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Noise sensitivity for the top eigenvector of a sparse random matrix*

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Abstract

We investigate the noise sensitivity of the top eigenvector of a sparse random symmetric matrix. Let v be the top eigenvector of an $N \times N$ sparse random symmetric matrix with an average of d non-zero centered entries per row. We resample k randomly chosen entries of the matrix and obtain another realization of the random matrix with top eigenvector $v^{[k]}$. Building on recent results on sparse random matrices and a noise sensitivity analysis previously developed for Wigner matrices, we prove that, if $d \geq N^{2/9}$, with high probability, when $k \ll N^{5/3}$, the vectors v and $v^{[k]}$ are almost collinear and, on the contrary, when $k \gg N^{5/3}$, the vectors v and $v^{[k]}$ are almost orthogonal. A similar result holds for the eigenvector associated to the second largest eigenvalue of the adjacency matrix of an Erdős-Rényi random graph with average degree $d \geq N^{2/9}$.

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1 Introduction

Noise sensitivity is an important phenomenon in probability theory that describes a function of many independent random variables whose output asymptotically decorrelates when only a small proportion of the random variables are resampled. It has deep connections with threshold phenomena and it has been extensively studied since the pioneering work of Benjamini, Kalai, and Schramm [3]. It has found many applications in theoretical computer science and statistical mechanics where it commonly appears in large systems in a critical state such as critical percolation. We refer to the monographs [8] and [14] for references and background.

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Recently, the authors of [4] have investigated the noise sensibility of Wigner random matrices, that is a $N \times N$ symmetric matrix with i.i.d. centered entries with unit variance above the diagonal. Calling an unit eigenvector corresponding to the largest eigenvalue *top eigenvector*, they studied how the direction of the top eigenvector varies when *resampling* a number k of uniformly chosen entries of a Wigner matrix. Under an exponential tail assumption on the entries, they proved a threshold phenomenon as N goes to infinity: if $k \ll N^{5/3}$, with high probability, the top eigenvectors remain nearly aligned while if $k \gg N^{5/3}$ their are almost orthogonal. Since $N^{5/3}$ is much smaller than $N(N+1)/2$, the number of independent random variables in the matrix, the latter result can be interpreted as a noise sensitivity statement. On the random matrix side, the proofs in [4] built on many outstanding results which have been proved on the spacing and fluctuations of eigenvalues and on the delocalization of their eigenvectors, we refer to [2, 13] for lecture notes on this topic.

In this paper, we extend the results of [4] to a large class of *sparse symmetric random matrices* with an average of d non-zero entries per row. In the regime $d \geq N^{\epsilon}$ for some $\epsilon > 0$, many remarkable results have recently been achieved for such sparse random matrices including eigenvector delocalization and Tracy-Widom or Gaussian fluctuation of the extremal eigenvalues, including [5, 10, 11, 16, 18, 22, 23, 24, 25]. One thus might expect to observe the same threshold phenomenon for the top eigenvector in sparse random matrix ensemble as it was shown for Wigner matrices. Indeed, we prove this phenomenon assuming a certain condition on the parameter d. Our work notably builds upon [11, 18] for local laws of the resolvent and [18, 23] for eigenvalue spacings.

Sparse random matrices have many applications in computer sciences and statistics. One of canonical models for a such matrices is the sparse Erdős-Rényi graph, which is often used to describe random networks. In view of the graph, resampling an entry (of the adjacency matrix) can be regarded as creating or deleting an edge on the graph with some probability so that we can generate a random perturbation to some given networks through resampling. Since eigenvectors tend to be more informative than eigenvalue, it might be expected that the above-described phase transition of top eigenvector find some opportunities to be applied in other disciplines.

1.1 Definition and main results

We first introduce the main model of random matrices which we will consider.

Definition 1.1 (Sparse random matrices)**.** *Let $\vartheta > 0$ be a fixed number and $q = q(N) \in (0, \sqrt{N}]$ be a sparsity parameter. Let $H = (h_{ij})$ be an $N \times N$ random matrix where all entries are real and independent up to the symmetry constraint $h_{ij} = h_{ji}$. We assume that h_{ij} is the product*

$$h_{ij} = \frac{x_{ij} y_{ij}}{q},$$

where $\{x_{ij} : i \leq j\}$ and $\{y_{ij} : i \leq j\}$ are independent and satisfy the following conditions: for all i, j

(i) $\mathbb{E} x_{ij} = 0$, $\mathbb{E} x_{ij}^2 = 1$ *and* $\mathbb{E} \exp(\vartheta x_{ij}^2) \leq \vartheta^{-1}$.

(ii) $\mathbb{P}(y_{ij} = 1) = 1 - \mathbb{P}(y_{ij} = 0) = q^2/N$.

The condition $\mathbb{E} \exp(\vartheta x_{ij}^2) \leq \vartheta^{-1}$ asserts that the entries of the matrix are uniformly sub-Gaussian. Our condition ensures that

$$\mathbb{E} h_{ij} = 0 \quad \text{and} \quad \mathbb{E} h_{ij}^2 = \frac{1}{N}.$$

The order of magnitude of a non-zero entry is of order $1/q$. More precisely, for any integer $k \geq 1$, there exists a constant $C = C(k, \vartheta) \geq 1$ such that,

$$q^{2-2k} N^{-1} \leq \mathbb{E} h_{ij}^{2k} \leq C q^{2-2k} N^{-1}. \tag{1.1}$$

In this paper, we will use the following notation in the asymptotic $N \to \infty$: The symbols $O(\cdot)$ and $o(\cdot)$ are used for the standard big-O and little-o notation. For nonnegative functions f and g of parameter N, we write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \leq Cg$, and $f \asymp g$ if $f \gtrsim g$ and $g \gtrsim f$. Finally, we use the less standard notation $f \ll g$ if there exists a constant $\epsilon > 0$ such that $N^{\epsilon} f = O(g)$. Beware that the underlying constants could depend implicitly on the parameter ϑ which is fixed throughout the paper.

We now describe the resampling procedure. Let $(i_k, j_k), 1 \leq k \leq N(N+1)/2$, be a random uniformly chosen ordering of the set $S = \{(i,j) : 1 \leq i \leq j \leq N\}$, independently of H. For a positive integer $k \leq N(N+1)/2$, the set $S_k = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ is thus a random set of k distinct pairs (with $i_m \leq j_m$) which is chosen uniformly from the family of all sets of k distinct elements in S. By convention S_0 is the empty set.

Definition 1.2 (Resampling procedure). *Let $H' = (h'_{ij})$ be an independent copy of H. For integer $0 \leq k \leq N(N+1)/2$, we define $H^{[k]} = (h_{ij}^{[k]})$ as the random symmetric matrix generated from the given random matrix H, by resampling entries in S_k: for $i \leq j$,*

$$h_{ij}^{[k]} = \begin{cases} h'_{ij} & (i,j) \in S_k, \\ h_{ij} & (i,j) \notin S_k. \end{cases}$$

The remaining entries of $H^{[k]}$ below the diagonal are determined by symmetry.

Let $\lambda_1 \geq \cdots \geq \lambda_N$ be the ordered eigenvalues of H. We consider an orthonormal basis of eigenvectors of H by $\{\mathbf{v}_1, \cdots, \mathbf{v}_N\}$, i.e., $H\mathbf{v}_i = \lambda_i \mathbf{v}_i$ and $\|\mathbf{v}_i\| = 1$ for each i. Note that Luh and Vu recently showed that sparse random matrices have simple spectrum [24] with probability tending to one as N goes to infinity. This implies that $\lambda_1 > \cdots > \lambda_N$ and the eigenvectors are uniquely determined up to a sign. We call $\mathbf{v}_1$ the top eigenvector of H. Similarly, we use the notation $\lambda_1^{[k]} \geq \cdots \geq \lambda_N^{[k]}$ and $\mathbf{v}_1^{[k]}, \cdots, \mathbf{v}_N^{[k]}$ to denote the ordered eigenvalues and the associated unit eigenvectors of $H^{[k]}$.

The usual scalar product in $\mathbb{R}^N$ is denoted by $\langle \cdot, \cdot \rangle$ and $\|v\|_\infty = \max_i |v_i|$ is the ℓ^∞-norm of a vector. Our main results are the following two complementary claims.

Theorem 1.3 (Noise sensitivity). *If $q \gtrsim N^{1/9}$ and $k \gg N^{5/3}$ then*

$$\mathbb{E} \left| \left\langle \mathbf{v}_1, \mathbf{v}_1^{[k]} \right\rangle \right| = o(1).$$

Theorem 1.4 (Noise stability). *If $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$ then*

$$\mathbb{E} \min_{s \in \{\pm 1\}} \sqrt{N} \|\mathbf{v}_1 - s\mathbf{v}_1^{[k]}\|_\infty = o(1).$$

As a result, $\mathbb{E} \left| \left\langle \mathbf{v}_1, \mathbf{v}_1^{[k]} \right\rangle \right| = 1 - o(1)$.

If $q \gtrsim \sqrt{N}$, then Theorem 1.3 and Theorem 1.4 are contained in [4]. To explain the threshold at $N^{5/3+o(1)}$ and the technical condition $q \gtrsim N^{1/9}$ on the sparsity parameter we may repeat the heuristic first explained in [4]. First, from [11], the eigenvectors of H are delocalized in the sense that $\|\mathbf{v}_m\|_\infty = N^{-1/2+o(1)}$ with high probability for any $1 \leq m \leq N$. Recall that $\lambda_1^{[k]}$ is the largest eigenvalue of $H^{[k]}$ with eigenvector $\mathbf{v}_1^{[k]}$. We might guess from the derivative of a simple eigenvalue as the function of the matrix entries that

$$\lambda_1^{[k]} - \lambda_1^{[k-1]} \simeq (1 + \mathbb{1}(i_k \neq j_k)) v_{i_k} (h'_{i_k j_k} - h_{i_k j_k}) v_{j_k} \simeq \frac{h'_{i_k j_k} - h_{i_k j_k}}{N^{1+o(1)}}, \tag{1.2}$$

where v_i is the i-th coordinate of the top eigenvector $\mathbf{v}^{[k]}$. Assuming that v_i is nearly independent of the matrix entries h_{ij} and h'_{ij}, since h_{ij} is centered with variance $1/N$, we would get from the central limit theorem that

$$\lambda_1^{[k]} - \lambda_1 = \sum_{t=0}^{k-1} (\lambda_1^{[t+1]} - \lambda_1^{[t]}) \simeq \frac{\sqrt{k}}{N^{3/2+o(1)}} \ .$$

On the other hand, if $q \gtrsim N^{1/9}$ then [18, Theorem 1.6] implies that $\lambda_1 - \lambda_2$ is of order $N^{-2/3}$. Hence as long as $\sqrt{k}/N^{3/2+o(1)}$ is much smaller than $N^{-2/3}$, it is believable that the approximation (1.2) is valid and that $\mathbf{v}_1^{[k]}$ is a small perturbation of $\mathbf{v}_1$. This explains the threshold at $k = N^{5/3+o(1)}$. In some sense, the proof of Theorem 1.4 makes rigorous the above heuristics. As it is usual in (non-integrable) random matrix theory, instead of working directly with eigenvalues, we will instead study the resolvent matrix of $H^{[k]}$ to shadow the behavior of $\mathbf{v}_1^{[k]}$ and $\lambda_1^{[k]}$ and interpret it as a stochastic process where k plays the role of time.

Remark (Noise sensitivity for other eigenvectors)**.** Following the above heuristic argument, for the eigenvector associated with the j-th largest eigenvalue, λ_j, we expect that the threshold is of order

$$N^{5/3+o(1)} \min(j, N-j)^{-2/3},$$

since the rigidity bound for λ_j is given as $N^{-2/3}\min(j, N-j)^{-1/3}$. However we note that an important modification is required to show the noise sensitivity of the other eigenvectors: the argument surrounding (3.3), Lemma 3.1 and Lemma 3.5 are tailored to the case of the top eigenvector.

Theorem 1.3 is proved by considering the variance of the largest eigenvalue λ_1 of H. The main inequality we prove is that

$$\mathbb{E}\left|\langle \mathbf{v}_1, \mathbf{v}_1^{[k]}\rangle\right|^2 \lesssim \frac{N^3 \mathrm{Var}(\lambda_1 - \mathcal{X})}{k}, \tag{1.3}$$

where $\mathcal{X}$ is defined as

$$\mathcal{X} = \frac{1}{N} \sum_{1\le i,j\le N} \left(h_{ij}^2 - \frac{1}{N}\right) = \frac{1}{N}\mathrm{Tr}(H^2) - 1. \tag{1.4}$$

It is a consequence of [18, Theorem 1.4] that $\mathrm{Var}(\lambda_1 - \mathcal{X})$ is of order $N^{-4/3+o(1)}$ provided that $q \gtrsim N^{1/9}$. We then deduce Theorem 1.3. As in [4], the proof of the inequality (1.3) is based on a variance formula for general functions of independent random variables due to Chatterjee [6]. The inequality (1.3) shows that small variance implies noise sensitivity of the top eigenvector.

We note that (1.3) is also true with $\mathcal{X}$ replaced by 0 (as done in [4]). It is immediate to check from (1.1) that $\mathrm{Var}(\mathcal{X}) \asymp 1/(Nq^2)$ which is larger than $N^{-4/3}$ for $q \le N^{1/6}$. Moreover, it follows from [15, 18] that $\mathrm{Var}(\lambda_1)$ is of the same order than $\mathrm{Var}(\mathcal{X})$ for $1 \ll q \le N^{1/6}$. Hence, the presence of $\mathcal{X}$ in (1.3) was necessary to conclude in the regime $N^{1/9} \lesssim q \ll N^{1/6}$.

We conjecture that Theorem 1.3 and Theorem 1.4 remains true as long as $q \gg 1$. With the current bounds available in [15, 18, 23] and the techniques of proof in the present paper, it is possible to obtain the following statements for $1 \ll q \le N^{1/9}$: the conclusion of Theorem 1.3 is true for $k \gg \min(N^{7/3}q^{-6}, N^2q^{-2})$ while the conclusion of Theorem 1.4 is true for $k \ll Nq^2$. Since we expect that these bounds are not optimal, we shall only focus in this paper on the case $q \gtrsim N^{1/9}$.

Remark (Higher order fluctuations of extremal eigenvalues)**.** When $1 \ll q \ll N^{1/9}$, we can recover the edge rigidity by introducing higher order random correction terms introduced in the recent preprint [21] posted after the first version of the present work. Thus it may be possible to show the conclusion of Theorem 1.3 under the condition that $q \gg 1$ and $k \gg N^{5/3}$ if we replace the term $\mathcal{X}$ with a new correction term $\widetilde{\mathcal{L}}$, in the main inequality (1.3). (See [21, Lemma 2.5] and [21, Theorem 2.10] for the precise definition of $\widetilde{\mathcal{L}}$.) This $\widetilde{\mathcal{L}}$ captures higher (sub-leading) oder fluctuations of extremal eigenvalues (of sparse random matrices) whereas the term $\mathcal{X}$ only governs the leading order fluctuation of those. We note that the argument associated with (3.2) should be modified to establish this extension rigorously. If we denote by $\widetilde{\mathcal{L}}_{st}$ the correction term corresponding to the matrix $H_{(st)}$ obtained from H by a single entry resampling at random. (See Section 3.1 for more detail.), we expect to have

$$\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}_{st} \prec N^{-1-\epsilon},$$

which will be beneficial to make some desired estimates. Similarly, for the extension of Theorem 1.4, the shift of the resolvent in Section 5.2 must be justified with some proper modifications.

Our definition of sparse random matrices was dictated by the use of [23] in the proof of Theorem 1.3. The proof of Theorem 1.4 does not use [23]. Thus Theorem 1.4 remains true for the more general sparse random matrix model considered in [18].

Remark. From the definition of our sparse random matrices, after k resampled entries there are only around kq^2/N entries which have been actually modified (most of the resampled entries simply replace a null entry by a null entry). Hence the threshold at $N^{5/3+o(1)}$ occurs after only $q^2N^{2/3+o(1)}$ visible changes of the matrix entries. With this point of view, as q gets smaller, we see that the top eigenvector gets more noise sensitive.

The proofs of Theorem 1.3 and Theorem 1.4 follow the same general strategy as [4]. It should be noted however that new technical challenges appear as the sparsity parameter q gets smaller. The dependency of the spectrum on a single matrix entry is larger and concentration inequalities are much weaker. As a consequence some bounds used in [4] where not good enough in the sparse regime. We had to modify substantially some technical arguments and also to improve some resolvent estimates on sparse random matrices from the current literature, they are gathered in the Section 6.

1.2 Extension to edge resampling in Erdős-Rényi random graphs

There is a natural extension of our main results to adjacency matrices of Erdős-Rényi random graphs. This adjacency matrix is the $N \times N$ random symmetric matrix whose diagonal entries are zero and whose entries above diagonal are independent Bernoulli random variables with mean q^2/N. We define the resampling procedure as in Definition 1.2 with the random sets S_k and an independent copy of Erdős-Rényi random graph. The resampling procedure describes a process where some randomly chosen edges of the graphs are added and other are removed (of order kq^2/N after k steps).

Let us denote an orthonormal basis of eigenvectors of Erdős-Rényi random graph by $\{\mathbf{w}_1, \cdots, \mathbf{w}_N\}$ where each $\mathbf{w}_i$ corresponds with the i-th largest eigenvalue. Similarly we denote by $\{\mathbf{w}_1^{[k]}, \cdots, \mathbf{w}_N^{[k]}\}$ the orthonormal eigenvector basis of Erdős-Rényi random graph after the resampling procedure.

In the regime $q \gg 1$, it is standard that the largest eigenvalue is close to q^2 and the top eigenvector is aligned the unit vector $\mathbf{e}$ with constant coordinates: $\mathbf{e}_i = 1/\sqrt{N}$ for all i, see [11, Theorem 2.16, Theorem 6.2] for precise statements. These results imply that $\mathbb{E}|\langle \mathbf{w}_1, \mathbf{w}_1^{[k]} \rangle| = 1 + o(1)$ for all k. There is thus a strong noise stability in this case. As one might expect, for the second largest eigenvalue and its corresponding eigenvector the situation is different and is parallel to sparse random matrices with mean zero entries.

Theorem 1.5 (Noise sensitivity)**.** *Fix $\ell \in \{2, N\}$. If $q \gtrsim N^{1/9}$ and $k \gg N^{5/3}$ then*

$$\mathbb{E}\left|\left\langle \mathbf{w}_\ell, \mathbf{w}_\ell^{[k]} \right\rangle\right| = o(1).$$

Theorem 1.6 (Noise stability)**.** *Fix $\ell \in \{2, N\}$. If $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$ then*

$$\mathbb{E} \min_{s\in\{\pm 1\}} \sqrt{N}\|\mathbf{w}_\ell - s\mathbf{w}_\ell^{[k]}\|_\infty = o(1).$$

The proofs of these results will follow from an adaptation of the proofs of Theorem 1.3 and Theorem 1.4. With a different perspective, the noise sensibility of the spectrum under edge resampling has already been considered in [9]. Our results suggest that in real-world networks, a heuristic to discriminate eigenvectors containing an information on the structure of the network from less relevant eigenvectors, could be through random uniform resampling of the edges: noise sensitive eigenvectors should not contain meaningful information.

Organization of the paper. In the next section, we shall cover some necessary tools used in the proof of the main results. In Section 3, we describe the high-level proofs of Theorem 1.3 and Theorem 1.4. The remaining sections, Section 4 and Section 5, are devoted to the details, for Theorem 1.3 and Theorem 1.4, respectively. The proofs of Theorem 1.5 and Theorem 1.6 are explained in Section 7. Finally, Section 6 contains some new resolvent estimates on sparse random matrices.

2 Preliminaries

In this section, we collect some necessary tools for the proof of main results.

2.1 Variance and noise sensitivity

For any positive integer i, denote $[i] = \{1, \cdots, i\}$. Let $Y_1, \cdots, Y_n$ be i.i.d. random variables taking values in a set $\mathcal{Y}$ equipped with a σ-algebra. Consider the random vector $Y = (Y_1, \cdots, Y_n)$ and let $Y' = (Y_1', \cdots, Y_n')$ be an independent copy of Y. We shall use the following notation,

$$Y^{(i)} = (Y_1, \cdots, Y_{i-1}, Y_i', Y_{i+1}, \cdots, Y_n)$$

For $\mathcal{I} \subset [n]$, we define $Y^{\mathcal{I}} = (Y_1^{\mathcal{I}}, \cdots, Y_n^{\mathcal{I}})$ by setting

$$Y_i^{\mathcal{I}} = \begin{cases} Y_i & \text{if } i \notin \mathcal{I}, \\ Y_i' & \text{if } i \in \mathcal{I}. \end{cases}$$

Let $\sigma = (\sigma(1), \cdots, \sigma(n))$ be a permutation in the symmetric group $\mathcal{S}_n$. For $i \in [n]$, we set $\sigma[i] = \{\sigma(1), \cdots, \sigma(i)\}$ and $\sigma[0] = \emptyset$. Let Y'' and Y''' be independent copies of Y. We assume Y, Y', Y'' and Y''' are independent. For $j \in [n]$, let $Y^{(j)\circ\sigma[i-1]}$ be the vector obtained from $Y^{\sigma[i-1]}$ by replacing j-th component of $Y^{\sigma[i-1]}$ as follows:

$$Y_j^{(j)\circ\sigma[i-1]} = \begin{cases} Y_j'' & j \in \sigma[i-1], \\ Y_j''' & j \notin \sigma[i-1]. \end{cases}$$

For example, if $n = 5$, $i = j = 3$ and $\sigma = (2, 3, 1, 5, 4)$, we have $\sigma([i-1]) = \{2, 3\}$,

$$Y^{\sigma[i-1]} = (Y_1, Y_2', Y_3', Y_4, Y_5) \quad \text{and} \quad Y^{(j)\circ\sigma[i-1]} = (Y_1, Y_2', Y_3'', Y_4, Y_5).$$

On the other hand, if $j = 1$, we have

$$Y^{(j)\circ\sigma[i-1]} = (Y_1''', Y_2', Y_3', Y_4, Y_5).$$

Lemma 2.1 (Variance and noise sensitivity). *Assume $f : \mathcal{Y}^n \to \mathbb{R}$ is a measurable function. Let j be a random variable uniformly distributed on $[n]$ independently of (Y, Y', Y'') and σ be uniformly distributed in $\mathcal{S}_n$ independently of (Y, Y', Y'', j). For any $k \in [n]$, define I_k by*

$$I_k = \mathbb{E}\left[\Big(f(Y) - f(Y^{(j)})\Big)\Big(f(Y^{\sigma([k-1])}) - f(Y^{(j)\circ\sigma([k-1])})\Big)\right].$$

Then, we have for any $k \in [n]$,

$$I_k \le \left(\frac{n+1}{n}\right)\left(\frac{2\mathrm{Var}\big(f(Y)\big)}{k}\right).$$

This is a small modification of [4, Lemma 3]. We refer to [4] for a proof and other similar statements.

2.2 Local laws and universality in sparse random theory

We start by introducing two handy probabilistic notions.

Definition 2.2 (Overwhelming probability). *Let $\{E_N\}$ be a sequence of events. We say E_N holds with* overwhelming probability *if for any $D > 0$, there exists $N_0(D)$ such that we have for $N \ge N_0(D)$*

$$\mathbb{P}\big(E_N^c\big) \le N^{-D}. \tag{2.1}$$

If $\{F_N\}$ is another sequence of events, we say that, on F_N, E_N holds with overwhelming probability, if $E_N \cup F_N^c$ has overwhelming probability.

Definition 2.3 (Stochastic domination). *Let (U_N) and (V_N) be two sequences of nonnegative random variables. U is said to be* stochastically dominated *by V if for all $\epsilon > 0$ and $D > 0$ there exists $N_0(\epsilon, D)$ such that we have for $N \ge N_0(\epsilon, D)$*

$$\mathbb{P}[U_N > N^{\epsilon} V_N] \le N^{-D}.$$

If U is stochastically dominated by V, we use the notation $U \prec V$. If (E_N) is a sequence of events, we say that on (E_N), $U \prec V$, if $U' \prec V$ with $U_N' = \mathbb{1}_{E_N} U_N$. Finally, if $(U_N(t))$ and $(V_N(t))$ are two families of sequences of non-negative random variables indexed by $t \in T$, we say that $U_t \prec V_t$ uniformly in $t \in T$ if the above integer $N_0(\epsilon, D)$ can be taken independent of $t \in T$.

Note that if U_N and V_N are deterministic then $U \prec V$ means $U_N \le N^{o(1)} V_N$.

Let $H = (h_{ij})$ be as in Definition 1.1. Recall that $\lambda_1 \ge \ldots \ge \lambda_N$ are the eigenvalues of H and $(\mathbf{v}_1, \ldots, \mathbf{v}_N)$ is an orthonormal basis of eigenvectors. A first key ingredient is the proof is the delocalization of eigenvectors.

Lemma 2.4 (Delocalization of eigenvectors [11, Theorem 2.16, Remark 2.18]). *Assume $q \gg 1$. We have*

$$\max_{1 \le i \le N} \|\mathbf{v}_i\|_\infty \prec \frac{1}{\sqrt{N}}.$$

A second ingredient is a non-asymptotic bound on the eigenvalue spacings of H.

Lemma 2.5 (Tail bounds for the gaps between eigenvalues [23, Theorem 2.2]). *Assume $q \gg 1$. There exists a constant $c > 0$ such that the following holds for any $\delta \ge N^{-c}$,*

$$\sup_{1 \le i \le N-1} \mathbb{P}\left(\lambda_i - \lambda_{i+1} \le \frac{\delta}{N}\right) = O(\delta \log N).$$

To be precise, in the above statement, the constant c depends on the sub-Gaussian tail parameter ϑ. Also, the $O(\cdot)$ on the right-hand side depends on ϑ and on a uniform lower bound on $\log q/\log N$ (which is positive by the assumption $q \gg 1$). There exists a non-quantitative result which is optimal on the scaling which is contained in [23, Theorem 1.6].

Lemma 2.6 (Tracy-Widom scaling for the gap [18, Theorem 1.6])**.** *Assume $q \gtrsim N^{1/9}$. For any $\epsilon > 0$, there exists a constant $c > 0$ such that*

$$\mathbb{P}(\lambda_1 - \lambda_2 \geq cN^{-2/3}) \geq 1 - \epsilon.$$

We now describe the location of the eigenvalues. Recall that if μ is a finite measure on $\mathbb{R}$, its *Cauchy-Stieltjes transform* is defined as the holomorphic function on $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by

$$z \mapsto \int \frac{d\mu(\lambda)}{\lambda - z}.$$

A measure is characterized by its Cauchy-Stieltjes transform and tools like Helffer-Sjöstrand formula allow to infer precise information on the measure through its Cauchy-Stieltjes transform, see e.g. [2] for its use in random matrix theory. We denote by $m(z)$ the Cauchy-Stieltjes transform of the empirical measure of eigenvalues of H:

$$m(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z}.$$

For an arbitrarily small constant $\epsilon > 0$, we define the shifted spectral domain

$$\mathcal{D}(\epsilon) = \left\{ w = \kappa + \mathrm{i}\eta \in \mathbb{C}_+ : |\kappa| \leq 3, 0 \leq \eta \leq 1, |\kappa| + \eta \geq N^{\epsilon}\left(\frac{1}{q^3 N^{1/2}} + \frac{1}{q^3 N\eta} + \frac{1}{(N\eta)^2}\right)\right\}.$$

Lemma 2.7 (Local law, Theorem 2.1 of [18])**.** *Assume $q \gg 1$ and let $\epsilon > 0$. There exists an explicit random symmetric measure $\rho_\star$ with random support $[-\mathcal{L}, \mathcal{L}]$ whose Stieltjes transform $m_\star$ satisfies the following. Uniformly for any $z = \mathcal{L} + w$, with $w = \kappa + \mathrm{i}\eta \in \mathcal{D}(\epsilon)$, we have,*

– *If $\kappa \geq 0$,*

$$|m(z) - m_\star(z)| \prec \frac{1}{\sqrt{|\kappa| + \eta}}\left(\frac{1}{N\eta^{1/2}} + \frac{1}{q^{3/2}N^{1/2}} + \frac{1}{q^3 N\eta} + \frac{1}{(N\eta)^2}\right),$$

– *If $\kappa \leq 0$,*

$$|m(z) - m_\star(z)| \prec \frac{1}{N\eta} + \frac{1}{q^{3/2}N^{1/2}\eta^{1/2}}.$$

We note that in [18], the set $\mathcal{D}(\epsilon)$ is restricted to $|\kappa| \leq 1$ but their local law holds for any κ taking value in any fixed interval (the focus in [18] is on the edge behavior). In Section 6, we will prove an improvement on Lemma 2.7 when κ and η are sufficiently small.

The random measure $\rho_\star$ has positive density on $(-\mathcal{L}, \mathcal{L})$ and it is a small deformation of the semi-circular law. From [18, Proposition 2.6], the measure $\rho_\star$ satisfies a polynomial equation whose coefficients depend on the moments of the entries of H and on the random variable $\mathcal{X}$ defined by (1.4), see Section 6 for details. Let us mention that there exists a deterministic real $L = 2 + O(1/q^2)$ such that

$$|\mathcal{L} - L - \mathcal{X}| \prec N^{-1/2}q^{-3}, \tag{2.2}$$

see [18, Proposition 2.6] for details. We have for all $z = E + \mathrm{i}\eta \in \mathbb{C}_+$,

$$\mathrm{Im}[m_\star(E+\mathrm{i}\eta)] \asymp \begin{cases} \sqrt{\kappa+\eta}, & \text{if } E \in [-\mathcal{L},\mathcal{L}] \\ \frac{\eta}{\sqrt{\kappa+\eta}}, & \text{if } E \notin [-\mathcal{L},\mathcal{L}], \end{cases} \tag{2.3}$$

where κ is the distance of E to $\{-\mathcal{L},\mathcal{L}\}$, the boundary of the support of $\rho_\star$.

Lemma 2.7 can be used to establish a rigidity estimate of the eigenvalues of H. For integer $1 \le i \le N$, we define the typical location of λ_i as the number γ_i such that

$$\rho_\star([\gamma_i,\mathcal{L}]) = \frac{i-1}{N},$$

that is, γ_i is associated with the $(i-1)$-th $1/N$-quantile of $\rho_\star$. See [15, Lemma 2.12] for the asymptotic value of γ_i in terms of $\mathcal{X}$ and the corresponding $1/N$-quantile of the semi-circular law.

If $q \gg 1$, there exists $\epsilon > 0$ such that $q \gg N^\epsilon$. Then, the proof of Theorem 1.4 in [18] with $\eta = N^{-2/3}$ and $\kappa = N^{\mathfrak{a}}(N^{-1/3}q^{-3} + N^{-2/3})$ (instead of $\kappa = N^{\mathfrak{a}}(q^{-6} + N^{-2/3})$ in [18]) gives

$$|\lambda_1 - \mathcal{L}| \prec N^{-1/3}q^{-3} + N^{-2/3}. \tag{2.4}$$

More generally, armed with Lemma 2.7, we can obtain the next lemma by following the standard argument using Helffer-Sjöstrand formula such as [2, Section 1.8] with cut on the imaginary axis at $\eta = N^{-2/3}$:

Lemma 2.8 (Eigenvalue rigidity). *Assume $q \gg 1$. For all $1 \le i \le N$, we have*

$$|\lambda_i - \gamma_i| \prec N^{-1/3}q^{-3} + N^{-2/3}.$$

From (2.3), we find easily $\gamma_i \gtrsim i^{2/3}N^{-2/3}$ uniformly in $1 \le i \le N$. Moreover by Lemma 2.8, if $q \gtrsim N^{1/9}$, we have $|\lambda_i - \gamma_i| \prec N^{-2/3}$. Hence, by Lemma 2.5, the next corollary follows.

Corollary 2.9. *Let $\epsilon > 0$ and assume $q \gtrsim N^{1/9}$. There exist $c > 0$ such the following holds for any $\delta \ge N^{-c}$, for all N large enough, with probability at least $1 - \delta \log N$:*

$$\lambda_1 - \lambda_i \ge \begin{cases} c\delta N^{-1} & \text{if } 2 \le i \le N^\epsilon \\ c i^{2/3} N^{-2/3} & \text{if } N^\epsilon < i \le N. \end{cases}$$

Moreover, on the event $\{\lambda_1 - \lambda_2 \ge c\delta N^{-1}\}$, the above inequalities holds with overwhelming probability.

3 High-level proof of the main results

We adapt the method of proof in [4] by applying recent results for the sparse Erdős-Rényi graph model, in order to establish Theorem 1.3 and Theorem 1.4.

3.1 High-level proof of Theorem 1.3

For any $1 \le i \le j \le N$, denote by $H_{(ij)}$ the symmetric matrix obtained from H by replacing the entries h_{ij} and h_{ji} with h''_{ij}, where h''_{ij} is an independent copy of h_{ij}. Similarly, we write $H^{[k]}_{(ij)}$ for the symmetric matrix obtained from $H^{[k]}$ by replacing $h^{[k]}_{ij}$ and $h^{[k]}_{ji}$ as follows:

- If $(i,j) \in S_k$, then $h^{[k]}_{ij}$ and $h^{[k]}_{ji}$ are replaced with h''_{ij}.
- If $(i,j) \notin S_k$, then $h^{[k]}_{ij}$ and $h^{[k]}_{ji}$ are replaced with h'''_{ij}, where h'''_{ij} is another independent copy of h_{ij}.

Denote by (st) a random pair of indices chosen uniformly from $\{(i,j) : 1 \le i \le j \le N\}$. Note that

$$|\{(i,j) : 1 \le i \le j \le N\}| = N(N+1)/2$$

Let $\mu_1 \ge \cdots \ge \mu_N$ be the ordered eigenvalues of $H_{(st)}$ and, let $\mathbf{u}_1, \cdots, \mathbf{u}_N$ be the associated unit eigenvectors of $H_{(st)}$. Similarly, we define $\mu_1^{[k]} \ge \cdots \ge \mu_N^{[k]}$ and $\mathbf{u}_1^{[k]}, \cdots, \mathbf{u}_N^{[k]}$ for $H_{(st)}^{[k]}$. We apply Lemma 2.1 with $Y = H$ and $f(H) = \lambda_1 - L - \mathcal{X}$:

$$\mathbb{E}\left[\left(\lambda_1 - \mu_1 - Q_{st}\right)\left(\lambda_1^{[k]} - \mu_1^{[k]} - Q_{st}^{[k]}\right)\right] \le \frac{2\mathrm{Var}(\lambda_1 - L - \mathcal{X})}{k} \cdot \frac{N(N+1)+2}{N(N+1)}, \tag{3.1}$$

where

$$\begin{aligned} Q_{st} &:= \frac{1}{N}(h_{st}^2 - (h_{st}'')^2)(1 + \mathbb{1}(s \ne t)), \\ Q_{st}^{[k]} &:= \begin{cases} \frac{1}{N}((h_{st}')^2 - (h_{st}'')^2)(1 + \mathbb{1}(s \ne t)) & \text{if } (st) \in S_k, \\ \frac{1}{N}(h_{st}^2 - (h_{st}''')^2)(1 + \mathbb{1}(s \ne t)) & \text{if } (st) \notin S_k. \end{cases} \end{aligned} \tag{3.2}$$

By the spectral theorem, we have

$$\langle \mathbf{u}_1, H\mathbf{u}_1 \rangle = \sum_{i=1}^N \lambda_i |\langle \mathbf{u}_1, \mathbf{v}_i \rangle|^2 \le \lambda_1 \sum_{i=1}^N |\langle \mathbf{u}_1, \mathbf{v}_i \rangle|^2 = \lambda_1 = \langle \mathbf{v}_1, H\mathbf{v}_1 \rangle. \tag{3.3}$$

Similarly, it follows that

$$\langle \mathbf{v}_1, H_{(st)}\mathbf{v}_1 \rangle \le \langle \mathbf{u}_1, H_{(st)}\mathbf{u}_1 \rangle.$$

Combining the two above inequalities, we obtain

$$\langle \mathbf{u}_1, (H - H_{(st)})\mathbf{u}_1 \rangle \le \lambda_1 - \mu_1 \le \langle \mathbf{v}_1, (H - H_{(st)})\mathbf{v}_1 \rangle.$$

Also, by the same argument, we have

$$\langle \mathbf{u}_1^{[k]}, (H^{[k]} - H_{(st)}^{[k]})\mathbf{u}_1^{[k]} \rangle \le \lambda_1^{[k]} - \mu_1^{[k]} \le \langle \mathbf{v}_1^{[k]}, (H^{[k]} - H_{(st)}^{[k]})\mathbf{v}_1^{[k]} \rangle.$$

Let us write $\mathbf{v}_1 = (v_1, \cdots, v_N)$, $\mathbf{u}_1 = (u_1, \cdots, u_N)$, $\mathbf{v}_1^{[k]} = (v_1^{[k]}, \cdots, v_N^{[k]})$, and $\mathbf{u}_1^{[k]} = (u_1^{[k]}, \cdots, u_N^{[k]})$. Then, we find

$$Z_{st} u_s u_t \le \lambda_1 - \mu_1 \le Z_{st} v_s v_t,$$

where

$$Z_{st} := (h_{st} - h_{st}'')(1 + \mathbb{1}(s \ne t)).$$

Similarly,

$$Z_{st}^{[k]} u_s^{[k]} u_t^{[k]} \le \lambda_1^{[k]} - \mu_1^{[k]} \le Z_{st}^{[k]} v_s^{[k]} v_t^{[k]},$$

where

$$Z_{st}^{[k]} := \begin{cases} (h_{st}' - h_{st}'')(1 + \mathbb{1}(s \ne t)) & \text{if } (st) \in S_k, \\ (h_{st} - h_{st}''')(1 + \mathbb{1}(s \ne t)) & \text{if } (st) \notin S_k. \end{cases}$$

We set $T_1 = (Z_{st} v_s v_t - Q_{st})(Z_{st}^{[k]} v_s^{[k]} v_t^{[k]} - Q_{st}^{[k]})$, $T_2 = (Z_{st} v_s v_t - Q_{st})(Z_{st}^{[k]} u_s^{[k]} u_t^{[k]} - Q_{st}^{[k]})$, $T_3 = (Z_{st} u_s u_t - Q_{st})(Z_{st}^{[k]} v_s^{[k]} v_t^{[k]} - Q_{st}^{[k]})$, $T_4 = (Z_{st} u_s u_t - Q_{st})(Z_{st}^{[k]} u_s^{[k]} u_t^{[k]} - Q_{st}^{[k]})$. We have

$$\min(T_1,T_2,T_3,T_4) \le \big(\lambda_1-\mu_1-Q_{st}\big)\big(\lambda_1^{[k]}-\mu_1^{[k]}-Q_{st}^{[k]}\big) \le \max(T_1,T_2,T_3,T_4). \tag{3.4}$$

The next key lemma asserts that after one resample the top eigenvectors are close in ℓ^∞-norm. Its proof will use the delocalization of eigenvectors and the rigidity of the eigenvalues.

Lemma 3.1. *Assume $q \gtrsim N^{1/9}$ and let $c,\delta>0$ be such that $N^{c+\delta} \ll q$. For $1\le i\le j\le N$, let $\mathbf{u}_1^{(ij)}$ be the top eigenvector of $H_{(ij)}$. Then, on the event $\{\lambda_1-\lambda_2 \ge N^{-1-c}\}$, the event*

$$\bigcap_{1\le i\le j\le N}\left\{\inf_{s\in\{\pm1\}}\|s\mathbf{v}_1-\mathbf{u}_1^{(ij)}\|_\infty \le N^{-1/2-\delta}\right\}$$

holds with overwhelming probability. The analogous result for $H_{(ij)}^{[k]}$ also holds.

Next, let $0<\delta<1/9$ and $0<\epsilon<\delta/3$ to be defined later, we define the events

$$\mathcal{E}_1 := \left\{\max\left(\|\mathbf{v}_1\|_\infty, \|\mathbf{u}_1\|_\infty, \|\mathbf{v}_1^{[k]}\|_\infty, \|\mathbf{u}_1^{[k]}\|_\infty\right) \le N^{\epsilon-1/2}\right\}, \tag{3.5}$$

$$\mathcal{E}_2 := \left\{\max\left(\|\mathbf{v}_1-\mathbf{u}_1\|_\infty, \|\mathbf{v}_1^{[k]}-\mathbf{u}_1^{[k]}\|_\infty\right) \le N^{-1/2-\delta}\right\}. \tag{3.6}$$

Set the event $\mathcal{E} := \mathcal{E}_1\cap\mathcal{E}_2$. Let $c>0$ such that $c+\delta<1/9$. According to Lemma 2.4, Lemma 2.5 and Lemma 3.1, we have $\mathbb{P}(\mathcal{E}^c)=O(N^{-c}\log N)$ by choosing the $\pm$-phase properly for $\mathbf{u}_{(ij)}$ and $\mathbf{u}_{(ij)}^{[k]}$. On the event $\mathcal{E}$, we observe that $v_sv_tu_s^{[k]}u_t^{[k]}$, $u_su_tv_s^{[k]}v_t^{[k]}$ and $u_su_tu_s^{[k]}u_t^{[k]}$ can be replaced with

$$v_sv_tv_s^{[k]}v_t^{[k]} + O\left(N^{3\epsilon-2-\delta}\right).$$

Thus, on the event $\mathcal{E}$, it follows from (3.4)

$$\begin{aligned}\big(\lambda_1-\mu_1-Q_{st}\big)\big(\lambda_1^{[k]}-\mu_1^{[k]}-Q_{st}^{[k]}\big) \ge Z_{st}Z_{st}^{[k]}v_sv_tv_s^{[k]}v_t^{[k]} - O\left(|Z_{st}Z_{st}^{[k]}|N^{3\epsilon-2-\delta}\right)\\ -|Q_{st}Z_{st}^{[k]}|N^{2\epsilon-1}-|Q_{st}^{[k]}Z_{st}|N^{2\epsilon-1}-|Q_{st}Q_{st}^{[k]}|.\end{aligned} \tag{3.7}$$

We shall check the following decorrelation lemma between the event $\mathcal{E}$ and our random variables of interest.

Lemma 3.2. *If $4\epsilon+\delta<1/9$, we have*

$$\mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_sv_tv_s^{[k]}v_t^{[k]}\mathbb{1}_{\mathcal{E}^c}\right] = o\left(\frac{1}{N^3}\right),$$

and

$$\mathbb{E}\left[(\lambda_2-\mu_2-Q_{st})(\lambda_2^{[k]}-\mu_2^{[k]}-Q_{st}^{[k]})\mathbb{1}_{\mathcal{E}^c}\right] = o\left(\frac{1}{N^3}\right).$$

Since $\mathbb{E}|Z_{st}Z_{st}^{[k]}| = O(N^{-1})$, $\mathbb{E}|Q_{st}Z_{st}^{[k]}| = O(N^{-2}q^{-1})$ and $\mathbb{E}|Q_{st}Q_{st}^{[k]}| = O(N^{-3}q^{-2})$, we deduce that the inequality

$$\mathbb{E}\left[(\lambda_2-\mu_2-Q_{st})(\lambda_2^{[k]}-\mu_2^{[k]}-Q_{st}^{[k]})\right] \ge \mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_sv_tv_s^{[k]}v_t^{[k]}\right] + o\left(\frac{1}{N^3}\right) \tag{3.8}$$

follows from (3.7) and Lemma 3.2.

Using that (s,t) is uniformly distributed on $\{(i,j):1\le i\le j\le N\}$ and that $\mathbb{E}[Z_{ij}Z_{ij}^{[k]}|S_k]=4/N$ if $i<j$, we will prove the following lemma.

Lemma 3.3. *We have*

$$\mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_sv_tv_s^{[k]}v_t^{[k]}\right] = \frac{2}{N^3}\mathbb{E}\left[\langle \mathbf{v}_1, \mathbf{v}_1^{[k]}\rangle^2\right] + o\left(\frac{1}{N^3}\right).$$

Now we are ready to prove the main statement. From (3.1) and (3.8), we find

$$\mathbb{E}\left[\langle \mathbf{v}_1, \mathbf{v}_1^{[k]}\rangle^2\right] \leq \frac{N^3\mathrm{Var}(\lambda_1 - L - \mathcal{X})}{k}(1+o(1)) + o(1).$$

Using (2.4), we have for any $\epsilon > 0$,

$$\mathrm{Var}(\lambda_1 - L - \mathcal{X}) = O(N^{\epsilon-4/3}).$$

It remains to use Jensen's inequality: $\left(\mathbb{E}|\langle \mathbf{v}_1, \mathbf{v}_1^{[k]}\rangle|\right)^2 \leq \mathbb{E}\left[\langle \mathbf{v}_1, \mathbf{v}_1^{[k]}\rangle^2\right]$ and the assumption $k \gg N^{5/3}$ to conclude that $\mathbb{E}\left[|\langle \mathbf{v}_1, \mathbf{v}_1^{[k]}\rangle|\right] = o(1)$.

This concludes the proof of Theorem 1.3 with Lemma 3.1, Lemma 3.2 and Lemma 3.3 granted. These lemmas are proved in Section 4. □

3.2 High-level proof of Theorem 1.4

For $z = E + \mathrm{i}\eta$ with $\eta > 0$ and $E \in \mathbb{R}$, we introduce the resolvent matrix

$$R(z) = (H - zI)^{-1},$$

where I denotes the identity matrix. We denote by $R^{[k]}(z)$ the resolvent of $H^{[k]}$. As already advertised in the introduction, the proof of Theorem 1.4 relies on a fine study of the functional process $R^{[k]}$ where k plays the role of time. The domain of the parameter z will be tuned to follow the evolution of the largest eigenvalue $\lambda_1^{[k]}$ and the top eigenvector $\mathbf{v}^{[k]}$ as k evolves.

The main technical result is the following.

Lemma 3.4. *Assume $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$. Then, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists $c > 0$ such that, with overwhelming probability,*

$$\sup_{z}\max_{1\leq i,j\leq N} N\eta|\mathrm{Im}R_{ij}^{[k]}(z) - \mathrm{Im}R_{ij}(z)| \leq N^{-c},$$

where the supremum is over all $z = E + \mathrm{i}\eta$ with $|E - \mathcal{L}| \leq N^{-2/3+\delta}$ and $\eta = N^{-2/3-\delta}$.

We write $\mathbf{v}_1 = (v_1, \cdots, v_N)$ and $\mathbf{v}_1^{[k]} = (v_1^{[k]}, \cdots, v_N^{[k]})$. The next lemma connects the entries of the resolvent with the coordinates of the top eigenvector for z close enough to the largest eigenvalue.

Lemma 3.5. *Assume $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$. Let $0 < \delta < \delta_0$ be as in Lemma 3.4. There exists $c' > 0$ such that with probability $1 - o(1)$ it holds that*

$$\max_{1\leq i,j\leq N} N|\eta\mathrm{Im}R_{ij}(z) - v_iv_j| \leq N^{-c'} \quad \text{and} \quad \max_{1\leq i,j\leq N} N|\eta\mathrm{Im}R_{ij}^{[k]}(z) - v_i^{[k]}v_j^{[k]}| \leq N^{-c'},$$

with $z = \lambda_1 + \mathrm{i}\eta$ and $\eta = N^{-2/3-\delta}$.

In the proof of Lemma 3.5, we will also show that λ_1 and $\lambda_1^{[k]}$ are close as long as $k \ll N^{5/3}$, see Lemma 5.5 below. The proofs of Lemma 3.4 and Lemma 3.5 are postponed to Section 5.

We now explain the proof of Theorem 1.4 with Lemma 3.4 and Lemma 3.5 granted. According to (2.4), we have

$$|\lambda_1 - \mathcal{L}| \prec N^{-2/3}.$$

Thus, with overwhelming probability $z = \lambda_1 + i\eta$ with $\eta = N^{-2/3-\delta}$ is in the domain considered in Lemma 3.4. In particular, since

$$|v_i v_j - v_i^{[k]} v_j^{[k]}| \le |v_i v_j - \eta \mathrm{Im} R_{ij}(z)| + \eta |\mathrm{Im} R_{ij}(z) - \mathrm{Im} R_{ij}^{[k]}(z)| + |\eta \mathrm{Im} R_{ij}^{[k]}(z) - v_i^{[k]} v_j^{[k]}|,$$

the combination of Lemma 3.4 and Lemma 3.5 implies the following claim:

Lemma 3.6. *Assume $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$. There exists $c > 0$ such that with probability $1 - o(1)$, it holds that:*

$$\max_{1 \le i,j \le N} N |v_i v_j - v_i^{[k]} v_j^{[k]}| \le N^{-c}. \tag{3.9}$$

Let $0 < c' < c$ with c as Lemma 3.6. To prove Theorem 1.4, we prove that with probability $1 - o(1)$,

$$\sqrt{N} \|\mathbf{v}_1 - \mathbf{v}_1^{[k]}\|_\infty \le N^{-c'}, \tag{3.10}$$

for a proper choice of the $\pm$-phase for the top eigenvectors. Let $\epsilon > 0$ such that $c' + \epsilon < c$. Let us call $\mathcal{E}_0$ the event that (3.9) holds and $\|\mathbf{v}_1^{[k]}\|_\infty \le N^{\epsilon - 1/2}$. By Lemma 3.6 and Lemma 2.4, it is sufficient to prove that for N large enough (3.10) holds on the event $\mathcal{E}_0$.

Let i be such that $|v_i| \ge 1/\sqrt{N}$. We choose the phases of $\mathbf{v}$ and $\mathbf{v}^{[k]}$ such that $v_i, v_i^{[k]}$ are non-negative. Then, we get on $\mathcal{E}_0$,

$$|v_i - v_i^{[k]}| = \frac{|v_i^2 - (v_i^{[k]})^2|}{v_i + v_i^{[k]}} \le \frac{|v_i^2 - (v_i^{[k]})^2|}{v_i} \le N^{-c-1/2}.$$

For any integer $1 \le j \le N$, we write:

$$|v_j - v_j^{[k]}| = \frac{1}{v_i} |v_i v_j - v_i v_j^{[k]}| \le \frac{1}{v_i} |v_i v_j - v_i^{[k]} v_j^{[k]}| + \frac{|v_j^{[k]}|}{v_i} |v_i - v_i^{[k]}|.$$

Hence, on the event $\mathcal{E}_0$, we find

$$|v_j - v_j^{[k]}| \le N^{-c-1/2} + N^{\epsilon - c - 1/2}.$$

For our choice of ϵ, we deduce that (3.10) holds for all N large enough. Theorem 1.4 is proved. □

4 Noise sensitivity of the top-eigenvector

4.1 Proof of Lemma 3.1

Let $\mu_1^{(ij)} \ge \cdots \ge \mu_N^{(ij)}$ be the ordered eigenvalues of $H_{(ij)}$ and, let $\mathbf{u}_1^{(ij)}, \cdots, \mathbf{u}_N^{(ij)}$ be the associated unit eigenvectors of $H_{(ij)}$. Using (3.3), we find

$$\lambda_1 \ge \langle \mathbf{u}_1^{(ij)}, H \mathbf{u}_1^{(ij)} \rangle = \mu_1^{(ij)} + \langle \mathbf{u}_1^{(ij)}, (H - H_{(ij)}) \mathbf{u}_1^{(ij)} \rangle \ge \mu_1^{(ij)} - 2(|h_{ij}| + |h''_{ij}|) \|\mathbf{u}_1^{(ij)}\|_\infty^2.$$

Similarly, reversing the role of H and $H_{(ij)}$, we get

$$\mu_1^{(ij)} \ge \lambda_1 - 2(|h_{ij}| + |h''_{ij}|) \|\mathbf{v}_1\|_\infty^2.$$

From (1.1) we have $|h_{ij}| \prec 1/q$. Hence, by Lemma 2.4 and $q \gg N^{c+\delta}$, we obtain

$$\max_{1 \le i \le j \le N} |\lambda_1 - \mu_1^{(ij)}| \prec \frac{1}{qN} \ll \frac{1}{N^{1+c}}. \tag{4.1}$$

We decompose $\mathbf{u}_1^{(ij)}$ in the eigenvector basis of H:

$$\mathbf{u}_1^{(ij)} = \sum_{\ell=1}^N \alpha_\ell \mathbf{v}_\ell.$$

We write two expressions for $H\mathbf{u}_1^{(ij)}$:

$$H\mathbf{u}_1^{(ij)} = \sum_{\ell=1}^{N} \lambda_\ell \alpha_\ell \mathbf{v}_\ell = (H - H_{(ij)})\mathbf{u}_1^{(ij)} + (\mu_1^{(ij)} - \lambda_1)\mathbf{u}_1^{(ij)} + \lambda_1 \mathbf{u}_1^{(ij)}.$$

We deduce that

$$\lambda_1 \mathbf{u}_1^{(ij)} = \sum_{\ell=1}^{N} \lambda_\ell \alpha_\ell \mathbf{v}_\ell + (H_{(ij)} - H)\mathbf{u}_1^{(ij)} + (\lambda_1 - \mu_1^{(ij)})\mathbf{u}_1^{(ij)}.$$

Next, by taking an inner product with $\mathbf{v}_\ell$ for $\ell \neq 1$, we obtain

$$\lambda_1 \alpha_\ell = \lambda_1 \langle \mathbf{v}_\ell, \mathbf{u}_1^{(ij)} \rangle = \langle \mathbf{v}_\ell, \lambda_1 \mathbf{u}_1^{(ij)} \rangle = \lambda_\ell \alpha_\ell + \langle \mathbf{v}_\ell, (H_{(ij)} - H)\mathbf{u}_1^{(ij)} \rangle + (\lambda_1 - \mu_1^{(ij)})\alpha_\ell.$$

In other words,

$$\Big((\lambda_1 - \lambda_\ell) + (\mu_1^{(ij)} - \lambda_1) \Big) \alpha_\ell = \langle \mathbf{v}_\ell, (H_{(ij)} - H)\mathbf{u}_1^{(ij)} \rangle. \tag{4.2}$$

Let $\epsilon > 0$. According to Corollary 2.9, there exists $c' > 0$ such that on the event $\{\lambda_1 - \lambda_2 \geq N^{-1-c}\}$, with overwhelming probability:

$$\lambda_1 - \lambda_\ell \geq \begin{cases} c' N^{-1-c} & 2 \leq \ell \leq N^\epsilon, \\ c' \ell^{2/3} N^{-2/3} & N^\epsilon < \ell \leq N. \end{cases} \tag{4.3}$$

Since Lemma 2.4 implies

$$\Big| \langle \mathbf{v}_\ell, (H_{(ij)} - H)\mathbf{u}_1^{(ij)} \rangle \Big| \leq 4 |h''_{ij} - h_{ij}| \|\mathbf{v}_\ell\|_\infty \|\mathbf{u}_1^{(ij)}\|_\infty \prec \frac{1}{qN},$$

we deduce from (4.2) and (4.1) that, on the event $\{\lambda_1 - \lambda_2 \geq N^{-1-c}\}$,

$$(\lambda_1 - \lambda_\ell) \cdot |\alpha_\ell| \prec \frac{1}{qN}.$$

Thus, combining this last inequality with (4.3), we obtain, on the event $\{\lambda_1 - \lambda_2 \geq N^{-1-c}\}$,

$$|\alpha_\ell| \prec \begin{cases} q^{-1} N^c & 2 \leq \ell \leq N^\epsilon \\ q^{-1} \ell^{-2/3} N^{-1/3} & N^\epsilon < \ell \leq N. \end{cases} \tag{4.4}$$

On the other hand, by setting $s = \alpha_1 / |\alpha_1|$, we have

$$\begin{aligned} \|s\mathbf{v}_1 - \mathbf{u}_1^{(ij)}\|_\infty &= \|(s - \alpha_1)\mathbf{v}_1 + \textstyle\sum_{\ell \neq 1} \alpha_\ell \mathbf{v}_\ell\|_\infty \\ &\leq (1 - |\alpha_1|)\|\mathbf{v}_1\|_\infty + \textstyle\sum_{\ell \neq 1} |\alpha_\ell| \|\mathbf{v}_\ell\|_\infty \\ &\prec N^{-1/2} \textstyle\sum_{\ell \neq 1} |\alpha_\ell|, \end{aligned}$$

where on the last line, we have used that $1 - |\alpha_1| = 1 - \sqrt{1 - \sum_{\ell \neq 1} \alpha_\ell^2} \leq \sum_{\ell \neq 1} |\alpha_\ell|$. Using (4.4), we finally obtain

$$\begin{aligned} \|s\mathbf{v}_1 - \mathbf{u}_1^{(ij)}\|_\infty &\prec N^{-1/2} q^{-1} N^{c+\epsilon} + N^{-1/2} q^{-1} N^{-1/3} \textstyle\sum_{N^\epsilon < \ell \leq N} \ell^{-2/3} \\ &\prec q^{-1} N^{-1/2+c+\epsilon} + q^{-1} N^{-1/2}. \end{aligned}$$

Therefore we complete the proof by choosing $\epsilon > 0$ small enough so that $q \gg N^{c+\delta+\epsilon}$. We can handle the case $H_{(ij)}^{[k]}$ similarly since $H^{[k]}$ and H have the same law. □

4.2 Proof of Lemma 3.2

Recall that $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$ where the events $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined in (3.5) and (3.6). We start by proving the first statement of Lemma 3.2. We split the expectation into two parts.

$$\mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_s v_t v_s^{[k]}v_t^{[k]}\mathbb{1}_{\mathcal{E}^c}\right] = \mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_s v_t v_s^{[k]}v_t^{[k]}\mathbb{1}_{\mathcal{E}_1\cap\mathcal{E}_2^c}\right] + \mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_s v_t v_s^{[k]}v_t^{[k]}\mathbb{1}_{\mathcal{E}_1^c}\right]. \tag{4.5}$$

Using Cauchy-Schwarz inequality, we find

$$\left|\mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_s v_t v_s^{[k]}v_t^{[k]}\mathbb{1}_{\mathcal{E}_1^c}\right]\right| \le \mathbb{E}\left[|Z_{st}Z_{st}^{[k]}|\mathbb{1}_{\mathcal{E}_1^c}\right] \le \sqrt{\mathbb{E}\left[|Z_{st}Z_{st}^{[k]}|^2\right]\mathbb{P}(\mathcal{E}_1^c)}.$$

By Lemma 2.4 the event $\mathcal{E}_1$ holds with overwhelming probability and by Equation (1.1), we have $\mathbb{E}|Z_{st}Z_{st}^{[k]}|^2 \lesssim 1/(Nq^2)$. It follows that for any $C > 0$,

$$\left|\mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_s v_t v_s^{[k]}v_t^{[k]}\mathbb{1}_{\mathcal{E}_1^c}\right]\right| = O(N^{-C}).$$

We now turn to the first term on the right-hand side of (4.5). With $\epsilon > 0$ as in the definition of $\mathcal{E}_1$, we want to show that

$$\mathbb{E}\left[\left|Z_{st}Z_{st}^{[k]}\right|\mathbb{1}_{\mathcal{E}_2^c}\right] \ll N^{-1-4\epsilon}, \tag{4.6}$$

which implies

$$\mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_s v_t v_s^{[k]}v_t^{[k]}\mathbb{1}_{\mathcal{E}_1\cap\mathcal{E}_2^c}\right] \le N^{4\epsilon-2}\mathbb{E}\left[\left|Z_{st}Z_{st}^{[k]}\right|\mathbb{1}_{\mathcal{E}_2^c}\right] \ll N^{-3}. \tag{4.7}$$

There is a dependence between $Z_{st}Z_{st}^{[k]}$ and the event $\mathcal{E}_2$. To circumvent this difficulty, we introduce some new events. Let $c > 4\epsilon$ such that $c + \delta < 1/9$ (and δ as in the definition of $\mathcal{E}_2$). we consider the event $\mathcal{E}_3 = \mathcal{E}_{3,0} \cup \mathcal{E}_{3,1}$ where

$$\begin{aligned}\mathcal{E}_{3,0} &= \left\{\min\left(\lambda_1 - \lambda_2, \lambda_1^{[k]} - \lambda_2^{[k]}\right) \ge N^{-1-c}\right\} \quad \text{and} \\ \mathcal{E}_{3,1} &= \left\{\min\left(\mu_1 - \mu_2, \mu_1^{[k]} - \mu_2^{[k]}\right) \ge N^{-1-c}\right\}.\end{aligned} \tag{4.8}$$

By Lemma 3.1, for any $C > 0$, we have $\mathbb{P}(\mathcal{E}_2^c \cap \mathcal{E}_3) = O(N^{-C})$. Therefore, arguing as above, it is sufficient to prove that

$$\mathbb{E}\left[\left|Z_{st}Z_{st}^{[k]}\right|\mathbb{1}_{\mathcal{E}_3^c}\right] \ll N^{-1-4\epsilon}.$$

We note that

$$\begin{aligned}\mathbb{E}\left[\left|Z_{st}Z_{st}^{[k]}\right|\mathbb{1}_{\mathcal{E}_3^c}\right] &\le \frac{1}{2}\mathbb{E}\left[\left(Z_{st}^2 + (Z_{st}^{[k]})^2\right)\mathbb{1}_{\mathcal{E}_3^c}\right] \\ &\lesssim \mathbb{E}\left[\left(h_{st}^2 + (h'_{st})^2 + (h''_{st})^2 + (h'''_{st})^2\right)\mathbb{1}_{\mathcal{E}_3^c}\right] \\ &\lesssim \mathbb{E}\left[(h_{st}^2 + (h'_{st})^2)\mathbb{1}_{\mathcal{E}_{3,1}^c}\right] + \mathbb{E}\left[((h''_{st})^2 + (h'''_{st})^2)\mathbb{1}_{\mathcal{E}_{3,0}^c}\right].\end{aligned} \tag{4.9}$$

We now use by construction the variables (h_{st}, h'_{st}) are independent of the event $\mathcal{E}_{3,1}$. We get by Lemma 2.5 that

$$\mathbb{E}\left[(h_{st}^2 + (h'_{st})^2)\mathbb{1}_{\mathcal{E}_{3,1}^c}\right] = O(N^{-1-c}\log N).$$

Similarly, h''_{st} and h'''_{st} are independent of the event $\mathcal{E}_{3,0}$ and

$$\mathbb{E}\left[((h''_{st})^2 + (h'''_{st})^2)\mathbb{1}_{\mathcal{E}_{3,0}^c}\right] = O(N^{-1-c}\log N).$$

Since we choose $c > 4\epsilon$, it concludes the proof of (4.6) and of the first claim of Lemma 3.2.

We now prove the second statement of Lemma 3.2. As above we decompose $\mathcal{E}^c$ as the disjoint union of $\mathcal{E}_1^c$ and $\mathcal{E}_1 \cap \mathcal{E}_2^c$. We have $|Q_{ij}| \prec 1/(Nq^2)$ and $|Z_{ij}| \prec 1/q$. Since $\mathcal{E}_1$ holds with overwhelming probability, we find from (3.4), that for any $C > 0$,

$$\mathbb{E}\left[\left|(\lambda_2 - \mu_2 - Q_{st})(\lambda_2^{[k]} - \mu_2^{[k]} - Q_{st}^{[k]})\right|\mathbb{1}_{\mathcal{E}_1^c}\right] = O(N^{-C}).$$

We now deal with the event $\mathcal{E}_1 \cap \mathcal{E}_2^c$. From (3.4) and arguing as in (4.7), we find

$$\begin{aligned}
&\mathbb{E}\left[\left|(\lambda_2 - \mu_2 - Q_{st})(\lambda_2^{[k]} - \mu_2^{[k]} - Q_{st}^{[k]})\right|\mathbb{1}_{\mathcal{E}_1\cap\mathcal{E}_2^c}\right] \\
&\quad\leq N^{4\epsilon-2}\mathbb{E}\left[\max(|Z_{st}|, N|Q_{st}|)\cdot\max(|Z_{st}^{[k]}|, N|Q_{st}^{[k]}|)\mathbb{1}_{\mathcal{E}_2^c}\right].
\end{aligned}$$

We observe that

$$\max(|Z_{st}|, N|Q_{st}|) \leq |h_{st}| + |h''_{st}| + |h_{st}|^2 + |h''_{st}|^2,$$

the right-hand side is at most $2(|h_{st}| + |h''_{st}|)$ with overwhelming probability since $|h_{ij}| \prec 1/q$. The same comment applies to $\max(|Z_{st}^{[k]}|, N|Q_{st}^{[k]}|)$. It follows that

$$\mathbb{E}\left[\max(|Z_{st}|, N|Q_{st}|)\cdot\max(|Z_{st}^{[k]}|, N|Q_{st}^{[k]}|)\mathbb{1}_{\mathcal{E}_2^c}\right] \lesssim \mathbb{E}\left[\left(h_{st}^2 + (h'_{st})^2 + (h''_{st})^2 + (h'''_{st})^2\right)\mathbb{1}_{\mathcal{E}_2^c}\right].$$

Finally, by Lemma 3.1 on the event $\mathcal{E}_3$, $\mathcal{E}_2$ has overwhelming probability. We thus may substitute in the above inequality $\mathbb{1}_{\mathcal{E}_2^c}$ by $\mathbb{1}_{\mathcal{E}_3^c}$. We are then back to the upper bound in (4.9). The conclusion follows. ☐

4.3 Proof of Lemma 3.3

Integrating over the random pair (st), we have

$$\mathbb{E}\left[Z_{st}Z_{st}^{[k]}v_s v_t v_s^{[k]} v_t^{[k]}\right] = \frac{2}{N(N+1)}\mathbb{E}\left[\sum_{1\leq i\leq j\leq N} Z_{ij}Z_{ij}^{[k]}v_i v_j v_i^{[k]} v_j^{[k]}\right].$$

For brevity, we set $V_{ij} = v_i v_j v_i^{[k]} v_j^{[k]}$. We split the above sum on the right-hand side into two parts,

$$\sum_{(ij)\in S_k} Z_{ij}Z'_{ij}V_{ij} + \sum_{(ij)\notin S_k} Z_{ij}Z''_{ij}V_{ij},$$

where $1 \leq i \leq j \leq N$ in both sums, $Z_{ij} = (h_{ij} - h''_{ij})(1 + \mathbb{1}(i \neq j))$, $Z'_{ij} = (h'_{ij} - h''_{ij})(1 + \mathbb{1}(i \neq j))$ and $Z''_{ij} = (h_{ij} - h'''_{ij})(1 + \mathbb{1}(i \neq j))$. Note that

$$\mathbb{E}\left[Z_{ij}Z'_{ij}\right] = \mathbb{E}\left[Z_{ij}Z''_{ij}\right] = \begin{cases} \frac{4}{N} & \text{if } i < j, \\ \frac{1}{N} & \text{if } i = j. \end{cases}$$

Due to the dependence, it is tricky to compute $\mathbb{E}[Z_{ij}Z_{ij}^{[k]}V_{ij}]$ directly. Thus, we introduce the conditional expectation $\mathbb{E}[\cdot|S_k]$ for given S_k to avoid this issue. We shall first estimate

$$\mathbb{E}\left[\sum_{1\leq i\leq j\leq N}\mathbb{E}\left[Z_{ij}Z_{ij}^{[k]}|S_k\right]V_{ij}\right],$$

and then show the contribution of

$$\mathbb{E}\left[\sum_{1\leq i\leq j\leq N}\left(Z_{ij}Z_{ij}^{[k]} - \mathbb{E}\left[Z_{ij}Z_{ij}^{[k]}|S_k\right]\right)V_{ij}\right]$$

is negligible. We start by computing

$$\sum_{1\le i\le j\le N}\mathbb{E}\left[Z_{ij}Z_{ij}^{[k]}|S_k\right]V_{ij}=\sum_{(ij)\in S_k}\mathbb{E}\left[Z_{ij}Z'_{ij}\right]V_{ij}+\sum_{(ij)\notin S_k}\mathbb{E}\left[Z_{ij}Z''_{ij}\right]V_{ij}.$$

Using the explicit expression for the expectations, we obtain

$$\begin{aligned}\sum_{i\le j}\mathbb{E}\left[Z_{ij}Z_{ij}^{[k]}|S_k\right]V_{ij}&=\frac{4}{N}\sum_{i<j}V_{ij}+\frac{2}{N}\sum_{i=j}V_{ij}-\frac{1}{N}\sum_{i=j}V_{ij}\\&=\frac{2}{N}\langle\mathbf{v},\mathbf{v}^{[k]}\rangle^2+O\left(\frac{1}{N}\sum_i|V_{ii}|\right).\end{aligned}$$

The last sum of the above equation is negligible. Indeed, using the delocalization of eigenvectors (Lemma 2.4), we have

$$\sum_i|V_{ii}|\prec N^{-1}.$$

Since $|V_{ij}|\le 1$, we deduce in particular that

$$\mathbb{E}\left[\sum_{1\le i\le j\le N}\mathbb{E}\left[Z_{ij}Z_{ij}^{[k]}|S_k\right]V_{ij}\right]=\frac{2}{N}\mathbb{E}\langle\mathbf{v},\mathbf{v}^{[k]}\rangle^2+o\left(\frac{1}{N}\right).$$

To conclude the proof of the lemma, what remains to show is

$$\mathbb{E}\left[\sum_{1\le i\le j\le N}W_{ij}V_{ij}\right]=o\left(\frac{1}{N}\right),$$

where we have set $W_{ij}:=Z_{ij}Z_{ij}^{[k]}-\mathbb{E}\left[Z_{ij}Z_{ij}^{[k]}|S_k\right]$. For the remainder of this proof, we fix a pair (i,j), $1\le i\le j\le N$. It is sufficient to check that $\mathbb{E}[W_{ij}V_{ij}]=o(1/N^3)$ where the $o(\cdot)$ is uniform over the choice of the pair (i,j).

Let h''''_{ij} be a independent copy of h_{ij} which is also independent of (H,H',H'',H'''). Similarly to $H_{(ij)}$ and $H^{[k]}_{(ij)}$, we can define analogously $\tilde{H}_{(ij)}$ and $\tilde{H}^{[k]}_{(ij)}$ by replacing (i,j)-element with h''''_{ij}. Denote by $\tilde{\mathbf{u}}_1=(\tilde{u}_1,\dots,\tilde{u}_N)$ and $\tilde{\mathbf{u}}_1^{[k]}=(\tilde{u}_1^{[k]},\dots,\tilde{u}_N^{[k]})$ the top eigenvectors of $\tilde{H}_{(ij)}$ and $\tilde{H}^{[k]}_{(ij)}$ respectively. To ease the notation, we define

$$U_{ij}:=\tilde{u}_i\tilde{u}_j\tilde{u}_i^{[k]}\tilde{u}_j^{[k]}.$$

By construction, we have

$$\mathbb{E}\left[W_{ij}U_{ij}\right]=\mathbb{E}\left[\mathbb{E}\left[W_{ij}|S_k\right]\cdot\mathbb{E}\left[U_{ij}|S_k\right]\right]=0,$$

because, given S_k, the pair $(Z_{ij},Z_{ij}^{[k]})$ only depends on $(h_{ij},h'_{ij},h''_{ij},h'''_{ij})$ while $(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_1^{[k]})$ is independent of $(h_{ij},h'_{ij},h''_{ij},h'''_{ij})$. Thus, it is enough to show

$$\mathbb{E}\left[W_{ij}\left(V_{ij}-U_{ij}\right)\right]=o\left(\frac{1}{N^3}\right).\tag{4.10}$$

The proof of (4.10) is performed as the proof of Lemma 3.2. Let $\delta>0$, $0<\epsilon<\delta/3$, $c>4\epsilon$ be such that $c+\delta<1/9$. We consider the event $\tilde{\mathcal{E}}_1$ defined as $\mathcal{E}_1$ but with $\tilde{H}_{(st)}$ and $\tilde{H}^{[k]}_{(st)}$ replacing $H_{(st)}$ and $H^{[k]}_{(st)}$:

$$\tilde{\mathcal{E}}_1=\left\{\max\left(\|\mathbf{v}_1\|_\infty,\|\tilde{\mathbf{u}}_1\|_\infty,\|\mathbf{v}_1^{[k]}\|_\infty,\|\tilde{\mathbf{u}}_1^{[k]}\|_\infty\right)\le N^{\epsilon-1/2}\right\}.$$

Similarly, if $\{\tilde{\mu}_i\}_{i=1}^N$ and $\{\tilde{\mu}_i^{[k]}\}_{i=1}^N$ are the eigenvalues of $\tilde{H}$ and $\tilde{H}^{[k]}$ respectively, we consider the event $\tilde{\mathcal{E}}_3 = \mathcal{E}_{3,0} \cup \tilde{\mathcal{E}}_{3,1}$ with $\mathcal{E}_{3,0}$ defined by (4.8) and

$$\tilde{\mathcal{E}}_{3,1} = \left\{ \min \left(\tilde{\mu}_1 - \tilde{\mu}_2, \tilde{\mu}_1^{[k]} - \tilde{\mu}_2^{[k]} \right) \geq N^{-1-c} \right\},$$

As in the proof of Lemma 3.2, we use Lemma 3.1 to deduce that on the event $\tilde{\mathcal{E}}_3$, we have

$$\|\mathbf{v}_1 - \tilde{\mathbf{u}}_1\|_\infty \leq N^{-\frac{1}{2}-\delta} \quad \text{and} \quad \|\mathbf{v}_1^{[k]} - \tilde{\mathbf{u}}_1^{[k]}\|_\infty \leq N^{-\frac{1}{2}-\delta},$$

with overwhelming probability after choosing the phases of $\tilde{\mathbf{u}}_{(ij)}$ and $\tilde{\mathbf{u}}_{(ij)}^{[k]}$ properly. Hence, on the event $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_1 \cap \tilde{\mathcal{E}}_3$, we find that the bound

$$|V_{ij} - U_{ij}| \leq C N^{3\epsilon - 2 - \delta}$$

holds with overwhelming probability for some $C > 0$. Also, by Lemma 2.4, the event $\tilde{\mathcal{E}}_1$ holds with overwhelming probability. Using that $|V_{ij}|, |U_{ij}| \leq 1$, we deduce that for any $C > 0$,

$$|\mathbb{E}\left[W_{ij}\left(V_{ij} - U_{ij}\right)\right]| \lesssim N^{3\epsilon-2-\delta}\mathbb{E}|W_{ij}| + N^{4\epsilon-2}\mathbb{E}[|W_{ij}|\mathbb{1}_{\tilde{\mathcal{E}}_3^c}] + N^{-C}.$$

Since $\mathbb{E}[|Z_{ij} Z_{ij}^{[k]}|] = O(1/N)$, we have $\mathbb{E}|W_{ij}| = O(1/N)$ and thus the first term on the right-hand side of the above equation is $o(1/N^3)$. For the second term, we simply write that

$$\mathbb{E}[|W_{ij}|\mathbb{1}_{\tilde{\mathcal{E}}_3^c}] \lesssim \mathbb{E}\left[\left(h_{ij}^2 + (h'_{ij})^2 + (h''_{ij})^2 + (h'''_{ij})^2 + \frac{1}{N}\right)\mathbb{1}_{\tilde{\mathcal{E}}_{3,1}^c}\right] = O\left(N^{-1-c}\log N\right),$$

where we have used the independence of $\tilde{\mathcal{E}}_{3,1}$ and $(h_{ij}, h'_{ij}, h''_{ij}, h'''_{ij})$ and invoked Lemma 2.5. Since $4\epsilon < c$, this concludes the proof of (4.10). ☐

5 Noise stability of the top-eigenvector

5.1 Preliminaries on the resolvent matrix

Our first lemma is used to detect the largest eigenvalues from the diagonal entries of the resolvent $R(z) = (H - zI)^{-1}$ for z close enough to $\mathcal{L}$.

Lemma 5.1. *Assume $q \gtrsim N^{1/9}$. For any integer $1 \leq j \leq N$, there exists a random integer $1 \leq i \leq N$ such that for all E and $\eta > 0$*

$$\big(\max(\eta, |\lambda_j - E|)\big)^{-2} \leq 2N\eta^{-1}\mathrm{Im}R(E + \mathrm{i}\eta)_{ii}.$$

The other way around, let $\epsilon > 0$. With overwhelming probability, for all integers $1 \leq i \leq N$ and all E such that $|E - \mathcal{L}| \leq N^{-2/3+\epsilon}$, we have

$$N\eta^{-1}\mathrm{Im}R(E + \mathrm{i}\eta)_{ii} \leq N^{4\epsilon}\left(\min_{1\leq j\leq N}|\lambda_j - E|\right)^{-2}.$$

Proof. From the spectral theorem,

$$N\eta^{-1}\mathrm{Im}R(E + \mathrm{i}\eta)_{ii} = \sum_{p=1}^N \frac{N(\mathbf{v}_p(i))^2}{(\lambda_p - E)^2 + \eta^2} \geq \frac{N(\mathbf{v}_j(i))^2}{(\lambda_j - E)^2 + \eta^2} \geq \frac{N(\mathbf{v}_j(i))^2}{2\big(\max(\eta, |\lambda_j - E|)\big)^2},$$

The first statement follows since there exists i such that $|\mathbf{v}_j(i)| \geq N^{-1/2}$.

Next, we prove the second statement. Fix $\epsilon > 0$ and consider E satisfying $|E - \mathcal{L}| \leq N^{-2/3+\epsilon}$. From (2.4), with overwhelming probability $|\lambda_1 - E| \leq 2N^{-2/3+\epsilon}$. Thus, from

Lemma 2.4 and Lemma 2.8, for some $c > 0$, with overwhelming probability the following event holds: (i) $\max_{1\le p\le N}\|\mathbf{v}_p\|_\infty^2 \le N^{-1+\epsilon}$, (ii) $|\lambda_1 - E| \le 2N^{-2/3+\epsilon}$ and (iii) for all such E with $|E-\mathcal{L}| \le N^{-2/3+\epsilon}$, we have $E - \lambda_p \ge cp^{2/3}N^{-2/3}$ for all integer $p > N' := \lfloor N^{2\epsilon}\rfloor$. On this event, from (i) and (iii), we have for some $C > 0$

$$\sum_{p=N'+1}^{N} \frac{N(\mathbf{v}_p(i))^2}{(\lambda_p - E)^2+\eta^2} \le \sum_{p=N'+1}^{N} \frac{N^\epsilon}{(\lambda_p - E)^2} \le CN^\epsilon (N')^{-1/3}N^{4/3},$$

and

$$\sum_{p=1}^{N'} \frac{N(\mathbf{v}_p(i))^2}{(\lambda_p - E)^2+\eta^2} \le \frac{N^\epsilon N'}{\left(\min_{1\le j\le N}|\lambda_j - E|\right)^2}.$$

Finally from (ii), for all N large enough, we have $CN^\epsilon(N')^{-1/3}N^{4/3} \le N^\epsilon N'(\min_{1\le j\le N} |\lambda_j - E|)^{-2}$. This proves the second statement. ☐

The following lemma on the resolvent of sparse random matrices will be crucial to study the resolvent process indexed by the successive resampled entries. Below, we use the Kronecker delta symbol: $\delta_{ij} = \mathbb{1}_{i=j}$.

Lemma 5.2. *Assume $q \gtrsim N^{1/9}$ and let $0 < \delta < 1/3$. We have*

$$\sup_z \max_{1\le i,j\le N} \Big||R(z)_{ij}| - \delta_{ij}\Big| \prec \frac{1}{q} + \frac{1}{N\eta},$$

and

$$\sup_z \max_{1\le i,j\le N} \left|\mathrm{Im}R(z)_{ij}\right| \prec \frac{1}{N\eta},$$

where the two suprema are over all $z = E + \mathrm{i}\eta$ with $|E-\mathcal{L}| \le N^{-2/3+\delta}$ and $\eta = N^{-2/3-\delta}$.

The first statement of the lemma is a consequence of [11, Theorem 2.8] and the norm estimate of $m_{\mathrm{sc}}(z)$. The second statement is new, it uses notably the improved local law for the Cauchy-Stieltjes transform $m(z)$ given in Lemma 6.1. We postpone its proof to Subsection 6.2. We are now ready to prove Lemma 3.4.

5.2 Proof of Lemma 3.4

Step 1: net argument. We have $|R_{ij}(z) - R_{ij}(z')| \le |z-z'|/\eta^2$ where $\eta = \min(\mathrm{Im}(z), \mathrm{Im}(z'))$. Hence, by a standard net argument where we partition the interval $[-N^{-2/3+\delta}, N^{-2/3+\delta}]$ into N^2 sub-intervals, it suffices to prove the conclusion of Lemma 3.4 for any fixed κ real with $|\kappa| \le N^{-2/3+\delta}$, $z = E + \mathrm{i}\eta$ where $E = \mathcal{L} + \kappa$ and $\eta = N^{-2/3-\delta}$. Moreover, from (2.2) and Lemma 5.2, it is sufficient to prove that for any deterministic real κ with $|\kappa| \le 2N^{-2/3+\delta}$,

$$N\eta|\mathrm{Im}R_{ij}^{[k]}(\tilde z) - \mathrm{Im}R_{ij}(\tilde z)| \prec N^{-c}, \tag{5.1}$$

uniformly in $1 \le i, j \le N$, with

$$\tilde z = \kappa + L + \mathcal{X} + \mathrm{i}\eta, \tag{5.2}$$

and L deterministic as in (2.2). In the remainder of the proof, we fix such κ and corresponding random $\tilde z$.

Step 2: shifted resolvent matrix. The random variable $\tilde{z}$ depends on the entries h_{i_t,j_t}, $1 \le t \le k$. To avoid this, we set

$$\hat{z} = \kappa + L + \hat{\mathcal{X}} + \mathrm{i}\eta,$$

where

$$\hat{\mathcal{X}} = \mathcal{X} - \frac{1}{N}\sum_{t=1}^{k}(1 + \mathbb{1}(i_t \neq j_t))\left(h^2_{i_t j_t} - \frac{1}{N}\right).$$

By construction, from (1.1), we have

$$|\hat{z} - \tilde{z}| \prec \max\left(\frac{1}{Nq^2}, \frac{\sqrt{k}}{N^{3/2}q}\right).$$

Recall the resolvent identity.

$$(X - zI)^{-1} = (Y - zI)^{-1} + (Y - zI)^{-1}(Y - X)(X - zI)^{-1}$$

and the Ward identity for the resolvent: for any integers i, j,

$$\sum_{l=1}^{N} R_{il}(z)\bar{R}_{jl}(z) = (R(z)R^*(z))_{ij} = \frac{\mathrm{Im}R_{ij}(z)}{\mathrm{Im}(z)}. \tag{5.3}$$

It implies that

$$\begin{aligned}|R_{ij}(\tilde{z}) - R_{ij}(\hat{z})| \le& |\hat{z} - \tilde{z}| \sum_l |R(\tilde{z})_{il}||R(\hat{z})_{lj}| \\ \le& |\hat{z} - \tilde{z}| \sqrt{\sum_l |R(\tilde{z})_{il}|^2}\sqrt{\sum_l |R(\tilde{z})_{lj}|^2} \\ \prec& \max\left(\frac{1}{Nq^2}, \frac{\sqrt{k}}{N^{3/2}q}\right)\frac{1}{N\eta^2},\end{aligned}$$

where we have used Cauchy-Schwarz inequality, (5.3) and Lemma 5.2. Since $k \ll N^{5/3}$, we have

$$\frac{\sqrt{k}}{N^{3/2}q\eta} \ll \frac{N^\delta}{q} \ll 1,$$

provided that $N^\delta \ll q$. We deduce that $N\eta|R_{ij}(\tilde{z}) - R_{ij}(\hat{z})| \prec N^{-c}$ for some $c > 0$. The same conclusion holds for $R^{[k]}_{ij}(\tilde{z}) - R^{[k]}_{ij}(\hat{z})$. It follows that to prove (5.1), it is sufficient to prove that

$$N\eta|\mathrm{Im}R^{[k]}_{ij}(\hat{z}) - \mathrm{Im}R_{ij}(\hat{z})| \prec N^{-c}, \tag{5.4}$$

uniformly in $1 \le i, j \le N$.

Step 3: fluctuation of the resolvent process. Now, for $0 \le t \le N(N+1)/2$, we define $R^{[t]}(z) = (H^{[t]} - z)^{-1}$ as the resolvent of $H^{[t]}$. Since no other value of the resolvent will be considered, for ease of notation, we omit the parameter $\hat{z}$ and simply write $R^{[t]}$ in place of $R^{[t]}(\hat{z})$. From the resolvent identity, we get

$$R^{[k]}_{ij} - R_{ij} = \sum_{t=1}^{k}\left(R^{[t]}_{ij} - R^{[t-1]}_{ij}\right) = \sum_{t=1}^{k}(h_{i_t j_t} - h'_{i_t j_t})(R^{[t]}E_{i_t j_t}R^{[t-1]})_{ij},$$

where $E_{ij} = \mathbf{e}_i \mathbf{e}_j^T + \mathbf{e}_j \mathbf{e}_i^T \mathbb{1}(i \neq j)$ where $\mathbf{e}_i$ denotes the canonical basis of $\mathbb{R}^n$ such that the i-th entry is equal to 1 and the other entries are equal to 0. We set

$$h_t = h_{i_t j_t}, \quad h'_t = h'_{i_t j_t}, \quad E_t = E_{i_t j_t} \quad \text{and}$$
$$G_t = N\eta \mathrm{Im}\left((R^{[t]} E_t R^{[t-1]})_{ij} + (R^{[t]} E_t R^{[t-1]})_{ji} \right),$$

(G_t depends implicitly on $\{i,j\}$). Since $R_{ij} = R_{ji}$, we get that, by construction,

$$N\eta(\mathrm{Im} R_{ij}^{[k]} - \mathrm{Im} R_{ij}) = \frac{1}{2} \sum_{t=1}^{k} (h_t - h'_t) G_t.$$

The main technical ingredient in the proof of Lemma 3.4 is the following statement (note that for deterministic sequences of non-negative numbers, $U \prec V$ is equivalent to $U \leq N^{o(1)} V$).

Lemma 5.3. *Assume $q \gtrsim N^{1/9}$ and $k \ll N^2$. With the above notation, for any integer $r \geq 1$, uniformly in i, j,*

$$\mathbb{E}\left(\sum_{t=1}^{k} (h_t - h'_t) G_t \right)^{2r} \prec \left(\frac{k}{N^3 \eta^2} \right)^r + \left(\frac{k}{N^3 \eta^2} \right) q^{2-2r}. \tag{5.5}$$

Before proving Lemma 5.3 in the next subsection, let us conclude the proof of Lemma 3.4. Since $\eta = N^{-2/3-\delta}$, we have $k/(N^3\eta^2) = N^{2\delta} k / N^{5/3}$. Moreover, if $k \ll N^{5/3}$, we may find a small $\delta_0 > 0$ such that $k \ll N^{5/3 - 3\delta_0}$. We set $c = \delta_0$ and assume that $0 < \delta < \delta_0$. From Markov inequality, this concludes the proof of (5.4) and Lemma 3.4.

5.3 Proof of Lemma 5.3

Step 1: moment expansion and symmetry. We set $y_t = h_t - h'_t$ and write

$$\mathbb{E}\left(\sum_{t=1}^{k} y_t G_t \right)^{2r} = \sum_{t_1, \ldots, t_{2r}} \mathbb{E}\big[y_{t_1} G_{t_1} \cdots y_{t_{2r}} G_{t_{2r}} \big].$$

Combining the terms with equal indices, we get

$$\mathbb{E}\left(\sum_{t=1}^{k} y_t G_t \right)^{2r} = \sum_{m=1}^{2r} \sum_{\rho} \sum_{T} \frac{(2r)!}{\prod_l \rho_l !} \mathbb{E}\left[\prod_{l=1}^{m} y_{t_l}^{\rho_l} G_{t_l}^{\rho_l} \right],$$

where the second sum is over vectors $\rho = (\rho_l)$, with $\rho_1 + \ldots + \rho_m = 2r$, $\rho_l \geq 1$ and the last sum is over all sequences $T = (t_1, \ldots, t_m)$ pairwise disjoint t_l in $\{1, \ldots, k\}$. Since r is fixed, it is enough to fix in the remainder of the proof an integer m in the above sum.

Despite the fact that $(G_t)_{t \in T}$ is not independent of $(y_t)_{t \in T}$, we start by checking that the contribution of vectors $\rho = (\rho_l)$ such that $\min_l \rho_l = 1$ is zero. More precisely, assume without loss of generality that $\rho_m = 1$. We set

$$K(m, \rho) = \sum_{T} \mathbb{E}\left[\prod_{l=1}^{m} y_{t_l}^{\rho_l} G_{t_l}^{\rho_l} \right] = \mathbb{E}\left[y_{t_m} G_{t_m} \prod_{l=1}^{m-1} y_{t_l}^{\rho_l} G_{t_l}^{\rho_l} \right].$$

We claim that $K(m, \rho) = 0$ if $\rho_m = 1$. Indeed, we can realize our random variables by considering the m-tuple $((h''_1, h'''_1), \ldots, (h''_m, h'''_m))$ of iid copies of h_{ij}, independent of a uniform m-tuple $((i'_1, j'_1), \ldots, (i'_m, j'_m))$ of distinct elements in $\{(i,j) : i \leq j\}$. Then for a given T as in the above sum and $1 \leq l \leq m$, we set $(h_{t_l}, h'_{t_l}) = (h''_l, h'''_l)$ and $(i_{t_l}, j_{t_l}) = (i'_l, j'_l)$. As a function of (h''_m, h'''_m), G_{t_m} is symmetric (because switching the values of h_m and h'_m maps $R^{[t_m]}$ to $R^{[t_m-1]}$ and maps $R^{[t_m-1]}$ to $R^{[t_m]}$). Moreover, as a function of (h''_m, h'''_m), for $l \le m-1$, G_{t_l} is a function of $h''_m \mathbb{1}(t_l < t_m) + h'''_m \mathbb{1}(t_l > t_m)$. Summing over T, it follows that

$$\sum_T \prod_{l=1}^{m-1} y_{t_l}^{\rho_l} G_{t_l}^{\rho_l}$$

is a symmetric function of (h''_m, h'''_m). Indeed, consider the map $(t_1, \dots, t_m) \mapsto (k+1-t_1, \dots, k+1-t_m)$. This maps defines an involution on the set of T in the above sum and its image on $\prod_{l=1}^{m-1} y_{t_l}^{\rho_l} G_{t_l}^{\rho_l}$ is symmetric in (h''_m, h'''_m). Therefore recalling that h_t and h'_t have the same distribution, we get

$$\sum_T \mathbb{E}\left[h_{t_m} G_{t_m} \prod_{l=1}^{m-1} y_{t_l}^{\rho_l} G_{t_l}^{\rho_l} \right] = \sum_T \mathbb{E}\left[h'_{t_m} G_{t_m} \prod_{l=1}^{m-1} y_{t_l}^{\rho_l} G_{t_l}^{\rho_l} \right].$$

Since $y_t = h_t - y'_t$, we get that $K(m, \rho) = 0$.

We thus restrict ourselves to vectors $\rho = (\rho_l)$ such that

$$\rho_l \ge 2, \quad \text{for all } 1 \le l \le m.$$

Our goal is then to prove that, uniformly over all T and such vectors ρ, we have

$$\mathbb{E}\left[\prod_{l=1}^{m} y_{t_l}^{\rho_l} G_{t_l}^{\rho_l} \right] \prec \left(\frac{1}{N^3 \eta^2} \right)^m q^{2(m-r)}. \tag{5.6}$$

This immediately implies the statement of the lemma since (i) $\min_l \rho_l \ge 2$ implies that $m \le r$ and (ii) there are at most k^m choices for the elements of T.

Step 2: resolvent bound. In order to extract the moments of y_t in (5.6), we shall use a decoupling argument using the resolvent expansion. For $0 \le s \le k$, we define $\hat{H}^{[s]}$ as the symmetric matrix obtained from $H^{[s]}$ by setting the entries $(i_t j_t)_{t \in T}$ and $(j_t i_t)_{t \in T}$ to 0. The resolvent of $\hat{H}^{[s]}$ at $\hat{z}$ is denoted by $\hat{R}^{[s]} = (\hat{H}^{[s]} - \hat{z})^{-1}$. We note that given $(i_t j_t)_{t \in T}$, the matrix $\hat{H}^{[s]}$ is independent of $(y_t)_{t \in T}$. For ease of notation, we also set

$$\alpha = \frac{1}{N\eta} \quad \text{and} \quad \beta = \frac{1}{q} + \frac{1}{N\eta}.$$

Iterating the resolvent identity, we get

$$\hat{R}^{[s]} - R^{[s]} = \sum_{p=1}^{8} \left(R^{[s]} (H^{[s]} - \hat{H}^{[s]}) \right)^p R^{[t]} + \left(R^{[s]} (H^{[s]} - \hat{H}^{[s]}) \right)^9 \hat{R}^{[s]}. \tag{5.7}$$

We have

$$H^{[s]} - \hat{H}^{[s]} = \sum_{t \in T} (h_{i_t j_t} \mathbb{1}(t > s) + h'_{i_t j_t} \mathbb{1}(t \le s)) E_{i_t j_t}$$

Recall the fact that $|h_{ij}| \prec q^{-1}$, $|R_{ij}^{[s]}| \prec 1$ (by Lemma 5.2) and $\|\hat{R}^{[s]}\| \le \eta^{-1}$. Since $q^9 \gtrsim N$, we deduce that

$$|\hat{R}_{ij}^{[s]} - R_{ij}^{[s]}| \prec \sum_{p=1}^{8} \frac{1}{q^p} + \frac{1}{q^9 \eta} \prec \beta.$$

Similarly, using $|\mathrm{Im}(ab)| \le |\mathrm{Im}(a)||b| + |a||\mathrm{Im}(b)|$, we find, by Lemma 5.2,

$$\left| \mathrm{Im} \hat{R}_{ij}^{[s]} - \mathrm{Im} R_{ij}^{[s]} \right| \prec \sum_{p=1}^{8} \frac{\alpha}{q^p} + \frac{1}{q^9 \eta} \prec \alpha.$$

Therefore, using again Lemma 5.2 and, we obtain

$$\max_{1\le i,j\le N}\left||\hat{R}^{[s]}_{ij}|-\delta_{ij}\right|\prec\beta\quad\text{and}\quad\max_{1\le i,j\le N}\left|\mathrm{Im}\hat{R}^{[s]}_{ij}\right|\prec\alpha. \tag{5.8}$$

We are ready for the decoupling argument.

Step 3: decoupled resolvent. The following lemma on stochastic domination is elementary.

Lemma 5.4. *Let (U_N), (V_N) be two sequences of non-negative random variables and (u_N) be a non-negative sequence such that $U_N\prec u_N$. If there exist $C>0$ and $p,q>0$ such that $1/p+1/q<1$ and $(\mathbb{E}U_N^p)^{1/p}\prec N^Cu_N$ and $(\mathbb{E}V_N^q)^{1/q}\prec N^C\mathbb{E}V_N$ then $\mathbb{E}[U_NV_N]\prec u_N\mathbb{E}[V_N]$.*

Proof. Set r such that $1/p+1/q+1/r=1$. From Hölder inequality, for any event $\mathcal{E}$

$$\mathbb{E}[U_NV_N]-\mathbb{E}[U_NV_N\mathbb{1}_{\mathcal{E}}]=\mathbb{E}[U_NV_N\mathbb{1}_{\mathcal{E}^c}]\le(\mathbb{E}U_N^p)^{1/p}(\mathbb{E}V_N^q)^{1/q}\mathbb{P}(\mathcal{E}^c)^{1/r}.$$

For a fixed $\epsilon>0$, we consider the event $\mathcal{E}=\{U_N\le N^\epsilon u_N\}$. Since $\mathcal{E}$ has overwhelming probability, we deduce from the assumptions that $\mathbb{E}[U_NV_N]\le N^\epsilon u_N\mathbb{E}[V_N\mathbb{1}_{\mathcal{E}}]+o(u_N\mathbb{E}V_N)$. The conclusion follows. □

We set

$$\hat{G}_t=N\eta\,\mathrm{Im}\left((\hat{R}^{[t]}E_t\hat{R}^{[t-1]})_{ij}+(\hat{R}^{[t]}E_t\hat{R}^{[t-1]})_{ji}\right).$$

In this paragraph, we prove that (5.6) holds when G_t is replaced by $\hat{G}_t$. In the next and final step, we will prove that G_t and $\hat{G}_t$ are close. From (5.8), we observe that for $t\in T$,

$$|\hat{G}_t|\prec 1,$$

Given $(i_tj_t)_{t\in T}$, y_t is independent of $(\hat{R}^{[s]})_{0\le s\le k}$. We deduce from (1.1), Lemma 5.4 and the assumption $\min_l\rho_l\ge 2$ that

$$\left|\mathbb{E}\left[\prod_{t\in T}y_t^{\rho_t}\hat{G}_t^{\rho_t}\right]\right|\lesssim\frac{q^{2(m-r)}}{N^m}\mathbb{E}\left[\prod_{t\in T}\hat{G}_t^2\right]. \tag{5.9}$$

Note that in the above expression, we have set $\rho_t:=\rho_l$ if $t=t_l\in T$.

Next, we estimate $\mathbb{E}\left[\prod_{t\in T}\hat{G}_t^2\right]$ in (5.9). We first observe that $\prod_{t\in T}(\alpha^2\hat{G}_t^2)$ is a sum of products of the form

$$\prod_{t\in T}\mathrm{Im}\left(\hat{R}^{[s_{1t}]}_{a_{1t}i_t}\hat{R}^{[s_{2t}]}_{a_{2t}j_t}\right)\mathrm{Im}\left(\hat{R}^{[s_{3t}]}_{a_{3t}i_t}\hat{R}^{[s_{4t}]}_{a_{4t}j_t}\right),$$

with $(a_{1t},a_{2t}),(a_{3t},a_{4t})\in\{(i,j),(j,i)\}$ and $(s_{1t},s_{2t}),(s_{3t},s_{4t})\in\{(t,t-1),(t-1,t)\}$. Using $|\mathrm{Im}(ab)|\le|\mathrm{Im}(a)||b|+|a||\mathrm{Im}(b)|$, we deduce from (5.8) that

$$\prod_{t\in T}\hat{G}_t^2\prec\sum\prod_{t\in T}\left|\hat{R}^{[s_t]}_{a_tb_t}\right|\left|\hat{R}^{[s'_t]}_{a'_tb'_t}\right|, \tag{5.10}$$

where the sum is over possible choices of a_t,a'_t in $\{i,j\}$, b_t,b'_t in $\{i_t,j_t\}$ and s_t,s'_t in $\{t-1,t\}$.

We now bound the right-hand side of (5.10). Since $2|ab|\le|a|^2+|b|^2$, it suffices to treat the case $(a_t,b_t,s_t)=(a'_t,b'_t,s'_t)$. We denote by $\mathbb{E}_T$ the conditional expectation with respect to $\mathcal{F}_T$, the σ- algebra generated by H, H' and $(i_s j_s)_{s\notin T}$ (in other words, we integrate only on $(i_t j_t)_{t\in T}$ given the rest of the variables). Since r is fixed, we have

$$\mathbb{E}_T \prod_{t\in T} \left|\hat{R}^{[s_t]}_{a_t b_t}\right|^2 \lesssim \frac{1}{N^{2m}} \sum_w \prod_{t\in T} |\hat{R}^{[s_t]}_{a_t u_t}[w]|^2,$$

where the sum is over all $w = ((u_t, v_t))_{t\in T}$ with $1 \le u_t, v_t \le N$ and $\hat{R}^{[s]}[w]$ is the resolvent of the symmetric matrix $H^{[s]}[w]$ obtained from $H^{[s]}$ by setting the entries $(u_t v_t)_{t\in T}$ and $(v_t u_t)_{t\in T}$ to 0 (that is $\hat{R}^{[s]} = \hat{R}^{[s]}[(i_t, j_t)_{t\in T}]$). We would like to apply Ward identity of the resolvent (5.3) in the above expression but the matrix $\hat{R}^{[s]}[w]$ depends on the summation index.

To overcome this difficulty, we approximate $\hat{R}^{[s]}[w]$ by the resolvent of another carefully chosen matrix. For $T_0 \subset T$, let W_{T_0} be the set of $w = ((u_t, v_t))_{t\in T}$ as above such that $\{u_t, v_t\} \cap \{i, j\} \neq \emptyset$ if and only if $t \in T_0$. If $w \in W_{T_0}$, we set $w_0 = ((u_t, v_t))_{t\in T_0}$ and $w_1 = ((u_t, v_t))_{t\notin T_0}$. We write

$$\mathbb{E}_T \prod_{t\in T} \left|\hat{R}^{[s_t]}_{a_t b_t}\right|^2 \lesssim \frac{1}{N^{2m}} \sum_{T_0\subset T} \sum_{w_0} \sum_{w_1} \prod_{t\in T} |\hat{R}^{[s_t]}_{a_t u_t}[w]|^2. \tag{5.11}$$

We next define $\hat{R}^{[s]}[w_0]$ as the resolvent of the symmetric matrix $H^{[s]}[w_0]$ obtained from $H^{[s]}$ by setting to the entries $(u_t v_t)_{t\in T_0}$ and $(v_t u_t)_{t\in T_0}$ to 0 and, for $t \in T\backslash T_0$, the entries $(u_t v_t)$ and $(v_t u_t)$ are set to $h_{u_t v_t}$ (irrespectively of the value of s). The computation leading to (5.8) gives

$$\max_{1\le i,j\le N} \left| \left|\hat{R}^{[s]}_{ij}[w_0]\right| - \delta_{ij} \right| \prec \beta \quad \text{and} \quad \max_{1\le i,j\le N} \left|\mathrm{Im}\hat{R}^{[s]}_{ij}[w_0]\right| \prec \alpha, \tag{5.12}$$

uniformly over all choices of w_0. Moreover, the resolvent identity implies

$$\hat{R}^{[s]}[w] = \hat{R}^{[s]}[w_0] + \hat{R}^{[s]}[w_0](H^{[s]}[w_0] - H^{[s]}[w])\hat{R}^{[s]}[w].$$

In particular, since u_t, v_t is different from i, j for all $t \notin T_0$, we deduce from (5.8)-(5.12) that for $t \notin T_0$ and $a \in \{i, j\}$, $|\hat{R}^{[s]}_{a u_t}[w]|, |\hat{R}^{[s]}_{a v_t}[w]| \prec \beta$ and similarly for $\hat{R}^{[s]}[w_0]$. Using (1.1), we find

$$|\hat{R}^{[s]}[w]_{a u_t} - \hat{R}^{[s]}[w_0]_{a u_t}| \prec \beta^2 q^{-1} \lesssim \alpha,$$

where the last inequality comes from $q \gtrsim N^{1/9}$. We note that the bound $|\hat{R}^{[s]}[w]_{a u_t} - R^{[s]}_{a u_t}| \prec q^{-1}$ would have been too large for our purposes for $q \lesssim N^{1/3}$.

In (5.11), we use for $t \in T_0$, $|\hat{R}^{[s_t]}_{a_t u_t}[w]| \prec 1$ and for $t \notin T_0$, $|\hat{R}^{[s_t]}_{a_t u_t}[w]| \prec |\hat{R}^{[s_t]}_{a_t u_t}[w_0]| + \alpha$. We obtain

$$\mathbb{E}_T \prod_{t\in T} \left|\hat{R}^{[s_t]}_{a_t b_t}\right|^2 \prec \frac{1}{N^{2m}} \sum_{T_0\subset T} \sum_{w_0} \sum_{w_1} \prod_{t\notin T_0} \left(|\hat{R}^{[s_t]}_{a_t u_t}[w_0]|^2 + \alpha^2 \right).$$

We next observe that for $t \notin T_0$, the matrix $\hat{R}^{[s_t]}[w_0]$ does not depend on $w_1 = ((u_s, v_s))_{s\notin T_0}$. We get

$$\begin{aligned}
\mathbb{E}_T \prod_{t\in T} \left|\hat{R}^{[s_t]}_{a_t b_t}\right|^2 &\prec \frac{1}{N^{2m}} \sum_{T_0\subset T} \sum_{w_0} \prod_{t\notin T_0} \left(\sum_{u,v} \left(|\hat{R}^{[s_t]}_{a_t u}[w_0]|^2 + \alpha^2 \right) \right) \\
&\prec \frac{1}{N^{2m}} \sum_{T_0\subset T} \sum_{w_0} \prod_{t\notin T_0} \left(N\alpha/\eta + N^2\alpha^2 \right) \\
&= \sum_{T_0\subset T} \frac{1}{N^{2|T_0|}} \sum_{w_0} \left(2\alpha^2\right)^{m-|T_0|},
\end{aligned}$$

where we have used Ward identity (5.3), $\alpha = 1/(N\eta)$ and (5.12). The number of possibilities for w_0 is at most $(4N)^{|T_0|}$. Hence, since $N\alpha^2 = N^{2\delta+1/3} \gg 1$, the above expression is maximized for $T_0 = \emptyset$ for all N large enough. Therefore, we finally obtain in (5.9) the bound,

$$\mathbb{E}\left[\prod_{t\in T} y_t^{\rho_t}\hat{G}_t^{\rho_t}\right] \prec \frac{q^{2(m-r)}}{N^m}\alpha^{2m}, \tag{5.13}$$

which is precisely the aimed bound in (5.6).

Step 4: resolvent expansion. In this final step, we prove (5.6). The proof is a slightly more complicated version of the argument leading to (5.13). To do this, in view of (5.13), it is sufficient to compare G_t and $\hat{G}_t$. From the resolvent identity, we have

$$R^{[s]} = \hat{R}^{[s]} + \sum_{p=1}^{n-1}\left(\hat{R}^{[s]}(\hat{H}^{[s]} - H^{[s]})\right)^p \hat{R}^{[s]} + \left(\hat{R}^{[s]}(\hat{H}^{[s]} - H^{[s]})\right)^n R^{[s]}.$$

By Lemma 5.2 and (5.8), we have

$$\alpha\left|\left(\left(\hat{R}^{[s]}(\hat{H}^{[s]} - H^{[s]})\right)^n R^{[s]}\right)_{ij}\right| \prec q^{-n}.$$

Hence, if n is large enough, this term can be made smaller than the right-hand side of (5.13). Recall $G_t = N\eta\mathrm{Im}((R^{[t]}E_tR^{[t-1]})_{ij} + (R^{[t]}E_tR^{[t-1]})_{ji})$. By the resolvent expansion,

$$\begin{aligned} R^{[t]}E_tR^{[t-1]} = &\left(\sum_{p=0}^{n-1}\left(\hat{R}^{[t]}(\hat{H}^{[t]} - H^{[t]})\right)^p \hat{R}^{[t]} + \left(\hat{R}^{[t]}(\hat{H}^{[t]} - H^{[t]})\right)^n R^{[t]}\right) \\ &\times E_t\left(\sum_{p'=0}^{n-1}\left(\hat{R}^{[t-1]}(\hat{H}^{[t-1]} - H^{[t-1]})\right)^{p'} \hat{R}^{[t-1]} + \left(\hat{R}^{[t-1]}(\hat{H}^{[t-1]} - H^{[t-1]})\right)^n R^{[t-1]}\right). \end{aligned}$$

Thus, we find that $G_t - \hat{G}_t$ can be written, up to negligible terms of order smaller than q^{-n}, as a finite sum of terms of the form

$$J = (N\eta)h^{\epsilon_1}_{x_1y_1}h^{\epsilon_2}_{x_2y_2}\cdots h^{\epsilon_{p+p'}}_{x_{p+p'}y_{p+p'}}\mathrm{Im}\left(\hat{R}^{[t]}_{ix_1}\hat{R}^{[t]}_{y_1x_2}\cdots\hat{R}^{[t]}_{y_pi_t^\epsilon}\hat{R}^{[t-1]}_{j_t^\epsilon x_{p+1}}\hat{R}^{[t-1]}_{y_{p+1}x_{p+2}}\cdots\hat{R}^{[t-1]}_{y_{p+p'}j}\right), \tag{5.14}$$

where $p+p' \geq 1$, $(x_ly_l) \in \{(i_sj_s),(j_si_s)\}_{s\in T}$, h^{ϵ_s} is either h or h' and $(i_t^\epsilon j_t^\epsilon) \in \{(i_tj_t),(j_ti_t)\}$. We call $\tau = p + p' \geq 1$ the length of the expansion. We define T_0 as the set of $t \in T$ such that $\{i_t,j_t\}\cap\{i,j\} \neq \emptyset$. We claim that

$$|J| \prec q^{-\tau}\left(\sum_{a,b,s}|\hat{R}^{[s]}_{ab}| + \beta^2 + \delta_t\right), \tag{5.15}$$

where the sum is over $s \in \{t,t-1\}$, $a \in \{i,j,i_s,j_s,s\in T_0\}$, $b \in \{i_t,j_t\}$ and $\delta_t \in \{0,1\}$ is the indicator that $\{i_t,j_t\}$ has a non-empty intersection with $\{i,j\}\cup\{i_s,j_s : s \in T\backslash t\}$. Indeed the factor $q^{-\tau}$ comes from (1.1). Next, we use (5.8) and $|\mathrm{Im}(ab)| \leq |\mathrm{Im}(a)||b| + |a||\mathrm{Im}(b)|$. If $\delta_t = 1$, we use $|\hat{R}^{[s]}_{kl}| \prec 1$ and the claimed bound follows. Assume otherwise that $\delta_t = 0$. Then, in (5.14), by assumption we have $\{x_l,y_l\} = \{x_{t_l},y_{t_l}\}$ for some $t_l \in T$. If there is at least one l such that $t_l \notin T_0$ then, since $\delta_t = 0$, there are at least 3 resolvent terms in (5.14) of the form $\hat{R}^{[s]}_{kl}$ with $k \neq l$. From (5.8), we then obtain the bound $J \prec q^{-\tau}\beta^2$. In the final case, we have $\delta_t = 0$ and for all l, $t_l \in T_0$ and the claimed bound (5.15) follows.

We deduce that

$$\prod_{t\in T} G_t^{\rho_t} - \prod_{t\in T} \hat{G}_t^{\rho_t} = \sum_{*} \prod_{t\in T} \hat{G}_t^{\sigma_t} \prod_{l=1}^{\rho_t-\sigma_t} J_{tl} + R,$$

where R is a remainder term with $|R| \prec q^{-n}$ and the sum is a finite sum over $0 \le \sigma_t \le \rho_t$ and terms J_{tl} as above of length $1 \le \tau_{tl} \le n$ such that

$$\tau = \sum_{t,l} \tau_{tl} \ge \sum_t (\rho_t - \sigma_t) \ge 1.$$

As in (5.10), we observe that for $t \in T$,

$$|\hat{G}_t| \prec \sum_{a,b,s} |\hat{R}^{[s]}_{ab}| \prec 1,$$

with $a \in \{i,j\}$, $b \in \{i_t, j_t\}$ and $s \in \{t-1, t\}$.

Using $2|ab| \le |a|^2 + |b|^2$ and the conditional independence of (y_t) and $(\hat{G}_t)$ given $(i_s j_s)_{s\in T}$, we deduce

$$\begin{aligned}\mathbb{E}\left|\prod_{t\in T} y^{\rho_t} G_t^{\rho_t} - \prod_{t\in T} y^{\rho_t}\hat{G}_t^{\rho_t}\right| &\prec \sum_{*} \frac{q^{2(m-r)}}{N^m}\mathbb{E}\prod_{t\in T} |\hat{G}_t|^{\sigma_t} \prod_{l=1}^{\rho_t-\sigma_t} J_{tl} + R' \\ &\prec \sum_{**} \frac{q^{2(m-r)}}{N^m}\mathbb{E}\prod_{t\in T}\left(|\hat{R}^{[s_t]}_{a_t b_t}|^2 + q^{-2\tau_t}\beta^4 + \delta_t\right) + R',\end{aligned}$$

where R' is negligible and the last sum is over the finitely many possibilities for $\tau_t \ge 1$, $a_t \in \{i, j, i_s, j_s, s \in T_0\}$ and $b_t \in \{i_t, j_t\}$.

We may now essentially repeat the argument in the previous step to argue that

$$\mathbb{E}_T \prod_{t\in T}\left(|\hat{R}^{[s_t]}_{a_t b_t}|^2 + q^{-2}\beta^4 + \delta_t\right) \prec \alpha^{2m},$$

where as above, $\mathbb{E}_T$ is the conditional expectation with respect to $\mathcal{F}_T$. This will conclude the proof of (5.6).

With the notation of (5.11) and the computation which follows, we write

$$\mathbb{E}_T \prod_{t\in T}\left(|\hat{R}^{[s_t]}_{a_t b_t}|^2 + q^{-2}\beta^4 + \delta_t\right) \prec \frac{1}{N^{2m}} \sum_{T_0\subset T}\sum_{w_0}\sum_{w_1}\prod_{t\notin T_0}\left(|\hat{R}^{[s_t]}_{a_t u_t}[w_0]|^2 + \alpha^2 + \delta_t[w]\right),$$

where $\delta_t[w]$ is the indicator that $\{u_t, v_t\}$ has a non-empty intersection with $\{i, j, u_s, v_s, s \in T\backslash t\}$. Note that we have used that $q^{-2}\beta^4 \le \alpha^2$ for our choice of q. We further decompose $T\backslash T_0$ as:

$$\prod_{t\notin T_0}\left(|\hat{R}^{[s_t]}_{a_t u_t}[w_0]|^2 + \alpha^2 + \delta_t[w]\right) = \sum_{T_1\subset T\backslash T_0}\prod_{t\in T_1}\left(|\hat{R}^{[s_t]}_{a_t u_t}[w_0]|^2 + \alpha^2\right)\prod_{t\notin T_1\cup T_0}\delta_t[w].$$

We observe that once all indices $(u_s, v_s)_{s\neq t}$ are chosen there at most $4mN$ choices of (u_t, v_t) such that $\delta_t[w] = 1$. It follows that

$$\begin{aligned}\sum_{w_1}\prod_{t\notin T_0}\left(|\hat{R}^{[s_t]}_{a_t u_t}[w_0]|^2 + \alpha^2 + \delta_t[w]\right) &\lesssim \sum_{T_1\subset T\backslash T_0} N^{m-|T_0|-|T_1|}\prod_{t\in T_1}\left(\sum_{u,v}\left(|\hat{R}^{[s_t]}_{a_t u}[w_0]|^2 + \alpha^2\right)\right) \\ &\prec \sum_{T_1\subset T\backslash T_0} N^{m-|T_0|-|T_1|}\prod_{t\in T_1}\left(N\alpha/\eta + N^2\alpha^2\right) \\ &\lesssim \sum_{T_1\subset T\backslash T_0} N^{m-|T_0|+|T_1|}\alpha^{2|T_1|},\end{aligned}$$

where we have used Ward identity (5.3), $\alpha = 1/(N\eta)$ and (5.12). We note that $N\alpha^2 \gg 1$. Thus for all N large enough, the above sum is maximized for $T_1 = T\backslash T_0$ and $N^{m-|T_0|+|T_1|}\alpha^{2|T_1|} = (N\alpha)^{2m-2|T_0|}$. Since the number of possibilities for w_0 is at most $(4N)^{|T_0|}$, we deduce that

$$\mathbb{E}_T \prod_{t\in T} \left(|\hat{R}^{[s_t]}_{a_t b_t}|^2 + q^{-2}\beta^4 + \delta_t \right) \prec \frac{1}{N^{2m}} \sum_{T_0 \subset T} N^{2m-|T_0|}\alpha^{2m-2|T_0|} \prec \alpha^{2m},$$

where we have again used that $N\alpha^2 \gg 1$. This concludes the proof of (5.6) and the proof of Lemma 5.3. □

5.4 Proof of Lemma 3.5

In order to show Lemma 3.5, we need to estimate the effect of the resampling to λ_1. The following proposition provide us the upper bound of the difference between λ_1 and $\lambda_1^{[k]}$.

Lemma 5.5. *Assume $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$. Then, if $0 < \delta < \delta_0$ with δ_0 as in Lemma 3.4, we have*

$$|\lambda_1 - \lambda_1^{[k]}| \prec N^{-2/3-\delta}.$$

Proof. If $\lambda_1 = \lambda_1^{[k]}$, we are done. Thus, suppose $\lambda_1^{[k]} < \lambda_1$. We set $\eta = N^{-2/3-\delta}$. According to Lemma 5.1, we can find $1 \le i \le N$ such that

$$\frac{1}{2\eta^2} \le N\eta^{-1}\mathrm{Im}R(\lambda_1 + \mathrm{i}\eta)_{ii}.$$

Since we have $|\lambda_1 - \mathcal{L}| \prec N^{-2/3}$, it follows from Lemma 5.1 that

$$N\eta^{-1}\mathrm{Im}R^{[k]}(\lambda_1 + \mathrm{i}\eta)_{ii} \prec \left(\min_{1\le j\le N} \left| \lambda_1 - \lambda_j^{[k]} \right| \right)^{-2}.$$

Since $\lambda_1 > \lambda_1^{[k]} \ge \lambda_2^{[k]} \ge \cdots \ge \lambda_N^{[k]}$, we observe

$$\min_{1\le j\le N} \left| \lambda_1 - \lambda_j^{[k]} \right| = \left| \lambda_1 - \lambda_1^{[k]} \right|.$$

Moreover, we can apply Lemma 3.4. For $c > 0$ as in Lemma 3.4, we obtain

$$\begin{aligned} N\eta^{-1}\mathrm{Im}R^{[k]}(\lambda_1 + \mathrm{i}\eta)_{ii} &\ge N\eta^{-1} \left(\mathrm{Im}R(\lambda_1 + \mathrm{i}\eta)_{ii} - \left| \mathrm{Im}R^{[k]}(\lambda_1 + \mathrm{i}\eta)_{ii} - \mathrm{Im}R(\lambda_1 + \mathrm{i}\eta)_{ii} \right| \right) \\ &\ge \frac{1}{2\eta^2} - \frac{1}{N^c\eta^2} \gtrsim \frac{1}{\eta^2}. \end{aligned}$$

As a result, we obtain

$$\frac{1}{\eta^2} \prec \left| \lambda_1 - \lambda_1^{[k]} \right|^{-2}.$$

In other words, $\left| \lambda_1 - \lambda_1^{[k]} \right| \prec \eta$. We have the same conclusion in the other case $\lambda_1^{[k]} > \lambda_1$ by reversing the role H and $H^{[k]}$. □

Proof of Lemma 3.5. We fix $0 < \delta < \delta_0$ and set $\eta = N^{-2/3-\delta}$. We write $\mathbf{v}_m = (\mathbf{v}_m(1), \dots, \mathbf{v}_m(N))$ and $\mathbf{v}_m^{[k]} = (\mathbf{v}_m^{[k]}(1), \dots, \mathbf{v}_m^{[k]}(N))$ for $m = 2, \dots, N$. By the spectral theorem, we have

$$N\eta\mathrm{Im}R(z)_{ij} = \frac{N\eta^2 v_i v_j}{(\lambda_1 - E)^2 + \eta^2} + \sum_{m=2}^{N} \frac{N\eta^2 \mathbf{v}_m(i)\mathbf{v}_m(j)}{(\lambda_m - E)^2 + \eta^2}.$$

Let $\epsilon > 0$ and let $N' := \lfloor N^{2\epsilon} \rfloor$. In the proof of Lemma 5.1, we have checked that with overwhelming probability: for all E satisfying $|E - \mathcal{L}| \le N^{-2/3+\epsilon}$, we have, for some $C > 0$,

$$\left| \sum_{m=N'+1}^{N} \frac{N\mathbf{v}_m(i)\mathbf{v}_m(j)}{(\lambda_m - E)^2 + \eta^2} \right| \le CN^{\epsilon}(N')^{-1/3}N^{4/3}. \tag{5.16}$$

By Lemma 2.6 and Lemma 2.4, we can find $c_0 > 0$ such that

$$\mathbb{P}\left(\mathcal{E}\right) \ge 1 - \epsilon/2,$$

where $\mathcal{E}$ is the event that (5.16) holds, $\{\lambda_1 - \lambda_2 > c_0 N^{-2/3}\}$ and $\max_m \|\mathbf{v}_m\|_\infty^2 \le N^{\epsilon-1}$. On the event $\mathcal{E}$, we find for all E with $|\lambda_1 - E| \le (c/2)N^{-2/3}$ that for some $C > 0$,

$$\left| \sum_{m=2}^{N'} \frac{Nv_{m,i}v_{m,j}}{(\lambda_m - E)^2 + \eta^2} \right| \le CN^{\epsilon}N'N^{4/3}.$$

We fix $\delta' > 0$ such that $\delta + \delta' < \delta_0$. On the event $\mathcal{E}$, for any E such that $|\lambda_1 - E| \le \eta N^{-\delta'}$, we have

$$\left| \frac{N\eta^2 v_i v_j}{(\lambda_1 - E)^2 + \eta^2} - Nv_iv_j \right| \le N^{\epsilon} \left| \frac{\eta^2}{(\lambda_1 - E)^2 + \eta^2} - 1 \right| \le N^{\epsilon - 2\delta'}.$$

Recall $\eta = N^{-2/3-\delta}$. Combining all of the above estimates and choosing $0 < \epsilon \le \min(\delta', \delta/3)$, we conclude that for all E satisfying $|\lambda_1 - E| \le \eta N^{-\delta'}$, for some $C > 0$,

$$\max_{1 \le i \le j \le N} |N\eta \mathrm{Im} R(E + \mathrm{i}\eta)_{ij} - Nv_{1,i}v_{1,j}| \le CN^{-\min(\delta,\delta')},$$

on the event $\mathcal{E}$. Now we repeat the above argument for $R^{[k]}$. We define the event $\mathcal{E}^{[k]}$ similarly for $H^{[k]}$. It provides us an event $\mathcal{E}^{[k]}$ of probability at least $1 - \epsilon/2$ such that for all E satisfying $|\lambda_1^{[k]} - E| \le \eta N^{-\delta'}$,

$$\max_{1 \le i \le j \le N} |N\eta \mathrm{Im} R^{[k]}(E + \mathrm{i}\eta)_{ij} - Nv_{1,i}^{[k]}v_{1,j}^{[k]}| \le CN^{-\min(\delta,\delta')}.$$

According to Lemma 5.5, we have $|\lambda_1 - \lambda_1^{[k]}| \le \eta N^{-\delta} = N^{-2/3-\delta-\delta'}$ with overwhelming probability (since $\delta + \delta' < \delta_0$). Since $\mathbb{P}(\mathcal{E} \cap \mathcal{E}^{[k]}) \ge 1 - \epsilon$ and ϵ can be made arbitrarily small, this concludes the proof of the lemma by picking any $0 < c' < \min(\delta, \delta')$. ☐

6 Resolvent of sparse random matrices

In this section, we have gathered the proofs of some estimates on the resolvent of H which have been used.

6.1 Cauchy-Stieltjes transform near the edge

Recall that for $z \in \mathbb{C}_+$, we have set $R(z) = (H - z)^{-1}$ and

$$m(z) = \frac{1}{N}\mathrm{Tr}R(z).$$

The following local law improves on Lemma 2.7 when κ and η are both small. We fix $\epsilon_0 > 0$, for example $\epsilon_0 = 1/4$ is sufficient for our purposes. We define the spectral domains:

$$\mathcal{D}_0 := \left\{ w = \kappa + \mathrm{i}\eta \in \mathbb{C}_+ : |\kappa| \le 3, N^{\epsilon_0 - 1} \le \eta \le 1 \right\}.$$

and we let $\bar{\mathcal{D}}_0 = \{w = \kappa + \mathrm{i}\eta : |\kappa| \le 3, \eta \ge N^{\epsilon_0 - 1}\}$ be the infinite half-strip containing $\mathcal{D}_0$.

Lemma 6.1. *Assume $q \gg 1$. Let $m_\star$ be as in Lemma 2.7. Uniformly on $w = \kappa + \mathrm{i}\eta \in \mathcal{D}_0$, we have, with $z = \mathcal{L} + w$,*

$$|m(z) - m_\star(z)| \prec \frac{1}{N\eta} + \frac{1}{q^3} + (\kappa+\eta)^{1/4}\left(\frac{1}{N\eta} + \frac{1}{q^3}\right)^{1/2}.$$

Before proving Lemma 6.1, we first prove a bound between $m_\star$ and m_{sc}, the Cauchy-Stieltjes transform of the semi-circular law.

Lemma 6.2. *Assume $q \gg 1$. Let $\epsilon > 0$ and set $\bar{q} = \min(q, N^{1/2-\epsilon})$. There exists $C > 0$ such that with overwhelming probability:*

$$\sup_{z \in \mathbb{C}_+} |m_{sc}(z) - m_\star(z)| \le \frac{C}{\bar{q}}.$$

Proof. By [18, Proposition 2.6], there exists a deterministic even polynomial

$$Q(y) = \frac{a_2}{q^2}y^4 + \frac{a_3}{q^4}y^6 + \cdots$$

whose coefficients a_i depend on the moments of h_{ij} and are uniformly bounded such that the random multivariate polynomial

$$P(z,y) := 1 + zy + y^2 + Q(y) + \mathcal{X}y^2 \tag{6.1}$$

satisfies

$$P(z, m_\star(z)) = 0.$$

We set $P_0(z,y) = 1 + zy + y^2$. We have $P_0(z, m_{\mathrm{sc}}(z)) = 0$. We set $f(z) = P_0(z, m_\star(z))$ and $g(z) = m_\star(z) - m_{\mathrm{sc}}(z)$.

We have $|\mathcal{X}| \prec 1/(q\sqrt{N})$. Hence the event $\mathcal{E} = \{|\mathcal{X}| \le 1/\bar{q}^2\}$ has overwhelming probability. On the event $\mathcal{E}$, uniformly in $z \in \mathbb{C}_+$, we have $|m_\star(z)| \le C$. Since, $f(z) = -Q(m_\star(z)) - m_\star(z)^2\mathcal{X}$, we deduce that if $\mathcal{E}$ holds, for all $z \in \mathbb{C}_+$,

$$|f(z)| \le \frac{C}{\bar{q}^2}. \tag{6.2}$$

By Taylor expansion, we have

$$f(z) = g(z)(z + 2m_{\mathrm{sc}}(z)) + g(z)^2.$$

With $\sqrt{\cdot}$ is the principal branch of the square root function, we have $z + 2m_{\mathrm{sc}}(z) = \sqrt{z^2-4}$. Hence

$$2g(z) = -\sqrt{z^2-4} \pm \sqrt{z^2 - 4 + 4f(z)}.$$

Since $g(z)$ is the difference of two Cauchy-Stieltjes transforms of probability measures, as $\mathrm{Im}(z)$ goes to infinity, $|g(z)|$ must vanish. From (6.2), this forces the choice of the above $\pm$-sign to be $+$ for all large z and thus for all $z \in \mathbb{C}_+$ since $g(z)$ is analytic on $\mathbb{C}_+$.

The remainder of the proof is obvious by decomposing in two possibilities: if $|f(z)| \ge |z^2-4|$ then $|g(z)| \le \sqrt{|f(z)|} + \sqrt{5|f(z)|}$. If $|f(z)| \le |z^2-4|$, then, by Taylor expansion,

$$\begin{aligned}|2g(z)| &= |-\sqrt{z^2-4} + \sqrt{z^2-4+4f(z)}| \\ &= \left|-\sqrt{z^2-4}\left(-1 + 1 + O\left(\frac{|f(z)|}{|z^2-4|}\right)\right)\right| \lesssim \frac{|f(z)|}{\sqrt{|z^2-4|}}.\end{aligned}$$

It concludes the proof since $1/\sqrt{|z^2-4|} \le 1/\sqrt{|f(z)|}$. □

By [11, Theorem 2.8], Lemma 6.2 implies the following weak local law.

Corollary 6.3. *Assume $q \gg 1$. For any $\epsilon > 0$, with overwhelming probability,*

$$\sup_{w \in \bar{\mathcal{D}}_0} \max_{1 \le i,j \le N} |R_{ij}(z) - \delta_{ij} m_\star(z)| \le N^\epsilon \left(\frac{1}{q} + \frac{1}{\sqrt{N\eta}} \right),$$

where $z = w + \mathcal{L}$ and $w = E + \mathrm{i}\eta$.

Proof. From [11, Theorem 2.8], the result holds with $m_{\mathrm{sc}}(z)$ in place of $m_\star(z)$ ([11, Theorem 2.8] is stated in $\mathcal{D}_0$ but the case $\eta \ge 1$ extends obviously). It remains to use Lemma 6.2 to bound the difference $m_{\mathrm{sc}}(z) - m_\star(z)$. ☐

All ingredients are gathered to prove Lemma 6.1.

Proof of Lemma 6.1. We fix $w \in \mathcal{D}_0$ and let $z = \mathcal{L} + w$. We set

$$g(z) = m(z) - m_\star(z) \quad \text{and} \quad \Lambda(z) = |g(z)|.$$

Let $P(z, y)$ be as in (6.1). Applying Taylor expansion, we have, from (6.1) and $P(z, m_\star(z)) = 0$,

$$P(z, m(z)) = \partial_2 P(z, m_\star(z))(g(z)) + \frac{1}{2}\partial_2^2 P(z, m_\star(z)) g(z)^2 + R(g(z)),$$

where $R(y) = b_1 y^3 + b_2 y^4 + \cdots$ is a deterministic polynomial whose coefficients are less than C/q^2. We set

$$f(z) = P(z, m(z)) - R(g(z)), \quad b(z) = \partial_2 P(z, m_\star(z)), \quad a(z) = \frac{1}{2}\partial_2^2 P(z, m_\star(z))$$

We get

$$a(z)g(z) = -b(z) \pm \sqrt{b(z)^2 + 4f(z)a(z)}.$$

By [18, Proposition 2.6], with overwhelming probability, the following event holds: for some $C > 0$, for all $z \in \mathbb{C}_+$, $|a(z) - 1| \le Cq^{-2}$ and $|b(z)| \ge \sqrt{||\kappa| + \eta|}/C$. Moreover, by Corollary 6.3, for some $C > 0$, with overwhelming probability, the following event: for all $z \in \mathcal{D}_0$, $\Lambda(z) \le 1/\log N$ and $|R(g(z))| \le C\Lambda(z)^3/q^2 \le \Lambda(z)^2/q^2$ (for N large enough). On the intersection of these two last events, say $\mathcal{E}$, since $|f(z)a(z)|$ is bounded uniformly on $\bar{\mathcal{D}}_0$ and $g(z)$ is analytic and vanishes as $\mathrm{Im}(z)$ goes to infinity, the only possibility for the $\pm$-sign is $+$. Arguing as in the proof of Lemma 6.2, we deduce that, if $\mathcal{E}$ holds, for some new $C > 0$,

$$\Lambda(z) = |g(z)| \le C\sqrt{|f(z)|}.$$

Since $|f(z)| \le |P(z, m(z))| + \Lambda(z)^2/q^2$, So finally, since $\Lambda(z) \le 1/\log N$ on $\mathcal{E}$, if N is large enough we get for some new $C > 0$,

$$\Lambda(z) \le C\sqrt{|P(z, m(z))|}. \tag{6.3}$$

The other way around, we now estimate $|P(z, m(z)|$ in terms of $\Lambda(z)$. By [18, Proposition 2.9], we have

$$\mathbb{E}\left[|P(z, m(z))|^{2r}\right] \prec \max_{1 \le s_1 + s_2 \le 2r}$$
$$\times \mathbb{E}\left[\left\{ |\partial_2 P(z, m(z))| \left(\frac{1}{q^3} + \frac{1}{N\eta} \right) \frac{\mathrm{Im}(m(z))}{N\eta} \right\}^{s_1/2} \left(\frac{\mathrm{Im}(m(z))}{N\eta} \right)^{s_2} |P(z, m(z))|^{2r - s_1 - s_2} \right],$$

(in [18], Proposition 2.9 is stated for $w \in \mathcal{D}(\epsilon)$ but their proof holds in the larger domain $\mathcal{D}_0$). From (2.3), it follows that

$$\mathrm{Im}(m_\star(z)) \lesssim \sqrt{|\kappa| + \eta},$$

which gives us

$$\mathrm{Im}(m(z)) \lesssim \sqrt{|\kappa|+\eta}+\Lambda.$$

Also, from [18, Proposition 2.6],

$$|\partial_2 P(z,m(z))| = |\partial_2 P(z,m_\star(z))| + O(\Lambda) \lesssim \sqrt{|\kappa|+\eta}+\Lambda.$$

By Young's inequality, we obtain, for any $\epsilon>0$,

$$\begin{aligned}
&\mathbb{E}\left[\left\{|\partial_2 P(z,m(z))|\left(\frac{1}{q^3}+\frac{1}{N\eta}\right)\frac{\mathrm{Im}(m(z))}{N\eta}\right\}^{s_1/2}\left(\frac{\mathrm{Im}(m(z))}{N\eta}\right)^{s_2}|P(z,m(z))|^{2r-s_1-s_2}\right]\\
&\qquad\lesssim N^\epsilon\mathbb{E}\left[|\partial_2 P(z,m(z))|^r\left(\frac{1}{q^3}+\frac{1}{N\eta}\right)^r\left(\frac{1}{N\eta}\right)^r\left\{(|\kappa|+\eta)^{r/2}+\Lambda^r\right\}\right]\\
&\qquad\qquad+N^\epsilon\left(\frac{1}{N\eta}\right)^{2r}\left\{(|\kappa|+\eta)^r+\mathbb{E}\Lambda^{2r}\right\}+N^{-\epsilon/(2r-1)}\mathbb{E}|P(z,m(z))|^{2r}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[|P(z,m(z))|^{2r}\right]\\
&\qquad\prec\left(\frac{1}{q^3}+\frac{1}{N\eta}\right)^r\left(\frac{1}{N\eta}\right)^r\left\{(|\kappa|+\eta)^r+\mathbb{E}\Lambda^{2r}\right\}+\left(\frac{1}{N\eta}\right)^{2r}\left\{(|\kappa|+\eta)^r+\mathbb{E}\Lambda^{2r}\right\}.
\end{aligned}$$

If $\gamma=1/q^3+1/(N\eta)$, we find

$$\mathbb{E}\left[|P(z,m(z))|^{2r}\right]\prec\left(\gamma\sqrt{|\kappa|+\eta}\right)^{2r}+\gamma^{2r}\mathbb{E}\Lambda^{2r}.$$

From (6.3), we deduce that

$$(\mathbb{E}\Lambda^{2r})^2\le\mathbb{E}\Lambda^{4r}\lesssim\mathbb{E}\left[|P(z,m(z))|^{2r}\right]\prec\left(\gamma\sqrt{|\kappa|+\eta}\right)^{2r}+\gamma^{2r}\mathbb{E}\Lambda^{2r}.$$

Since $x^2\le a+bx$, $a,b,x\ge 0$ implies that $x\le\sqrt{2a}+b$, we have established that

$$\mathbb{E}[\Lambda^{2r}|\prec\left(\gamma+\left(\gamma\sqrt{|\kappa|+\eta}\right)^{1/2}\right)^{2r}.$$

Since r is arbitrary, from Markov inequality, the proof is complete. □

6.2 Proof of Lemma 5.2

It is enough to only show the second equality on $\mathrm{Im}(R_{ij})$, which is a consequence of the following lemma.

Lemma 6.4. *Assume $q\gtrsim N^{1/9}$ and let $0\le\delta<1/3$. We have*

$$\sup_z\max_{1\le i,j\le N}|\mathrm{Im}R_{ij}(z)-\delta_{ij}\mathrm{Im}(m_\star(z))|\prec\frac{1}{N\eta},$$

where the supremum is over all $z=E+\mathrm{i}\eta$ with $|E-\mathcal{L}|\le 2N^{-2/3+\delta}$ and $\eta=N^{-2/3-\delta}$.

Under the assumptions of the above lemma, we get for $i\neq j$,

$$\mathrm{Im}R_{ij}(z)\prec\frac{1}{N\eta},$$

and, since by Equation (2.3), $\mathrm{Im}(m_\star(z)) \lesssim N^{-1/3+\delta/2}$,

$$\mathrm{Im}R(z)_{ii} \le |\mathrm{Im}R(z)_{ii} - \mathrm{Im}(m_\star(z))| + \mathrm{Im}(m_\star(z)) \prec \frac{1}{N\eta}.$$

The second statement of Lemma 5.2 follows. The remainder of this subsection is dedicated to the proof of Lemma 6.4. It relies on an iterative self-improving error bound on resolvent estimates. The proof is an adaptation of [11, Section 3], there are however some new difficulties coming from the randomness of $\mathcal{L}$.

Step 1: net argument. Arguing as in Step 1 of the proof of Lemma 3.4, it is sufficient to prove that for any deterministic real κ with $|\kappa| \le N^{-2/3+\delta}$,

$$\max_{1\le i,j\le N} |\mathrm{Im}R_{ij}(\tilde z) - \delta_{ij}\mathrm{Im}(m_\star(\tilde z))| \prec \frac{1}{N\eta}, \tag{6.4}$$

with $\tilde z$ defined by (5.2). In the remainder of the proof, we fix such κ and corresponding random $\tilde z$.

Step 2: inductive events. For $\eta = N^{-2/3-\delta}$, we set $\mathcal{D}_1 = \{z = E + \mathrm{i}\eta \in \mathbb{C}_+ : |E - \mathcal{L}| \le 2N^{-2/3+\delta}\}$. We introduce the following variables

$$\Lambda_e := \sup_{z\in\mathcal{D}_1} \max_{i,j} |R_{ij}(z) - \delta_{ij} m_\star(z)|, \quad \Lambda := \sup_{z\in\mathcal{D}_1} |m(z) - m_\star(z)|.$$
$$\Lambda_o^{\mathrm{Im}} := \max_{i\ne j} \mathrm{Im}R_{ij}(\tilde z), \quad \Lambda_d^{\mathrm{Im}} := \max_i |\mathrm{Im}R_{ii}(\tilde z) - \mathrm{Im}(m_\star(\tilde z))|.$$

Note that Λ_o^{Im} and Λ_d^{Im} depend implicitly on κ (which is fixed). For $\alpha > 0$ such that $(N\eta)^{-1} \le \alpha \le 1/q$, we introduce the events

$$\Omega := \left\{\Lambda \le N^\epsilon (N\eta)^{-1} \;;\; \Lambda_e \le N^\epsilon q^{-1}\right\} \quad \text{and} \quad \Omega(\alpha) := \Omega \cap \left\{\Lambda_o^{\mathrm{Im}} + \Lambda_d^{\mathrm{Im}} \le N^\epsilon \alpha\right\},$$

where $\epsilon > 0$ is an arbitrarily small constant to be chosen later. We note that by Lemma 6.1 and Corollary 6.3, the event $\Omega(1/q)$ has overwhelming probability. By an inductive argument, we will prove that $\Omega(N^\epsilon/(N\eta))$ has overwhelming probability (if $N\eta < q$, that is $q > N^{1/3-\delta}$, there is nothing more to prove and the proof of the lemma is complete). Then, if $\Omega(N^\epsilon/(N\eta))$ has overwhelming probability for all fixed $0 < \epsilon < 1/9$ then (6.4) holds and the proof of the lemma is complete.

Step 3: resolvent of minors. For ease of notation, in the sequel, we often omit $\tilde z$ and write m, $m_\star$ and R in place of $m(\tilde z)$, $m_\star(\tilde z)$ and $R(\tilde z)$. We have $|m_\star| \asymp 1$, hence on $\Omega(\alpha)$, we find $|R_{ii}| \asymp 1$. Similarly, from Equation (2.3), on $\Omega(\alpha)$,

$$\mathrm{Im}R_{ii} \le \mathrm{Im}(m_\star) + |\mathrm{Im}R_{ii} - \mathrm{Im}(m_\star)| \lesssim N^{-1/3+\delta/2} + N^\epsilon\alpha \lesssim N^\epsilon\alpha,$$

where have used that $(N\eta)^{-1} = N^{-1/3+\delta} \lesssim \alpha$. In summary, on $\Omega(\alpha)$, for all $i \ne j$,

$$|R_{ii}| \asymp 1, \quad |R_{ij}| \le \frac{N^\epsilon}{q} \quad \text{and} \quad \mathrm{Im}R_{ii} + \mathrm{Im}R_{ij} \lesssim N^\epsilon\alpha. \tag{6.5}$$

To be precise, the underlying constants in $\asymp$ and $\lesssim$ in the above expressions depend only on the measure $\rho_\star$ through Equation (2.3). We will use this convention in the rest of the proof.

For $\mathbb{T} \subset \{1, \cdots, N\}$, let $H^{(\mathbb{T})}$ be the $(N - |\mathbb{T}|) \times (N - |\mathbb{T}|)$ minor of H obtained by removing all rows and columns of H indexed by $i \in \mathbb{T}$. In addition, we set $R^{(\mathbb{T})}(z) =$

$(H^{(\mathbb{T})} - zI)^{-1}$. Our first goal is extend the bounds in (6.5) to $R^{(\mathbb{T})} = R^{(\mathbb{T})}(\tilde{z})$ when $\Omega(\alpha)$ holds uniformly over sets $\mathbb{T}$ with $|\mathbb{T}| \leq 2$.

For $i, j \neq k$, we have the following identity

$$R_{ij}^{(k)} = R_{ij} - \frac{R_{ik}R_{kj}}{R_{kk}},$$

(see e.g. [2, Lemma 3.5]). Thus, since $|\mathrm{Im}(ab)| \leq |\mathrm{Im}(a)b| + |\mathrm{Im}(b)a|$, we get

$$\left|\mathrm{Im}\left(\frac{ab}{c}\right)\right| \leq \frac{|\mathrm{Im}(a)bc| + |\mathrm{Im}(b)ac| + |\mathrm{Im}(c)ab| + |\mathrm{Im}(a)\mathrm{Im}(b)\mathrm{Im}(c)|}{|c|^2}$$

and

$$\begin{aligned}&|\mathrm{Im}R_{ij}^{(k)} - \mathrm{Im}R_{ij}|\\ &\leq \frac{\mathrm{Im}(R_{ik})|R_{kj}R_{kk}| + \mathrm{Im}(R_{kj})|R_{ik}R_{kk}| + \mathrm{Im}(R_{kk})|R_{ik}R_{kj}| + \mathrm{Im}(R_{ik})\mathrm{Im}(R_{kj})\mathrm{Im}(R_{kk})}{|R_{kk}|^2}.\end{aligned}$$

On $\Omega(\alpha)$, from (6.5), we obtain

$$\left|\mathrm{Im}R_{ij}^{(k)} - \mathrm{Im}R_{ij}\right| \lesssim \frac{N^{2\epsilon}\alpha}{q} \ll N^{\epsilon}\alpha.$$

Similarly, for all $z \in \mathcal{D}_1$,

$$\left|R_{ij}^{(k)}(z) - R_{ij}(z)\right| \leq \frac{|R_{ik}(z)R_{kj}(z)|}{|R_{kk}(z)|} \lesssim \frac{N^{2\epsilon}}{q^2} \ll \frac{N^{\epsilon}}{q}.$$

We may repeat the above computation for $\mathbb{T} = \{k, l\}$ and $l \neq k$. It follows that, if $\Omega(\alpha)$ holds, for all $\mathbb{T}$ with $|\mathbb{T}| \leq 2$, for all $i, j \notin \mathbb{T}$ with $i \neq j$,

$$\sup_{z\in\mathcal{D}_1} |R_{ii}^{(\mathbb{T})}(z)| \asymp 1, \quad \sup_{z\in\mathcal{D}_1} |R_{ij}^{(\mathbb{T})}(z)| \lesssim \frac{N^{\epsilon}}{q} \quad \text{and} \quad \mathrm{Im}R_{ii}^{(\mathbb{T})} + \mathrm{Im}R_{ij}^{(\mathbb{T})} \lesssim N^{\epsilon}\alpha. \tag{6.6}$$

Step 4: concentration inequality. Next, if $\mathbb{T} \subset \{1, \ldots, N\}$, we use the notation

$$\sum_{i}^{(\mathbb{T})} = \sum_{i:i\notin(\mathbb{T})}.$$

Using classical resolvent identities, the following variables are used in the next step to control $R_{ii}(z)$ and $R_{ij}(z)$:

$$Z_{ij}(z) := \sum_{k,l}^{(ij)} h_{ik}R_{kl}^{(ij)}(z)h_{lj},$$

$$Z_i(z) := \sum_{k,l}^{(i)} \left(h_{ik}h_{li} - \frac{1}{N}\delta_{kl}\right) R_{kl}^{(i)}(z).$$

For a fixed $z \in \mathbb{C}_+$, we note that $R^{(i)}(z)$ is independent of the vector $(h_{ik})_k$ and similarly for $R^{(ij)}(z)$ with $(h_{ik}, h_{jl})_{k,l}$. We are however interested in $R^{(i)} = R^{(i)}(\tilde{z})$ and $R^{(ij)} = R^{(ij)}(\tilde{z})$ with $\tilde{z}$ defined by (5.2), this breaks the above independence property. To circumvent this difficulty, we define $\tilde{z}_i = \kappa + L + \mathcal{X}_i + \mathrm{i}\eta$ and $\tilde{z}_{ij} = \kappa + L + \mathcal{X}_{ij} + \mathrm{i}\eta$ with

$$\mathcal{X}_i = \frac{1}{N}\sum_{k,l}^{(i)} \left(h_{kl}^2 - \frac{1}{N}\right) \quad \text{and} \quad \mathcal{X}_{ij} = \frac{1}{N}\sum_{k,l}^{(ij)} \left(h_{kl}^2 - \frac{1}{N}\right).$$

The independence of $R^{(i)}(\tilde{z}_i)$ and $(h_{il})_{l\neq i}$ is now restored, and similarly for $R^{(ij)}(\tilde{z}_{ij})$. The next lemma relies on the concentration of the variables Z_{ij} and Z_i.

Lemma 6.5. *Assume* $q \gg 1$ *and* $1/(N\eta) \leq \alpha \leq 1/q$. *We have on* $\Omega(\alpha)$,

$$|Z_i| \prec N^\epsilon \left(\frac{1}{q} + \sqrt{\frac{\alpha}{N\eta}}\right), \quad |Z_{ij}| \prec N^\epsilon \left(\frac{1}{q^2} + \sqrt{\frac{\alpha}{N\eta}}\right)$$
$$|\mathrm{Im}(Z_i)| \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}}\right), \quad |\mathrm{Im}(Z_{ij})| \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}}\right).$$

Proof. We start by controlling $\tilde{R}^{(i)} = R^{(i)}(\tilde{z}_i)$ and $\tilde{R}^{(ij)} = R^{(ij)}(\tilde{z}_{ij})$. From the resolvent identity, we have

$$R^{(i)} - \tilde{R}^{(i)} = -(\tilde{z} - \tilde{z}_i) R^{(i)} \tilde{R}^{(i)}.$$

Moreover,

$$|\tilde{z} - \tilde{z}_i| \prec \frac{1}{Nq},$$

and on $\Omega(\alpha)$, from (6.6), for any k, l

$$\left|\left(R^{(i)} \tilde{R}^{(i)}\right)_{kl}\right| \leq \sum_a^{(i)} |R^{(i)}_{ka} \tilde{R}^{(i)}_{al}| \leq \sum_a^{(i)} \left((R^{(i)}_{ka})^2 + (\tilde{R}^{(i)}_{al})^2\right) | \lesssim \frac{N^\epsilon \alpha}{\eta}.$$

So finally,

$$\left|R^{(i)}_{kl} - \tilde{R}^{(i)}_{kl}\right| \prec \frac{N^\epsilon \alpha}{qN\eta} \tag{6.7}$$

The same bound holds for $\left|R^{(ij)}_{kl} - \tilde{R}^{(ij)}_{kl}\right|$.

Now, using the independence of $\tilde{R}^{(i)}$ and $(h_{il})_l$, the large deviation estimate [11, Lemma 3.8 (ii)] and (6.6)-(6.7), we obtain on $\Omega(\alpha)$,

$$\begin{aligned} |Z_i| &\prec \left|\sum_k^{(i)} \left(|h_{ik}|^2 - \frac{1}{N}\right) \tilde{R}^{(i)}_{kk}\right| + \left|\sum_{k \neq l}^{(i)} h_{ik} \tilde{R}^{(i)}_{kl} h_{li}\right| + \frac{N^\epsilon \alpha}{qN\eta} \\ &\prec \frac{\max_k |\tilde{R}^{(i)}_{kk}|}{q} + \frac{\max_{k \neq l} |\tilde{R}^{(i)}_{kl}|}{q} + \left(\frac{1}{N^2} \sum_{k,l}^{(ij)} |\tilde{R}^{(i)}_{kl}|^2\right)^{1/2} + \frac{N^\epsilon \alpha}{qN\eta} \\ &\prec N^\epsilon \left(\frac{1}{q} + \frac{1}{q^2} + \sqrt{\frac{\alpha}{N\eta}}\right), \end{aligned}$$

where we have used Ward identity (5.3). The first claim follows.

Similarly, since for $i \neq j$, the random variables $\{h_{ik}\}_{k:k \neq j}$ are independent of $\{h_{lj}\}_{l:l \neq i}$, from [11, Lemma 3.8 (iii)] and (6.6)-(6.7), on $\Omega(\alpha)$, we have

$$\begin{aligned} \left|\sum_{k,l}^{(ij)} Z_{ij}\right| &\prec \frac{\max_k |\tilde{R}^{(ij)}_{kk}|}{q^2} + \frac{\max_{k \neq l} |\tilde{R}^{(ij)}_{kl}|}{q} + \left(\frac{1}{N^2} \sum_{k,l}^{(ij)} |\tilde{R}^{(ij)}_{kl}|^2\right)^{1/2} + \frac{N^\epsilon \alpha}{qN\eta} \\ &\prec N^\epsilon \left(\frac{1}{q^2} + \sqrt{\frac{\alpha}{N\eta}}\right). \end{aligned}$$

The same argument gives, on $\Omega(\alpha)$,

$$|\mathrm{Im}(Z_{ij})| = \left|\sum_{k,l}^{(ij)} h_{ik} \mathrm{Im}(\tilde{R}^{(ij)}_{kl}) h_{lj}\right| \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}}\right).$$

Finally, we obtain similarly, on $\Omega(\alpha)$

$$\mathrm{Im} Z_i = \sum_k^{(i)} \left(|h_{ik}|^2 - \frac{1}{N}\right) \mathrm{Im} \tilde{R}^{(i)}_{kk} + \sum_{k \neq l}^{(i)} h_{ik} \mathrm{Im}(\tilde{R}^{(i)}_{kl}) h_{li} \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}}\right).$$

The proof is complete. ☐

Step 5: Iteration of the error bounds. Our next lemma improves the bound for the off-diagonal entries of $\mathrm{Im}(R)$ when $\Omega(\alpha)$ holds.

Lemma 6.6. *Assume $q \gg N^\epsilon$ and $1/(N\eta) \le \alpha \le 1/q$. We have on $\Omega(\alpha)$,*

$$\Lambda_o^{\mathrm{Im}} \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}} \right).$$

Proof. Let $i \neq j$. Using

$$R_{ij} = -R_{ii} R_{jj}^{(i)} (h_{ij} - Z_{ij}), \tag{6.8}$$

(see e.g. [2, Lemma 3.5]), it follows from (6.6) and Lemma 6.5 that on $\Omega(\alpha)$,

$$\begin{aligned} \mathrm{Im} R_{ij} &\le \mathrm{Im} R_{ii} |R_{jj}^{(i)}| |h_{ij} - Z_{ij}| + |R_{ii}| \mathrm{Im} R_{jj}^{(i)} |h_{ij} - Z_{ij}| + |R_{ii}| |R_{jj}^{(i)}| \mathrm{Im} Z_{ij} \\ &\prec N^{2\epsilon} \alpha \left(\frac{1}{q} + \sqrt{\frac{\alpha}{N\eta}} \right) + N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}} \right). \end{aligned}$$

Since $\alpha \le 1/q \ll N^{-\epsilon}$, the second term is dominant. Thus, by taking the maximum over $i \neq j$, the statement of the lemma follows. □

It remains to control the diagonal entries of $\mathrm{Im}(R)$ when $\Omega(\alpha)$ holds.

Lemma 6.7. *Assume $q \gtrsim N^\epsilon$ and $1/(N\eta) \le \alpha \le 1/q$. We have on $\Omega(\alpha)$,*

$$\Lambda_d^{\mathrm{Im}} \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}} + \frac{1}{N\eta} \right).$$

Proof. We will prove that if $\Omega(\alpha)$ holds then

$$\max_i \left| \mathrm{Im}(R_{ii}) - \mathrm{Im}(m) \right| \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}} \right). \tag{6.9}$$

Since, by assumption, on $\Omega(\alpha)$, $\Lambda = |m - m_\star| \le N^\epsilon/(N\eta)$, it will conclude the proof.

From [11, Lemma 3.10], the following identity holds: for all $z \in \mathbb{C}_+$,

$$-\frac{1}{R_{ii}(z)} = z + m(z) - \Upsilon_i(z),$$

where

$$\Upsilon_i(z) = h_{ii} - Z_i(z) + \mathcal{A}_i(z) \quad \text{and} \quad \mathcal{A}_i(z) = \frac{1}{N} \sum_j \frac{R_{ij}(z) R_{ji}(z)}{R_{ii}(z)}.$$

In particular, $R_{ii} - R_{jj} = R_{ii} R_{jj} (\Upsilon_j - \Upsilon_i)$ and consequently, from (6.6), on $\Omega(\alpha)$,

$$\begin{aligned} |\mathrm{Im} R_{ii} - \mathrm{Im} R_{jj}| &\lesssim |\mathrm{Im} R_{ii}| |R_{jj}| |\Upsilon_i - \Upsilon_j| + |R_{ii}| |\mathrm{Im} R_{jj}| |\Upsilon_i - \Upsilon_j| + |R_{ii}| |R_{jj}| |\mathrm{Im}(\Upsilon_i - \Upsilon_j)| \\ &\lesssim N^\epsilon \alpha |\Upsilon_i - \Upsilon_j| + |\mathrm{Im}(\Upsilon_i - \Upsilon_j)|. \end{aligned}$$

From Ward identity (5.3),

$$|\mathcal{A}_i| \le \frac{1}{N} \sum_j \frac{|R_{ij}(\tilde{z}_i)|^2}{|R_{ii}(\tilde{z}_i)|} \le \frac{C N^\epsilon \alpha}{N\eta}.$$

Therefore, it follows from Lemma 6.5 that on $\Omega(\alpha)$,

$$|\Upsilon_i - \Upsilon_j| \prec N^\epsilon \left(\frac{1}{q} + \sqrt{\frac{\alpha}{N\eta}} \right).$$

Similarly, we obtain from Lemma 6.5 that on $\Omega(\alpha)$,

$$|\mathrm{Im}(\Upsilon_i - \Upsilon_j)| \prec N^{\epsilon}\left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}}\right).$$

Therefore, it follows that

$$|\mathrm{Im}R_{ii} - \mathrm{Im}R_{jj}| \prec N^{\epsilon}\left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}}\right).$$

Since

$$\mathrm{Im}R_{ii} - \mathrm{Im}(m) = \frac{1}{N}\sum_j (\mathrm{Im}R_{ii} - \mathrm{Im}R_{jj}),$$

Equation (6.9) is established. □

We are now ready to complete the proof of (6.4) by proving that $\Omega(N^{\epsilon}/(N\eta))$ has overwhelming probability. Let $\alpha_1 = 1/q$. As already pointed, $\Omega(\alpha_1)$ has overwhelming probability. If $\alpha_1 \ll N^{\epsilon}/(N\eta)$, we are done. Otherwise, by Lemma 6.6 and Lemma 6.7, we have

$$\Lambda_o^{\mathrm{Im}} + \Lambda_d^{\mathrm{Im}} \prec N^{\epsilon}\left(\frac{\alpha_1}{q} + \sqrt{\frac{\alpha_1}{N\eta}} + \frac{1}{N\eta}\right).$$

In particular, setting

$$\alpha_2 = \frac{\alpha_1}{q} + \sqrt{\frac{\alpha_1}{N\eta}} + \frac{1}{N\eta},$$

we have that $\Omega(\alpha_2)$ has overwhelming probability. If $\alpha_2 \ll N^{\epsilon}/(N\eta)$, we are done. Otherwise we continue. This process reach below $N^{\epsilon}/(N\eta)$ after a finite number of iterations because $q \gg 1$ and we have $\alpha_{k+1} \lesssim \alpha_k \max\left(1/q, \sqrt{1/(\alpha_k N\eta)}\right)$ as long as $\alpha_k \gg 1/(N\eta)$. □

7 Erdős-Rényi graphs

Let A be the normalized adjacency matrix of Erdős-Rényi graph with edge density $p = q^2/N$. Each entry of the matrix $A = (a_{ij})_{1\le i,j\le N}$ is distributed as follows. Every diagonal entry a_{ii} is zero. If $i < j$,

$$a_{ij} = \begin{cases} \zeta/q & \text{with probability } q^2/N, \\ 0 & \text{with probability } 1 - q^2/N, \end{cases}$$

where

$$\zeta := (1 - q^2/N)^{-1/2}.$$

The resampling procedure is defined as in Definition 1.2 with the random sets S_k and an independent copy $A' = (a'_{ij})$ of A. For each integer $0 \le k \le N(N+1)/2$, we then obtain a matrix $A^{[k]}$ whose entries in S_k above the diagonal are equal to the entries of A' and whose entries in S_k^c are equal to the corresponding entries in A.

Let $\nu_1 \ge \cdots \ge \nu_N$ be the ordered eigenvalues of A. We denote an orthonormal basis of eigenvectors of A by $\{\mathbf{w}_1, \cdots, \mathbf{w}_N\}$, i.e., $A\mathbf{w}_i = \nu_i \mathbf{w}_i$ and $\|\mathbf{w}_i\| = 1$ for each i. Again from [24], with probability tending to one as N goes to infinity, $\nu_1 > \cdots > \nu_N$ and the eigenvectors are uniquely determined up to a sign. Similarly, we use the notation

$\nu_1^{[k]} \geq \cdots \geq \nu_N^{[k]}$ and $\mathbf{w}_1^{[k]}, \cdots, \mathbf{w}_N^{[k]}$ to denote the ordered eigenvalues and the associated unit eigenvectors of $A^{[k]}$.

In this section, we explain the proof of Theorem 1.5 and Theorem 1.6. Let us fix $\ell = 2$. The case $\ell = N$ can be handled in the almost similar argument. We define $N \times N$ matrix $\mathring{A} = (\mathring{a}_{ij})$ by extracting the mean from the adjacency matrix A,

$$\mathring{a}_{ij} := a_{ij} - \mathbb{E}a_{ij}.$$

We find that

$$A = \mathring{A} + f\mathbf{e}\mathbf{e}^* - aI,$$

where $f := \zeta q$, $\mathbf{e} := N^{-1/2}(1, 1, \cdots, 1)^T \in \mathbb{R}^N$ and $a := f/N$. The random correction term $\mathcal{X}$ is again defined by setting

$$\mathcal{X} = \frac{1}{N} \sum_{1 \leq i,j \leq N} \left(\mathring{a}_{ij}^2 - \frac{1}{N} \right). \tag{7.1}$$

We note that $\mathring{A}$ satisfies most properties of the sparse random matrix H such as Lemma 2.4, Lemma 2.6 and Lemma 2.7, see [10, 11, 18].

7.1 Local laws and universality of Erdős-Rényi graphs

The delocalization of eigenvectors is valid for A.

Lemma 7.1 (Theorem 2.16 of [11])**.** *Assume $q \gg 1$. We have*

$$\max_{1 \leq i \leq N} \|\mathbf{w}_i\|_\infty \prec \frac{1}{\sqrt{N}}.$$

A non-asymptotic bound on the eigenvalue spacings of A is given as follows.

Lemma 7.2 (Theorem 2.6 of [23])**.** *Assume $q \gg 1$. There exists a constant $c > 0$ such that the following holds for any $\delta \geq N^{-c}$,*

$$\sup_{1 \leq i \leq N-1} \mathbb{P}\left(\nu_i - \nu_{i+1} \leq \frac{\delta}{N} \right) = O(\delta \log N).$$

Let $\mathring{\nu}_1 \geq \cdots \geq \mathring{\nu}_N$ be the ordered eigenvalues of $\mathring{A}$. The following lemma explains the eigenvalue sticking between ν_{i+1} and $\mathring{\nu}_i$.

Lemma 7.3 (Eigenvalue sticking [10, Lemma 6.2])**.** *Assume $q \gg 1$. There exits $\delta > 0$ such that we have for all $1 \leq i \leq \delta N$*

$$|\nu_{i+1} - (\mathring{\nu}_i - a)| \prec \frac{1}{N}.$$

Similarly, if $N(1-\delta) \leq i \leq N$, it follows that

$$|\nu_i - (\mathring{\nu}_i - a)| \prec \frac{1}{N}.$$

Using the eigenvalue sticking lemma, we can show there is a gap of the order $N^{-2/3}$ between the extremal eigenvalues.

Lemma 7.4 (Tracy-Widom scaling for the gap)**.** *Assume $q \gtrsim N^{1/9}$. For any $\epsilon > 0$, there exists a constant $c > 0$ such that*

$$\mathbb{P}(\nu_2 - \nu_3 \geq cN^{-2/3}) \geq 1 - \epsilon.$$

Proof. According to [18, Theorem 1.6], we have for some constant

$$\mathbb{P}(\mathring{\nu}_1 - \mathring{\nu}_2 \geq (c/3)N^{-2/3}) \geq 1 - \epsilon/3.$$

(Setting diagonal entries to zeros does not harm the main argument of [18].) Thanks to Lemma 7.3, it follows that

$$\nu_2 - \nu_3 = \mathring{\nu}_1 - \mathring{\nu}_2 + O_{\prec}(N^{-1}),$$

and the lemma directly follows. □

We denote by $m_A(z)$ and $m_{\mathring{A}}(z)$ the Cauchy-Stieltjes transforms of the empirical measures of eigenvalues of A and $\mathring{A}$ respectively:

$$m_A(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\nu_i - z}, \quad m_{\mathring{A}}(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\mathring{\nu}_i - z}.$$

The local law estimates for Erdős-Rényi graphs holds.

Lemma 7.5. *Assume $q \gg 1$. Let $m_\star$ be as in Lemma 2.7. Uniformly on $w = \kappa + \mathrm{i}\eta \in \mathcal{D}_0$, we have, with $z = \mathcal{L} + w$,*

$$|m_A(z-a) - m_\star(z)| \prec \frac{1}{N\eta} + \frac{1}{q^3} + (\kappa+\eta)^{1/4}\left(\frac{1}{N\eta} + \frac{1}{q^3}\right)^{1/2}.$$

Proof. Since $A + aI - \mathring{A} = f\mathbf{e}^*$ has rank one, we have

$$|m_A(z-a) - m_{\mathring{A}}(z)| = |m_{A+aI}(z) - m_{\mathring{A}}(z)| \leq \frac{\pi}{N\eta},$$

the last inequality being a standard consequence of the interlacing inequality, see e.g. [11, Lemma 7.1]. The conclusion of the lemma then follows from Lemma 6.1 (local law) applied to $\mathring{A}$. □

Next, by the standard argument using Helffer-Sjöstrand formula, the following statements immediately follow as a consequence of Lemma 7.5.

Lemma 7.6 (Eigenvalue rigidity)**.** *Assume $q \gg 1$. For all $2 \leq i \leq N$, we have*

$$|\nu_i - (\gamma_i - a)| \prec N^{-1/3}q^{-3} + N^{-2/3}.$$

Corollary 7.7. *Let $\epsilon > 0$ and assume $q \gtrsim N^{1/9}$. There exists $c > 0$ such the following holds for any $\delta > 0$, for all N large enough, with probability at least $1 - \delta \log N$:*

$$\nu_2 - \nu_i \geq \begin{cases} c\delta N^{-1} & \text{if } 3 \leq i \leq N^\epsilon \\ ci^{2/3}N^{-2/3} & \text{if } N^\epsilon < i \leq N. \end{cases}$$

Moreover, on the event $\{\nu_1 - \nu_2 \geq c\delta N^{-1}\}$, the above inequalities holds with overwhelming probability.

7.2 Proof of Theorem 1.5

The outline of the proof is essentially same with that for the case of (centered) sparse random matrices. The adjacency matrix of Erdős-Rényi graph, A, can be regarded as a rank-one perturbation of a sparse random matrix so an adaptation is required. Since we now consider the second top eigenvector, not the top eigenvector, the argument of (3.3) and Lemma 3.1 should be modified. See Lemma 7.8 and Lemma 7.9 for detail.

Recall that $\nu_1 \geq \cdots \geq \nu_N$ are the ordered eigenvalues of A and $\mathbf{w}_1, \cdots, \mathbf{w}_N$ are the associated unit eigenvectors of A. For any $1 \leq i, j \leq N$, denote by $A_{(ij)}$ the symmetric matrix obtained from A by replacing the entry a_{ij} and a_{ji} with a''_{ij} and a''_{ji} respectively. We define $A^{[k]}_{(ij)} = (a^{[k]}_{(ij)})$ by

$$a^{[k]}_{(ij)} = \begin{cases} a''_{ij} & (i,j) \in S_k, \\ a'''_{ij} & (i,j) \notin S_k, \end{cases}$$

where a'''_{ij} is another independent copy of a_{ij}. Note that $\mathring{A}' = (\mathring{a}'_{ij})$, $\mathring{A}'' = (\mathring{a}''_{ij})$ and $\mathring{A}''' = (\mathring{a}'''_{ij})$ are also independent copies. We denote the ordered eigenvalues of $A^{[k]}$ and their associated eigenvectors by $\nu_1 \geq \cdots \geq \nu_N$ and $\mathbf{w}_1, \cdots, \mathbf{w}_N$. Denote by (st) a random pair of indices chosen uniformly from $\{(i,j) : 1 \leq i \leq j \leq N\}$. Note that

$$|\{(i,j) : 1 \leq i \leq j \leq N\}| = N(N+1)/2$$

Let $\mu_1 \geq \cdots \geq \mu_N$ be the ordered eigenvalues of $A_{(st)}$ and, let $\mathbf{u}_1, \cdots, \mathbf{u}_N$ be the associated unit eigenvectors of $A_{(st)}$. Similarly, we define $\mu^{[k]}_1 \geq \cdots \geq \mu^{[k]}_N$ and $\mathbf{u}^{[k]}_1, \cdots, \mathbf{u}^{[k]}_N$ for $A^{[k]}_{(st)}$. We apply Lemma 2.1 with $Y = A$ and $f(A) = \nu_2 - L - \mathcal{X}$:

$$\mathbb{E}\left[\left(\nu_2 - \mu_2 - Q_{st}\right)\left(\nu^{[k]}_2 - \mu^{[k]}_2 - Q^{[k]}_{st}\right)\right] \leq \frac{2\mathrm{Var}(\nu_2 - L - \mathcal{X})}{k} \cdot \frac{N(N+1)+2}{N(N+1)}, \tag{7.2}$$

where

$$Q_{st} := \frac{2}{N}(\mathring{a}^2_{st} - (\mathring{a}''_{st})^2) \quad \text{and}$$
$$Q^{[k]}_{st} := \begin{cases} \frac{2}{N}((\mathring{a}'_{st})^2 - (\mathring{a}''_{st})^2) & \text{if } (st) \in S_k, \\ \frac{2}{N}(\mathring{a}^2_{st} - (\mathring{a}'''_{st})^2) & \text{if } (st) \notin S_k. \end{cases}$$

Lemma 7.8. *Let us write $\mathbf{w}_2 = (w_1, \cdots, w_N)$ and $\mathbf{u}_2 = (u_1, \cdots, u_N)$. There exists $\epsilon > 0$ such that the following holds with overwhelming probability:*

$$Z_{st} u_s u_t - \frac{N^\epsilon}{q^3 N^2} \leq \nu_2 - \mu_2 \leq Z_{st} w_s w_t + \frac{N^\epsilon}{q^3 N^2}$$

where

$$Z_{st} := 2(\mathring{a}_{st} - \mathring{a}''_{st}).$$

Similarly, with overwhelming probability, we have

$$Z^{[k]}_{st} u^{[k]}_s u^{[k]}_t - \frac{N^\epsilon}{q^3 N^2} \leq \nu^{[k]}_2 - \mu^{[k]}_2 \leq Z^{[k]}_{st} w^{[k]}_s w^{[k]}_t + \frac{N^\epsilon}{q^3 N^2},$$

where $\mathbf{w}^{[k]}_2 = (w^{[k]}_1, \cdots, w^{[k]}_N)$, $\mathbf{u}^{[k]}_2 = (u^{[k]}_1, \cdots, u^{[k]}_N)$ and

$$Z^{[k]}_{st} := \begin{cases} 2(\mathring{a}' - \mathring{a}'') & \textit{if } (st) \in S_k, \\ 2(\mathring{a} - \mathring{a}''') & \textit{if } (st) \notin S_k. \end{cases}$$

Proof. By spectral theorem, we have

$$\langle \mathbf{u}_2, A\mathbf{u}_2 \rangle = \nu_1 |\langle \mathbf{u}_2, \mathbf{w}_1 \rangle|^2 + \sum_{i=2}^N \nu_i |\langle \mathbf{u}_2, \mathbf{w}_i \rangle|^2 \leq (\nu_1 - \nu_2)|\langle \mathbf{u}_2, \mathbf{w}_1 \rangle|^2 + \langle \mathbf{w}_2, A\mathbf{w}_2 \rangle.$$

We write

$$\mathbf{w}_1 = \alpha \mathbf{u}_2 + \beta \mathbf{x}$$

where $\alpha = \langle \mathbf{u}_2, \mathbf{w}_1 \rangle$, $\mathbf{x} \in \text{span}(\mathbf{u}_1, \mathbf{u}_3, \cdots, \mathbf{u}_N)$ and $\|\mathbf{x}\|^2 = 1 - \alpha^2$. Since

$$A_{(st)}\mathbf{w}_1 = A\mathbf{w}_1 + (A_{(st)} - A)\mathbf{w}_1 = \nu_1 \mathbf{w}_1 + (A_{(st)} - A)\mathbf{w}_1$$

and also

$$A_{(st)}\mathbf{w}_1 = \alpha\mu_2 \mathbf{u}_2 + \beta A_{(st)}\mathbf{x},$$

it follows that

$$\nu_1 \mathbf{w}_1 = \alpha\mu_2 \mathbf{u}_2 + \beta A_{(st)}\mathbf{x} + (A - A_{(st)})\mathbf{w}_1.$$

Then,

$$\nu_1 \alpha = \nu_1 \langle \mathbf{u}_2, \mathbf{w}_1 \rangle = \langle \mathbf{u}_2, \nu_1 \mathbf{w}_1 \rangle = \mu_2 \alpha + \langle \mathbf{u}_2, (A - A_{(st)})\mathbf{w}_1 \rangle.$$

By the eigenvector delocalization,

$$|(\nu_1 - \mu_2)\alpha| = |\langle \mathbf{u}_2, (A - A_{(st)})\mathbf{w}_1 \rangle| \prec \frac{1}{qN}.$$

According to [11, Theorem 6.2], we have $\nu_1 \sim \zeta q + (\zeta q)^{-1}$ with overwhelming probability. Also, by the eigenvalue rigidity, we find $\mu_2 \le C$ with overwhelming probability. Finally, we obtain

$$|\alpha| \prec \frac{1}{q^2 N},$$

which implies

$$\langle \mathbf{u}_2, A\mathbf{u}_2 \rangle \le \frac{N^\epsilon}{q^3 N^2} + \langle \mathbf{w}_2, A\mathbf{w}_2 \rangle. \tag{7.3}$$

Similarly, we have with overwhelming probability

$$\langle \mathbf{w}_2, A_{(st)}\mathbf{w}_2 \rangle \le \frac{N^\epsilon}{q^3 N^2} + \langle \mathbf{u}_2, A_{(st)}\mathbf{u}_2 \rangle.$$

As a result, it follows with overwhelming probability

$$\langle \mathbf{u}_2, (A - A_{(st)})\mathbf{u}_2 \rangle - \frac{N^\epsilon}{q^3 N^2} \le \nu_2 - \mu_2 \le \langle \mathbf{w}_2, (A - A_{(st)})\mathbf{w}_2 \rangle + \frac{N^\epsilon}{q^3 N^2}.$$

Using the same argument, we observe with overwhelming probability

$$\langle \mathbf{u}_2^{[k]}, (A^{[k]} - A_{(st)}^{[k]})\mathbf{u}_2^{[k]} \rangle - \frac{N^\epsilon}{q^3 N^2} \le \nu_2^{[k]} - \mu_2^{[k]} \le \langle \mathbf{w}_2^{[k]}, (A^{[k]} - A_{(st)}^{[k]})\mathbf{w}_2^{[k]} \rangle + \frac{N^\epsilon}{q^3 N^2} \square$$

We set $T_1 = (Z_{st} w_s w_t + N^\epsilon/q^3N^2 - Q_{st})(Z_{st}^{[k]} w_s^{[k]} w_t^{[k]} + N^\epsilon/q^3N^2 - Q_{st}^{[k]})$, $T_2 = (Z_{st} w_s w_t + N^\epsilon/q^3N^2 - Q_{st})(Z_{st}^{[k]} u_s^{[k]} u_t^{[k]} - N^\epsilon/q^3N^2 - Q_{st}^{[k]})$, $T_3 = (Z_{st} u_s u_t - N^\epsilon/q^3N^2 - Q_{st})(Z_{st}^{[k]} w_s^{[k]} w_t^{[k]} + N^\epsilon/q^3N^2 - Q_{st}^{[k]})$, $T_4 = (Z_{st} u_s u_t - N^\epsilon/q^3N^2 - Q_{st})(Z_{st}^{[k]} u_s^{[k]} u_t^{[k]} - N^\epsilon/q^3N^2 - Q_{st}^{[k]})$. We have

$$\min(T_1, T_2, T_3, T_4) \le \big(\nu_2 - \mu_2 - Q_{st}\big)\big(\nu_2^{[k]} - \mu_2^{[k]} - Q_{st}^{[k]}\big) \le \max(T_1, T_2, T_3, T_4). \tag{7.4}$$

Lemma 7.9. *Assume $q \gtrsim N^{1/9}$ and let $c, \delta > 0$ be such that $N^{c+\delta} \ll q$. For $1 \le i \le j \le N$, let $\mathbf{u}_2^{(ij)}$ be a unit eigenvector of $A_{(ij)}$ associated with the second largest eigenvalue of $A_{(ij)}$. Then, on the event $\{\nu_2 - \nu_3 \ge N^{-1-c}\}$, the event*

$$\bigcap_{1\le i\le j\le N} \left\{ \inf_{s\in\{\pm1\}} \|s\mathbf{w}_2 - \mathbf{u}_2^{(ij)}\|_\infty \le N^{-1/2-\delta} \right\}$$

holds with overwhelming probability. The analogous result for $H_{(ij)}^{[k]}$ also holds.

Proof. We shall modify the proof of Lemma 3.1. Let $\mu_1^{(ij)} \ge \cdots \ge \mu_N^{(ij)}$ be the ordered eigenvalues of $A_{(ij)}$ and, let $\mathbf{u}_1^{(ij)}, \cdots, \mathbf{u}_N^{(ij)}$ be the associated unit eigenvectors of $A_{(ij)}$. According to (7.3), we have with overwhelming probability

$$\begin{aligned}
\nu_2 &\ge \langle \mathbf{u}_2^{(ij)}, A\mathbf{u}_2^{(ij)} \rangle - \frac{N^\epsilon}{q^3 N^2} \\
&= \mu_2^{(ij)} + \langle \mathbf{u}_2^{(ij)}, (A - A_{(ij)})\mathbf{u}_2^{(ij)} \rangle - \frac{N^\epsilon}{q^3 N^2} \\
&\ge \mu_2^{(ij)} - 2|\mathring{a}_{ij} - \mathring{a}''_{ij}| \|\mathbf{u}_2^{(ij)}\|_\infty^2 - \frac{N^\epsilon}{q^3 N^2} \\
&\ge \mu_2^{(ij)} - \frac{N^\epsilon}{qN}.
\end{aligned}$$

Reversing the role of A and $A_{(ij)}$, we also have with overwhelming probability

$$\mu_2^{(ij)} \ge \nu_2 - \frac{N^\epsilon}{qN}.$$

Thus, it follows that with overwhelming probability

$$\max_{1\le i\le j\le N} |\nu_2 - \mu_2^{(ij)}| \le \frac{N^\epsilon}{qN}. \tag{7.5}$$

We write

$$\mathbf{u}_2^{(ij)} = \sum_{\ell=1}^N \alpha_\ell \mathbf{w}_\ell,$$

and get

$$\nu_2 \mathbf{u}_2^{(ij)} = \sum_{\ell=1}^N \nu_\ell \alpha_\ell \mathbf{w}_\ell + (A_{(ij)} - A)\mathbf{u}_2^{(ij)} + (\nu_2 - \mu_2^{(ij)})\mathbf{u}_2^{(ij)}.$$

Next, by taking an inner product with $\mathbf{v}_\ell$ for $\ell \ne 2$, we obtain

$$\Big((\nu_2 - \nu_\ell) + (\mu_2^{(ij)} - \nu_2) \Big) \alpha_\ell = \langle \mathbf{w}_\ell, (A_{(ij)} - A)\mathbf{u}_2^{(ij)} \rangle.$$

According to [11, Theorem 6.2] and Corollary 7.7, the following holds with overwhelming probability on the event $\{\nu_2 - \nu_3 \ge N^{-1-c}\}$:

$$|\nu_2 - \nu_\ell| \gtrsim \begin{cases} q & \ell = 1, \\ N^{-1-c} & 3 \le \ell \le N^\epsilon, \\ \ell^{2/3} N^{-2/3} & N^\epsilon < \ell \le N. \end{cases}$$

Due to (7.5), we have with overwhelming probability

$$|\nu_2 - \nu_\ell| \gg |\mu_2^{(ij)} - \nu_2|,$$

for every $\ell \in \{1, \cdots, N\}$. Since the eigenvector delocalization implies

$$\left|\langle \mathbf{w}_\ell, (A_{(ij)} - A)\mathbf{u}_2^{(ij)}\rangle\right| \prec \frac{1}{qN},$$

we can observe

$$|\alpha_\ell| \prec \begin{cases} q^{-2}N^{-1} & \ell = 1, \\ q^{-1}N^c & 3 \le \ell \le N^\epsilon, \\ q^{-1}\ell^{-2/3}N^{-1/3} & N^\epsilon < \ell \le N. \end{cases} \tag{7.6}$$

What remains can be done similarly as we did in Section 4.1. □

Next, let $0 < \delta < 1/9$ and $0 < \epsilon < \delta/3$ to be defined later, we define the events

$$\mathcal{E}_1 := \left\{\max\left(\|\mathbf{w}_2\|_\infty, \|\mathbf{u}_2\|_\infty, \|\mathbf{w}_2^{[k]}\|_\infty, \|\mathbf{u}_2^{[k]}\|_\infty\right) \le N^{\epsilon-1/2}\right\}, \tag{7.7}$$

$$\mathcal{E}_2 := \left\{\max\left(\|\mathbf{w}_2 - \mathbf{u}_2\|_\infty, \|\mathbf{w}_2^{[k]} - \mathbf{u}_2^{[k]}\|_\infty\right) \le N^{-1/2-\delta}\right\}. \tag{7.8}$$

Set the event $\mathcal{E} := \mathcal{E}_1 \cap \mathcal{E}_2$. Let $c > 0$ such that $c + \delta < 1/9$. According to Lemma 2.4, [23, Theorem 2.6] and Lemma 7.9, we have $\mathbb{P}(\mathcal{E}^c) = O(N^{-c}\log N)$ by choosing the $\pm$-phase properly for $\mathbf{u}_{(ij)}$ and $\mathbf{u}_{(ij)}^{[k]}$. On the event $\mathcal{E}$, it follows that from (7.4)

$$\begin{aligned} \big(\nu_2 - \mu_2 - Q_{st}\big)\big(\nu_2^{[k]} - \mu_2^{[k]} - Q_{st}^{[k]}\big) &\ge Z_{st}Z_{st}^{[k]}w_s w_t w_s^{[k]} w_t^{[k]} - O\left(|Z_{st}Z_{st}^{[k]}|N^{3\epsilon-2-\delta}\right) \\ &\quad - |Q_{st}Z_{st}^{[k]}|N^{2\epsilon-1} - |Q_{st}^{[k]}Z_{st}|N^{2\epsilon-1} - |Q_{st}Q_{st}^{[k]}| - o(N^{-3}). \end{aligned} \tag{7.9}$$

The proof is done by the following two lemmas.

Lemma 7.10. *If* $4\epsilon + \delta < 1/9$*, we have*

$$\mathbb{E}\left[Z_{st}Z_{st}^{[k]}w_s w_t w_s^{[k]} w_t^{[k]}\mathbb{1}_{\mathcal{E}^c}\right] = o\left(\frac{1}{N^3}\right), \tag{7.10}$$

and

$$\mathbb{E}\left[(\nu_2 - \mu_2 - Q_{st})(\nu_2^{[k]} - \mu_2^{[k]} - Q_{st}^{[k]})\mathbb{1}_{\mathcal{E}^c}\right] = o\left(\frac{1}{N^3}\right). \tag{7.11}$$

Lemma 7.11. *We have*

$$\mathbb{E}\left[Z_{st}Z_{st}^{[k]}w_s w_t w_s^{[k]} w_t^{[k]}\right] = \frac{2}{N^3}\mathbb{E}\left[\langle \mathbf{w}_2, \mathbf{w}_2^{[k]}\rangle^2\right] + o\left(\frac{1}{N^3}\right).$$

The above two lemma can be shown in the very similar way of Lemma 3.2 and Lemma 3.3 so we omit the detail. Applying (3.1), we establish

$$\mathbb{E}\left[\langle \mathbf{w}_2, \mathbf{w}_2^{[k]}\rangle^2\right] \le \frac{N^3\mathrm{Var}(\nu_2 - L - \mathcal{X})}{k}(1 + o(1)) + o(1).$$

Using (2.4) and Cauchy interlacing, we have for any $\epsilon > 0$,

$$\mathrm{Var}(\nu_2 - L - \mathcal{X}) = O(N^{\epsilon-4/3}),$$

which concludes the proof. □

7.3 Proof of Theorem 1.6

As in the previous subsection, we shall rely on the same strategy described in Section 3 and focus on explaining how to modify some details in regard to rank-one perturbation.

For $z = E + \mathrm{i}\eta$ with $\eta > 0$ and $E \in \mathbb{R}$, we define (with an abuse of notation)

$$R(z) := (A - zI)^{-1},$$

where I denotes the identity matrix. We denote by $R^{[k]}(z)$ the resolvent of $A^{[k]}$. Then, as we showed in Section 3.2, the desired result follows from the following two lemmas.

Lemma 7.12. *Assume $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$. Let $R(z)$ be the resolvent of A. Then, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists $c > 0$ such that, with overwhelming probability,*

$$\sup_{z} \max_{1 \le i,j \le N} N\eta |\mathrm{Im} R^{[k]}_{ij}(z-a) - \mathrm{Im} R_{ij}(z-a)| \le N^{-c},$$

where the supremum is over all $z = E + \mathrm{i}\eta$ with $|E - \mathcal{L}| \le N^{-2/3+\delta}$ and $\eta = N^{-2/3-\delta}$, and the term $\mathcal{L}$ is defined as in Lemma 2.7 with setting $\mathcal{X}$ as in (7.1).

Proof. We notice that

$$\begin{aligned} R^{[k]}_{ij} - R_{ij} &= \sum_{t=1}^{k} \left(R^{[t]}_{ij} - R^{[t-1]}_{ij} \right) \\ &= \sum_{t=1}^{k} (a_{i_t j_t} - a'_{i_t j_t})(R^{[t]} E_{i_t j_t} R^{[t-1]})_{ij} \\ &= \sum_{t=1}^{k} (\mathring{a}_{i_t j_t} - \mathring{a}'_{i_t j_t})(R^{[t]} E_{i_t j_t} R^{[t-1]})_{ij}. \end{aligned}$$

We also find that the resolvent estimates of H, Lemma 5.2, still holds for the resolvent of A.

Lemma 7.13. *Assume $q \gtrsim N^{1/9}$ and let $0 < \delta < 1/3$. Let $R(z)$ be the resolvent of A. We have*

$$\sup_{z} \max_{1 \le i,j \le N} \Big| |R(z-a)_{ij}| - \delta_{ij} \Big| \prec \frac{1}{q} + \frac{1}{N\eta},$$

and

$$\sup_{z} \max_{1 \le i,j \le N} \left| \mathrm{Im} R(z-a)_{ij} \right| \prec \frac{1}{N\eta},$$

where the two suprema are over all $z = E + \mathrm{i}\eta$ with $|E - \mathcal{L}| \le N^{-2/3+\delta}$ and $\eta = N^{-2/3-\delta}$, and the term $\mathcal{L}$ is defined as in Lemma 2.7 with setting $\mathcal{X}$ as in (7.1).

The first statement of the lemma immediately follows from [11, Theorem 2.9]. We shall prove the second statement in Subsection 7.4. Lemma 7.13 is an essential ingredient. What remains would be a straightforward modification of the proof of Lemma 3.4. Note that we used the trivial inequality $h_{ij} \prec q^{-1}$ in the proof of Lemma 3.4 and it still holds that $a_{ij} \prec q^{-1}$. □

Lemma 7.14. *We write $\mathbf{w}_2 = (w_1, \cdots, w_N)$ and $\mathbf{w}_2^{[k]} = (w_1^{[k]}, \cdots, w_N^{[k]})$. Assume $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$. Let $0 < \delta < \delta_0$ be as in Lemma 7.12. There exists $c' > 0$ such that with probability $1 - o(1)$ it holds that*

$$\max_{1 \le i,j \le N} N|\eta \mathrm{Im} R_{ij}(z) - w_i w_j| \le N^{-c'} \quad \textit{and} \quad \max_{1 \le i,j \le N} N|\eta \mathrm{Im} R^{[k]}_{ij}(z) - w^{[k]}_i w^{[k]}_j| \le N^{-c'},$$

with $z = \nu_2 + \mathrm{i}\eta$ and $\eta = N^{-2/3-\delta}$.

Proof of Lemma 7.14. The next lemma is a modification of Lemma 5.5. See the following lemma.

Lemma 7.15. *Assume $q \gtrsim N^{1/9}$ and $k \ll N^{5/3}$. Then, if $0 < \delta < \delta_0$ with δ_0 as in Lemma 7.12, we have*

$$|\nu_2 - \nu_2^{[k]}| \prec N^{-2/3-\delta}.$$

Proof. If $\nu_2 = \nu_2^{[k]}$, we are done. Thus, suppose $\nu_2^{[k]} < \nu_2$. We set $\eta = N^{-2/3-\delta}$. According to Lemma 5.1, we can find $1 \le i \le N$ such that

$$\frac{1}{2\eta^2} \le N\eta^{-1}\mathrm{Im}R(\nu_2 + \mathrm{i}\eta)_{ii}.$$

Since we have $|(\nu_2 + a) - \mathcal{L}| \prec N^{-2/3}$, it follows from Lemma 5.1 that

$$N\eta^{-1}\mathrm{Im}R^{[k]}(\nu_2 + \mathrm{i}\eta)_{ii} \prec \left(\min_{1\le j\le N}\left|\nu_2 - \nu_j^{[k]}\right|\right)^{-2}.$$

With overwhelming probability, $\nu_1^{[k]} \gg \nu_2 > \nu_2^{[k]} \ge \nu_3^{[k]} \ge \cdots \ge \nu_N^{[k]}$ so we have

$$\min_{1\le j\le N}\left|\nu_2 - \nu_j^{[k]}\right| = \left|\nu_2 - \nu_2^{[k]}\right|.$$

Applying Lemma 7.12, we get the desired result by showing

$$N\eta^{-1}\mathrm{Im}R^{[k]}(\nu_1 + \mathrm{i}\eta)_{ii} \gtrsim \frac{1}{\eta^2}.$$

The other case $\nu_2^{[k]} > \nu_2$ can be proven by reversing the role A and $A^{[k]}$. □

We fix $0 < \delta < \delta_0$ and set $\eta = N^{-2/3-\delta}$. We write $\mathbf{w}_m = (\mathbf{w}_m(1), \dots, \mathbf{w}_m(N))$ and $\mathbf{w}_m^{[k]} = (\mathbf{w}_m^{[k]}(1), \dots, \mathbf{w}_m^{[k]}(N))$ for $m \neq 2$. By the spectral theorem, we have

$$N\eta\mathrm{Im}R(z)_{ij} = \frac{N\eta^2 w_i w_j}{(\nu_2 - E)^2 + \eta^2} + \sum_{m\neq 2}^{N} \frac{N\eta^2 \mathbf{w}_m(i)\mathbf{w}_m(j)}{(\nu_m - E)^2 + \eta^2}.$$

Let $\epsilon > 0$ and let $N' := \lfloor N^{2\epsilon} \rfloor$. We see that with overwhelming probability: for all E satisfying $|E - (\mathcal{L} - a)| \le N^{-2/3+\epsilon}$, we have, for some $C > 0$,

$$\left|\sum_{m=N'+1}^{N} \frac{N\mathbf{w}_m(i)\mathbf{w}_m(j)}{(\nu_m - E)^2 + \eta^2}\right| \le CN^{\epsilon}(N')^{-1/3}N^{4/3}. \tag{7.12}$$

We can find $c_0 > 0$ such that

$$\mathbb{P}(\mathcal{E}) \ge 1 - \epsilon/2,$$

where $\mathcal{E}$ is the event that (7.12) holds, $\nu_1 - \nu_2 \ge c_0 q$, $\nu_2 - \nu_3 > c_0 N^{-2/3}$ and $\max_m \|\mathbf{w}_m\|_\infty^2 \le N^{\epsilon-1}$. On the event $\mathcal{E}$, we find for all E with $|\nu_2 - E| \le (c_0/2)N^{-2/3}$ that for some $C > 0$,

$$\left|\sum_{m=3}^{N'} \frac{N\mathbf{w}_m(i)\mathbf{w}_m(j)}{(\nu_m - E)^2 + \eta^2}\right| \le CN^{\epsilon}N'N^{4/3},$$

and

$$\left|\frac{N\mathbf{w}_1(i)\mathbf{w}_1(j)}{(\nu_1 - E)^2 + \eta^2}\right| \le N^{\epsilon}q^{-2}.$$

We fix $\delta' > 0$ such that $\delta + \delta' < \delta_0$. On the event $\mathcal{E}$, for any E such that $|\nu_2 - E| \le \eta N^{-\delta'}$, we have

$$\left| \frac{N\eta^2 w_i w_j}{(\nu_2 - E)^2 + \eta^2} - N w_i w_j \right| \le N^{\epsilon - 2\delta'}.$$

The proof is done by following the argument of the proof of Lemma 3.5 and applying Lemma 7.15. □

7.4 Resolvent of Erdős-Rényi graph: Proof of Lemma 7.13

The second statement of Lemma 7.13 will be shown in this subsection. It is enough to prove the following lemma.

Lemma 7.16. *Assume $q \gtrsim N^{1/9}$ and let $0 \le \delta < 1/3$. $R(z)$ be the resolvent of A. We have*

$$\sup_z \max_{1 \le i,j \le N} |\mathrm{Im} R_{ij}(z - a) - \delta_{ij} \mathrm{Im}(m_\star(z - a))| \prec \frac{1}{N\eta},$$

where the supremum is over all $z = E + \mathrm{i}\eta$ with $|E - \mathcal{L}| \le 2N^{-2/3+\delta}$ and $\eta = N^{-2/3-\delta}$, and the term $\mathcal{L}$ is defined as in Lemma 2.7 with setting $\mathcal{X}$ as in (7.1)*.*

Proof. We can prove this lemma by using the same argument in Subsection 6.2 with some additional ingredients, Lemma 7.17 and Lemma 7.18. We already know it is sufficient to prove that for any deterministic real κ with $|\kappa| \le N^{-2/3+\delta}$,

$$\max_{1 \le i,j \le N} |\mathrm{Im} R_{ij}(\tilde{z} - a) - \delta_{ij} \mathrm{Im}(m_\star(\tilde{z} - a))| \prec \frac{1}{N\eta}, \tag{7.13}$$

with $\tilde{z}$ defined by $\tilde{z} = \kappa + L + \mathcal{X} + \mathrm{i}\eta$.

For $\eta = N^{-2/3-\delta}$, we set $\mathcal{D}_1 = \{z = E + \mathrm{i}\eta \in \mathbb{C}_+ : |E - \mathcal{L}| \le 2N^{-2/3+\delta}\}$. We introduce the following variables

$$\Lambda_e := \sup_{z \in \mathcal{D}_1} \max_{i,j} |R_{ij}(z - a) - \delta_{ij} m_\star(z)|, \quad \Lambda_A := \sup_{z \in \mathcal{D}_1} |m_A(z - a) - m_\star(z)|.$$
$$\Lambda_o^{\mathrm{Im}} := \max_{i \ne j} \mathrm{Im} R_{ij}(\tilde{z} - a), \quad \Lambda_d^{\mathrm{Im}} := \max_i |\mathrm{Im} R_{ii}(\tilde{z} - a) - \mathrm{Im}(m_\star(\tilde{z}))|.$$

For $\alpha > 0$ such that $(N\eta)^{-1} \le \alpha \le 1/q$, we introduce the events

$$\Omega := \left\{ \Lambda_A \le N^\epsilon (N\eta)^{-1} \, ; \, \Lambda_e \le N^\epsilon q^{-1} \right\} \quad \text{and} \quad \Omega(\alpha) := \Omega \cap \left\{ \Lambda_o^{\mathrm{Im}} + \Lambda_d^{\mathrm{Im}} \le N^\epsilon \alpha \right\},$$

where $\epsilon > 0$ is an arbitrarily small constant to be chosen later. By Lemma 7.5, [11, Theorem 2.9] and Lemma 6.2, the event $\Omega(1/q)$ holds with overwhelming probability. If $\Omega(N^\epsilon/(N\eta))$ has overwhelming probability for all fixed $0 < \epsilon < 1/9$ then (7.13) holds and the proof of the lemma is done.

Note that, if $\Omega(\alpha)$ holds, for all $\mathbb{T}$ with $|\mathbb{T}| \le 2$, for all $i, j \notin \mathbb{T}$ with $i \ne j$,

$$\sup_{z \in \mathcal{D}_1} |R_{ii}^{(\mathbb{T})}(z)| \asymp 1, \quad \sup_{z \in \mathcal{D}_1} |R_{ij}^{(\mathbb{T})}(z)| \lesssim \frac{N^\epsilon}{q} \quad \text{and} \quad \mathrm{Im} R_{ii}^{(\mathbb{T})} + \mathrm{Im} R_{ij}^{(\mathbb{T})} \lesssim N^\epsilon \alpha. \tag{7.14}$$

For ease of notation, in the sequel, we often omit $\tilde{z}$ and write m, $m_\star$ and R in place of $m(\tilde{z} - a)$, $m_\star(\tilde{z})$ and $R(\tilde{z} - a)$. We define $Z_{ij}(\tilde{z} - a)$ by setting

$$Z_{ij} := \sum_{k,l}^{(ij)} a_{ik} R_{kl}^{(ij)} a_{lj}.$$

We set

$$\begin{aligned} Z_i &:= Z_{ii} - \frac{1}{N}\sum_k^{(i)} R_{kk}^{(i)} \\ &= \sum_k^{(i)} \left(|a_{ik}|^2 - \frac{1}{N}\right) R_{kk}^{(i)} + \sum_{k\neq l}^{(i)} a_{ik} R_{kl}^{(i)} a_{li}. \end{aligned}$$

In addition, let us define $\tilde{z}_i = \kappa + L + \mathcal{X}_i + \mathrm{i}\eta$ and $\tilde{z}_{ij} = \kappa + L + \mathcal{X}_{ij} + \mathrm{i}\eta$ with

$$\mathcal{X}_i = \frac{1}{N}\sum_{k,l}^{(i)} \left(\mathring{a}_{ij}^2 - \frac{1}{N}\right) \quad \text{and} \quad \mathcal{X}_{ij} = \frac{1}{N}\sum_{k,l}^{(ij)} \left(\mathring{a}_{ij}^2 - \frac{1}{N}\right).$$

Let us set $\tilde{R}^{(i)} := R^{(i)}(\tilde{z}_i - a)$ and $\tilde{R}^{(ij)} := R^{(ij)}(\tilde{z}_{ij} - a)$. The following lemmas are new inputs to show the desired results.

Lemma 7.17. *Assume $q \gg 1$ and $1/(N\eta) \le \alpha \le 1/q$. We have on $\Omega(\alpha)$,*

$$\sum_{k,l}^{(i)} \frac{f}{N} \mathrm{Im}\left(\tilde{R}_{kl}^{(i)}\right) \mathring{a}_{li} \prec \frac{f^2}{N}\frac{1}{N\eta},$$

and

$$\sum_{k,l}^{(i)} \frac{f^2}{N^2} \mathrm{Im}\left(\tilde{R}_{kl}^{(i)}\right) \prec \frac{f^2}{N}\frac{1}{N\eta}.$$

Proof. Using the spectral decomposition of $R^{(i)}$, we have

$$\sum_{k,l}^{(i)} \frac{f}{N}\mathrm{Im}\left[\tilde{R}_{kl}^{(i)}\right] \mathring{a}_{li} = \frac{f\sqrt{N-1}}{N}\left(\sum_\alpha \frac{\eta\langle \mathbf{e}_{N-1}, \mathbf{w}_\alpha^{(i)}\rangle}{(\nu_\alpha^{(i)} - \kappa - L - \mathcal{X}_i - a)^2 + \eta^2} \sum_l^{(i)} \mathbf{w}_\alpha^{(i)}(l)\mathring{a}_{li}\right).$$

Thus, it is enough to estimate

$$\frac{f}{\sqrt{N}}\sum_\alpha \frac{\eta\langle \mathbf{e}_{N-1}, \mathbf{w}_\alpha^{(i)}\rangle}{(\nu_\alpha^{(i)} - \kappa - L - \mathcal{X}_i - a)^2 + \eta^2} \sum_l^{(i)} \mathbf{w}_\alpha^{(i)}(l)\mathring{a}_{li}. \tag{7.15}$$

We have $|\langle \mathbf{e}_{N-1}, \mathbf{w}_\alpha^{(i)}\rangle| \le 1$, $|\mathbf{w}_\alpha^{(i)}(l)| \prec N^{-1/2}$. Moreover, from the large deviation estimate [11, Lemma 3.8 (ii)], we obtain

$$\sum_l |\mathring{a}_l| = N\mathbb{E}|\mathring{a}| + \sum_l (|\mathring{a}_l| - \mathbb{E}|\mathring{a}_l|) \prec q \asymp f.$$

Note also that

$$\sum_\alpha \frac{\eta}{(\nu_\alpha^{(i)} - \kappa - L - \mathcal{X}_i - a)^2 + \eta^2} = \mathrm{Im}(m_{A^{(i)}}(\tilde{z}_i - a)).$$

It follows from the local law that

$$\frac{f}{\sqrt{N}}\sum_\alpha \frac{\eta\langle \mathbf{e}_{N-1}, \mathbf{w}_\alpha^{(i)}\rangle}{(\nu_\alpha^{(i)} - \kappa - L - \mathcal{X}_i - a)^2 + \eta^2} \sum_l^{(i)} \mathbf{w}_\alpha^{(i)}(l)\mathring{a}_{li} \prec \frac{f^2}{N}\frac{1}{N\eta}. \tag{7.16}$$

Similarly, the second statement follows from

$$\frac{f}{\sqrt{N}}\sum_\alpha \frac{\eta\langle \mathbf{e}_{N-1}, \mathbf{w}_\alpha^{(i)}\rangle}{(\nu_\alpha^{(i)} - \kappa - L - \mathcal{X}_i - a)^2 + \eta^2} \sum_l^{(i)} \mathbf{w}_\alpha^{(i)}(l)\frac{f}{N} \prec \frac{f^2}{N}\frac{1}{N\eta}, \tag{7.17}$$

as claimed. □

Lemma 7.18. *Assume* $q \gg 1$ *and* $1/(N\eta) \le \alpha \le 1/q$. *We have on* $\Omega(\alpha)$,

$$|Z_i| \prec N^\epsilon \left(\frac{1}{q} + \sqrt{\frac{\alpha}{N\eta}}\right), \quad |Z_{ij}| \prec N^\epsilon \left(\frac{1}{q^2} + \sqrt{\frac{\alpha}{N\eta}}\right)$$
$$|\mathrm{Im}(Z_i)| \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}}\right), \quad |\mathrm{Im}(Z_{ij})| \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}}\right).$$

Proof. From the resolvent identity, we have

$$R^{(i)} - \tilde{R}^{(i)} = -(\tilde{z} - \tilde{z}_i) R^{(i)} \tilde{R}^{(i)}.$$

Moreover,

$$|\tilde{z} - \tilde{z}_i| \prec \frac{1}{Nq},$$

and on $\Omega(\alpha)$, from (7.14), for any k, l

$$\left|\left(R^{(i)}\tilde{R}^{(i)}\right)_{kl}\right| \le \sum_a^{(i)} |R^{(i)}_{ka}\tilde{R}^{(i)}_{al}| \le \sum_a^{(i)} \left((R^{(i)}_{ka})^2 + (\tilde{R}^{(i)}_{al})^2\right) | \lesssim \frac{N^\epsilon \alpha}{\eta}.$$

So finally,

$$\left|R^{(i)}_{kl} - \tilde{R}^{(i)}_{kl}\right| \prec \frac{N^\epsilon \alpha}{qN\eta}. \tag{7.18}$$

The same bound holds for $\left|R^{(ij)}_{kl} - \tilde{R}^{(ij)}_{kl}\right|$.

We write

$$Z_i(\tilde{z}) = \sum_k^{(i)} \left(|a_{ik}|^2 - \frac{1}{N}\right) \tilde{R}^{(i)}_{kk} + \sum_{k\neq l}^{(i)} a_{ik} \tilde{R}^{(i)}_{kl} a_{li}$$
$$+ \sum_k^{(i)} \left(|a_{ik}|^2 - \frac{1}{N}\right)\left(R^{(i)}_{kk} - \tilde{R}^{(i)}_{kk}\right) + \sum_{k\neq l}^{(i)} a_{ik}\left(R^{(i)}_{kl} - \tilde{R}^{(i)}_{kl}\right) a_{li}.$$

Note that

$$\left|\sum_k^{(i)} \left(|a_{ik}|^2 - \frac{1}{N}\right)\left(R^{(i)}_{kk} - \tilde{R}^{(i)}_{kk}\right)\right| \prec \frac{N^\epsilon \alpha}{qN\eta} \sum_k^{(i)} \left||a_{ik}|^2 - \frac{1}{N}\right|$$
$$\lesssim \frac{N^\epsilon \alpha}{qN\eta} \sum_k^{(i)} \left(\mathring{a}_{ik}^2 + \frac{f}{N}|\mathring{a}_{ik}| + \frac{f^2}{N^2} + \frac{1}{N}\right)$$
$$\lesssim \frac{N^\epsilon \alpha}{N\eta},$$

and

$$\left|\sum_{k\neq l}^{(i)} a_{ik}\left(R^{(i)}_{kl} - \tilde{R}^{(i)}_{kl}\right) a_{li}\right| \prec \frac{N^\epsilon \alpha}{qN\eta} \sum_{k\neq l}^{(i)} |a_{ik}a_{li}|$$
$$\lesssim \frac{N^\epsilon \alpha}{qN\eta} \sum_{k\neq l}^{(i)} \left(|\mathring{a}_{ik}\mathring{a}_{li}| + \frac{f}{N}|\mathring{a}_{ik}| + \frac{f}{N}|\mathring{a}_{li}| + \frac{f^2}{N^2}\right)$$
$$\lesssim \frac{N^\epsilon q\alpha}{N\eta}.$$

Then we have

$$|Z_i| \prec \left| \sum_k^{(i)} \left(|a_{ik}|^2 - \frac{1}{N} \right) \tilde{R}_{kk}^{(i)} + \sum_{k \neq l}^{(i)} a_{ik} \tilde{R}_{kl}^{(i)} a_{li} \right| + \frac{N^\epsilon q \alpha}{N\eta}.$$

Using the large deviation estimate [11, Lemma 3.8 (ii)], it follows that

$$\left| \sum_k^{(i)} \left(\mathring{a}_{ik}^2 - \frac{1}{N} \right) \tilde{R}_{kk}^{(i)} + \sum_{k \neq l}^{(i)} \mathring{a}_{ik} \tilde{R}_{kl}^{(i)} \mathring{a}_{li} \right| \prec \frac{\max_k |\tilde{R}_{kk}^{(i)}|}{q} + \frac{\max_{k \neq l} |\tilde{R}_{kl}^{(i)}|}{q} + \left(\frac{1}{N^2} \sum_{k,l}^{(ij)} |\tilde{R}_{kl}^{(i)}|^2 \right)^{1/2}.$$

Applying [11, Lemma 7.5] and [11, Inequality (7.18)], we find

$$\left| \sum_{k,l}^{(i)} \left(\frac{f}{N} \mathring{a}_{ik} + \frac{f}{N} \mathring{a}_{li} + \frac{f^2}{N^2} \right) \tilde{R}_{kl}^{(i)} \right| \lesssim \frac{1}{q} + \frac{1}{N\eta}.$$

In sum, we establish on $\Omega(\alpha)$,

$$|Z_i| \prec N^\epsilon \left(\frac{1}{q} + \sqrt{\frac{\alpha}{N\eta}} \right),$$

where we have used Ward identity (5.3). The first claim follows.

Similarly, since for $i \neq j$, the random variables $\{h_{ik}\}_{k:k\neq j}$ are independent of $\{h_{lj}\}_{l:l\neq i}$, from (7.14)-(7.18), [11, Lemma 3.8 (iii)], [11, Lemma 7.5] and [11, Inequality (7.18)], on $\Omega(\alpha)$, we have

$$\begin{aligned} \left| \sum_{k,l}^{(ij)} Z_{ij} \right| &\prec \frac{\max_k |\tilde{R}_{kk}^{(ij)}|}{q^2} + \frac{\max_{k \neq l} |\tilde{R}_{kl}^{(ij)}|}{q} + \left(\frac{1}{N^2} \sum_{k,l}^{(ij)} |\tilde{R}_{kl}^{(ij)}|^2 \right)^{1/2} + \frac{1}{q} + \frac{1}{N\eta} + \frac{N^\epsilon q \alpha}{N\eta} \\ &\prec N^\epsilon \left(\frac{1}{q^2} + \sqrt{\frac{\alpha}{N\eta}} \right). \end{aligned}$$

The same argument gives with aid of Lemma 7.17, on $\Omega(\alpha)$,

$$|\mathrm{Im}(Z_{ij})| = \left| \sum_{k,l}^{(ij)} a_{ik} \mathrm{Im}(R_{kl}^{(ij)}) a_{lj} \right| \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}} \right).$$

Finally, we obtain similarly, on $\Omega(\alpha)$

$$\mathrm{Im} Z_i = \sum_k^{(i)} \left(|a_{ik}|^2 - \frac{1}{N} \right) \mathrm{Im} R_{kk}^{(i)} + \sum_{k \neq l}^{(i)} a_{ik} \mathrm{Im}(R_{kl}^{(i)}) a_{li} \prec N^\epsilon \left(\frac{\alpha}{q} + \sqrt{\frac{\alpha}{N\eta}} \right),$$

as claimed. □

Following Step 5 (iteration of the error bounds) of Subsection 6.2, we can complete the proof with the above technical lemmas. We omit the details. □

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