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The Fleming-Viot process with McKean-Vlasov dynamics*

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Abstract

The Fleming-Viot particle system consists of N identical particles diffusing in an open domain $D \subset \mathbb{R}^d$. Whenever a particle hits the boundary ∂D, that particle jumps onto another particle in the interior. It is known that this system provides a particle representation for both the Quasi-Stationary Distribution (QSD) and the distribution conditioned on survival for a given diffusion killed at the boundary of its domain. We extend these results to the case of McKean-Vlasov dynamics. We prove that the law conditioned on survival of a given McKean-Vlasov process killed on the boundary of its domain may be obtained from the hydrodynamic limit of the corresponding Fleming-Viot particle system. We then show that if the target killed McKean-Vlasov process converges to a QSD as $t \to \infty$, such a QSD may be obtained from the stationary distributions of the corresponding N-particle Fleming-Viot system as $N \to \infty$.

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1 Introduction

The long-term behaviour of Markovian processes with an absorbing boundary has been studied since the work of Yaglom on sub-critical Galton-Watson processes [32], a review of which can be found in [27]. The long-time limits we obtain are quasi-stationary distributions (QSDs). In this paper we study the behavior of a system of interacting diffusion processes, known as a Fleming-Viot particle system, which is known to provide a particle representation for these long time limits [9], [27, Section 6].

Given an open set $D \subset \mathbb{R}^d$, we consider $N \geq 2$ particles diffusing in the domain D. The particle positions are denoted by $X_t^1, \ldots, X_t^N \in D$, so that $\vec{X}_t^N = (X_t^1, \ldots, X_t^N)$ is

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a D^N-valued stochastic process. A drift acting on the particles will depend on their empirical measure. Let $\mathcal{P}(D)$ be the set of Borel probability measures on D, and let $\vartheta^N : D^N \to \mathcal{P}(D)$ be the map which takes the points $x_1, \dots, x_N \in D$ to their empirical measure,

$$\vartheta^N(x_1, \dots, x_N) = \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k}, \tag{1.1}$$

which is invariant under permutation of the indices. Given $\vec{X}_t^N$, $\vartheta^N(\vec{X}_t^N)$ is the empirical measure of the N particles (at time t), a random probability measure supported on D. We further define a measurable drift

$$b : \mathcal{P}(D) \times D \to \mathbb{R}^d. \tag{1.2}$$

We now define the particle system which is the subject of this paper.

Definition 1.1 (Fleming-Viot Particle System with McKean-Vlasov Dynamics)**.** *Let υ^N be a probability measure on D^N, and let $\{W_t^i\}_{i=1}^N$ be a collection of independent Brownian motions on $\mathbb{R}^d$. Then the particle system $\{X^i\}_{i=1}^N \subset D$ with initial distribution υ^N is defined up to a time τ_{WD} by*

$$\begin{cases} (i) & \vec{X}_0^N \sim \upsilon^N. \\ (ii) & \textit{For } t \in [0, \tau_{\text{WD}}) \textit{ and between jump times (while } \{X_t^i\}_{i=1}^N \subset D\textit{), the particles} \\ & \textit{evolve according to the system} \\ & dX_t^i = b(\vartheta^N(\vec{X}_t^N), X_t^i)dt + dW_t^i, \quad i = 1, \dots, N. \\ (iii) & \textit{Whenever a particle } X^i \textit{ hits the boundary } \partial D\textit{, } X^i \textit{ instantly jumps to the location} \\ & \textit{of another particle chosen independently and uniformly at random.} \end{cases} \tag{1.3}$$

The time τ_{WD} up to which the system is well-defined could be finite if multiple particles hit the boundary simultaneously at some finite time, or if infinitely many jumps occur in finite time, or if (in the case of unbounded domain) some particles "escape to infinity" in finite time. For $k = 1, 2, \dots$, we write τ_k for the k^{th} time at which any particle jumps (upon hitting ∂D), and moreover $\tau_\infty = \lim_{k \to \infty} \tau_k$, after which the particle system is not well-defined. We also define $\tau_{\text{stop}} = \inf\{t > 0 : \ \exists\, j \neq k \text{ such that } X_{t-}^j,\ X_{t-}^k \in \partial D\}$ after which the particle system is not well-defined. We also define $\tau_{\max} = \inf\big\{t > 0 : \sup_{\substack{t' \le t \\ 1 \le i \le N}} |X_{t'}^i| = \infty\big\}$. Thus, the particle system is well-defined only up to the time

$$\tau_{\text{WD}} = \tau_\infty \wedge \tau_{\text{stop}} \wedge \tau_{\max}.$$

Although the Brownian motions $\{W_i\}_{i=1}^N$ are independent, the drift b in the motion of the i^{th} particle X_t^i may depend on X^i and on the empirical measure $\vartheta^N(\vec{X}_t^N)$ of all N particles. The particles also depend on each other through the rule for relocating a particle when it hits the boundary ∂D. Because we do not make strong regularity assumptions on the drift b, we will interpret the SDE in (1.3) in the weak sense, which we make precise in Definition 2.1.

This system is a generalisation of the Fleming-Viot system introduced in the foundational papers of Burdzy, Holyst, Ingerman, and March [8, 9]. Their work involved the particular case of purely Brownian dynamics (i.e. $b \equiv 0$) on a bounded domain D. Even if $b \equiv 0$, it is not clear that the system (1.3) should be well-defined for all $t > 0$. In particular, the following problem remains open.

Problem 1.2 ([7])**.** Consider the $b \equiv 0$ case. Is it true that $\tau_\infty = \infty$, almost surely, for any bounded open connected set $D \subset \mathbb{R}^d$?

In [9, 20, 5], conditions for the global well-posedness ($\mathbb{P}(\tau_{\text{WD}} = +\infty) = 1$) of this system were established for the case $b \equiv 0$ when D is bounded (and the boundary satisfies various additional conditions). As discussed in [5], the proof given in [9, Theorem 1.1] has an irreparable error; however, implicit in [9, Theorem 1.4] is another proof that works when the domain satisfies an interior ball condition. These are complemented by [21, 29, 30], providing well-posedness for general diffusions on possibly unbounded domains (satisfying various additional conditions). We provide a similar result (Theorem 2.6) for bounded b of the form (1.2) and D being a possibly unbounded domain satisfying the uniform interior ball condition, which is not covered by previous results.

In [9, 19], Burdzy, et al. also consider the limits $N \to \infty$ and $t \to \infty$. They established that the empirical measure of the particle system converges to the solution of the heat equation renormalised to have constant mass 1, corresponding to the distribution of Brownian motion killed at the boundary of its domain, conditioned on survival. The notion of convergence was later strengthened by Grigorescu and Kang in [19]. In particular if $\frac{1}{N}\sum_{i=1}^{N}\delta_{X_0^i} \to \nu$ weakly in probability then

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i} \to \frac{u}{|u|_*} = \mathcal{L}_\nu(B_t|\tau_\partial > t) \quad \text{weakly in probability,}$$

where $|u|_*$ is the mass of u, which is a solution of the heat equation with Dirichlet boundary condition

$$\partial_t u = \frac{1}{2}\Delta u, \quad u_{|\partial D} = 0, \quad u_0 = \nu = 0,$$

and where $(B_t)_{0\le t\le\tau_\partial}$ is a Brownian motion with initial condition $B_0 \sim \nu$ stopped at the time $\tau_\partial = \inf\{t > 0 : B_t \in \partial D\}$. Note that, by abuse of notation, we are using functions interchangeably with the measures having their density.

Moreover for fixed N, Burdzy, et al. [9, Theorem 1.4] prove that $\vec{X}_t^N$ has a stationary distribution $\mathbf{M}^N$ on D^N to which the distribution of $\vec{X}_t^N$ converges exponentially fast as $t \to \infty$. Furthermore the corresponding stationary random empirical measure $\chi_{\mathbf{M}}^N \sim \vartheta_{\#}^N\mathbf{M}^N$ converges weakly in probability as $N \to \infty$ to a function $\phi(x)$ which is the principal Dirichlet eigenfunction of the Laplacian on D,

$$\frac{1}{2}\Delta\phi + \lambda\phi = 0, \quad \phi > 0 \quad \text{on} \quad D, \quad \phi = 0 \quad \text{on} \quad \partial D, \tag{1.4}$$

normalised to have integral 1. This normalized eigenfunction corresponds to the quasi-stationary distribution (QSD) for Brownian motion killed on the boundary of its domain

$$\mathcal{L}_\phi(B_t|\tau_\partial > t) = \phi, \quad 0 \le t < \infty.$$

This QSD is the unique quasi-limiting distribution (QLD) for Brownian motion killed at the boundary of its domain. That is, for any initial condition ν,

$$\mathcal{L}_\nu(B_t|\tau_\partial > t) \to \phi \quad \text{as} \quad t \to \infty.$$

Similar results have been established for a variety of other Fleming-Viot particle systems with Markovian dynamics: for instance by Ferrari and Maric [15] in the case of countable state spaces and by Villemonais [30] in the case of general strong Markov processes. These are complemented by generic long-time convergence criteria for the conditional distribution of killed Markov processes [27, Theorem 7], [12, 13]. Campi and Fischer [11] have also considered a similar mean field game with particles killed at the boundary of their domain (corresponding to bankruptcy) and interacting with the renormalised empirical measure (their setup did not feature branching, so the mass decreases over time).

Summary of results

In the present paper, we extend the results in the Markovian case to the more general system (1.3) which includes dynamics of McKean-Vlasov type whereby the particles interact through the dependence of the drift b on the empirical measure. Throughout the paper, we assume that the open set $D \subset \mathbb{R}^d$ satisfies the interior ball condition with radius $r > 0$: for every $x \in D$ there exists a point $y \in D$ such that $x \in B(y, r) \subseteq D$. We also assume that the drift

$$b : \mathcal{P}(D) \times D \to \mathbb{R}^d$$

is measurable with respect to the Borel sigma algebra on $\mathcal{P}(D) \times D$ and uniformly bounded by $B < \infty$, where $\mathcal{P}(D)$ is endowed with the topology of weak convergence of measures.

We begin by establishing in Theorem 2.6 global well-posedness of the system (1.3), meaning that $\mathbb{P}(\tau_{\mathrm{WD}} = \infty) = 1$. At the same time, we establish some estimates on the N-particle system which shall be used throughout this paper.

We then seek to characterise the behaviour as $t \to \infty$ for fixed $N < \infty$. Here we must impose an additional assumption: that the domain D is bounded and path-connected. Under these conditions, we establish in Theorem 2.7 that the system (1.3) is ergodic, having a unique stationary distribution ψ^N on D^N. The reason that boundedness is assumed is that on unbounded domains we have the possibility of mass escaping to infinity over infinite time horizons. We conjecture that a Lyapunov criterion should exist allowing our large time results to be extended to the setting of unbounded domains. In the Markovian case, such Lyapunov criteria have been established in [14, 13].

We then consider the behaviour of the system (1.3) as $N \to \infty$. We no longer need to impose the assumption that D is bounded and path-connected. We will establish a hydrodynamic limit theorem – Theorem 2.9 – which will be the main result of this paper. As we will show, the limit behavior of $\vec{X}^N$ as $N \to \infty$ can be described in terms of the following conditional McKean-Vlasov system

$$\begin{cases} (i) & (X_t : 0 \leq t \leq \tau_\partial) \text{ is a continuous process defined up to the} \\ & \text{stopping time } \tau_\partial = \inf\{t > 0 : X_t \in \partial D\}, \\ (ii) & X_t \in D \text{ for } t < \tau_\partial,\ X_{\tau_\partial} = \lim\limits_{t \nearrow \tau_\partial^-} X_t \in \partial D, \\ (iii) & \text{Initial condition: } X_0 \sim \nu \in \mathcal{P}(D), \\ (iv) & X_t \text{ satisfies } dX_t = b(\mathcal{L}(X_t | \tau_\partial > t), X_t)dt + dW_t, \quad 0 \leq t \leq \tau_\partial, \\ & \text{where } W \text{ is a Brownian motion,} \end{cases} \tag{1.5}$$

which gives rise to the flow of conditional laws

$$(\mathcal{L}(X_t | \tau_\partial > t) : 0 \leq t < \infty). \tag{1.6}$$

In the SDE, the drift is a function of $\mathcal{L}(X_t|\tau_\partial > t) \in \mathcal{P}(D)$, the law of X_t conditioned on $\{\tau_\partial > t\}$, where τ_∂ is the first time X_t hits the boundary ∂D. For convenience, we also define

$$m_t = \mathcal{L}(X_t | \tau_\partial > t) \in \mathcal{P}(D), \quad t \geq 0 \tag{1.7}$$

and

$$J_t = -\ln \mathbb{P}(\tau_\partial > t), \quad t \geq 0. \tag{1.8}$$

These are only well-defined for as long as $\mathbb{P}(\tau_\partial > t) > 0$. We therefore define the following.

Definition 1.3 (Global Weak Solution to (1.5))**.** *If a weak solution to* (1.5) *satisfies* $\mathbb{P}(\tau_\partial > t) > 0$ *for all* $t \in [0, \infty)$ *we say it is a global weak solution.*

Remark 1.4. Strictly speaking we should define X_t as occupying some cemetary state for all $t \geq \tau_\partial$. This state could be some point seperate from $\mathbb{R}^d$ or it could be the point on the boundary that X_t hits at time τ_∂. Nevertheless it shall be more convenient for our purposes for killed processes to be defined only up the killing time τ_∂. Thus, the notation $\mathcal{L}(X_t|\tau_\partial > t)$ is equivalent to $\mathcal{L}(X_{t\wedge\tau_\partial}|\tau_\partial > t)$, which is an element of $\mathcal{P}(D)$. Abusing notation, we write $\mathcal{L}(X_t)$ for the sub-probability measure $\mathcal{L}(X_t|\tau_\partial > t)\mathbb{P}(\tau_\partial > t)$ – so in particular $\mathcal{L}(X_t)$ assigns mass only to D and not to any “cemetary state”.

Similar processes have been studied over finite time horizons for instance by Caines, Ho and Song [10]; Hambly, Ledger and Søjmark [22]; and in the context of Mean Field Games by Campi and Fischer [11].

We establish in Proposition 2.8 that all weak solutions are global weak solutions along with the existence, uniqueness in law and time continuity of such solutions. This allows us to uniquely define the following semigroup,

$$\begin{gathered} G_t(\nu) := \mathcal{L}_\nu(X_t|\tau_\partial > t) \text{ where } (X, \tau_\partial, W) \text{ is a} \\ \text{global weak solution to (1.5) with initial condition } X_0 \sim \nu, \end{gathered} \tag{1.9}$$

which we later show in Proposition 2.11 is jointly continuous in $[0, \infty) \times \mathcal{P}(D)$. The density $u = m_t e^{-J_t} = \mathcal{L}(X_t)$ corresponds to a weak solution of the following nonlinear transport equation

$$\partial_t u + \nabla \cdot \left(b\left(\frac{u}{|u|_*}, x\right) u \right) = \frac{1}{2}\Delta u, \qquad u\big|_{\partial D} = 0,$$

where $|u|_*$ is the mass of u on D.

Returning to our Fleming-Viot system of N particles, we define the empirical measure of the N-particle system

$$m_t^N = \vartheta^N(\vec{X}_t^N), \tag{1.10}$$

which has initial distribution

$$m_0^N \sim \xi^N := \vartheta^N_\# \upsilon^N. \tag{1.11}$$

Thus m_0^N is a random probability measure on D; ξ^N is the law of this random measure and is the pushforward of υ^N under the map ϑ^N. We further define

$$J_t^N = \frac{1}{N}\sup\{k \in \mathbb{N} \mid \tau_k \leq t\}, \tag{1.12}$$

which is the number of jumps of the N-particle process up to time t, normalized by $1/N$. In Theorem 2.9 we establish $(m_t^N, J_t^N)_{0\leq t<\infty}$ converges uniformly on compacts in probability to $(m_t, J_t)_{0\leq t<\infty}$.

Having established ergodicity for fixed N and hydrodynamic convergence to the flow of conditional laws (1.6) for the system (1.5), it is natural to ask whether we might obtain convergence in large time for (1.6). We recall the semigroup (1.9) and ask when the limit

$$\lim_{t\to\infty} G_t(\nu) = \lim_{t\to\infty} \mathcal{L}_\nu(X_t|\tau_\partial > t)$$

exists. We now extend the definitions given in the Markovian case in [27, Section 2].

Definition 1.5 (McKean-Vlasov QLDs and QSDs)**.** *For a domain $D \subset \mathbb{R}^d$ and drift b satisfying the assumptions of Proposition 2.8, let $G_t : \mathcal{P}(D) \to \mathcal{P}(D)$ be the unique associated semigroup as in* (1.9)*. Let π be a Borel probability measure on D. We say that π is a quasi-limiting distribution (QLD) for (b, D) if there is a probability measure ν on D such that*

$$G_t(\nu) \to \pi \quad \textit{in} \quad \mathcal{P}(D) \quad \textit{as} \quad t \to \infty, \tag{1.13}$$

in which case we say that π is the Yaglom limit for initial condition ν. By defining a McKean-Vlasov QLD in terms of weak convergence of probability measures, we are using a weaker notion of convergence here as compared to the Markovian case [27, Definition 1] using setwise convergence, as this definition is more natural for our purposes.

We define π to be a quasi-stationary distribution (QSD) for (b, D) if

$$G_t(\pi) = \pi, \quad 0 \leq t < \infty. \tag{1.14}$$

We then define the set of QSDs to be

$$\Pi = \{\pi \in \mathcal{P}(D) : \pi \text{ is a QSD for } (b, D)\}. \tag{1.15}$$

We will ask in Problem 2.12 when we have that

$$\text{the Yaglom limit } \lim_{t\to\infty} G_t(\nu) \text{ exists for every } \nu \in \mathcal{P}(D) \tag{1.16}$$

(but we do not require the same limit for different $\nu \in \mathcal{P}(D)$). This is the most significant issue left unresolved in this paper; in our later theorems we assume we are working with a case where (1.16) does hold. We would not have (1.16) if, for example, $G_t(\nu)$ converges to a limit cycle as $t \to \infty$ for some $\nu \in \mathcal{P}(D)$.

Whereas we are not able to resolve Problem 2.12, we are able to extend [27, Proposition 1] from the Markovian case to the McKean-Vlasov case: establishing in Proposition 2.13 that π is a QSD if and only it is a QLD, that QSDs can be characterised as the solutions of a nonlinear eigenproblem, that Π is a non-empty compact set (in particular, at least one QSD exists) and that the killing time τ_∂ at quasi-equilibrium is exponentially distributed with rate given by the corresponding eigenvalue. This and all of our later results require the domain D be bounded. Whilst Proposition 2.13 implies that Π, the set of QSDs, is a non-empty compact set, we shall demonstrate in Example 2.15 that it need not be a singleton; there may be more than one QSD.

In Theorem 2.7 we establish that $\vec{X}_t^N$ is ergodic with stationary distribution we call ψ^N. We may therefore associate to this an empirical measure-valued stationary distribution

$$\Psi^N := \vartheta^N_{\#}\psi^N, \tag{1.17}$$

which is the stationary distribution for the empirical measure-valued process m_t^N. We associate to each Ψ^N a random variable distributed like Ψ^N,

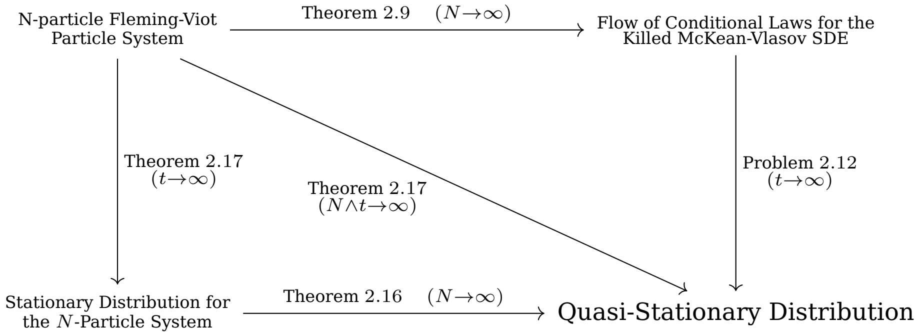
$$\pi^N \sim \Psi^N. \tag{1.18}$$

In Theorem 2.16 we establish that if we do have (1.16) then the π^N converge in probability to Π. In other words, if we sample a random empirical measure from Ψ^N for large N, then with large probability our random empirical measure is close to some QSD $\pi \in \Pi$. This is an extension of [9, Theorem 1.4 (ii)] which dealt with the $b = 0$ case.

Whilst we show π^N is close to the set Π with large probability, we do not show that it is close to all of Π with large probability. When the QSDs are non-unique – when Π contains more than one element – one may ask which QSDs are "selected" by the Fleming-Viot particle system? We conjecture that this should correspond to the stability of the semigroup G_t, so that in particular the stability of the QSDs could be determined by sampling π^N sufficiently many times and observing which QSDs are "selected".

If we drop the assumption (1.16), we shall see that the distribution of π^N converges to the set of invariant measures for the semigroup G_t. Thus at least one of the invariant measures can be obtained from the Fleming-Viot particle system. More broadly, due to the McKean-Vlasov interaction, the semigroup G_t could have more interesting dynamical systems properties than in the Markovian case. In the case without killing, this is a well-studied problem, a thorough treatment of which can be found in [16]. We therefore ask what about the dynamical system G_t can be deduced from the corresponding Fleming-Viot particle system?

The following diagram summarises the relationship between our results:



2 Statement of results

We begin with a more precise description of the particle system (1.3) which is the subject of this paper.

Definition 2.1 (Weak Solution to (1.3)). *Let $\vec{W}_t = (W_t^1,\dots,W_t^N)$ be a collection of independent Brownian motions on $\mathbb{R}^d$ with respect to a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let υ^N be a probability measure on D^N. We say that $(\vec{X}_t,\vec{W}_t,\mathcal{F}_t)$ is a weak solution to* (1.3) *with initial condition υ^N if $\vec{X}_0\sim\upsilon^N$, and there is an increasing sequence of $\mathcal{F}_t$-stopping times $\{\tau_k\}_{k=0}^{\infty}$ with $\tau_0=0$ such that the following hold.*

1. *$\vec{X}_t$ is a càdlàg process. For each k, $\vec{X}_t$ is continuous on $[\tau_k,\tau_{k+1})$ and satisfies*

$$X_t^i = X_{\tau_k}^i + \int_{\tau_k}^t b(\vartheta^N(\vec{X}_s),X_s^i)ds + W_t^i - W_{\tau_k}^i, \qquad i=1,\dots,N;\;\; t\in[\tau_k,\tau_{k+1}). \tag{2.1}$$

   For all $k\geq 1$, and with probability one, there is a unique particle index $\ell(k)\in\{1,\dots,N\}$ such that

$$\tau_k = \min_{i=1,\dots,N}\inf\{t>\tau_{k-1}\mid \lim_{s\to t^-}X_s^i\in D^c\} = \inf\{t>\tau_{k-1}\mid \lim_{s\to t^-}X_s^{\ell(k)}\in D^c\}. \tag{2.2}$$

2. *For all $k\geq 1$,*

$$\lim_{t\to\tau_k^-}X_t^j = X_{\tau_k}^j\in D, \quad \forall\; j\in\{1,\dots,N\}\setminus\{\ell(k)\} \tag{2.3}$$

   and

$$\mathbb{P}(X_{\tau_k}^{\ell(k)} = X_{\tau_k}^j \mid \mathcal{F}_{\tau_k^-}) = \frac{1}{N-1}, \quad \forall\; j\in\{1,\dots,N\}\setminus\{\ell(k)\} \tag{2.4}$$

   hold with probability one.

This is no longer well-defined once two particles hit the boundary at the same time,

$$\tau_{stop} = \inf\{t>0:\ \exists\, j\neq k \text{ such that } X_t^j,\ X_t^k\in\partial D\}.$$

Moreover if there are an infinite number of stopping times τ_k in finite time, this is no longer well-defined after the time

$$\tau_\infty = \lim_{k\to\infty}\tau_k. \tag{2.5}$$

Furthermore if the domain D is unbounded, the particles may escape to infinity in finite time, after which time the particle system is not well-defined. We write

$$\tau_{\max}=\inf\big\{t>0:\ \sup_{\substack{t'\leq t\\ 1\leq i\leq N}}|X^i_{t'}|=\infty\big\}. \tag{2.6}$$

Therefore $(\vec{X}_t,\vec{W}_t)_{0\leq t<\tau_{WD}}$ is defined only up to the time

$$\tau_{WD}:=\tau_{stop}\wedge\tau_{\infty}\wedge\tau_{\max}. \tag{2.7}$$

The index $\ell(k)$ in (2.2) is the index of the unique particle that hits the boundary ∂D at time τ_k; the statement (2.3) means that the paths of the other particles are continuous at time τ_k; the statement (2.2) means that at time τ_k, the particle with index $\ell(k)$ jumps to the location of another particle chosen uniformly at random from the other $N-1$ particles.

It will be convenient to define a the family of random variables

$$U^i_k\in\{1,\dots,N\}\setminus\{i\} \tag{2.8}$$

to be the index of the particle that X^i jumps onto at its k^{th} jump time. We will refer to U^i_k as the target index or target particle of particle for X^i at its k^{th} jump time. Thus, $\{\{U^i_k\}_{k=1}^{\infty}\}_{i=1}^N$ are a family of independent random variables such that for each $i\in\{1,\dots,N\}$ the variables $\{U^i_k\}_{k=1}^{\infty}$ are all uniformly distributed on the set $\{1,\dots,N\}\setminus\{i\}$.

Before stating our results, we define the spaces of measures we employ throughout this paper.

Definition 2.2 (Spaces of Measures). *Given a metric space (χ,d), we equip (χ,d) with the Borel sigma algebra $\mathcal{B}(\chi)$ and define $\mathcal{P}(\chi)$ to be the space of probability measures on χ equipped with the topology of convergence of probability measures. We write $\mathcal{M}(\chi)$ for the space of Borel measures on (χ,d) equipped with the topology of weak convergence of measures. We further define $\mathcal{M}_+(\chi)=\mathcal{M}(\chi)\setminus\{0\}$ to be those measures with positive total mass (equipped with the same topology).*

For separable metric spaces (χ,d) we equip $\mathcal{P}(\chi)$ with the Wasserstein-1 metric on $\mathcal{P}$ using the bounded metric $d^1(x,y)=1\wedge d(x,y)$ on the underlying space χ, which metrises the topology of weak convergence of measures [17]. We denote this metric W (unless there is a possible confusion as to the underlying metric space χ, in which case we write $d_{\mathcal{P}_W(\chi)}$) and write $\mathcal{P}_W(\chi)=(\mathcal{P}(\chi),W)$.

We shall establish hydrodynamic convergence in the sense of uniform convergence in $\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0}$ on compact subsets of time in probability. We metrize this as follows. We firstly define the uniform metric over finite time horizons,

$$\begin{aligned}d^{\infty}_{[0,T]}:\mathcal{D}([0,T];\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0})\times\mathcal{D}([0,T];\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0})\rightarrow\mathbb{R}_{\geq 0}\\ d^{\infty}_{[0,T]}((\mu^1,y^1),(\mu^2,y^2))=\sup_{0\leq t\leq T}(W(\mu^1_t,\mu^2_t)+|y^1_t-y^2_t|).\end{aligned} \tag{2.9}$$

We then define the metric

$$\begin{aligned}d^{\infty}:\mathcal{D}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0})\times\mathcal{D}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0})\rightarrow\mathbb{R}_{\geq 0}\\ d^{\infty}(f,g)=\sum_{T=1}^{\infty}2^{-T}(d^{\infty}_{[0,T]}((f_t)_{0\leq t\leq T},(g_t)_{0\leq t\leq T})\wedge 1).\end{aligned} \tag{2.10}$$

This metrises uniform convergence on compacts, which means that

$$d^{\infty}((\mu^n_t,y^n_t)_{0\leq t<\infty},(\mu,y)_{0\leq t<\infty})\rightarrow 0\quad\textit{as}\quad n\rightarrow\infty$$

if and only if

$$\sup_{t\leq T} W(\mu^n_t,\mu_t)\to 0 \quad \text{and} \quad \sup_{t\leq T}|y^n_t-y_t|\to 0 \quad \text{as} \quad n\to\infty \quad \text{for every } T<\infty.$$

Thus the random $\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0}$-valued càdlàg processes (μ^N,y^N) converge to (μ,y) uniformly in $\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0}$ on compacts in probability if and only if $d^\infty((\mu^N,y^N),(\mu,y))\to 0$ in probability.

We shall also make use of the total variation norm, which we label $||.||_{TV}$.

Finally, we note that since the metric d^1 on χ is bounded by 1, the induced Wasserstein distance W is then bounded by the total variation distance,

$$W(\mu,\nu)\leq \frac{1}{2}||\mu-\nu||_{TV} \quad \text{for all} \quad \mu,\nu\in\mathcal{P}(\chi).$$

We will always assume D is an open subdomain of $\mathbb{R}^d$ whose boundary ∂D satisfies the following interior ball condition.

Condition 2.3. *The boundary ∂D satisfies the uniform interior ball condition: there is a fixed radius $r>0$ such that for every $x\in D$ there exists a point $y\in D$ such that $x\in B(y,r)\subseteq D$.*

Regarding the drift b, we will always assume that $(\mu,x)\mapsto b(\mu,x)$ is measurable with respect to the Borel sigma algebra on $\mathcal{P}(D)\times D$ and uniformly bounded with $|b|\leq B<\infty$. For some results, we will also assume the following condition.

Condition 2.4. *The boundary ∂D is C^∞. Moreover, in addition to being measurable and uniformly bounded, the drift $b:\mathcal{P}_W(D)\times D\to\mathbb{R}^d$ is jointly continuous, and is Lipschitz in the first variable: there is $C>0$ such that*

$$|b(\mu_1,x)-b(\mu_2,x)|\leq CW(\mu_1,\mu_2), \quad \forall x\in D, \quad \mu_1,\mu_2\in\mathcal{P}_W(D). \tag{2.11}$$

Remark 2.5. The Lipschitz assumption (2.11) may be replaced with the strictly weaker assumption that b is uniformly Lipschitz with respect to the total variation metric. This does not require changes to the proof, however for simplicity we assume b is uniformly Lipschitz with respect to the W metric.

Moreover the Lipschitz condition (2.11) is used only to establish uniqueness in law of global weak solutions to (1.5) for given initial conditions; for all our results this Lipschitz condition may be replaced with any other condition providing for uniqueness in law of global weak solutions to (1.5).

Furthermore in Proposition 2.8 and theorems 2.9 and 2.10 we could without changes to the proofs assume b to be time-inhomogeneous; so that $b:[0,\infty)\times\mathcal{P}_W(D)\times D\to\mathbb{R}^d$ is measurable, and for Lebesgue-almost every t and uniform $C<\infty$, $(m,x)\mapsto b(t,m,x)$ is jointly continuous and satisfies (2.11).

We firstly establish the particle system is defined over an infinite time horizon.

Theorem 2.6 (Global Well-Posedness of the N-Particle System (1.3))**.** *Under Condition 2.3, there exists a weak solution of* (1.3) *for which $\mathbb{P}(\tau_{WD}=\infty)=1$; any weak solution of* (1.3) *satisfies $\mathbb{P}(\tau_{WD}=\infty)=1$ and weak solutions of* (1.3) *are unique in law.*

We now address the large time properties of the system for fixed finite N. We must impose the additional assumption that the domain D is bounded and path-connected. The boundedness assumption is needed as we do not currently have a good way of preventing the mass "escaping to infinity" over an infinite time horizon when the domain is unbounded. We establish ergodicity of the system for fixed N.

Theorem 2.7 (Ergodicity of the N-Particle System (1.3))**.** *In addition to Condition 2.3, assume that D is path-connected and bounded. Then we have that for every N fixed,* *there exists a unique stationary distribution $\psi^N \in \mathcal{P}(D^N)$ of the N-particle system (Definition 1.1). Moreover there exist constants $C_N, \lambda_N > 0$ such that for every initial distribution $X_0 \sim \upsilon^N$ we have $||\mathcal{L}(\vec{X}^N_t) - \psi^N||_{TV} \leq C_N e^{-\lambda_N t}$.*

We now turn to the question of extracting a hydrodynamic limit. We no longer need to impose the assumption that the domain D is bounded or path-connected. Our hydrodynamic limit will be given by the flow of conditional laws (1.6) corresponding to solutions of (1.5), and so before stating our hydrodynamic limit theorem we firstly give the properties of (1.5). We recall that where a weak solution (X, τ_∂, W) to (1.5) satisfies $\mathbb{P}(\tau_\partial > t) > 0$ for all $t \in [0, \infty)$, we say it is a global weak solution.

Proposition 2.8 (Properties of the McKean-Vlasov Process (1.5))**.** *Assume Condition 2.4. For every $\nu \in \mathcal{P}(D)$ there exists a unique in law weak solution (X, τ_∂, W) to* (1.5) *with initial condition $X_0 \sim \nu$. Moreover this weak solution to* (1.5) *is a global weak solution and satisfies:*

(i) $\mathbb{P}(\tau_\partial > t)$ *is continuous and positive on* $[0, \infty)$*;*

(ii) $\mathcal{L}(X_t|\tau_\partial > t) \in \mathcal{C}([0, \infty); \mathcal{P}_W(D)) \cap \mathcal{C}((0, \infty); L^1(D))$.

We let $(\vec{X}^N_t : 0 \leq t < \infty) = ((X^{N,1}_t, \dots, X^{N,N}_t) : 0 \leq t < \infty)$ be a sequence of weak solutions to (1.3) with initial conditions $\vec{X}^N_0 \sim \upsilon^N$. We define m^N_t, ξ^N and J^N_t as in (1.10), (1.11) and (1.12). We have the following hydrodynamic limit theorem.

Theorem 2.9 (Hydrodynamic Limit Theorem)**.** *Assume Condition 2.4. Let $\nu \in \mathcal{P}(D)$ and assume that $W(m^N_0, \nu) \to 0$ in probability as $N \to \infty$. Let (X, τ_∂, W) be a (unique in law) global weak solution to* (1.5) *with initial distribution $X_0 \sim \nu$, and define as in* (1.7) *and* (1.8)

$$m_t = \mathcal{L}(X_t|\tau_\partial > t) \quad \textit{and} \quad J_t = -\ln \mathbb{P}(\tau_\partial > t).$$

Then, as $N \to \infty$, we have uniform convergence on compacts in probability,

$$(m^N_t, J^N_t)_{0 \leq t < \infty} \to (m_t, J_t)_{0 \leq t < \infty} \quad \textit{in } d^\infty \textit{ in probability.}$$

The existence part of Proposition 2.8 and theorems 2.9, 2.16 and 2.17 are essentially all corollaries of the following generalised hydrodynamic limit theorem – Theorem 2.10. Relying on the machinery of sections 4, 6 and 7, this theorem will be proven in Section 8.

This hydrodynamic limit theorem is valid when the initial conditions are only known to constitute a tight family of random measures (as opposed to convergent weakly in probability to a deterministic initial profile as in Theorem 2.9). We define

$$\begin{aligned}\Xi = \{(\mathcal{L}(X_t|\tau_\partial > t), -\ln \mathbb{P}(\tau_\partial > t))_{0 \leq t < \infty} \in \mathcal{C}([0, \infty); \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}) :\\ (X, \tau_\partial, W) \text{ is a global weak solution of (1.5)}\\ \text{for some initial condition } X_0,\ \mathcal{L}(X_0) \in \mathcal{P}_W(D)\}.\end{aligned} \tag{2.12}$$

For $T < \infty$ we define $d^{\mathcal{D}}_{[0,T]}$ to be the Skorokhod metric on $\mathcal{D}([0, T]; \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0})$. We then define the metric

$$\begin{aligned}d^{\mathcal{D}} : \mathcal{D}([0, \infty); \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}) \times \mathcal{D}([0, \infty); \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0}\\ d^{\mathcal{D}}(f, g) = \sum_{T=1}^{\infty} 2^{-T}(d^{\mathcal{D}}_{[0,T]}((f_t)_{0 \leq t \leq T}, (g_t)_{0 \leq t \leq T}) \wedge 1).\end{aligned} \tag{2.13}$$

Note that convergence with respect to $d^{\mathcal{D}}$ to a continuous function implies convergence with respect to d^∞ to the same continuous function.

Theorem 2.10. *Assume Condition 2.4 and that $\{\xi^N\}$ is a tight family of measures in $\mathcal{P}(\mathcal{P}_W(D))$. Then the laws of the processes $\{(m_t^N, J_t^N)_{0\le t\le T}\}$ are a tight family of measures on $(\mathcal{D}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\ge 0}), d^{\mathcal{D}})$. Moreover if along some subsequence $(m_t^N, J_t^N)_{0\le t<\infty} \xrightarrow{d} (m_t, J_t)_{0\le t<\infty}$, then*

$$(m_t, J_t)_{0\le t<\infty} \in \Xi \cap \mathcal{C}((0,\infty); L^1(D)\times\mathbb{R}_{\ge 0}) \subseteq \mathcal{C}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\ge 0})$$

holds almost surely.

Proposition 2.8 guarantees for us that the semigroup G_t on $\mathcal{P}_W(D)$ given in (1.9) is well-defined. We will establish in Section 9 the following properties of the semigroup G_t.

Proposition 2.11 (Properties of the Semigroup G_t)**.** *Assume Condition 2.4. Then the semigroup G_t is jointly continuous in $[0,\infty)\times\mathcal{P}_W(D)$,*

$$[0,\infty)\times\mathcal{P}_W(D) \ni (t,\nu) \mapsto G_t(\nu)\in\mathcal{P}_W(D) \quad \textit{is continuous.} \tag{2.14}$$

Furthermore if the domain D is bounded, then for all $t_0>0$, G_{t_0} has pre-compact image Image$(G_{t_0}) \subset\subset \mathcal{P}_W(D)$.

Having established ergodicity for fixed N and hydrodynamic convergence to the flow of conditional laws (1.6) for the system (1.5), we ask when the limit $\lim_{t\to\infty} G_t(\nu)$ exists. We henceforth assume the domain is bounded. The following represents the most significant issue left to resolve from this paper.

Problem 2.12 (Convergence to Quasi-Equilibrium)**.** Under what conditions does

$$\text{the Yaglom limit } \lim_{t\to\infty} G_t(\nu) \text{ exist with respect to } W \text{ for every } \nu\in\mathcal{P}_W(D) \tag{2.15}$$

(with the limit possibly depending on $\nu\in\mathcal{P}_W(D)$)? Can we find conditions under which there exists $\pi\in\mathcal{P}_W(D)$ such that

$$G_t(\nu)\to\pi \text{ uniformly in } W \text{ as } t\to\infty? \tag{2.16}$$

Although we are unable to resolve Problem 2.12, we shall establish the following.

Proposition 2.13 (Existence and Properties of QSDs)**.** *In addition to conditions 2.3 and 2.4, we assume that D is bounded. Then we have the following.*

1. *The following are equivalent:*
   - *(a) π is a QSD for* (1.5)*;*
   - *(b) π is a QLD for* (1.5)*;*
   - *(c) $\pi\in L^1(D)$ is a probability density such that*

   $$\langle \pi(\cdot), \lambda\varphi(\cdot) + b(\pi,\cdot)\cdot\nabla\varphi(\cdot) + \frac{1}{2}\Delta\varphi(\cdot)\rangle = 0, \qquad \forall\, \varphi\in C_0^\infty(\bar{D}) \tag{2.17}$$

   for some $\lambda>0$, whereby we define the test functions

   $$C_0^\infty(\bar{D}) = \{\varphi\in C_c^\infty(\bar{D}) : \varphi = 0 \textit{ on } \partial D\}. \tag{2.18}$$

2. *For any weak solution (X,τ_∂,W) to* (1.5) *with quasi-stationary initial condition π we have $\tau_\partial\sim$ Exp(λ) where λ is the constant such that (π,λ) is a solution to* (2.17)*.*
3. *Π is a non-empty compact subset of $\mathcal{P}_W(D)$.*

Remark 2.14. The equation (2.17) is the weak formulation of the following nonlinear PDE,

$$\lambda\pi - \nabla\cdot(b(\pi,\cdot)\pi) + \frac{1}{2}\Delta\pi = 0 \quad \text{on} \quad D, \quad \pi = 0 \quad \text{on} \quad \partial D. \tag{2.19}$$

Whereas Π must be a non-empty compact set (in $\mathcal{P}_W(D)$), we demonstrate in the following example that it need not be a singleton.

Example 2.15. We assume $D=(-1,1)$ and the drift is given by the first moment

$$dX_t = \gamma\mathbb{E}[X_t|\tau_\partial > t]dt + dW_t.$$

This satisfies the conditions of Proposition 2.13, so we may check using Part 1c of Proposition 2.13 that the QSDs are given by

$$\pi_b = Ae^{\gamma b x}\cos\left(\frac{\pi}{2}x\right) \quad \text{where } b \text{ is a solution of} \quad b = \tanh(\gamma b) - \frac{8\gamma b}{4\gamma^2 b^2 + \pi^2}.$$

For all values of γ, $\pi_0 = A\cos(\frac{\pi}{2}x)$ is a QSD, which for small γ is the only QSD. Moreover by calculating the derivative of

$$F(b) = \tanh(\gamma b) - \frac{8\gamma b}{4\gamma^2 b^2 + \pi^2}$$

at 0 we see that $b \mapsto F(b)$ exhibits a pitchfork bifurcation at $\gamma = \frac{\pi^2}{\pi^2 - 8}$, so that for $\gamma > \frac{\pi^2}{\pi^2-8}$ there are multiple QSDs π_0 and $\pi_\pm$.

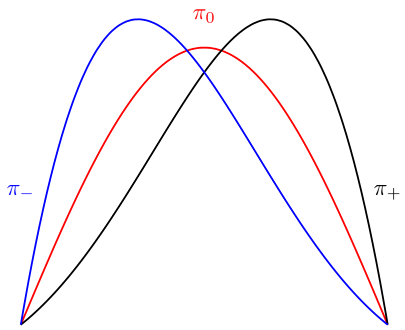


Figure 1: QSDs for Example 2.15

We now show that, if we don't have (2.15), then the stationary distributions for our N-particle system (given by Theorem 2.7) converge to the set of QSDs Π.

Theorem 2.16 (Convergence of the N-Particle Stationary Distributions to QSDs)**.** *In addition to conditions 2.3 and 2.4, we assume that D is bounded. Moreover we assume that*

$$\text{the Yaglom limit } \lim_{t\to\infty} G_t(\nu) \text{ exists with respect to } W \text{ for every } \nu \in \mathcal{P}_W(D). \tag{2.15}$$

We take the stationary empirical measures $\Psi^N = \vartheta^N_{\#}\psi^N$ and take a sequence of $\mathcal{P}_W(D)$-valued random variables π^N with distribution $\pi^N \sim \Psi^N$ as in (1.18)*. Then we have*

$$W(\pi^N, \Pi) \to 0 \quad \text{in probability as} \quad N \to \infty. \tag{2.20}$$

Since we do not necessarily have (2.15), it is worthwhile asking what happens when we don't have (2.15). In general we shall see that we obtain the invariant measures for G_t,

$$M_G = \{\mathbb{P} \in \mathcal{P}(\mathcal{P}_W(D)) : (G_t)_{\#}\mathbb{P} = \mathbb{P} \quad \text{for all} \quad t \geq 0\}. \tag{2.21}$$

Then by propositions 2.11 and 2.13, M_G is a non-empty compact subset of $\mathcal{P}(\mathcal{P}_W(D))$. Furthermore it is clear from the proof of Theorem 2.16 that under the same assumptions as Theorem 2.16, except for (1.16), we have

$$W(\mathcal{L}(\pi^N), M_G) \to 0 \quad \text{as} \quad N \to \infty. \tag{2.22}$$

Therefore the Fleming-Viot particle system allows us to obtain at least one of the invariant measures of G_t.

Finally, under an additional assumption on the semigroup G_t, we establish convergence as the number of particles and the time horizon converge to infinity together. We prove the following theorem.

Theorem 2.17. *In addition to conditions 2.3 and 2.4, we assume that D is bounded. Moreover we assume that there exists a QSD π such that*

$$W(G_t(\nu), \pi) \to 0 \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad \nu \in \mathcal{P}_W(D). \tag{2.16}$$

Then by Proposition 2.13 there exists $\lambda > 0$ such that (π, λ) is a solution of (2.17). *We take a sequence of weak solutions $(\vec{X}^N_t : 0 \leq t < \infty) = ((X^{N,1}_t, \dots, X^{N,N}_t) : 0 \leq t < \infty)$ to* (1.3) *with arbitrary initial conditions $\vec{X}^N_0 \sim \upsilon^N$. We define m^N_t and J^N_t as in* (1.10) *and* (1.12)*,*

$$m^N_t = \vartheta^N(\vec{X}^N_t), \quad J^N_t = \frac{1}{N} \sup\{k \in \mathbb{N} \mid \tau_k \leq t\}.$$

Then we have

$$(m^N_{t_0+t}, J^N_{t_0+t} - J^N_{t_0})_{0 \leq t < \infty} \to (\pi, \lambda t)_{0 \leq t < \infty} \text{ in } d^\infty \text{ in probability as } t_0 \wedge N \to \infty. \tag{2.23}$$

3 Proof strategy for sections 4 and 6-8

The results of sections 4, 6 and 7 shall be used in Section 8 to establish our hydrodynamic limit theorem, as we shall explain here.

The proof of our hydrodynamic limit theorem shall require defining the following Fleming-Viot particle systems with generalised dynamics – therefore we establish the results of Section 4 and 6 for such generalised systems. Whenever we consider Fleming-Viot particle systems with generalised dynamics, we shall assume the domain D is an open subdomain of $\mathbb{R}^d$ satisfying Condition 2.3 and the drifts satisfy the following condition.

Condition 3.1. *The drifts b^i_t are $(\mathcal{F}_t)_{t \geq 0}$-adapted and uniformly bounded $|b^i_t| \leq B$ ($i = 1, \dots, N$).*

Otherwise, the Fleming-Viot particle system with generalised dynamics has the same definition as the particle system with McKean-Vlasov dynamics.

We establish in Section 4 estimates on the N-particle system which shall be used throughout this paper along with global well-posedness of the N-particle system with generalised dynamics (and hence for the system with McKean-Vlasov dynamics – Theorem 2.6). These estimates, in particular, will allow us control the mass close to the boundary, uniformly in N. This will be an essential ingredient in our proof of hydrodynamic convergence in Section 8.

The estimates of Section 4 hinge on constructing – in a completely different manner – a family of Bessel processes similar to those constructed by Burdzy, Holyst and March [9, Proof of Theorem 1.4] to deal with the $b = 0$ case. These are N i.i.d. Bessel processes coupled to the N-particle system in such a way so as to provide controls on the mass close to the boundary. While the Bessel processes we obtain are very similar to the Bessel processes constructed in [9], the method of construction is more similar to the construction used to establish well-posedness in [29] by constructing a different family of processes. In [9] their construction begins by taking the Bessel processes and then using a classical skew-product decomposition [23] to construct the particle system with $b \equiv 0$. This has no hope of working however in the $b \neq 0$ case as such a skew-product decomposition is not available. Similarly to [29], we instead start with the particle system and from there construct the Bessel processes. We use a Doob-Meyer decomposition piecewise between a family of stopping times to construct an associated Brownian motion for each particle, and then use these Brownian motions to drive our Bessel processes.

In [9, 20, 5], conditions for the global well-posedness ($\mathbb{P}(\tau_{\text{WD}} = +\infty) = 1$) of this system were established for the case $b \equiv 0$ when D is bounded (and satisfies various additional conditions). These are complemented by [21, 30, 29], providing well-posedness for general diffusions on possibly unbounded domains (satisfying various additional conditions). The closest of these to our setup is [29]. For such domains, one could obtain the global well-posedness for the system with generalised dynamics from the $b \equiv 0$ case using Girsanov's theorem – they can be related via a Girsanov transform, which preserves $\{\tau_{\text{WD}} < \infty\}$ as a null event. None of these, however, apply to general unbounded domains satisfying the uniform interior ball condition. Nevertheless, the Bessel processes we construct allow us to establish well-posedness for the system with generalised dynamics and possibly unbounded domains satisfying only the uniform interior ball condition.

We shall prove Lemmas 6.1 and 6.2 in Section 6. Lemma 6.1 will be crucial in our proof of hydrodynamic convergence as it will make available to us a uniqueness theorem for the linear Fokker-Planck equation [28, Theorem 1.1]. It guarantees that subsequential limits of the empirical measure valued process almost surely has a density. The proof of Lemma 6.1 hinges on an analysis of the dynamical historical processes introduced by Bieniek and Burdzy [4]. The machinery we construct to prove Lemma 6.1 then enables us to prove Lemma 6.2, which constrains the number of particles far away from the boundary over fixed time horizons.

We then prove Proposition 7.2 in Section 7, establishing that we may couple the N-particle system on an infinite domain with an appropriately constructed Fleming-Viot N-particle system with generalised dynamics on a large but finite subdomain. Moreover we obtain uniform controls on the difference between the two N-particle systems. This coupled particle system having generalised dynamics is the reason we established the previous estimates of sections 4 and 6 for such generalised systems. As we will explain in the proof of Theorem 2.10, this will allow us to circumvent the problem that the uniqueness theorem we use [28, Theorem 1.1] for the linear Fokker-Planck equation only applies on bounded domains.

Having established these estimates, we are in a position to prove Proposition 2.8 and Theorem 2.9 by way of Theorem 2.10. Theorem 2.10 characterises subsequential limits of the N-particle system as corresponding to solutions of the McKean-Vlasov SDE (1.5) – but this doesn't assume the existence of such solutions. Therefore by choosing a sequence of N-particle systems with the appropriate initial conditions we are able to construct a weak solution to (1.5) in the $N \to \infty$ limit. We establish uniqueness of weak solutions to (1.5) by a contraction argument using Girsanov's theorem similar to the proof of [11, Proposition C.1], completing the proof of Proposition 2.8. Theorem 2.9 then follows by a compactness-uniqueness argument.

The estimates of Section 4 and Lemma 6.2 are used to establish tightness in the proof of Theorem 2.10; the former preventing mass from accumulating on the boundary and the latter preventing mass "escaping to infinity" over a finite time horizon.

We then employ martingale methods to characterise subsequential limits $(m_t, J_t)_{0 \le t < \infty}$ as being supported on the solution set of a nonlinear Fokker-Planck equation. We note that martingale methods have also been used to establish hydrodynamic convergence in the Markovian case ([19] and [30]). We then show that these nonlinear Fokker-Planck solutions correspond to global weak solutions of our McKean-Vlasov SDE (1.5) by verifying they satisfy the same linear Fokker-Planck equation and using a uniqueness theorem [28, Theorem 1.1]. Availing ourselves of this uniqueness theorem requires Lemma 6.1 and – in the case of unbounded domains – combining Proposition 7.2 with a change to our notion of solution to the nonlinear Fokker-Planck equation.

We note this is where the assumption that D has C^∞ boundary becomes necessary, as [28, Theorem 1.1] assumes the domain has C^∞ boundary. Were a more general uniqueness theorem available, this would enable a corresponding generalisation of our results: to more general boundaries, the particles having non-constant diffusivities or the incorporation of "soft killing" (killing according to a Poisson clock).

4 Well-posedness of and estimates for the N-particle system

The goal of this section is to establish Theorem 2.6 along with some estimates for the N-Particle System. We shall prove estimates on the jump times $\{\tau_k\}_{k=0}^\infty$ and on the empirical measure m_t^N of the N-particle process. The estimates in particular will prevent mass accumulating on the boundary when we take various limits in later sections. Theorem 2.6 will be seen to be a consequence of these estimates.

As discussed in Section 3, we establish well-posedness and our estimates for Fleming-Viot particle systems with generalised dynamics, which is defined as follows.

Definition 4.1 (Weak Solution to the Fleming-Viot Particle System with Generalised Dynamics)**.** *Let $\vec{W}_t = (W_t^1, \dots, W_t^N)$ be a collection of independent Brownian motions on $\mathbb{R}^d$ with respect to a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let υ^N be a probability measure on D^N. We say that $(\vec{X}_t, \vec{W}_t, \mathcal{F}_t)$ is a Fleming-Viot particle system with generalised dynamics with initial condition υ^N having drift processes $\vec{b}_t = (b_t^1, \dots, b_t^N)$ if $\vec{X}_0 \sim \upsilon^N$, if $\vec{b}_t$ satisfies Condition 3.1: the drifts b_t^i are $(\mathcal{F}_t)_{t\geq 0}$-adapted and uniformly bounded $|b_t^i| \leq B$ ($i = 1, \dots, N$), and if there is an increasing sequence of $\mathcal{F}_t$-stopping times $\{\tau_k\}_{k=0}^\infty$ with $\tau_0 = 0$ such that the following hold:*

1. *$\vec{X}_t$ is a càdlàg process. For each k, $\vec{X}_t$ is continuous on $[\tau_k, \tau_{k+1})$ and satisfies*

$$X_t^i = X_{\tau_k}^i + \int_{\tau_k}^t b_s^i ds + W_t^i - W_{\tau_k}^i, \qquad i = 1, \dots, N; \;\; t \in [\tau_k, \tau_{k+1}). \tag{4.1}$$

For all $k \geq 1$, and with probability one, there is a unique particle index $\ell(k) \in \{1, \dots, N\}$ such that

$$\tau_k = \min_{i=1,\dots,N} \inf\{t > \tau_{k-1} \mid \lim_{s\to t^-} X_s^i \in D^c\} = \inf\{t > \tau_{k-1} \mid \lim_{s\to t^-} X_s^{\ell(k)} \in D^c\}. \tag{4.2}$$

2. *For all $k \geq 1$,*

$$\lim_{t\to\tau_k^-} X_t^j = X_{\tau_k}^j \in D, \quad \forall\; j \in \{1, \dots, N\} \setminus \{\ell(k)\}, \tag{4.3}$$

and

$$\mathbb{P}(X_{\tau_k}^{\ell(k)} = X_{\tau_k}^j \mid \mathcal{F}_{\tau_k^-}) = \frac{1}{N-1}, \quad \forall\; j \in \{1, \dots, N\} \setminus \{\ell(k)\} \tag{4.4}$$

hold with probability one.

This is no longer well-defined once two particles hit the boundary at the same time:

$$\tau_{stop} = \inf\{t > 0 : \exists\, j \neq k \text{ such that } X_t^j,\; X_t^k \in \partial D\}.$$

Moreover if there are an infinite number of stopping times τ_k in finite time, this is no longer well-defined after the time

$$\tau_{\infty} = \lim_{k\to\infty} \tau_k. \tag{4.5}$$

Furthermore if the domain D is unbounded, the particles may "escape to infinity" in finite time, after which time the particle system is not well-defined. We write

$$\tau_{\max} = \inf\big\{t > 0 : \sup_{\substack{t'\le t\\ 1\le i\le N}} |X^i_{t'}| = \infty\big\}. \tag{4.6}$$

Therefore $(\vec{X}_t, \vec{W}_t)_{0\le t<\tau_{\text{WD}}}$ is defined only up to the time

$$\tau_{\mathit{WD}} := \tau_{\mathit{stop}} \wedge \tau_{\infty} \wedge \tau_{\max}. \tag{4.7}$$

Throughout this section,

$$(\vec{X}_t, \vec{W}_t, \vec{b}_t)_{0\le t<\tau_{\text{WD}}} = ((X^1,\dots,X^N_t),(W^1,\dots,W^N_t),(b^1,\dots,b^N_t))_{0\le t<\tau_{\text{WD}}}$$

will refer to a weak solution to the Fleming-Viot particle system ($N\ge 2$) with generalised dynamics having drift processes bounded by $|b^i_t|\le B$. We further define m^N_t and ξ^N as in (1.10) and (1.11),

$$m^N_t = \vartheta^N(\vec{X}^N_t), \quad m^N_0 \sim \xi^N.$$

We will couple the particles $X^1,\dots,X^N$ to appropriately constructed independent strong solutions $(\eta^1,\tilde{W}^1),\dots,(\eta^N,\tilde{W}^N)$ of the following SDE,

$$d\eta_t = \begin{cases} d\tilde{W}_t + Bdt + \frac{d-1}{2\eta_t}dt - dL^{r-\eta}_t, & d>1 \\ d\tilde{W}_t + Bdt + dL^{\eta}_t - dL^{r-\eta}_t, & d=1 \end{cases}, \quad \eta_0 = r, \tag{4.8}$$

where $r>0$ is the constant from the global interior ball condition, Condition 2.3. Here L^{η} and $L^{r-\eta}$ are the local times

$$L^{\eta}_t := \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t \mathbb{1}(|\eta_s|<\epsilon)d[\eta]_s, \quad L^{r-\eta}_t := \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t \mathbb{1}(|r-\eta_s|<\epsilon)d[\eta]_s. \tag{4.9}$$

We will then use this coupling to obtain estimates on the N-particle system.

Proposition 4.2. *We assume the Brownian motions W^i are jointly independent and defined up to time ∞. There exists on the same probability space a family $(\eta^1_t,\tilde{W}^1_t)_{0\le t<\infty},\dots,(\eta^N_t,\tilde{W}^N_t)_{0\le t<\infty}$ of strong solutions to* (4.8) *which are jointly independent, but coupled to $X^1,\dots,X^N$ up to time $\tau_{\mathit{WD}} = \tau_{\infty}\wedge\tau_{\mathit{stop}}\wedge\tau_{\max}$ so that*

$$d(X^i_t,\partial D) \ge r - \eta^i_t \in [0,r], \quad 0\le t<\tau_{\mathit{WD}}. \tag{4.10}$$

Remark 4.3. The coupling (4.10) only holds up to time τ_{WD}, although $(\eta^i_t,\tilde{W}^i_t)$ are defined for all $t\ge 0$.

We then establish the following lemma.

Lemma 4.4. *If $(\eta^1,\tilde{W}^1)$, $(\eta^2,\tilde{W}^2)$ are two independent solutions to* (4.8) *on the same probability space, then*

$$\mathbb{P}(\exists\, t>0 \text{ such that } \eta^1_t=\eta^2_t=r) = 0. \tag{4.11}$$

For the case of Brownian dynamics ($b\equiv 0$) with bounded domain D, the authors of [9] established controls analogous to Proposition 4.2 with $b=0$ and D bounded. The method of construction they used, however, was quite different. As outlined in Section 3, their approach does not work in our case. On the other hand, a different family of processes was constructed in [29] to establish the well-posedness of the Fleming-Viot particle system they considered there. Their construction cannot be used to establish the estimates of this section, however, without having to invoke a rather unwieldy condition. As we explained in Section 3, here we construct a family of Bessel processes similar to those constructed in [9] using a strategy similar to that implemented in [29].

Proposition 4.5. *For any weak solution to the Fleming-Viot particle system with generalised dynamics, $\tau_{WD} = \tau_\infty = \tau_{stop} = \tau_{\max} = \infty$ almost surely. In particular, the coupling defined in Proposition 4.2 holds for all $t \geq 0$.*

Having established $\tau_{\text{WD}} = \infty$ almost surely in the case of generalised dynamics, we have $\tau_{\text{WD}} = \infty$ in the case of McKean-Vlasov dynamics, giving the proof of Theorem 2.6.

Proof of Theorem 2.6. It is clearly possible to construct a weak solution of the driftless system up to time τ_{WD}, so that between jump times and for $t < \tau_{\text{WD}}$ particle X^i satsifies $dX^i_t = dW^i_t$. Therefore by Girsanov's theorem we obtain the existence of a weak solution $(\vec{X}_t, \vec{W}_t)_{0\leq t<\tau_{\text{WD}}}$ to the N-particle system with McKean-Vlasov dynamics (1.3) up to time τ_{WD}. This and every other weak solution to (1.3) defined up to time τ_{WD} is defined for all time with $\tau_{\text{WD}} = \infty$ almost surely by Proposition 4.5.

Uniqueness of the law of $(\vec{X}_t)_{0\leq t<\infty}$ follows from uniqueness for the driftless system, by change of measure (by the same argument that weak solutions to SDEs with bounded measurable coefficients are unique in law; see [25, Proposition 3.10 of Section 5.3]). □

We shall then establish tightness for the laws of the empirical measure valued process at times bounded away from 0, when the domain D is bounded.

Proposition 4.6. *We assume D is bounded. For any $T_0 > 0$ there exists a compact set $\mathcal{K}_{T_0} \subseteq \mathcal{P}(\mathcal{P}_W(D))$ dependent only upon the upper bound on the drift B and the domain D such that the empirical measure $m^N_t := \vartheta^N(\vec{X}^N_t)$ must satisfy $\mathcal{L}(m^N_t) \in \mathcal{K}_{T_0}$ for all $t \geq T_0$.*

We henceforth fix a finite time horizon $T < \infty$, but no longer assume D is bounded. We establish the following proposition.

Proposition 4.7. *Define for $c > 0$ the closed set $V_c = \{x \in D : d(x, \partial D) \geq c\}$. Then we have:*

1. *For every $\epsilon > 0$, $T_0 > 0$ there exists a constant c, dependent only upon ϵ, T_0, the time horizon T, the upper bound B for the drift, and the constant $r > 0$ of the interior ball condition, such that $K_{\epsilon,T_0} = V_c \subseteq D$ must satisfy*

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{t\in[T_0,T]} m^N_t(K^C_{\epsilon,T_0}) \geq \epsilon\right) = 0. \tag{4.12}$$

2. *We now assume $\xi^N := \vartheta^N_{\#}\upsilon^N$ is tight in $\mathcal{P}(\mathcal{P}_W(D))$ (i.e. as a tight family of random measures on the open set D) – so that mass does not concentrate on the boundary. Fix $\epsilon, \delta > 0$. Then there exists a constant $\tilde{c} > 0$ depending on ϵ, δ, B, r, and T such that $\hat{K}_{\epsilon,\delta} = V_{\tilde{c}} \subseteq D$ satisfies*

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} m^N_t(\hat{K}^C_{\epsilon,\delta}) \geq \epsilon\right) < \delta. \tag{4.13}$$

Remark 4.8. In Part 1 of Proposition 4.7, we do not assume that the initial random measures $\xi^N := \vartheta^N_{\#}\upsilon^N$ are tight as random measures on D. In particular we may have ξ^N converging weakly in probability to an atom on ∂D or the mass could escape to infinity.

Remark 4.9. There are two conventions as to the definition of a geometric random variable. Throughout we use the definition in which the distribution is supported on $\{1, 2, \ldots\}$, with distribution given by

$$\mathbb{P}(G \geq k) = (1-p)^{k-1}.$$

Our final estimate controls the number of jumps by any particle over a finite time horizon:

Proposition 4.10. *Assume that $\{\mathcal{L}(m_0^N)\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D))$. Let $J_t^{N,i}$ be the number of jumps of the i^{th} particle in the N-particle system up to time t. Then for every $\epsilon > 0$, there exists a stopping time τ_ϵ^N and constants $M_\epsilon < \infty$, $p_\epsilon > 0$ (all dependent upon T) such that for all N large enough:*

1. *The number of jumps $J^{N,i}_{\tau_\epsilon^N \wedge T}$ by particle i up to time $T \wedge \tau_\epsilon^N$ is stochastically bounded by the sum of M_ϵ i.i.d. Geom(p_ϵ) distributions.*
2. *The stopping times τ_ϵ^N satisfy*

$$\limsup_{N\to\infty} \mathbb{P}(\tau_\epsilon^N \leq T) \leq \epsilon.$$

4.1 Proof of Proposition 4.2

The proof proceeds in the follow steps:

1. We fix for the time being X_t^i (with driving Brownian motion W^i) and seek to construct a family $(\eta_t^i, \tilde{W}^i)_{0\leq t<\infty}$, $i \in \{1, \ldots, N\}$, of independent solutions of (4.8) and some càdlàg processes D_t^i, $i \in \{1, \ldots, N\}$, such that

$$\eta_t^i \geq D_t^i \geq r - d(X_t^i, \partial D), \quad 0 \leq t < \tau_{\text{WD}}. \tag{4.14}$$

   For clarity, we will usually drop the superscript i in what follows: η_t, $\tilde{W}_t$, D_t, τ_ω will refer to quantities that depend on the particle index i. Our construction of $(\eta_t, \tilde{W}_t)_{0\leq t<\infty}$ proceeds as follows:

   (a) We define stopping times $\tau_{(j,k,\ell)}$ for every triple $(j, k, \ell) \in \mathbb{N}_0^3$, thereby obtaining a collection of random subintervals $[\tau_{(j,k,\ell)}, \tau_{(j,k,\ell+1)})$ of $[0, \tau_\infty \wedge \tau_{\text{stop}})$. We write ω_0 for the order-type of the natural numbers, associate to the ordinal $\omega = j\omega_0^2 + k\omega_0 + \ell < \omega_0^3$ the triple (j, k, ℓ) and write τ_ω for the stopping time $\tau_{(j,k,\ell)}$. The use of ordinals will enable us to use ordinal induction. Moreover we write $\tau_{\omega_0^3} := \tau_{\text{WD}}$ and write I_ω for the interval $[\tau_\omega, \tau_{\omega+1}) = [\tau_{(j,k,\ell)}, \tau_{(j,k,\ell+1)})$ (whereby $[t, t) := \emptyset$). The ordering

$$\tau_{\omega_1} \leq \tau_{\omega_2} \text{ for } \omega_1 \leq \omega_2 \leq \omega_0^3 \tag{4.15}$$

   shall be immediate from the construction. Moreover we shall establish the following lemma.

   Lemma 4.11. *For limit ordinals $\omega \leq \omega_0^3$ we have*

$$\tau_{\omega'} \uparrow \tau_\omega \quad \textit{as} \quad \omega' \uparrow \omega \quad \textit{for every } \omega \leq \omega_0^3 \textit{ a limit ordinal.} \tag{4.16}$$

   By ordinal induction the random subintervals I_ω form a disjoint cover of $[0, \tau_{\omega_0^3}) = [0, \tau_{\text{WD}})$. Moreover on each interval I_ω, X_t will be contained in the ball $B(v_\omega, r)$ (where $r > 0$ is the constant we assume to exist in the interior ball condition and $v_\omega = X_{\tau_\omega}$).

(b) We use our construction in part 1a to define

$$
\begin{aligned}
E_t = \sum_{\omega} \mathbb{1}_{I_\omega}(t) d(X_t, v_\omega), \quad 0 \le t < \tau_{\mathrm{WD}}, \\
D_t^\omega = \big(d(X_{(t\wedge\tau_{\omega+1})^-}, v_\omega) - d(X_{t\wedge\tau_\omega}, v_\omega)\big)\mathbb{1}(\tau_\omega < \tau_{\omega+1})\mathbb{1}(t > \tau_\omega)
\end{aligned}
\tag{4.17}
$$

We observe that D^ω is a continuous semimartingale, with $dD_t^\omega = dE_t$ for $t \in I_\omega$. We employ a Doob-Meyer decomposition of D_t^ω on each interval I_ω to construct a Brownian motion $(\tilde{W}_t)_{0\le t<\infty}$ such that $(E_t)_{0\le t<\tau_{\mathrm{WD}}}$ is a $[0,r]$-valued process which satisfies

$$
dE_t = \begin{cases} d\tilde{W}_t + Bdt + \frac{d-1}{2E_s}dt - dH_t, & d > 1 \\ d\tilde{W}_t + Bdt + dL_s^E ds - dH_t, & d = 1 \end{cases}, \tag{4.18}
$$

where H_t is a non-decreasing, adapted process. Moreover there exists a càdlàg adapted process n_t such that

$$
\tilde{W}_t = \int_0^t n_s \cdot dW_s, \quad 0 \le t < \infty. \tag{4.19}
$$

(c) We establish that (4.8) has strong solutions for this driving motion $\tilde{W}_t$, and that $\eta_t = \eta_t^i$ satisfies (4.14).

(d) We then compare (4.18) and (4.8) establish

$$
E_t \le \eta_t, \quad 0 \le t < \tau_{\mathrm{WD}}, \tag{4.20}
$$

and therefore we have (4.14).

2. We repeat the above construction for each X^i, writing $(\eta^i, \tilde{W}^i)$ for the strong solutions we construct. By examining the quadratic covariation of the Brownian motions $\tilde{W}^i$ (using (4.19)) we establish the $(\eta^i, \tilde{W}^i)$ are jointly independent.

Step 1a

We now define functions ρ and v as in [9]. With $r > 0$ being the constant assumed to exist by the interior ball condition (Condition 2.3), define

$$
\rho(x) = \sup_{\substack{B(y,r) \text{ such that} \\ D \supseteq B(y,r) \ni x}} d(x, \partial B(y,r)).
$$

We claim there exists $v : D \to D$ measurable such that:

1. $B(v(x), r) \subseteq D$ for every $x \in D$;
2. $d(x, \partial B(v(x), r)) \ge \frac{\rho(x)}{2}$.

The construction of v is fairly elementary. We firstly take an ascending sequence of compact sets $K_1, K_2, \ldots$ with union D. We fix K_i and seek to define on K_i a suitable function v^i satisfying 1 and 2. It is easy to see that for every $x \in K_i$ we can choose $y(x)$ such that $d(x, \partial B(y,r)) > \frac{\rho(x)}{2}$. Then on an open neighbourhood $V_x \ni x$ we have $d(x', \partial B(y,r)) > \frac{\rho(x')}{2}$ as both $x' \mapsto d(x', \partial B(y(x), r))$ and $x' \mapsto \rho(x')$ are continuous functions. We may cover K_i with open sets V_x, $x \in K_i$, and take a finite subcover $V_{x_1}, \ldots, V_{x_n}$ (for some n). We now define

$$
v^i(x') := \begin{cases} x_1, & x' \in K_i \cap V_{x_1} \\ x_j, & x' \in K_i \cap (V_{x_j} \setminus (V_{x_1} \cup \ldots \cup V_{x_{j-1}})) \end{cases}.
$$

Then v^i is piecewise constant (and hence measurable) and satisfies 1 and 2 on K^i. Therefore defining v by

$$v(x') := \begin{cases} v^1(x'), & x' \in K_1 \\ v^i(x'), & x' \in K_i \setminus (K_1 \cup \ldots \cup K_{j-1}) \end{cases}$$

we are done. We now turn to the construction of the stopping times $\tau_{(j,k,\ell)}$, for triples $(j,k,\ell) \in \mathbb{N}_0^3$.

1. $\tau_{(0,0,0)} := 0$.
2. $\tau_{(j+1,0,0)} := \inf\{t > \tau_{(j,0,0)} : X_{t-} \in \partial D\} \wedge \tau_{\mathrm{WD}}$, for all $j \in \mathbb{N}_0$.
3. With $j \in \mathbb{N}_0$ fixed, we now define $\tau_{(j,0,\ell)} \in [\tau_{(j,0,0)}\, , \, \tau_{(j+1,0,0)}]$ for every $\ell \in \mathbb{N}$. We proceed inductively, having already defined $\tau_{(j,0,\ell)}$ for $\ell = 0$ in the previous step. We suppose that $\tau_{(j,0,\ell)} \in [\tau_{(j,0,0)}\, , \, \tau_{(j+1,0,0)}]$ has been defined for some $\ell \in \mathbb{N}_0$. If $\tau_{(j,0,\ell)} = \tau_{(j+1,0,0)}$, we set $\tau_{(j,0,\ell+1)} := \tau_{(j,0,\ell)}$. Otherwise, $\tau_{(j,0,\ell)} < \tau_{(j+1,0,0)}$ holds and $X_{\tau_{(j,0,\ell)}} \in D$. Therefore, we may define $X_{(j,0,\ell)} := X_{\tau_{(j,0,\ell)}}$ and $v_{(j,0,\ell)} := v(X_{(j,0,\ell)})$ which satisfies
$$B(v_{(j,0,\ell)}, r) \subseteq D. \tag{4.21}$$
We then define
$$\tau_{(j,0,\ell+1)} = \tau_{\mathrm{WD}} \wedge \begin{cases} \inf\{t > \tau_{(j,0,\ell)} : d(X_{t^-}, v_{(j,0,\ell)}) \geq r\}, & \text{if } \rho(X_{\tau_{(j,0,\ell))}}) > 2^{-0} \\ \tau_{(j,0,\ell)}, \quad \text{if } \rho(X_{\tau_{(j,0,\ell)}}) \leq 2^{-0} & \text{or } \quad \tau_{(j,0,\ell)} = \tau_{(j+1,0,0)} \end{cases},$$
which satisfies $\tau_{(j,0,\ell+1)} \leq \tau_{(j+1,0,0)}$ by (4.21). By induction on ℓ, this defines $\tau_{(j,0,\ell)}$ for all $\ell \in \mathbb{N}_0$ and we have $\tau_{(j,0,0)} \leq \tau_{(j,0,\ell)} \leq \tau_{(j,0,\ell+1)} \leq \ldots \leq \tau_{(j+1,0,0)}$.
4. We then establish (Lemma 4.12) that either $\tau_{(j+1,0,0)} = \infty$ and $\tau_{(j,0,\ell)} \uparrow \tau_{(j+1,0,0)}$ as $\ell \uparrow \infty$, or else $\tau_{(j+1,0,0)} < \infty$ and there exists some random $\ell_{(j,0)} < \infty$ such that either $\rho(X_{(j,0,\ell_{(j,0)})}) \leq 2^{-0}$ or $\tau_{(j,0,\ell(j,0))} = \tau_{(j+1,0,0)}$. In the former case ($\tau_{(j+1,0,0)} = \infty$) we define
$$\tau_{(j,1,0)} := \tau_{(j+1,0,0)}.$$
Otherwise we have $\tau_{(j,0,\ell)} := \tau_{(j,0,\ell_{(j,0)})}$ for all $\ell \geq \ell_{(j,0)}$ so that we may define
$$\tau_{(j,1,0)} := \tau_{(j,0,\ell_{(j,0)})} \leq \tau_{(j+1,0,0)}.$$
5. We repeat the above inductively. We fix k and assume we have defined $\tau_{(j,0,0)} \leq \tau_{(j,k,0)} \leq \tau_{(j+1,0,0)}$. We seek to define
$$\tau_{(j,0,0)} \leq \tau_{(j,k,0)} \leq \tau_{(j,k,1)} \leq \ldots \leq \tau_{(j,k+1,0)} \leq \tau_{(j+1,0,0)}.$$
Proceeding as in Step 3, if $\tau_{(j,k,\ell)} = \tau_{(j+1,0,0)}$ we define
$$\tau_{(j,k,\ell+1)} = \tau_{(j+1,0,0)}.$$
Otherwise $\tau_{(j,k,\ell)} < \tau_{(j+1,0,0)}$ so we may define as before $v_{(j,k,\ell)} = v(X_{\tau_{(j,k,\ell)}})$ and $X_{(j,k,\ell)} := X_{\tau_{(j,k,\ell)}}$. We may then define
$$\tau_{(j,k,\ell+1)} = \tau_{\mathrm{WD}} \wedge \begin{cases} \inf\{t > \tau_{(j,k,\ell)} : d(X_t, v_{(j,k,\ell)}) \geq r\}, & \text{if } \rho(X_{\tau_{(j,k,\ell))}}) > 2^{-k} \\ \tau_{(j,k,\ell)}, \quad \text{if } \rho(X_{\tau_{(j,k,\ell)}}) \leq 2^{-k} & \text{or } \quad \tau_{(j,k,\ell)} = \tau_{(j+1,0,0)} \end{cases}.$$
Having defined $\tau_{(j,k,\ell)}$ for $\ell = 0, 1, \ldots$ we now turn to defining $\tau_{(j,k+1,0)}$. We establish the following lemma.

Lemma 4.12. *Either* $\tau_{(j+1,0,0)} = \infty$ *and* $\tau_{(j,k,\ell)} \uparrow \infty$ *as* $\ell \uparrow \infty$*, or else* $\tau_{(j+1,0,0)} < \infty$ *and there exists some random* $\ell_{(j,k)} < \infty$ *such that either* $\rho(X_{(j,k,\ell_{(j,k)})}) \leq 2^{-k}$ *or* $\tau_{(j,k,\ell(j,k))} = \tau_{(j+1,0,0)}$.

In the former case ($\tau_{(j+1,0,0)} = \infty$) we define

$$\tau_{(j,k+1,0)} := \tau_{(j+1,0,0)}.$$

Otherwise we have $\tau_{(j,k,\ell)} := \tau_{(j,k,\ell_{(j,k)})}$ for all $\ell \geq \ell_{(j,k)}$ so that we may define

$$\tau_{(j,k+1,0)} := \tau_{(j,k,\ell_{(j,k)})} \leq \tau_{(j+1,0,0)}.$$

Repeating inductively in k we have defined $\tau_{(j,k,\ell)}$ for $(j,k,\ell) \in \mathbb{N}_0^3$, subject to proving Lemma 4.12.

Proof of Lemma 4.12. We fix j and k. We consider sub-intervals $[mh,(m+1)h]$ ($m = 0,1,\ldots$) of length $h > 0$ to be determined, over each of which the diffusion term dominates the drift term. We write $N_m := |\{\ell' : \tau_{(j,k,\ell')} \in [mh,(m+1)h] \cap [0,\tau_{\text{WD}})$ and $\tau_{(j,k,\ell')} < \tau_{(j+1,0,0)}\}|$ (for our j and k fixed). Then it is sufficient to show that $h > 0$ may be chosen so that

$$\mathbb{P}(N_m = \infty) = 0 \quad \text{for all} \quad 0 \leq m < \infty.$$

We recall that we have fixed i, and moreover $X_t = X_t^i$ has driving Brownian motion $W_t = W_t^i$ which satisfies

$$|(X_{t_2} - X_{t_1}) - (W_{t_2} - W_{t_1})| \leq B(t_2 - t_1) \tag{4.22}$$

if X_t does not hit the boundary during the time interval $[t_1, t_2]$.

We observe therefore that if our distance to the boundary is bounded from below then in order for our particle to die within a sufficiently small time interval, the driving Brownian motion W_t must travel a distance bounded from below in this small time interval. In particular we suppose that we have $\tau_{(j,k,\ell)} \in [mh,(m+1)h] \cap [0,\tau_{\text{WD}})$ with $\tau_{(j,k,\ell)} < \tau_{(j+1,0,0)}$ and $\rho(v_{(j,k,\ell)}) \geq 2^{-k}$. Then in order to also have $\tau_{(j,k,\ell+1)} \leq (m+1)h$ it must be the case that X_t hits $\partial B(v(X_{\tau_{(j,k,\ell)}}), r))$ before time $(m+1)h$. We now recall

$$d(X_{\tau_{(j,k,\ell)}}, \partial B(v(X_{\tau_{(j,k,\ell)}}), r)) \geq \frac{\rho(X_{\tau_{(j,k,\ell)}})}{2} \geq 2^{-k-1}.$$

Therefore if $\tau_{(j,k,\ell)} \in [mh,(m+1)h] \cap [0,\tau_{\text{WD}})$ with $\tau_{(j,k,\ell)} < \tau_{(j+1,0,0)}$ and $\rho(v_{(j,k,\ell)}) \geq 2^{-k}$, then in order to also have $\tau_{(j,k,\ell+1)} < \tau_{(j+1,0,0)}$ and $\rho(v_{(j,k,\ell+1)}) \geq 2^{-k}$ we must have $|X_{(j,k,\ell+1)} - X_{(j,k,\ell)}| \geq 2^{-k-1}$, which requires the driving Brownian motion satisfy

$$|W_{\tau_{(j,k,\ell+1)}} - W_{\tau_{(j,k,\ell)}}| \geq 2^{-k-1} - Bh.$$

We note that this latter event happening is independent of $\mathcal{F}_{\tau_{(j,k,\ell)}}$, and for $h < \frac{2^{-(k+2)}}{B}$ has probability at most some $p < 1$. Therefore at time $\tau_{(j,k,\ell)} \in [mh,(m+1)h] \cap [0,\tau_\infty \wedge \tau_{\text{stop}})$, the probability this is the final such stopping time in the interval $[rh,(r+1)h]$ is at least $1 - p > 0$. Recalling Remark 4.9, we see that N_m is stochastically dominated by a $\text{Geom}(1-p)$ distribution for $h < \frac{2^{-(k+2)}}{B}$. □

We have left to prove Lemma 4.11.

Proof of Lemma 4.11. We begin with the $\omega = \omega_0^3$ case. This is true by definition.

Next, consider the case that $\omega = (j+1)\omega_0^2$. If $\tau_{(j+1,0,0)} = \infty$ then $\tau_{(j,k,0)} = \infty$ for all $k \geq 1$ and we are done. We may therefore assume $\tau_{(j,k+1,0)} < \tau_{(j+1,0,0)} < \infty$ for all $k \in \mathbb{N}_0$

otherwise we are done. Then Lemma 4.12 gives that $\rho(X_{\tau_{(j,k+1,0)}}) = \rho(X_{\tau_{(j,k,\ell_{(j,k)})}}) \leq 2^{-k} \to 0$ as $k \to \infty$ and so $d(X_{\tau_{j\omega_0^2+k\omega_0}}, \partial D) \to 0$ as $k \to \infty$. Therefore by the almost-sure continuity of the path X_t and the fact that $\tau_{(j+1)\omega_0^2} < \infty$ we have $\lim_{k\to\infty} X_{\tau_{(j,k,0)}} \in \partial D$. Therefore we have $\tau_{j\omega_0^2+k\omega_0} \to \tau_{(j+1)\omega_0^2}$ as $k \to \infty$.

Finally, in the case that $\omega = j\omega_0^2 + (k+1)\omega_0$, this is an immediate consequence of Lemma 4.12. □

Step 1b

We begin by constructing $\tilde{W}_t$ and showing that it can be written in the form (4.19) for a càdlàg adapted process n_t which we also construct. We recall (4.17) where we define for $\omega < \omega_0^3$

$$\begin{aligned} E_t &= \sum_{\omega} \mathbb{1}_{I_\omega}(t) d(X_t, v_\omega), \quad 0 \leq t < \tau_{\text{WD}}, \\ D_t^\omega &= \big(d(X_{(t\wedge\tau_{\omega+1})^-}, v_\omega) - d(X_{t\wedge\tau_\omega}, v_\omega)\big)\mathbb{1}(\tau_\omega < \tau_{\omega+1})\mathbb{1}(t > \tau_\omega). \end{aligned} \tag{4.23}$$

After adding a positive drift $B\mathbb{1}_{I_\omega}(t)$, D_t^ω becomes a submartingale, so we may take the Doob-Meyer decomposition, obtaining a mean zero Martingale term $\tilde{W}_t^\omega$ with quadratic variation $\int_0^t \mathbb{1}_{I_\omega}(s)ds$ (i.e. a Brownian motion started at time τ_ω and stopped at time $\tau_{\omega+1}$). Indeed we can write

$$\begin{aligned} D_t^\omega = &\Big(\sqrt{(X_{(t\wedge\tau_{\omega+1})^-} - v_\omega)\cdot(X_{(t\wedge\tau_{\omega+1})^-} - v_\omega)} - \sqrt{(X_{t\wedge\tau_\omega} - v_\omega)\cdot(X_{t\wedge\tau_\omega} - v_\omega)}\Big) \\ &\times \mathbb{1}(\tau_\omega < \tau_{\omega+1})\mathbb{1}(t > \tau_\omega) \end{aligned}$$

so that we have

$$dD_t^\omega = \mathbb{1}_{I_\omega}(t)\frac{X_t - v_\omega}{|X_t - v_\omega|}\cdot dW_t + \text{finite variation terms}$$

and therefore

$$d\tilde{W}_t^\omega = \mathbb{1}_{I_\omega}(t)\frac{X_t - v_\omega}{|X_t - v_\omega|}\cdot dW_t.$$

We fix $\hat{n} \in \mathbb{R}^d$ such that $|\hat{n}| = 1$ so that $\hat{n}\cdot W_t$ is a Brownian motion. We now write

$$\tilde{W}_t = \int_0^{t\wedge\tau_{\text{WD}}} \sum_{\omega} \mathbb{1}_{I_\omega}(s) d\tilde{W}_s^\omega + \int_{t\wedge\tau_{\text{WD}}}^t \hat{n}\cdot dW_s, \quad 0 \leq t < \infty \tag{4.24}$$

which is clearly a Brownian motion, since the I_ω form a countable partition of $[0, \tau_{\text{WD}})$. We recall that we want to define $\tilde{W}_t$ beyond time τ_{WD} if $\tau_{\text{WD}} < \infty$. In particular we can write

$$\tilde{W}_t = \int_0^t \underbrace{\Big(\sum_{\omega} \mathbb{1}_{I_\omega}(s)\frac{X_s - v_\omega}{|X_t - v_\omega|} + \mathbb{1}_{[\tau_{\text{WD}},\infty)}(s)\Big)\hat{n}}_{=:n_s}\cdot dW_s$$

and hence we have (4.19).

We now claim

$$H_t := E_0 - E_t + \int_0^t \begin{cases} d\tilde{W}_s + Bds + \frac{d-1}{2E_s}ds, & d > 1 \\ d\tilde{W}_s + Bds + dL_s^E, & d = 1 \end{cases}, \quad 0 \leq t < \tau_{\text{WD}}, \tag{4.25}$$

is non-decreasing.

Proof (4.25) *is non-decreasing.* It will be convenient here to extend the definition of E_t by defining $E_{\tau_{\rm WD}} = r$ if $\tau_{\rm WD} < \infty$.

We proceed by ordinal induction. We inductively claim

$$H_t \text{ is non-decreasing on } [0,\tau_\omega]\cap[0,\tau_{\rm WD}) \text{ for } \omega\leq\omega_0^3, \tag{4.26}$$

which immediately implies (4.25) by Lemma 4.11.

The $\omega=0$ case is immediate.

If $\omega=\omega'+1$ is a successor ordinal, then it is sufficient to show that H_t is non-decreasing on $[\tau_{\omega'},\tau_\omega]$. We may assume $\tau_{\omega'}<\tau_\omega$, otherwise we are done.

$$dD_t^{\omega'} = \mathbb{1}(t\in I_{\omega'})\begin{cases} d\tilde{W}_t^{\omega'}+\bar{b}_t^{\omega'}dt+\frac{d-1}{2E_t}dt, & d>1\\ d\tilde{W}_t^{\omega'}+\bar{b}_t^{\omega'}dt+dL^E, & d=1\end{cases}$$

for some process $\bar{b}_t^{\omega'}\leq B$. Therefore for $\tau_{\omega'}\leq t<\tau_\omega$,

$$H_t-H_{\tau_{\omega'}} = E^{\omega'}_{\tau_{\omega'}}-E_t^{\omega'}+\int_{\tau_{\omega'}}^t \begin{cases} d\tilde{W}_s+Bds+\frac{d-1}{2E_s}ds, & d>1\\ d\tilde{W}_s+Bds+dL_s^E, & d=1\end{cases} = \int_{\tau_{\omega'}}^t (B-\bar{b}_s^{\omega'})ds,$$

which is non-decreasing.

Moreover we note by construction that $\limsup_{t\uparrow\tau_\omega}E_t=r$ so that $\limsup_{t\uparrow\tau_\omega}(H_{\tau_\omega}-H_t)\geq 0$ if $\tau_\omega<\tau_{\rm WD}$. Thus we have dealt with the case where ω is a successor ordinal.

We finally consider the case whereby $\omega\leq\omega_0^3$ is a limit ordinal. If $\tau_{\omega'}=\tau_\omega$ for some $\omega'<\omega$ we are done by our induction hypothesis. Moreover $(H_t)_{0\leq t<\tau_\omega}$ is non-decreasing by our induction hypothesis. Therefore if $\tau_\omega=\tau_{\rm WD}$ we are done.

We now assume otherwise, so that for $\omega'<\omega$ we have $\tau_{\omega'}<\tau_\omega<\tau_{\rm WD}$. It is sufficient to show that $\limsup_{t\uparrow\tau_\omega}H_t\leq H_{\tau_\omega}$. We take a sequence of successor ordinals $\omega_n\uparrow\omega$ with $\omega_n<\omega$. For each n we have some $\omega_n\leq\omega_n'<\omega$ such that $\tau_{\omega_n'}<\tau_{\omega_n'+1}$. However we know by construction that $E_{\tau_{\omega_n'+1}-}=r$ so by the same calculation as in the case of successor ordinals, $\limsup_{t\uparrow\tau_\omega}(H_{\tau_\omega}-H_t)\geq 0$ hence we are done. □

Thus we have established (4.19) whereby H_t defined in (4.18) is a non-decreasing adapted process.

Step 1c

Theorem 1.3 of [3] gives the existence and uniqueness of strong solutions to reflected SDEs in convex domains where the drift is C^1 and Lipschitz. That theorem applies directly to (4.8) in the $d=1$ case. In the $d>1$ case, the only issue is that the drift is locally Lipschitz but not globally Lipschitz. Here we must stop the process η_t when it hits $\epsilon>0$, then take ϵ to zero and note that on any fixed finite time horizon the probability of hitting this barrier goes to zero as $\epsilon\to 0$.

Step 1d

We have constructed a solution $(\eta,\tilde{W})$ to (4.8) and claim that

$$\eta_t\geq E_t\geq r-d(X_t,\partial D),\quad 0\leq t<\tau_{\rm WD}. \tag{4.14}$$

The second inequality is obvious, we now establish the first.

We recall that $(\eta,\tilde{W})$ satisfies

$$d\eta_t=\begin{cases} d\tilde{W}_t+\frac{d-1}{2\eta_t}dt+Bdt-dL_t^{r-\eta}, & d>1\\ d\tilde{W}_t+Bdt+dL^\eta-dL_t^{r-\eta}, & d=1\end{cases},\quad 0\leq t<\infty,\quad \eta_0=r, \tag{4.8}$$

whereas E_t is a $[0,r]$-valued process which satisfies

$$dE_t = \begin{cases} d\tilde{W}_t + \frac{d-1}{2E_s}dt + Bdt + dH_t, & d>1 \\ d\tilde{W}_t + Bdt + dL^E_s ds + dH_t, & d=1 \end{cases}, \quad 0 \le t < \tau_{\mathrm{WD}} \tag{4.18}$$

for some non-decreasing adapted process H_t. Therefore we have

$$\begin{aligned} d(\eta_t - E_t) &= dH_t + \mathbb{1}_{d>1}\Big(\frac{d-1}{2\eta_t} - \frac{d-1}{2E_t}\Big)dt \\ &+ \mathbb{1}_{d=1}\big(dL^\eta_t - dL^E_t\big) - dL^{r-\eta}_t, \quad 0 \le t < \tau_{\mathrm{WD}}. \end{aligned} \tag{4.27}$$

We fix $\delta > 0$ and assume for contradiction there exists $t_1 < \tau_{\mathrm{WD}}$ such that $\eta_{t_1} - E_{t_1} \le -2\delta$. We define $t_0 = \sup\{t' < t_1 : \eta'_t - E_{t'} \ge -\delta\}$. Then since H_t is non-decreasing we have $(\eta_{t_0} - E_{t_0}) \ge \limsup_{t'\uparrow t_1}(\eta'_t - E_{t'}) \ge -\delta$. Therefore $t_0 < t_1$ and we must have $\eta_t < E_t - \delta$ for $t \in (t_0, t_1]$. Thus as H_t is non-decreasing $(\eta_{t_0} - E_{t_0}) \le \liminf_{t'\downarrow t_1}(\eta'_t - E_{t'}) \le -\delta$ and therefore $(\eta_{t_0} - E_{t_0}) = -\delta$. Therefore we must have

$$L^{r-\eta}_{t_1} - L^{r-\eta}_{t_0} = 0, \quad L^E_{t_1} - L^E_{t_0} = 0, \quad \mathbb{1}_{d>1}\Big(\frac{d-1}{2\eta_t} - \frac{d-1}{2E_t}\Big)dt \ge 0 \quad \text{for} \quad t \in [t_0, t_1].$$

Therefore we have

$$\begin{aligned} -\delta \ge (\eta_{t_1} - E_{t_1}) - (\eta_{t_0} - E_{t_0}) &= \underbrace{\int_{t_0}^{t_1} \mathbb{1}_{d>1}\Big(\frac{d-1}{2\eta_s} - \frac{d-1}{2E_s}\Big)ds}_{\ge 0} \\ + \underbrace{H_{t_1} - H_{t_0}}_{\ge 0} + \underbrace{\mathbb{1}_{d=1}(L^\eta_{t_1} - L^\eta_{t_0})}_{\ge 0} - \underbrace{(L^E_{t_1} - L^E_{t_0})}_{=0} &- \underbrace{(L^{r-\eta}_{t_1} - L^{r-\eta}_{t_0})}_{=0} \ge 0. \end{aligned}$$

This is a contradiction, hence we must have $\eta_t \ge E_t$ for $t < \tau_{\mathrm{WD}}$.

We have now completed Step 1d.

Step 2

From (4.19) we can write

$$d\tilde{W}^i_t = n^i_t \cdot dW^i_t$$

for some processes n^i_t. We write $n^i_t(k)$ and $W^i_t(k)$ for the component of n^i and W^i in the k^{th} dimension respectively. Therefore we have for $i \ne j$ that

$$d[\tilde{W}^i, \tilde{W}^j]_t = \sum_{k,l} n^i_t(k) n^j_t(l) d[W^i(k), W^j(l)]_t = 0.$$

Thus the Brownian motions $\tilde{W}^i$ and $\tilde{W}^j$ have zero covariance, so $\tilde{W}^1, \ldots, \tilde{W}^N$ are jointly independent. Since each $(\eta^i, \tilde{W}^i)$ is a measurable function of $\tilde{W}^i$, they must also be independent.

Thus we have constructed independent identically distributed strong solutions $(\eta^1, \tilde{W}_1), \ldots,$
$(\eta^N, \tilde{W}^N)$ of (4.8) satisfying (4.10) so have established Proposition 4.2.

□

4.2 Proof of Lemma 4.4

We consider on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ two independent strong solutions $(\eta^k, \tilde{W}^k)$ $(k = 1, 2)$ to (4.8), which we recall is given by

$$d\eta_t = \begin{cases} d\tilde{W}_t + \frac{d-1}{2\eta_t}dt + Bdt - dL^{r-\eta}_t, & d>1 \\ d\tilde{W}_t + Bdt + dL^\eta_t - dL^{r-\eta}_t, & d=1 \end{cases}, \quad \eta_0 = r, \tag{4.8}$$

such that $(\eta^1,\tilde{W}^1)$ and $(\eta^2,\tilde{W}^2)$ are independent of each other. Given $t_0\in\mathbb{Q}_{\geq 0}$ we define

$$\tau_{t_0}=\inf\{t>t_0:\min(\eta^1_t,\eta^2_t)\leq\frac{r}{2}\}.$$

We define for $k=1,2$,

$$\overline{W}^k_t=\tilde{W}^k_t+\begin{cases}0, & t\leq t_0\\ B(t-t_0)+\int_{t_0}^t\frac{d-1}{2\eta^k_s}ds, & t_0\leq t\leq\tau_{t_0}\\ B(\tau_{t_0}-t_0)+\int_{t_0}^{\tau_{t_0}}\frac{d-1}{2\eta^k_s}ds, & t\geq\tau_{t_0}\end{cases}.$$

By Girsanov's theorem there is an equivalent probability measure $\overline{\mathbb{P}}$ under which $\overline{W}^1$ and $\overline{W}^2$ are Brownian motions, which by examining the covariation we see must be independent. Now we observe that $(\eta^k_t)_{t_0\leq t\leq\tau_{t_0}}$ must satisfy

$$d\eta^k_t=d\overline{W}^k_t-dL^{r-\eta^k}_t,\quad t_0\leq t\leq\tau_{t_0},\quad\eta^k_{t_0}=r.$$

We have the existence of a strong solution $\hat{\eta}^k_t=\eta^k_{t_0}+\overline{W}^k_t-\sup_{t_0\leq t'\leq t}\overline{W}^k_{t'}$ which by computing $d(\hat{\eta}^k-\eta^k)^2\leq 0$ we see must be equal to η^k (i.e. we have pathwise uniqueness). Therefore η^k is a measurable function of $\overline{W}^k$, hence $r-\eta^1$ and $r-\eta^2$ are independent and distributed under $\overline{\mathbb{P}}$ like the absolute value of a 1-dimensional Brownian motion. Therefore by Pythagoras $\sqrt{(r-\eta^1_t)^2+(r-\eta^2_t)^2}$ must be distributed under $\overline{\mathbb{P}}$ like the absolute value of a 2-dimensional Brownian motion. Therefore $\overline{\mathbb{P}}(\exists t_0<t<\tau_{t_0}$ such that $\eta^1_t=\eta^2_t=r)=0$ hence $\mathbb{P}(\exists t_0<t<\tau_{t_0}$ such that $\eta^1_t=\eta^2_t=r)=0$. Taking the union over $t_0\in\mathbb{Q}_{\geq 0}$ we are done.

□

4.3 Proof of Proposition 4.5

We now use Proposition 4.2 and Lemma 4.4 to establish that $\tau_{\text{WD}}=\infty$. The main idea is that by Proposition 4.2 the event $\tau_{\text{WD}}<\infty$ corresponds to the event that two of the η^i hit r at the same time, which almost surely doesn't happen by Lemma 4.4.

In [9] they justified that $\tau_{\text{stop}}\geq\tau_\infty$ on the basis of the hitting time of a Brownian motion in an arbitrary domain having a continuous density. However, (4.10) and Lemma 4.4 give us that $\tau_{\text{stop}}\geq\tau_\infty\wedge\tau_{\max}$ for free. Indeed, if $\tau_{\text{stop}}<\tau_\infty\wedge\tau_{\max}$ then two particles (say X^i and X^j) hit the boundary at time τ_{stop}, so that by (4.10) $\eta^i_{\text{stop}}=\eta^j_{\text{stop}}=r$. Therefore by Lemma 4.4, $\mathbb{P}(\tau_{\text{stop}}<\tau_\infty\wedge\tau_{\max})=0$.

We now have $\tau_{\text{stop}}\geq\tau_\infty\wedge\tau_{\max}$ almost surely. Since between killing times τ_k, the particles can't travel an infinite distance over a finite time horizon $T<\infty$, we may inductively in k see that $\tau_{\max}\geq\tau_k\wedge T$. Since $T<\infty$ is arbitrary, $\tau_{\max}\geq\tau_\infty$.

Thus $\tau_\infty\leq\tau_{\max}\wedge\tau_{\text{stop}}$, so we now seek to show $\tau_\infty=\infty$ almost surely. We assume for the sake of contradiction $\tau_\infty<\infty$ with positive probability. We write τ^i_k for the k^{th} jump time of particle i. Then there exists $1\leq i\leq N$ such that $\tau^i_k\uparrow\tau_\infty<\infty$ as $k\to\infty$ with positive probability. If this is the case, then i must jump an infinite number of times up to time τ_∞. Therefore by the pigeonhole principle, for some $j\neq i$, i jumps infinitely many times onto j before time $\tau_\infty<\infty$ with positive probability

We therefore assume i jumps onto j infinitely many times up to time $\tau_\infty<\infty$. Since the drift is bounded and $\tau_\infty=\tau_{\text{WD}}<\infty$ we almost surely have

$$\sum_{\substack{k\text{ such that}\\ \text{i jumps onto j}}}(X^i_{\tau^i_{k+1}-}-X^i_{\tau^i_k})^2<\infty. \tag{4.28}$$

We write $\tau_k^{i,j}$ for the k^{th} time particle i hits the boundary and jumps to particle j. Then by (4.28) we have

$$d(X^j_{\tau_k^{i,j}}, \partial D) = d(X^i_{\tau_k^{i,j}}, \partial D) \to 0 \quad \text{as} \quad k \to \infty.$$

Thus $\limsup_{t\uparrow\tau_\infty} \eta^i_t = \limsup_{t\uparrow\tau_\infty} \eta^j_t = r$ by (4.10). Thus if $\tau_\infty = \tau_{\text{WD}} < \infty$ with positive probability then $\eta^i_{\tau_\infty} = \eta^j_{\tau_\infty} = r$ with positive probability, which is not the case by Lemma 4.4.

Therefore $\tau_{\text{WD}} = \infty$ almost surely. □

4.4 Proof of Proposition 4.6

It is sufficient by [24, Theorem 4.10] to show that the expected mean measures,

$$\{\chi_t : \chi_t(A) := \mathbb{E}[m^N_t(A)] \text{ whereby } m^N_t := \vartheta^N(\vec{X}^N_t) \text{ for some weak solution } \vec{X}^N_t \text{ to (1.3)} \\ \text{for any initial condition } \vec{X}^N_0 \sim \upsilon^N \in \mathcal{P}(D^N), \text{ any } N \text{ and any } t \geq T_0\},$$

are tight. We define $V_\delta = \{x \in D : d(x, \partial D) \geq \delta\}$. Then we have $\chi_t(V^c_\delta) = \mathbb{P}(d(X^{N,1}_t, \partial D) < \delta) \leq \mathbb{P}(r - \eta^{N,1}_t < \delta) = \mathbb{P}(\eta^{N,1}_t > r - \delta)$ by Tonelli's theorem and Proposition 4.2. This bound is uniform over all weak solutions for all N, all initial conditions, and all $T \geq T_0$, hence we are done.

□

4.5 Proof of part 1 of Proposition 4.7

We henceforth fix $T_0 > 0$ and $\epsilon > 0$. We shall take $K_{\epsilon,T_0} = V_c = \{x : d(x, \partial D) \geq c\}$ for $c > 0$ to be determined. We may by Proposition 4.2 construct i.i.d. solutions of (4.8) $(\eta^1, \tilde{W}^1), \ldots, (\eta^N, \tilde{W}^N)$ such that

$$\{m^N_t(K^c) \geq \epsilon \text{ for some } T_0 \leq t \leq T\} \subseteq \Big\{\sup_{T_0 \leq t \leq T} \frac{1}{N} \sum_{j=1}^N \mathbb{1}(d(X^j, \partial D) \leq c) \geq \epsilon\Big\} \\ \subseteq \Big\{\sup_{T_0 \leq t \leq T} \frac{1}{N} \sum_{j=1}^N \mathbb{1}(\eta^j_t \geq r - c) \geq \epsilon\Big\}.$$

Therefore it is sufficient to show that we may take $c > 0$ small enough such that

$$\limsup_{N\to\infty} \mathbb{P}(\sup_{T_0 \leq t \leq T} \frac{1}{N} \sum_{j=1}^N \mathbb{1}(\eta^j_t \geq r - c) \geq \epsilon) = 0. \tag{4.29}$$

Our strategy will be to implement Kingman's Subadditive Ergodic Theorem. We will establish (4.29) with $\eta^1, \ldots, \eta^N$ constructed on a different probability space (which is sufficient). We consider a strong solution (η, W) of (4.8) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We thereby, by taking an infinite product, construct i.i.d. solutions η^i of (4.8) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})^{\otimes\infty}$. It is classical that the map

$$\mathcal{T} : (\Omega, \mathcal{F}, \mathbb{P})^{\otimes\infty} \ni (\omega_1, \omega_2, \ldots) \mapsto (\omega_2, \omega_3, \ldots) \in (\Omega, \mathcal{F}, \mathbb{P})^{\otimes\infty}$$

is ergodic. For $c > 0$ to be determined and every $n \in \mathbb{N}$ we let $g_n(\omega) = \sup_{T_0 \leq t \leq T} \sum_{1 \leq i \leq n} \mathbb{1}(\eta^i_t \geq r - c)$. Then it is easy to see g_n satisfies

$$g_{n+m}(\omega) \leq g_n(\omega) + g_m(\mathcal{T}^n(\omega)). \tag{4.30}$$

Therefore by Kingman's Subadditive Ergodic Theorem we have

$$\lim_{n\to\infty} \frac{1}{n} g_n(\omega) = \inf_{m \geq 1} \mathbb{E}[\frac{1}{m} g_m] \qquad \mathbb{P}^{\otimes\infty}\text{-almost surely.}$$

Thus it is sufficient to establish that there exists $c > 0$ and $n < \infty$ such that $\mathbb{E}[\frac{1}{n}g_n] < \epsilon$. We fix $n > \frac{2}{\epsilon}$ and note that

$$\mathbb{E}[\frac{1}{n}g_n] \leq \frac{1}{n}\mathbb{P}(g_n \leq 1) + \mathbb{P}(g_n \geq 2) \leq \frac{\epsilon}{2} + \mathbb{P}(g_n \geq 2).$$

Therefore it is sufficient to show $\mathbb{P}(g_n \geq 2) < \frac{\epsilon}{2}$ for some $c > 0$ small enough. We may consider the ranked particles $\eta_t^{(1)} \geq \eta_t^{(2)} \geq \ldots \geq \eta_t^{(n)}$, in particular we consider the second ranked particle

$$\eta_t^{(2)} = \sup\{\eta_t^i : \exists j \neq i \text{ such that } \eta_t^j \geq \eta_t^i\},$$

which we note has continuous sample paths. Then we have

$$\{g_n \geq 2\} = \{\sup_{T_0 \leq t \leq T} \eta^{(2)} \geq r - c\}.$$

Our goal is to show the probability of this event is less than $\frac{\epsilon}{2}$ for $c > 0$ small enough. Recall that $\eta^i \leq r$. Since $\eta^{(2)}$ has continuous sample paths and $[T_0, T]$ is compact we have

$$\begin{aligned}&\{\eta_t^i = \eta_t^j = r \text{ for some } i \neq j \text{ and } T_0 \leq t \leq T\}\\ &= \{\eta_t^{(2)} = r \text{ for some } T_0 \leq t \leq T\} = \cap_{c>0}\{\eta_t^{(2)} \geq r - c \text{ for some } T_0 \leq t \leq T\}.\end{aligned}$$

The probability of this event is zero by Lemma 4.4 hence we have

$$\lim_{c \to 0} \mathbb{P}(\sup_{T_0 \leq t \leq T} \eta^{(2)} \geq r - c) = 0.$$

Therefore $\mathbb{P}(g_n \geq 2) = \mathbb{P}(\sup_{T_0 \leq t \leq T} \eta^{(2)} \geq r - c) < \frac{\epsilon}{2}$ for $c > 0$ small enough. Therefore we have

$$\lim_{N \to \infty} \sup_{T_0 \leq t \leq T} \frac{1}{N} \sum_{j=1}^N \mathbb{1}(\eta_t^j \geq r - c) < \epsilon \quad \mathbb{P}^{\otimes \infty}\text{-almost surely.}$$

Thus we have (4.29) on our original probability space. We finally note that the choice of $c > 0$ is dependent only upon the parameters of the Bessel processes, hence dependent only upon $T_0,\ T,\ \epsilon,\ B$ and r.

□

4.6 Proof of part 2 of Proposition 4.7

We recall $\xi^N = \vartheta^N_{\#}\upsilon^N$. Since $\{\xi^N\}$ are tight as a family of random measures, for every $\epsilon,\ \delta > 0$ there exists $c' > 0$ such that

$$\mathbb{P}(\xi^N(V_{c'}^c) \geq \frac{\epsilon}{10}) < \delta.$$

So, by bounding the distance travelled by a particle in time T_0 for small enough $T_0 > 0$, we have that for some smaller $c'' > 0$ and all N large enough that

$$\mathbb{P}(m_t^N(V_{c''}^c) \geq \epsilon \text{ for some } t \leq T_0) < \delta.$$

We now take $\hat{c}(\epsilon, \delta) = c(\epsilon, T_0) \wedge c''$ so that $\hat{K}_{\epsilon,\delta} = V_{\hat{c}}$ satisfies (4.13).

□

4.7 Proof of Proposition 4.10

Here we adopt a strategy similar to Part 1 of the proof of [9, Theorem 1.3] (where they considered the Brownian case). There they argued that a positive proportion of specially selected particles stay within a given set with probability converging to 1. Then they argued that each time some particle dies there is a probability bounded away from 0 of this particle jumping onto one of these specially selected particles. If that is the case, then the probability of not dying off is bounded away from 0 as the distance between the given set and the boundary is bounded away from 0. Thus each time a particle hits the boundary, there is a probability bounded away from 0 of this being the last death time of the particle so long as the specially selected particles are within the given set.

Their proof that a positive proportion of specially selected particles stay within a given set with probability converging to 1 relies on the independence of the particles in the Brownian case. This does not apply in our case, so instead we must use the closed set we constructed in Part 2 of Proposition 4.7. Moreover we break $[0, T]$ into a large number of sub-intervals, over each of which the diffusive term dominates the drift term (this is not necessary in the $b = 0$ case as there is no drift).

We recall that $\text{Geom}(p)$ refers to the geometric distribution on $\{1, 2, \dots\}$ with distribution given by $\mathbb{P}(G \geq k) = (1-p)^{k-1}$ (Remark 4.9). We now set

$$\tau_\epsilon^N = \inf\{t \geq 0 : m_t^N(\hat{K}^c_{\frac{1}{2},\epsilon}) \geq \frac{1}{2}\}, \tag{4.31}$$

so that we have $\limsup_{N\to\infty} \mathbb{P}(\tau_\epsilon^N \leq T) \leq \epsilon$. We break $[0, T]$ into M to be determined sub-intervals $[rh, (r+1)h]$ ($r = 0, \dots, M-1$) of length $h = \frac{T}{M}$ and define

$$J_r := |\{k : \tau_k^i \in [rh, (r+1)h] \text{ and } \tau_i^i \leq \tau_\epsilon^N\}|.$$

We recall that X_t^i has driving Brownian motion W_t^i and satisfies

$$|(X^i_{t_2} - X^i_{t_1}) - (W^i_{t_2} - W^i_{t_1})| \leq B(t_2 - t_1) \tag{4.22}$$

if X_t^i does not hit the boundary during the time interval $[t_1, t_2]$. We recall the observation that if our distance to the boundary is bounded from below then in order for our particle to die within a sufficiently small time interval, the driving Brownian motion W_t must travel a distance bounded from below in this small time interval. Using Part 2 of Proposition 4.7 take $\delta = \frac{\hat{c}(\epsilon,\delta)}{3} > 0$ so that $d(\hat{K}_{\frac{1}{2},\epsilon}, \partial D) = 3\delta$, and further take $M > \frac{TB}{\delta}$. Thus if $rh \leq \tau_k^i \leq \tau_{k+1}^i \leq (r+1)h$ and $X^i_{\tau_k^i} \in \hat{K}_{\frac{1}{2},\epsilon}$ we must have

$$|W_{(r+1)h\wedge\tau^i_{k+1}-} - W_{\tau^i_k}| \geq 3\delta - Bh \geq 2\delta.$$

Moreover $\mathbb{P}(|W_{(r+1)h\wedge\tau^i_{k+1}-} - W_{\tau^i_k}| \geq 2\delta | \mathcal{F}_{\tau^i_k}) < p$ for some $p < 1$. Therefore at each death time $\tau_k^i \in [rh, (r+1)h]$ with $\tau_k^i \leq \tau_\epsilon^N$ there is a probability at least $\frac{1}{2}$ of jumping to a particle in $\hat{K}_{\frac{1}{2},\epsilon}$ and if this is the case there is then a probability of at least $1 - p > 0$ of this being the final time particle i jumps during the interval $[rh, (r+1)h]$. Therefore J_r can be coupled to a Geometric random variable of success probability $(1-p) \times \frac{1}{2}$ which is independent of $\mathcal{F}_{hr}$ and dominates J_r.

□

5 Ergodicity of the N-particle System (1.3) – Theorem 2.7

The goal of this section is to establish Theorem 2.7, giving ergodicity of the particle system for fixed N. We recall that in Theorem 2.7 we assume D is bounded and path-connected, which we therefore assume in this section. Since N is fixed, we neglect to write it for convenience. We write $G = D^N$ and P_t for the transition semigroup for $\vec{X}$. We recall the Doeblin condition in continuous time

Definition 5.1 (Doeblin Condition, [26]). *There exists $t_* > 0$, $\alpha > 0$ and a probability measure ν such that for any $x \in G$, $P_{t_*}(x, dy) \geq \alpha\nu(dy)$.*

We now recall [26, Corollary 2.7].

Theorem 5.2 ([26]). *Assume that the Doeblin condition in continuous time holds. Then there exists a unique invariant distribution ψ, and moreover we have*

$$||P_t(x,\cdot) - \psi(\cdot)||_{TV} \leq (1-\alpha)^{\lfloor \frac{t}{t_*} \rfloor}, \quad \forall x \in G.$$

Thus it is sufficient to establish the Doeblin condition holds.

Step 1

We define $V_\epsilon = \{x \in D : d(x, \partial D) \geq \epsilon\}$ and $G_\epsilon = V_\epsilon^N$. We fix $0 < \epsilon_1 < \frac{r}{2}$. We shall construct a relatively compact, open and path connected domain K with smooth (C^∞) boundary ∂K such that

$$G_{\epsilon_1} \subseteq K \subset\subset G.$$

In particular if $\vec{x} = (x_1, \ldots, x_N) \in D^N$ then $d(x_i, \partial D) \geq \epsilon_1$ for all i implies $\vec{x} \in K$.

We fix $\vec{x}^* \in G$ and define the function

$$p(\vec{x}) := \sup_{\substack{\gamma:\vec{x}^* \to \vec{x} \\ \text{a path in } G}} d(\gamma, \partial G).$$

Then since p is continuous and positive, there exists $\epsilon' > 0$ such that $p > \epsilon' > 0$ on G_{ϵ_1}. We then define the relatively compact, open, path-connected set K',

$$G_{\epsilon_1} \subseteq K' := \{\vec{x} \in G : p(\vec{x}) > \epsilon'\} \subset\subset G.$$

We now expand K' a bit to obtain an open, path-connected, relatively compact domain K with smooth boundary. There exists $\epsilon'' > 0$ such that $d(K', \partial G) > \epsilon''$. We take $\varphi \in C_c^\infty(\mathbb{R}^{Nd})$ a mollifier supported on the ball $B(0, \frac{\epsilon''}{4})$ so that by Sard's theorem there exists $0 < c < 1$ such that

$$K'' = \{x : \varphi * \mathbb{1}_{K' + B(0, \frac{\epsilon''}{2})} > c\} \supseteq K' \supseteq G_{\epsilon_1}$$

is a relatively compact, open domain with smooth boundary. Thus taking K to be the path-connected component of K'' containing K', we obtain our desired domain.

Step 2

We recall that D satisfies the interior ball condition with radius r. We may by Proposition 4.2 define N i.i.d. Bessel processes, with positive drift B, $\eta^1, \ldots, \eta^N$, such that $r - \eta^i \leq d(X^i, \partial D)$ for each i. Then with probability at least p_1 for some $p_1 > 0$, $\eta_1^1, \ldots, \eta_1^N \leq r - 2\epsilon_1$. This gives us that there exists $p_1 > 0$ such that $P_1(\vec{x}, G_{2\epsilon_1}) \geq p_1$ for all $\vec{x} \in G = D^N$.

Step 3

For $\vec{u} = (u_1, \ldots, u_{Nd}) \in G$ and $\epsilon > 0$ we define

$$F(\vec{u}, \epsilon) = \{(u'_1, \ldots, u'_{Nd}) : |u'_i - u_i| < \epsilon\}.$$

We take $\vec{u} \in G$ and $\epsilon_2 > 0$ such that $F(\vec{u}, 5\epsilon_2) \subseteq G_{2\epsilon_1}$ and fix $C = F(\vec{u}, \epsilon_2)$. We claim that there exists $\delta_2 > 0$ and $p_2 > 0$ such that $P_{\delta_2}(\vec{x}, C) \geq p_2$ for all $\vec{x} \in G_{2\epsilon_1}$.

We let $Q_t(x, \cdot)$ be the transition kernel for Brownian motion started at $\vec{x} \in K$ and killed when it hits ∂K. We can write the SDE for $\vec{X}_t$ between jump times as

$$d\vec{X}_t = \vec{b}(\vec{X}_t)dt + d\vec{W}_t.$$

Since the drift is bounded, and both $d(C^c, F(\vec{u}, \frac{\epsilon_2}{2}))$ and $d(K, \partial G)$ are bounded away from 0, there exists $\delta_2 > 0$ small enough such that for all $x \in K$

$$\vec{x} + \vec{W}_{\delta_2} \in F(\vec{u}, \frac{\epsilon_2}{2}) \text{ and } \vec{x} + \vec{W}_{t'} \text{ does not leave } K \text{ for } t' \leq \delta_2$$
$$\Rightarrow \ \vec{X}_{\delta_2} \in C \text{ and } \vec{X}_{t'} \text{ does not hit } \partial G \text{ for } t' \leq \delta_2.$$

Therefore we have

$$P_{\delta_2}(\vec{x}, C) \geq Q_{\delta_2}(\vec{x}, F(\vec{u}, \frac{\epsilon_2}{2})) \quad \text{for all} \quad x \in G_{2\epsilon_1}.$$

Taking a smooth function $\mathbb{1}_{F(\vec{u}, \frac{\epsilon_2}{4})} \leq \Phi \leq \mathbb{1}_{F(\vec{u}, \frac{\epsilon_2}{2})}$ we see $(t, \vec{x}) \mapsto Q_t(\vec{x}, \Phi)$ is a smooth solution of the heat equation on K with Dirichlet boundary conditions, so by the Maximum principle $\vec{x} \mapsto Q_1(\vec{x}, \mathbb{1}_{F(\vec{u}, \frac{\epsilon_2}{2})})$ is bounded away from 0 on $G_{2\epsilon_1}$. Thus $P_{\delta_2}(\vec{x}, C)$ is bounded away from 0 on $G_{2\epsilon_1}$.

Step 4

Lemma C.1 then implies there exists $p_3 > 0$ such that $P_1(\vec{x}, \cdot) \geq p_3 \text{Leb}_{|C}(\cdot)$ for all $\vec{x} \in C$. Setting $t_* = 1 + \delta_2 + 1$, $\alpha = p_1 p_2 p_3 \text{Leb}(C)$ and $\nu = \frac{1}{\text{Leb}(C)} \text{Leb}_{|C}$ we have established Doeblin's condition.

This completes our proof of Theorem 2.7.

□

6 Density estimate for the Proof of Theorem 2.9

Unlike the previous section, we no longer assume D is path-connected or bounded; here we assume only that D is an open subdomain of $\mathbb{R}^d$ satisfying the uniform interior ball condition – Condition 2.3. We take a sequence of weak solutions to the Fleming-Viot particle system with generalised dynamics

$$(\vec{X}_t^N, \vec{W}_t^N, \vec{b}_t^N)_{0 \leq t < \infty} = ((X_t^{N,1}, \dots, X_t^{N,N}), (W_t^{N,1}, \dots, W_t^{N,N}), (b_t^{N,1}, \dots, b_t^{N,N}))_{0 \leq t < \infty}$$

and with initial conditions $\vec{X}_0^N \sim \upsilon^N$. Moreover the drifts $b_t^{N,i}$ are uniformly bounded with $|b_t^{N,i}| \leq B < \infty$. We define m_t^N and J_t^N as in (1.10) and (1.12)

$$m_t^N = \vartheta^N(\vec{X}_t^N), \quad J_t^N = \frac{1}{N} \sup\{k \in \mathbb{N} \mid \tau_k^N \leq t\}.$$

The goal of this section is to establish the following lemma, which provides for controls on possible subsequential limits.

Lemma 6.1. *For fixed $T < \infty$ we assume that laws of $\{(m_t^N, J_t^N)_{0 \leq t \leq T})\}_{N \geq 2}$ are a tight family of measures on $\mathcal{D}([0,T]; \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0})$ with limit distributions supported on $\mathcal{C}([0,T]; \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0})$. Then for every subsequential limit in distribution $(m_t, J_t)_{0 \leq t \leq T} \in \mathcal{C}([0,T]; \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0})$ we have:*

1. *The random measure m defined by $dm = dm_t dt$ is almost surely absolutely continuous with respect to $\text{Leb}_{D \times [0,T]}$.*
2. *For every $0 < t \leq T$ we almost surely have m_t is absolutely continuous with respect to Leb_D.*

Note that we are not claiming here that almost surely m_t is absolutely continuous with respect to Leb_D for all $0 < t \leq T$.

We focus on the proof of Part 1 of Lemma 6.1 – the proof of Part 2 is the same. We then use the machinery we construct to prove Lemma 6.1 to prove the following lemma.

Lemma 6.2. *We assume that $\{\mathcal{L}(m_0^N)\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D))$. Then for any $T \in [0,\infty)$ we have*

$$\limsup_{N\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} m_t^N(B(0,R)^c)\right] \to 0 \quad \text{as} \quad R \to \infty. \tag{6.1}$$

The proofs of this section shall rely on an analysis of the "Dynamical Historical Processes" defined in [4].

6.1 Dynamical historical processes

The Dynamical Historical Process $(\mathcal{H}_s^{N,i,t})_{0\leq s\leq t}$ is the unique continuous path from time 0 to time t which is equal to one of the particles at all times and equal to X_t^i at time t. We provide a definition of "Dynamical Historical Process" (DHP) which is equivalent to that found in [4], but which will be more useful for our purposes.

We shall define the set of "Chains" $\mathcal{C}^N$ and associate to each $\alpha \in \mathcal{C}^N$ a solution (X^α, W^α) of

$$dX_t^\alpha = b(X_t^\alpha, m_t^N)dt + dW_t^\alpha, \quad 0 \leq t < \tau = \inf\{t : X_{t^-}^\alpha \in \partial D\}, \quad X_0^\alpha = X_0^{i_0(\alpha)},$$

whereby $i_0(\alpha) \in \{1,\dots,N\}$. Each $\alpha \in \mathcal{C}^N$ provides a recipe for a continuous path made from the trajectories of the particle system, killed at the first time it hits ∂D. The index $i_0(\alpha) \in \{1,\dots,N\}$ is the index of the particle whose trajectory X^α "follows" at time 0.

We shall then define for each $1 \leq i \leq N$ a càdlàg $\mathcal{C}^N$-valued process $\alpha_t^{N,i}$ which provides a recipe for the unique continuous path made from the trajectories of the particles finishing with $X_t^{N,i}$ at time t.

Definition 6.3 (Set of Chains $\mathcal{C}^N$)**.** *We define $\mathcal{C}^N$ to be the collection of all "Chains", which we define as*

$$\begin{aligned}\mathcal{C}^N = \{((j_\ell,0),(j_{\ell-1},k_{\ell-1}),\dots,(j_1,k_1),(j_0,k_0)) : j_{\ell'} \in \{1,\dots,N\} \text{ for } \ell' \leq \ell,\\ k_{\ell'} \in \mathbb{N}, j_{\ell'} \neq j_{\ell'+1} \text{ for } \ell' < \ell \text{ and } 0 \leq \ell < \infty\}.\end{aligned}$$

Given $\alpha = ((j_\ell,0),(j_{\ell-1},k_{\ell-1}),\dots,(j_0,k_0)) \in \mathcal{C}^N$ we write $|\alpha| = \ell$ for the "length" of the chain. Thus $\alpha = ((i,0))$ is defined to have length $|\alpha| = 0$.

We now construct (X^α, W^α) for $\alpha \in \mathcal{C}^N$ as follows. Recall from (2.8) that U_k^i is the target index of particle i at its k^{th} jump time. We firstly define the càdlàg processes $(\mathcal{I}_t^\alpha, \Lambda_t^\alpha)_{0\leq t<\infty}$ for $\alpha = ((j_\ell,0),(j_{\ell-1},k_{\ell-1}),\dots,(j_0,k_0))$ according to

$$\begin{cases}\text{Initial Condition:} \quad (\mathcal{I}_0^\alpha, \Lambda_0^\alpha) = (j_\ell, \ell),\\ (\mathcal{I}_t^\alpha, \Lambda_t^\alpha) : (j_r, r) \mapsto (j_{r-1}, r-1) \quad \text{if } t = \tau_{k_{r-1}}^{j_{r-1}} \text{ and } j_r = U_{k_{r-1}}^{j_{r-1}},\\ (\mathcal{I}_t^\alpha, \Lambda_t^\alpha) \quad \text{is constant otherwise.}\end{cases}$$

We then define

$$\begin{aligned}X_t^\alpha = X_t^{\mathcal{I}_t^\alpha}, \quad 0 \leq t < \tau^\alpha = \inf\{t > 0 : X_{t^-}^\alpha \in \partial D\},\\ dW_t^\alpha := dW_t^{\mathcal{I}_t^\alpha}, \quad 0 \leq t < \infty, \quad W_0^\alpha = 0.\end{aligned}$$

We see that X^α must satisfy the SDE

$$dX_t^\alpha = b(m_t^N, X_t^\alpha)dt + dW_t^\alpha, \quad 0 \leq t < \tau^\alpha = \inf\{t > 0 : X_{t^-}^\alpha \in \partial D\}, \quad X_0^\alpha := X_0^{i_0(\alpha)}, \tag{6.2}$$

whereby $i_0(\alpha) := \mathcal{I}_0^\alpha$.

We now define the Dynamical Historical Process.

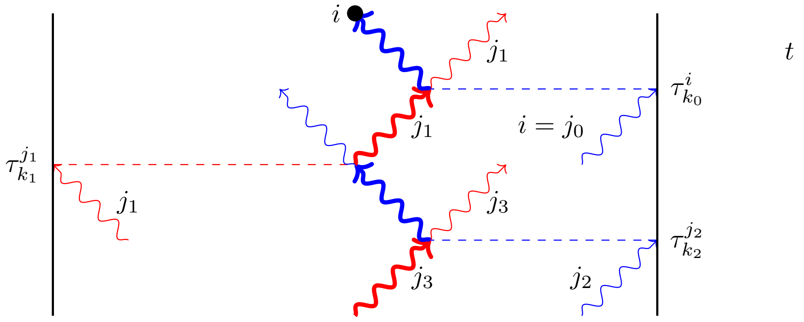


Figure 2: The continuous thick path denotes the path of the DHP corresponding to particle $X^{N,i}$ at time t.

Definition 6.4 (Dynamical Historical Processes)**.** *Given* $\alpha = ((j_\ell, 0), (j_{\ell-1}, k_{\ell-1}), \ldots, (j_1, k_1))$ $\in \mathcal{C}^N$ *and* (j_0, k_0) *with* $j_0 \neq j_1$ *we set*

$$\alpha \oplus (j_0, k_0) = ((j_\ell, 0), (j_{\ell-1}, k_{\ell-1}), \ldots, (j_1, k_1), (j_0, k_0)).$$

We then define the $\mathcal{C}^N$*-valued processes* $\alpha^{N,i}_t$ *(*$i = 1, \ldots, N$*):*

1. *At time* 0 *we define*
$$\alpha^{N,i}_0 = (i, 0).$$
2. *Between death times of* X^i*,* $\alpha^{N,i}_t$ *is constant:*
$$\alpha^{N,i}_t = \alpha^{N,i}_{\tau^i_k}, \quad \tau^i_k \leq t < \tau^i_{k+1}.$$
3. *At time* τ^i_k *if* $U^i_k = j$ *then we set*
$$\alpha^{N,i}_{\tau^i_k} = \alpha^{N,j}_{\tau^i_k -} \oplus (i, k). \tag{6.3}$$

Then we note that by construction $\tau^{\alpha^{N,i}_t} > t$*. We may now define the Dynamical Historical Processes of* $X^{N,1}, \ldots, X^{N,N}$ *by*

$$\mathcal{H}^{N,i,t}_s := X^{\alpha^{N,i}_t}_s, \quad 0 \leq s \leq t. \tag{6.4}$$

We say that the DHP $\mathcal{H}^{N,i,t}$ follows particle j at time s if $\mathcal{I}^{\alpha^{N,i}_t}_s = j$. Thus in Figure 2 the DHP $\mathcal{H}^{N,i,t}$ follows particle j_3 at time 0 and particle i at time t. We let $R^i_t \geq 0$ be the index of the most recent jump time of particle i:

$$R^i_t = \max\{k \geq 0 \mid \tau^i_k \leq t\},$$

with the convention that $\tau^i_0 = 0$ so that $\mathcal{H}^{N,i,t}_s$ follows particle i at time s for $s \in [\tau^i_{R^i_t}, t]$.

6.2 Proof of part 1 of Lemma 6.1

Without loss of generality, suppose that $(m^N_t, J^N_t)_{0 \leq t \leq T}$ converges in distribution on $\mathcal{D}([0, T]; \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0})$ to $(m_t, J_t)_{0 \leq t \leq T} \in \mathcal{C}([0, T]; \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0})$, as $N \to \infty$ (or along a subsequence). We will write

$$m = m_t \otimes dt, \quad dm = dm_t dt, \quad m^N = m^N_t \otimes dt, \quad dm^N = dm^N_t dt. \tag{6.5}$$

Our goal is to show that, $\mathbb{P}$-almost surely, the random measure $m = m_t \otimes dt$ is absolutely continuous with respect to $\text{Leb}_{D\times[0,T]}$.

For $\vec{h} = (h_1, \dots, h_d) \in \mathbb{R}^d_{>0}$ and $\vec{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ we define the rectangle

$$R_{\vec{h}}(\vec{x}) = (x_1 - h_1, x_1 + h_1) \times \dots \times (x_d - h_d, x_d + h_d). \tag{6.6}$$

Define $\mathcal{A} = \{R_{\vec{h}}(\vec{x}) \times [t_0, t_1] : t_0, t_1 \in \mathbb{Q},\ 0 < t_0 \le t_1,\ \vec{x} \in \mathbb{Q}^d,\ \vec{h} \in \mathbb{Q}^d_{>0}\}$ and take $\mathcal{R}$ to be the set of finite unions of sets in $\mathcal{A}$ (note that $\mathcal{R}$ is a countable collection of sets). For $E \in \mathcal{B}(D \times (0,T]) \setminus \{\emptyset\}$, define $T_{\min}(E) = \inf\{t : (x,t) \in E\}$. For $\rho \in \mathcal{R}$ we define

$$\rho_t := \{x : (x,t) \in \rho\}.$$

Our proof of the almost-sure absolute continuity of the random measure m begins with the following two lemmas.

Lemma 6.5. *Fix $T < \infty$ and suppose that we have a random measure $m \in \mathcal{P}(D \times [0,T])$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $m(D \times \{0\}) = 0$ holds $\mathbb{P}$-almost surely. We further assume that for every $\epsilon > 0$ there exists a non-increasing function $C_\epsilon : (0,T] \to \mathbb{R}_{\ge 0}$ such that*

$$\mathbb{E}\left[0 \vee \sup_{\rho\in\mathcal{R}} \big(m(\rho) - C_\epsilon(T_{\min}(\rho))\mathit{Leb}(\rho)\big)\right] \le \epsilon. \tag{6.7}$$

Then $m << \mathit{Leb}_{D\times[0,T]}$ holds $\mathbb{P}$-almost surely.

The proof of Lemma 6.5 is given later in Appendix A. We note that (6.7) is a property of the law of the random measure m. Therefore, by Skorokhod's representation theorem, we could assume the convergence of $(m^N_t, J^N_t)_{0\le t\le T}$ to $(m_t, J_t)_{0\le t\le T}$ holds almost surely on a possibly different probability space $(\Omega^{a.s.}, \mathcal{F}^{a.s.}, \mathbb{P}^{a.s.})$.

Lemma 6.6. *Suppose that, on the probability space $(\Omega^{a.s.}, \mathcal{F}^{a.s.}, \mathbb{P}^{a.s.})$, some random variables $\{(m^N_t)_{0\le t\le T}\}$ converge in $\mathcal{D}([0,T]; \mathcal{P}_W(D))$ as $N \to \infty$, $\mathbb{P}^{a.s.}$-almost surely, to $(m_t)_{0\le t\le T} \in \mathcal{C}([0,T]; \mathcal{P}_W(D))$. Then, for all $\rho \in \mathcal{R}$, we $\mathbb{P}^{a.s.}$-almost surely have*

$$\int_0^T m_t(\rho_t)dt \le \liminf_{N\to\infty} \int_0^T m^N_t(\rho_t)\, dt.$$

Proof. Since $(m_t)_{0\le t\le T} \in \mathcal{C}([0,T]; \mathcal{P}_W(D))$, by assumption, we know that $(m^N_t)_{0\le t\le T}$ converges to $(m_t)_{0\le t\le T}$ with respect to the uniform (in W) metric. So, $\mathbb{P}^{a.s.}$-almost surely we have

$$m_t(\rho_t) \le \liminf_{N\to\infty} m^N_t(\rho_t)$$

for every $t > 0$ by the Portmenteau Theorem and the fact ρ_t is an open set. From this fact and Fatou's lemma, we infer that, $\mathbb{P}^{a.s}$-almost surely,

$$\int_0^T m_t(\rho_t)dt \le \int_0^T \liminf_{N\to\infty} m^N_t(\rho_t)dt \le \liminf_{N\to\infty} \int_0^T m^N_t(\rho_t)dt. \tag{6.8}$$

□

So, to verify the condition (6.7) for the limit measure m, we turn our attention to estimating $m^N(\rho)$. Whereas Lemma 6.6 requires almost-sure convergence, the construction we will use to obtain controls on $m^N(\rho)$ doesn't necessarily make sense on such a new probability space obtained with Skorokhod's representation theorem. We will therefore obtain controls on $m^N(\rho)$ working on our original filtered probability space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We will then transfer these controls to controls on the limit by way of Skorokhod's representation theorem and Lemma 6.6.

Working for the time being on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, we now turn our attention to estimating:

$$m^N(\rho) = \int_0^T m_t^N(\rho_t)\,dt = \frac{1}{N}\sum_{i=1}^N \int_0^T \mathbb{1}(X_t^{N,i} \in \rho_t)\,dt.$$

Estimating this quantity involves bounding the number of particles in a given set ρ_t at time t. It is straightforward to do this with pure diffusions. In our system, however, the jumps make this estimate more difficult.

Recalling the definition of the Dynamical Historical Process $\mathcal{H}_s^{N,i} = X_s^{\alpha_t^{N,i}}$, for $s \in [0,t]$, we let $G_t^{\ell,n,i}$ be the event that

$$|\alpha_t^{N,i}| \leq \ell \tag{6.9}$$

and

$$J_s^{N,\mathcal{I}_s^{\alpha_t^{N,i}}} \leq n, \quad \forall\ \ s \in [0,t]. \tag{6.10}$$

The first condition says that the DHP makes no more than ℓ "transfers", and the second says that if the DHP $\mathcal{H}_s^{N,i,t}$ is following particle j at time s, then particle j has made no more than n jumps up to time s. We recall that

$$X_t^{N,i} = \mathcal{H}_t^{N,i,t}, \quad 0 \leq t \leq T, \quad 1 \leq i \leq N. \tag{6.11}$$

Now we bound $m^N(\rho)$ by

$$m^N(\rho) = \frac{1}{N}\sum_{i=1}^N \int_0^T \mathbb{1}(X_t^{N,i} \in \rho_t)\mathbb{1}(G_t^{\ell,n,i})\,dt + \frac{1}{N}\sum_{i=1}^N \int_0^T \mathbb{1}(X_t^{N,i} \in \rho_t)\mathbb{1}((G_t^{\ell,n,i})^C)dt \tag{6.12}$$

$$\leq \frac{1}{N}\sum_{i=1}^N \int_0^T \mathbb{1}(\mathcal{H}_t^{N,i,t} \in \rho_t)\mathbb{1}(G_t^{\ell,n,i})\,dt + \sup_{t\in[0,T]} \frac{T}{N}\sum_{i=1}^N \mathbb{1}((G_t^{\ell,n,i})^C). \tag{6.13}$$

We write $S_1^{N,\ell,n}(\rho)$ and $S_2^{N,\ell,n}$ for the two terms in (6.13):

$$S_1^{N,\ell,n}(\rho) = \frac{1}{N}\sum_{i=1}^N \int_0^T \mathbb{1}(\mathcal{H}_t^{N,i,t} \in \rho_t)\mathbb{1}(G_t^{\ell,n,i})dt$$

and

$$S_2^{N,\ell,n} = \sup_{t\in[0,T]} \frac{T}{N}\sum_{i=1}^N \mathbb{1}((G_t^{\ell,n,i})^C). \tag{6.14}$$

In particular, notice that $S_2^{N,\ell,n}$ does not depend on the set ρ.

For $\ell, n \in \mathbb{N}$ fixed, we will prove later (Section 6.2.1) that there exists $C_{\ell,n} : (0,T] \to \mathbb{R}_{\geq 0}$ a non-increasing function such that for all $\rho \in \mathcal{R}$

$$C_{\ell,n}(T_{\min}(\rho))\text{Leb}(\rho) \vee S_1^{N,\ell,n}(\rho) \xrightarrow{p} C_{\ell,n}(T_{\min}(\rho))\text{Leb}(\rho) \tag{6.15}$$

as $N \to \infty$. In addition to this, we will prove later (Section 6.2.2) that for any $\epsilon > 0$, we may choose $\ell = \ell(\epsilon)$ and $n = n(\epsilon)$ such that

$$\limsup_{N\to\infty} \mathbb{E}[S_2^{N,\ell,n}] \leq \epsilon. \tag{6.16}$$

Clearly, the random variables $S_2^{N,\ell,n}$ are uniformly bounded: $|S_2^{N,\ell,n}| \leq T$. In particular, for fixed ℓ and n, the laws of $\{S_2^{N,\ell,n}\}_{N\geq 2}$ are a tight family. Therefore, there is a random variable G^ϵ so that along a subsequence, $S_2^{N,\ell,n} \to G^\epsilon$ in distribution as $N \to \infty$. By (6.16), $\mathbb{E}[G^\epsilon] \leq \epsilon$ must hold.

By the Skorokhod representation theorem, we may for fixed $\epsilon > 0$ assume that both

$$(m_t^N)_{0\leq t\leq T} \to (m_t)_{0\leq t\leq T} \quad \text{and} \quad S_2^{N,\ell,n} \to G^\epsilon \tag{6.17}$$

hold almost surely (perhaps on a new probability space $(\Omega^{a.s.}, \mathcal{F}^{a.s.}, \mathbb{P}^{a.s.})$, which does not depend on ρ). From (6.13) and (6.15) we have for any $\rho \in \mathcal{R}$ and $\delta > 0$ that

$$\mathbb{P}^{a.s.}(m^N(\rho) - S_2^{N,\ell,n} \geq C_{\ell,n}\text{Leb}(\rho) + \delta) \to 0 \quad \text{as} \quad N \to \infty. \tag{6.18}$$

(The quantities in (6.18) are all defined on the probability space $(\Omega^{a.s}, \mathcal{F}^{a.s.}, \mathbb{P}^{a.s.})$). Using Lemma 6.6 and (6.17) we have

$$\liminf_{N\to\infty} \big(m^N(\rho) - S_2^{N,\ell,n}\big) \geq m(\rho) - G^\epsilon, \quad \mathbb{P}^{a.s.}\text{-almost surely.}$$

Therefore for every $\rho \in \mathcal{R}$ and $\delta > 0$, using (6.18) and Fatou's lemma we have

$$\mathbb{P}^{a.s.}(m(\rho) - G^\epsilon > C_{\ell,n}\text{Leb}(\rho) + \delta) \leq \liminf_{N\to\infty} \mathbb{P}^{a.s.}(m^N(\rho) - S_2^{N,\ell,n} > C_{\ell,n}\text{Leb}(\rho) + \delta) = 0. \tag{6.19}$$

Therefore, since $\delta > 0$ is arbitrary and $\mathcal{R}$ is countable, this implies

$$\sup_{\rho\in\mathcal{R}} \big(m(\rho) - C_{\ell,n}\text{Leb}(\rho)\big) \leq G^\epsilon \quad \mathbb{P}^{a.s.}\text{-almost surely.}$$

We finally note that

$$\mathbb{E}^{\mathbb{P}^{a.s.}}\big[\sup_{\rho\in\mathcal{R}} \big(m(\rho) - C_{\ell,n}(T_{\min}(\rho))\text{Leb}(\rho)\big)\big] \leq \epsilon$$

is a statement about the distribution of m, so must also hold true under $\mathbb{P}$. Except for the proof of (6.15) and (6.16), this establishes condition (6.7) in Lemma 6.5 and completes the proof of of Part 1 of Lemma 6.1. The rest of this section is devoted to the proofs of (6.15) and (6.16).

6.2.1 Proof of (6.15)

The following lemma will be a key tool for controlling the density of diffusions with bounded drift. We write $\vec{n}(\vec{x})$ ($\vec{x} \in \partial\mathbb{R}^d_{>0}$) for the inward normal of the positive orthant $\mathbb{R}^d_{>0}$ and consider strong solutions of the SDE

$$dY_t = (-B, \ldots, -B)dt + d\tilde{W}_t + \vec{n}(Y_t)dL_t^Y, \quad Y_0 = 0, \tag{6.20}$$

where L_t^Y is the local time of Y_t with the boundary $\partial\mathbb{R}^d_{>0}$. This is a normally reflected diffusion in $\mathbb{R}^d_{>0}$ with constant drift. Recall $R_{\vec{h}}(\vec{0})$ defined at (6.6).

Lemma 6.7. *Consider on some filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}')$ the family of $\mathbb{R}^d$-valued weak solutions (X^γ, W^γ) $(\gamma \in \Gamma)$ of the SDE*

$$dX_t^\gamma = b_t^\gamma dt + dW_t^\gamma, \quad 0 \leq t < \infty, \tag{6.21}$$

whereby $|b^\gamma| \leq B$ is $(\mathcal{F}'_t)_{t\geq 0}$-adapted. Then there exists on $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}')$ a family of identically distributed strong solutions $(Y^\gamma, \tilde{W}^\gamma)$ to (6.20) which satisfy the following:

1. *X^γ dominates Y^γ so that*

$$X_t^\gamma \in R_{\vec{h}}(\vec{0}) \;\Rightarrow\; Y_t^\gamma \in R_{\vec{h}}(\vec{0}), \quad t \geq 0,\ \vec{h} \in \mathbb{R}^d_{>0} \tag{6.22}$$

whereby $\vec{0} = (0, \ldots, 0)$.

2. *We have explicit controls on the density of Y^{γ} so that there exists $C:(0,\infty)\to\mathbb{R}_{>0}$ non-increasing such that*

$$\mathbb{P}(Y^{\gamma}_t\in R_{\vec{h}}(\vec{0}))\leq C_t\textit{Leb}(R_{\vec{h}}(\vec{0})). \tag{6.23}$$

3. *For any event $A\in\mathcal{F}'_0$ and $\gamma_1,\gamma_2\in\Gamma$: if conditional upon the event A, W^{γ_1} and W^{γ_2} are conditionally independent, then so too are Y^{γ_1} and Y^{γ_2}.*

We will use Lemma 6.7 in Appendix C to prove Lemma C.1, providing controls on the density of a diffusion for generic bounded drift, which shall be used throughout this paper.

We consider the possibilities for $\alpha^{N,i}_t$ given $G^{\ell,n,i}_t$. The condition that $J^{N,\mathcal{I}^{\alpha^{N,i}_t}}_s\leq n$ for all $0\leq s\leq t$ then allows us to see that for each transfer of the DHP from particle j to particle k, j is within the first n particles k jumps onto. Therefore to obtain all possibilities for $\alpha^{N,i}_t$ given $G^{\ell,n,i}_t$, it is sufficient to consider the first n particles i jumps onto, the first n particles each of these children jumps onto, and repeating this ℓ times to obtain all possibilities for $\alpha^{N,i}_t$; these possibilities form a tree structure. We take $\Pi^{\ell,n}$ to be a perfect n-ary tree of length ℓ and construct a random injective function

$$\hat{\alpha}^{N,\ell,n}_i:\Pi^{\ell,n}\to\mathcal{C}^N$$

with image $\mathcal{C}^{N,\ell,n}_i\subseteq\mathcal{C}^N$. This random function shall be such that

$$G^{\ell,n,i}_t\subseteq\{\alpha^{N,i}_t=\hat{\alpha}^{N,\ell,n}_i(v)\quad\text{for some}\quad v\in\Pi^{\ell,n}\} \tag{6.24}$$

and such that

$$\hat{\alpha}^{N,\ell,n}_i(v)\quad\text{is}\quad\sigma(U^i_k:1\leq i\leq N,\ k\geq 0)\text{-measurable.} \tag{6.25}$$

We then define $\mathcal{T}^{N,\ell,n}_i$ to be the following $\{1,\dots,N\}^{\Pi^{\ell,n}}$-valued random variable

$$\mathcal{T}^{N,\ell,n}_i(v)=j_r\quad\text{whereby}\quad\hat{\alpha}^{N,\ell,n}_i(v)=((j_r,0),\dots). \tag{6.26}$$

$\mathcal{T}^{N,\ell,n}_i$ assigns the root of $\Pi^{\ell,n}$ to i, assigns the k^{th} child of the root to the k^{th} particle i jumps onto, and so forth. We then define

$$\mathcal{G}^{N,\ell,n}_i=\text{Image}(\mathcal{T}^{N,\ell,n}_i)$$

to be the collection of all particles given by $\mathcal{T}^{N,\ell,n}_i$ at some branch of $\Pi^{\ell,n}$. Thus $\mathcal{G}^{N,\ell,n}_i$ is the collection of all particles which may be followed by $X^{\hat{\alpha}^{N,\ell,n}_i(v)}$ for some $v\in\Pi^{\ell,n}$.

We define a new filtered probability space $(\Omega,\mathcal{F},(\bar{\mathcal{F}}_t)_{t\geq 0},\mathbb{P})$ given by the initial enlargement

$$\bar{\mathcal{F}}_t=\mathcal{F}_t\wedge\sigma(U^i_k:1\leq i\leq N,\ k\geq 0). \tag{6.27}$$

We note the following:

1. This new filtered probability space has the same sigma-algebra as our previous probability space $(\Omega,\mathcal{F},\mathbb{P})$. Thus any random variable we define on this new sigma-algebra is defined on our previous probability space and vice-versa – only the adaptedness properties with respect to the filtration may change.
2. Since $(\mathcal{F}_t)_{t\geq 0}$ is a subfiltration of $(\bar{\mathcal{F}}_t)_{t\geq 0}$ any $(\mathcal{F}_t)_{t\geq 0}$-adapted process is $(\bar{\mathcal{F}}_t)_{t\geq 0}$-adapted.
3. The Brownian motion W^i is independent of $\sigma(U^i_k:1\leq i\leq N,\ k\geq 0)$, hence an $(\bar{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion. Moreover since $\hat{\alpha}^{N,\ell,n}_i(v)$ is $\bar{\mathcal{F}}_0$-measurable we have

$$W^{\hat{\alpha}^{N,\ell,n}_i(v)}_t:=\int_0^t dW^{\mathcal{I}^{\hat{\alpha}^{N,\ell,n}_i(v)}_s}_s,\quad 0\leq t<\infty,\quad W^{\hat{\alpha}^{N,\ell,n}_i(v)}_0=0$$

is an $(\bar{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion.

4. If the set of particles $\hat{\alpha}_i^{N,\ell,n}(v)$ follows is disjoint from those followed by $\hat{\alpha}_j^{N,\ell,n}(v')$ – hence if $\mathcal{G}_i^{N,\ell,n}\cap\mathcal{G}_j^{N,\ell,n}=\emptyset$ – then $W^{\hat{\alpha}_i^{N,\ell,n}(v)}$ and $W^{\hat{\alpha}_j^{N,\ell,n}(v')}$ have zero covariation. Therefore conditional on the event
$$A_{i,j}^{N,\ell,n}=\{\mathcal{G}_i^{N,\ell,n}\cap\mathcal{G}_j^{N,\ell,n}=\emptyset\}\in\bar{\mathcal{F}}_0,$$
$W^{\hat{\alpha}_i^{N,\ell,n}(v)}$ and $W^{\hat{\alpha}_j^{N,\ell,n}(v')}$ must be independent.

For every $\rho\in\mathcal{R}$ we fix a finite index set ι_ρ such that ρ is given by the union
$$\rho=\cup_{\beta\in\iota_\rho}[t_0^\beta,t_1^\beta]\times R_{\vec{h}_\beta}(\vec{x}_\beta) \tag{6.28}$$
whereby
$$\sum_{\beta\in\iota_\rho}\text{Leb}([t_0^\beta,t_1^\beta]\times R_{\vec{h}_\beta}(\vec{x}_\beta))\leq 2\text{Leb}(\rho). \tag{6.29}$$

For each $1\leq i\leq N$, $v\in\Pi^{\ell,n}$, $\beta\in\iota_\rho$ we apply Lemma 6.7 to $\{X^{\hat{\alpha}_i^{N,\ell,n}(v)}-\vec{x}_\beta\}$ to construct $Y_t^{i,v,\beta}$ and define
$$\eta_i^{N,\ell,n,\rho}:=\sum_{\substack{\beta\in\iota_\rho\\ v\in\Pi^{\ell,n}}}\int_{t_0^\beta}^{t_1^\beta}\mathbb{1}(Y_t^{i,v,\beta}\in R_{\vec{h}_\beta}(\vec{0}))dt,\quad \rho\in\mathcal{R}. \tag{6.30}$$

We therefore have
$$\begin{aligned}S_1^{N,l,n}(\rho)&=\frac{1}{N}\sum_{i=1}^N\int_0^T\mathbb{1}(X_t^{\alpha_t^{N,i}}\in\rho_t)\mathbb{1}_{G_t^{\ell,n,i}}dt\leq\frac{1}{N}\sum_{i=1}^N\sum_{\beta\in\iota_\rho}\int_{t_0^\beta}^{t_1^\beta}\mathbb{1}(X_t^{\alpha_t^{N,i}}\in R_{\vec{h}_\beta}(\vec{x}_\beta))\mathbb{1}_{G_t^{\ell,n,i}}dt\\&\underbrace{\leq}_{\text{by (6.24)}}\frac{1}{N}\sum_{i=1}^N\sum_{\beta\in\iota_\rho}\frac{1}{N}\sum_{i=1}^N\sum_{v\in\Pi^{\ell,n}}\int_{t_0^\beta}^{t_1^\beta}\mathbb{1}(X_t^{\hat{\alpha}_i^{N,\ell,n}(v)}\in R_{\vec{h}_\beta}(\vec{x}_\beta))\mathbb{1}_{G_t^{\ell,n,i}}dt\\&\underbrace{\leq}_{\text{by (6.22)}}\frac{1}{N}\sum_{i=1}^N\sum_{\substack{\beta\in\iota_\rho\\ v\in\Pi^{\ell,n}}}\int_{t_0^\beta}^{t_1^\beta}\mathbb{1}(Y_t^{i,v,\beta}\in R_{\vec{h}_\beta}(\vec{0}))dt=\frac{1}{N}\sum_{i=1}^N\eta_i^{N,\ell,n,\rho}.\end{aligned}$$

We conclude our proof of (6.15) by establishing the following lemma and verifying $\{\eta_i^{N,l,n,\rho}:1\leq i\leq N\}$ satisfies the conditions of this lemma with $M=C_{\ell,n}(T_{\min}(\rho))\text{Leb}(\rho)$.

Lemma 6.8. *Let $\{\gamma_k^N:1\leq k\leq N\in\mathbb{N}\}$ be a triangular array of random variables, and let $S_N=\sum_{k\leq N}\gamma_k^N$. We suppose that the γ_k^N are uniformly bounded, that $\sup_{j\neq k}$ Cov$(\gamma_j^N,\gamma_k^N)\to 0$ as $N\to\infty$, and that $\limsup_{N\to\infty}\sup_{1\leq j\leq N}\mathbb{E}[\gamma_j^N]\leq M$. Then we have $\frac{S_N}{N}\vee M\to M$ in probability.*

6.2.2 Proof of Lemma 6.7

We firstly construct $Y^\gamma,\tilde{W}^\gamma$ for $\gamma\in\Gamma$. We write $X_t^{\gamma,d'}$ for the $d'^{\,th}$ coordinate of $X_t^{\gamma,d'}$ for $1\leq d'\leq d$. We take the Doob-Meyer decomposition of $|X_t^{\gamma,d'}|$, obtaining it as the sum of a Brownian motion $\tilde{W}_t^{\gamma,d'}$, a drift ($\leq B$) term and a local time term up to the time τ^γ. We then write
$$\tilde{W}_t^\gamma=(\tilde{W}_t^{\gamma,1},\dots,\tilde{W}_t^{\gamma,d})$$
and continue $\tilde{W}_t^\gamma$ after the time τ^γ by setting $d\tilde{W}_t^\gamma=dW_t^\gamma$. It is then immediate that there exists an $(\mathcal{F}'_t)_{t\geq 0}$-adapted $d\times d$ signature matrix-valued process K_t^γ such that $\tilde{W}^\gamma$ satisfies
$$d\tilde{W}_t^\gamma=K_t^\gamma dW_t^\gamma. \tag{6.31}$$

Having constructed $(\mathcal{F}'_t)_{t\geq 0}$-Brownian motions $\tilde{W}^{\gamma,d'}$, we have $(\mathcal{F}'_t)_{t\geq 0}$- adapted strong solutions $(Y^{\gamma,d'},\tilde{W}^{\gamma,d'})$ of the SDE

$$dY_t = d\tilde{W}_t - Bdt + dL^Y_t,\quad Y_0=0 \tag{6.32}$$

(which exists by [3, Theorem 1.3]). Thus $(Y^{\gamma},W^{\gamma})=((Y^{\gamma,1},\dots,Y^{\gamma,d}),W^{\gamma})$ is a strong solution to (6.20). Now we observe that for some $|b^{\gamma,d'}_t|\leq B$ we have

$$d(|X^{\gamma,d'}_t|-Y^{\gamma,d'}_t)=(B-b^{\gamma,d'}_t)dt+dL^{|X^{\gamma,d'}|}-dL^{Y^{\gamma,d'}}_t,\quad t<\tau^{\gamma}.$$

Hence by the same proof that $\eta_t\geq E_t$ in the proof of Step 1d of Proposition 4.2 we have $|X^{\gamma,d'}_t|\geq Y^{\gamma,d'}_t$ for all $t<\tau^{\gamma}$. This immediately implies (6.22).

We now control the expectation, showing that there exists $C:(0,T]\rightarrow\mathbb{R}_{\geq 0}$ non-increasing such that for all $\vec{h}\in\mathbb{R}^d_{>0}$ and $\gamma\in\Gamma$ we have (6.23). We have [1, Equation (1.1)] an explicit expression for the cumulative density function of reflected Brownian motion with constant negative drift reflected at 0. Differentiating [1, Equation (1.1)] in y we have that for some $c<\infty$ the transition density satisfies

$$p_t(x,y)\leq\frac{c}{\sqrt{t}}.$$

Therefore $\mathbb{P}(Y^{\gamma,d'}_t\in[0,h])\leq\frac{c}{\sqrt{t}}h$ for $t>0$, $h\geq 0$ and $1\leq d'\leq d$.

We use (6.31) to see that $\tilde{W}^{\gamma,d_1}$ and $\tilde{W}^{\gamma,d_2}$ are pairwise independent Brownian motions for $d_1\neq d_2$ and hence jointly independent. Therefore $\{Y^{\gamma,d'}:1\leq d'\leq d\}$ are independent as they are measurable functions of independent Brownian motions. Thus we have

$$\mathbb{P}(Y^{\gamma}_t\in R_{\vec{h}}(\vec{0}))=\prod_{1\leq d'\leq d}\mathbb{P}(Y^{\gamma,d'}_t\in[0,h_{d'}])\leq\frac{c^d}{t^{\frac{d}{2}}}\mathrm{Leb}(R_{\vec{h}}(\vec{0})).$$

Finally we observe that for any event $A\in\mathcal{F}'_0$ and $\gamma_1,\gamma_2\in\Gamma$; if conditional upon the event A, W^{γ_1} and W^{γ} are conditionally independent; then they must have zero covariation. Using (6.31) we see that $\tilde{W}^{\gamma_1}$ and $\tilde{W}^{\gamma_2}$ must also have zero covariation, hence be conditionally independent. Therefore upon the event A, Y^{γ_1} and Y^{γ_2} are independent as they are measurable functions of independent Brownian motions.

□

Construction of $\hat{\alpha}^{N,l,n}_i$

We define the random function $\hat{\alpha}^{N,l,n}_i$ by firstly defining its image:

Definition 6.9. *We define $\mathcal{C}^{N,\ell,n}_i\subseteq\mathcal{C}^N$ by*

$$\begin{aligned}\mathcal{C}^{N,\ell,n}_i=\Big\{&((j_{\ell'},0),(j_{\ell'-1},k_{\ell'-1}),\dots,(j_1,k_1),(j_0,k_0))\in\mathcal{C}^N:\\ &U^{j_r}_{k_r}=j_{r+1}\text{ whereby }k_r\leq n\text{ for all }r<\ell',\ \ell'\leq\ell,\ j_0=i\Big\}.\end{aligned}\tag{6.33}$$

We now parametrise the elements of $\mathcal{C}^{N,\ell,n}_i$ as follows. We define $\Pi^{\ell,n}$ to be a perfect n-ary tree of length ℓ.

Definition 6.10 ($\Pi^{\ell,n}$)**.** *We define $\Pi^{\ell,n}$ to be a perfect n-ary tree of length ℓ (so that each leaf is of depth ℓ with the root defined to be of depth 0). We adopt standard Ulam-Harris notation, writing $\emptyset$ for the root of $\Pi^{\ell,n}$, (k_0) for the k_0^{th} child of $\emptyset$ ($k_0\leq n$) and recursively defining $(k_0,\dots,k_r,k_{r+1})$ to be the k_{r+1}^{th} child of $(k_0,\dots,k_r)$ (for $r\leq\ell-2$ and $k_r\leq n$).*

Note that the leaves of this tree terminate with an $\ell-1$ subscript: $(k_0, k_1, \ldots, k_{\ell-1})$. Then we see that the random map

$$\iota_i^{N,\ell,n} : \mathcal{C}_i^{N,\ell,n} \to \Pi^{\ell,n}, \quad (i,0) \mapsto \emptyset$$
$$((j_r,0),(j_{r-1},k_{r-1}),\ldots,(j_1,k_1),(i,k_0)) \mapsto (k_0,k_1,\ldots,k_{r-1}), \quad 1 \le r \le \ell,$$

is bijective. To see that $\iota_i^{N,\ell,n}$ is surjective, fix some $(k_0,\ldots,k_{\ell'-1}) \in \Pi^{\ell,n}$ and recursively define $j_{r+1} = U_{k_r}^{j_r}$ ($r < \ell'$), $j_0 = i$. Then we see $\iota_i^{N,\ell,n}(((j_{\ell'},0),(j_{\ell'-1},k_{\ell'-1}),\ldots,(j_1,k_1),(i,k_0))) = (k_0,k_1,\ldots,k_{\ell'-1})$ whereby $((j_{\ell'},0),(j_{\ell'-1},k_{\ell'-1}),\ldots,(j_1,k_1),(i,0)) \in \mathcal{C}_i^{N,\ell,n}$. To see that $\iota_i^{N,\ell,n}$ is injective, suppose that $\iota_i^{N,\ell,n}(((j_{\ell'},0),(j_{\ell'-1},k_{\ell'-1}),\ldots,(j_1,k_1),(i,k_0))) = (k_0,k_1,\ldots,k_{\ell'-1})$ ($\ell' \le \ell$). Then we must have $j_1 = U_{k_0}^i$ and $j_{r+1} = U_{k_r}^{j_r}$ for $r < \ell'$. This uniquely defines

$$((j_{\ell'},0),(j_{\ell'-1},k_{\ell'-1}),\ldots,(j_1,k_1),(i,k_0)).$$

Thus we can take the inverse of $\iota_i^{N,\ell,n}$, parametrising the elements of $\mathcal{C}_i^{N,\ell,n}$ with $\Pi^{\ell,n}$,

$$\Pi^{\ell,n} \to \mathcal{C}_i^{N,\ell,n}, \quad v \mapsto \hat{\alpha}_i^{N,\ell,n}(v), \quad \hat{\alpha}_i^{N,\ell,n}(\emptyset) = (i,0)$$
$$\hat{\alpha}_i^{N,\ell,n}((k_0,k_1,\ldots,k_{\ell'-1})) = ((j_{\ell'},0),(j_{\ell'-1},k_{\ell'-1}),\ldots,(j_1,k_1),(i,k_0)).$$

This shows that $\hat{\alpha}_i^{N,\ell,n} : \Pi^{\ell,n} \to \mathcal{C}^N$ is a random injection with image $\mathcal{C}_i^{N,\ell,n}$.

Proving and verifying the conditions of Lemma 6.8

Proof of Lemma 6.8. Clearly $\frac{S_N}{N} - \frac{1}{N}\sum_{j=1}^N \mathbb{E}[\gamma_j^N]$ has zero expectation, so we now show it has variance converging to zero. Since the γ_k^N are uniformly bounded, so are $\mathrm{Var}(\gamma_k^N)$. We therefore have

$$\begin{aligned}\mathrm{Var}\Big(\frac{S_N}{N} - \frac{1}{N}\sum_{j=1}^N \mathbb{E}[\gamma_j^N]\Big) &= \frac{1}{N^2}\sum_{k=1}^N \mathrm{Var}(\gamma_k^N) + \sum_{j\neq k}\mathrm{Cov}(\gamma_j^N,\gamma_k^N) \\ &\le \underbrace{\frac{N}{N^2}\sup_k \mathrm{Var}(\gamma_k^N)}_{\to 0} + \underbrace{\frac{N^2-N}{N^2}\sup_{j\neq k}\mathrm{Cov}(\gamma_j^N,\gamma_k^N)}_{\to 0} \to 0.\end{aligned}$$

Therefore $\frac{S_N}{N} - \frac{1}{N}\sum_{j=1}^N \mathbb{E}[\gamma_j^N] \to 0$ in probability. Since $\limsup_{N\to\infty}\sup_{1\le j\le N}\mathbb{E}[\gamma_j^N] \le M$ we have $\frac{S_N}{N} \vee M \to M$ in probability as $N \to \infty$. □

Clearly the $\eta_i^{N,\ell,n,\rho}$ are uniformly bounded in N, so it is sufficient to control the expectation and covariance as in Lemma 6.8. We do this using Lemma 6.7.

We start by controlling the expectation, using Tonelli's theorem and (6.23) to see that we have C_t non-increasing such that

$$\mathbb{E}[\eta_i^{N,\ell,n,\rho}] \le \sum_{\substack{\beta\in\iota_\rho \\ v\in\Pi^{\ell,n}}} C_{t_0^\beta}(t_1^\beta - t_0^\beta)\mathrm{Leb}(R_{\vec{h}_\beta}(\vec{0}))$$
$$\le C_{T_{\min}(\rho)}|\Pi^{\ell,n}|\sum_{\beta\in\iota_\rho}\mathrm{Leb}([t_0^\beta,t_1^\beta]\times R_{\vec{h}_\beta}(\vec{x}_\beta)) \le 2C_{T_{\min}(\rho)}|\Pi^{\ell,n}|\mathrm{Leb}(\rho).$$

We therefore define $C_{\ell,n}(t) = 2|\Pi^{\ell,n}|C_t$ so that $\mathbb{E}[\eta_i^{N,\ell,n,\rho}] \le 2|\Pi^{\ell,n}|C_{T_{\min}(\rho)}$. We now seek to show that

$$\sup_{i\neq j}\mathrm{Cov}(\eta_i^{N,\ell,n,\rho},\eta_j^{N,\ell,n,\rho}) \to 0 \quad \text{as} \quad N\to\infty.$$

We recall that conditional on the event $A^{N,\ell,n}_{i,j}$ the Brownian motions $W^{\hat{\alpha}^{N,\ell,n}_i(v)}$ and $W^{\hat{\alpha}^{N,\ell,n}_j(v')}$ are independent. Thus using Lemma 6.7, conditional on the event $A^{N,\ell,n}_{i,j}$, Y^{i,v,β_1} and Y^{j,v',β_2} are independent for $\beta_1,\beta_2\in\iota_\rho$. Therefore it is sufficient to show that

$$\inf_{i\neq j}\mathbb{P}(A^{N,\ell,n}_{i,j})\to 1\quad\text{as}\quad N\to\infty.$$

We calculate

$$\mathbb{P}(\mathcal{G}^{N,\ell,n}_i\cap\mathcal{G}^{N,\ell,n}_j\neq\emptyset)\leq\frac{|\Pi^{\ell,n}|^2}{N-1}\to 0\quad\text{as}\quad N\to\infty.$$

To see this, we see that the elements of $\mathcal{G}^{N,\ell,n}_i$ and $\mathcal{G}^{N,\ell,n}_j$ are chosen independently and uniformly at random, so that each element of $\mathcal{G}^{N,\ell,n}_j$ has a probability at most $\frac{|\mathcal{G}^{N,\ell,n}_i|}{N-1}\leq\frac{|\Pi^{\ell,n}|}{N-1}$ of being in $\mathcal{G}^{N,\ell,n}_i$. Therefore by a union bound we are done.

We have concluded our proof of (6.15). □

Proof of (6.16)

We recall τ^N_ϵ is the stopping time defined in Proposition 4.10, and $J^{N,i}_t$ is the number of jumps by particle X^i in time t. We shall now bound the probability of $(G^{\ell,n,i}_t)^c$ by decomposing it into events $A^{N,\ell,n,\epsilon}_i$, $B^{N,i,\ell,\epsilon}_t$ and $\{\tau^N_\epsilon\leq T\}$, whereby

$$A^{N,\ell,n,\epsilon}_i=\cup_{v\in\Pi^{\ell,n}}\{J^{N,\mathcal{T}^{N,\ell,n}_i(v)}_{T\wedge\tau^N_\epsilon}\geq n+1\},\tag{6.34}$$

$$B^{N,\ell,\epsilon,i}_t=\{|\alpha^{N,i}_{t\wedge\tau^N_\epsilon}|\geq\ell+1\}.\tag{6.35}$$

Step 1

We begin by decomposing $(G^{\ell,n,i}_t)^c$ into the events

$$(G^{\ell,n,i}_t)^c\subseteq A^{N,\ell,n,\epsilon}_i\cup B^{N,\ell,\epsilon,i}_t\cup\{\tau^N_\epsilon\leq T\},\tag{6.36}$$

none of which are dependent upon any choice of $\rho\in\mathcal{R}$, only $B^{N,\ell,\epsilon,i}_t$ being dependent upon $t\leq T$, and whereby $B^{N,\ell,\epsilon,i}_t$ is not dependent upon n.

We may decompose $(G^{l,n,i}_t)^c$ into

$$\{(G^{\ell,n,i}_t)^c\}\subseteq\{|\alpha^{N,i}_t|\geq\ell+1\}\cup\{J^{N,\mathcal{I}^{N,\alpha^{N,i}_t}_s}_s>n\text{ for some }s\in[0,t]\text{ and }|\alpha^{N,i}_t|\leq\ell\}.$$

Since Proposition 4.10 gives controls on the number of jumps only up to time τ^N_ϵ, it is necessary to localise up to time τ^N_ϵ so that

$$\{(G^{\ell,n,i}_t)^c\}\subseteq\{|\alpha^{N,i}_{t\wedge\tau^N_\epsilon}|\geq\ell+1\}\cup\{\tau^N_\epsilon\leq T\}$$
$$\cup\{J^{N,\mathcal{I}^{N,\alpha^{N,i}_t}_s}_{T\wedge\tau^N_\epsilon}>n\text{ for some }s\in[0,t]\text{ and }|\alpha^{N,i}_t|\leq\ell\}.$$

Focusing on the third term on the right hand side, since $|\alpha^{N,i}_t|\leq l$ we can write

$$\alpha^{N,i}_t=((j_{\ell'},0),(j_{\ell'-1},k_{\ell'-1}),\dots,(i,k_0))$$

for some $\ell'\leq\ell$, so that we may take r minimal such that $J^{N,j_r}_{T\wedge\tau^N_\epsilon}>n$. Therefore, $k_0,\dots,k_{r-1}\leq n$ and $r\leq\ell$ so that we have

$$((j_r,0),(j_{r-1},k_{r-1}),\dots,(i,k_0))\in\mathcal{C}^{\ell,n}_i.$$

Thus $j_r=\mathcal{T}^{N,\ell,n}_i(v)$ for $v=(k_0,k_1,\dots,k_{r-1})\in\Pi^{\ell,n}$. Therefore we have (6.36) so that

$$\{(G^{\ell,n,i}_t)^c\}\subseteq\{|\alpha^{N,i}_{t\wedge\tau^N_\epsilon}|\geq\ell+1\}\cup\cup_{v\in\Pi^{\ell,n}}\{J^{N,\mathcal{T}^{\ell,n}_i(v)}_{T\wedge\tau^N_\epsilon}\geq n+1\}\cup\{\tau^N_\epsilon\leq T\}.$$

Step 2

We now show that we may choose $\ell = \ell(\epsilon)$ large enough so that

$$\limsup_{N\to\infty} \mathbb{E}^{\mathbb{P}}\big[\sup_{t\leq T} \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(B_t^{N,\ell,\epsilon,i})dt\big] \leq \epsilon. \tag{6.37}$$

There exists (by Proposition 4.10) $\bar{J} < \infty$ such that

$$\limsup_{N\to\infty} \mathbb{P}(J^N_{T\wedge\tau_\epsilon^N} \geq \bar{J}) \leq \frac{\epsilon}{3}. \tag{6.38}$$

We define $\mathcal{S}_N = \inf\{t : J_t^N \geq \bar{J}\}$ and $L_t^N := \frac{1}{N}(1+\sum_i |\alpha_t^{N,i}|)$. We fix for the time being $1 \leq i \leq N$. We see from (6.3) that if i jumps at time t, the expected value of $|\alpha_t^{N,i}|$ is at most

$$\frac{1}{N-1}\sum_{j\neq i}|\alpha_{t-}^{N,j}| + 1 = \frac{1}{N-1}\sum_{j\neq i}(|\alpha_{t-}^{N,j}|+1) \leq \frac{N}{N-1}L_{t-}^N.$$

Moreover the length $|\alpha_{t-}^{N,i}|$ immediately prior to the jump must be non-negative, hence the expected increase in $|\alpha_t^{N,i}|$ at time t is at most $\frac{N}{N-1}L_{t-}^N$. Therefore the expected value of L_t^N immediatly after the jump at time t is at most $\frac{N}{N-1}L_{t-}^N$. Further, the length of $|\alpha_t^{N,j}|$ does not change for $j \neq i$ and the $|\alpha_t^{N,i}|$ are bounded by $N(\bar{J}+1)+1$ up to time $\mathcal{S}_N$. Thus we see that

$$\big(1+\frac{1}{N-1}\big)^{-NJ^N_{t\wedge\mathcal{S}_N}} L^N_{t\wedge\mathcal{S}_N} = \big(\frac{N}{N-1}\big)^{-NJ^N_{t\wedge\mathcal{S}_N}} L^N_{t\wedge\mathcal{S}_N} \tag{6.39}$$

is a supermartingale, which takes the value 1 at time 0. We now observe that

$$\frac{\ell}{N}\sum_{i=1}^{N}\mathbb{1}(B_t^{N,\ell,n,\epsilon,i}) = \frac{\ell}{N}\sum_{i=1}^{N}\mathbb{1}(|\alpha^{N,i}_{t\wedge\tau_\epsilon^N}| \geq \ell+1) \leq L^N_{t\wedge\tau_\epsilon^N}.$$

Thus, since (6.39) is a supermartingale, we have for all N and $t \leq T$ that

$$\begin{aligned}
\mathbb{P}\big(\sup_{t\leq T\wedge\mathcal{S}_N} \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}(B_t^{N,\ell,n,\epsilon,i}) \geq \frac{\epsilon}{3}\big) &\leq \mathbb{P}(\sup_{t\leq T} L^N_{t\wedge\tau_\epsilon^N\wedge\mathcal{S}_N} \geq \frac{\ell\epsilon}{3}) \\
&\leq \mathbb{P}\big(\sup_{t\leq T}\big(\frac{N}{N-1}\big)^{N(\bar{J}-J^N_{t\wedge\mathcal{S}_N\wedge\tau_\epsilon^N})} L^N_{t\wedge\tau_\epsilon^N\wedge\mathcal{S}_N} \geq \frac{\ell\epsilon}{3}\big) \\
= \mathbb{P}\big(\sup_{t\leq T}\big(\frac{N}{N-1}\big)^{-NJ^N_{t\wedge\mathcal{S}_N\wedge\tau_\epsilon^N}} L^N_{t\wedge\tau_\epsilon^N\wedge\mathcal{S}_N} &\geq \frac{\ell\epsilon}{3}\big(\frac{N}{N-1}\big)^{-N\bar{J}}\big) \\
&\leq \frac{3}{\epsilon\ell}\big(\frac{N}{N-1}\big)^{N\bar{J}} \leq \frac{3e^{2\bar{J}}}{\epsilon\ell}.
\end{aligned}$$

Therefore for some $\ell = \ell(\epsilon)$ large enough we have for all N that

$$\mathbb{P}\big(\sup_{t\leq T\wedge\mathcal{S}_N} \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}(B_t^{N,\ell,n,\epsilon,i}) \geq \frac{\epsilon}{3}\big) \leq \frac{\epsilon}{3}.$$

Combining this with (6.38) and observing that $\sup_{t\leq T\wedge\tau_\epsilon^N}\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}(B_t^{N,l,n,\epsilon,i}) \leq 1$ we have (6.37).

Step 3

Having fixed $\ell = \ell(\epsilon)$ we may choose $n = n(\epsilon)$ large enough such that we have

$$\limsup_{N\to\infty} \mathbb{E}^{\mathbb{P}}\big[\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(A_i^{N,\ell,n,\epsilon})\big] \leq \epsilon. \tag{6.40}$$

Recalling $\mathcal{T}_i^{N,\ell,n}$ defined at (6.26), we define the initial enlargement

$$\mathcal{F}_t^v = \mathcal{F}_t \vee \sigma(U_k^{\mathcal{T}_i^{N,\ell,n}(v')},\ k\geq 0,\ v'\in \Pi^{\ell,n,v}),\quad t\geq 0 \tag{6.41}$$

whereby we write $\Pi^{\ell,n,v}$ for $\Pi^{\ell,n}$ with all descendents of v removed (we remove v itself). We then observe that:

1. $\mathcal{T}_i^{N,\ell,n}(v)$ is $\mathcal{F}_0^v$ measurable.
2. Conditional upon $\mathcal{T}_i^{N,\ell,n}(v') \neq \mathcal{T}_i^{N,\ell,n}(v)$ for $v'\in\Pi^{\ell,n,v}$, the jumps $U_k^{\mathcal{T}_i^{N,\ell,n}(v)}$ are chosen independently and uniformly at random at the times $\tau_k^{\mathcal{T}_i^{N,\ell,n}(v)}$.
3. $W^{\mathcal{T}_i^{N,\ell,n}(v)}$ is an $(\mathcal{F}_t^v)_{t\geq 0}$-Brownian motion as with the argument that $W^{\hat{\alpha}_i^{N,\ell,n}(v)}$ is an $(\bar{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion in the proof of (6.15).

We fix for the time being $1\leq i\leq N$ and now work on $(\Omega,\mathcal{F},(\mathcal{F}_t^v)_{t\geq 0},\mathbb{P})$. We see that with probability at most $\frac{|\Pi^{\ell,n,v}|}{N-1}\leq \frac{(n+1)^\ell}{N-1}$, $\mathcal{T}_i^{N,\ell,n}(v)\neq \mathcal{T}_t^{N,\ell,n}(v')$ for all $v'\in\Pi^{\ell,n,v}$. Otherwise $W^{\mathcal{T}_i^{N,\ell,n}(v)}$ is an $\mathbb{F}^v$-Brownian motion and $U_k^{\mathcal{T}_i^{N,\ell,n}(v)}$ $(k\geq 1)$ are chosen independently and uniformly at random at time $\tau_k^{\mathcal{T}_i^{N,\ell,n}(v)}$, so that we can repeat the argument of the proof of Proposition 4.10 in order to obtain

$$\begin{aligned}\mathbb{P}(J_{T\wedge\tau_\epsilon}^{\mathcal{T}_i^{N,\ell,n}(v)}\geq n+1) &\leq \mathbb{P}(J_{T\wedge\tau_\epsilon}^{\mathcal{T}_i^{N,\ell,n}(v)}\geq n+1|\mathcal{T}_i(v)\neq\mathcal{T}_t^{N,\ell,n}(v') \text{ for } v'\in\Pi^{\ell,n,v})\\ &+\mathbb{P}(\mathcal{T}_i(v)=\mathcal{T}_t^{N,\ell,n}(v') \text{ for some } v'\in\Pi^{\ell,n,v})\leq M_\epsilon p_\epsilon^{\lfloor\frac{n+1}{M_\epsilon}\rfloor}+\frac{|\Pi^{\ell,n}|}{N-1}\end{aligned}$$

for some $0<p_\epsilon<1$ and $M_\epsilon<\infty$. Whereas we may have established this using a new filtration, our probability space $(\Omega,\mathcal{F},\mathbb{P})$ has been kept fixed. Therefore we have

$$\mathbb{P}(A_i^{N,\ell,n,\epsilon})\leq |\Pi^{\ell,n}|M_\epsilon p_\epsilon^{\lfloor\frac{n+1}{M_\epsilon}\rfloor}+\frac{|\Pi^{\ell,n}|^2}{N-1}.$$

Thus we have (using Tonelli's theorem and that $|\Pi^{\ell,n}|$ grows polynomially in n for fixed ℓ)

$$\limsup_{N\to\infty}\mathbb{E}[\frac{1}{N}\sum_{i=1}^N \mathbb{1}(A_i^{N,\ell,n,\epsilon})]\leq |\Pi^{\ell,n}|M_\epsilon p_\epsilon^{\lfloor\frac{n+1}{M_\epsilon}\rfloor}\to 0\quad\text{as}\quad n\to\infty.$$

Having fixed $\ell=\ell(\epsilon)$ we may therefore choose $n=n(\epsilon)$ such that

$$\limsup_{N\to\infty}\mathbb{E}[\frac{1}{N}\sum_{i=1}^N \mathbb{1}(A_i^{N,\ell,n,\epsilon})]<\epsilon,$$

so that we have (6.40).

From (6.37), (6.40) and Proposition 4.10 we may conclude that for all $\epsilon>0$ there exists $\ell=\ell(\epsilon), n=n(\epsilon)$ such that

$$\limsup_{N\to\infty}\mathbb{E}^{\mathbb{P}}\big[\sup_{t\leq T}\frac{1}{N}\sum_{i=1}^N \mathbb{1}((G_t^{\ell,n,i})^c)\big]\leq\epsilon. \tag{6.42}$$

This completes the proof of (6.16) and therefore of Part 1 of Lemma 6.1.

6.3 Proof of part 2 of Lemma 6.1

We may observe that the proof of Part 1 may be repeated with $\mathcal{A}$ replaced by $\{(a^1,b^1)\times\ldots\times(a^d,b^d):a^i,b^i\in\mathbb{Q}\}$, and $\mathcal{R}$ adjusted accordingly to obtain a proof of Part 2.

We have now concluded our proof of Lemma 6.1. □

6.4 Proof of Lemma 6.2

The $T=0$ case is an immediate consequence of the assumption that $\{\mathcal{L}(m_0^N)\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D))$, so we may henceforth assume that $T>0$. We now prove Lemma 6.2 using the machinery we constructed to prove Lemma 6.1. We take $R<\infty$ to be determined and write $F_R=B(0,R)^c$. As with (6.13) we have

$$m_t^N(F_R)\leq\frac{1}{N}\sum_{i=1}^N\mathbb{1}(\mathcal{H}_t^{N,i,t}\in F_R)\mathbb{1}_{G_t^{\ell,n,i}}+\frac{1}{N}\sum_{i=1}^N\mathbb{1}((G_t^{\ell,n,i})^C).$$

We then use (6.24) to see that

$$\sup_{t\leq T}m_t^N(F_R)\leq\frac{1}{N}\sum_{i=1}^N\sum_{v\in\Pi^{\ell,n}}\mathbb{1}\big(\sup_{t\leq T}|X_t^{\hat{\alpha}_i^{N,\ell,n}(v)}|\geq R\big)+\frac{1}{T}S_2^{N,\ell,n}$$

where $S_2^{N,\ell,n}$ was defined at (6.14). We now fix $\ell=\ell(\epsilon)$ and $n=n(\epsilon)$ as in (6.16) so that $\limsup_{N\to\infty}\mathbb{E}[S_2^{N,\ell,n}]\leq\epsilon$.

These are then random variables on the filtered probability space $(\Omega,\mathcal{F},(\bar{\mathcal{F}}_t)_{t\geq 0},\mathbb{P})$ defined in (6.27) with respect to which $W^{\hat{\alpha}_i^{N,\ell,n}(v)}$ is an $(\bar{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion, $X^{\hat{\alpha}_i^{N,\ell,n}(v)}$ is adapted and is a solution of the SDE (6.2):

$$\begin{aligned}dX_t^{\hat{\alpha}_i^{N,\ell,n}(v)}&=b(m_t^N,X_t^{\hat{\alpha}_i^{N,\ell,n}(v)})dt+dW_t^{\alpha},\quad 0\leq t<\tau^{\hat{\alpha}_i^{N,\ell,n}(v)},\\ \tau^{\hat{\alpha}_i^{N,\ell,n}(v)}&=\inf\{t>0:X_{t^-}^{\hat{\alpha}_i^{N,\ell,n}(v)}\in\partial D\}.\end{aligned}\tag{6.43}$$

Using the fact that (6.1) holds for $T=0$, and the fact the drift is bounded, we have

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N\sum_{v\in\Pi^{\ell,n}}\mathbb{1}\big(\sup_{t\leq T}|X_t^{\hat{\alpha}_i^{N,\ell,n}(v)}|\geq R\big)\Big]=0.$$

Therefore we have

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{E}[\sup_{t\leq T}m_t^N(F_R)]\leq\frac{\epsilon}{T}.$$

Since $\epsilon>0$ is arbitrary, we are done.

□

7 Coupling to a particle system on a large but bounded subdomain

We construct here a coupling which will allow us in Section 8 to establish our hydrodynamic limit theorem on unbounded domains. We prove the following lemma in Appendix B:

Lemma 7.1. *Let $D\subseteq\mathbb{R}^d$ be a non-empty open domain with C^∞ boundary ∂D. Then for every $R>R_{\min}:=\inf\{R'>0:B(0,R')\cap D\neq\emptyset\}$ there exists a non-empty open bounded domain D_R with C^∞ boundary such that $D\cap B(0,R)\subseteq D_R\subseteq D$.*

For all $R > R_{\min}$ we let D_R be such a subdomain of D. Since D_R is a smooth bounded domain there exists $r_R > 0$ such that D_R satisfies the interior ball condition with radius $r > r_R > 0$: for every $x \in D_R$ there exists $y \in D_R$ such that $x \in B(y, r_R) \subseteq D_R$.

Given the Fleming-Viot particle system $\vec{X}^N$ with McKean-Vlasov dynamics constructed on the filtered probability space $(\Omega^N, \mathcal{F}^N, (\mathcal{F}^N_t)_{t\geq 0}, \mathbb{P}^N)$ and associated empirical measure processes

$$m^N_t = \frac{1}{N}\sum_i \delta_{X^{N,i}_t},$$

we now define a coupling, on an enlarged filtered probability space $(\Omega^{N,R}, \mathcal{F}^{N,R}, (\mathcal{F}^N_t)_{t\geq 0}, \mathbb{P}^{N,R})$, between $\vec{X}^N$ and another Fleming-Viot particle system with general dynamics $\vec{X}^{N,R}$ on the subdomain D_R having drift processes $b^{N,R,i}_t = b(m^N_t, X^{N,R,i}_t)$ (defined in Definition 4.1). In particular, we prove the following proposition.

Proposition 7.2. *For $R > R_{\min} + 1$, the Fleming-Viot particle system $\vec{X}^N$ can be coupled with another particle system $\vec{X}^{N,R}$ on an enlarged filtered probability space $(\Omega^{N,R}, \mathcal{F}^{N,R}, (\mathcal{F}^{N,R}_t)_{t\geq 0}, \mathbb{P}^{N,R})$ such that the following properties hold:*

1. *The particle system $\vec{X}^{N,R}$ is a Fleming-Viot N-particle system with generalised dynamics (defined in Definition 4.1) on the domain D_R having drift processes $b^{N,R,i}_t = b(m^N_t, X^{N,R,i}_t)$, which is well-defined up to time ∞.*
2. *Assuming $\{\mathcal{L}(m^N_0)\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D))$, then the empirical measure processes*

$$m^{N,R}_t = \frac{1}{N}\sum_i \delta_{X^{N,R,i}_t},$$

and the jump processes

$$J^N_t = \frac{1}{N}\#\{\text{jumps up to time } t \text{ by } \vec{X}^N\}, \quad J^{N,R}_t = \frac{1}{N}\#\{\text{jumps up to time } t \text{ by } \vec{X}^{N,R}\},$$

satisfy

(a) *$\{\mathcal{L}(m^{N,R}_0) : N \in \mathbb{N}\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D_R))$;*
(b) *for any $T < \infty$,*

$$\limsup_{N\to\infty} \mathbb{E}\Big[\sup_{t\leq T}||m^N_t - m^{N,R}_t||_{TV} + 1 \wedge \sup_{t\leq T}|J^N_t - J^{N,R}_t|\Big] \to 0 \quad \text{as} \quad R \to \infty. \tag{7.1}$$

Note that by Theorem 2.6, $(\vec{X}^N_t)_{0\leq t<\infty}$ has the same distribution under $\mathbb{P}^N$ as under $\mathbb{P}^{N,R}$. We shall firstly construct the coupling before establishing that this coupling satisfies (7.1).

7.1 Construction of the coupling

Since N is fixed in this construction, we neglect the N superscript for the sake of notation. We fix a point $x^* \in D \cap B(0, R-1)$. We then take a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ on which are defined the jointly independent Brownian motions $(\tilde{W}^i_t)_{t\geq 0}$ $(i = 1, \dots, N)$ and whereby $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ is the natural filtration of the Brownian motions $\tilde{W}^i$. We then define a probability space $(\Omega^V, \mathcal{F}^V, \mathbb{P}^V)$ on which the jointly independent uniform $\{1, \dots, N\} \setminus \{i\}$-valued random variables V^i_k $(i = 1, \dots, N;\ k \geq 1)$ are defined. We shall firstly define our construction on the measurable space (which we shall later equip with the appropriate filtration and probability measure)

$$(\Omega^R, \mathcal{F}^R) = (\Omega \times \tilde{\Omega} \times \Omega^V, \mathcal{F} \otimes \tilde{\mathcal{F}} \otimes \sigma^V).$$

We shall partition $\{1, \dots, N\}$ into "blue" indices $\mathcal{B}_t$ and "yellow" indices $\mathcal{Y}_t$ at each time t – we shall say the i^{th} particles, both X^i_t and $X^{R,i}_t$, are blue (or yellow) at time t if

$i \in \mathcal{B}_t$ (or $i \in \mathcal{Y}_t$). We shall refer to particles in the particle system $\vec{X}^R$ as "R-particles". At time 0 we assign indices $i \in \{1, \dots, N\}$ to be blue if $|X_0^i| \le R-1$ and otherwise yellow:

$$\mathcal{B}_0 := \{i : |X_0^i| \le R-1\}, \quad \mathcal{Y}_0 := \{i : |X_0^i| > R-1\}. \tag{7.2}$$

The initial condition for the R-particles is given by

$$X_0^{R,i} = \begin{cases} X_0^i, & i \in \mathcal{B}_0 \\ x^*, & i \in \mathcal{Y}_0 \end{cases} . \tag{7.3}$$

Having defined these initial conditions, we now summarize the properties of the coupling construction; we give a more precise construction below:

- All R-particles, whether blue or yellow, will satisfy $X_t^{R,i} \in D_R$ for all $t \ge 0$.
- Once an index turns yellow, it will remain yellow: if $i \in \mathcal{Y}_s$, then $i \in \mathcal{Y}_t$ for all $t \ge s$. Between jump times, the yellow R-particles are driven by the Brownian motion $\tilde{W}^i$ (independent of W_t), according to

$$dX_t^{R,i} = d\tilde{W}_t^i, \quad i \in \mathcal{Y}_t.$$

  Upon hitting ∂D_R, a yellow R-particle jumps onto another R-particle chosen uniformly at random.
- As long as an index i is blue, the particle $X^{R,i}$ follows X^i, meaning $X_t^{R,i} = X_t^i$ if $i \in \mathcal{B}_t$. In particular, between jump times a blue R-particle satisfies

$$dX^{R,i} = b(m_t^N, X_t^{R,i})\, dt + dW_t^i, \quad i \in \mathcal{B}_t.$$

  Upon hitting ∂D_R, a blue R-particle jumps onto another R-particle chosen uniformly at random. However, the blue particle may turn to yellow upon hitting ∂D.
- An index i can change from blue to yellow only when X^i hits the boundary ∂D_R. There are two ways this can happen. First, a blue index i turns yellow whenever the associated particle $X^{R,i} = X^i$ hits $\partial D_R \setminus \partial D$; at this point, $X^{R,i}$ jumps onto another R-particle (in D_R), but X^i does not jump at this time (because it has not hit ∂D). A blue index turns yellow also in the case that $X^{R,i} = X^i$ hits $\partial D \setminus \partial D_R$ if X^i happens to jump onto a particle with yellow index $j \in \mathcal{Y}_t$. In this case, the index i is turned yellow, and the associate R-particle jumps onto the yellow R-particle $X^{R,j} \in D_R$ (which may be at a location different from X^j). When a blue R-particle hits $\partial D \setminus \partial D_R$ and jumps onto another blue particle, then it remains blue. In particular, $X^{R,i} = X^i$ jump to the same location and the relation $X_t^{R,i} = X_t^i$ continues as long as $i \in \mathcal{B}_t$.

A precise construction of the coupling can be carried out inductively. We will define the times $(\tau_k^R)_{k=0}^\infty$ corresponding to the k^{th} death time of any of the R-particles (k^{th} time at which an R particle hits ∂D). Our coupling is constructed up to time τ_k^R, inductively in k. We proceed as follows.

Step 1

Assuming that for some $k \ge 0$ we have defined the random times $0 = \tau_0^R < \ldots < \tau_k^R$ and $\vec{X}_t^R$, $\mathcal{B}_t$ and $\mathcal{Y}_t$ for $t \le \tau_k^R$ we define τ_{k+1}^R and $\vec{X}_t^R$, $\mathcal{B}_t$ and $\mathcal{Y}_t$ for $t < \tau_{k+1}^R$ according to

$$\tau_{k+1}^R = \inf\{t > \tau_k^R \;:\; X_{t^-}^{R,i} \in \partial D_R \quad \text{for some } i\},$$

$$X_t^{R,i} = \begin{cases} X_t^i, & i \in \mathcal{B}_{\tau_k^R} \\ X_{\tau_k^R}^{R,i} + \tilde{W}_t^i - \tilde{W}_{\tau_k^R}^i, & i \in \mathcal{Y}_{\tau_k^R} \end{cases} , \quad t \in [\tau_k^R, \tau_{k+1}^R).$$

This means that the R-particles $X^{R,i}$ which are blue at time τ_k^R track the corresponding X^i, whilst the yellow R-particles track the path of the corresponding $(\tilde{F}_t)_{t\geq 0}$-Brownian motion $\tilde{W}_t^i$, up to the next time τ_{k+1}^R one of the R-particles the boundary ∂D_R. Furthermore, we define

$$\mathcal{B}_t := \mathcal{B}_{\tau_k^R}, \quad \mathcal{Y}_t := \mathcal{Y}_{\tau_k^R}, \quad t \in [\tau_k^R, \tau_{k+1}^R),$$

so that between hitting times the colors of the R-particles do not change.

Step 2

We now define the construction at time τ_{k+1}^R. It may be the case that two or more of the R-particles hit the boundary ∂D_R at the same time (when we equip our construction with a probability measure this will turn out to be a null event), if this is the case we halt our construction at the time we call $\tau_{\text{stop}}^R := \tau_{k+1}^R$.

Otherwise there is only one R-particle which hits the boundary at time τ_{k+1}^R, with unique index $\ell(k+1) \in \{1, \dots, N\}$ such that $X^{R,\ell(k+1)}_{\tau_{k+1}^R -} \in \partial D_R$. There are three distinct possibilities:

1. It could be that index $\ell(k+1)$ is yellow immediately prior to the hitting time. In this case the index $V_{k+1}^{\ell(k+1)}$ is chosen and $X^{R,\ell(k+1)}$ jumps onto $X^{R,V_{k+1}^{\ell(k+1)}}$:

$$X^{R,\ell(k+1)}_{\tau_{k+1}^R} := X^{R,V_{k+1}^{\ell(k+1)}}_{\tau_{k+1}^R -}.$$

   In this case $\ell(k+1)$ remains yellow: $\ell(k+1) \in \mathcal{Y}_{\tau_{k+1}^R}$, and none of the other indices change colour:

$$\mathcal{Y}_{\tau_{k+1}^R} := \mathcal{Y}_{\tau_{k+1}^R -}, \quad \mathcal{B}_{\tau_{k+1}^R} := \mathcal{B}_{\tau_{k+1}^R -}.$$

2. It could be the case that $\ell(k+1)$ is blue immediately prior to the hitting time τ_{k+1}^R, and $X^{R,\ell(k+1)}$ hits $\partial D_R \setminus \partial D$ at this time. Thus $X^{R,\ell(k+1)}$ was tracking $X^{\ell(k+1)}$ up to the hitting time, but of course $X^{\ell(k+1)}$ cannot jump at this time as it did not hit the boundary of D. In this case only the R-particle jumps, choosing the index $V_{k+1}^{\ell(k+1)}$ to jump onto, and the index $\ell(k+1)$ switches to yellow:

$$X^{R,\ell(k+1)}_{\tau_{k+1}^R} := X^{R,V_{k+1}^{\ell(k+1)}}_{\tau_{k+1}^R -}, \quad \mathcal{Y}_{\tau_{k+1}^R} := \mathcal{Y}_{\tau_{k+1}^R -} \cup \{\ell(k+1)\}, \quad \mathcal{B}_{\tau_k^R} := \mathcal{B}_{\tau_{k+1}^R -} \setminus \{\ell(k+1)\}.$$

3. The final possibility is that $\ell(k+1)$ is blue immediately prior to time τ_{k+1}^R, at which time $X^{R,\ell(k+1)}$ hits $\partial D_R \cap \partial D$. If this is the case $X^{\ell(k+1)}$ and $X^{R,\ell(k+1)}$ hit $\partial D \cap \partial D_R$ together, so that τ_{k+1}^R corresponds to $\tau_{k'}^{\ell(k+1)}$ for some $k' \leq k+1$ (i.e. the k'^{th} hitting time of particle $X^{\ell(k+1)}$ with the boundary ∂D). Recall that the particle $X^{\ell(k+1)}$ then jumps onto the particle $X^{U_{k'}^{\ell(k+1)}}$ at this time $\tau_{k'}^{\ell(k+1)} = \tau_k^R$. We then define the R-particle $X^{R,\ell(k+1)}$ as jumping onto the R-particle with the same index $U_{k'}^{\ell(k+1)}$. Thus, if that index is blue (if $U_{k'}^{\ell(k+1)} \in \mathcal{B}_{\tau_{k+1}^R -}$), then both $X^{\ell(k+1)}$ and $X^{R,\ell(k+1)}$ jump onto the same location (which, by induction, is in D_R) and remain blue:

$$X^{R,\ell(k+1)}_{\tau_{k+1}^R} := X^{\ell(k+1)}_{\tau_{k+1}^R}, \quad \mathcal{Y}_{\tau_{k+1}^R} := \mathcal{Y}_{\tau_{k+1}^R -}, \quad \mathcal{B}_{\tau_{k+1}^R} := \mathcal{B}_{\tau_{k+1}^R -}.$$

   Otherwise, the target index $U_{k'}^{\ell(k+1)}$ is yellow, $U_{k'}^{\ell(k+1)} \in \mathcal{Y}_{\tau_{k+1}^R -}$, and we set

$$X^{R,\ell(k+1)}_{\tau_{k+1}^R} := X^{R,U_{k+1}^{\ell(k+1)}}_{\tau_{k+1}^R -}, \quad \mathcal{Y}_{\tau_{k+1}^R} := \mathcal{Y}_{\tau_{k+1}^R -} \cup \{\ell(k+1)\}, \quad \mathcal{B}_{\tau_k^R} := \mathcal{B}_{\tau_{k+1}^R -} \setminus \{\ell(k+1)\}.$$

In all three cases none of the other R-particles jump at the time τ_{k+1}^R.

Step 3

This is well-defined on $(\Omega^R, \mathcal{F}^R)$ up to the time $\tau_{\text{WD}}^R := \tau_\infty^R \wedge \tau_{\text{stop}}^R$, where

$$\tau_\infty^R = \lim_{k\to\infty} \tau_k^R, \quad \tau_{\text{stop}}^R = \inf\{t > 0 : \exists\, j \neq k \text{ such that } X_{t^-}^{R,j},\ X_{t^-}^{R,k} \in \partial D_R\}.$$

We now equip our measurable space with a filtration and probability measure. We define the filtration

$$\mathcal{F}_t^R := \mathcal{F}_t \otimes \tilde{\mathcal{F}}_t \otimes \sigma(V_k^{\ell(k)} : \tau_k^R \leq t), \quad 0 \leq t < \infty$$

and the probability measure

$$\hat{\mathbb{P}} = \mathbb{P} \otimes \tilde{\mathbb{P}} \otimes \mathbb{P}^V.$$

We see that on the filtered probability space $(\Omega^R, \mathcal{F}^R, (\mathcal{F}_t^R)_{t\geq 0}, \hat{\mathbb{P}})$ our original N-particle system $(\vec{X}^N, \vec{W}^N)$ has the same distribution, and moreover $\vec{X}^{N,R}$ is a generalised Fleming-Viot particle system with drift

$$b_t^{R,i} = \mathbb{1}(i \in \mathcal{B}_t^R) b(m^N, X_t^{R,i}).$$

In particular, in between the jump times $\{\tau_k^R\}$ (which are $\mathcal{F}_t^R$ stopping times), $X_t^{R,i}$ satisfies

$$dX_t^{R,i} = \mathbb{1}(i \in \mathcal{B}_t)\big(b(m_t^N, X_t^{R,i})dt + dW_t^i\big) + \mathbb{1}(i \in \mathcal{Y}_t)d\tilde{W}_t^i. \tag{7.4}$$

Therefore by Proposition 4.5, $\hat{\mathbb{P}}(\tau_{\text{WD}} = \infty) = 1$. Since the Brownian motions $\{\tilde{W}^i\}$ are independent of the Brownian motions $\{W^i\}$, we may use Girsanov's theorem to tilt the probability measure $\hat{\mathbb{P}}$, obtaining a probability measure $\mathbb{P}^R$ under which both

$$W_t^{R,i} := \int_0^t \mathbb{1}(i \in \mathcal{B}_s) dW_s^i + \int_0^t \mathbb{1}(i \in \mathcal{Y}_s)(d\tilde{W}_s^{R,i} - b(m^N, X_s^{R,i})ds), \quad 1 \leq i \leq N$$

and W^i $(1 \leq i \leq N)$ are $\mathbb{P}^R$-Brownian motions. We see that between jumps $X_t^{R,i}$ is a solution of the SDE

$$dX_t^{R,i} = b(m_t^N, X_t^{R,i})dt + dW_t^{R,i}. \tag{7.5}$$

Since $\hat{\mathbb{P}}$ and $\mathbb{P}^R$ are equivalent, $\{\tau_{\text{WD}} < \infty\}$ remains a null event. By considering the covariation, we see $\{W^i\}$ and $\{W^{R,i}\}$ both remain families of independent Brownian motions (though not independent of each other). We have therefore finished our construction of $\vec{X}_t^R$.

$\{\mathcal{L}(m_0^{N,R}) : N \in \mathbb{N}\}$ **is tight in** $\mathcal{P}(\mathcal{P}_W(D_R))$**.**

We note that a family of random measures being tight in $\mathcal{P}(\mathcal{P}_W(D_R))$ is equivalent to their mean measures being tight in $\mathcal{P}(D_R)$ [24, Theorem 4.10]. Using (7.2) and (7.3) we can write $m_0^{R,N}(A) = m_0^N(A \cap \bar{B}(0, R-1)) + m_0^N((\bar{B}(0, R-1))^c)\delta_{x^*}(A)$. Therefore the expected mean measures $E_0^{N,R}(A) = \mathbb{E}[m_0^{N,R}(A)]$ and $E_0^N(A) = \mathbb{E}[m_0^N(A)]$ satisfy

$$E_0^{N,R}(A) \leq E_0^N(A \cap B(0, R-1)) + \delta_{x^*}(A).$$

Since $\{\mathcal{L}(m_0^N)\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D))$, $\{E_0^N\}$ is tight in $\mathcal{P}(D)$, so that $\{(A \mapsto E_0^N(A \cap B(0, R-1)))\}$ is tight in $D \cap B(0, R-1) \subseteq D_R$. Therefore $\{E_0^{N,R}\}$ is tight in $\mathcal{P}(D_R)$ hence $\{\mathcal{L}(m_0^{N,R})\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D_R))$.

7.2 Proof the coupling satisfies (7.1)

Step 1: We first show that

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{E}[\sup_{t\leq T}||m^{N,R}-m^N||_{\rm TV}]=0. \tag{7.6}$$

Since $\sup_{t\leq T}||m^{N,R}-m^N||_{\rm TV}\leq\frac{|\mathcal{Y}_T|}{N}$ almost surely, it suffices to estimate $|\mathcal{Y}_T|$, the number of yellow indices at time T.

For the sake of notation, we write X^i for $X^{N,i}$ and W^i for $W^{N,i}$. Recall that a blue index i turns yellow if the particle $X^{R,i}=X^i$ hits $\partial D\setminus\partial D_R$ and jumps to a yellow particle, or if it hits the boundary $\partial D_R\setminus\partial D$. By Lemma 6.2 we know that

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{P}(\sup_{t\leq T}m^N_t(B(0,\frac{R}{3})^c)\geq\epsilon')=0$$

holds for any $\epsilon'>0$, which implies that any blue particle hitting ∂D will jump, with high probability, to a particle within $B(0,R/3)$. This, the fact that the drift is bounded (we may assume $BT<R/3$) and the fact that

$$\lim_{R\to\infty}\mathbb{P}(\sup_{t',t''\leq T}|W_{t'}-W_{t''}|\geq R/3)=0 \tag{7.7}$$

for a Brownian motion W, will give control on the possibility that a blue particle hits $\partial D_R\setminus\partial D$.

We write τ^i_k for the $k^{\rm th}$ death time of particle X^i – at which time it jumps onto the particle with index U^i_k – and τ_k for the $k^{\rm th}$ death time of any of the particles X^j (for any j). These stopping times are also dependent upon N, but again we suppress the superscript for the sake of notation. We define the initial enlargement

$$\mathcal{F}^W_t:=\mathcal{F}^R_t\wedge\sigma((W^i_s)_{0\leq s\leq T}:\ i=1,\dots,N).$$

Under this filtration, the target indices U^i_k are still chosen independently and uniformly at the corresponding hitting times. We then define the $(\mathcal{F}^W_t)_{0\leq t<\infty}$-adapted processes

$$\begin{aligned}&\tilde{\mathcal{B}}_t:=\{i\in\mathcal{B}_t:\sup_{t',t''\leq T}|W^i_{t'}-W^i_{t''}|<\frac{R}{3}\text{ and }|X^i_{\tau^i_k}|<\frac{R}{3}\text{ for all }\tau^i_k\leq t\}\subseteq\mathcal{B}_t,\\&\bar{\mathcal{Y}}_t:=(\tilde{\mathcal{B}}_t)^c\supseteq\mathcal{Y}_t.\end{aligned}$$

The time at which a blue particle hits $\partial D\cap\partial D_R$ must coincide with one of the death times of the original particles X^i. We claim that if $i\in\tilde{\mathcal{B}}_{t-}\setminus\tilde{\mathcal{B}}_t$, then X^i must hit $\partial D\cap\partial D_R$ at time t. It is sufficient to show that it does not hit $\partial D_R\setminus\partial D$. We take the largest k such that $\tau^i_k<t$, so that we have

$$|X^i_{t-}|\leq|W^i_t-W^i_{\tau^i_k}|+B|t-\tau^i_k|+|X^i_{\tau^i_k}|<R$$

since $i\in\tilde{\mathcal{B}}_{t-}$. Thus $X^i_{t-}\notin\partial D_R\setminus\partial D$. Thus $\mathbb{1}(i\in\tilde{\mathcal{B}}_t)$ is constant on $[\tau^i_k,\tau^i_{k+1})$ for all $k\geq 0$ and non-increasing on $[0,\infty)$. Although $|\mathcal{Y}_t|$ may increase at times when an R particle hits $\partial D_R\setminus\partial D$ (which is not one of the hitting times τ^i_k), we have $\mathbb{1}(i\in\mathcal{Y}_t)\leq\mathbb{1}(i\in\bar{\mathcal{Y}}_t)$, and the latter can increase only at a hitting time τ^i_k. So, our goal now is to control the growth of $|\bar{\mathcal{Y}}_t|$.

If $i\in\mathcal{Y}_{\tau^i_{k+1}}\cap\tilde{\mathcal{B}}_{\tau^i_k}$ and $\tau^i_k<\tau^i_{k+1}\leq T$, then it must be the case that $U^i_{k+1}\in\mathcal{Y}_{\tau^i_{k+1}-}$, meaning that the blue particle X^i jumps onto a yellow particle at time τ^i_{k+1}. Therefore if $i\in\bar{\mathcal{Y}}_{\tau^i_{k+1}}\setminus\bar{\mathcal{Y}}_{\tau^i_{k+1}-}=\bar{\mathcal{Y}}_{\tau^i_{k+1}}\cap\tilde{\mathcal{B}}_{\tau^i_k}$ and $\tau^i_{k+1}\leq T$ then it must be that either $U^i_{k+1}\in\mathcal{Y}_{\tau^i_{k+1}-}\subseteq\bar{\mathcal{Y}}_{\tau^i_{k+1}-}$ or the particle i jumps to a location $|X^i_{\tau^i_{k+1}}|\geq\frac{R}{3}$ so that $i\notin\tilde{\mathcal{B}}_{\tau^i_{k+1}}$, even if $i\in\mathcal{B}_{\tau^i_{k+1}}$.

We also note that $\mathbb{P}(i \in \bar{\mathcal{Y}}_T \setminus \bar{\mathcal{Y}}_{T-}) = 0$, a consequence of Corollary C.2 in Appendix C. Conditioned on $\mathcal{F}^W_{\tau^i_{k+1}\wedge T-}$, the target index U^i_{k+1} is chosen uniformly from the remaining $N-1$ indices. Therefore we have

$$\mathbb{P}(i \in \bar{\mathcal{Y}}_{\tau^i_{k+1}\wedge T} \setminus \bar{\mathcal{Y}}_{\tau^i_{k+1}\wedge T-} | \mathcal{F}^W_{\tau^i_{k+1}\wedge T-}) \leq \frac{N}{N-1} m^N_{\tau^i_{k+1}\wedge T-}(B(0,\frac{R}{3})^c) + \frac{|\bar{\mathcal{Y}}_{\tau^i_{k+1}\wedge T-}|}{N-1}. \quad (7.8)$$

For $\epsilon' > 0$ and $\bar{J} < \infty$ to be determined, we define

$$\hat{\tau} = \inf\{t > 0 : J^N_t \geq \bar{J} \quad \text{or} \quad m^N_t(B(0,\frac{R}{3})^c) \geq \epsilon'\} \wedge T, \qquad E_k = \frac{\mathbb{E}[|\bar{\mathcal{Y}}_{\tau_k\wedge\hat{\tau}}|]}{N}.$$

Since at most one particle is killed at time τ_k (almost surely) we have

$$\begin{aligned} E_{k+1} - E_k &= \frac{1}{N}\mathbb{E}\left[|\bar{\mathcal{Y}}_{\tau_{k+1}\wedge\hat{\tau}} \setminus \bar{\mathcal{Y}}_{\tau_{k+1}\wedge\hat{\tau}-}|\mathbb{1}(\tau_{k+1} \leq \hat{\tau})\right] \\ &= \frac{1}{N}\mathbb{E}\big[\mathbb{E}[|\bar{\mathcal{Y}}_{\tau_{k+1}\wedge\hat{\tau}} \setminus \bar{\mathcal{Y}}_{\tau_{k+1}\wedge\hat{\tau}-}||\mathcal{F}^W_{\tau_{k+1}\wedge\hat{\tau}-}] \underbrace{\mathbb{1}(\tau_{k+1} \leq \hat{\tau})}_{\mathcal{F}^W_{\tau_{k+1}\wedge\hat{\tau}-}\text{-measurable}} \big] \\ &\leq \frac{1}{N}\mathbb{E}\Big[\big(\frac{N}{N-1} m^N_{\tau_{k+1}-}(B(0,\frac{R}{3})^c) + \frac{|\bar{\mathcal{Y}}_{\tau_{k+1}-}|}{N-1}\big)\mathbb{1}(\tau_{k+1} \leq \hat{\tau})\Big] \quad \text{by} \quad (7.8) \\ &\leq \frac{\epsilon'}{N-1} + \frac{1}{N-1}E_k. \end{aligned}$$

Thus $E_{k+1} \leq (1 + \frac{1}{N-1})E_k + \frac{\epsilon'}{N-1}$ holds for all $k \geq 0$, so that

$$E_{k+1} + \epsilon' \leq \big(1 + \frac{1}{N-1}\big)(E_k + \epsilon'),$$

and then

$$E_{N\bar{J}} \leq E_{N\bar{J}} + \epsilon' \leq \big(1 + \frac{1}{N-1}\big)^{N\bar{J}}(E_0 + \epsilon').$$

We note that if $\hat{\tau} = T$, then $\tau_{N\bar{J}} \wedge \hat{\tau} = T$ so that we have

$$\begin{aligned} \mathbb{E}\big[\frac{|\mathcal{Y}_T|}{N}\big] \leq \mathbb{E}\big[\frac{|\bar{\mathcal{Y}}_T|}{N}\big] &\leq \mathbb{E}\big[\frac{|\bar{\mathcal{Y}}_{\tau_{N\bar{J}}\wedge\hat{\tau}}|}{N}\mathbb{1}(\hat{\tau} = T)\big] + \mathbb{E}\big[\frac{|\bar{\mathcal{Y}}_T|}{N}\mathbb{1}(\hat{\tau} < T)\big] \\ &\leq E_{N\bar{J}} + \mathbb{P}(\hat{\tau} < T) \leq \big(1 + \frac{1}{N-1}\big)^{N\bar{J}}(E_0 + \epsilon') + \mathbb{P}(\hat{\tau} < T). \end{aligned}$$

Recalling (7.2), observe that

$$E_0 = \frac{1}{N}\mathbb{E}[\bar{\mathcal{Y}}_0] \leq \mathbb{P}\left(\sup_{t',t''\leq T}|W_{t'} - W_{t''}| \geq R/3\right) + \mathbb{E}[m^N_0(B(0,R/3)^c)].$$

Therefore, by (7.7), we see that $\limsup_{R\to\infty}\limsup_{N\to\infty} E_0 = 0$. By Lemma 6.2, we see that $\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{P}(\sup_{t\leq T} m^N_t(B(0,\frac{R}{3})^c) \geq \epsilon') = 0$. From this we conclude

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\frac{1}{N}\mathbb{E}[|\mathcal{Y}_T|] \leq e^{\bar{J}}\epsilon' + \limsup_{N\to\infty}\mathbb{P}(J^N_T \geq \bar{J}).$$

We now fix $\epsilon > 0$ and use Proposition 4.10 to take $\bar{J} < \infty$ such that $\limsup_{N\to\infty}\mathbb{P}(J^N_T \geq \bar{J}) < \epsilon$. Then taking $0 < \epsilon' < \frac{\epsilon}{e^{\bar{J}}}$ we have $\limsup_{R\to\infty}\limsup_{N\to\infty}\frac{1}{N}\mathbb{E}[|\mathcal{Y}_T|] \leq 2\epsilon$ for arbitrary $\epsilon > 0$. Hence, we have now proved that

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{E}[\sup_{t\leq T}||m^{N,R} - m^N||_{\mathrm{TV}}] \leq \limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{E}\big[\frac{|\mathcal{Y}_T|}{N}\big] = 0, \qquad (7.9)$$

which is (7.6).

Step 2: Next, we will prove that

$$\limsup_{N\to\infty}\mathbb{E}[1\wedge\sup_{t\leq T}|J^{N,R}-J^N|]\to 0 \quad \text{as} \quad R\to\infty. \tag{7.10}$$

The main idea is that as long as an index i is blue, the jumps of X^i and $X^{R,i}$ coincide. Therefore, in estimating $|J^{N,R}-J^N|$ we only need to count the jumps once i turns yellow. We return to our original filtered probability space $(\Omega^{N,R},\mathcal{F}^{N,R},(\mathcal{F}^{N,R}_t)_{t\geq 0},\mathbb{P}^{N,R})$. For $i\in\{1,\dots,N\}$ we define the stopping times at which a given index becomes yellow,

$$\tau_i^{\mathcal{Y}}:=\inf\{t>0:i\in\mathcal{Y}_t\},$$

so that we have

$$\sup_{t\leq T}|J^{N,R}_t-J^N_t|\leq\frac{1}{N}\sum_{i=1}^N\big(|J^{N,R,i}_T-J^{N,R,i}_{\tau_i^{\mathcal{Y}}\wedge T}|+|J^{N,i}_T-J^{N,i}_{\tau_i^{\mathcal{Y}}\wedge T}|\big).$$

We fix $\epsilon>0$ and define

$$V_c=\{x\in D:d(x,\partial D)\geq c\},\quad V_c^R=\{x\in D_R:d(x,\partial D_R)\geq c\}$$

for $c>0$ to be determined as in Proposition 4.7. We define the stopping times

$$\tau_c^N=\inf\{t>0:m^N(V_c)\leq\frac{1}{2}\},\quad \tau_c^{N,R}=\inf\{t>0:m^{N,R}(K_c^R)\leq\frac{1}{2}\},\quad \mathcal{T}_c=\tau_c^N\wedge\tau_c^{N,R}.$$

By Proposition 4.7 there is $\tilde{c}=\tilde{c}(\frac{1}{10},\epsilon)>0$ such that

$$\limsup_{N\to\infty}\mathbb{P}(\sup_{t\in[0,T]}m_t^N(V_{\tilde{c}}^c)\geq\frac{1}{10})<\epsilon,$$

so that applying (7.6) we have

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{P}(\sup_{t\in[0,T]}m_t^{N,R}((V_{\tilde{c}})^c)\geq\frac{2}{10})\leq 2\epsilon.$$

Therefore $\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{P}(\mathcal{T}_{\tilde{c}}<T)\leq 3\epsilon$. We therefore have

$$\begin{aligned}\mathbb{E}\big[1\wedge\sup_{t\leq T}|J^{N,R}_t-J^N_t|\big]&\leq\mathbb{P}(\mathcal{T}_{\tilde{c}}<T)\\&+\frac{1}{N}\sum_{i=1}^N\mathbb{E}\big[|J^{N,R,i}_{T\wedge\mathcal{T}_{\tilde{c}}}-J^{N,R,i}_{\tau_i^{\mathcal{Y}}\wedge T\wedge\mathcal{T}_{\tilde{c}}}|+|J^{N,i}_{T\wedge\mathcal{T}_{\tilde{c}}}-J^{N,i}_{\tau_i^{\mathcal{Y}}\wedge T\wedge\mathcal{T}_{\tilde{c}}}|\big|\tau_i^{\mathcal{Y}}<T\wedge\mathcal{T}_{\tilde{c}}\big]\mathbb{P}(\tau_i^{\mathcal{Y}}<T\wedge\mathcal{T}_{\tilde{c}}).\end{aligned}\tag{7.11}$$

We note that $(\vec{X}^N_{\tau_i^{\mathcal{Y}}+t})_{t\geq 0}$ and $(\vec{X}^{N,R}_{\tau_i^{\mathcal{Y}}+t})_{t\geq 0}$ are Fleming-Viot particle systems with generalised dynamics. We recall from (4.31) that the stopping time τ_ϵ^N given in Proposition 4.10 is given by $\inf\{t>0:m_t^N(V_{\tilde{c}(\frac{1}{2},\epsilon)})<\frac{1}{2}\}$. Moreover the constants M_ϵ and p_ϵ obtained in that proof were dependent only upon the upper bound on the drift $B<\infty$, and the value of $\tilde{c}(\frac{1}{2},\epsilon)$. We therefore see that we may apply (the proof of) Proposition 4.10 to see that there exists $C_\epsilon<\infty$ dependent only upon $\tilde{c}(\frac{1}{10},\epsilon)$ such that

$$\mathbb{E}\big[|J^{N,R,i}_{T\wedge\mathcal{T}_{\tilde{c}}}-J^{N,R,i}_{\tau_i^{\mathcal{Y}}\wedge T\wedge\mathcal{T}_{\tilde{c}}}|+|J^{N,i}_{T\wedge\mathcal{T}_{\tilde{c}}}-J^{N,i}_{\tau_i^{\mathcal{Y}}\wedge T\wedge\mathcal{T}_{\tilde{c}}}|\big|\tau_i^{\mathcal{Y}}<T\wedge\mathcal{T}_{\tilde{c}}\big]\leq C_\epsilon.$$

We therefore have

$$\mathbb{E}\big[1\wedge\sup_{t\leq T}|J^{N,R}_t-J^N_t|\big]\leq\frac{C_\epsilon}{N}\mathbb{E}[\mathcal{Y}_T]+\mathbb{P}(\mathcal{T}_{\tilde{c}}<T).$$

Taking $\limsup_{R\to\infty}\limsup_{N\to\infty}$ of both sides, using (7.6) and noting $\epsilon>0$ was arbitrary, we conclude that (7.10) holds.

8 Hydrodynamic limit theorem

In this section we shall establish Theorem 2.10. We shall then prove the uniqueness in law of weak solutions to the McKean-Vlasov SDE (1.5), before combining this with Theorem 2.10 to prove Theorem 2.9 along with the existence part of Proposition 2.8 – completing its proof.

However the proof of Theorem 2.10 relies on Lemma 8.1, which provides a partial result for Proposition 2.8 along with compactness for families of global weak solutions to the McKean-Vlasov SDE (1.5) whose initial conditions belong to a compact set. This lemma will also be used in Section 9. Therefore we firstly prove Lemma 8.1.

Throughout this section we assume Condition 2.4. For $\kappa \subseteq \mathcal{P}_W(D)$ we define

$$\begin{aligned}\Xi(\kappa) = \{(\mathcal{L}(X_t|\tau_\partial > t), -\ln \mathbb{P}(\tau_\partial > t))_{0\leq t<\infty} \in \mathcal{C}([0,\infty);\mathcal{P}_W(D)\times \mathbb{R}_{\geq 0}) :\\ (X,\tau_\partial,W) \text{ is a global weak solution of (1.5) with initial condition } \mathcal{L}(X_0)\in\kappa\},\end{aligned} \tag{8.1}$$

which we equip with the metric $d^{\mathcal{D}}$. Therefore (2.12) is given by $\Xi = \Xi(\mathcal{P}_W(D))$.

Lemma 8.1. *Every weak solution (X,τ_∂,W) to (1.5) is a global weak solution such that*

$$(\mathcal{L}(X_t|\tau_\partial > t))_{0\leq t<\infty} \in \mathcal{C}([0,\infty);\mathcal{P}_W(D)) \quad \textit{and} \quad (\mathbb{P}(\tau_\partial > t))_{0\leq t<\infty} \in \mathcal{C}([0,\infty);\mathbb{R}_{>0}).$$

Moreover $\Xi(\kappa)$ is a compact subset of $(\mathcal{C}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0}),d^{\infty})$ for $\kappa\subseteq\mathcal{P}_W(D)$ compact.

Note that Lemma 8.1 allows for the possibility that $\Xi(\kappa)$ is the compact set $\emptyset$.

8.1 Proof of Lemma 8.1

We begin by showing that every weak solution (X,τ_∂,W) to (1.5) is a global weak solution with W-continuous in time laws. We suppose $(X_t,W_t)_{0\leq t\leq\tau_\partial}$ is a weak solution to (1.5). Lemma C.1, which we establish in Appendix C, automatically implies that (X,τ_∂,W) is a global weak solution.

We now turn to proving that global weak solutions to (1.5) have W-continuous in time conditional laws. We fix some weak solution (X,τ_∂,W) of (1.5) (which is a global weak solution) and $\phi\in C_b(D)$. Corollary C.2 (established in Appendix C) gives that $\mathbb{P}(\tau_\partial = t) = 0$ for $t\geq 0$. Therefore we have

$$\lim_{t'\uparrow t}\phi(X_{t'})\mathbb{1}(\tau_\partial > t') = \lim_{t'\downarrow t}\phi(X_{t'})\mathbb{1}(\tau_\partial > t') = \phi(X_t)\mathbb{1}(\tau_\partial > t) \quad \text{almost surely.}$$

Thus $\mathcal{L}(X_{t'})\to\mathcal{L}(X_t)$ in $\mathcal{M}(D)$ as $t'\to t$. Moreover since $\mathbb{P}(\tau_\partial = t) = 0$, $\mathbb{P}(\tau_\partial > t')\to\mathbb{P}(\tau_\partial > t)$ as $t'\to t$. Therefore we have

$$\mathcal{L}(X_t|\tau_\partial > t)\in\mathcal{C}([0,\infty);\mathcal{P}(D)), \quad \mathbb{P}(\tau_\partial > t)\in\mathcal{C}([0,\infty);\mathbb{R}_{>0}),$$

where $\mathcal{P}(D)$ is equipped with the topology of weak convergence of probability measures. Since W generates this same topology, we have $(\mathcal{L}(X_t|\tau_\partial > t))_{0\leq t<\infty}\in\mathcal{C}([0,\infty);\mathcal{P}_W(D))$.

Compactness of $\Xi(\kappa)$

We now turn to establishing that $\Xi(\kappa)$ is a compact subset of $(\mathcal{C}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0},d^{\infty})$ for $\kappa\subseteq\mathcal{P}_W(D)$ compact. Since the empty set is compact, we may assume without loss of generality that $\Xi(\kappa)\neq\emptyset$. We take $(X^k,\tau^k_\partial,W^k)$ a sequence of global weak solutions to (1.5) with initial conditions $\mathcal{L}(X^k_0)\in\kappa$.

Step 1. We define

$$F^k_t := \int_0^t b(\mathcal{L}(X^k_s|\tau^k_\partial > s),X^k_s)ds$$

and the metric of uniform convergence on compact intervals of time:

$$\bar{d}_{\infty}((x^1_t, f^1_t, w^1_t)_{0\leq t<\infty}, (x^2_t, f^2_t, w^2_t)_{0\leq t<\infty})$$
$$= \sum_{n=1}^{\infty} 2^{-n}(\sup_{t\leq n}(|x^1_t - x^2_t| + |f^1_t - f^2_t| + |w^1_t - w^2_t|) \wedge 1).$$

We claim that $\{\mathcal{L}((X^k_{t\wedge\tau^k_\partial}, F^k_{t\wedge\tau^k_\partial}, W^k_{t\wedge\tau^k_\partial})_{0\leq t<\infty})\}$ is tight in $\mathcal{P}((\mathcal{C}([0,\infty);\bar{D}\times\mathbb{R}^d\times\mathbb{R}^d), \bar{d}_\infty))$.

Aldous' condition [2, Theorem 1] gives that $\{\mathcal{L}((X^k_{t\wedge\tau^k_\partial}, F^k_{t\wedge\tau^k_\partial}, W^k_{t\wedge\tau^k_\partial})_{0\leq t\leq T})\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\bar{D}\times\mathbb{R}^d\times\mathbb{R}^d))$ hence in $\mathcal{P}(\mathcal{C}([0,T];\bar{D}\times\mathbb{R}^d\times\mathbb{R}^d))$ (equipped with the uniform metric) for any $T<\infty$. We now fix $\epsilon>0$. Then there exists for each $T\in\mathbb{N}$ some $K_T\subseteq \mathcal{C}([0,T];\bar{D}\times\mathbb{R}^d\times\mathbb{R}^d)$ compact such that $\mathbb{P}((X^k_{t\wedge\tau^k_\partial}, F^k_{t\wedge\tau^k_\partial}, W^k_{t\wedge\tau^k_\partial})_{0\leq t\leq T}\notin K_T)<\epsilon 2^{-T}$. We therefore define

$$\mathcal{K} = \{f\in(\mathcal{C}([0,\infty);\bar{D}\times\mathbb{R}^d\times\mathbb{R}^d), d^\infty) : (f_t)_{0\leq t\leq T}\in K_T \quad \text{for all} \quad T\in\mathbb{N}\}.$$

We see that $\mathcal{K}$ is clearly compact in $(\mathcal{C}([0,\infty);\bar{D}\times\mathbb{R}^d\times\mathbb{R}^d), \bar{d}^\infty)$, and moreover $\mathbb{P}((X^k_{t\wedge\tau^k_\partial}, F^k_{t\wedge\tau^k_\partial}, W^k_{t\wedge\tau^k_\partial})_{0\leq t<\infty}\notin\mathcal{K})\leq\sum_{T=1}^\infty \epsilon 2^{-T}\leq\epsilon$. This establishes the claim.

Step 2. We equip $[0,\infty]$ with the topology given by the one-point compactification of $[0,\infty)$, metrised with the metric $d_{[0,\infty]}(x,y) = |\frac{1}{x+1}-\frac{1}{y+1}|$. Then $\{\mathcal{L}(\tau^k_\partial)\}$ must be tight after compactification, hence the joint laws are tight, so that

$$\{\mathcal{L}(((X^k_{t\wedge\tau^k_\partial}, F^k_{t\wedge\tau^k_\partial}, W^k_t)_{0\leq t<\infty}, \tau^k_\partial))\}$$

is tight in

$$\mathcal{P}((\mathcal{C}([0,\infty);\bar{D}\times\mathbb{R}^d\times\mathbb{R}^d), \bar{d}_\infty)\times([0,\infty], d_{[0,\infty]})).$$

We consider any convergent in distribution subsequential limit

$$((X_{t\wedge\tau_\partial}, F_{t\wedge\tau_\partial}, W_{t\wedge\tau_\partial})_{0\leq t<\infty}, \tau_\partial),$$

so that on some new probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ we have $\mathbb{P}'$-almost sure convergence in $\bar{d}^\infty$ by Skorokhod's representation theorem. Having almost sure convergence (rather than convergence in distribution) shall become useful in Step 4. We equip $(\Omega',\mathcal{F}',\mathbb{P}')$ with the filtration $\mathcal{F}'_t := \cap_{h>0}\sigma((X_{s\wedge\tau_\partial}, F_{s\wedge\tau_\partial}, W_s)_{0\leq s\leq t+h}, \tau_\partial\wedge(t+h))$. We see that $(W_t)_{t\geq 0}$ must be an $(\mathcal{F}'_t)_{t\geq 0}$-Brownian motion and τ_∂ an $(\mathcal{F}'_t)_{t\geq 0}$-stopping time. It is now sufficient to show that:

1. $(\mathcal{L}(X^k_t|\tau^k_\partial>t), -\ln\mathbb{P}(\tau^k_\partial>t))_{0\leq t<\infty}\to(\mathcal{L}(X_t|\tau_\partial>t), -\ln\mathbb{P}(\tau_\partial>t))_{0\leq t<\infty}$ in d^∞;
2. (X, W, τ_∂) is a global weak solution of (1.5).

Step 3. Next, we establish that there exists an $\mathbb{F}'$-adapted and uniformly bounded process b_t such that

$$dX_t = b_t dt + dW_t, \quad 0\leq t<\tau_\partial = \inf\{t>0 : X_{t-}\in\partial D\}.$$

Corollary C.2 then gives that $\mathbb{P}(\tau_\partial = t)=0$ whilst Lemma C.1 gives that $\mathbb{P}(\tau_\partial>t)>0$ for all $t\geq 0$.

We note that (X, W, τ_∂, F) $\mathbb{P}'$-almost surely satisfies

$$dX_t = dF_t + dW_t, \quad t\leq\tau_\partial, \tag{8.2}$$

whereby W_t is a Brownian motion up to time τ_∂ and moreover F has B-Lipschitz paths. We now define $b_t = \lim_{h\to 0}\frac{F_{t+h}-F_t}{h}\in[-B,B]$ when the limit exists and $b_t=0$ otherwise. Since F_t is Lipschitz, Rademacher's theorem allows us to see that

$$dX_t = b_t dt + dW_t, \quad t\leq\tau_\partial.$$

We now seek to show that $\mathbb{P}'(\tau_\partial = \inf\{t > 0 : X_{t-} \in \partial D\}) = 1$. We let $\tau'_\partial = \inf\{t : X_{t-} \in \partial D\}$. Clearly we must have $X_{\tau_\partial -} \in \partial D$ if $\tau_\partial < \infty$ hence it is sufficient to show $\mathbb{P}'(\tau'_\partial < \tau_\partial) = 0$. We must have $X_t \in \bar{D}$ for every $t \leq \tau_\partial$. Since ∂D is smooth and $\mathbb{P}'$-almost surely X_t satisfies (8.2), if $\tau'_\partial < \tau_\partial$ then $\mathbb{P}'$-almost surely there exists $\tau'' \in (\tau'_\partial, \tau_\partial)$ such that $X_{\tau''} \notin \bar{D}$. This is impossible, thus $\mathbb{P}'(\tau'_\partial < \tau_\partial) = 0$.

Step 4. We establish that $\mathcal{L}(X^k_t|\tau^k_\partial > t) \to \mathcal{L}(X_t|\tau_\partial > t)$ in W and $\mathbb{P}(\tau^k_\partial > t) \to \mathbb{P}(\tau_\partial > t)$ pointwise in t.

Since $\mathbb{P}(\tau_\partial = t) = 0$, $\mathbb{P}(\tau^k_\partial > t) \to \mathbb{P}(\tau_\partial > t)$. We now take $\phi \in C_b(D)$ and extend ϕ to a $\bar{D}$ by setting $\phi(x) = 0$ for $x \in \partial D$. We have $X^k_{t\wedge\tau^k_\partial} \to X_{t\wedge\tau_\partial}$ $\mathbb{P}'$-almost surely. Unless $\tau^k_\partial > t$ for arbitrarily large k and $\tau_\partial \leq t$ we must have $\phi(X^k_{t\wedge\tau^k_\partial}) \to \phi(X_{t\wedge\tau_\partial})$. However since $\tau^k_\partial \to \tau_\partial$, $\mathbb{P}'(\limsup_{k\to\infty} \tau^k_\partial \geq t \geq \tau_\partial) \leq \mathbb{P}'(\tau_\partial = t) = 0$ hence $\phi(X^k_{t\wedge\tau^k_\partial}) \to \phi(X_{t\wedge\tau_\partial})$ $\mathbb{P}'$-almost surely. Therefore we have $\mathcal{L}(X^k_t) \to \mathcal{L}(X_t)$ in $\mathcal{M}(D)$ hence $\mathcal{L}(X^k_t|\tau^k_\partial > t) \to \mathcal{L}(X_t|\tau_\partial > t)$ in W.

Step 5. We now establish that $(\mathcal{L}(X^k_t|\tau^k_\partial > t), -\ln\mathbb{P}(\tau^k_\partial > t))_{0\leq t<\infty} \to (\mathcal{L}(X_t|\tau_\partial > t) - \ln\mathbb{P}(\tau_\partial > t))_{0\leq t<\infty}$ in d^∞.

We begin by establishing that for all $T < \infty$ we have

$$\sup_{t\leq T}|-\ln\mathbb{P}(\tau^k_\partial > t) + \ln\mathbb{P}(\tau_\partial > t)| \to 0 \quad \text{as} \quad k \to \infty. \tag{8.3}$$

We will then establish that for all $T < \infty$ we have

$$\sup_{t\leq T} W(\mathcal{L}(X^k_t|\tau^k_\partial > t), \mathcal{L}(X_t|\tau_\partial > t)) \to 0 \quad \text{as} \quad k \to \infty. \tag{8.4}$$

These would then imply $(\mathcal{L}(X^k_t|\tau^k_\partial > t), -\ln\mathbb{P}(\tau^k_\partial > t))_{0\leq t<\infty} \to (\mathcal{L}(X_t|\tau_\partial > t) - \ln\mathbb{P}(\tau_\partial > t))_{0\leq t<\infty}$ in d^∞.

$\mathbb{P}(\tau^k_\partial > t)$ and $\mathbb{P}(\tau_\partial > t)$ are continuous, non-negative, non-increasing in t, and uniformly (in $k \in \mathbb{N}$, $t \leq T$) bounded away from 0. This and Step 4 imply (8.3) and that

$$\sup_{\substack{0\leq t\leq t+h\leq T\\ k\in\mathbb{N}}} \mathbb{P}(t < \tau^k_\partial \leq t+h) \to 0 \quad \text{as} \quad h \to 0 \tag{8.5}$$

by elementary analysis. We now turn to establishing (8.4). We calculate

$$\begin{aligned} &W(\mathcal{L}(X^k_{t+h}|\tau^k_\partial > t+h), \mathcal{L}(X^k_t|\tau^k_\partial > t)) \\ &\leq W(\mathcal{L}(X^k_{t+h}|\tau^k_\partial > t+h), \mathcal{L}(X^k_t|\tau^k_\partial > t+h)) + W(\mathcal{L}(X^k_t|\tau^k_\partial > t+h), \mathcal{L}(X^k_t|\tau^k_\partial > t)). \end{aligned} \tag{8.6}$$

We begin by bounding $W(\mathcal{L}(X^k_t|\tau^k_\partial > t+h), \mathcal{L}(X^k_t|\tau^k_\partial > t))$. We observe that

$$\begin{aligned} \mathcal{L}(X^k_t) = \mathcal{L}(X^k_t|\tau^k_\partial > t+h) \quad &\underbrace{\mathbb{P}(\tau^k_\partial > t+h)}_{=\mathbb{P}(\tau^k_\partial > t) - \mathbb{P}(t<\tau^k_\partial\leq t+h)} \\ +\mathcal{L}(X^k_t|t < \tau^k_\partial \leq t+h)\mathbb{P}(t < \tau^k_\partial \leq t+h) &= \mathcal{L}(X^k_t|\tau^k_\partial > t+h)\mathbb{P}(\tau^k_\partial > t) \\ &+\big(\mathcal{L}(X^k_t|t < \tau^k_\partial \leq t+h) - \mathcal{L}(X^k_t|\tau^k_\partial > t+h)\big)\mathbb{P}(t < \tau^k_\partial \leq t+h). \end{aligned}$$

Therefore we have

$$\begin{aligned} &\mathcal{L}(X^k_t|\tau^k_\partial > t) = \mathcal{L}(X^k_t|\tau^k_\partial > t+h) \\ &+\big(\mathcal{L}(X^k_t|t < \tau^k_\partial \leq t+h) - \mathcal{L}(X^k_t|\tau^k_\partial > t+h)\big)\frac{\mathbb{P}(t < \tau^k_\partial \leq t+h)}{\mathbb{P}(\tau^k_\partial > t)}, \end{aligned}$$

so that using (8.5) we have

$$W(\mathcal{L}(X^k_t|\tau^k_\partial > t), \mathcal{L}(X^k_t|\tau^k_\partial > t+h)) \leq ||\mathcal{L}(X^k_t|\tau^k_\partial > t) - \mathcal{L}(X^k_t|\tau^k_\partial > t+h)||_{\text{TV}} \leq \frac{\mathbb{P}(t < \tau^k_\partial \leq t+h)}{\mathbb{P}(\tau^k_\partial > t)}.$$

We have $W(\mathcal{L}(X^k_t|\tau^k_\partial > t+h), \mathcal{L}(X^k_{t+h}|\tau^k_\partial > t+h)) \leq \frac{\mathbb{E}[|X^k_{(t+h)\wedge\tau^k_\partial} - X^k_{t\wedge\tau^k_\partial}|]}{\mathbb{P}(\tau^k_\partial > t+h)}$ so that using (8.6) we have

$$\sup_{\substack{k\in\mathbb{N}\\ 0\leq t\leq t+h\leq T}} W(\mathcal{L}(X^k_{t+h}|\tau^k_\partial > t+h), \mathcal{L}(X^k_t|\tau^k_\partial > t))$$

$$\leq \sup_{\substack{k\in\mathbb{N}\\ 0\leq t\leq t+h\leq T}} \Big(\frac{\mathbb{P}(t < \tau^k_\partial \leq t+h)}{\mathbb{P}(\tau^k_\partial > t)} + \frac{\mathbb{E}[|X^k_{(t+h)\wedge\tau^k_\partial} - X^k_{t\wedge\tau^k_\partial}|]}{\mathbb{P}(\tau^k_\partial > t+h)}\Big) \to 0 \quad \text{as} \quad h \to 0.$$

Therefore using Step 4 we have (8.4).

Step 6. By considering the martingale problem, we see that $b_t = b(\mathcal{L}(X_t|\tau_\partial > t), X_t)$ for $t < \tau_\partial$, hence (X, τ_∂, W) must be a global weak solution of (1.5). Since $\kappa \ni \mathcal{L}(X^k_0) \overset{W}{\to} \mathcal{L}(X_0)$ and κ is compact in $\mathcal{P}_W(D)$, $\mathcal{L}(X_0) \in \kappa$. Thus $(\mathcal{L}(X_t|\tau_\partial > t), -\ln\mathbb{P}(\tau_\partial > t))_{0\leq t<\infty} \in \Xi(\kappa)$. Using Step 5 we have established that for any sequence $(\mathcal{L}(X^k_t|\tau^k_\partial > t), -\ln\mathbb{P}(\tau^k_\partial > t))_{0\leq t<\infty}$ in $\Xi(\kappa)$ there is a further subsequence converging in d^∞ to an element $(\mathcal{L}(X_t|\tau_\partial > t), -\ln\mathbb{P}(\tau_\partial > t))_{0\leq t<\infty}$ of $\Xi(\kappa)$. This concludes our proof of Lemma 8.1.

□

8.2 Proof of Theorem 2.10

Our goal is to establish tightness of $\{\mathcal{L}((m^N_t, J^N_t)_{0\leq t<\infty})\}$ and characterise the limit distributions as being supported on Ξ – the set of flows of laws of a stochastic process.

To characterise subsequential limits the strategy we would like to employ is to use martingale methods to chararacterise subsequential limits as being supported on the solution set of a nonlinear Fokker-Planck PDE, then to show that these PDE solutions correspond to global weak solutions of (1.5).

Formally speaking, subsequential limits should correspond to weak solutions of the nonlinear Fokker-Planck equation:

$$\partial_t u + \nabla\cdot\left(b\left(\frac{u}{\int_D u(y)dy}, x\right)u\right) = \frac{1}{2}\Delta u, \qquad u\big|_{\partial D} = 0$$

renormalised to have mass 1. We may rigorously show that subsequential limits of $\{(m^N_t, J^N_t)_{0\leq t<\infty}\}$ correspond to weak solutions of this PDE. However on unbounded domains we cannot directly show these PDE solutions correspond to solutions of the McKean-Vlasov SDE (1.5) as we need to make use of a uniqueness theorem [28, Theorem 1.1] for solutions of the linear Fokker-Planck equation which requires boundedness of the domain.

We will instead consider a notion of solution which satisfies a certain approximation condition upon truncation of the domain to a large but bounded subdomain D_R of D. Proposition 7.2 allows us to couple our N-particle system $\vec{X}^N$ to an N-particle system $\vec{X}^{N,R}$ on D_R and obtain uniform controls on the difference between the two N-particle systems. Thus we show subsequential limits satisfy this approximation condition, and by martingale methods are solutions of our PDE.

We then show that such approximable PDE solutions correspond to solutions of the McKean-Vlasov SDE (1.5).

Overview

For $R > R_{\min} + 1$, we take $\vec{X}^{N,R}$ to be the particle system on the subdomain $D_R \subseteq D$ whose existence is guaranteed by Proposition 7.2 with associated empirical measure valued process and jump process respectively given by

$$m_t^{N,R} = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^{N,R,i}}, \quad J_t^{N,R} = \frac{1}{N}\#\{\text{jumps up to time } t \text{ by } \vec{X}^{N,R}\}.$$

We define for $1 + R_{\min} < R < \infty$ the test functions

$$C_0^\infty(\bar{D}_R \times [0,\infty)) = \{\varphi \in C_c^\infty(\bar{D}_R \times [0,\infty)) : \varphi_{|_{\partial D_R \times [0,\infty)}} \equiv 0\} \tag{8.7}$$

and define $C_0^\infty(\bar{D} \times [0,\infty))$ in the same manner, with D_R replaced with D.

We define

$$\begin{aligned} M_t^{\varphi,N,R} &:= \big(1 - \frac{1}{N}\big)^{NJ_t^{N,R}}\langle m_t^{N,R}(\cdot), \varphi(\cdot,t)\rangle - \langle m_0^{N,R}(\cdot,0), \varphi(\cdot,0)\rangle \\ &\quad - \int_0^t \big(1 - \frac{1}{N}\big)^{NJ_s^{N,R}}\langle m_s^{N,R}(\cdot), \partial_s\varphi(\cdot,s) + b(m_s^N,\cdot)\cdot\nabla\varphi(\cdot,s) \\ &\quad + \frac{1}{2}\Delta\varphi(\cdot,s)\rangle ds, \quad 0 \le t \le T, \quad \varphi \in C_0^\infty(\bar{D}_R \times [0,\infty)), \end{aligned} \tag{8.8}$$

and define $M_t^{\varphi,N}$ in the same manner, with D_R replaced with D and $m^{N,R}$ replaced with m^N.

By showing these are martingales and using the Martingale Central Limit Theorem [31, Theorem 2.1] we establish the following proposition.

Proposition 8.2. *For $R > 1 + R_{\min}$, $T < \infty$ and for fixed test function $\varphi \in C_0^\infty(\bar{D}_R \times [0,\infty))$, $(M_t^{\varphi,N,R})_{0\le t\le T}$ (and similarly $M_t^{\varphi,N})_{0\le t\le T}$ for $\varphi \in C_0^\infty(\bar{D} \times [0,\infty))$) converges to zero uniformly in probability,*

$$\sup_{t\le T}|M_t^{\varphi,N,R}| \to 0 \quad \textit{in probability} \quad \textit{as} \quad N \to \infty. \tag{8.9}$$

We then establish tightness of $\{\mathcal{L}((m_t^{N,R}, J_t^{N,R})_{0\le t\le T})\}$ by combing Proposition 8.2 with the estimates of Section 4 (which prevent mass accumulating on the boundary).

Proposition 8.3. *We show for $R > 1 + R_{\min}$ and $T < \infty$ that $\{\mathcal{L}((m_t^{N,R}, J_t^{N,R})_{0\le t\le T})\}$ (similarly $\{\mathcal{L}((m_t^N, J_t^N)_{0\le t\le T})\}$) is tight in $\mathcal{P}(\mathcal{D}([0,T]; \mathcal{P}_W(D_R) \times \mathbb{R}_{\ge 0}))$ (respectively $\mathcal{P}(\mathcal{D}([0,T]; \mathcal{P}_W(D) \times \mathbb{R}_{\ge 0}))$) with almost surely continuous limit distributions.*

It is then simple to use Proposition 8.3 to establish the following proposition.

Proposition 8.4. *$\{\mathcal{L}((m_t^N, J_t^N)_{0\le t<\infty})\}$ is tight in $\mathcal{P}((\mathcal{D}([0,\infty); \mathcal{P}_W(D) \times \mathbb{R}_{\ge 0}), d^{\mathcal{D}}))$ with almost surely continuous limit distributions.*

Along subsequential limits we have $(1 - \frac{1}{N})^{NJ_t^N} \to e^{-J_t}$ and $m_t^N \to m_t$ so that Proposition 8.2 gives us that $y_t := e^{-J_t}m_t$ almost surely corresponds to a weak solution of

$$\langle \varphi, y_t\rangle - \langle \varphi, y_0\rangle = \int_0^t \langle y_s(\cdot), \partial_s\varphi(\cdot,s) + b(m_s,\cdot)\cdot\nabla\varphi(\cdot,s) + \frac{1}{2}\Delta\varphi(\cdot,s)\rangle ds = 0,$$
$$0 \le t \le T, \quad \varphi \in C_0^\infty(\bar{D} \times [0,\infty)).$$

We would then like to show that such a PDE solution corresponds to a solution of the McKean-Vlasov SDE (1.5) by constructing a diffusion killed at the boundary ∂D with drift $b(y_t, X_t)$ and showing that $\mathcal{L}(X_t) = y_t$. This final step requires a uniqueness result of Porretta [28, Theorem 1.1] for weak solutions of the linear Fokker-Planck PDE (both y_t and the $\mathcal{L}(X_t)$ satisfy the same linear Fokker-Planck PDE with fixed drift $b(y_t,\cdot)$). Availing ourselves of this uniqueness theorem, however, requires the following:

1. We require $y = y_t \otimes dt$ to have a density with respect to $\mathrm{Leb}_{D\times[0,\infty)}$. Lemma 6.1 allows us to see that this is the case.
2. We require y_0 to have a density with respect to Leb_D. Lemma 6.1 allows us to see that this is the case after arbitrarily small time intervals. This issue may be overcome by arguing after a small time interval $t_0 > 0$, showing that $(y_{t_0+t})_{t\geq 0}$ corresponds to a McKean-Vlasov solution, then taking a limit as $t_0 \to 0$ using Lemma 8.1.
3. We require D to be bounded, whereas we wish to include the case where D is unbounded. To address this issue, we employ the coupling of Section 7. Since D_R is bounded, we may apply the above strategy to the coupled particle system $\vec{X}^{N,R}$. By then employing the uniform controls of Proposition 7.2 and changing our notion of PDE solution, we are able to circumvent this problem.

We now introduce our notion of PDE solution. Given $y \in \mathcal{C}([0,\infty);\mathcal{P}_W(D))$ and $R > 1 + R_{\min}$ we define

$$\begin{aligned}\mathcal{H}_{R,T}(y) = \{z \in \mathcal{C}([0,T];\mathcal{M}(D_R)) \cap L^1(D_R\times[0,T]) : z_t \in L^1(D_R) \quad \text{for all} \quad t \in \mathbb{Q}_{>0} \quad \text{and}\\ \langle z_t(\cdot),\varphi(\cdot,t)\rangle - \langle z_0(\cdot),\varphi(\cdot,0)\rangle - \int_0^t \langle z_s(\cdot), \partial_s\varphi(\cdot,s) + b(y_s,\cdot)\cdot\nabla\varphi(\cdot,s) + \frac{1}{2}\Delta\varphi(\cdot,s)\rangle ds = 0,\\ 0\leq t\leq T, \quad \varphi\in C_0^\infty(\bar{D}_R\times[0,\infty))\}.\end{aligned} \tag{8.10}$$

This is the solution set of the linear Fokker-Planck equation on the truncated domain and truncated time interval with drift given by $b(y_s,\cdot)$. We now define the following notion of approximable PDE solution for the nonlinear Fokker-Planck equation as

$$\begin{aligned}\mathcal{S} = \{(y,f) \in \mathcal{C}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0}) : \text{ for every } \epsilon > 0 \text{ and } T<\infty \text{ we have for } R<\infty\\ \text{arbitrarily large that there exists } z\in\mathcal{H}_{R,T}(y) \text{ with } \sup_{t\leq T}\lvert\lvert y_t e^{-f_t} - z_t\rvert\rvert_{\mathrm{TV}} \leq \epsilon\}.\end{aligned} \tag{8.11}$$

Note that at this point, we have not established existence of either PDE solutions or McKean-Vlasov solutions for given initial data. We will combine Proposition 8.2 with Lemma 6.1 to show that any subsequential limit of our Fleming-Viot particle system must meet the criteria pathwise to being a PDE solution.

Proposition 8.5. *We suppose that some subsequence of* $\{(m^N_t, J^N_t)_{0\leq t<\infty}\}$ *converges in* $(\mathcal{D}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0}), d^{\mathcal{D}})$ *in distribution to* $(m_t,J_t)_{0\leq t<\infty}$*. Then* $(m_t,J_t)_{0\leq t<\infty}\in\mathcal{S}$ *almost surely.*

We then show that any such PDE solution must correspond to a solution of our McKean-Vlasov SDE (1.5).

Proposition 8.6. *Approximable PDE solutions correspond to McKean-Vlasov solutions:*

$$\mathcal{S} \subseteq \Xi \cap \mathcal{C}((0,\infty);L^1(D)). \tag{8.12}$$

Taken together, these give Theorem 2.10.

Proof of Proposition 8.2

We provide here the proof for $M^{\varphi,N,R}_t$. The proof for $M^{\varphi,N}_t$ is identical with D_R, $m^{N,R}$, $C_0^\infty(\bar{D}_R\times[0,\infty))$ and $\tau^{N,R}_k$ replaced with D, m^N, $C_0^\infty(\bar{D}\times[0,\infty))$ and τ^N respectively.

We fix $\varphi\in C_0^\infty(\bar{D}_R\times[0,\infty))$, $1+R_{\min} < R \leq \infty$ and establish $(M^{\varphi,N,R}_t)_{0\leq t\leq T}$ converges to zero in distribution.

It is trivial that $M_t^{\varphi,N,R}$ is integrable for all t. We recall that $\tau_k^{N,R}$ is the k^{th} death time of any particle in the coupled system with $\tau_0^{N,R} := 0$. Inducting in k, we shall establish that $M_t^{\varphi,N,R,k} := M_{t\wedge\tau_k}^{\varphi,N,R}$ is a martingale. This is trivially true for $k = 0$.

We note that $J_t^{N,R,k}$ is constant on $[\tau_k, \tau_{k+1})$, and moreover the infinitesimal generator of $\langle m_t^{N,R}(\cdot), \varphi(\cdot,t)\rangle$ is $\langle m_t^{N,R}(\cdot), \partial_t\varphi(\cdot,t)+b(m_t^N,\cdot)\cdot\nabla\varphi(\cdot,t)+\frac{1}{2}\Delta\varphi(\cdot,t)\rangle$. Therefore we have $M_{t\wedge\tau_{k+1}^{N,R}-}^{\varphi,N,R,k+1} = \mathbb{1}(t<\tau_{k+1})M_t^{\varphi,N,R,k+1} + \mathbb{1}(t\geq\tau_{k+1}^{N,R})M_{\tau_{k+1}^{N,R}-}^{\varphi,N,R,k+1}$ is a martingale.

At time $\tau_{k+1}^{N,R}$, the particle which dies (let's say particle i) jumps to a uniformly chosen different particle (let's say particle j). Since φ vanishes on the boundary ∂D_R, the value of $\varphi(X_t^{R,i},t)$ jumps from 0 to $\varphi(X_{\tau_{k+1}-}^{R,j}, \tau_{k+1}-)$, the expected value of which must be

$$\frac{1}{N-1}\sum_{j\neq i}\varphi(X_t^j,t) = \frac{N}{N-1}\langle m_{\tau_{k+1}-}^{N,R}, \varphi(\cdot,\tau_{k+1}-)\rangle.$$

Thus we have

$$\begin{aligned}\mathbb{E}[\langle m_{\tau_{k+1}}^{N,R}(\cdot),\varphi(\cdot,\tau_{k+1})\rangle|\mathcal{F}_{\tau_{k+1}-}] &= \frac{1}{N}\Big[\frac{N}{N-1}\langle m_{\tau_{k+1}-}^{N,R},\varphi(\cdot,\tau_{k+1}-)\rangle\\ +N\langle m_{\tau_{k+1}-}^{N,R},\varphi(\cdot,\tau_{k+1}-)\rangle\Big] &= \Big(1-\frac{1}{N}\Big)^{-1}\langle m_{\tau_{k+1}-}^{N,R},\varphi(\cdot,\tau_{k+1}-)\rangle.\end{aligned}$$

Thus $\mathbb{E}[M_{\tau_{k+1}^{N,R}}^{k+1}|\mathcal{F}_{\tau_{k+1}-}] = M_{\tau_{k+1}^{N,R}-}^{k+1}$. Therefore we have $M_t^{\varphi,N,R,k+1}$ is a martingale. Thus $M_t^{\varphi,N,R}$ is a martingale.

We shall now employ the Martingale Central Limit Theorem [31, Theorem 2.1] to obtain convergence to 0 in probability as $N\to\infty$, by obtaining controls on the quadratic variation. We note that the control we obtain on the quadratic variation is similar to that obtained in [30]. There the author established convergence of the Fleming-Viot process driven by a killed strong Markov process. They used a martingale given in terms of the semigroup, whereas here the martingales $M_t^{\varphi,N,R}$ and $M_t^{\varphi,N}$ are given in terms of the infinitesimal generator. The use of the semigroup allowed them to obtain in [30, Theorem 1] a quantitative rate of convergence given in terms of arbitrary bounded measurable test functions – the use of martingales given in terms of the infinitesimal generator could only provide a rate of convergence given in terms of test functions belonging to the domain of the infinitesimal generator. Such an approach based on the linear semigroup becomes problematic here, however, due to the presence of the mean-field term.

Between times $\tau_k^{N,R}$ and $\tau_{k+1}^{N,R}$, we have

$$dM_t^{\varphi,N,R} = \Big(1-\frac{1}{N}\Big)^{NJ_t^{N,R}}\frac{1}{N}\sum_{i=1}^N\nabla\varphi(X_t^{R,i},t)\cdot dW_t^{R,i} + \text{drift terms}.$$

Hence we have $[M^{N,R,\varphi}]_{T\wedge\tau_{k+1}-} - [M^{N,R,\varphi}]_{T\wedge\tau_k} \leq \frac{1}{N}||\nabla\varphi||_\infty^2(\tau_{k+1}^{N,R}-\tau_k^{N,R})$. Moreover, at each jump time, the jumps of $M^{\varphi,N,R}$ are bounded by

$$\begin{aligned}&|M_{T\wedge\tau_{k+1}^{N,R}}^{\varphi,N,R} - M_{T\wedge\tau_{k+1}^{N,R}-}^{\varphi,N,R}| \leq \big(1-\frac{1}{N}\big)^k\big|\big\langle m_{T\wedge\tau_{k+1}^{N,R}}^{N,R}(\cdot) - m_{T\wedge\tau_{k+1}^{N,R}-}^{N,R}(\cdot),\varphi(\cdot,T\wedge\tau_{k+1}^{N,R})\big\rangle\big|\\ &+\big(1-\frac{1}{N}\big)^k\big|\big\langle(1-\frac{1}{N})m_{T\wedge\tau_{k+1}^{N,R}}^{N,R}(\cdot) - m_{T\wedge\tau_{k+1}^{N,R}}^{N,R}(\cdot),\varphi(\cdot,T\wedge\tau_{k+1}^{N,R})\big\rangle\big| \leq 3\big(1-\frac{1}{N}\big)^k\frac{||\varphi||_\infty^2}{N}.\end{aligned}$$

Therefore the jumps of $[M^{\varphi,N,R}]_t$ are bounded by

$$[M^{\varphi,N,R}]_{T\wedge\tau_{k+1}^{N,R}} - [M^{\varphi,N,R}]_{T\wedge\tau_{k+1}^{N,R}-} \leq \Big(1-\frac{1}{N}\Big)^{2k}\frac{9||\varphi||_\infty^2}{N^2} \leq \Big(1-\frac{1}{N}\Big)^k\frac{9||\varphi||_\infty^2}{N^2}.$$

Therefore summing the geometric series we have

$$[M^{N,R,\varphi}]_T \leq \frac{1}{N}||\nabla\varphi||_{\infty}^2 T + \frac{9||\varphi||_{\infty}^2}{N} \to 0 \quad \text{as} \quad N \to \infty.$$

Thus we have $[M^{N,R,\varphi}]_T$ converges to zero in probability as $N \to \infty$. Moreover it is trivial that $\mathbb{E}[\sup_{t\leq T}|M_t^{N,R,\varphi} - M_{t-}^{N,R,\varphi}|] \to 0$ in probability as $N \to \infty$. Thus using the Martingale Central Limit Theorem [31, Theorem 2.1] we have $(M_t^{\varphi,N,R})_{0\leq t\leq T} \to 0$ uniformly in probability.

□

Proof of Proposition 8.3

We provide here the proof for $\{\mathcal{L}((m_t^{N,R}, J_t^{N,R})_{0\leq t\leq T})\}$. The proof for $\{\mathcal{L}((m_t^N, J_t^N)_{0\leq t\leq T})\}$ is identical, but with D_R, $m^{N,R}$, $C_0^\infty(\bar{D}_R \times [0,\infty))$ and $\tau_k^{N,R}$ replaced with D, m^N, $C_0^\infty(\bar{D} \times [0,\infty))$ and τ^N, respectively, aside from two places where Lemma 6.2 must be invoked.

The proof can be broken down into the following steps:

1. We begin by establishing that $\{\mathcal{L}((J_t^{N,R})_{0\leq t\leq T})\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathbb{R}_{\geq 0}))$, and moreover any limit distribution is supported on the space of continuous functions.
2. We then show $\{\mathcal{L}((m_t^{N,R})_{0\leq t\leq T})\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(D_R)))$.
3. Having shown that $\{\mathcal{L}(((m_t^{N,R})_{0\leq t\leq T},(J_t^{N,R})_{0\leq t\leq T}))\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(D_R)) \times \mathcal{D}([0,T];\mathbb{R}_{\geq 0}))$ with limit distributions supported on $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(D_R)) \times \mathcal{C}([0,T];\mathbb{R}_{\geq 0}))$, we establish that $\{\mathcal{L}((m_t^{N,R}, J_t^{N,R})_{0\leq t\leq T})\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(D_R)\times\mathbb{R}_{\geq 0}))$ with almost surely continuous limit distributions.

Step 1

Markov's inequality and Proposition 4.10 give that $\{\mathcal{L}(J_T^{N,R} : N \in \mathbb{N})\}$ is tight. Thus it is enough to show the set of laws of $\varsigma_t^N := \big(1-\frac{1}{N}\big)^{NJ_t^{N,R}}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathbb{R}))$ with limit distributions supported on $\mathcal{C}([0,T];\mathbb{R})$. We will employ Aldous' condition [2, Theorem 1]. Since we have $0 \leq \big(1-\frac{1}{N}\big)^{NJ_t^{N,R}} \leq 1$ then $\{\mathcal{L}(\varsigma_t^N)\}$ must be tight for each fixed t. We therefore need to establish [2, Condition A].

We fix $\epsilon, \delta > 0$. As in Part 2 of Proposition 4.7 we take $\hat{K}_{\frac{\epsilon}{2},\frac{\delta}{2}} = V_{\hat{c}(\frac{\epsilon}{2},\frac{\delta}{2})} \subseteq D_R$ such that we have $\limsup_{N\to\infty}\mathbb{P}(\sup_{t\leq T} m_t^{N,R}(\hat{K}^c_{\frac{\epsilon}{2},\frac{\delta}{2}}) \geq \frac{\epsilon}{2}) \leq \frac{\delta}{2}$. Since D_R is bounded, $\hat{K}_{\frac{\epsilon}{2},\frac{\delta}{2}}$ is compact. Here the proof for $\{\mathcal{L}((m_t^N, J_t^N)_{0\leq t\leq T})\}$ diverges from the present proof as D is not necessarily bounded. In this case we use Lemma 6.2 to obtain $R'_{\epsilon,\delta} < \infty$ such that $\limsup_{N\to\infty}\mathbb{P}(\sup_{t\leq T} m_t^{N,R}(B(0,R'_{\epsilon,\delta})^c) \geq \frac{\epsilon}{2}) \leq \frac{\delta}{2}$. In either case we obtain $\tilde{K}_{\epsilon,\delta} = \hat{K}_{\frac{\epsilon}{2},\frac{\delta}{2}} \cap \bar{B}(0,R'_{\epsilon,\delta}) \subseteq D_R$ compact such that

$$\limsup_{N\to\infty}\mathbb{P}(\sup_{t\leq T} m_t^{N,R}(\tilde{K}^c_{\epsilon,\delta}) \geq \epsilon) \leq \delta.$$

We now take $\varphi_{\epsilon,\delta} \in C_c^\infty(D_R)$ such that $\mathbb{1}_{\tilde{K}_{\epsilon,\delta}} \leq \varphi_{\epsilon,\delta} \leq 1$. Thus we have

$$\limsup_{N\to\infty}\mathbb{P}\Big(\sup_{0\leq t\leq T}|1-\langle m_t^{N,R},\varphi_{\epsilon,\delta}(\cdot)\rangle| \geq \epsilon\Big) \leq \delta. \tag{8.13}$$

We then take $M_t^{\varphi_{\epsilon,\delta},N,R}$ as in (8.8) and observe

$$\varsigma_{t+h}^N - \varsigma_t^N = (\varsigma_{t+h}^N - \varsigma_{t+h}^N\langle m_{t+h}^{N,R},\varphi_{\epsilon,\delta}(\cdot)\rangle) - (\varsigma_t^N - \varsigma_t^N\langle m_t,\varphi_{\epsilon,\delta}(\cdot)\rangle)$$
$$+(M_{t+h}^{\varphi_{\epsilon,\delta},N,R} - M_t^{\varphi_{\epsilon,\delta},N,R}) + \int_t^{t+h}\varsigma_s^N\langle m_s^{N,R}(\cdot), b(m_s^N,\cdot)\cdot\nabla\varphi_{\epsilon,\delta} + \frac{1}{2}\Delta\varphi_{\epsilon,\delta}\rangle ds.$$

We bound the first two terms on the right hand side using (8.13), the third term converges to zero in probability using Proposition 8.2 whilst the integrand in the fourth term is bounded (by $C_{\epsilon,\delta} < \infty$ say). Therefore we have

$$\liminf_{N\to\infty} \mathbb{P}\big(\sup_{\substack{0\le t\le t+h' \\ \le t+h\le T}} |\varsigma^N_{t+h'} - \varsigma^N_t| \le 3\epsilon + C_{\epsilon,\delta} h \big) \ge 1 - 2\delta.$$

This establishes [2, Condition A]. Moreover for any subsequential limit in distribution ς^∞ and $\epsilon, \delta > 0$ there exists some $h_{\epsilon,\delta} = \frac{\epsilon}{C_{\epsilon,\delta}} > 0$ such that

$$\mathbb{P}(\sup_{\substack{h'\le h_{\epsilon,\delta} \\ 0\le t\le T-h'}} |\varsigma^\infty_{t+h'} - \varsigma^\infty_t| \ge 5\epsilon) \le 2\delta.$$

Thus as $\delta > 0$ is arbitrary there exists some random $h(\epsilon) > 0$ such that

$$\sup_{\substack{h'\le h(\epsilon) \\ 0\le t\le T-h'}} |\varsigma^\infty_{t+h'} - \varsigma^\infty_t| \le \epsilon \quad \text{almost surely.}$$

Since $\epsilon > 0$ is arbitrary, $\zeta^\infty \in \mathcal{C}([0,T];\mathbb{R})$ almost surely.

Step 2

We show $\{\mathcal{L}((m^{N,R}_t)_{0\le t\le T})\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(\bar{D}_R)))$, then extend this to showing $\{\mathcal{L}((m^{N,R}_t)_{0\le t\le T})\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(D_R)))$.

Since D_R is bounded, [18, Theorem 2.1] gives us the following lemma.

Lemma 8.7 ([18])**.** *We suppose that for every $\varphi \in C^\infty_c(\mathbb{R}^d)$ the laws of $\varrho^N_t := \langle \varphi(\cdot), m^{N,R}_t(\cdot)\rangle$ are tight in $\mathcal{P}(\mathcal{D}([0,T];\mathbb{R}))$. Then $\{\mathcal{L}((m^{N,R}_t)_{0\le t\le T})\}$ must be tight in $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(\bar{D})))$.*

Here the proof for $\{\mathcal{L}((m^N_t, J^N_t)_{0\le t\le T})\}$ diverges from the present proof as D is not necessarily bounded. In this case we obtain Lemma 8.7 by combining [18, Theorem 2.1] with Lemma 6.2.

We now verify the assumptions of Lemma 8.7. We fix $\varphi \in C^\infty_c(\mathbb{R}^d)$ and establish that $\{\mathcal{L}(\varrho^N_t)\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathbb{R}))$ by way of Aldous' criterion [2, Theorem 1].

Since φ is bounded $\{\mathcal{L}(\varrho^N_t)\}$ is tight on the line for fixed t, so it is sufficient to check [2, Condition A]. We let τ^N be a sequence of stopping times and δ_N a sequence of constants as defined in [2, Condition 1]. We write $\varrho^N = \varrho^{N,C} + \varrho^{N,J}$ whereby $\varrho^{N,C}$ is continuous and $\varrho^{N,J}_t = \sum_{t'\le t} \varrho^N_{t'} - \varrho^N_{t'-}$. Then $\varrho^{N,C}$ is a diffusion process with uniformly bounded drift and diffusivity hence we have

$$\varrho^{N,C}_{\tau_N+\delta_N} - \varrho^{N,C}_{\tau_N} \xrightarrow{p} 0.$$

We note that the jumps of ϱ are of magnitude bounded by $\frac{C}{N}$ for some $C < \infty$. Therefore to verify

$$\varrho^{N,J}_{\tau_N+\delta_N} - \varrho^{N,J}_{\tau_N} \xrightarrow{p} 0$$

it is enough to check

$$J^{N,R}_{\tau_N+\delta_N} - J^{N,R}_{\tau_N} \xrightarrow{p} 0.$$

We have this since $\{\mathcal{L}((J^{N,R}_t)_{0\le t\le T})\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathbb{R}))$ with limit distributions supported on $\mathcal{C}([0,T];\mathbb{R})$ (Step 1). Thus we have verified [2, Condition A]

$$\varrho^N_{\tau_N+\delta_N} - \varrho^N_{\tau_N} \xrightarrow{p} 0,$$

and hence have verified the assumption of Lemma 8.7.

Having established $\{\mathcal{L}((m^{N,R}_t)_{0\le t\le T})\}$ is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(\bar{D}_R)))$, we now show it is tight in $\mathcal{P}(\mathcal{D}([0,T];\mathcal{P}_W(D_R)))$. Using Skorokhod's representation theorem, we consider along any subsequence a further subsequence converging on a possibly different probability $(\Omega',\mathcal{F}',\mathbb{P}')$ space in $\mathcal{D}([0,T];\mathcal{P}_W(\bar{D}_R))$ $\mathbb{P}'$-almost surely to $(m^R_t)_{0\leq t\leq T}$. It is sufficient to show $(m^R_t)_{0\leq t\leq T}\in\mathcal{D}([0,T];\mathcal{P}_W(D_R))$ $\mathbb{P}'$-almost surely.

For each $\epsilon,T_0>0$, Part 1 of Proposition 4.7 implies that $m^R_t(K^c_{\epsilon,T_0})<\epsilon$ for every $T_0\leq t\leq T$ $\mathbb{P}'$-almost surely. Therefore $m^R_t(\partial D)=0$ for every $T_0\leq t\leq T$ $\mathbb{P}'$-almost surely. Since T_0 can be made arbitrarily small and $\{\mathcal{L}(m^{N,R}_0)\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D_R))$ we have $m^R_t(\partial D)=0$ for all $0\leq t\leq T$ $\mathbb{P}'$-almost surely.

Step 3

It is sufficient to consider some subsequence on which $((m^{N,R}_t)_{0\leq t\leq T},(J^{N,R}_t)_{0\leq t\leq T})$ converges in $\mathcal{D}([0,T];\mathcal{P}_W(D_R))\times\mathcal{D}([0,T];\mathbb{R}_{\geq 0})$ in distribution, then establish along this subsequence convergence in $\mathcal{D}([0,T];\mathcal{P}_W(D_R)\times\mathbb{R}_{\geq 0})$ in distribution with limit distributions supported on $\mathcal{C}([0,T];\mathcal{P}_W(D_R)\times\mathbb{R}_{\geq 0})$.

Indeed by the Skorokhod Representation Theorem on a possibly different probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ we have along this subsequence $\mathbb{P}'$-almost sure convergence of $((m^{N,R}_t)_{0\leq t\leq T},(J^{N,R}_t)_{0\leq t\leq T})$ to a limit we call $((m^R_t)_{0\leq t\leq T},(J^R_t)_{0\leq t\leq T})$. By Step 1 we have J^R_t is continuous, and hence $\mathbb{P}'$-almost surely $J^{N,R}_t$ converges uniformly to J^R_t.

From the definition of the Skorokhod metric [6, Equation (12.13), Page 124] it is trivial that this implies $(m^{N,R}_t,J^{N,R}_t)_{0\leq t\leq T}$ converges to $(m^R_t,J^R_t)_{0\leq t\leq T}$ $\mathbb{P}'$-almost surely in $\mathcal{D}([0,T];\mathcal{P}_W(D_R)\times\mathbb{R}_{\geq 0})$. We have

$$\Big|M^{N,R,\varphi}_{t+h}-M^{N,R,\varphi}_t-\big(\langle(1-\frac{1}{N})^{NJ^{N,R}_{t+h}}m^{N,R}_{t+h}(\cdot),\varphi(\cdot)\rangle-\langle(1-\frac{1}{N})^{NJ^{N,R}_t}m^{N,R}_t(\cdot),\varphi(\cdot)\rangle\big)\Big| \leq C_\varphi h,\quad \varphi\in C^\infty_c(D),$$

where C_φ is a constant dependent only upon φ. Note that we are viewing φ both as a function in $C^\infty_c(D_R)$ and a function in $C^\infty_0(\bar{D}_R\times[0,\infty))$ which is constant in time up to time T by abuse of notation. Proposition 8.2 then implies that almost surely $(m^R_t)_{0\leq t\leq T}$ satisfies for all $0\leq t\leq t+h\leq T$:

$$|\langle e^{-J^R_{t+h}}m^R_{t+h}(\cdot),\varphi(\cdot)\rangle-\langle e^{-J^R_t}m^R_t(\cdot),\varphi(\cdot)\rangle|\leq C_\varphi h,\quad \varphi\in C^\infty_c(D_R\times[0,T]).$$

We know $m^R_te^{-J^R_t}\in\mathcal{D}([0,T];\mathcal{M}(D_R))$, so that we have

$$|\langle e^{-J^R_t}m^R_t(\cdot),\varphi(\cdot)\rangle-\langle e^{-J^R_{t-}}m^R_{t-}(\cdot),\varphi(\cdot)\rangle|=0,\quad \varphi\in C^\infty_c(D_R),\quad 0\leq t\leq T.$$

This implies $e^{-J^R_t}m^R_t=e^{-J^R_{t-}}m^R_{t-}$ for all $t\leq T$ hence $e^{-J^R_t}m^R_t\in\mathcal{C}([0,T];\mathcal{M}(D_R))$. Thus almost surely $m^R\in\mathcal{C}([0,T];\mathcal{P}(D_R))$. Since W metrises the topology of weak convergence of probability measures [17], we are done. □

Proof of Proposition 8.4

We fix $\epsilon>0$. Then by Proposition 8.3 there exists for each $T\in\mathbb{N}$ some $K_T\subseteq\mathcal{D}([0,T];\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0})$ compact such that $\mathbb{P}((m^N_t,J^N_t)_{0\leq t\leq T}\notin K_T)<\epsilon 2^{-T}$. We therefore define

$$\mathcal{K}=\{f\in\mathcal{D}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0}),d^{\mathcal{D}}):(f_t)_{0\leq t\leq T}\in K_T\quad\text{for all}\quad T\in\mathbb{N}\}.$$

We see that $\mathcal{K}$ is clearly compact in $(\mathcal{D}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0}),d^{\mathcal{D}})$, and moreover $\mathbb{P}((m^N_t,J^N_t)_{0\leq t<\infty}\notin\kappa)\leq\sum_T\epsilon 2^{-T}\leq\epsilon$.

□

Proof of Proposition 8.5

We write $(\Omega', \mathcal{F}', \mathbb{P}')$ for the probability space on which our subsequential limit $(m_t, J_t)_{0\leq t<\infty}$ is defined. We define

$$\begin{aligned}\mathcal{S}_{\epsilon,R,T} = \{(y,f) \in \mathcal{C}([0,\infty); \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}) :\ &\text{there exists} \\ &z \in \mathcal{H}_{R,T}(y) \text{ with } \sup_{t\leq T} ||y_t e^{-f_t} - z_t||_{\text{TV}} \leq \epsilon\}.\end{aligned} \tag{8.14}$$

We claim that

$$\mathbb{P}'((m_t, J_t)_{0\leq t<\infty} \in \mathcal{S}^C_{\epsilon,R,T}) \to 0 \quad \text{as} \quad R \to \infty \tag{8.15}$$

for all $\epsilon > 0$ and $T < \infty$ fixed.

We fix $R < \infty$ for the time being. We take, on the probability space $(\Omega^{N,R}, \mathcal{F}^{N,R}, \mathbb{P}^{N,R})$, the particle system $\vec{X}^{N,R}$ on D_R coupled to $\vec{X}^N$ whose existence is guaranteed by Proposition 7.2. We have by propositions 8.3 and 8.4 that $\{\mathcal{L}(((m^N_t, J^N_t)_{0\leq t<\infty}, (m^{N,R}_t, J^{N,R}_t)_{0\leq t\leq T}))\}$ is tight in $\mathcal{P}((\mathcal{D}([0,\infty); \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}), d^{\mathcal{D}}) \times \mathcal{D}([0,T]; \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}))$ with limit distributions supported on $\mathcal{C}([0,\infty); \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}) \times \mathcal{C}([0,T]; \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0})$. We may therefore take a further subsequence along which $\{((m^N_t, J^N_t)_{0\leq t<\infty}, (m^{N,R}_t, J^{N,R}_t)_{0\leq t\leq T})\}$ is convergent in distribution. Using Skorokhod's representation theorem, these may be supported on a probability space $(\Omega'', \mathcal{F}'', \mathbb{P}'')$ along which $\{((m^N_t, J^N_t)_{0\leq t<\infty}, (m^{N,R}_t, J^{N,R}_t)_{0\leq t\leq T})\}$ is $\mathbb{P}'$-almost surely convergent, to a limit we call $((m_t, J_t)_{0\leq t<\infty}, (m^R_t, J^R_t)_{0\leq t\leq T})$.

Note that we are abusing notation here, writing $(m_t, J_t)_{0\leq t<\infty}$ both for a random variable on $(\Omega', \mathcal{F}', \mathbb{P}')$ and for a random variable on $(\Omega'', \mathcal{F}'', \mathbb{P}'')$. Nevertheless, by construction they have the same law, hence

$$\mathbb{P}'((m_t, J_t)_{0\leq t<\infty} \in \mathcal{S}^C_{\epsilon,R,T}) = \mathbb{P}''((m_t, J_t)_{0\leq t<\infty} \in \mathcal{S}^C_{\epsilon,R,T}).$$

For $t \in \mathbb{Q}_{>0}$ we have by Lemma 6.1 that $m^R_t \in L^1(D_R)$ $\mathbb{P}''$-almost surely. Therefore $m^R_t \in L^1(D_R)$ for all $t \in \mathbb{Q}_{>0}$, $\mathbb{P}''$-almost surely. Moreover Lemma 6.1 gives that $m^R = m^R_t \otimes dt$ satisfies $m^R \in L^1(D_R \times [0,T])$, $\mathbb{P}''$-almost surely. Therefore, by Proposition 8.2 we have

$$(m^R_t e^{-J^R_t})_{0\leq t\leq T} \in \mathcal{H}_{R,T}(m) \quad \mathbb{P}''\text{-almost surely.}$$

Since convergence in Skorokhod space to a continuous function implies uniform convergence, $(m^N_t, J^N_t)_{0\leq t\leq T} \to (m_t, J_t)_{0\leq t\leq T}$ and $(m^{N,R}_t, J^{N,R}_t)_{0\leq t\leq T} \to (m^R_t, J^R_t)_{0\leq t\leq T}$ in $d^\infty_{[0,T]}$ $\mathbb{P}''$-almost surely. Therefore we have

$$\sup_{t\leq T} ||m^R_t - m_t||_{\text{TV}} \leq \liminf_{N\to\infty} \sup_{t\leq T} ||m^{N,R}_t - m^N_t||_{\text{TV}} \quad \mathbb{P}''\text{-almost surely.}$$

Therefore we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}''}\Big[\sup_{t\leq T}||m^R_t e^{-J^R_t} - m_t e^{-J_t}||_{\text{TV}}\Big] &\leq \mathbb{E}^{\mathbb{P}''}\Big[\sup_{t\leq T}||m^R_t - m_t||_{\text{TV}} + 1 \wedge \sup_{t\leq T}|J^R_t - J_t|\Big] \\ &\leq \mathbb{E}^{\mathbb{P}''}\Big[\liminf_{N\to\infty}\big(\sup_{t\leq T}||m^{N,R}_t - m^N_t||_{\text{TV}} + 1 \wedge \sup_{t\leq T}|J^{N,R}_t - J^N_t|\big)\Big] \\ &\overset{\text{Fatou's Lemma}}{\leq} \liminf_{N\to\infty} \mathbb{E}^{\mathbb{P}''}\Big[\sup_{t\leq T}||m^N_t - m^{N,R}_t||_{\text{TV}} + 1 \wedge \sup_{t\leq T}|J^N_t - J^{N,R}_t|\Big] \\ = \liminf_{N\to\infty} \mathbb{E}^{\mathbb{P}^{N,R}}\Big[\sup_{t\leq T}||m^N_t - m^{N,R}_t||_{\text{TV}} &+ 1 \wedge \sup_{t\leq T}|J^N_t - J^{N,R}_t|\Big] \to 0 \quad \text{as} \quad R \to \infty\end{aligned}$$

by Proposition 7.2. Therefore using Markov's inequality we have

$$\begin{aligned}\mathbb{P}'((m_t, J_t)_{0\leq t<\infty} \in \mathcal{S}^C_{\epsilon,R,T}) &= \mathbb{P}''((m_t, J_t)_{0\leq t<\infty} \in \mathcal{S}^C_{\epsilon,R,T}) \\ &\leq \frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}''}\Big[\sup_{t\leq T}||m^R_t e^{-J^R_t} - m_t e^{-J_t}||_{\text{TV}}\Big] \to 0 \quad \text{as} \quad R \to \infty.\end{aligned}$$

Therefore we have (8.15) so that for all $R_0 < \infty$,

$$(m_t, J_t)_{0 \le t < \infty} \in \cup_{R \ge R_0} \mathcal{S}_{\epsilon,R,T} \quad \mathbb{P}'\text{-almost surely.}$$

Therefore we have

$$(m_t, J_t)_{0 \le t < \infty} \in \cap_{\epsilon > 0} \cap_{T \in \mathbb{N}} \cap_{R_0 \in \mathbb{N}} \cup_{R \ge R_0} \mathcal{S}_{\epsilon,R,T} = \mathcal{S} \quad \mathbb{P}'\text{-almost surely.} \quad \square$$

Proof of Proposition 8.6

Step 1

We fix deterministic $(m_t, J_t)_{0 \le t < \infty} \in \mathcal{S}$ and use Girsanov's theorem to construct (X, τ_∂, W) a global weak solution of the SDE

$$dX_t = b(m_t, X_t)dt + dW_t, \quad 0 \le t \le \tau_\partial = \inf\{t > 0 : X_t \in \partial D\}. \tag{8.16}$$

Step 2

For the time being we fix $R, T < \infty$ and assume that there exists $z \in \mathcal{H}_{R,T}(m)$ such that $z_0 \in L^1(D_R)$ and $(z_t)_{0 \le t \le T}$ is a solution of

$$\begin{aligned} \langle z_t(\cdot), \varphi(\cdot, t)\rangle - \langle z_0(\cdot), \varphi(\cdot, 0)\rangle - \int_0^t \langle z_s(\cdot), \partial_s \varphi(\cdot, s) + b(m_s, \cdot) \cdot \nabla \varphi(\cdot, s) \\ + \frac{1}{2}\Delta \varphi(\cdot, s)\rangle ds = 0, \quad 0 \le t \le T, \quad \varphi \in C_0^\infty(\bar{D}_R \times [0, \infty)). \end{aligned} \tag{8.17}$$

Then defining $\tau_\partial^R = \inf\{t > 0 : X_t \in \partial D_R\} \le \tau_\partial$ and $(X_t^R)_{0 \le t \le \tau_\partial^R} := (X_t)_{0 \le t \le \tau_\partial}$ we obtain $(X^R, \tau_\partial^R, W)$ a weak solution of the SDE

$$dX_t^R = b(m_t, X_t^R)dt + dW_t, \quad 0 \le t \le \tau_\partial^R = \inf\{t > 0 : X_t^R \in \partial D_R\} \tag{8.18}$$

such that

$$\sup_{t \le T} ||\mathcal{L}(X_t) - \mathcal{L}(X_t^R)||_{\text{TV}} \le \mathbb{P}(\tau_\partial^R < \tau_\partial \wedge T). \tag{8.19}$$

We now establish that

$$\mathcal{L}(X_t^R) = z_t \quad \text{for} \quad t \le T, \quad (z_t)_{0 \le t \le T} \in \mathcal{C}([0, T); L^1(D_R)). \tag{8.20}$$

Indeed we observe that

$$\varphi(X^R_{t \wedge \tau_\partial^R}, t \wedge \tau_\partial^R) - \varphi(X_0^R, 0) - \int_0^{t \wedge \tau_\partial^R} (\partial_s + b(m_s, X_s^R) \cdot \nabla + \frac{1}{2}\Delta)\varphi(X_s^R, s)ds, \quad 0 \le t \le T$$

is a martingale for every $\varphi \in C_0^\infty(\bar{D} \times [0, \infty))$.

Taking expectation, we see that $\mathcal{L}(X_t)$ – must satisfy the PDE (8.17). Moreover we have $z_0, \mathcal{L}(X_0) \in L^1(D_R)$ and $z_t \otimes dt, \mathcal{L}(X_t) \otimes dt \in L^1(D_R \times [0, T))$. We therefore have $\mathcal{L}(X_t^R) = z_t$ by the uniqueness results of [28, Theorem 1.1], and by [28, Theorem 3.6] we also have $(z_t)_{0 \le t \le T} \in \mathcal{C}([0, T); L^1(D_R))$.

Step 3

We suppose that for all $\epsilon > 0$ and $T < \infty$ there exists $R < \infty$ arbitrarily large such that there exists $z \in \mathcal{H}_{R,T}(m)$ with $\sup_{t \le T} ||z_t - m_t e^{-J_t}||_{\text{TV}} \le \epsilon$ and $z_0 \in L^1(D_R)$. Then we claim

$$(m_t, J_t)_{0 \le t < \infty} \in \Xi \cap \mathcal{C}([0, \infty); L^1(D) \times \mathbb{R}_{>0}). \tag{8.21}$$

We have from Step 2 the sequence of solutions $(X^{R_n},\tau_{\partial}^{R_n},W)$ to (8.18) on the domains D_{R_n} with $R_n\to\infty$ as $n\to\infty$ such that

$$\sup_{t\leq T}||\mathcal{L}(X_t)-m_te^{-J_t}||_{\text{TV}}\leq\sup_{t\leq T}||\mathcal{L}(X_t^{R_n})-\mathcal{L}(X_t)||_{\text{TV}}+\sup_{t\leq T}||\mathcal{L}(X_t^{R_n})-m_te^{-J_t}||_{\text{TV}}$$
$$\leq\frac{1}{n}+\mathbb{P}(\tau_{\partial}^{R_n}<\tau_{\partial}\wedge T).$$

Since $D\cap B(0,R_n)=D_{R_n}\cap B(0,R_n)$, $\mathbb{P}(\tau_{\partial}^{R_n}<\tau_{\partial}\wedge T)\to 0$ as $n\to\infty$ hence

$$\sup_{t\leq T}||m_te^{-J_t}-\mathcal{L}(X_t|\tau_{\partial}>t)e^{\ln\mathbb{P}(\tau_{\partial}>t)}||_{\text{TV}}=0.$$

Thus (X,τ_{∂},W) is a global weak solution of (1.5) and therefore $(m_t,J_t)\in\Xi$. Moreover since $(\mathcal{L}(X_t^{R_n}))_{0\leq t\leq T}\in\mathcal{C}([0,T];L^1(D))$ for all n, $(\mathcal{L}(X_t))_{0\leq t\leq T}\in\mathcal{C}([0,T];L^1(D))$. We have established (8.21).

Step 4

We therefore have that if $(m_t,J_t)_{0\leq t<\infty}\in\mathcal{S}$ then $(m_{t_0+t},J_{t_0+t})_{0\leq t<\infty}\in\Xi$ for all $t_0\in\mathbb{Q}_{>0}$. We have that

$$(m_{t_0+t},J_{t_0+t})_{0\leq t<\infty}\to(m_t,J_t)_{0\leq t<\infty}\quad\text{in}\quad d^{\infty}\quad\text{as}\quad t_0\to 0.$$

Since $m_{t_0}\to m_0$ in W, Lemma 8.1 allows us to extract a subsequence converging to an element of Ξ, hence $(m_t,J_t)_{0\leq t<\infty}\in\Xi$. Moreover since $(m_{t_0+t},J_{t_0+t})_{0\leq t<\infty}\in\mathcal{C}([0,\infty);L^1(D)\times\mathbb{R}_{\geq 0})$ for all $t_0\in\mathbb{Q}_{>0}$ we have $(m_t,J_t)_{0\leq t<\infty}\in\mathcal{C}((0,\infty);L^1(D)\times\mathbb{R}_{\geq 0})$. □

8.3 Uniqueness in law of weak solutions to (1.5)

We implement a strategy similar to the proof of [11, Proposition C.1]. We fix $\nu\in\mathcal{P}(D)$ and firstly seek to show

$$\begin{gathered}(\mathcal{L}^{\nu}(X_t|\tau_{\partial}>t))_{0\leq t<\infty}\quad\text{is unique amongst}\\ \text{all weak solutions to (1.5) with initial condition } X_0\sim\nu.\end{gathered}\tag{8.22}$$

We take weak solutions to (1.5) $(X^1,W^1,\tau_{\partial}^1)$ and $(X^2,W^2,\tau_{\partial}^2)$ of (1.5) on the possibly different probability spaces $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$ and $(\Omega^2,\mathcal{F}^2,\mathbb{P}^2)$. We note by our earlier result that these must be global weak solutions. We then define $\mathcal{L}(X_t^k)=u_t^k$ for $k=1,2$ and $t<\infty$.

We recall that b is uniformly Lipschitz in the measure argument with respect to the W metric. Since this metric is dominated by the total variation metric (up to a constant), b is uniformly Lipschitz in the measure argument with respect to the total variation metric.

By abuse of notation we write

$$b:\mathcal{M}_+(D)\times D\ni(u,x)\mapsto b(\frac{u}{|u|_*},x)\in\mathbb{R}^d,\quad |u|_*=u(D)$$

where $|u|_*$ is the mass of u on D.

Therefore since $|u_t^1|,|u_t^2|\geq|u_1^1|\wedge|u_1^1|>0$ for $0\leq t\leq 1$ there exists $C_{\text{Lip}}<\infty$ such that

$$|b(u_t^1,x)-b(u_t^2,x)|\leq C_{\text{Lip}}||u_t^1-u_t^2||_{\text{TV}},\quad x\in D,\quad t\leq 1.$$

We now define

$$d_t=\sup_{s\leq t}||u_s^2-u_s^1||_{TV}$$

and drifts

$$b^1(x,t)=b(u_t^1,x)\text{ and }b^2(x,t)=b(u_t^2,x).$$

We consider weak solutions of the SDE

$$dX_t = b^1(X_t,t)dt + dW_t, \quad 0 \le t \le \tau_\partial = \inf\{t : X_t \in \partial D\}, \quad X_0 \sim \nu. \tag{8.23}$$

Weak solutions to (8.23) are unique in law by the same change of measure argument giving that weak solutions to SDEs without killing with bounded measurable coefficients are unique in law; see [25, Proposition 3.10].

Clearly $(X^1, W^1, \tau^1_\partial)$ on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ is a weak solution of (8.23). We have by Girsanov's theorem (since b^1, b^2 are bounded Novikov's condition is satisfied) that

$$W'_t = W^2_t - \int_0^t \left(b^1(X^2_s, s) - b^2(X^2_s, s)\right) ds$$

is a $\mathbb{P}'$-Brownian motion whereby

$$Z_t = \frac{d\mathbb{P}'}{d\mathbb{P}^2}_{|_{\mathcal{F}^2_t}} = \varepsilon(Y)_t, \quad Y_t = \int_0^t \left(b^1(X^2_s, s) - b^2(X^2_s, s)\right) dW^2_s.$$

Therefore we have

$$dX^2_t = b^2(X^2_t, t)dt + dW^2_t = b^1(X^2_t, t)dt + dW'_t,$$

so that $(X^2_t, W'_t, \tau^2_\partial)$ on $(\Omega^2, \mathcal{F}^2, \mathbb{P}')$ is also a weak solution of (8.23). By uniqueness in law of weak solutions to (8.23) we have

$$\mathcal{L}^{\mathbb{P}^1}(X^1_t) = \mathcal{L}^{\mathbb{P}'}(X^2_t), \quad t \le 1. \tag{8.24}$$

We now fix some measurable set $A \subseteq \mathbb{R}$ and see that

$$\begin{aligned} |u^1_t(A) - u^2_t(A)| &= |\mathbb{P}^1(X^1_t \in A) - \mathbb{P}^2(X^2_t \in A)| = \left|\mathbb{P}'(X^2_t \in A) - \mathbb{P}^2(X^2_t \in A)\right| \\ &= \left|\mathbb{E}^{\mathbb{P}^2}\left[\mathbb{1}_{X_t \in A}\big(Z_t - 1\big)\right]\right| \underbrace{\le}_{\substack{\text{Holder's} \\ \text{inequality}}} ||Z_t - 1||_{L^2(\mathbb{P}^2)} \sqrt{u_1(A)} \le ||Z_t - 1||_{L^2(\mathbb{P}^2)}. \end{aligned}$$

Taking the supremum over measurable sets $A \subseteq \mathbb{R}$ we have

$$||u^1_t - u^2_t||^2_{TV} \le \mathbb{E}^{\mathbb{P}^2}[(Z_t - 1)^2] = \mathbb{E}^{\mathbb{P}^2}[Z^2_t] - 2 \underbrace{\mathbb{E}^{\mathbb{P}^2}[Z_t]}_{\substack{=1 \text{ as } Z_t \text{ is a} \\ \mathbb{P}^2\text{-martingale}}} + 1 = \mathbb{E}^{\mathbb{P}^2}[Z^2_t] - 1.$$

We calculate the first term on the right using Ito's formula,

$$\mathbb{E}^{\mathbb{P}^2}[Z^2_t] = 1 + \int_0^t \mathbb{E}^{\mathbb{P}^2}[(b^1 - b^2)^2(X_s, s)Z^2_s]ds \le 1 + \int_0^t (C_{\text{Lip}} d_s)^2 \mathbb{E}^{\mathbb{P}^2}[Z^2_s]ds.$$

By Gronwall's inequality, using that $d_t \le 1$ and $e^{rt} \le 1 + rte^r$ for $0 \le t \le 1$ we have

$$\mathbb{E}^{\mathbb{P}^2}[Z^2_t] \underbrace{\le}_{\text{Gronwall}} e^{\int_0^t (C_{\text{Lip}} d_s)^2} \le e^{C^2_{\text{Lip}} d^2_t t} \le 1 + C^2_{\text{Lip}} d^2_t e^{C^2_{\text{Lip}}} t \quad \text{for} \quad 0 \le t \le 1.$$

Therefore we have

$$||u^1_t - u^2_t||^2_{TV} \le C^2_{\text{Lip}} d^2_t e^{C^2_{\text{Lip}}} t \quad \text{for} \quad 0 \le t \le 1.$$

Therefore for some $C < \infty$ we have

$$d_t \le C\sqrt{t} d_t.$$

Thus for $t < \frac{1}{2C^2} \wedge 1$ we have $u^2_t = u^1_t$. By iteration we have $u^1_t = u^2_t$ for $t \le 1$. Repeating inductively we have $u^1_t = u^2_t$ for all $t < \infty$. This implies (8.22).

This then implies uniqueness in law. Indeed (8.22) implies that both $(X^1, W^1, \tau^1_\partial)$ and $(X^2, W^2, \tau^2_\partial)$ are weak solutions to (8.23) and hence are equal in law.

□

8.4 Proof of Proposition 2.8 and Theorem 2.9

Given $\nu \in \mathcal{P}_W(D)$, let $\{\vec{X}_t^N\}_{N\geq 2}$ be any sequence of weak solutions to (1.3) with initial conditions $\vec{X}_0^N$ such that the (random) empirical measures $m_0^N = \vartheta_{\#}^N \vec{X}_0^N$ converge in $\mathcal{P}_W(D)$ to ν, in probability as $N \to \infty$. This can be achieved, for example, by taking $\vec{X}_0^N \sim \nu^{\otimes N}$.

Next, we define m_t^N and J_t^N as in (1.10) and (1.12),

$$m_t^N = \vartheta^N(\vec{X}_t^N), \quad J_t^N = \frac{1}{N}\sup\{k \in \mathbb{N} \mid \tau_k \leq t\}.$$

Theorem 2.10 and the fact that $m_0^N \to \nu$ in probability imply that the laws of $(m_t^N, J_t^N)_{0\leq t<\infty}$ are tight in $\mathcal{P}((\mathcal{D}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\geq 0})), d^{\mathcal{D}}))$ and every limit distribution of this family is supported on $\Xi(\{\nu\})\cap\mathcal{C}((0,\infty);L^1(D)\times\mathbb{R}_{\geq 0})$. In particular, $\Xi(\{\nu\})\cap\mathcal{C}((0,\infty);L^1(D)\times\mathbb{R}_{\geq 0})$ is non-empty. We have already proved uniqueness in law of weak solutions to (1.5); therefore this limit distribution is uniquely determined. Taken together with Lemma 8.1, this establishes Proposition 2.8

The fact that the limit distribution is unique implies convergence along the entire sequence $N \to \infty$ in probability to the same element of $\Xi(\{\nu\})\cap\mathcal{C}((0,\infty);L^1(D)\times\mathbb{R}_{\geq 0})$. Furthermore, since convergence in $d^{\mathcal{D}}$ to a continuous function implies convergence in d^{∞}, we have convergence in d^{∞} in probability. This proves Theorem 2.9.

□

9 Properties of the semigroup G_t – Proposition 2.11

Our goal in this section is to establish Proposition 2.11. We begin with a proof of (2.14). We take $(t_n,\nu_n)\to(t,\nu)$ and $T>\sup_n t_n$. Then Lemma 8.1 and Proposition 2.8 imply that $(G_t(\nu_n))_{0\leq t\leq T}\to(G_t(\nu))_{0\leq t\leq T}$ in $d^{\infty}_{[0,T]}$ as $n\to\infty$. Therefore

$$W(G_{t_n}(\nu_n),G_t(\nu)) \leq W(G_{t_n}(\nu_n),G_{t_n}(\nu)) + W(G_{t_n}(\nu),G_t(\nu)) \to 0 \quad \text{as} \quad n\to\infty.$$

We have thus established (2.14).

We now assume D is bounded, fix $t_0>0$ and combine the estimates on the N-particle system we established in Part 1 of Proposition 4.7 with the hydrodynamic convergence theorem (Theorem 2.9) to prove that Image$(G_{t_0}) \subset\subset \mathcal{P}_W(D)$.

Let $(\vec{X}_t^N : 0\leq t<\infty) = ((X_t^{N,1},\dots,X_t^{N,N}) : 0\leq t<\infty)$ be a sequence of weak solutions to (1.3) with initial conditions $\vec{X}_0^N\sim\nu^{\otimes N}$. We define $m_t^N=\vartheta^N(\vec{X}_t^N)$ as in 1.10. Therefore Part 1 of Proposition 4.7 gives that for all $\epsilon>0$ there exists $c=c(\epsilon,t_0)$ dependent only upon t_0, $\epsilon>0$, the upper bound on the drift $B<\infty$ and the constant of the interior ball condition $r>0$ such that the compact set $K_{\epsilon,t_0}=V_{c(\epsilon,t_0)}$ satisfies

$$\lim_{N\to\infty}\mathbb{P}(m_{t_0}^N(K_{\epsilon,t_0}^c)\geq\epsilon)=0.$$

Therefore by our hydrodynamic convergence theorem (Theorem 2.9) we have

$$G_{t_0}(\nu)(K_{\epsilon,t_0}^c)\leq\epsilon.$$

Since K_{ϵ,t_0} was not dependent upon ν, $G_{t_0}(\nu)(K_{\epsilon,t_0}^c)\leq\epsilon$ for all $\nu\in\mathcal{P}(D)$. Therefore

$$G_{t_0}(\nu)\in\{\mu\in\mathcal{P}(D):\mu(K_{2^{-n},t_0}^c)\leq 2^{-n},\quad n\in\mathbb{N}\},$$

which is a tight family of measures on D.

□

10 Existence and properties of QSDs – Proposition 2.13

Parts 1 and 2 of Proposition 2.13

We firstly establish $1a \Leftrightarrow 1b$. It is trivial to see that a QSD is a QLD. In the opposite direction we consider a QLD π which is the Yaglom limit for initial condition ν: $G_t(\nu) \to \pi$ in W as $t \to \infty$. We define the continuous map

$$p : (\mathcal{C}([0,\infty);\mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}), d^{\infty}) \ni (f,g) \mapsto f \in \mathcal{C}([0,1];\mathcal{P}_W(D)).$$

We further define

$$\zeta_1(\kappa) := \{G_t(\mu)_{0\leq t\leq 1} : \mu \in \kappa\} = p(\Xi(\kappa)).$$

We have by Lemma 8.1 that $\Xi(\kappa)$ is compact in $(\mathcal{C}([0,\infty);\mathcal{P}_W(D) \times \mathbb{R}_{\geq 0}), d^{\infty})$ for compact κ, hence $\zeta_1(\kappa)$ is compact in $\mathcal{C}([0,1];\mathcal{P}_W(D))$ as it is the continuous image of a compact set. We now take $\kappa = \{G_n(\nu) : n \in \mathbb{N}\} \cup \{\pi\}$ which is compact in $\mathcal{P}_W(D)$. Thus we have

$$\zeta_1(\kappa) = \{(G_{n+t}(\nu))_{0\leq t\leq 1} : n \in \mathbb{N}\} \cup \{G_t(\pi)_{0\leq t\leq 1}\}$$

is compact in $\mathcal{C}([0,1];\mathcal{P}_W(D))$. We note that

$$(G_{n+t}(\nu))_{0\leq t\leq 1} \to (\pi)_{0\leq t\leq 1} \quad \text{in} \quad \mathcal{C}([0,1];\mathcal{P}_W(D)) \quad \text{as} \quad n \to \infty.$$

Thus we must have

$$(\pi)_{0\leq t\leq 1} \in \zeta_1(\kappa).$$

Therefore $G_t(\pi) = \pi$ for $0 \leq t \leq 1$. Thus π is a QSD.

We now establish $1a \Rightarrow 1c$ along with Part 2 of Proposition 2.13. We take π a QSD, (X, τ_∂, W) a global weak solution to (1.5) with initial condition $X_0 \sim \pi$ and $(m_t, J_t)_{0\leq t\leq 1} = (\mathcal{L}(X_t|\tau_\partial > t), -\ln \mathbb{P}(\tau_\partial > t))_{0\leq t<\infty} \in \Xi(\{\pi\})$. By considering the martingale problem we see that $m_t e^{-J_t} = \pi e^{-J_t} = \mathcal{L}(X_t)$ satisfies

$$\begin{aligned} e^{-J_t}\langle \pi(\cdot), \varphi(\cdot)\rangle - \langle \pi(\cdot), \varphi(\cdot)\rangle = \int_0^t e^{-J_s}\langle \pi, b(\pi,\cdot)\cdot\nabla\varphi \\ +\frac{1}{2}\Delta\varphi\rangle ds, \quad 0 \leq t < \infty, \quad \varphi \in C_0^{\infty}(\bar{D}). \end{aligned} \tag{10.1}$$

Clearly the right hand side is differentiable in time, so the left hand side must be also and so we have

$$-\langle \pi(\cdot), \varphi(\cdot)\rangle e^{-J_t}\frac{d}{dt}J_t = e^{-J_t}\langle \pi, b(\pi,\cdot)\cdot\nabla\varphi + \frac{1}{2}\Delta\varphi\rangle, \quad \varphi \in C_0^{\infty}(\bar{D}).$$

Thus $\frac{d}{dt}J_t$ must be constant and so equal to some $\lambda \geq 0$. Since we can't have $\mathbb{P}(\tau_\partial > t) = 1$ for all $t > 0$, we must have $\lambda > 0$. Moreover we must have $\mathcal{L}_\pi(X_1) \in L^1(D)$ since $\mathcal{L}_\pi(X_1)$ can be related to the distribution at time 1 of Brownian motion killed at the boundary by a Girsanov transformation – thus we must have $\pi \in L^1(D)$. Thus (π, λ) satisfies (2.17) and hence $1a \Rightarrow 1c$. Moreover $e^{-\lambda t} = e^{-J_t} = |\mathcal{L}(X_t)| = \mathbb{P}(\tau_\partial > t)$ so that $\tau_\partial \sim \exp(\lambda)$ and hence we have Part 2 of Proposition 2.13.

We now establish $1c \Rightarrow 1a$. We take $(\pi, \lambda) \in L^1(D) \times (0, \infty)$ a solution of (2.17) and take (X, τ_∂, W) a weak solution of the SDE

$$dX_t = b(\pi, X_t)dt + dW_t, \quad 0 \leq t \leq \tau_\partial = \inf\{t : X_{t-} \in \partial D\}, \quad X_0 \sim \pi$$

(which exists by Girsanov's theorem). We have both $\mathcal{L}_\pi(X_t) = \mathcal{L}_\pi(X_t|\tau_\partial > t)\mathbb{P}(\tau_\partial > t)$ and $\pi e^{-\lambda t}$ must be $L^1_{\text{loc}}(\bar{D}\times[0,\infty))$ solutions to the PDE (for every $T<\infty$)

$$\langle y_t(\cdot),\varphi(\cdot,t)\rangle - \langle \pi(\cdot),\varphi(\cdot,0)\rangle = \int_0^t \langle y_s(\cdot),\partial_s\varphi(\cdot,s)$$
$$+b(\pi,\cdot)\cdot\nabla\varphi(\cdot,s) + \frac{1}{2}\Delta\varphi(\cdot,s)\rangle ds,\quad 0\le t<\infty,\quad \varphi\in C_0^\infty(\bar{D}\times[0,\infty)).$$

Therefore by [28, Theorem 1.1] we have $\pi e^{-\lambda t} = \mathcal{L}(X_t)$, thus $\mathcal{L}(X_t|\tau_\partial > t) = \pi$ and hence (X,τ_∂,W) satisfies (1.5). Thus π is a QSD.

Part 3 of Proposition 2.13

We define $\Pi_n = \{\pi\in\mathcal{P}(D) : G_{2^{-n}}(\pi) = \pi\}$. We recall that Proposition 2.11 gives that $G_{2^{-n}} : \mathcal{P}(D)\to\mathcal{P}(D)$ is continuous with tight image. Since the convex hull of a tight family of measures is tight, the closed convex hull $F_{2^{-n}} := \overline{\text{Conv}}(\text{Image}(G_{t_0}))$ is compact in $\mathcal{P}(D)$. Therefore Π_n corresponds to the fixed points of the map

$$G_{2^{-n}} : F_{2^{-n}} \to F_{2^{-n}},$$

which is a continuous map from a compact convex subset of a locally convex topological vector space ($\mathcal{M}(D)$) to itself. Thus Schauder's fixed point theorem implies Π_n is a non-empty compact subset of $\mathcal{P}(D)$. It is therefore sufficient to prove

$$\Pi = \cap_n \Pi_n \tag{10.2}$$

as the intersection of a descending sequence of non-empty compact sets must be non-empty and compact.

We clearly have that $\Pi\subseteq\Pi_n$ for all n, so it is sufficient to establish $\cap_n\Pi_n\subseteq\Pi$. We suppose $\pi\in\cap_n\Pi_n$ and fix $t>0$. We take a sequence of dyadic rationals $t_n\to t$. We have $\pi = G_{t_n}(\pi)\to G_t(\pi)$ by Proposition 2.11 so that we have $G_t(\pi)=\pi$. Since t is arbitrary, $\pi\in\Pi$.

□

11 QSDs as limits of the Fleming-Viot particle system

The goal of this section is to establish that QSDs may be obtained as limits of the N-particle system. In Theorem 2.16 we show that the stationary distributions of the N-particle system converge to the set of QSDs. In Theorem 2.17 we then establish under an additional assumption on the semigroup G_t convergence as N and T go to infinity together.

Proof of Theorem 2.16

We take the N-particle stationary distributions ψ^N, associated to which are the corresponding stationary empirical measures $\Psi^N = \vartheta^N_{\#}\psi^N$ as in (1.17) and $\mathcal{P}_W(D)$-valued random variables $\pi^N\sim\Psi^N$. We consider a sequence of stationary solutions $\vec{X}^N$ to (1.3) with initial distributions ψ^N. We write $m^N_t := \vartheta^N(\vec{X}^N_t)$ and $J^N_t = \frac{1}{N}\sup\{k\in\mathbb{N}\mid\tau_k\le t\}$ as in (1.10) and (1.12).

Since $m^N_1\sim\Psi^N$, Proposition 4.6 gives that $\{\Psi^N\}$ are a tight family of random measures. We may then use Theorem 2.10 to establish that $\{\mathcal{L}((m^N_t, J^N_t)_{0\le t<\infty})\}$ is tight in $\mathcal{P}((\mathcal{D}([0,\infty);\mathcal{P}_W(D)\times\mathbb{R}_{\ge 0}), d^{\mathcal{D}}))$. We then consider an arbitrary convergent subsequence, along which $\Psi^N\to\Psi^\infty$ in $\mathcal{P}(\mathcal{P}_W(D))$ and $(m^N_t, J^N_t)_{0\le t<\infty}\to(m^\infty_t, J^\infty_t)_{0\le t<\infty}$ in distribution, which must satisfy $(m^\infty_t, J^\infty_t)_{0\le t<\infty}\in\Xi$ almost surely by Theorem 2.10. We take a random variable $\pi^\infty \sim \Psi^\infty$ so that we have

$$\Psi^\infty \sim m_t^\infty = G_t(m_0^\infty), \quad t \geq 0,$$

so that in particular Ψ^∞ is an invariant measure for the semigroup G_t. We calculate

$$\mathbb{E}[W(\pi^\infty, \Pi)] = \mathbb{E}[W(G_t(\pi^\infty), \Pi)] \to 0 \quad \text{as} \quad t \to \infty$$

by Lebesgue's Dominated Convergence Theorem and our assumption (2.15).

Thus $\pi^\infty \in \Pi$ almost surely, so Ψ^∞ is supported on Π. Thus along every subsequence, there is a further subsequence along which $W(\pi^N, \Pi) \to 0$ in probability, hence we have convergence in probability along the original sequence.

□

Proof of Theorem 2.17

We take an arbitrary sequence $t_N \to \infty$ and fix $\epsilon > 0$. We take (using the assumption (2.16)) $T < \infty$ such that $W(G_T(\nu), \pi) \leq \epsilon$ for all $\nu \in \mathcal{P}_W(D)$.

Then by Proposition 4.6, $\{\mathcal{L}(m^N_{t_N - T})\}$ is tight in $\mathcal{P}(\mathcal{P}_W(D))$ and hence by Theorem 2.10 and Skorokhod's representation theorem we may take a subsequence and possibly different probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ on which $(m^N_{t_N-T+t}, J^N_{t_N-T+t})_{0\leq t<\infty}$ converges in d^∞ $\mathbb{P}'$-almost surely to $(m_t, J_t)_{0\leq t<\infty} \in \Xi \subseteq \mathcal{C}([0,\infty); \mathcal{P}_W(D) \times \mathbb{R}_{\geq 0})$. Then $m_T = G_T(m_0)$ so that on this subsequence

$$\limsup_{N\to\infty} W(m^N_{t_N}, \pi) \leq \epsilon \quad \mathbb{P}'\text{-almost surely.}$$

This subsequence was arbitrary as was $\epsilon > 0$, so we have $W(m^N_{t_0}, \pi) \to 0$ in probability as $N \wedge t_0 \to \infty$. Using Theorem 2.9 and Proposition 2.13 we are done.

□

A Proof of Lemma 6.5

We define

$$G_\epsilon = 0 \vee \sup_{\rho \in \mathcal{R}} \big(m(\rho) - C_\epsilon(T_{\min}(\rho))\mathrm{Leb}(\rho)\big).$$

Since we have bounded m on sets in $\mathcal{R}$ in terms of Leb, we may bound the corresponding outer measure m^* in terms of the outer measure Leb^*. Specifically

$$m^*(E) \leq C_\epsilon(T_{\min}(E))\mathrm{Leb}^*(E) + G_\epsilon \quad \text{for all} \quad E \in \mathcal{B}(D \times [0,T]) \setminus \{\emptyset\}$$

holds almost surely. Since $m \leq m^*$ and $\mathrm{Leb} = \mathrm{Leb}^*$, this implies that

$$m(E) \leq C_\epsilon(T_{\min}(E))\mathrm{Leb}(E) + G_\epsilon \quad \text{for all} \quad E \in \mathcal{B}(D \times [0,T]) \setminus \{\emptyset\} \tag{A.1}$$

holds $\mathbb{P}^\epsilon$-almost surely. We define

$$\mathcal{N}_{\delta,T_0} = \{\mu \in \mathcal{P}(D \times [0,T]) : \mu(N) \leq \delta \quad \text{for all} \quad N \in \mathcal{B}(D \times (T_0, T]) \\ \text{such that } \mathrm{Leb}_{D\times(0,T]}(N) = 0\}$$

for $\delta, T_0 \geq 0$. Then (A.1) implies that for $\delta, T > 0$ we have

$$\mathbb{P}(m \in \mathcal{N}_{\delta,T_0}) \geq \mathbb{P}(G_\epsilon \leq \delta) \geq 1 - \frac{\epsilon}{\delta} \text{ by Markov's inequality.}$$

Since $\epsilon > 0$ is arbitrary, we have $\mathbb{P}(m \in \mathcal{N}_{\delta,T_0}) = 1$ for all $\delta, T > 0$. We now note that $\mathcal{N}_{0,0} = \cap_{T_0>0} \cap_{\delta>0} \mathcal{N}_{\delta,T_0}$ so that

$$\mathbb{P}(m \ll \mathrm{Leb}_{D\times(0,T]}) = \mathbb{P}(m \in \mathcal{N}_{0,0}) = 1.$$

Moreover we have $\mathbb{P}(m(D \times \{0\}) = 0) = 1$. Therefore $\mathbb{P}(m \ll \mathrm{Leb}_{D\times[0,T]}) = 1$.

□

B Proof of Lemma 7.1

We fix φ a positive mollifier supported on $B(0,1)$ and take $\rho \in C^{\infty}(\mathbb{R}^d)$ such that

$$D = \{\rho > 0\}, \quad \partial D = \{\rho = 0\}, \quad \nabla \rho \neq 0 \quad \text{on} \quad \partial D.$$

We define $g = (\varphi * \mathbb{1}_{B(0,R+3)})\rho \in C_c^{\infty}(\mathbb{R}^d)$ which we note satisfies:

1. $D \cap \bar{B}(0,R+2) = \{g > 0\} \cap \bar{B}(0,R+2)$;
2. $\partial D \cap \bar{B}(0,R+2) = \{g = 0\} \cap \bar{B}(0,R+2)$;
3. $\nabla g \neq 0$ on $\partial D \cap \bar{B}(0,R+2)$.

We then define $h = \varphi * \mathbb{1}_{B(0,R+1)^c} \in C_c^{\infty}(\mathbb{R}^d)$ which we note satisfies:

1. $h \equiv 0$ on $\bar{B}(0,R)$;
2. $0 \leq h \leq 1$ on $\bar{B}(0,R+2) \setminus \bar{B}(0,R)$;
3. $h \equiv 1$ on $B(0,R+2)^c$.

We define $f = g - \epsilon h \in C^{\infty}(\mathbb{R}^d)$ for $\epsilon > 0$ to be determined and claim that for some $\epsilon > 0$ small enough, $D_R := \{f > 0\}$ gives a domain with our desired values. We firstly observe that for all $\epsilon > 0$,

$$D \cap \bar{B}(0,R) \subseteq \{f > 0\} \subseteq D \cap \bar{B}(0,R+4).$$

Therefore by the implicit function theorem it is sufficient to show that

$$f = 0 \;\Rightarrow\; \nabla f \neq 0 \tag{B.1}$$

for some $\epsilon > 0$ small enough. Sard's theorem allows us to take $\epsilon_n \downarrow 0$ such that

$$g = \epsilon_n \;\Rightarrow\; \nabla g \neq 0.$$

Therefore $f_n = g - \epsilon_n h$ satisfies

$$f_n(x) = 0 \quad \text{and} \quad |x| \geq R+2 \;\Rightarrow\; \nabla f_n(x) \neq 0.$$

We now assume for contradiction that for all n there exists $|x_n| \leq R+2$ such that $f_n(x_n) = 0$ and $\nabla f_n(x_n) = 0$. We take a convergent subsequence $x_{n_k} \to x \in \bar{B}(0,R+2)$, so that $0 = f_{n_k}(x_{n_k}) \to g(x)$ and $0 = \nabla f_{n_k}(x_{n_k}) \to \nabla g(x)$. This is a contradiction, hence we may choose ϵ_n such that

$$g - \epsilon_n h = 0 \;\Rightarrow\; \nabla(g - \epsilon_n h) \neq 0. \quad \square$$

C Controls on the density and hitting time of generic diffusions

Lemma C.1. *Let (X, τ_{∂}, W) be a weak solution of the SDE*

$$dX_t = b_t dt + dW_t, \quad 0 \leq t \leq \tau_{\partial} = \inf\{t > 0 : X_t \in \partial D\} \tag{C.1}$$

on the domain $D \subseteq \mathbb{R}^d$ and filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where b_t is $(\mathcal{F}_t)_{t\geq 0}$-adapted and uniformly bounded $|b_t| \leq B$. For $\vec{x} = (x_1, \dots, x_d) \in D \subseteq \mathbb{R}^d$ and $h > 0$ we define the open cube

$$F(\vec{x}, h) = \{\vec{y} = (y_1, \dots, y_d) \in D : |y_i - x_i| < h\}.$$

Throughout we write $\mathcal{L}(X_t) = \mathcal{L}(X_t | \tau_{\partial} > t)\mathbb{P}(\tau_{\partial} > t)$ for the law of the killed process restricted to D. Then we have the following:

1. *There exists a non-increasing function* $C:(0,\infty)\to\mathbb{R}_{>0}$ *such that*

$$\mathcal{L}(X_t)(\cdot)\leq C_t\textit{Leb}_D(\cdot),\quad 0<t<\infty. \tag{C.2}$$

2. *If* $h,t>0$ *and* $F(\vec{x},5h)\subseteq D$ *there exists* $c>0$ *dependent only upon the upper bound on the drift* $B<\infty$, $t>0$ *and* $h>0$ *such that*

$$\mathcal{L}(X_t)_{|_{F(\vec{x},h)}}(\cdot)\geq c\mathbb{P}(X_t\in F(\vec{x},h))\textit{Leb}_{|_{F(\vec{x},h)}}(\cdot). \tag{C.3}$$

We obtain from this the following corollary.

Corollary C.2. *For every* $t\geq 0$ *we have* $\mathbb{P}(\tau_\partial=t)=0$.

Proof of Lemma C.1. We firstly establish (C.2). We may apply Lemma 6.7 to the family of processes $\{X_t-\vec{x}:\vec{x}\in D\}$ to see that

$$\mathbb{P}(X_t\in F(\vec{x},h))\leq C_t\text{Leb}(F(\vec{x},h))$$

where C_t is the function given by Lemma 6.7. By considering the outer measure generated by the open cubes, we see that

$$\mathcal{L}(X_t)(\cdot)\leq\mathcal{L}(X_t)^*(\cdot)\leq C_t\text{Leb}^*(\cdot)=C_t\text{Leb}(\cdot),\quad 0<t<\infty.$$

We now establish (C.3). We consider on the probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ a family of weak solutions (X^γ,W^γ) ($\gamma\in\Gamma$) on the domains $D^\gamma\supseteq F(\vec{0},4h)$ to the SDEs

$$dX_t^\gamma=b_t^\gamma dt+dW_t^\gamma,\quad 0\leq t\leq\tau_\partial^{X,\gamma}=\inf\{t>0:X_t^\gamma\in\partial D^\gamma\},$$

where $\{b^\gamma\}$ are bounded $|b_t^\gamma|\leq B$ and $(\mathcal{F}_t)_{t\geq 0}$-adapted processes. We take $\vec{h}=(h,\dots,h)$ and write $\vec{n}(\vec{x})$ for the inward normal of the positive orthant $\mathbb{R}^d_{>0}$. If we repeat the proof of Lemma 6.7 (on page 37) with strong solutions of the SDE (6.32) replaced with strong solutions of the 1-dimensional SDE (which exists by [3, Theorem 1.3])

$$dZ_t=d\tilde{W}_t+Bdt+dL_t^Z,\quad Z_0=2h,$$

we obtain for each γ a strong solution $(Z^\gamma,\tilde{W}^\gamma)$ of the d-dimensional SDE

$$dZ_t^\gamma=(B,\dots,B)dt+\vec{n}(Z_t^\gamma)dL_t^{Z^\gamma},\ 0\leq t\leq\tau_\partial^{Z^\gamma}=\inf\{t>0:Z_t^\gamma\in\partial F(\vec{0},4h)\},\ Z_0^\gamma=2\vec{h},$$

where $L_t^{Z^\gamma}$ is the local time of Z_t^γ with the boundary $\partial\mathbb{R}^d_{>0}$ and which satisfies

$$X_0^\gamma\in F(\vec{0},2h)\text{ and }Z_t^\gamma\in F(\vec{0},\epsilon)\ \Rightarrow\ X_t^\gamma\in F(\vec{0},\epsilon),\quad 0<\epsilon<h. \tag{C.4}$$

Moreover we may take $c>0$ such that for all $h>\epsilon>0$ and $\gamma\in\Gamma$ we have $\mathbb{P}(Z_t^\gamma\in F(\vec{0},\epsilon))\geq c\text{Leb}(F(\vec{0},\epsilon))$. We consider the processes $\{X_t-\vec{y}:\vec{y}\in F(\vec{x},h)\}$ so that

$$\mathbb{P}(X_t\in F(\vec{y},\epsilon))\geq\mathbb{P}(X_0\in F(\vec{y},2h))c\text{Leb}(F(\vec{0},\epsilon))\geq\mathbb{P}(X_0\in F(\vec{x},h))c\text{Leb}(F(\vec{0},\epsilon)).$$

Therefore by considering the inner measure generated by the open cubes, we see that

$$\mathcal{L}(X_t)(\cdot)\geq\mathcal{L}(X_t)_*(\cdot)\geq c\text{Leb}_*(\cdot)=c\text{Leb}(\cdot)$$

□

Proof of Corollary C.2. The $t=0$ case is trivial, so we assume $t>0$. Indeed by the continuity of the paths of X_t, we have for all $R<\infty$ and $\epsilon>0$,

$$\mathbb{P}(\tau_\partial=t)\leq\limsup_{h\to\infty}\mathbb{P}(d(X_{t-h},\partial D)\leq\epsilon\quad\text{and}\quad|X_{t-h}|\leq R+1)+\mathbb{P}(|X_t|\geq R).$$

Applying (C.2) we have

$$\mathbb{P}(\tau_\partial=t)\leq C_{\frac{t}{2}}\text{Leb}(\{x:d(x,\partial D)\leq\epsilon\quad\text{and}\quad|x|\leq R+1\})+\mathbb{P}(|X_t|\geq R).$$

Taking $\limsup_{R\to\infty}\limsup_{\epsilon\to 0}$ of both sides we are done. □

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