde{\sigma}_{s-\frac{t}{2}}$ and $\sigma_{\frac{t}{2}}$ be semicircular elements with variances $s - t/2$ and $t/2$ in a W^*-probability space $(\mathscr{A}, \tau)$, which is a finite von Neumann algebra $\mathscr{A}$ with a faithful, normal, tracial state τ. Suppose also that all three of them are freely independent. Throughout the paper, we let ν be the law (or distribution) of y_0, which is the unique compactly supported probability measure on $\mathbb{R}$ such that

$$\int x^n \, d\nu(x) = \tau(y_0^n), \quad \text{for all } n \in \mathbb{N}.$$

Recall that, in this paper, we compute the Brown measure of the element

$$y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}} \in \mathscr{A}.$$

Background information of free probability and Brown measure is reviewed in Section 2. The choice of the parameters s, t comes from the context of the two-parameter Segal–Bargmann transform [11, 18, 23]. It is a interpolation between the self-adjoint element $y_0 + \sigma_s$ and the element $y_0 + i\sigma_s$ studied in [21].

We make the following standing assumption about the element $y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$. We use $\mathrm{Law}(a)$ to denote the law of any self-adjoint random variable $a \in \mathscr{A}$ and $\mathrm{Brown}(a)$ to denote the Brown measure of any non-self-adjoint random variable $a \in \mathscr{A}$.

Assumption 1.1. *Throughout the paper, we assume either $s > \frac{t}{2}$ or ν is not a Dirac measure, so that $\mathrm{Law}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}})$ is not a Dirac measure.*

When this assumption does not hold, that is, if $\mathrm{Law}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}})$ is a Dirac measure, then one cannot apply the results from [21]. However, in this case, the element $y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$ has the form $u\mathbf{1} + i\sigma_{\frac{t}{2}}$ for some constant $u \in \mathbb{R}$ (where $\mathbf{1}$ is the identity element in $\mathscr{A}$). The Brown measure is then a semicircular distribution centered at u with variance $t/2$ on the vertical line through the point u. Under Assumption 1.1, by the results in [21], the Brown measure is absolutely continuous with respect to the Lebesgue measure on the plane.

The following theorem summarizes Theorems 3.3 and 3.7; the proofs can be found in Sections 3.2 and 3.3. The results in [25] and [21] show that both $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}})$

and $\mathrm{Brown}(y_0+c_s)$ can be pushed forward to $\mathrm{Law}(y_0+\sigma_s)$. Points 2 and 3 of the following theorem are proved by comparing these two push-forward maps. We then use the push-forward result to compute the density of $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-\frac{t}{2}}+i\sigma_{\frac{t}{2}})$ given in Point 1 of the following theorem.

Theorem 1.2. *1. For each $s\geq\frac{t}{2}>0$, there is a continuous function $b_{s,t}:\mathbb{R}\to[0,\infty)$ such that the Brown measure of $y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2}$ is supported in the closure of the set*

$$\Omega_{s,t}=\{a+ib\in\mathbb{C}|\,|b|<b_{s,t}(a)\}.$$

The boundary of $\Omega_{s,t}$ is of measure zero with respect to the Brown measure. The Brown measure is absolutely continuous with respect to the Lebesgue area measure on $\mathbb{C}$, with density

$$w_{y_0,s,t}(a+ib)=\frac{1}{2\pi t}\left(1+t\frac{d}{da}\int_{\mathbb{R}}\frac{(\alpha_{s,t}(a)-x)\,d\nu(x)}{(\alpha_{s,t}(a)-x)^2+v_{y_0,s}(\alpha_{s,t}(a))^2}\right),$$

for $|b|<b_{s,t}(a)$, where $\alpha_{s,t}$ is a certain homeomorphism on $\mathbb{R}$ and $v_{y_0,s}$ is a certain nonnegative continuous function on $\mathbb{R}$ such that $\alpha_{s,t}$ and $v_{y_0,s}\circ\alpha_{s,t}$ are differentiable in $\Omega_{s,t}\cap\mathbb{R}$. In particular, the density is constant in the vertical direction.

2. *The Brown measure of $y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2}$ is the push-forward measure of the Brown measure of y_0+c_s by the homeomorphism $U_{s,t}:\mathbb{C}\to\mathbb{C}$,*

$$U_{s,t}(\alpha+i\beta)=a_{s,t}(\alpha)+i\frac{t}{s}\beta$$

where $a_{s,t}$ is the inverse function of $\alpha_{s,t}$.

3. *The push-forward measure of the Brown measure of $y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2}$ by the map, constant in the vertical directions,*

$$Q_{s,t}(a+ib):=\frac{1}{s-t}[sa-t\alpha_{s,t}(a)]$$

is the law of the self-adjoint element $y_0+\sigma_s$.

We now describe briefly how to compute the functions $\alpha_{s,t}$, $b_{s,t}$, and $v_{y_0,s}\circ\alpha_{s,t}$ from the above theorem in $\Omega_{s,t}\cap\mathbb{R}$. Given $a\in\mathbb{R}$, we try to solve for $\alpha\in\mathbb{R}$ and $v>0$ the equations

$$\begin{aligned}\int\frac{d\nu(x)}{(\alpha-x)^2+v^2}&=\frac{1}{s}\\ \frac{(2s-t)\alpha}{s}-(s-t)\int\frac{x\,d\nu(x)}{(\alpha-x)^2+v^2}&=a.\end{aligned}\tag{1.1}$$

The following proposition shows that $a\in\Omega_{s,t}\cap\mathbb{R}$ is precisely when (1.1) has a unique pair of solution. It also shows how the functions $\alpha_{s,t}$, $v_{y_0,s}\circ\alpha_{s,t}$ and $b_{s,t}$ in Theorem 1.2 are computed using the solution. This proposition is proved in Corollary 3.8.

Proposition 1.3. *Given any $a\in\mathbb{R}$,* (1.1) *has a pair of solution $\alpha\in\mathbb{R}$ and $v>0$ if and only if $a\in\Omega_{s,t}\cap\mathbb{R}$. In this case, the solution is unique, and $\alpha_{s,t}(a)=\alpha$, $v_{y_0,s}(\alpha_{s,t}(a))=v$ and $b_{s,t}(a)=\frac{t}{s}v$.*

In the special case $s=t$, we obtain $\alpha_{s,t}(a)=a$ and, by Theorem 1.2,

$$w_{y_0,s,s}(a+ib)=\frac{1}{\pi s}\left(1-\frac{t}{2}\frac{d}{da}\int_{\mathbb{R}}\frac{x\,d\nu(x)}{(a-x)^2+v_{y_0,s}(a)^2}\right)$$

which reduces to the results in [25]. In another special case $t = 2s$, the equations in (1.1) reduces to (1.4) and (1.5) in [21]; the function $\alpha_{s,t}$ is the function a_0^s in [21] and the density is given by

$$\frac{1}{2\pi s}\left(\frac{da_0^s}{da} - \frac{1}{2}\right).$$

Thus, in the case, Theorem 1.2 reduces to the results in [21].

In Sections 4 and 5, we also investigate the asymptotic behaviors of the Brown measure of $y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$, which are summarized in the following theorem; roughly speaking, the Brown measure of $y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$ behaves like the Brown measure of $\tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$. Point 1 of the following theorem is proved in Theorems 5.1 and 5.2; Point 2 is proved in Theorem 5.3 and 5.4; and Point 3 is proved in Theorem 5.5. See these theorems for the precise statements.

Theorem 1.4. *In all of the following three limiting regimes, the function $b_{s,t}$ is unimodal for all large enough s.*

1. *As $s, t \to \infty$ such that the ratio s/t remains as a constant $> \frac{1}{2}$: the domain $\Omega_{s,t}$ is asymptotically equivalent to a region bounded an ellipse centered at $(\tau(y_0), 0)$ with horizontal semi-axis of length $\frac{2s-t}{\sqrt{s}}$ and vertical semi-axis of length $\frac{t}{\sqrt{s}}$. The density $w_{y_0,s,t}$ converges to the constant*

$$\frac{1}{\pi}\frac{s}{(2s-t)t}.$$

   Both convergences are uniform outside any neighborhood of the endpoints of $\Omega_{s,t} \cap \mathbb{R}$.

2. *As $s \to \infty$ and t is kept fixed: the domain $\Omega_{s,t}$ is asymptotically equivalent to a region bounded by a long and thin ellipse centered at $(\tau(y_0), 0)$, with horizontal semi-axis of length $2\sqrt{s}$ and vertical semi-axis of length $\frac{t}{\sqrt{s}}$. The density converges to the constant*

$$\frac{1}{2\pi t}.$$

   Both convergences are uniform outside any neighborhood of the endpoints of $\Omega_{s,t} \cap \mathbb{R}$.

3. *As $s, t \to \infty$ such that the ratio $s/t = \frac{1}{2}$: the domain $\Omega_{s,t}$ is asymptotically equivalent to a region bounded a narrow and tall ellipse centered at $(\tau(y_0), 0)$, with vertical semi-axis of length $2\sqrt{s}$. The set $\Omega_{s,t} \cap \mathbb{R}$ concentrates around $\tau(y_0)$; more precisely, given any $c > 1$, we have*

$$-\frac{4c\tau(y_0^2)}{\sqrt{s}} < \inf(\Omega_{s,t} \cap \mathbb{R}) - \tau(y_0) < 0 < \sup(\Omega_{s,t} \cap \mathbb{R}) - \tau(y_0) < \frac{4c\tau(y_0^2)}{\sqrt{s}}.$$

   for all large enough s.

We do not have a density estimate for the last case.

2 Background and previous results

2.1 Free random variables

Definition 2.1. 1. *We call $(\mathscr{A}, \tau)$ a W^*-**probability space** if $\mathscr{A}$ is a von Neumann algebra and τ is a normal, faithful tracial state on $\mathscr{A}$. The elements in $\mathscr{A}$ are called **non-commutative random variables**, or simply random variables.*

2. *The $*$-subalgebras $A_1, \ldots A_n \subset \mathscr{A}$ are said to be **freely independent** if given an $i_1, i_2, \ldots i_m \in \{1, \ldots, n\}$ with $i_k \neq i_{k+1}$, $a_{i_j} \in \mathscr{A}_{i_j}$ are centered, then we also have $\tau(a_{i_1}a_{i_2}\ldots a_{i_m}) = 0$. The random variables $a_1, \ldots, a_m$ are freely independent if the $*$-algebras they generate are freely independent.*

3. *For a self-adjoint element $a \in \mathscr{A}$, the **distribution**, or the **law**, of a is a compactly supported measure μ on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} f\, d\mu = \tau(f(a))$$

for all continuous function f. We denote by $\mathrm{Law}(a)$ the law of a.

We now introduce the random variables that are key to this paper. The **semicircular element** σ_t has the **semicircular distribution**, or the **semicircle law** of variance t, supported on $[-2\sqrt{t}, 2\sqrt{t}]$ with density

$$\frac{\sqrt{4t - x^2}}{2\pi t}\, dx.$$

The **circular element** c_s has the form $\tilde{\sigma}_{\frac{s}{2}} + i\sigma_{\frac{s}{2}}$ where $\tilde{\sigma}_{\frac{s}{2}}$ and $\sigma_{\frac{s}{2}}$ are freely independent semicircular elements. The **elliptic element** has the form $\tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$ where $\tilde{\sigma}_{s-\frac{t}{2}}$ and $\sigma_{\frac{t}{2}}$ are freely independent semicircular elements.

2.1.1 The R-transform

Let $a \in \mathscr{A}$ be a self-adjoint element with law μ. Then we consider the **Cauchy transform**

$$G_a(z) = \int \frac{1}{z - x}\, d\mu(x)$$

defined outside the spectrum of a. The Cauchy transform G_a is univalent around ∞. Denote by K_a the inverse of G_a at ∞, and let

$$R_a(z) = K_a(z) - \frac{1}{z}.$$

We call K_a the K**-transform** of a and R_a the R**-transform** of a.

Theorem 2.2 ([32])**.** *If $a_1, a_2 \in \mathscr{A}$ are freely independent self-adjoint random variables, then the R-transform of the random variable $a = a_1 + a_2$ is given by*

$$R_a = R_{a_1} + R_{a_2}.$$

Using the notations in the theorem, the distribution of a is called the **free convolution** of $a_1 + a_2$.

2.2 The Brown measure

In this section, we review the definition of the Brown measure, which was introduced by Brown [9]. Let $a \in \mathscr{A}$. We define a function S by

$$S(\lambda, \varepsilon) = \tau[\log(|a - \lambda|^2 + \varepsilon)], \quad \lambda \in \mathbb{C}, \varepsilon > 0.$$

Then

$$S(\lambda, 0) = \lim_{\varepsilon \to 0^+} S(\lambda, \varepsilon)$$

exists as a subharmonic function on $\mathbb{C}$, with value in $\mathbb{R} \cup \{-\infty\}$. The **Brown measure** of a, denoted by $\mathrm{Brown}(a)$, is defined to be

$$\mathrm{Brown}(a) = \frac{1}{4\pi} \Delta_\lambda S(\lambda, 0)$$

where the Laplacian is in distributional sense.

One can see that $S(\lambda, 0)$ does define a harmonic function *outside* the spectrum of a; the Brown measure of a is a probability measure supported on the spectrum of a. The support of $\mathrm{Brown}(a)$, however, can be a proper subset of the spectrum of a.

The Brown measure of an $N \times N$ matrix is the empirical eigenvalue distribution of the matrix. If a sequence of random matrices A_N converges in $*$-distribution to an element a in a non-commutative probability space, one generally expects that the empirical eigenvalue distribution of A_N converges to the Brown measure of a; this, however, is not always the case. A counter-example is the nilpotent matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix};$$

this sequence of matrices converges to the Haar unitary element in $*$-distribution but the empirical eigenvalue distribution is always the Dirac measure at 0.

The Brown measure of the circular element $c_s = \tilde{\sigma}_{\frac{s}{2}} + i\sigma_{\frac{s}{2}}$ is called the **circular law** and is supported in the disk of radius $\sqrt{s}$ centered at the origin. The density is the constant

$$\frac{1}{\pi s}$$

in the support. The circular element is an R-diagonal element. The Brown measure of the circular element can be computed by the method developed by Haagerup and Larsen [16] and Haagerup and Schultz [17].

The Brown measure of the elliptic element $\tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$ is called the **elliptic law** and is supported in an ellipse with semi-axes on the real and imaginary axes of length $\frac{2s-t}{\sqrt{s}}$ and $\frac{t}{\sqrt{s}}$ respectively. The density is the constant

$$\frac{1}{\pi}\frac{s}{2s-t}$$

in the support. The elliptic law was computed by Biane and Lehner [8].

2.3 Biane's free convolution formula

In this section, we review the results of the distribution of the free convolution of a self-adjoint element and a semicircular element established by Biane [7]; several functions and a domain also come up in our study of Brown measure. Given a self-adjoint random variable x_0 with law μ, we consider the function

$$v_{x_0,t}(u) = \inf\left\{ v > 0 \,\middle|\, \int_{\mathbb{R}} \frac{d\mu(x)}{(x-u)^2 + v^2} > \frac{1}{t} \right\}.$$

That is, if

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2} > \frac{1}{t},$$

then $v_{x_0,t}(u)$ is defined to be the unique positive number such that

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_{x_0,t}(u)^2} = \frac{1}{t}; \tag{2.1}$$

otherwise, if

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2} \leq \frac{1}{t},$$

then we set $v_{x_0,t}(u) = 0$. It is noted in [7] that the function $v_{x_0,t}$ is continuous on $\mathbb{R}$ and is differentiable at the points u where $v_{x_0,t}(u) > 0$.

Definition 2.3. *We introduce the following notations.*

1. $\Delta_{x_0,t} = \{u + iv \in \mathbb{C} | v > v_{x_0,t}(u)\}$ *is the region above the graph of* $v_{x_0,t}$ *in the upper half plane.*
2. $H_{x_0,t}(z) = z + tG_{x_0}(z)$*,* $z \in \Delta_{x_0,t}$*.*

Theorem 2.4 ([7])**.** 1. *The function* $H_{x_0,t}$ *is an injective conformal map, from* $\Delta_{x_0,t}$ onto *the upper half plane* $\mathbb{C}^+$*; the function* $H_{x_0,t}$ *extends to a homeomorphism from the closure* $\overline{\Delta_{x_0,t}}$ *of* $\Delta_{x_0,t}$ *onto* $\mathbb{C}^+ \cup \mathbb{R}$*. In particular,* $H_{x_0,t}(u + iv_{x_0,t}(u))$ *is real.*

2. *The function* $H_{x_0,t}$ *satisfies*

$$G_{x_0+\sigma_t}(H_{x_0,t}(z)) = G_{x_0}(z).$$

3. *The measure* $\mathrm{Law}(x_0 + \sigma_t)$ *is absolutely continuous with respect to the Lebesgue measure; its density* $p_{x_0,t}$ *can be computed by the function* $\psi_{x_0,t}(u) := H_{x_0,t}(u + iv_{x_0,t}(u))$*. The function* $\psi_{x_0,t} : \mathbb{R} \to \mathbb{R}$ *is a homeomorphism, and*

$$p_{x_0,t}(\psi_{x_0,t}(u)) = \frac{v_{x_0,t}(u)}{\pi t}.$$

4. *As a consequence, the support of* $\mathrm{Law}(x_0 + \sigma_t)$ *is the closure of the open set* $\{\psi_{x_0,t}(u) | v_{x_0,t}(u) > 0\}$*.*

Remark 2.5. Let $\Lambda_{x_0,t} = \{u + iv \in \mathbb{C} | \, |v| < v_{x_0,t}(u)\}$. The map $H_{x_0,t}$ can be extended to an injective conformal map on $(\overline{\Lambda_{x_0,t}})^c$ by Schwarz reflection with a continuous extension to $\Lambda^c_{x_0,t}$. From now on, $H_{x_0,t}$ means the extension defined on $\Lambda^c_{x_0,t}$. If $v_{x_0,t}(u) > 0$, $H_{x_0,t}$ maps both boundary points $u \pm iv_{x_0,t}(u)$ of $\Lambda_{x_0,t}$ to the same point in the support of $\mathrm{Law}(x_0 + \sigma_t)$.

We then define the right inverse $H^{-1}_{x_0,t}$ of $H_{x_0,t}$ as follows. Outside the interior of the support of $\mathrm{Law}(x_0 + \sigma_t)$, which is the closure of an open set by Theorem 2.4(4), $H^{-1}_{x_0,t}$ is defined to be the inverse of $H_{x_0,t}$. Given any q in the interior of the support of $\mathrm{Law}(x_0 + \sigma_t)$, we define

$$H^{-1}_{x_0,t}(q) = u + iv_{x_0,t}(u)$$

where u is chosen such that $H_{x_0,t}(u + iv_{x_0,t}(u)) = q$. Thus, the restriction of $H^{-1}_{x_0,t}(q)$ to $\mathbb{C}^+ \cup \mathbb{R}$ is the inverse of $H_{x_0,t}$ on $\overline{\Delta_{x_0,t}}$.

2.4 Sum of a self-adjoint and a circular elements

In [25], the author and Zhong computed the Brown measure of $x_0 + c_t$, where x_0 is a self-adjoint element freely independent of the circular element c_t, using the method introduced by Driver, Hall and Kemp [12]. Interestingly, the support of the Brown measure is bounded by the graph of Biane's function $v_{x_0,t}$ introduced in Section 2.3 and the density is closely related to the law of the self-adjoint element $x_0 + \sigma_t$. In this section, we review the results established in [25].

Theorem 2.6. *Let*

$$\Lambda_{x_0,t} = \{u + iv \in \mathbb{C} | \, |v| < v_{x_0,t}(u)\}. \tag{2.2}$$

Then $\Lambda_{x_0,t}$ *is a set of full measure with respect to* $\mathrm{Brown}(x_0 + c_t)$*, and its density* $w_{x_0,t}$ *has the form*

$$w_{x_0,t}(u + iv) = \frac{1}{2\pi t} \frac{d\psi_{x_0,t}(u)}{du}, \quad u + iv \in \Lambda_{x_0,t}$$

where $\psi_{x_0,t}$ *is defined in Theorem 2.4. The density is constant along the vertical segments.*

Furthermore, the push-forward of $\mathrm{Brown}(x_0+c_t)$ *by*

$$\Psi_{x_0,t}(u+iv)=H_{x_0,t}(u+iv_{x_0,t}(u)),\quad u+iv\in\Lambda_{x_0,t}$$

which is independent of v*, is the law of* $x_0+\sigma_t$*.*

2.5 Sum of a self-adjoint and an imaginary multiple of semicircular elements

Hall and the author computed in [21] the Brown measure of $x_0+i\sigma_t$, a sum of a self-adjoint element and an imaginary multiple of semicircular element. The computation of the Brown measure of elements of the form $x_0+i\sigma_t$ covers the case x_0+c_t which has the same $*$-moments as $x_0+\sigma_{t/2}+i\tilde{\sigma}_{t/2}$ where $\sigma_{\frac{t}{2}}$ and $\tilde{\sigma}_{\frac{t}{2}}$ are freely independent semicircular elements, both freely independent of x_0. The results in [21] show that there is a connection between the Brown measure of $x_0+i\sigma_t$, that of x_0+c_t as well as the law of $x_0+\sigma_t$, for the *same* self-adjoint element x_0.

We need the following notations to describe the results in [21].

Definition 2.7. *Let* x_0 *be a self-adjoint element.*

1. *Given any* $r\in\mathbb{R}$*, let* $H_{x_0,r}(z)=z+rG_{x_0}(z)$*,* $z\in\Delta_{x_0,|r|}$*. Compared to the holomorphic function* H *in Definition 2.3, we allow* r *negative in this notation. By the results in [21], for* $t>0$*, the map* $H_{x_0,-t}(z)$ *is an injective conformal map on* $\Delta_{x_0,t}$ *(see Definition 2.3 using* x_0 *and the positive* t*, not* $-t$*). In [21], the authors use the notation* J_t *instead of* $H_{x_0,-t}$*. Furthermore,* $H_{x_0,r}$ *can be extended on* $\Lambda^c_{x_0,s}$ *by Schwarz reflection.*
2. *Define* $h_{x_0,t}(u)=\mathrm{Re}[H_{x_0,-t}(u+iv_{x_0,t}(u))]$ *on* $\mathbb{R}$*. This function* $h_{x_0,t}$ *is a homeomorphism from* $\mathbb{R}$ *to* $\mathbb{R}$*; it is a strictly increasing function. If* $v_{x_0,t}(u)>0$*, we have* $h'_{x_0,t}(u)>0$*.*
3. *Denote by* $h^{-1}_{x_0,t}$ *the inverse of* $h_{x_0,t}$*.*

The following theorem established in [21] computes the Brown measure of $x_0+i\sigma_t$.

Theorem 2.8. *Let*

$$\Omega_{x_0,t}=[H_{x_0,-t}(\Lambda^c_{x_0,t})]^c.$$

Then we can write $\Omega_{x_0,t}$ *as*

$$\Omega_{x_0,t}=\{a+ib\in\mathbb{C}|\,|b|<b_{x_0,t}(a)\}$$

where $b_{x_0,t}(a)=2v_{x_0,t}(h^{-1}_{x_0,t}(a))$ *is a nonnegative function on* $\mathbb{R}$*. The set* $\Omega_{x_0,t}$ *itself is a set of full measure with respect to* $\mathrm{Brown}(x_0+i\sigma_t)$*.*

Inside $\Omega_{x_0,t}$*,* $\mathrm{Brown}(x_0+i\sigma_t)$ *is absolutely continuous with respect to the Lebesgue measure on the plane with a strictly positive density; the density has the form*

$$\frac{1}{2\pi t}\left(\frac{dh^{-1}_{x_0,t}(a)}{da}-\frac{1}{2}\right),\quad a+ib\in\Omega_{x_0,t}.$$

In particular, the density is independent of b *and is constant along the vertical segments.*

We now describe the connections of $\mathrm{Brown}(x_0+c_t)$, $\mathrm{Brown}(x_0+i\sigma_t)$, and $\mathrm{Law}(x_0+\sigma_t)$. Let $U_{x_0,t}:\overline{\Lambda_{x_0,t}}\to\overline{\Omega_{x_0,t}}$ be a homeomorphism defined by

$$U_{x_0,t}(u+iv)=h_{x_0,t}(u)+2iv.$$

Note that the map $U_{x_0,t}$ takes the vertical line segments in $\overline{\Lambda_{x_0,t}}$ *linearly* to vertical line segments in $\overline{\Omega_{x_0,t}}$. Also, recall that $\Lambda_{x_0,t}$ defined in (2.2) is an open set of full measure of $\mathrm{Brown}(x_0+c_t)$. The following theorem establishes the push-forward relations between $\mathrm{Brown}(x_0+c_t)$, $\mathrm{Brown}(x_0+i\sigma_t)$ and $\mathrm{Law}(x_0+\sigma_t)$. It is proved in [21].

Theorem 2.9. *1. The push-forward measure of* $\mathrm{Brown}(x_0 + c_t)$ *under* $U_{x_0,t}$ *is the Brown measure* $\mathrm{Brown}(x_0 + i\sigma_t)$.

2. The push-forward of $\mathrm{Brown}(x_0 + i\sigma_t)$ *under the map*

$$Q_{x_0,t}(a + ib) := 2h^{-1}_{x_0,t}(a) - a \tag{2.3}$$

is the law of $x_0 + \sigma_t$*. The map* $Q_{x_0,t}$ *agrees with* $\Psi_{x_0,t} \circ U^{-1}_{x_0,t}$ *where* $\Psi_{x_0,t}$ *is defined in Theorem 2.6. Alternatively, by Definition 8.1 of [21], we can write*

$$Q_{x_0,t}(a + ib) = H_{x_0,t} \circ H^{-1}_{x_0,-t}(a + ib_{x_0,t}(a)), \quad a \in \Omega_{x_0,t}.$$

Moreover, $Q_{x_0,t}$ *is a diffeomorphism on* $\Omega_{x_0,t} \cap \mathbb{R}$.

Although $Q_{x_0,t}$ is not an invertible map, Point 2 of Theorem 2.9 characterizes the probability measure on $\Omega_{x_0,t}$ whose density is constant along vertical segments. Similar results of the following proposition for the Brown measures of different random variables can be found in [12, 25].

Proposition 2.10. *The Brown measure of* $x_0 + i\sigma_t$ *is the unique measure* m *on* $\overline{\Omega_{x_0,t}}$ *that is absolutely continuous with respect to the Lebesgue measure such that the density is constant along vertical segments and the push-forward of* m *by* $Q_{x_0,t}$ *is* $\mathrm{Law}(x_0 + \sigma_t)$.

Proof. Suppose that $dm(a + ib) = g(a)\, da\, db$ on $\Omega_{x_0,t}$. Write $u = Q_{x_0,t}(a)$. Since $\Omega_{x_0,t}$ has the form described in Theorem 2.8, the push-forward of m by $Q_{x_0,t}$ has the form

$$4v_t(h^{-1}_{x_0,t}(a))g(a)\, da = 4v_t(h^{-1}_{x_0,t}(a))g(a)\frac{da}{du}du, \quad u \in Q_{x_0,t}(\Omega_{x_0,t} \cap \mathbb{R}). \tag{2.4}$$

By the definition (2.3) of $Q_{x_0,t}$ and Theorem 2.8, the density of $\mathrm{Brown}(x_0 + i\sigma_t)$ has the form $(1/4\pi t)(du/da)$ that is strictly positive.

By Point 2 of Theorem 2.9, taking $g(a) = (1/4\pi t)(du/da)$ to be the density of $\mathrm{Brown}(x_0 + i\sigma_t)$ gives $\mathrm{Law}(x_0 + \sigma_t)$; that is, $\mathrm{Law}(x_0 + \sigma_t)$ has the form

$$\frac{1}{\pi t}v_t(h^{-1}_{x_0,t}(a))\, du, \quad u \in Q_{x_0,t}(\Omega_{x_0,t}).$$

Since du/da is positive, the only $g(a)$ that makes the measure in (2.4) equal to $\mathrm{Law}(x_0 + i\sigma_t)$ is $(1/4\pi t)(du/da)$. This shows that $\mathrm{Brown}(x_0 + i\sigma_t)$ is the only measure on $\overline{\Omega_{x_0,t}}$ that is absolutely continuous with respect to the Lebesgue measure such that the density is constant along vertical segments and the push-forward of m by $Q_{x_0,t}$ is $\mathrm{Law}(x_0 + \sigma_t)$. □

3 The Brown measure computation

Let y_0 be a self-adjoint element, $\tilde{\sigma}_{s-\frac{t}{2}}$ and $\sigma_{\frac{t}{2}}$ be two semicircular elements, all freely independent. Denote the law of y_0 by ν. We study the Brown measure of

$$y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$$

with $0 < \frac{t}{2} \leq s$.

If the law of $y_0 + \tilde{\sigma}_{s-\frac{t}{2}}$ is a Dirac mass at one point, then the Brown measure of $y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$ is singular with respect to the Lebesgue measure on the plane, and is a semicircular distribution along a vertical segment. Thus, we recall our standing assumption (Assumption 1.1) that either $s > \frac{t}{2}$ or ν is not a Dirac mass, so that $\mathrm{Law}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}})$ is not a Dirac mass.

For convenience, we define

$$x_0 = y_0 + \tilde{\sigma}_{s-\frac{t}{2}}.$$

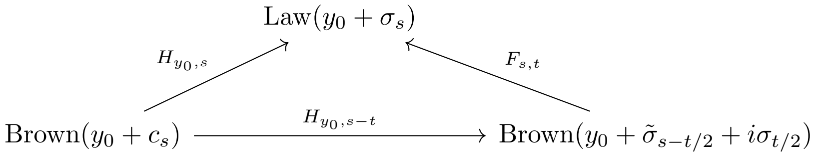

Figure 1: Holomorphic maps between the complements of the supports of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$, $\mathrm{Brown}(y_0 + c_s)$, and $\mathrm{Law}(y_0 + \sigma_s)$

By Theorem 2.8, $\Omega_{x_0,t/2}$ is an open set of full measure of $\mathrm{Brown}(x_0+i\sigma_{t/2})$. Since $x_0+i\sigma_{t/2}$ depends on both parameters s and t, we write

$$\Omega_{s,t} = \Omega_{x_0,t/2}.$$

We also write the boundary of $\Omega_{s,t}$ as $a + ib_{s,t}(a)$ instead of $a + ib_{x_0,t/2}(a)$. We recall from Remark 2.5 that given any q in the support of $\mathrm{Law}(y_0 + \sigma_s)$, $H^{-1}_{y_0,s}(q)$ means the unique point $a_0 + iv_{y_0,s}(a_0)$ on the boundary of $\Lambda_{y_0,s}$.

3.1 The domain of the Brown measure

By Theorem 2.4 and the definition of $\Omega_{x_0,t/2}$ (see Theorem 2.8), the map

$$F_{s,t}(z) = H_{x_0,t/2} \circ H^{-1}_{x_0,-t/2}(z) \tag{3.1}$$

is an injective conformal mapping from $(\overline{\Omega_{s,t}})^c$ to the complement of the support of $\mathrm{Law}(y_0 + \sigma_s)$.

We want to establish a push-forward result that the push-forward measure of $\mathrm{Brown}(y_0 + c_s)$ by a map constructed by $H_{y_0,s-t}$ is $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$. The main theorem in this section establishes the connection between the domains $\Omega_{s,t}$ and $\Lambda_{y_0,s}$ of $\mathrm{Brown}(y_0 + c_s)$ and $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$ respectively. The strategy is to show that $F_{s,t}$, originally defined using $H_{x_0,t/2}$ and $H_{x_0,-t/2}$, can be written in terms of $H_{y_0,s}$ and $H_{y_0,s-t}$ as in Proposition 3.2. Figure 1 demonstrates the connections of the complements of the supports of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$, $\mathrm{Brown}(y_0 + c_s)$, and $\mathrm{Law}(y_0 + \sigma_s)$, where $x_0 = y_0 + \tilde{\sigma}_{s-t/2}$, by the holomorphic functions $F_{s,t}$, $H_{y_0,s-t}$ and $H_{y_0,s}$. We remark that the parameters s and t satisfy $0 < t \le 2s$; the parameter $s - t$ in the subscript of $H_{y_0,s-t}$ can be negative.

Theorem 3.1. *The function $H_{y_0,s-t}$ is an injective conformal map on $(\overline{\Lambda_{y_0,s}})^c$ and extends to a homeomorphism on $\Lambda^c_{y_0,s}$. We also have*

$$\Omega^c_{s,t} = H_{y_0,s-t}(\Lambda^c_{y_0,s}). \tag{3.2}$$

In particular, $\Omega_{s,s} = \Lambda_{y_0,s}$, recovering the domain in Theorem 2.6.

Proposition 3.2. *The inverse $F^{-1}_{s,t}$ of $F_{s,t}$ can be written as*

$$F^{-1}_{s,t}(z) = (H_{y_0,s-t} \circ H^{-1}_{y_0,s})(z) \tag{3.3}$$

for all z outside the support of $\mathrm{Law}(y_0 + \sigma_s)$.

This shows that, when $y_0 = 0$, $F_{s,t}$ is the additive analogue of the function $f_{s,t}$ introduced in [23] in the context of free Segal–Bargmann–Hall transform.

Proof. Recall that we denote $y_0 + \tilde{\sigma}_{s-\frac{t}{2}}$ by x_0. By Theorem 2.4,

$$G_{y_0+\sigma_s}\left(H_{x_0,t/2}(z)\right) = G_{x_0+\sigma_{t/2}}(H_{x_0,t/2}(z)) = G_{x_0}(z) = G_{y_0+\sigma_{s-t/2}}(z) \tag{3.4}$$

because $\tilde{\sigma}_{s-t/2} + \sigma_{t/2}$ has the same distribution as σ_s. When $|z|$ large, (3.4) becomes

$$H^{-1}_{x_0,t/2}(z) = K_{y_0+\sigma_{s-t/2}}(G_{y_0+\sigma_s}(z)). \tag{3.5}$$

Since the R-transform of the sum of two freely independent variables is the sum of the R-transforms of each variable (See Section 2.1.1),

$$R_{y_0+\sigma_{s-t/2}}(z) = R_{y_0}(z) + R_{\sigma_{s-t/2}}(z) = R_{y_0}(z) + \left(s - \frac{t}{2}\right) z.$$

Substracting by $\frac{1}{z}$ gives us

$$K_{y_0+\sigma_{s-t/2}}(z) = K_{y_0}(z) + \left(s - \frac{t}{2}\right) z. \tag{3.6}$$

Therefore,

$$K_{y_0+\sigma_{s-t/2}}\left(G_{y_0+\sigma_s}(z)\right) = K_{y_0}(G_{y_0+\sigma_s}(z)) + \left(s - \frac{t}{2}\right) G_{y_0+\sigma_s}(z). \tag{3.7}$$

By the definition of $F^{-1}_{s,t}$ in (3.1),

$$F^{-1}_{s,t}(z) = H_{x_0,-t/2}\left(H^{\langle -1\rangle}_{x_0,t/2}(z)\right) = H^{\langle -1\rangle}_{x_0,t/2}(z) - \frac{t}{2} G_{x_0+\sigma_{s-t/2}}\left(H^{-1}_{x_0,t/2}(z)\right) \tag{3.8}$$

Using (3.5) and (3.7), the above becomes

$$\begin{aligned} F^{-1}_{s,t}(z) =& K_{y_0+\sigma_{s-t/2}}(G_{y_0+\sigma_s}(z)) - \frac{t}{2} G_{y_0+\sigma_s}(z) \\ =& K_{y_0}(G_{y_0+\sigma_s}(z)) + \left(s - \frac{t}{2}\right) G_{y_0+\sigma_s}(z) - \frac{t}{2} G_{y_0+\sigma_s}(z) \\ =& K_{y_0}(G_{y_0+\sigma_s}(z)) + (s-t) G_{y_0+\sigma_s}(z). \end{aligned} \tag{3.9}$$

Now, since $H_{y_0,s}$ satisfies $G_{y_0+\sigma_s}(H_{y_0,s}(z)) = G_{y_0}(z)$, we have

$$H^{-1}_{y_0,s}(z) = K_{y_0}(G_{y_0+\sigma_s}(z))$$

for all large enough $|z|$. It follows from (3.9) that $F^{-1}_{s,t}$ can be written as

$$F^{-1}_{s,t}(z) = H^{-1}_{y_0,s}(z) + (s-t) G_{y_0}(H^{-1}_{y_0,s}(z)) = (H_{y_0,s-t} \circ H^{-1}_{y_0,s})(z)$$

for all large enough z. Since both sides of the above expression are defined on the complement of the support of $\mathrm{Law}(y_0 + \sigma_s)$, (3.3) holds for all z in the complement of the support of $\mathrm{Law}(y_0 + \sigma_s)$ by analytic continuation. □

Proof of Theorem 3.1. The function $F^{-1}_{s,t}$ is an injective conformal map on the complement of the support of $\mathrm{Law}(y_0 + \sigma_s)$. Thus, by Proposition 3.3

$$H_{y_0,s-t}(z) = F^{-1}_{s,t} \circ H_{y_0,s}(z), \quad z \in \Delta_{y_0,s}$$

is an injective conformal map onto

$$\{a + ib \in \mathbb{C} | \, |b| > b_{s,t}(a)\}.$$

Now, that the function $H_{y_0,s-t}$ extends to a homeomorphism on $\overline{\Delta_{y_0,s}}$ follows from an elementary topological argument by regarding $\Delta_{y_0,s} \cup \{\infty\}$ and $\{a + ib \in \mathbb{C} | \, |b| > b_{x_0,t}(a)\} \cup \{\infty\}$ as two disks in the Riemann sphere. Thus, $H_{y_0,s-t}$ is an injective conformal map on $(\overline{\Lambda_{y_0,s}})^c$ and extends to a homeomophism on $\Lambda^c_{y_0,s}$ by Schwarz reflection about the real axis.

Equation (3.2) is a restatement of Proposition 3.3. If $s = t$, the holomorphic function $H_{y_0,s-t}$ is the identity map; therefore, $\Omega_{s,s} = \Lambda_{y_0,s}$ by (3.2). □

3.2 Two push-forward properties

In Section 3.1, we establish the connection between $\Lambda_{y_0,s}$ and $\Omega_{s,t}$ through the map $H_{y_0,s-t}$. In this section, we prove that the push-forward measure of $\mathrm{Brown}(y_0+c_s)$ by a canonical map constructed using $H_{y_0,s-t}$ is $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$. The main observation is that both $\mathrm{Brown}(y_0+c_s)$ and $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$ can be pushed forward to $\mathrm{Law}(y_0+\sigma_s)$, by Theorems 2.6 and 2.9. These push-forward maps are not injective; nevertheless, Proposition 2.10 shows that they characterize $\mathrm{Brown}(y_0+c_s)$ and $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$.

For convenience, we use the notations $a+ib$ for the points in $\Omega_{s,t}$, $\alpha+i\beta$ for the points in $\Lambda_{y_0,s}$, and u for the points in the support of $\mathrm{Law}(y_0+\sigma_s)$.

Define the function $a_{s,t}:\mathbb{R}\to\mathbb{R}$ by

$$a_{s,t}(\alpha)=\mathrm{Re}[H_{y_0,s-t}(\alpha+iv_{y_0,s}(\alpha))],\quad \alpha\in\mathbb{R}.$$

Let $U_{s,t}:\Lambda_{y_0,s}\to\Omega_{s,t}$ be defined by

$$\begin{aligned}\mathrm{Re}\,U_{s,t}(\alpha+i\beta)&=a_{s,t}(\alpha)\\ \mathrm{Im}\,U_{s,t}(\alpha+i\beta)&=\frac{t\beta}{s}.\end{aligned}\tag{3.10}$$

We will prove that $a_{s,t}$ is a homeomorphism on $\mathbb{R}$ in Proposition 3.4. We can then immediately see that $U_{s,t}$ is indeed a homeomorphism on the complex plane $\mathbb{C}$. In this section, we prove the following two push-forward properties that are introduced in Points 2 and 3 of Theorem 1.2.

Theorem 3.3. *We have the following results about push-forward measures.*

1. *The push-forward of* $\mathrm{Brown}(y_0+c_s)$ *under the map* $U_{s,t}$ *is* $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$.
2. *The push-forward of* $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$ *by the map*

$$Q_{s,t}(a+ib)=\frac{1}{s-t}[sa-t\alpha_{s,t}(a)]$$

is $\mathrm{Law}(y_0+\sigma_s)$.

Recall that the function $F_{s,t}$ is defined in (3.1). By Theorems 2.6 and 2.9, the push-forward of $\mathrm{Brown}(y_0+c_s)$ by $\Psi_{y_0,s}$ defined by

$$\Psi_{y_0,s}(\alpha+i\beta)=H_{y_0,s}(\alpha+iv_{y_0,s}(\alpha)),\quad \alpha+i\beta\in\Lambda_{y_0,s}$$

and the push-forward of $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$ by $Q_{x_0,t/2}$ (where $x_0=y_0+\tilde{\sigma}_{s-t/2}$) defined by

$$Q_{x_0,t/2}(a+ib)=F_{s,t}(a+ib_{s,t}(a)),\quad a+ib\in\Omega_{s,t}$$

are both $\mathrm{Law}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$. In the proof of Theorem 3.3, we actually can see that $Q_{s,t}=Q_{x_0,t/2}$. Figure 2 illustrates the push-forward relations between all of these measures.

Before we prove this theorem, we first study the function $a_{s,t}$ in the definition of $U_{s,t}$.

Proposition 3.4. *The function* $a_{s,t}$ *is strictly increasing. It is a homeomorphism onto* $\mathbb{R}$. *In particular,* $a_{s,t}$ *has an inverse on* $\mathbb{R}$ *that is also strictly increasing. Furthermore,* $a'_{s,t}(\alpha)>0$ *for all* $\alpha\in\Lambda_{y_0,s}\cap\mathbb{R}$.

The upper boundary curve $a+ib_{s,t}(a)$ *of* $\Omega_{s,t}$ *can be parametrized by* $\alpha\in\Lambda_{y_0,s}\cap\mathbb{R}$. *The parameterization is*

$$a+ib_{s,t}(a)=a_{s,t}(\alpha)+\frac{it}{s}v_{y_0,s}(\alpha).\tag{3.11}$$

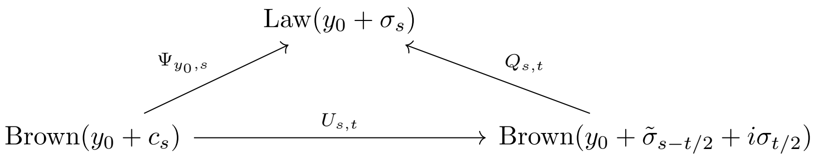

Figure 2: Push-forward relations between the probability measures $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$, $\mathrm{Brown}(y_0 + c_s)$, and $\mathrm{Law}(y_0 + \sigma_s)$, where $x_0 = y_0 + \tilde{\sigma}_{s-t/2}$.

Proof. By a direct computation,

$$a_{s,t}(\alpha) = \frac{s-t}{s}\left(\frac{t\alpha}{s-t} + \mathrm{Re}[H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha))]\right).$$

If $s > t$, then $a_{s,t}$ is strictly increasing because $\mathrm{Re}[H_{y_0,s}(\alpha+iv_{y_0,s}(\alpha))]$ is strictly increasing in $\alpha \in \mathbb{R}$ by Theorem 2.4. If $s < t$, then we write

$$a_{s,t}(\alpha) = \frac{t-s}{s}\left(\frac{(2s-t)\alpha}{t-s} + \mathrm{Re}[H_{y_0,-s}(\alpha + iv_{y_0,s}(\alpha))]\right)$$

which is a strictly increasing function since $\mathrm{Re}[H_{y_0,-s}(\alpha + v_{y_0,s}(\alpha))]$ is strictly increasing in $\alpha \in \mathbb{R}$, by Point 2 of Definition 2.7. If $s = t$, $a_{s,t}$ is just the identity function. In any case, if $v_{y_0,s}(\alpha) > 0$, $a_{s,t}$ is differentiable at α and $a'_{s,t}(\alpha) > 0$ by Point 2 of Definition 2.7.

By Theorem 3.1, $a + ib_{s,t}(a) = H_{s-t}(\alpha + iv_{y_0,s}(\alpha))$ for a unique $\alpha \in \Lambda_{y_0,s} \cap \mathbb{R}$. The imaginary part of $H_{s-t}(\alpha + iv_{y_0,s}(\alpha))$ is given by

$$v_{y_0,s}(\alpha)\left(1 - (s-t)\int \frac{1}{(\alpha - x)^2 + v_{y_0,s}(\alpha)^2}\, d\nu(x)\right) = \frac{t}{s} v_{y_0,s}(\alpha)$$

by (2.1). This proves the parametrization (3.11). □

Proposition 3.5. *The function $U_{s,t} : \Lambda_{y_0,s} \to \Omega_{s,t}$ defined by* (3.10) *is a diffeomorphism; it extends to a homeomorphism from $\overline{\Lambda_{y_0,s}}$ to $\overline{\Omega_{s,t}}$. Moreover, it agrees with $H_{y_0,s-t}$ on the boundary of $\Lambda_{y_0,s}$.*

Proof. By Point 1 of Theorem 3.7, $a_{s,t}$ is injective, strictly increasing and differentiable in $\Lambda_{y_0,s} \cap \mathbb{R}$ with nonzero derivative; therefore, $U_{s,t}$ is a diffeomorphism from $\Lambda_{y_0,s}$ onto $\Omega_{s,t}$. Since $a_{s,t}$ is a homeomorphism defined on $\mathbb{R}$, the map $U_{s,t}$ can be extended to a homeomorhism in $\mathbb{C}$; in particular, it is a homeomorphism from $\overline{\Lambda_{y_0,s}}$ to $\overline{\Omega_{s,t}}$.

It is clear from (3.11) that $U_{s,t}$ agrees with $H_{y_0,s-t}$ on the boundary of $\Lambda_{y_0,s}$. □

Before we prove Theorem 3.3, we write the function $\alpha_{s,t}$ in Theorem 3.7 as the solution of the following integral equation

$$a = a_{s,t}(\alpha_{s,t}(a)) = \alpha_{s,t}(a) + (s-t)\int \frac{(\alpha_{s,t}(a) - x)\, d\nu(x)}{(\alpha_{s,t}(a) - x)^2 + v_{y_0,s}(\alpha_{s,t}(a))^2}. \tag{3.12}$$

Proof of Theorem 3.3. Recall that the density of $\mathrm{Brown}(y_0 + c_s)$ is constant along vertial segments in $\Lambda_{y_0,s}$. By (3.10), the Jacobian matrix of $U_{s,t}$ on $\Lambda_{y_0,s}$ is diagonal and $\mathrm{Im}(U_{s,t}(\alpha + i\beta))$ depends linearly in β. Thus, the density of the push-forward measure of $\mathrm{Brown}(y_0 + c_s)$ by $U_{s,t}$ is again constant along vertical segments in $\Omega_{s,t}$.

We apply Proposition 2.10 to show that the push-forward of $\mathrm{Brown}(y_0 + c_s)$ by $U_{s,t}$ is $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$. By Proposition 3.2, for any $\alpha + i\beta \in \Lambda_{y_0,s}$,

$$\begin{aligned}Q_{x_0,t} \circ U_{s,t}(\alpha + i\beta) &= F_{s,t}(a_{s,t}(\alpha) + ib_{s,t}(a_{s,t}(\alpha))) \\ &= H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) \\ &= \Psi_{y_0,s}(\alpha + i\beta).\end{aligned}$$

This shows that if we further push forward by $Q_{x_0,t}$ the push-forward of $\mathrm{Brown}(y_0 + c_s)$ by $U_{s,t}$, we get the push-forward of $\mathrm{Brown}(y_0 + c_s)$ by $\Psi_{y_0,s}$, which is $\mathrm{Law}(y_0 + \sigma_s)$ by Theorem 2.6. This completes the proof of Point 1 of the theorem.

We now prove Point 2. By Point 1, $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$ is the push-forward measure of $\mathrm{Brown}(y_0 + c_s)$. Since $U_{s,t}$ is a diffeomorphism on $\Lambda_{y_0,s}$, the push-forward of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$ by $\Psi_{y_0,s} \circ U_{s,t}^{-1}$ is $\mathrm{Law}(y_0 + \sigma_s)$. (In fact, by the proof of Point 1, $\Psi_{y_0,s} \circ U_{s,t}^{-1} = Q_{x_0,t}$.) We then compute

$$\begin{aligned}\Psi_{y_0,s} \circ U_{s,t}^{-1}(a + ib) &= \Psi_{y_0,s}\left(\alpha_{s,t}(a) + i\frac{s}{t}b\right) \\ &= \alpha_{s,t}(a) + s\int \frac{\alpha_{s,t}(a) - x}{(\alpha_{s,t}(a) - x)^2 + v_{y_0,s}(\alpha_{s,t}(a))} d\nu(x) \\ &= \alpha_{s,t}(a) + \frac{s}{s-t}(a - \alpha_{s,t}(a))\end{aligned}$$

where we use (3.12) in the last equality. The above equation simplies to the definition of $Q_{s,t}$, completing the proof. □

The density $w_{y_0,s,t}$ of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$ can be computed in terms of the density $w_{y_0,s}$ of $\mathrm{Brown}(y_0 + c_s)$. We will give an alternative formula in the next section.

Corollary 3.6. *Let $r = t/s$ and write $a + ib = U_{s,t}(\alpha + i\beta)$ for all $\alpha + i\beta \in \Lambda_{y_0,s}$. Then we have*

$$w_{y_0,s,t}(a + ib) = \frac{1}{r}\frac{w_{y_0,s}(\alpha + i\beta)}{r + 2\pi(1-r)s \cdot w_{y_0,s}(\alpha + i\beta)}$$

for all $a + ib \in \Omega_{s,t}$.

Proof. Denote $r = t/s$. We can write the function $a_{s,t}(\alpha)$ defined in Proposition 3.4 as

$$\begin{aligned}a_{s,t}(\alpha) &= \alpha + (1-r)s\,\mathrm{Re}\left[\int \frac{d\nu(x)}{\alpha + iv_{y_0,s}(\alpha) - x}\right] \\ &= \alpha + (1-r)[H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) - \alpha] \\ &= (1-r)\psi_{y_0,s}(\alpha) + r\alpha.\end{aligned}$$

So, we have

$$\frac{da_{s,t}(\alpha)}{d\alpha} = r + 2\pi(1-r)s \cdot w_{y_0,s}(\alpha + i\beta).$$

By Theorem 3.3, we can compute the density $w_{y_0,s,t}(a + ib)\,da\,db$ in terms of $w_{y_0,s}$ as

$$\begin{aligned}w_{y_0,s,t}(a + ib)\,da\,db &= w_{y_0,s}(\alpha + i\beta)\,d\alpha\,d\beta \\ &= w_{y_0,s}(\alpha + i\beta)\,\frac{d\alpha}{da}\,\frac{d\beta}{db}\,da\,db \\ &= \frac{1}{r}\frac{w_{y_0,s}(\alpha + i\beta)}{r + 2\pi(1-r)s \cdot w_{y_0,s}(\alpha + i\beta)}\,da\,db,\end{aligned}$$

completing the proof. □

3.3 The density of the Brown measure

The main theorem of this section is to compute the density of $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$ stated in Point 1 of Theorem 1.2.

Theorem 3.7. *The Brown measure of $y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2}$ is absolutely continuous with respect to the Lebesgue measure on the plane and is supported on $\overline{\Omega_{s,t}}$. The open set $\Omega_{s,t}$ is a set of full measure of the Brown measure. The density of the Brown measure is given by*

$$w_{y_0,s,t}(a+ib)=\frac{1}{2\pi t}\left(1+t\frac{d}{da}\int\frac{\alpha_{s,t}(a)-x}{(\alpha_{s,t}(a)-x)^2+v_{y_0,s}(\alpha_{s,t}(a))^2}\,d\nu(x)\right)$$

on the set $\Omega_{s,t}$. In particular, the density is constant along the vertical segments.

Proof. We only need to compute the density. The proof uses the first push-forward property stated in Theorem 3.3. By Theorem 2.6, $\mathrm{Brown}(y_0+c_s)$ is given by

$$\begin{aligned}&\frac{1}{2\pi s}\frac{d}{d\alpha}H_{y_0,s}(\alpha+iv_{y_0,s}(\alpha))\,d\alpha\,d\beta\\&=\frac{1}{2\pi s}\frac{d}{d\alpha}\left(a_{s,t}(\alpha)+t\int\frac{(\alpha-x)\,d\nu(x)}{(\alpha-x)^2+v_{y_0,s}(\alpha)^2}\right)d\alpha\,d\beta\end{aligned}$$

for $\alpha+i\beta\in\Lambda_{y_0,s}$. The determinant of the Jacobian matrix of $U_{s,t}$ defined in (3.10) is $(t/s)(da_{s,t}/d\alpha)$. By the push-forward property in Point 1 of Theorem 3.3, we compute $\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})$ by doing a change of variable $a+ib=a_{s,t}(\alpha)+i(t/s)\beta$ to the above formula of $\mathrm{Brown}(y_0+c_s)$ and get

$$\mathrm{Brown}(y_0+\tilde{\sigma}_{s-t/2}+i\sigma_{t/2})=\frac{1}{2\pi t}\frac{d}{da}\left(a+t\int\frac{(\alpha_{s,t}(a)-x)\,d\nu(x)}{(\alpha_{s,t}(a)-x)^2+v_{y_0,s}(\alpha_{s,t}(a))^2}\right)da\,db$$

on $\Omega_{s,t}$. We have completed the proof. □

Before we end this section, we prove Proposition 1.3 in the following corollary.

Corollary 3.8. *Given any $a\in\mathbb{R}$, (1.1) has a pair of solution $\alpha\in\mathbb{R}$ and $v>0$ if and only if $a\in\Omega_{s,t}\cap\mathbb{R}$. In this case, the solution is unique; moreover, we have $\alpha_{s,t}(a)=\alpha$, $v_{y_0,s}(\alpha_{s,t}(a))=v$ and $b_{s,t}(a)=\frac{t}{s}v$.*

Proof. Let $a\in\Omega_{s,t}\cap\mathbb{R}$. Then, by (2.1) and (3.12), $\alpha=\alpha_{s,t}(a)$ and $v=v_{y_0,s}(\alpha_{s,t}(a))$ is a pair of solution of (1.1). This shows existence of the equation. We now show the solution is indeed unique. Suppose that $\alpha\in\mathbb{R}$ and $v>0$ is a pair of solution. We must show that $\alpha=\alpha_{s,t}(a)$ and $v=v_{y_0,s}(\alpha_{s,t}(a))$. By (2.1), the first equation of (1.1) says $v=v_{y_0,s}(\alpha)$. Using the first equation

$$\int\frac{d\nu(x)}{(\alpha-x)^2+v^2}=\frac{1}{s},$$

of (1.1), the second equation of (1.1) can be written as

$$a=\alpha+(s-t)\int\frac{(\alpha-x)\,d\nu(x)}{(\alpha-x)^2+v_{y_0,s}(\alpha)^2},$$

which shows $a=a_{s,t}(\alpha)$, and so $\alpha=\alpha_{s,t}(a)$.

Conversely, suppose that (1.1) has a pair of solution $\alpha\in\mathbb{R}$ and $v>0$. Then the arguemnt that shows uniqueness of solution in the preceding paragraph proves that $v=v_{y_0,s}(\alpha_{s,t}(a))$ and so $a=a_{s,t}(\alpha)$. Thus, (3.11) shows $b_{s,t}(a)=tv/s>0$, and so $a\in\Omega_{s,t}\cap\mathbb{R}$. □

4 Asymptotic behaviors of adding a circular element

4.1 The graph of $v_{y_0,s}$ as $s \to \infty$

In this section, we study the asymptotic behavior of $v_{y_0,s}$ and $\Lambda_{y_0,s}$ as $s \to \infty$. Below is the main theorem of this section.

Theorem 4.1. *The following asymptotic behaviors of the graph of $v_{y_0,s}$ hold.*

1. *Let $D_\nu = \sup\{|x-y|\,|x,y \in \operatorname{supp}\mu\}$. When $s \geq 4D_\nu^2$, the function $v_{y_0,s}$ is unimodal. In particular, $\Lambda_{y_0,s} \cap \mathbb{R}$ is an interval.*
2. *Given any $c > 1$, we have*

$$\left|\sup \Lambda_{y_0,s} \cap \mathbb{R} - (\tau(y_0) + \sqrt{s})\right| < \frac{3c\tau(y_0^2)}{2\sqrt{s}}$$

and

$$\left|\inf \Lambda_{y_0,s} \cap \mathbb{R} - (\tau(y_0) - \sqrt{s})\right| < \frac{3c\tau(y_0^2)}{2\sqrt{s}}$$

for all large enough s. In particular,

$$\Lambda_{y_0,s} \cap \mathbb{R} \subset \left(\tau(y_0) - \sqrt{s} - \frac{3c\tau(y_0^2)}{2\sqrt{s}}, \tau(y_0) + \sqrt{s} + \frac{3c\tau(y_0^2)}{2\sqrt{s}}\right)$$

for all large enough s.

3. *Given any $\varphi_0 \in (0, \pi/2)$, then for all large enough s, for all $|\cos\varphi| \leq \cos\varphi_0$, the unique $\alpha \in \mathbb{R}$ such that*

$$H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) = 2\sqrt{s}\cos\varphi.$$

satisfies

$$\left|\alpha + iv_{y_0,s}(\alpha) - \sqrt{s}e^{i\varphi}\right| < \frac{1}{(\sin\varphi_0)\sqrt{s}}.$$

Point 1 of Theorem 4.1 is a known result in [22, Theorem 3.2]. We state it here for completeness; it is also useful for us to understand the asymptotic behaviors of $\Lambda_{y_0,s}$.

We study the asymptotic behaviors of $v_{y_0,s}$ by looking at $v_{\frac{y_0}{\sqrt{s}},1}$, whose graph is scaled by $\sqrt{s}$ the graph of $v_{y_0,s}$. We look at

$$H_{\frac{y_0}{\sqrt{s}},1}(z) = z + G_{\frac{y_0}{\sqrt{s}}}(z).$$

If s is large enough, $H_{\frac{y_0}{\sqrt{s}},1}$ is defined for all $|z| > \frac{1}{2}$ since y_0 is assumed to be bounded.

We assume y_0 is centered and has unit variance until the proof of Theorem 4.1 for simplicity. The function $H_{\frac{y_0}{\sqrt{s}},1}$ is the inverse subordination function of the free convolution $\frac{y_0}{\sqrt{s}} + \sigma_1$. When s is large, $\frac{y_0}{\sqrt{s}} + \sigma_1$ behaves like σ_1; our strategy is to compare $\frac{y_0}{\sqrt{s}} + \sigma_1$ with σ_1. Denote by $k(z)$ the function $H_{0,1}(z)$; that is

$$k(z) = z + \frac{1}{z}.$$

The techniques in this section are similar to techniques in proving the supercovergence results in [5, 6, 34].

Lemma 4.2. *Assume y_0 is a bounded random variable with $\tau(y_0) = 0$ and $\tau(y_0^2) = 1$. Then given any $c > 1$, there exists $s_0 > 0$ such that*

$$\left|H_{\frac{y_0}{\sqrt{s}},1}(z) - k(z)\right| < \frac{c}{s|z|^3}, \quad |z| > \frac{1}{2}$$

for all $s \geq s_0$.

Proof. When s is large enough, we can write

$$H_{\frac{y_0}{\sqrt{s}},1}(z) = k(z) + \frac{1}{s}\sum_{n=2}^{\infty}\frac{\tau(y_0^n)}{s^{\frac{n}{2}-1}z^{n+1}}$$

for all $|z| > \frac{1}{2}$. Observe that

$$\left|\sum_{n=2}^{\infty}\frac{\tau(y_0^n)}{s^{\frac{n}{2}-1}z^{n+1}}\right| \leq \frac{\tau(y_0^2)}{|z|^3} + \frac{1}{|z|^3}\sum_{n=3}^{\infty}\frac{|\tau(y_0^n)|}{s^{\frac{n}{2}-1}(1/2)^{n-2}}$$

for all $|z| > \frac{1}{2}$. Since we assume $\tau(y_0^2) = 1$ and

$$\lim_{s\to\infty}\sum_{n=3}^{\infty}\frac{|\tau(y_0^n)|}{s^{\frac{n}{2}-1}(1/2)^{n-2}} = 0,$$

the result follows. □

We compute that $k'(z) = 1 - \frac{1}{z^2}$; the double zeros of k are 1 and -1. The next lemma shows that $H_{\frac{y_0}{\sqrt{s}},1}$ also has doubles zeros at a point close to 1 and a point close to -1. Since $v_{\frac{y_0}{\sqrt{s}},1}$ is unimodal for large s, these two points are the only double zeros of $H_{\frac{y_0}{\sqrt{s}},1}$. Since $H_{\frac{y_0}{\sqrt{s}},1}$ is symmetric about the real axis, these two double zeros must be real numbers. Again since $v_{\frac{y_0}{s},1}$ is unimodal for large s, $\Lambda_{\frac{y_0}{s},1}\cap\mathbb{R}$ is an open interval and the two double zeros of $H_{\frac{y_0}{\sqrt{s}},1}$ are the endpoints of $\Lambda_{\frac{y_0}{s},1}\cap\mathbb{R}$.

Lemma 4.3. *Given any $c > 1$, there exists s_0 such that*

$$\left|H'_{\frac{y_0}{\sqrt{s}},1}(\pm1 + re^{i\theta}) - k'(\pm1 + re^{i\theta})\right| < \frac{3c}{s(1-r)^4}$$

for all $s \geq s_0$ and $r < \frac{1}{2}$.

Proof. Recall that

$$H_{\frac{y_0}{\sqrt{s}},1}(z) = k(z) + \frac{1}{s}\sum_{n=2}^{\infty}\frac{\tau(y_0^n)}{s^{\frac{n}{2}-1}z^{n+1}};$$

we compute

$$H'_{\frac{y_0}{\sqrt{s}},1}(z) = 1 - \frac{1}{z^2} - \frac{1}{s}\left(\frac{3\tau(y_0^2)}{z^4} + \frac{1}{z^4}\sum_{n=3}^{\infty}\frac{(n+1)\tau(y_0^n)}{s^{\frac{n}{2}-1}z^{n-2}}\right) \tag{4.1}$$

Let $c > 1$ be given. If $z = 1 + re^{i\theta}$ with $r < 1/2$, then for all large enough s,

$$\left|\frac{3\tau(y_0^2)}{z^4} + \frac{1}{z^4}\sum_{n=3}^{\infty}\frac{(n+1)\tau(y_0^n)}{s^{\frac{n}{2}-1}z^{n-2}}\right| < \frac{3c}{(1-r)^4}$$

since $|z| > 1 - r > 1/2$ and $\tau(y_0^2) = 1$. The case for $z = 1 - re^{i\theta}$ is similar. □

Proposition 4.4. *We have*

$$1 - \frac{3c}{2s} < \sup\Lambda_{\frac{y_0}{\sqrt{s}},1}\cap\mathbb{R} < 1 + \frac{3c}{2s}$$

and

$$-1 - \frac{3c}{2s} < \inf\Lambda_{\frac{y_0}{\sqrt{s}},1}\cap\mathbb{R} < -1 + \frac{3c}{2s}$$

for all large enough s. In particular,

$$\Lambda_{\frac{y_0}{\sqrt{s}},1}\cap\mathbb{R} \subset \left(-1 - \frac{3c}{2s}, 1 + \frac{3c}{2s}\right)$$

for all large enough s.

Proof. Recall that $\sup \Lambda_{\frac{y_0}{s},1} \cap \mathbb{R}$ and $\inf \Lambda_{\frac{y_0}{s},1} \cap \mathbb{R}$ are the only double zeros for $H_{\frac{y_0}{\sqrt{s}},1}$ when s is large enough so that $v_{y_0,s}$ is unimodal.

Let $c > 1$. We compute, with $z = 1 + re^{i\theta}$,

$$\left|1 - \frac{1}{z^2}\right| = \left|\frac{r(2e^{i\theta} + re^{2i\theta})}{(1 + re^{i\theta})^2}\right| \geq \frac{r(2-r)}{(1+r)^2}.$$

Then, by choosing any $1 < c' < c$ in Lemma 4.3, $r = \frac{3c}{2s}$ satisfies

$$\left|H'_{\frac{y_0}{\sqrt{s}},1}(1 + re^{i\theta}) - k'(1 + re^{i\theta})\right| < \frac{3c'}{s(1-r)^4} < \frac{r(2-r)}{(1+r)^2} \leq \left|1 - \frac{1}{z^2}\right|$$

for all large enough s, because, if s is large enough

$$\frac{3c'(1+r)^2}{r(2-r)(1-r)^4} = \frac{3c'(1+r)^2 2s}{3c(2-r)(1-r)^4} < s.$$

By Rouché's theorem, we have

$$1 - \frac{3c}{2s} < \sup \Lambda_{\frac{y_0}{\sqrt{s}},1} \cap \mathbb{R} < 1 + \frac{3c}{2s}.$$

The proof of

$$-1 - \frac{3c}{2s} < \inf \Lambda_{\frac{y_0}{\sqrt{s}},1} \cap \mathbb{R} < -1 + \frac{3c}{2s}$$

is similar. □

Proposition 4.5. *Given any $\varphi_0 \in (0, \pi/2)$, then for all large enough s, for all $|\cos\varphi| \leq \cos\varphi_0$, the unique $\alpha \in \mathbb{R}$ such that*

$$H_{\frac{y_0}{\sqrt{s}},1}(\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)) = 2\cos\varphi.$$

satisfies

$$\left|\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha) - e^{i\varphi}\right| < \frac{1}{(\sin\varphi_0)s}.$$

Proof. Fix $\varphi_0 \in (0, \pi/2)$ and let $r = \frac{1}{(\sin\varphi_0)s}$. Then, given any $\varphi \in (0, \pi)$ such that $\sin\varphi \geq \sin\varphi_0$, we have, for large s,

$$\begin{aligned} \left|k(e^{i\varphi} + re^{i\theta}) - k(e^{i\varphi})\right| &= \left|re^{i\theta}\left(\frac{re^{i\theta} + 2i\sin\varphi}{e^{i\varphi} + re^{i\theta}}\right)\right| \\ &\geq \frac{1}{\sin\varphi_0 s}\frac{2\sin\varphi_0 - r}{1+r}. \end{aligned} \tag{4.2}$$

Fix any $1 < c < 2$. The lower bound in (4.2) of $s\left|k(e^{i\varphi} + re^{i\theta}) - k(e^{i\varphi})\right|$ converges to 2 as $s \to \infty$. It follows from Lemma 4.2 that, for all large enough s,

$$\begin{aligned} \left|H_{\frac{y_0}{\sqrt{s}},1}(e^{i\varphi} + re^{i\theta}) - k(e^{i\varphi} + re^{i\theta})\right| &< \frac{c}{s(1-r)^3} \\ &< \left|k(e^{i\varphi} + re^{i\theta}) - k(e^{i\varphi})\right| \\ &= \left|k(e^{i\varphi} + re^{i\theta}) - 2\cos\varphi)\right|; \end{aligned}$$

by Rouche's theorem, there exists a point $p_{\cos\varphi}$ such that $\left|p_{\cos\varphi} - e^{i\varphi}\right| < \frac{1}{(\sin\varphi_0)s}$ and

$$H_{\frac{y_0}{\sqrt{s}},1}(p_{\cos\varphi}) = 2\cos\varphi.$$

In particular, $H_{\frac{y_0}{\sqrt{s}},1}(p_{\cos\varphi}) \in \mathbb{R}$. The proposition now follows from the fact that $v_{\frac{y_0}{\sqrt{s}},1}(\alpha)$ is the unique positive number (if exists) such that

$$H_{\frac{y_0}{\sqrt{s}},1}(\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)) \in \mathbb{R}.$$

This completes the proof. □

Proof of Theorem 4.1. Point 1 is a result in [22, Theorem 3.2] which states that v_s is unimodal for $s \geq 4D_\nu^2$. This implies $\Lambda_{y_0,s} \cap \mathbb{R} = (\inf \Lambda_{y_0,s}, \sup \Lambda_{y_0,s})$ is an interval.

Let

$$Y = \frac{y_0 - \tau(y_0)}{\sqrt{\tau(y_0^2)}}$$

and write $t = s/\tau(y_0^2)$. By Theorem 2.6, $\Lambda_{y_0,s}$ is the domain of full measure of $\mathrm{Brown}(y_0 + c_s)$. Since $\mathrm{Brown}(y_0 + c_s)$ is the push-forward of $\mathrm{Brown}\left(\frac{Y}{\sqrt{t}} + c_1\right)$ by the function

$$z \mapsto \tau(y_0) + z\sqrt{t\tau(y_0^2)} = \tau(y_0) + z\sqrt{s}$$

by [17, Proposition 2.14]. Thus,

$$\Lambda_{y_0,s} = \left\{ \tau(y_0) + z\sqrt{s} \in \mathbb{C} \middle| \, z \in \Lambda_{\frac{Y}{\sqrt{t}},1} \right\}.$$

Points 2 and 3 then follow from applying Proposition 4.4 and Proposition 4.5 with $t = s/\tau(y_0^2)$ in place of s respectively; $\Lambda_{y_0,s}$ is obtained by scaling $\Lambda_{\frac{Y}{\sqrt{t}},1}$ by $\sqrt{s}$ and translating by $\tau(y_0)$. □

4.2 The density as $s \to \infty$

In this section, we estimate the density of $\mathrm{Brown}(y_0 + c_s)$ for large s. The Brown measure of c_s is the uniform measure on the disk of radius $\sqrt{s}$; that is, the density is the constant

$$\frac{1}{\pi s} \tag{4.3}$$

inside the unit disk. The following theorem states that for a fixed y_0, as $s \to \infty$, the density $w_{y_0,s}$ of $\mathrm{Brown}(y_0 + c_s)$ is approximately the same constant in (4.3).

Theorem 4.6. *Denote by $w_{y_0,s}$ the density of $\mathrm{Brown}(y_0 + c_s)$. Then, for any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, we have*

$$\left| w_{y_0,s}(\alpha + i\beta) - \frac{1}{\pi s} \right| < \frac{c\tau(y_0^2)}{2\pi s^2 \sin^2 \varphi_0} \left(3 + \frac{2}{\sin \varphi_0} \right), \quad |\psi_{y_0,s}(\alpha)| < 2\sqrt{s} \cos \varphi_0$$

for all large enough s.

To simplify the computation, we assume $\tau(y_0) = 0$ and $\tau(y_0^2) = 1$ until the proof of the theorem. The key is to estimate the difference between the complex derivatives $H'_{\frac{y_0}{\sqrt{s}},1}$ and k'; indeed the density is directly related to the real part of the complex derivative of the subordination function $H^{-1}_{\frac{y_0}{\sqrt{s}},1}$.

Lemma 4.7. *Given any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, for all sufficient large s, the unique α such that*

$$H_{\frac{y_0}{\sqrt{s}},1}(\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)) = 2\cos\varphi, \quad \sin\varphi > \sin\varphi_0$$

satisfies

$$\left| \frac{1}{\mathrm{Re}(1/k'(\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)))} - \frac{1}{\mathrm{Re}(1/k'(e^{i\varphi}))} \right| < \frac{2c}{s \sin^3 \varphi_0}.$$

Proof. Fix any $\varphi_0 \in (0, \pi/2)$ and $c > 1$. By Proposition 4.5, for any $\varphi \in (0, \pi)$ such that $\sin\varphi > \sin\varphi_0$, the unique $\alpha \in \mathbb{R}$ such that

$$H_{\frac{y_0}{\sqrt{s}},1}(\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)) = 2\cos\varphi.$$

satisfies

$$\left|\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha) - e^{i\varphi}\right| < \frac{1}{(\sin\varphi_0)s} \tag{4.4}$$

for all large enough s. We know that $\frac{1}{\mathrm{Re}(1/k'(z))} = 2$ because

$$\frac{1}{k'(z)} = \frac{e^{i\varphi}}{e^{i\varphi} - e^{-i\varphi}} = \frac{1}{2}(1 - i\cot\varphi). \tag{4.5}$$

Using (4.4) and (4.5), we have

$$\frac{1}{(1/2 - |\mathrm{Re}(1/k'(w)) - \mathrm{Re}(1/k'(e^{i\varphi}))|)^2} < 4\sqrt{c} \tag{4.6}$$

for all large enough s.

Write $z = e^{i\varphi}$ and $w = \alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)$. Observe that

$$\frac{1}{k'(w)} - \frac{1}{k'(z)} = \frac{w^2}{w^2-1} - \frac{z^2}{z^2-1} = \frac{(z-w)(z+w)}{(w^2-1)(z^2-1)}. \tag{4.7}$$

Also, it is straightforward to check that $\left|z^2-1\right| = \left|e^{2i\varphi}-1\right| = 2\sin\varphi$, and, by (4.4),

$$\left|w^2 - z^2\right| = |w-z|\,|w+z| < \frac{1}{(\sin\varphi_0)s}\left(2 + \frac{1}{(\sin\varphi_0)s}\right).$$

We have, for all large enough s,

$$\left|\frac{1}{k'(w)} - \frac{1}{k'(z)}\right| < \frac{1}{4\sin^2\varphi_0}\frac{2\sqrt{c}}{s(\sin\varphi_0)}.$$

Thus, by the mean value theorem (applied to the function $1/(\frac{1}{2}+x)$), and (4.4)-(4.7),

$$\begin{aligned}\left|\frac{1}{\mathrm{Re}(1/k'(w))} - \frac{1}{\mathrm{Re}(1/k'(e^{i\varphi}))}\right| &\le \frac{\left|\mathrm{Re}(1/k'(w)) - \mathrm{Re}(1/k'(e^{i\varphi}))\right|}{(1/2 - |\mathrm{Re}(1/k'(w)) - \mathrm{Re}(1/k'(e^{i\varphi}))|)^2} \\ &< 4\sqrt{c}\frac{\sqrt{c}}{2s\sin^3\varphi_0} = \frac{2c}{s\sin^3\varphi_0}\end{aligned}$$

for all large enough s, completing the proof. □

Lemma 4.8. *For any $c > 1$, we have*

$$\left|\frac{1}{\mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(z))} - \frac{1}{\mathrm{Re}(1/k'(z))}\right| < \frac{3c}{s\,|z|^4}\frac{1}{[\mathrm{Re}(1/k'(z))]^2}\frac{1}{|k'(z)|^2}, \quad |z| > \frac{1}{2}.$$

for all large enough s.

When $|z| = 1$ but $z \neq 1, -1$, the right hand side of the inequality does not divide by zero. More explicitly, if $z = e^{i\varphi}$, we have

$$|k'(z)| = \left|z^2 - 1\right| = 2\sin\varphi. \tag{4.8}$$

Proof. Let $c > 1$. By (4.1), for all $|z| > \frac{1}{2}$,

$$\left|H'_{\frac{y_0}{\sqrt{s}},1}(z) - k'(z)\right| \le \frac{1}{s\,|z|^4}\left(3\tau(y_0^2) + \sum_{n=3}^{\infty}\frac{(n+1)\,|\tau(y^n)|}{s^{\frac{n}{2}-1}(1/2)^{n-2}}\right) < \frac{3c^{1/3}\tau(y_0^2)}{s\,|z|^4} \tag{4.9}$$

for all large enough s. We then must have

$$\left|\frac{1}{\mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(z))}\right| < \frac{c^{1/3}}{\mathrm{Re}(1/k'(z))} \quad \text{and} \quad \left|\frac{1}{H'(\frac{y_0}{\sqrt{s}},1)(z)}\right| < \frac{c^{1/3}}{|k'(z)|}$$

for all large enough s. Therefore, we have

$$\begin{aligned}\left|\frac{1}{\mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(z))} - \frac{1}{\mathrm{Re}(1/k'(z))}\right| &= \frac{\left|\mathrm{Re}(1/k'(z)) - \mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(z))\right|}{\left|\mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(z))\mathrm{Re}(1/k'(z))\right|} \\ &\leq \frac{c^{1/3}}{|\mathrm{Re}(1/k'(z))|^2}\frac{c^{1/3}}{|k'(z)|^2}|H'(z) - k'(z)| \\ &< \frac{3c\tau(y_0^2)}{s|z|^4}\frac{1}{[\mathrm{Re}(1/k'(z))]^2}\frac{1}{|k'(z)|^2},\end{aligned}$$

which is the desired inequality since we assume $\tau(y_0^2) = 1$ until the proof of Theorem 4.6. □

Lemma 4.9. *Given any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, for all sufficient large s, the unique α such that*

$$H_{\frac{y_0}{\sqrt{s}},1}(\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)) = 2\cos\varphi, \quad \sin\varphi > \sin\varphi_0$$

satisfies

$$\left|\frac{1}{\mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(w))} - 2\right| < \frac{c}{s\sin^2\varphi_0}\left(3 + \frac{2}{\sin\varphi_0}\right)$$

where $w = \alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)$.

Proof. Let $c > 1$. Write $z = e^{i\varphi}$ and $w = \alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)$. Recall that $\frac{1}{\mathrm{Re}(1/k'(z))} = 2$ by (4.5). We estimate

$$\left|\frac{1}{\mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(w))} - 2\right| \leq \left|\frac{1}{\mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(w))} - \frac{1}{\mathrm{Re}(1/k'(w))}\right| + \left|\frac{1}{\mathrm{Re}(1/k'(w))} - \frac{1}{\mathrm{Re}(1/k'(z))}\right|. \tag{4.10}$$

We estimate the first term in (4.10) using Proposition 4.5 and Lemmas 4.7 and 4.8. Fix any $1 < c' < c$. For all large enough s, the first term is bounded by

$$\begin{aligned}\frac{3c'}{s|w|^4}\frac{1}{[\mathrm{Re}(1/k'(w))]^2}\frac{1}{|k'(w)|^2} &\leq \frac{3c'}{s[1 - 1/(\sin\varphi_0 s)^4]}\left(\frac{1}{\mathrm{Re}(1/k'(e^{i\varphi}))} + \frac{2c'}{s\sin^3\varphi_0}\right)^2\frac{1}{|k'(w)|^2} \\ &< \frac{12c}{s}\frac{1}{4\sin^2\varphi} \\ &\leq \frac{3c}{s\sin^2\varphi_0}\end{aligned}$$

by (4.8) and Lemmas 4.7 and 4.8.

By Lemma 4.7, the second term in (4.10) is bounded by

$$\left|\frac{1}{\mathrm{Re}(1/k'(w))} - \frac{1}{\mathrm{Re}(1/k'(z))}\right| < \frac{2c}{s\sin^3\varphi_0}.$$

The result then follows from adding these estimates. □

Proposition 4.10. *Denote by* $w_{\frac{y_0}{\sqrt{s}},1}$ *the density of* $\mathrm{Brown}(\frac{y_0}{\sqrt{s}}+c_1)$*. Then, for any* $c>1$ *and* $\varphi_0\in(0,\pi/2)$*, we have*

$$\left|w_{\frac{y_0}{\sqrt{s}},1}(\alpha+i\beta)-\frac{1}{\pi}\right|<\frac{c}{2\pi s\sin^2\varphi_0}\left(3+\frac{2}{\sin\varphi_0}\right),\quad \left|\psi_{\frac{y_0}{\sqrt{s}},1}(\alpha)\right|<2\cos\varphi_0$$

for all large enough s*.*

Proof. By Equation (3.31) of [25],

$$\mathrm{Re}\left(\frac{1}{H'_{\frac{y_0}{\sqrt{s}},1}(w)}\right)\frac{d\psi_{\frac{y_0}{\sqrt{s}},1}(\alpha)}{d\alpha}=1$$

where $w=\alpha+iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)$. (This formula appeals to the subordination function $H^{-1}_{\frac{y_0}{\sqrt{s}},1}$ of the free convolution $\frac{y_0}{\sqrt{s}}+\sigma_1$ has an analytic continuation in a neighborhood of any point $\psi_{\frac{y_0}{\sqrt{s}},1}(\alpha+iv_{\frac{y_0}{\sqrt{s}},1}(\alpha))$ if $v_{\frac{y_0}{\sqrt{s}},1}(\alpha)>0$; see [2, Theorem 3.3(1)].) Thus, we can express the real derivative through complex derivative

$$\frac{d\psi_{\frac{y_0}{\sqrt{s}},1}(\alpha)}{d\alpha}=\frac{1}{\mathrm{Re}(1/H'_{\frac{y_0}{\sqrt{s}},1}(w))}.$$

By Lemma 4.9, given any $c>1$ and $\varphi_0\in(0,\pi/2)$, for all sufficient large s, the unique α such that

$$\psi_{\frac{y_0}{\sqrt{s}},1}(\alpha)=2\cos\varphi,\quad \sin\varphi>\sin\varphi_0$$

satisfies

$$\left|\frac{d\psi_{\frac{y_0}{\sqrt{s}},1}(\alpha)}{d\alpha}-2\right|<\frac{c}{s\sin^2\varphi_0}\left(3+\frac{2}{\sin\varphi_0}\right).$$

The proposition now follows from Theorem 2.6 □

All the estimates in this section that we have done are under the assumption $\tau(y_0)=0$ and $\tau(y_0^2)$. We are now ready to prove the estimate of the density of $\mathrm{Brown}(y_0+c_s)$ for arbitrary $\tau(y_0)$ and $\tau(y_0^2)$.

Proof of Theorem 4.6. Without loss of generality, we assume $\tau(y_0)=0$, since otherwise we translate the density by $\tau(y_0)$.

We first assume $\tau(y_0^2)=1$. Let $w=\alpha+iv_{y_0,s}(\alpha)$ and $z=\frac{w}{\sqrt{s}}$. Then

$$z=\frac{\alpha}{\sqrt{s}}+iv_{\frac{y_0}{\sqrt{s}},1}\left(\frac{\alpha}{\sqrt{s}}\right).$$

Since $\mathrm{Brown}(y_0+c_s)$ is the push-forward measure of $\mathrm{Brown}\left(\frac{y_0}{\sqrt{s}}+c_1\right)$ by $z\mapsto\sqrt{s}z$,

$$w_{y_0,s}(\alpha+i\beta)=\frac{1}{s}\cdot w_{\frac{y_0}{\sqrt{s}},1}\left(\frac{1}{\sqrt{s}}(\alpha+i\beta)\right),\quad z\in\Lambda_{y_0,s}.$$

By Proposition 4.10, for any $c>1$ and $\varphi_0\in(0,\pi/2)$, we have

$$\left|w_{y_0,s}(\alpha+i\beta)-\frac{1}{\pi s}\right|<\frac{c}{2\pi s^2\sin^2\varphi_0}\left(3+\frac{2}{\sin\varphi_0}\right),\quad |\psi_{y_0,s}(\alpha)|<2\sqrt{s}\cos\varphi_0$$

for all large enough s. This establishes the result with $\tau(y_0^2)=1$.

For arbitrary $\tau(y_0^2)$, let $Y = \frac{y_0}{\sqrt{\tau(y_0^2)}}$. We consider the random variable $\frac{1}{\sqrt{\tau(y_0^2)}}(y_0 + c_s)$ which has the same $*$-moments, hence the same Brown measure, as $Y + c_t$, where $t = s/\tau(y_0^2)$.

By the result for $\tau(y_0^2) = 1$, given any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, we have

$$\left|w_{Y,t}(\alpha + i\beta) - \frac{1}{\pi t}\right| < \frac{c}{2\pi t^2 \sin^2 \varphi_0}\left(3 + \frac{2}{\sin \varphi_0}\right), \quad |\psi_{Y,t}(\alpha)| < 2\sqrt{t}\cos \varphi_0 \tag{4.11}$$

for all large enough t. Now, since $\mathrm{Brown}(y_0 + c_s)$ is the push-forward measure of $\mathrm{Brown}(Y + c_t)$ by $z \mapsto \sqrt{\tau(y_0^2)}z$, by (4.11), we must have

$$\left|w_{y_0,s}(\alpha + i\beta) - \frac{1}{\pi s}\right| < \frac{c\tau(y_0^2)}{2\pi s^2 \sin^2 \varphi_0}\left(3 + \frac{2}{\sin \varphi_0}\right), \quad |\psi_{y_0,s}(\alpha)| < 2\sqrt{s}\cos \varphi_0$$

for all large enough s. □

5 Asymptotic behaviors of adding an elliptic element

In this section, we study three limiting behaviors of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$ as $s \to \infty$. The first regime is to keep s and t at the same ratio $r = t/s$; the second regime is to keep t fixed; the last regime is to fix $s = t/2$.

5.1 Fix s/t and let $s, t \to \infty$

5.1.1 Domain behavior

In this section, we discuss the asymptotic behavior of the domain of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}})$ for a fixed $r = t/s$. When $y_0 = 0$, the domain of $\tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$ has the shape of an ellipse with boundary

$$\frac{2s - t}{\sqrt{s}}\cos\varphi + i\frac{t}{\sqrt{s}}\sin\varphi, \quad \varphi \in [0, 2\pi] \tag{5.1}$$

(See [8, Example 5.3]). As $s \to \infty$ with $r = t/s$ fixed, the random variable $y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$ behaves like the elliptic element $\tau(y_0) + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}}$. Roughly speaking, the domain $\Omega_{s,t}$ of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}})$ is asymptotically an ellipse with boundary as in (5.1) translated by $\tau(y_0)$. The following theorem states precisely the asymptotic behavior of the domain $\Omega_{s,t}$ of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}})$; the main tool is Theorem 4.1.

Theorem 5.1. *Fix the ratio $r = t/s$. The following asymptotic behaviors of the graph of $\Omega_{s,t}$ hold.*

1. *Let $D_\nu = \sup\{|x - y|\,|x, y \in \operatorname{supp}\mu\}$. When $s \geq 4D_\nu^2$, the function $b_{s,t}$ is unimodal. In particular, $\Omega_{s,t} \cap \mathbb{R}$ is an interval.*
2. *Given any $c > 1$, we have*

$$\left|\sup \Omega_{s,t} \cap \mathbb{R} - \left(\tau(y_0) + \frac{2s - t}{\sqrt{s}}\right)\right| < \frac{c(3r + 2\,|1 - r|)\tau(y_0^2)}{2\sqrt{s}}$$

and

$$\left|\inf \Omega_{s,t} \cap \mathbb{R} - \left(\tau(y_0) - \frac{2s - t}{\sqrt{s}}\right)\right| < \frac{c(3r + 2\,|1 - r|)\tau(y_0^2)}{2\sqrt{s}}$$

for all sufficiently large s. In particular, $\Lambda_{y_0,s} \cap \mathbb{R}$ is contained in

$$\left(\tau(y_0) - \frac{2s - t}{\sqrt{s}} - \frac{c(3r + 2\,|1 - r|)\tau(y_0^2)}{2\sqrt{s}}, \tau(y_0) + \frac{2s - t}{\sqrt{s}} + \frac{c(3r + 2\,|1 - r|)\tau(y_0^2)}{2\sqrt{s}}\right)$$

for all large enough s.

3. *Given any $\varphi_0 \in (0, \pi/2)$, then for all large enough s, for all $|\cos\varphi| \le \cos\varphi_0$, the unique $\alpha \in \mathbb{R}$ such that*

$$H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) = 2\sqrt{s}\cos\varphi.$$

satisfies

$$\left|U_{s,t}(\alpha + iv_{y_0,s}(\alpha)) - \left[\frac{2s-t}{\sqrt{s}}\cos\varphi + i\frac{t}{\sqrt{s}}\sin\varphi\right]\right| < \frac{r}{(\sin\varphi_0)\sqrt{s}}.$$

Proof. Point 1 follows directly from [22, Theorem 3.2] which states that $v_{y_0,s}$ is unimodal for $s \ge 4D_\nu^2$, because, by Proposition 3.4, we have

$$b_{s,t} = \frac{t}{s} v_{y_0,s}.$$

Fix $r = t/s$ throughout this proof. We now prove Point 2. Without loss of generality, we assume $\tau(y_0) = 0$. We first estimate $a_{1,r}(\alpha^*)$ where

$$\alpha^* = \sup \Lambda_{y_0/\sqrt{s},1} \cap \mathbb{R}.$$

We compute

$$a_{1,r}(\alpha^*)-(2-r) = (\alpha^*-1)\left(1 - \frac{1-r}{\alpha^*}\right) + \frac{(1-r)\tau(y_0^2)}{s(\alpha^*)^3} + \frac{(1-r)}{s^{3/2}}\sum_{n=3}^{\infty}\frac{\tau(y_0^n)}{s^{(n-3)/2}(\alpha^*)^{n+1}}. \quad (5.2)$$

By Proposition 4.4 (with s replaced by $s/\tau(y_0^2)$), given any $c > 1$, for all large enough s, we have

$$|a_{1,r}(\alpha^*) - (2-r)| < \frac{c(3r + 2|1-r|)\tau(y_0^2)}{2s}.$$

Since

$$\sup \Omega_{s,t} \cap \mathbb{R} = \sqrt{s} a_{1,r}(\alpha^*),$$

we have

$$\left|\sup \Omega_{s,t} \cap \mathbb{R} - \left(\tau(y_0) + \frac{2s-t}{\sqrt{s}}\right)\right| < \frac{c(3r + 2|1-r|)\tau(y_0^2)}{2\sqrt{s}}$$

for all sufficiently large s. The estimate for $\inf \Omega_{s,t} \cap \mathbb{R}$ is similar.

We prove Point 3 now. By Theorem 3.3, we know that

$$\Omega_{s,t} = U_{s,t}(\Lambda_{y_0,s}).$$

Suppose α is chosen such that $\psi_{y_0,s}(\alpha) = 2\sqrt{s}\cos\varphi$. We compute the upper boundary curve $a + ib_{s,t}(a) = U_{s,t}(\alpha + iv_{y_0,s}(\alpha))$ as

$$\begin{aligned} a_{s,t}(\alpha) &= (1-r)\psi_{y_0,s}(\alpha) + r\alpha = 2(1-r)\sqrt{s}\cos\varphi + r\alpha; \\ b_{s,t}(a) &= b_{s,t}(a_{s,t}(\alpha)) = rv_{y_0,s}(\alpha). \end{aligned}$$

So, we have

$$\left|a + ib_{s,t}(a) - \sqrt{s}[(2-r)\cos\varphi + ir\sin\varphi]\right| = r\left|\alpha + iv_{y_0,s}(\alpha) - \sqrt{s}e^{i\varphi}\right|. \quad (5.3)$$

Therefore, by Theorem 4.1, for any $\varphi_0 \in (0, \pi/2)$,

$$\begin{aligned} \left|a + ib_{s,t}(a) - \sqrt{s}[(2-r)\cos\varphi + ir\sin\varphi]\right| &= r\left|\alpha + iv_{y_0,s}(\alpha) - \sqrt{s}e^{i\varphi}\right| \\ &< \frac{r}{(\sin\varphi_0)\sqrt{s}} \end{aligned}$$

for all sufficiently large s. This proves Point 3. □

5.1.2 Density behavior

In this section, we investigate the asymptotic behavior of the density of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}})$ for a fixed $r = t/s$. In the case $y_0 = 0$, $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}})$ is the elliptic law, with constant density

$$\frac{1}{\pi}\frac{s}{(2s-t)t} \tag{5.4}$$

in domain $\Omega_{s,t}$, which is a region bounded by an ellipse in this case (See [8, Example 5.3]).

Denote by $w_{y_0,s,t}$ the density of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-\frac{t}{2}} + i\sigma_{\frac{t}{2}})$. We will prove that as s large and $r = t/s$ fixed, the density $w_{y_0,s,t}$ is approximately the same constant in (5.4). The main tool is the estimate of the density of $\mathrm{Brown}(y_0 + c_s)$ in Theorem 4.6.

Theorem 5.2. *Fix $r = t/s$. Given any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, we have*

$$\left| w_{y_0,s,t}(a+ib) - \frac{1}{\pi}\frac{s}{(2s-t)t} \right| < \frac{c\tau(y_0^2)}{2\pi\sin^2\varphi_0}\frac{1}{(2s-t)^2}\left(3 + \frac{2}{\sin\varphi_0}\right)$$

whenever $\psi_{y_0,s}(\alpha_{s,t}(a)) < 2\sqrt{s}\cos\varphi_0$, for all large enough s.

Proof. Let $c > 1$ be given. By Corollary 3.6, if we write $a + ib = U_{s,t}(\alpha + i\beta)$ for all $\alpha + i\beta \in \Lambda_{y_0,s}$. Then we have

$$w_{y_0,s,t}(a+ib) = \frac{1}{r}\frac{w_{y_0,s}(\alpha+i\beta)}{r + 2\pi(1-r)s\cdot w_{y_0,s}(\alpha+i\beta)}$$

for all $a + ib \in \Omega_{s,t}$.

Now, by the formula

$$\frac{1}{\pi s}\frac{1}{2-r} = \frac{1/(\pi s)}{r + 2\pi(1-r)s\cdot(1/\pi s)},$$

and Theorem 4.6, for any $1 < c' < c$, if $\psi_{y_0,s}(\alpha) < 2\sqrt{s}\cos\varphi_0$, then we have $\pi s w_{y_0,s}(\alpha + i\beta) \to 1$, and

$$\begin{aligned}\left| \frac{w_{y_0,s}(\alpha+i\beta)}{r + 2\pi(1-r)s\cdot w_{y_0,s}(\alpha+i\beta)} - \frac{1/(\pi s)}{2-r} \right| &= \frac{r\,|w_{y_0,s}(\alpha+i\beta) - 1/(\pi s)|}{[r + 2\pi(1-r)s\cdot w_s(\alpha+i\beta)][2-r]} \\ &< \frac{cr\tau(y_0^2)}{2\pi s^2\sin^2\varphi_0}\left(3 + \frac{2}{\sin\varphi_0}\right)\frac{1}{(2-r)^2}\end{aligned}$$

for all large enough s. The proof follows from dividing the above estimate by r. ☐

5.2 Fix t and let $s \to \infty$

In this section, we investigate the asymptotic behavior of $\mathrm{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$ with t fixed and $s \to \infty$.

5.2.1 Domain behavior

The following theorem states that $\Omega_{s,t}$ has the shape of an ellipse in the limit with fixed t as $s \to \infty$, except points close to the *endpoints* of $\Omega_{s,t} \cap \mathbb{R}$. The limiting ellipse has a very short minor axis; it is a long and thin ellipse.

Theorem 5.3. *Fix $t > 0$. The following asymptotic behaviors of the graph of $\Omega_{s,t}$ hold.*

1. *Let $D_\nu = \sup\{|x-y|\,|x,y \in \operatorname{supp}\mu\}$. When $s \geq 4D_\nu^2$, the function $b_{s,t}$ is unimodal. In particular, $\Omega_{s,t} \cap \mathbb{R}$ is an interval.*

2. *Given any* $c > 1$*, we have*

$$\left|\sup \Omega_{s,t} \cap \mathbb{R} - \left(\tau(y_0) + 2\sqrt{s}\right)\right| < \frac{c\left|\tau(y_0^2) - t\right|}{\sqrt{s}}$$

and

$$\left|\inf \Omega_{s,t} \cap \mathbb{R} - \left(\tau(y_0) - 2\sqrt{s}\right)\right| < \frac{c\left|\tau(y_0^2) - t\right|}{\sqrt{s}}$$

for all sufficiently large s*. In particular,*

$$\Lambda_{y_0,s} \cap \mathbb{R} \subset \left(\tau(y_0) - 2\sqrt{s} - \frac{c\left|\tau(y_0^2) - t\right|}{\sqrt{s}}, \tau(y_0) + 2\sqrt{s} + \frac{c\left|\tau(y_0^2) - t\right|}{\sqrt{s}}\right)$$

for all large enough s*.*

3. *Given any* $\varphi_0 \in (0, \pi/2)$*, then for all large enough* s*, for all* $|\cos\varphi| \leq \cos\varphi_0$*, the unique* $\alpha \in \mathbb{R}$ *such that*

$$H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) = 2\sqrt{s}\cos\varphi.$$

satisfies

$$\left|U_{s,t}(\alpha + iv_{y_0,s}(\alpha)) - \left[\frac{2s - t}{\sqrt{s}}\cos\varphi + i\frac{t}{\sqrt{s}}\sin\varphi\right]\right| < \frac{t}{(\sin\varphi_0)s^{3/2}}.$$

Furthermore, we have

$$\lim_{s\to\infty} \sup\{|\mathrm{Im}\, z|\, |z \in \Omega_{s,t}\} = 0.$$

Proof. Point 1 follows directly from Theorem 3.3 and [22, Theorem 3.2] which states that $v_{y_0,s}$ is unimodal for $s \geq 4D_\nu^2$, because, by (3.11), we have

$$b_{s,t} = \frac{t}{s} v_{y_0,s}.$$

Fix $t > 0$. We now prove Point 2. Without loss of generality, we assume $\tau(y_0) = 0$. We first estimate $a_{1,r}(\alpha^*)$ where

$$\alpha^* = \sup \Lambda_{y_0/\sqrt{s},1} \cap \mathbb{R}.$$

We calculate

$$\begin{aligned} a_{1,r}(\alpha^*) - 2 &= \alpha^* - 2 + (1 - r)\sum_{n=0}^{\infty} \frac{\tau(y_0^n)}{s^{\frac{n}{2}}(\alpha^*)^{n+1}} \\ &= \alpha^* - 1 + \frac{1 - \alpha^*}{\alpha^*} - \frac{t}{s\alpha^*} + \frac{\tau(y_0^2)}{s(\alpha^*)^3} + \sum_{n=3}^{\infty} \frac{\tau(y_0^n)}{s^{\frac{n}{2}}(\alpha^*)^{n+1}} \\ &= (\alpha^* - 1)\frac{\alpha^* - 1}{\alpha^*} + \frac{\tau(y_0^2) - t(\alpha^*)^2}{s(\alpha^*)^3} + \sum_{n=3}^{\infty} \frac{\tau(y_0^n)}{s^{\frac{n}{2}}(\alpha^*)^{n+1}} \end{aligned}$$

By Proposition 4.4 (with s replaced by $s/\tau(y_0^2)$), given any $c > 1$, for all large enough s, we have (by keeping the only order $1/s$ term)

$$|a_{1,r}(\alpha^*) - 2| < \frac{c\left|\tau(y_0^2) - t\right|}{s}.$$

It follows that

$$\left|\sup \Omega_{s,t} \cap \mathbb{R} - \left(\tau(y_0) + 2\sqrt{s}\right)\right| < \frac{c\left|\tau(y_0^2) - t\right|}{\sqrt{s}}$$

for all sufficiently large s. The estimate for $\inf \Omega_{s,t} \cap \mathbb{R}$ is similar.

We now prove Point 3. By (5.3),

$$\left|a + ib_{s,t}(a) - \sqrt{s}[(2-r)\cos\varphi + ir\sin\varphi]\right| = r\left|\alpha + iv_{y_0,s}(\alpha) - \sqrt{s}e^{i\varphi}\right|.$$

Therefore, by Theorem 4.1, for any $\varphi_0 \in (0, \pi/2)$,

$$\left|a + ib_{s,t}(a) - \sqrt{s}[(2-r)\cos\varphi + ir\sin\varphi]\right| < \frac{t}{(\sin\varphi_0)s^{3/2}} \tag{5.5}$$

for all sufficiently large s.

Let $\varphi_0 = \frac{\pi}{6}$ so that $\sin\varphi > 1/2$ for all φ such that $|\cos\varphi| < \cos\varphi_0$. We label by α_φ the unique $\alpha \in \mathbb{R}$ such that

$$H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) = 2\sqrt{s}\cos\varphi, \quad |\cos\varphi| \le \cos\varphi_0.$$

By (5.5), we have

$$\sup\{b_{s,t}(a_{s,t}(\alpha)) | \alpha_{\pi-\varphi_0} < \alpha < \alpha_{\varphi_0}\} > \frac{t}{\sqrt{s}} - \frac{2t}{s^{3/2}}.$$

Since

$$b_{s,t}(a_{s,t}(\alpha_{\varphi_0})) < \frac{t}{2\sqrt{s}} + \frac{2t}{s^{3/2}}$$

and, by Point 1, the function $b_{s,t}$ is unimodal,

$$b_{s,t}(a_{s,t}(\alpha)) < \frac{t}{2\sqrt{s}} + \frac{2t}{s^{3/2}}, \quad \alpha \ge \alpha_{\varphi_0} \text{ or } \alpha \le \alpha_{\pi-\varphi_0}. \tag{5.6}$$

For all $\alpha_{\pi-\varphi_0} < \alpha < \alpha_{\varphi_0}$,

$$\sup\{b_{s,t}(a_{s,t}(\alpha)) | \alpha_{\pi-\varphi_0} < \alpha < \alpha_{\varphi_0}\} < \frac{t}{\sqrt{s}} + \frac{2t}{s^{3/2}}. \tag{5.7}$$

Therefore, we conclude

$$\lim_{s\to\infty} \sup\{|\operatorname{Im} z| \,|z \in \Omega_{s,t}\} = 0$$

by (5.6) and (5.7). □

5.2.2 Density behavior

If we consider the special case of $y_0 = 0$, $\operatorname{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$ is just the elliptic law; as mentioned in (5.4), it has a constant density

$$\frac{1}{\pi}\frac{s}{(2s-t)t}.$$

If we fixed t and let $s \to \infty$, this density converges to the constant $1/(2\pi t)$.

The following theorem states that if we consider an arbitrary self-adjoint initial condition y_0, the density of $\operatorname{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$ also converges to $1/(2\pi t)$; the convergence is uniform away the endpoints of $\Omega_{s,t} \cap \mathbb{R}$.

Theorem 5.4. *Denote by $w_{y_0,s,t}$ the density of $\operatorname{Brown}(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})$. Then given any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, there is an $s_0 > 0$ such that*

$$\left|w_{y_0,s,t}(a+ib) - \frac{1}{2\pi t}\right| < \frac{c}{4\pi s}, \quad |\psi_{y_0,s}(\alpha_{s,t}(a))| < 2\sqrt{s}\cos\varphi_0$$

for all $s > s_0$.

Proof. Let $c > 1$ and $\varphi_0 \in (0, \pi/2)$ be given. By Corollary 3.6, if we write $(a, b) = U_{s,t}(\alpha, \beta)$ for all $\alpha + i\beta \in \Lambda_{y_0,s}$. Then we have

$$w_{y_0,s,t}(a+ib) = \frac{1}{2\pi t}\frac{s\pi w_{y_0,s}(\alpha+i\beta)}{t/(2s) + (1-t/s)\pi s\cdot w_{y_0,s}(\alpha+i\beta)} \tag{5.8}$$

for all $a + ib \in \Omega_{s,t}$.

By Theorem 4.6, given any $1 < c' < c$, we have

$$|\pi s\cdot w_{y_0,s}(\alpha+i\beta) - 1| < \frac{c'\tau(y_0^2)}{2s\sin^2\varphi_0}\left(3 + \frac{2}{\sin\varphi_0}\right), \quad |\psi_{y_0,s}(\alpha)| < 2\sqrt{s}\cos\varphi_0$$

for all large enough s. Then, we compute

$$\begin{aligned}\left|\frac{s\pi w_{y_0,s}(\alpha+i\beta)}{t/(2s) + (1-t/s)\pi s\cdot w_{y_0,s}(\alpha+i\beta)} - 1\right| &= \frac{t}{s}\left|\frac{\pi s\cdot w_{y_0,s}(\alpha+i\beta) - 1/2}{t/(2s) + (1-t/s)\pi s\cdot w_{y_0,s}(\alpha+i\beta)}\right| \\ &< \frac{c't}{s}\left[\frac{1}{2} + \frac{c'\tau(y_0^2)}{2s\sin^2\varphi_0}\left(3 + \frac{2}{\sin\varphi_0}\right)\right] \\ &< \frac{ct}{2s}.\end{aligned}$$

for all large enough s, since $t/(2s) + (1-t/s)\pi s\cdot w_{y_0,s}(\alpha+i\beta)$ converges to 1. Thus, using (5.8), we have the estimate (uniform for all $|\psi_{y_0,s}(\alpha_{s,t}(a))| < 2\sqrt{s}\cos\varphi_0$)

$$w_{y_0,s,t}(a+ib) - \frac{1}{2\pi t} = \frac{1}{2\pi t}\left|\frac{s\pi w_{y_0,s}(\alpha+i\beta)}{t/(2s) + (1-t/s)\pi s\cdot w_{y_0,s}(\alpha+i\beta)} - 1\right| < \frac{c}{4\pi s}$$

for all sufficiently large s. □

5.3 Set $s = t/2$ and let $s \to \infty$

In this section, we investigate the asymptotic behavior of $\mathrm{Brown}(y_0 + \sigma_{s-t/2} + i\tilde{\sigma}_{t/2})$ with $s = t/2$ and $s \to \infty$. Note that, when $s = t/2$, the random variable $y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2}$ is $y_0 + i\sigma_s$.

Theorem 5.5. *1. Let $D_\nu = \sup\{|x-y|\,|x, y \in \operatorname{supp}\mu\}$. When $s \geq 4D_\nu^2$, the function $b_{s,t}$ is unimodal. In particular, $\Omega_{s,t}\cap\mathbb{R}$ is an interval.*

2. We have

$$-\frac{4c\tau(y_0^2)}{\sqrt{s}} < \inf(\Omega_{s,t}\cap\mathbb{R}) - \tau(y_0) < 0 < \sup(\Omega_{s,t}\cap\mathbb{R}) - \tau(y_0) < \frac{4c\tau(y_0^2)}{\sqrt{s}}$$

for all s large enough. In particular,

$$\Omega_{s,t}\cap\mathbb{R} \subset \left(\tau(y_0) - \frac{4c\tau(y_0^2)}{\sqrt{s}}, \tau(y_0) + \frac{4c\tau(y_0^2)}{\sqrt{s}}\right)$$

for all s large enough.

3. We also have

$$\left|\sup\{|\mathrm{Im}\, z|\,|z\in\Omega_{s,t}\} - 2\sqrt{s}\right| < \frac{2c}{\sqrt{s}}$$

for all large enough s.

Proof. Point 1 follows directly from [22, Theorem 3.2] which states that $v_{y_0,s}$ is unimodal for $s \geq 4D_\nu^2$, because, (3.11), we have

$$b_{s,t} = 2v_{y_0,s}.$$

We now prove Point 2. Let $c > 1$ be given. Without loss of generality, we assume $\tau(y_0) = 0$. Denote

$$M_s = \sup(\Lambda_s \cap \mathbb{R}) \quad \text{and} \quad m_s = \inf(\Lambda_s \cap \mathbb{R}).$$

Then $\sup(\Omega_{y_0,s} \cap \mathbb{R}) = a_{y_0,s}(M_s)$ and $\inf(\Omega_{y_0,s} \cap \mathbb{R}) = a_{y_0,s}(m_s)$. First, $M_s > \sup(\operatorname{supp}\nu)$ by Point 1 of Theorem 4.1. Recall from Definition 2.7 that (since M_t is real)

$$\begin{aligned} a_{y_0,s}(M_s) &= H_{y_0,-s}(M_s) \\ &= M_s - s\int \frac{d\nu(x)}{M_s - x} \\ &= \frac{1}{M_s}(M_s^2 - s) - \frac{s}{M_s^3}\sum_{n=2}^{\infty}\frac{\tau(y_0^n)}{M_s^{n-2}}. \end{aligned} \tag{5.9}$$

Now, by Theorem 4.1, we have

$$\sqrt{s} - \frac{3c\tau(y_0^2)}{2\sqrt{s}} < M_s < \sqrt{s} + \frac{3c'\tau(y_0^2)}{2\sqrt{s}}$$

for all large enough s. Thus we can estimate $|a_{y_0,s}(M_t)|$ by (5.9)

$$\begin{aligned} |a_{y_0,s}(M_s)| &= \left|(M_s - \sqrt{s})\left(1 + \frac{\sqrt{s}}{M_s}\right) - \frac{s}{M_s^3}\sum_{n=2}^{\infty}\frac{\tau(y_0^n)}{M_s^{n-2}}\right| \\ &< \frac{3c\tau(y_0^2)}{\sqrt{s}} + \frac{c\tau(y_0^2)}{\sqrt{s}} \\ &= \frac{4c\tau(y_0^2)}{\sqrt{s}}. \end{aligned}$$

By that $\mathrm{Brown}(y_0 + i\sigma_s)$ is symmetric about the real axis and the *holomorphic* moments of $\mathrm{Brown}(y_0 + i\sigma_s)$ agree with the corresponding holomorphic moments of $y_0 + i\sigma_s$ [9],

$$\begin{aligned} \int a\, d\mathrm{Brown}(y_0 + i\sigma_s)(a + ib) &= \int (a + ib)\, d\mathrm{Brown}(y_0 + i\sigma_s)(a + ib) \\ &= \tau(y_0 + i\sigma_s) = 0. \end{aligned} \tag{5.10}$$

It is impossible that $a_{y_0,s}(M_s) \le 0$; otherwise, since $\Omega_{y_0,s}$ is not a subset of the imaginary axis, the integral in (5.10) is negative, contradicting that the integral is 0.

The estimate for $a_{y_0,s}(m_s)$ is similar.

To prove Point 3, we let $\varphi_0 \in (0, \pi/2)$ such that $1/(\sin\varphi_0) < c$. By Theorem 4.1, if we write α_φ the unique real number such that

$$H_{y_0,s}(\alpha_\varphi + iv_{y_0,s}(\alpha_\varphi)) = 2\sqrt{s}\cos\varphi, \quad |\cos\varphi| \le \cos\varphi_0,$$

then

$$\left|\alpha_\varphi + iv_{y_0,s}(\alpha_\varphi) - \sqrt{s}e^{i\varphi}\right| < \frac{1}{(\sin\varphi_0)\sqrt{s}}.$$

Thus, we have

$$\sqrt{s} - \frac{1}{(\sin\varphi_0)\sqrt{s}} < \sup\{v_{y_0,s}(\alpha_\varphi) |\, |\cos\varphi| < \cos\varphi_0\} < \sqrt{s} + \frac{1}{(\sin\varphi_0)\sqrt{s}}.$$

Also, for all $\alpha \ge \alpha_{\varphi_0}$ or $\alpha \le \alpha_{\pi-\varphi_0}$, we have, by unimodality of $v_{y_0,s}$,

$$\begin{aligned} v_{y_0,s}(\alpha) &< \sqrt{s}\sin\varphi_0 + \frac{1}{(\sin\varphi_0)\sqrt{s}} \\ &< \sqrt{s} - \frac{1}{\sqrt{s}\sin\varphi_0} \\ &< \sup\{v_{y_0,s}(\alpha_\varphi) |\, |\cos\varphi| < \cos\varphi_0\} \end{aligned}$$

for all large enough s. It follows that

$$\left|\sup_{\alpha\in\mathbb{R}} v_{y_0,s}(\alpha) - \sqrt{s}\right| < \frac{1/(\sin\varphi_0)}{\sqrt{s}} < \frac{c}{\sqrt{s}}$$

for all sufficiently large s. Because $b_{s,t} = 2v_{y_0,s}$, Point 3 of this theorem is established. □

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