SUSY transfer matrix approach for the real symmetric 1d random band matrices*

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Abstract
This paper adapts the recently developed rigorous application of the supersymmetric transfer matrix approach for the Hermitian 1d band matrices to the case of the orthogonal symmetry. We consider $N \times N$ block band matrices consisting of $W \times W$ random Gaussian blocks (parametrized by $j,k \in \Lambda = [1,n] \cap \mathbb{Z}$, $N = nW$) with a fixed entry’s variance $J_{jk} = W^{-1}(\delta_{j,k} + \beta \Delta_{j,k})$ in each block. Considering the limit $W, n \to \infty$, we prove that the behaviour of the second correlation function of characteristic polynomials of such matrices in the bulk of the spectrum exhibit a crossover near the threshold $W \sim \sqrt{N}$.

Keywords: SUSY; random band matrices; real symmetric case; universality; characteristic polynomials.

MSC2020 subject classifications: 60B20.

1 Introduction
Starting from the works of Erdős, Yau, Schlein with co-authors (see [16] and reference therein) and Tao and Vu (see, e.g., [35]), significant progress in understanding of universal behaviour of local eigenvalues statistics of many random graph and random matrix models were achieved. However, for the random matrices with spacial structure our understanding is much more limited.

One of the most important such models is the ensemble of random band matrices (RBM), i.e. $N \times N$ matrices having non-zero entries only in a strip of width $2W$ near the main diagonal. Such matrices interpolate between mean-field type Wigner matrices (Hermitian or real symmetric matrices with i.i.d. random entries) and random Schrödinger operators, which have only a random diagonal potential in addition to the deterministic

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Laplacian on a box in $\mathbb{Z}^d$. The density of states $\rho$ of a general class of RBM with $W \gg 1$ is given by the well-known Wigner semicircle law (see [3, 24]):

$$\rho(E) = \frac{1}{2\pi} \sqrt{4 - E^2}, \quad E \in [-2, 2].$$

(1.1)

The main long standing problem in the field is to prove a fundamental physical conjecture formulated in late 80s (see [10], [17]). The conjecture states that the eigenvectors of $N \times N$ RBM are completely delocalized and the local spectral statistics governed by the Wigner-Dyson statistics for large bandwidth $W$ (i.e. the local behaviour is the same as for Wigner matrices), and by Poisson statistics for a small $W$ (with exponentially localized eigenvectors). This is the analogue of the celebrated Anderson metal-insulator transition for random Schrödinger operators (see [34] for more details).

The transition (crossover) for RBM in one spacial dimension is conjectured to occur around the critical value $W = \sqrt{N}$. The conjecture is supported by physical derivation due to Fyodorov and Mirlin (see [17]), and also by the so-called Thouless scaling. On the mathematical level of rigour, localization of eigenvectors in the bulk of the spectrum was first shown for $W \ll N^{1/8}$ [27], and then the bound was improved to $N^{3/7}$ [26]. On the other side, by a development of the Erdős-Schlein-Yau approach to Wigner matrices (see [16]), there were obtained some results where the weaker form of delocalization was proved for $W \gg N^{6/7}$ in [14], $W \gg N^{4/5}$ in [15], $W \gg N^{7/9}$ in [18]. The combination of this approach with the new ideas based on quantum unique ergodicity gave first GUE/GOE gap distributions for RBM with $W \sim N$ [5], and then were developed in [6]–[7], [38] to obtain bulk universality and complete delocalization in the range $W \gg N^{3/4}$ (see review [4] for the details).

There is a completely different approach which allows to work with random operators with a non-trivial spacial structure based on supersymmetry techniques (SUSY). It is widely used in the physics literature (see e.g. reviews [13], [23]) but its rigorous mathematical application is usually quite difficult and it requires to incorporate various analytic and statistical mechanics techniques. However, for the 1d Hermitian RBM of a certain type it was successfully done both for correlation functions of characteristic polynomials and for usual correlation functions. More precisely, combining SUSY with a delicate steepest descent method and transfer matrix techniques, we were able to perform a complete study of the local regime of characteristic polynomials for Hermitian Gaussian 1d RBM (see [29] for the regime $W \gg \sqrt{N}$, [30] for the regime $W \ll \sqrt{N}$, and [32] for the regime $W \sim \sqrt{N}$), and also obtain the first rigorous universality result for the second order correlation function for the whole delocalized region $W \gg \sqrt{N}$ (see [31]).

Let us mention also that SUSY approach was also applied to obtain the detailed information of the density of states for the Hermitian RBM of higher dimensions (see [11], [12]).

There are much less rigorous applications of SUSY techniques for the case of real symmetric matrices, since the SUSY integral representations are more complicated for the case of orthogonal symmetry. However, the technique of [29] was successfully adapted in [33] to the study of characteristic polynomials for real symmetric Gaussian 1d RBM in the delocalized regime $W \gg \sqrt{N}$. In this paper we want to perform the complete study of characteristic polynomials for real symmetric Gaussian 1d RBM adapting the SUSY transfer matrix techniques of [30], [32] to the case of orthogonal symmetry. This is an important step towards the proof of the universality of the usual correlation functions for the case of real symmetric 1d RBM, as well as for the general development of rigorous application of SUSY approach for the real symmetric case.

The model we are going to consider is different from the model of 1d RBM considered in [29]–[30], [32] and in [33], but coincides with the model considered in [31]. Namely,
we consider real symmetric block band matrices, i.e. real symmetric matrices $H_N$, $N = nW$ with elements $H_{jk,\alpha\gamma}$, where $j, k \in 1, \ldots, n$ (they parametrize the lattice sites) and $\alpha, \beta = 1, \ldots, W$ (they parametrize the orbitals on each site). The entries $H_{jk,\alpha\gamma}$ are random Gaussian variables with mean zero such that

$$\langle H_{j_1k_1,\alpha_1\beta_1} H_{j_2k_2,\alpha_2\beta_2} \rangle = \delta_{j_1j_2} \delta_{k_1k_2} \delta_{\alpha_1\alpha_2} \delta_{\beta_1\beta_2} J_{j_1k_1}. \quad (1.2)$$

Here $J_{jk} \geq 0$ are matrix elements of the positive-definite symmetric $n \times n$ matrix $J$, such that $\sum_{j=1}^{n} J_{jk} = 1/W$.

The probability law of $H_N$ can be written in the form

$$P_N(dH_N) = \exp\left\{ -\frac{1}{4} \sum_{j,k} \sum_{\alpha,\gamma} W \frac{H_{jk,\alpha\gamma}^2}{J_{jk}} \right\} dH_N, \quad (1.3)$$

where

$$dH_N = \prod_{j=k} dH_{jk,\alpha\gamma} \prod_{j} \frac{dH_{jj,\alpha\alpha}}{\sqrt{2\pi J_{jj}}} \prod_{j} \frac{dH_{jj,\alpha\alpha}}{4\pi J_{jj}}.$$}

Such models were first introduced and studied by Wegner (see [28], [37]) (and sometimes they are also called Wegner’s orbital models).

As in [31], we consider the case

$$J = 1/W + \beta \Delta^{(0)}/W, \quad \beta < 1/4, \quad (1.4)$$

where $W \gg 1$ and $\Delta^{(0)}$ is the discrete Laplacian on $[1, n] \cap \mathbb{Z}$ with Neumann boundary conditions. Clearly, this model is one of the possible realizations of the Gaussian random band matrices with the band width $2W + 1$ (note that the model can be defined similarly in any dimensions $d > 1$ taking $j, k \in [1, n]^d \cap \mathbb{Z}^d$ in (1.2)).

The main interest of this paper is to study the behaviour of correlation functions (or the mixed moments) of characteristic polynomials which can be defined as

$$F_{2k}(\Lambda) = \int \prod_{s=1}^{2k} d(\lambda_s - H_N)P_n(dH_N), \quad (1.5)$$

where $P_n(dH_N)$ is defined in (1.3), and $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{2k}\}$ are real or complex parameters that may depend on $N$. As in the Hermitian case, correlation functions of characteristic polynomials of real symmetric 1d RBM are expected to exhibit a crossover near the threshold $W \sim \sqrt{N}$: it is expected that they the same local behaviour as for GOE for $W \gg \sqrt{N}$, and the different behaviour for $W \ll \sqrt{N}$.

The asymptotic local behaviour in the bulk of the spectrum of the 2k-point mixed moment for GOE is well-known. It was proved for $k = 1$ by Brézin and Hikami [8] (based on SUSY approach), and for general $k$ by Borodin and Strahov [9] (with a different techniques) that

$$F_{2k}(\Lambda_0 + \hat{\xi}/N\rho(E)) = C_{N,k} \text{Pf}\left\{DS(\pi(\xi_i - \xi_j))\right\}_{i,j=1}^{2k} (1 + o(1)), \quad (1.6)$$

where $C_{N,k}$ is some multiplicative constant depending on $N$, $k$, $\hat{\xi}$, and $\rho(E)$ is the density of states in the bulk.

$$DS(x) = -3 \frac{d}{dx} \frac{\sin x}{x} = 3\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right), \quad (1.6)$$

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where $C_{N,k}$ is some multiplicative constant depending on $N$, $k$, $\hat{\xi}$, and $\rho(E)$ is the density of states in the bulk.
$\Delta(\xi_1, \ldots, \xi_k)$ is the Vandermonde determinant of $\xi_1, \ldots, \xi_k$, and
$$\hat{\xi} = \text{diag}\{\xi_1, \ldots, \xi_{2k}\}, \quad \Lambda_0 = E \cdot I.$$  

In particular, for $k = 1$ we have
$$F_2\left(\lambda_0 + \frac{\xi}{2N \rho(E)}\right) = C_N\left(\frac{\sin(\pi(\xi_1 - \xi_2))}{\pi^2(\xi_1 - \xi_2)^2} - \frac{\cos(\pi(\xi_1 - \xi_2))}{\pi^2(\xi_1 - \xi_2)^2}\right)(1 + o(1)).$$

The last formula was proved also for real symmetric Wigner and general sample covariance matrices (see [20]).

Set
$$\lambda_1 = E + \frac{\xi}{2N \rho(E)}, \quad \lambda_2 = E - \frac{\xi}{2N \rho(E)}, \quad (1.7)$$

where $E \in (-2, 2)$, $\rho$ is defined in (1.1), and $\xi$ is a real parameter varying in any compact set $K \subset \mathbb{R}$, and define
$$D_2 = E_2^{1/2}(E, E). \quad (1.8)$$

The main result of the paper is the following theorem:

**Theorem 1.1.** For the real symmetric 1d block random band matrices $H_N$, $N = nW$ of (1.2)–(1.4) we have

$$\lim_{n \to \infty} F_2\left(E + \frac{\xi}{2N \rho(E)}, E - \frac{\xi}{2N \rho(E)}\right) = \begin{cases} DS(\pi \xi), & W \gg n \gg 1; \\ (e^{-C\Delta - i\pi \xi \hat{\nu}} \cdot 1, 1), & n = C_W \\ 1, & 1 \ll W \leq n/\log^2 n, \end{cases}$$

where $DS(x)$ is defined in (1.6), $C^* = C/\sqrt{2\pi \rho(E)}$ with $\rho(E)$ of (1.1), and $\varepsilon$ is any sufficiently small positive number. In this formula $\Delta$ is a Laplace-Bertrami operator on $\tilde{Sp}(2) = Sp(2)/Sp(1) \times Sp(1)$, $Sp(n)$ is a compact symplectic group of $2n \times 2n$ unitary symplectic matrices, and $(\cdot, \cdot)$ is an inner product on $L_2[Sp(2), d\mu]$, where $d\mu$ is the Haar measure on $Sp(2)$. $\hat{\nu}$ is an operator of multiplication by

$$\nu(Q) = 1 - 2(|Q_{12}|^2 + |Q_{14}|^2) \quad (1.9)$$

on $\tilde{Sp}(2)$. Notice that the since $N = nW$, the transition happens at $W \sim \sqrt{N}$.

**1.1 Notation**

We denote by $C, C_1$, etc. various $W$ and $N$-independent quantities below, which can be different in different formulas. Integrals without limits denote the integration (or the multiple integration) over the whole real axis, or over the Grassmann variables.

Moreover,

- $W$ is a size of the block, and $n$ is the number of blocks in a row, so $N = nW$ is the size of the matrix $H$ of (1.2);
- $E\{\ldots\}$ is an expectation with respect to the measure (1.3);
- $a_{\pm} = iE \pm \sqrt{4 - E^2} = e^{\pm i\alpha}$;
- $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
- $D_0 = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}$, $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\hat{\xi} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;
- $D_{0,4} = \begin{pmatrix} D_0 & 0 \\ 0 & D_0 \end{pmatrix}$, $\hat{\xi}_4 = \begin{pmatrix} \hat{\xi} \\ 0 \end{pmatrix}$, $L_4 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$;
Averaging over (1.3), we get

\[ Q \left( \begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right) Q^t = \left( \begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right). \]

\[ \hat{U}(2) = U(2)/(U(1) \times U(1)), \quad \hat{Sp}(2) = Sp(2)/(Sp(1) \times Sp(1)); \]

\[ \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}, \quad \omega_A = \{ z \in \mathbb{C} : |z| = 1 + A/n \}; \]

\[ d\mu \text{ is the Haar measure on } \hat{U}(2), \quad d\mu \text{ is the Haar measure on } \hat{Sp}(2); \]

\[ c_\pm = 1 + a_\pm^2; \quad t_\ast = (2\pi \rho(E))^2 \]

We denote by \( \hat{a} \) the vector \( (a_1, a_2) \);

2 Integral representation

The main aim of this section is to derive the following proposition

**Proposition 2.1.** The second correlation function (1.5) of the characteristic polynomials for 1d real symmetric Gaussian block band matrices (1.2)–(1.4) can be represented as follows:

\[
F_2(\lambda_0 + \frac{\xi}{2N \rho(E)}) = C_{n,W} \int \exp \left\{ \frac{\beta W}{4} \sum_{j=2}^{n} \text{Tr} (F_j - F_{j-1})^2 \right\} \left( \frac{W}{4} \sum_{j=1}^{n} \left( \text{Tr} F_j^2 - 2\text{Tr} F_j (\lambda_0 + \frac{\xi}{2N \rho(E)}) \right) \right) \prod_{j=1}^{n} (\det F_j)^{-W/2} dF_j,
\]

where \( \lambda_0,4 \) and \( \xi_4 \) are defined in Notation, \( N = nW, C_{n,W} \) is some constant depending on \( W \) and \( n \) but not on \( \xi \), and \( F_j \in Sp(2) \) are unitary symplectic \( 4 \times 4 \) matrices.

**Proof.** Introduce the following Grassmann fields:

\[ \Psi_l = (\psi_{jl}^T)_{j=1,\ldots,n}, \quad \psi_{jl} = (\psi_{j1l}, \psi_{j2l}, \ldots, \psi_{jWl})^T, \quad l = 1, 2. \]

Using (A.7) (see Appendix A) we obtain

\[
F_2(\lambda_1, \lambda_2) = \mathbb{E} \left\{ \int \exp \left\{ -\Psi_1^\ast (\lambda_1 - H_N) \Psi_1 - \Psi_2^\ast (\lambda_2 - H_N) \Psi_2 \right\} d\Psi \right\} = \int d\Psi \text{ exp } \left\{ -\lambda_1 \Psi_1^\ast \Psi_1 - \lambda_2 \Psi_2^\ast \Psi_2 \right\} \times \mathbb{E} \left\{ \exp \left\{ \sum_{j<k} \sum_{\alpha<\gamma} H_{jk,\alpha\gamma}(\eta_{jk,\alpha\gamma} + \eta_{kj,\gamma\alpha}) + \sum_{j} \sum_{\alpha<\gamma} H_{jk,\alpha\gamma}(\eta_{jk,\alpha\gamma} + \eta_{kj,\gamma\alpha}) \right\} \right\},
\]

where

\[
d\Psi = \prod_{j=1}^{n} \prod_{\alpha=1}^{W} \prod_{j'=1}^{2} d\psi_{j'\alpha} d\psi_{j\alpha},
\]

\[
\eta_{jk,\alpha\gamma} = \bar{\psi}_{j1\alpha} \psi_{k1\gamma} + \bar{\psi}_{j2\alpha} \psi_{k2\gamma}, \quad \text{if } j \neq k \text{ or } \alpha \neq \gamma;
\]

\[
\eta_{jj,\alpha\alpha} = (\bar{\psi}_{j1\alpha} \psi_{j1\alpha} + \bar{\psi}_{j2\alpha} \psi_{j2\alpha})/2.
\]

Averaging over (1.3), we get

\[
F_2(\lambda_1, \lambda_2) = \int d\Psi \text{ exp } \left\{ -\lambda_1 \Psi_1^\ast \Psi_1 - \lambda_2 \Psi_2^\ast \Psi_2 \right\} \times \exp \left\{ \frac{1}{2} \sum_{j<k,\alpha,\gamma} J_{jk} (\eta_{jk,\alpha\gamma} + \eta_{kj,\gamma\alpha})^2 + \frac{1}{2} \sum_{j,\alpha<\gamma} J_{jj} (\eta_{jj,\alpha\gamma} + \eta_{j\gamma,\gamma\alpha})^2 + \sum_{j,\alpha} J_{jj} \eta_{jj,\alpha\alpha}^2 \right\}.
\]
It is easy to see that
\[
\frac{1}{2} \sum_{\alpha, \gamma} (\eta_{jk, \alpha\gamma} + \eta_{kj, \gamma\alpha})^2 = \left(\psi_{1j}^+ \psi_{j2} - \psi_{2j}^+ \psi_{1j} - \psi_{1j}^+ \psi_{j1} - \psi_{j1}^+ \psi_{1j}\right)^2
\]
\[
= \frac{1}{2} \left(\psi_{1j}^+ \psi_{j2} - \psi_{2j}^+ \psi_{1j} - \psi_{1j}^+ \psi_{j1} - \psi_{j1}^+ \psi_{1j}\right)
\]
\[
= \frac{1}{2} \sum_{l=1,2} \lambda_l \Psi_l \Psi_l = \frac{1}{2} \sum_{j=1}^n \text{Tr} \tilde{F}_j A_n
\]

where
\[
\tilde{F}_j = \begin{pmatrix} \psi_{j1}^+ \psi_{1j} & \psi_{j2}^+ \psi_{2j} & 0 & \psi_{j1}^+ \psi_{j2} \\ \psi_{2j}^+ \psi_{j1} & \psi_{1j}^+ \psi_{j2} & 0 & \psi_{2j}^+ \psi_{1j} \\ 0 & \psi_{j1}^+ \psi_{j2} & \psi_{j1}^+ \psi_{j1} & 0 \\ \psi_{j2}^+ \psi_{j1} & 0 & \psi_{j2}^+ \psi_{1j} & \psi_{j2}^+ \psi_{j2} \end{pmatrix}, \quad A_n = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}.
\]

Applying the superbosonization formula (see Proposition A.1, Appendix A), we obtain
\[
F_2(\lambda_1, \lambda_2) = C_{nW} \int \exp \left\{ -\frac{1}{4} \sum_{j,k=1}^n J_{jk} \text{Tr} F_j F_k - \frac{1}{2} \sum_{j,k=1}^n \text{Tr} F_j A_4 \right\} \prod_{j=1}^n (\det F_j)^{-W/2} \prod_{j=1}^n d F_j,
\]

where \(\{F_j\}_{j=1}^n\) are unitary symplectic \(4 \times 4\) matrices from \(Sp(2)\), and \(C_{nW}\) is some constant depending on \(W\) and \(n\) but not on \(\lambda_1, \lambda_2\). Shifting \(F_j \rightarrow i W F_j\) and plugging in (1.7), we get Proposition 2.1.

\[\square\]

3 Representation in the operator form

To study (2.1), we are going to apply the transfer matrix approach.

Namely, introduce
\[
\mathcal{F}(X) = \exp \left\{ W \left( \frac{1}{8} \text{Tr} X^2 - \frac{iE}{4} \text{Tr} X - \frac{1}{4} \text{Tr} \log X - C_+ \right) \right\},
\]
\[
\mathcal{F}_\xi(X) = \mathcal{F}(X) \cdot \mathcal{F}_{n, \xi}(X), \quad \mathcal{F}_{n, \xi}(X) := \exp \left\{ -\frac{i}{8 n \rho(E)} \text{Tr} X \xi \right\},
\]

where
\[
C_+ = \frac{a_+^2}{2} - i E a_+ - \log a_+
\]
is chosen in such a way that \(|\mathcal{F}(X)| = 1\) in the saddle-points (see (4.2) later).

Let also \(K, K_\xi : Sp(2) \rightarrow Sp(2)\) be the operators with the kernels
\[
K(X,Y) = \frac{W^3}{2 \pi^3} \mathcal{F}(X) \exp \left\{ \frac{\beta W}{4} \text{Tr} (X - Y)^2 \right\} \mathcal{F}(Y); \quad (3.2)
\]
\[
K_\xi(X,Y) = \frac{W^3}{2 \pi^3} \mathcal{F}_\xi(X) \exp \left\{ \frac{\beta W}{4} \text{Tr} (X - Y)^2 \right\} \mathcal{F}_\xi(Y). \quad (3.3)
\]

Then Proposition 2.1 can be reformulated as
\[
F_2 \left( E + \frac{\xi}{2 N \rho(E)}, E - \frac{\xi}{2 N \rho(E)} \right) = \tilde{C}_{n,W} (K_{\xi}^{-1} \mathcal{F}_\xi, \mathcal{F}_\xi), \quad (3.4)
\]

where \((\cdot, \cdot)\) is a standard inner product in \(Sp(2)\) with respect to the Haar measure \(d\mu\), and \(\tilde{C}_{n,W}\) is some constant depending on \(W\) and \(n\) but not on \(\xi\).

For an arbitrary compact operator $M$ denote by $\lambda_j(M)$ the $j$th (by modulus) eigenvalue of $M$, so that $|\lambda_0(M)| \geq |\lambda_1(M)| \geq \ldots$.

Since $K_\xi$ is a compact operator, one can rewrite

$$
(K_n^{-1}f, \bar{g}) = \sum_{j=0}^{\infty} \lambda_j^{-1}(K_\xi)c_j, \quad \text{with} \quad c_j = (f, \bar{\psi}_j)(\bar{\phi}_\xi, \bar{\psi}_j),
$$

where \{\psi_j\} are eigenvectors corresponding to \{\lambda_j(K_\xi)\}, and \{\bar{\psi}_j\} are the eigenvectors of $K^*_\xi$. Similar equality is true if we replace $K_\xi$ and $F_\xi$ by $K$ and $F$. Hence, to study (2.1), it suffices to study the eigenvalues and eigenvectors of $K_\xi$, $K$.

### 4 Sketch of the proof of Theorem 1.1

As was mentioned above, we are interested in the analysis of the spectral properties of $K_\xi$ of (3.3) (see (3.4)). It appears that it is simpler to work with the resolvent analog of (3.4)

$$
(K_n^{-1}f, \bar{g}) = -\frac{1}{2\pi i} \oint_{\mathcal{L}} z^{-1}(G_\xi(z)f, \bar{g})dz, \quad G_\xi(z) = (K_\xi - z)^{-1},
$$

where $\mathcal{L}$ is any closed contour which encloses all eigenvalues of $K_\xi$.

The idea of the proof is very close to [30]–[32]. To outline it, we start with the following definition

**Definition 4.1.** We say that the operator $A_{n,W}$ is equivalent to $B_{n,W}$ ($A_{n,W} \sim B_{n,W}$) on some contour $\mathcal{L}$ if

$$
\int_{\mathcal{L}} z^{-1}((A_{n,W} - z)^{-1}f, \bar{g})dz = \int_{\mathcal{L}} z^{-1}((B_{n,W} - z)^{-1}f, \bar{g})dz (1 + o(1)), \quad n, W \to \infty,
$$

with some particular functions $f, g$ depending of the problem.

The aim is to find some operator equivalent to $K_\xi$ whose spectral analysis is more accessible. Now we are going to discuss how this was done on the ideological level. The specific choice of the contour $\mathcal{L}$ and functions $f, g$ for each step will be discussed in details in Section 6.

It is easy to check that the stationary points of the function $F$ of (3.1) are

$$
X_+ = a_+ \cdot I_4, \quad X_- = a_- \cdot I_4; \quad X_\pm (Q) = QD_{0,4}Q^*, \quad Q \in \hat{Sp}(2),
$$

where $a_\pm, D_{0,4}$ are defined in Notation. Recall also that the value of $|F|$ at points (4.2) is 1.

The first step in the proof of Theorem 1.1 is to apply the saddle-point approximation. Roughly speaking, we show that if we introduce the projection $Pr_s$ onto the $W^{-1/2} \log W$-neighbourhoods of the saddle points $X_+, X_-$ and the saddle “surface” $X_\pm$, then in the sense of Definition 4.1

$$
K_\xi \sim Pr_s K_\xi Pr_s =: K_{s,\xi}.
$$

Moreover, one can show that only the neighbourhood of the saddle “surface” $X_\pm$ gives the main contribution to the integral. The proof is based on a study of a quadratic approximation of a function $F$ of (3.1). Let us also emphasize, that for the block band matrices (1.2)–(1.4) this step is much simpler than for the model considered in [30]–[32] due to the large coefficient $W$ in the exponent of $F$. This analysis will be performed in details in Section 5.
To study the operator $K_{*,\xi}$ near the saddle "surface" $X_{\pm}$ we use the "polar coordinates". Namely, the matrices from $\Sp(2)$ have two eigenvalues $a_j, b_j \in \mathbb{T} = \{ z : |z| = 1 \}$ of the multiplicity two and can be considered as quaternion $2 \times 2$ matrices. In this language $F_j$ are quaternion unitary matrices, and so they can be diagonalized by the quaternion unitary $2 \times 2$ matrices from $\Sp(2)$ (see, e.g., [22], Chapter 2.4).

Change the variables to $F_j = Q_j^* A_j Q_j$, where $A_j, 4 = \text{diag} \{ a_j, 1, a_j, 1 \}$, eigenvalues $a_j, 2 \in \mathbb{T}$, and $Q_j \in \Sp(2)$. Then $dF_j$ of (2.1) becomes (see, e.g., [22])

$$\frac{\pi^2}{12} (a_{1j} - a_{2j})^4 da_j \, d\mu(Q_j),$$

where

$$da_j = \frac{da_{1j}}{2\pi i} \frac{da_{2j}}{2\pi i},$$

and $d\mu(Q_j)$ is the normalized to unity Haar measure on the symplectic group $\Sp(2)$. Thus we get

$$(K_{\xi}^{n-1} F_\xi, F_\xi) = \frac{\pi^{2n}}{12^n} \int (a_{11} - a_{21})^2 F_\xi(a_{11}, a_{21}, Q_1)(a_{1n} - a_{2n})^2 F_\xi(a_{1n}, a_{2n}, Q_n)$$

$$\times \prod_{j=1}^{n-1} (a_{1j} - a_{2j})^2 (a_{1,j+1} - a_{2,j+1})^2 K_\xi(F_j, F_{j+1}) \prod_{j=1}^{n} da_j \, d\mu(Q_j).$$

Introduce

$$t = (a_1 - a_2)(a_1' - a_2').$$

Then we obtain

$$F_2 \left( E + \frac{\xi}{2N\rho(E)}, E - \frac{\xi}{2N\rho(E)} \right) = \tilde{C}_{n,W}(K_{n-1}^{-1} f, f),$$

where now $(\cdot, \cdot)$ is a standard inner product in $L_2[\mathbb{T}^2] \times L_2[\Sp(2), d\mu(Q)]$, and $\tilde{C}_{n,W}$ is some constant depending on $W$ and $n$ but not on $\xi$. Here

$$f(a_1, a_2, Q) = (a_1 - a_2)^2 F_\xi(a_1, a_2, Q),$$

and $K_\xi = F_{n,\xi} K F_{n,\xi}$ is an integral operator in $L_2[\mathbb{T}^2] \times L_2[\Sp(2), d\mu(Q)]$ defined by the kernel

$$K_\xi(X, Y) = F_{n,\xi}(a_1, a_2, Q) K(a_1, a_2, Q; a_1', a_2', Q') F_{n,\xi}(a_1', a_2', Q'),$$

where

$$K(a_1, a_2, Q; a_1', a_2', Q') = A_{a}(\hat{a}, \hat{a}') K_*(t, Q_1, Q_2);$$

$$K_*(t, Q, Q') := \frac{\beta W^2}{6} \cdot \exp \{ -t \beta W S(Q(Q')^*) \}, \quad S(Q) = |Q_{12}|^2 + |Q_{14}|^2;$$

$$F_{n,\xi}(a, b, Q) = \exp \{ - i \xi \pi \cdot \nu(a - b, Q)/n \};$$

$$\nu(p, Q) = \frac{p}{4\pi \rho(E)} \text{Tr} Q L_d Q^* L_d = \frac{p}{2\pi \rho(E)} (1 - 2S(Q))$$

with $t$ of (4.3). $K_*$ here is a contribution of the symplectic group $\Sp(2)$ into operator $K$, and $\exp \{ -i \xi \pi \cdot \nu(x, Q)/n \}$ comes from the $1/n$-order perturbation $F_{n,\xi}$ of $F$ appearing in
\[ F_\xi \] (see (3.1)). Operator \( A_a \) is a contribution of eigenvalues \( a_1, a_2 \) and it has the form

\[
A_a(\bar{a}; \bar{a}') = A(a_1, a_1')A(a_2, a_2'),
\]

\[
A(a, a') = \left( \frac{W}{2\pi} \right)^{1/2} e^{-W\Phi(a, a')};
\]

\[
\Phi(x, y) = \frac{\beta}{2} (x - y)^2 - \frac{1}{2} \varphi_0(x) - \frac{1}{2} \varphi_0(y) + \Re \varphi_0(a_+);
\]

\[
\varphi_0(x) = x^{2}/2 - ixE - \log x. \quad (4.10)
\]

Observe that the operator \( K_s(t, Q, Q') \) with some \( t > 0 \) is self-adjoint and its kernel depends only on \( S(Q, Q')^* \). Thus by the standard representation theory arguments (see e.g. [19], [36]), its eigenfunctions are the the same as for Laplace-Bertrami operator on \( Sp(2) \). More precisely:

**Proposition 4.2.** Consider any self-adjoint integral operator \( M \) in \( L_2[\hat{S}p(2), d\mu(Q)] \). If its kernel \( M(Q, Q') \) depends only on \( Q, Q'(\phi) \), then its eigenvectors coincide with eigenvectors of Laplace-Bertrami operator on \( Sp(2) \). Moreover, if the subspace

\[ L_2[S, d\mu(Q)] \subset L_2[\hat{S}p(2), d\mu(Q)] \]

of the functions depending on \( S(Q) \) (see (4.7)) only is invariant under \( M \), then it can be diagonalized by the eigenfunctions

\[
\phi_j(Q) = (-1)^j P_{2j}(\sqrt{S(Q)}), \quad (4.11)
\]

where \( P_{2j}(x) \) are orthogonal with respect to the weight \((1 - x^2)x^3\) on \([0, 1]\) polynomials of degree \( 2j \), \( \phi_0(x) = 1 \) (polynomials \( P_2 \), can be written as \( P_{2j}(x) = c_j F_{0j}(-j, j + 3, 2; 1 - x^2) \), where \( F_{0j} \) is a hypergeometric function, and \( c_j \) is a normalization constant, see [19], Ch. 5). In addition, the following holds

\[
(2x^2 - 1)P_{2j}(x) = \frac{j + 3}{2j + 3} P_{2j+2}(x) + \frac{j}{2j + 3} P_{2j-2}(x), \quad (4.12)
\]

so the operator \( \hat{\nu} \) of multiplication on \( \nu(x, Q) \) of (4.8) is three diagonal in basis (4.11), and

\[
(\hat{\nu} \cdot \phi_0, \phi_0) = 0. \quad (4.13)
\]

If \( M(Q_1, Q_2) = K_s(t, Q_1, Q_2) \) of (4.7), then the corresponding eigenvalues \( \{\lambda_j(t)\}_{j=0}^\infty \), if \( t > d > 0 \), where \( d \) is some absolute positive constant, have the form

\[
\lambda_j(t) = 1 - \frac{(j + 1)(j + 2)}{Wt} + O((j^2/Wt)^2) + O(e^{-tW}). \quad (4.14)
\]

The proof of the proposition can be found in Appendix B.

Notice that, according to Proposition 4.2, since \( F(Q), F_\xi(Q) \) are the functions of \( S(Q) \) only, in what follows we can consider restrictions of \( K_\xi, K, \) and \( K_s \) of (4.7) to \( L_2[S, d\mu(Q)] \) (to simplify notations we will denote these restrictions by the same letters).

In addition, it follows from Proposition 4.2 that if we introduce the following basis in \( L_2[\mathbb{R}^2] \times L_2[S, d\mu(Q)] \)

\[
\Psi_{k,j}(\hat{a}, Q) = \Psi_{k}(\hat{a}) \phi_j(Q),
\]

\[
\Psi_k(\hat{a}) = \psi_{k_1}(a_1) \psi_{k_2}(a_2),
\]
We are going to show that only the top eigenvalue of $A$ where $P$ according to Proposition 4.2, $\phi$ the next step in the proof of Theorem 1.1 is to show that we can restrict the number of $P$ corresponding eigenvalues are (see (4.14)) recall that according to Proposition 4.2 the eigenvectors of $\hat{K}$ of (4.18) with $\xi = 0$.

Now (4.17), (4.1) and Definition 4.1 give

$$F_2 \left( E + \frac{\xi}{2N\rho(E)}, E - \frac{\xi}{2N\rho(E)} \right) = C_{n,W} \left( \hat{K}_{s,l}^{-1} f \xi, \hat{f} \xi \right) (1 + o(1)) = C_{n,W} \lambda_0(K_{s,l})^{-1} f \xi (\hat{K}_{s,l}^{-1} 1,1) (1 + o(1)),$$

where $f_0 = (f, \Psi_0)$, and we used that $f \xi$ asymptotically can be replaced by $f \otimes 1$, where $f$ does not depend on $\xi$ and $Q_l$. Similarly

$$D_2 = C_{n,W} \left( \hat{K}_{s,l}^{-1} f \xi, \hat{f} \xi \right) (1 + o(1)) = C_{n,W} \lambda_0(K_{s,l})^{-1} f \xi (\hat{K}_{s,l}^{-1} 1,1) (1 + o(1)).$$

According to Proposition 4.2, $\phi_0(Q) = 1$ is an eigenvector of $\hat{K}_{s,l}$ of (4.18) with $\xi = 0$ and the corresponding eigenvalue is 1, thus

$$\left( \hat{K}_{s,l}^{-1} 1,1 \right) = 1.$$

Hence

$$F_2 \left( E + \frac{\xi}{2N\rho(E)}, E - \frac{\xi}{2N\rho(E)} \right) = \left( \hat{K}_{s,l}^{-1} 1,1 \right) (1 + o(1)).$$

Recall that according to Proposition 4.2 the eigenvectors of $\hat{K}_{s,l}$ are (4.11) and the corresponding eigenvalues are (see (4.14))

$$\lambda_j := \lambda_j(t_*) = 1 - j(j + 3)/t_* W + O((j(j + 3)/W)^2), \quad j = 0, 1, \ldots, l - 1.$$
Moreover, it follows from (4.6)–(4.7) that
\[ \hat{K}_{\xi,1} = \hat{K}_{0,1} - n^{-1} \pi i \xi \hat{\nu} + o(n^{-1}), \quad \hat{\nu} = P_l \hat{\nu} P_l, \]
where \( \hat{\nu} \) is the operator of multiplication by (1.9), and \( o(1/n) \) means some operator whose norm is \( o(1/n) \). Thus the eigenvalues of \( \hat{K}_{\xi,1} \) are in the \( n^{-1} \)-neighbourhood of \( \lambda_j \).
In the localized regime \( W^{-1} \gg n^{-1} \) we have \( l = 1 \), thus only \( \lambda_0(K_{\xi}) \) contributes to (4.19). Since (see Proposition 4.2)
\[ (\hat{\nu} 1, 1) = 0, \]
we get
\[ \lambda_0(\hat{K}_{\xi}) = 1 + o(n^{-1}), \]
and so the limit of (4.19) is 1 (see the end of Section 6 for more details).
In the delocalized regime all eigenvalues of \( K_{\xi,1} \) contribute to (4.19), but \( K_{0,1} \to I \)
(roughly speaking, this means that the second term in the r.h.s. of (4.20) does not give a contribution). Hence we have
\[ \hat{K}_{\xi,1} \approx 1 - n^{-1} i \pi \hat{\nu} = (\hat{K}_{\xi,1}^{n-1} 1, 1) \to (e^{-i \pi \hat{\nu}} 1, 1) = DS(\pi \xi) \]
with \( DS(\pi \xi) \) of (1.6) (see (B.2) and the end of Section 6 for more details).
In the critical regime \( W^{-1} = C_s n^{-1} \) all eigenvalues of \( K_{\xi,1} \) contribute, but now both second term in the r.h.s. of (4.20) and \( 1/n \)-order term in the r.h.s. of (4.21) make an impact.
As it was mentioned above, the Laplace-Bertrami operator \( \Delta \) on \( L_2[S,d\mu] \) has eigenvalues
\[ \lambda_j^* = j(j + 3). \]
Thus, \( 1 - n^{-1} C^* \Delta \) with \( C^* = C_s / t_* \) has the same basis of eigenvectors with eigenvalues
\[ 1 - j(j + 3) / t_* W. \]
Recall that we are interested in \( j \leq l - 1 \sim \log W \) (since \( P_l \) is the projection on \( \{ \phi_j \}_{j \leq l-1} \). Hence, according to (4.20)–(4.21), in the regime \( W^{-1} = C_s n^{-1} \) we can write
\[ \hat{K}_{\xi,1} \approx P_l \left( 1 - n^{-1} (C^* \Delta + i \pi \nu) \right) P_l + o(n^{-1}), \]
which implies
\[ (\hat{K}_{\xi,1}^{n-1} 1, 1) \to (e^{-C^* \Delta - i \pi \nu} 1, 1), \]
and finishes the proof of Theorem 1.1. The detailed proof of (4.22) is given in Section 6 (see Lemma 6.5).

5 Saddle-point analysis

Recall that the stationary points of the function \( \mathcal{F} \) of (3.1) are defined in (4.2).
We start the proof from the restriction of the integration with respect to \( a, a' \) by the neighbourhood of \( a_\pm \). Set
\[
\Omega_+ = \{ x : |x - a_+| \leq \log W/W^{1/2} \}, \quad \Omega_- = \{ x : |x - a_-| \leq \log W/W^{1/2} \},
\]
\[
\tilde{\Omega}_\pm = \{ a_1, a'_1 \in \Omega_+, a_2, a'_2 \in \Omega_- \},
\]
\[
\tilde{\Omega}_+ = \{ a_1, a'_1, a_2, a'_2 \in \Omega_+ \}, \quad \tilde{\Omega}_- = \{ a_1, a'_1, a_2, a'_2 \in \Omega_- \}
\]
(5.1)
and let \( 1_{\tilde{\Omega}_+}, 1_{\tilde{\Omega}_-} \) be indicator functions of the above domains.
Lemma 5.1. Given $A(a, a')$ of (4.9), we have
\[
\int_{T\setminus (\Omega_+ \cup \Omega_-)} |A(a, a')||da'| \leq C e^{-c \log^2 W}.  \tag{5.2}
\]

**Proof.** Recall that
\[a_\pm = e^{\pm i \alpha_0},\]
and write for the parametrization $a = e^{i \varphi}, a' = e^{i \varphi'}$
\[
\begin{align*}
-\Re \Phi(e^{i \varphi}, e^{i \varphi'}) &= -\beta (\cos \varphi - \cos \varphi')^2/2 + \beta (\sin \varphi - \sin \varphi')^2/2 - \frac{\sin^2 \varphi + \sin^2 \varphi'}{2} \\
&\quad + E(\sin \varphi + \sin \varphi') + \sin^2 \alpha_0 - E \sin \alpha_0 \\
&= -\beta (\cos \varphi - \cos \varphi')^2/2 + \beta (\sin \varphi - \sin \varphi')^2/2 \\
&\quad - (\sin \varphi - \sin \alpha_0)^2/2 - (\sin \varphi' - \sin \alpha_0)^2/2 \\
&\leq -\beta (\sin \varphi - \sin \alpha_0)^2/2 - (\sin \varphi' - \sin \alpha_0)^2/2 \\
&\leq -(1 - 2 \beta)(\sin \varphi - \sin \alpha_0)^2/2 - (1 - 2 \beta)(\sin \varphi' - \sin \alpha_0)^2/2.
\end{align*}
\]
Here we have used $\sin \alpha_0 = E/2$. We have also for $a' \in T \setminus (\Omega_+ \cup \Omega_-)$
\[|\sin \varphi' - \sin \alpha_0| \geq C \log W/\sqrt{W}.
\]
Since $\beta < 1/4$, this implies (5.2) \(\Box\)

Lemma 5.1 yields that
\[
\int dQ'da'(1 - 1_{\Omega_+} - 1_{\Omega_+} - 1_{\Omega_-})\|K\| \leq e^{-c \log^2 W} \tag{5.3}
\]
Let us prove the following simple proposition

**Proposition 5.2.** Let the matrix $H(z)$ have the block form
\[
H(z) = \begin{pmatrix} H_{11}(z) & H_{12}(z) \\ H_{21}(z) & H_{22}(z) \end{pmatrix}.
\]

Then
\[
G(z) := H^{-1}(z) = \begin{pmatrix} G_{11} & -G_{11} H_{12} H_{22}^{-1} \\ -H_{22}^{-1} H_{21} G_{11} & H_{22}^{-1} + H_{22}^{-1} H_{21} G_{11} H_{12} H_{22}^{-1} \end{pmatrix} \tag{5.4}
\]
\[
G_{11} = (H_{11} - H_{12} H_{22}^{-1} H_{21})^{-1},
\]
If $H_{22}^{-1}$ is an analytic function for $|z| > 1 - \delta$, and $\|H_{22}^{-1}\| \leq C$, then
\[
\int_{\omega_A} z^{n-1}(G(z)f, g)dz = \int_{\omega_A} z^{n-1}(G_{11} f^{(1)}(z), g^{(1)}(z))dz + O(e^{-nc}) \tag{5.5}
\]
\[
f^{(1)}(z) = f_0 - H_{12} H_{22}^{-1} f_1, \quad g^{(1)}(z) = g_0 - H_{21}^T (H_{22}^{-1})^{-1} g_1
\]
where $\omega_A = \{z : |z| = 1 + A/n\}$, $f = (f_0, f_1)$, $g = (g_0, g_1)$ where $f_0$ and $g_0$ are the projection of $f$ and $g$ on the subspace corresponding to $H_{11}$, while $f_1$ and $g_1$ are the projection of $f$ and $g$ on the subspace corresponding to $H_{22}$.

**Proof.** Formula (5.4) is the well-known block matrix inversion formula. Now apply the formula (5.4) and write
\[
\int_{\omega_A} z^{n-1}(G(z)f, g)dz = \int_{\omega_A} z^{n-1}(G_{11} f^{(1)}(z), g^{(1)}(z))dz + \int_{\omega_A} z^{n-1}(H_{22}^{-1} f_1, g_1)dz.
\]
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For the second integral change the integration contour from $\omega_A$ to $|z| = 1 - \delta$. Then the inequality

$$|z|^{n-1} \leq (1 - \delta)^{n-1} \leq C e^{-nc}$$

yields (5.5).

Notice that since $\|K\| \leq 1$ and $|\mathcal{F}_{n,\xi}| \leq 1 + C/n$, we can find such $A$ that all eigenvalues of $K_\xi$ lie inside $\omega_A = \{z : |z| = 1 + A/n\}$.

Set

$$H_{11}(z) = H_{11} - z = (1_{\Omega_-} K_\xi 1_{\Omega_+}) \oplus (1_{\Omega_+} K_\xi 1_{\Omega_-}) \oplus (1_{\Omega_-} K_\xi 1_{\Omega_+}) - z = K_{\xi,\pm} \oplus K_{\xi,+} \oplus K_{\xi,-} - z.$$  

Then (5.3) yields

$$\|H_{22}\| + \|H_{12}\| + \|H_{21}\| \leq C e^{-c \log^2 W}.$$  

Therefore, for any $|z| > \frac{1}{2}$

$$\|H_{12}(H_{22} - z)^{-1} H_{21}\| \leq C e^{-c \log^2 W}.$$  

Moreover, it will be proven below that

$$\|(H_{11} - z)^{-1}\| \leq C n, \quad z \in \omega_A,$$

and so for $G_{11}$ of (5.4) we have

$$\|G_{11} - (H_{11} - z)^{-1}\| \leq e^{-c \log^2 W/2}.$$  

Here we have used $W \geq n^c$. Thus we obtain by Proposition 5.2

$$\oint_{\omega_A} z^{n-1}(G_\xi(z) f, g) dz = \oint_{\omega_A} z^{n-1}((H_{11} - z)^{-1} f, g) dz + O(e^{-c \log^2 W/2}) + O(e^{-nc}), \quad (5.6)$$

where $G_\xi(z)$ is a resolvent of $K_\xi$ (see (4.1)). In view of the block structure of $H_{11}$, its resolvent also has a block structure, hence

$$\oint_{\omega_A} z^{n-1}(G_\xi(z) f, g) dz = \oint_{\omega_A} z^{n-1}(G_{\xi,\pm}(z) f, g) dz + \oint_{\omega_A} z^{n-1}(G_{\xi,+}(z) f, g) dz + \oint_{\omega_A} z^{n-1}(G_{\xi,-}(z) f, g) dz = I_{\xi,\pm} + I_{\xi,+} + I_{\xi,-}, \quad (5.7)$$

where

$$G_{\xi,\pm} = (K_{\xi,\pm} - z)^{-1}, \quad G_{\xi,+} = (K_{\xi,+} - z)^{-1}, \quad G_{\xi,-} = (K_{\xi,-} - z)^{-1}$$

and $f, f, f, g, g, g, g$ are projections of $f$ and $g$ onto the subspaces corresponding to $K_{\xi,\pm}, K_{\xi,+}, K_{\xi,-}$. One can perform similar analysis for $K$ instead of $K_\xi$ and define $I_\pm, I_+, I_-$.

In the next sections we are going to study each integral $I_{\xi,\pm}, I_{\xi,+},$ and $I_{\xi,-}$ separately. It will be shown below (see Section 7) that that $I_{\xi,+}$ and $I_{\xi,-}$ are exponentially small comparable to $I_{\xi,\pm}$, so the main task is to study $I_{\xi,\pm}$.

6 Analysis of $I_{\xi,\pm}$

As was mentioned in Section 4, to analyze $K_\pm$ and $K_{\xi,\pm}$ we are going to use the polar decomposition (4.6)–(4.10).

We start with the analysis of operator $A_n$ of (4.9) in the domain $\tilde{\Omega}_\pm$ of (5.1).
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To this end, we are going to consider quadratic approximation of $A(a, a')$ defined in (4.9). Make a change of variables

$$a_{11} = a_+(1 + i\theta_+\bar{a}_{11}/\sqrt{W}), \quad a_{21} = a_-(1 + i\theta_-\bar{a}_{21}/\sqrt{W}), \quad (6.1)$$

where $\theta_\pm$ are some complex constants with $|\theta_\pm| = 1$ which will be determined later (see (6.5)). Notice that the Jacobian of (6.1) is a constant depending on $n, W$ but not on $\zeta$, thus it does not contribute to $C_{n,W}$ (see (4.4)). Define

$$A^+(\bar{a}, \bar{a}') = 1_{\Omega_+} A \left( a_+ (1 + i\theta_+ \bar{a}/\sqrt{W}), a_+(1 + i\theta_+ \bar{a}'/\sqrt{W}) \right) 1_{\Omega_+},$$

$$A^-(\bar{a}, \bar{a}') = 1_{\Omega_-} A \left( a_-(1 + i\theta_- \bar{a}/\sqrt{W}), a_-(1 + i\theta_- \bar{a}'/\sqrt{W}) \right) 1_{\Omega_-}. \quad (6.2)$$

Then

$$K_{\xi,\pm}(a_1, a_2, Q; a_1', a_2', Q') = A^+(\bar{a}_1, \bar{a}_1') A^-(\bar{a}_2, \bar{a}_2') K_+(t, Q, Q') e^{-\frac{i\xi}{\sqrt{2}}} (\nu(a_1-a_2, Q) + \nu(a_1'-a_2', Q')). \quad (6.3)$$

Since $\varphi_0'(a_+) = c_+$ (see (4.10) and (1.1)), it is easy to see that the kernel $A^+$ of (6.2) takes the form

$$A^+(\bar{a}, \bar{a}') = A^+(\bar{a}, \bar{a}) (1 + W^{-1/2} \tilde{\rho}_+(\tilde{a})(1 + W^{-1/2} \tilde{\rho}_+(\tilde{a}')) + O(e^{-c\log^2 W}), \quad (6.4)$$

$$A^+(\bar{a}, \bar{a}') = \frac{a_+\theta_+}{\sqrt{2\pi}} \exp \left\{ (a_+ \theta_+)^2 [\beta(\bar{a} - \bar{a}')^2/2 - c_+ \bar{a}^2/4 - c_+ (\bar{a}')^2/4] \right\}$$

$$\tilde{\rho}_+(\tilde{a}) = ic_{\xi} \bar{a}^3 - ic_{\xi} \bar{a} W^{-1/2} - ic_{\xi} \bar{a}^2 W^{-1} + \ldots$$

where the coefficients $c_{\xi}, c_+ , \ldots$ are expressed in terms of the derivatives of $\varphi_0$ at $a_+$. Similarly $A^-$ of (6.2) can be approximated via $A_-^*$ defined similarly to $A_+^*$ in (6.4).

It is easy to check that for $\beta < 1/4$ the real parts of the eigenvalues $a_{1,+}, a_{2,+}$ of the quadratic form

$$\left( \begin{array}{cc} a_+^2 (\frac{\xi_+}{\xi_0} - \beta) & a_+^2 \beta \\ a_+^2 \beta & a_+^2 (\frac{\xi_0}{\xi_+} - \beta) \end{array} \right)$$

in the exponent of $A_+^*$ of (6.4) are positive. Same is true for $A_-^*$. Denote

$$\theta_\pm = (|\kappa_\pm|/\kappa_\pm)^{1/2}, \quad |\kappa_\pm| = (a_{1,\pm} a_{2,\pm})^{1/2} = a_0^2 (c_0^2/4 - \beta c_\pm)^{1/2}, \quad (6.5)$$

with $c_\pm$ of (1.1). Notice that $\theta_\pm$ is defined in such a way that

$$\Re(\theta_\pm^2 \alpha_{1,\pm}) > 0, \quad \Re(\theta_\pm^2 \alpha_{2,\pm}) > 0.$$

Now introduce the orthonormal bases

$$\psi^\pm_k(\bar{a}) = |\kappa_\pm|^{1/4} H_k(|\kappa_\pm|^{1/2} \bar{a}) e^{-|\kappa_\pm| \bar{a}^2/2}. \quad (6.6)$$

where $\{H_k(x)\}$ are Hermite polynomials which are orthonormal with the weight $e^{-x^2}$:

$$H_k(x) = (2^{k-1/2}k!\sqrt{2\pi})^{-1/2} e^{x^2} \left( \frac{d}{dx} \right)^k e^{-x^2}. \quad (6.7)$$

Below we will need the following lemma

Lemma 6.1.  (i) Let $\kappa_+, \kappa_-$ be defined as in (6.5). Then the matrices of the operators $A_+^*$ and $A_-^*$ are diagonal in the basis $\{\psi_+^k\}$ and $\{\psi_+^k\}$, and corresponding eigenvalues have the form

$$
\lambda_k^\pm = \lambda_k(A_\pm^k) = \lambda_0^\pm \cdot q_k^\pm, \quad k = 0, 1, 2 \ldots
$$

with

$$
\begin{align*}
\lambda_0^\pm &= (\kappa_+/a_\pm^2 + c_\pm/2 - \beta)^{-1/2}, \\
q_\pm &= \frac{\beta}{\kappa_+/a_\pm^2 + c_\pm/2 - \beta}, \quad |q_\pm| < 1.
\end{align*}
$$

Notice that $|q_\pm| < 1$ implies

$$
|\lambda_0^\pm| \leq \beta^{-1/2}.
$$

The matrices of operators $A^+$ and $A^-$ of (6.2) have the form

$$
\begin{align*}
(A_\pm^k)_{k,k} &= \lambda_0^\pm \cdot q_k^\pm + O(1/W), \\
(A_\pm^k)_{k,k'} &= O(W^{-1/2})(\delta_{|k-k'|,1} + \delta_{|k-k'|,3}) + O(W^{-1})\delta_{|k-k'|,2} + O(W^{-1-k-k'}), \quad k \neq k'.
\end{align*}
$$

(ii) The eigenvalues of operator

$$
A_\pm = 1_{\tilde{\Omega}_\pm} (\lambda_0(t)A_\pm) 1_{\tilde{\Omega}_\pm}
$$

are $\lambda_0^+ \lambda_0^- q^k q^l + O(1/W)$, $k, l = 0, 1, \ldots$ and they are solutions of the equation

$$
(A_\pm)_{0,0} - z = (A_\pm)^{(12)}((A_\pm)^{(22)} - z)^{-1}(A_\pm)^{(21)} = 0,
$$

where

$$
A_\pm = \begin{pmatrix} A_{00} & A^{(12)} \\ A^{(21)} & A_{22} \end{pmatrix}
$$

according to the decomposition $\{\psi_+^k, \psi_-^k\} = \{\psi_0^k, \psi_0^-\} \oplus \{\psi_+^k, \psi_-^k\}_{k \neq 0}$ with $k = (k_1, k_2)$. Here $\lambda_0(t)$ is the top eigenvalue of $K_+(t, Q, Q')$ (see (4.14)).

The top eigenvalue of $K_\pm$ has the form

$$
\lambda_0(K_\pm) = \lambda_0(A_\pm) = \lambda_0^+ \lambda_0^- + O(1/W).
$$

Proof. To simplify formulas, we consider the kernel (see (6.4)–(6.5))

$$
M(x, y) = a_+(2\pi)^{-1/2} e^{-\langle A_\pm x, x \rangle/2}, \quad A = \begin{pmatrix} \mu & \nu \\ \nu & -\mu \end{pmatrix}, \quad \lambda_\pm = \mu \pm \nu, \quad \Re \lambda_\pm > 0.
$$

Then, taking $\kappa = \sqrt{\mu^2 - \nu^2} = \sqrt{\lambda_+ \lambda_-}$, we obtain

$$
a_+(2\pi)^{-1/2} \int e^{-\langle A_\pm x, x \rangle/2 + \kappa y^2/2} \left( \frac{d}{dy} \right)^k e^{-\kappa y^2} dy
\begin{align*}
&= q^k \cdot a_+(\mu + \kappa)^{-1/2} e^{\kappa x^2/2} \left( \frac{d}{dx} \right)^k e^{-\kappa x^2}, \\
&= q \frac{\nu}{\mu + \kappa},
\end{align*}
$$

so $e^{\kappa y^2/2} \left( \frac{d}{dy} \right)^k e^{-\kappa y^2}, \quad k = 0, 1, \ldots$ are the eigenvectors of $M$. Since $M$ is compact, we have $|q| < 1$. Notice also that

$$
a_+(\mu + \kappa_\pm)^{-1/2} = \lambda_0^\pm.
$$
Now if we change the variables
\[ x_1 = \theta x, \quad y_1 = \theta y, \quad \theta = e^{-i(\arg \lambda_+ + \arg \lambda_-)/4} = e^{-i\arg \kappa/2}, \]
then for the new matrix \( \hat{A} = \theta^2 A \) has eigenvalues \( \theta^2 \lambda_+, \theta^2 \lambda_- \), whose real parts are still positive, \( \tilde{\kappa} = |\kappa| \), and \( \tilde{q} = q \). This finishes the proof of (6.7)–(6.8).

Formula (6.10) follows directly from (6.4) and the fact that the Gaussian integral of \( x^{2k+1} \) is zero, and it immediately gives the statement about eigenvalues of \( A_k \) (it is easy to see that \( \lambda_0(t) \) does not change anything since it has only \( \tilde{a}/W^{3/2} \) and \( \tilde{a’}/W^{3/2} \)).

Equation (6.12) can be obtained from the standard Schur inversion formula. The rest of part (ii) follows directly from (i) and Proposition 4.2.

Now we are going to normalize \( K_\pm, K_{\xi,\pm} \) by \( \lambda_0(K_\pm) \):

\[ \hat{K}_\pm = \lambda_0(K_\pm)^{-1} K_\pm, \quad \hat{K}_{\xi,\pm} = \lambda_0(K_{\xi,\pm})^{-1} K_{\xi,\pm} \]

(6.13)

with \( K_{\xi,\pm} \) of (6.3). Notice that

\[ \hat{K}_\pm = \hat{A}_\pm \cdot \hat{K}_\ast \]

(6.14)

where

\[ \hat{A}_\pm = (\lambda_0(A_\pm))^{-1} A_\pm, \quad \hat{K}_\ast(t, Q, Q’) = (\lambda_0(t))^{-1} K_\ast(t, Q, Q’), \]

so both top eigenvalues of \( \hat{A}_\pm, \hat{K}_\ast \) are 1, and

\[ \hat{\lambda}_j(\hat{K}_\ast) = 1 - \frac{j(j + 3)}{tW} + O((j^2/tW)^2), \quad j = 1, 2, \ldots \]

(6.15)

Therefore, it is easy to see that all eigenvalues of \( \hat{K}_\pm, \hat{K}_{\xi,\pm} \) lie inside \( \omega_A = \{ z : |z| = 1 + A/n \} \). Thus, we get

\[ I_{\pm,\xi} = \int_{\omega_A} L_2(R^2) \times L_2(Sp(2), d\mu(Q)) = H_1 \oplus H_2, \]

(6.16)

and write

\[ \hat{K}_\pm = \begin{pmatrix} \hat{K}_{(11)} & \hat{K}_{(12)} \\ \hat{K}_{(21)} & \hat{K}_{(22)} \end{pmatrix}, \quad \hat{K}_{\xi,\pm} = \begin{pmatrix} \hat{K}_{(11)} & \hat{K}_{(12)} \\ \hat{K}_{(21)} & \hat{K}_{(22)} \end{pmatrix} \]

(6.17)

according to this decomposition. We will need the following simple lemma

**Lemma 6.2.** Given decomposition (6.18), we have

\[ \hat{K}_{(12)} = \hat{K}_{(21)} = 0, \quad \| \hat{K}_{(12)} \| \leq C/n, \quad \| \hat{K}_{(21)} \| \leq C/n, \]

(6.18)

and for \( |z| \geq 1 + A/n \) with big enough \( A \) we have

\[ \| (\hat{K}_{(11)} - z)^{-1} \| \leq Cn, \]

(6.19)

\[ \| (\hat{K}_{(22)} - z)^{-1} \| \leq CW/l^2, \]

(6.20)

and same is valid for \( \hat{K}_\ast \).
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Proof of Lemma 6.2. The bound (6.19) follows from the block-diagonal structure of \( \hat{K}_{\pm} \) with respect to the basis (6.16) (see (4.15)), and the fact that \( \hat{K}_{\pm, \xi} \) is 1/n-order perturbation of \( \hat{K}_{\pm} \).

The bound (6.20) follows from

\[
\| \hat{K}^{(11)}_{\xi} \| \leq 1 + c/n,
\]

since for big enough \( A \)

\[
\| (\hat{K}^{(11)}_{\xi} - z)^{-1} \| \leq |z|^{-1} \sum_{k=0}^{\infty} \left( \frac{\| \hat{K}^{(11)}_{\xi} \|}{|z|} \right)^k \leq Cn.
\]

Similarly, according to (4.14)–(4.15), we get

\[
\| \hat{K}^{(22)}_{\xi} \| \leq 1 - \frac{C l (l + 3)}{W}
\]

and, since \( l^2/W \sim \log^2 n/n \) for \( W \geq Cn \) and \( l^2/W \gg n^{-1} \) for \( W \ll n \),

\[
\| \hat{K}^{(22)}_{\xi} \| \leq 1 - \frac{C l^2}{W},
\]

which implies (6.21). \( \square \)

The next step is to prove that we can consider only the upper-left block \( \hat{K}^{(11)}_{\xi} \) of \( \hat{K}_{\xi} \) (see (6.18)). More precisely, we are going to prove

**Lemma 6.3.** We have

\[
\int_{\omega_A} z^{-n-1}(\hat{G}_{1, \xi}(z)f, \tilde{f})dz = \int_{\omega_A} z^{-n-1}((\hat{G}_{1, \xi}^{(11)}(z)^{-1}f_{11}, \tilde{f}_{11}) - O\left( W \log n \right)) dz + O\left( l^2 n \right),
\]

where

\[
\hat{G}_{1, \xi}(z) = (\hat{K}^{(11)}_{\xi} - z)^{-1},
\]

and we decomposed \( f = (f_{11}, f_{21}) \) with respect to the decomposition (6.17). Notice that

\[
\frac{W \log n}{l^2 n} \leq \frac{1}{\log n}.
\]

**Proof.** Using the well-known Schur inversion formula we get

\[
(\hat{K}_{\xi} - z)^{-1} = \begin{pmatrix}
\hat{G}_{1, \xi}^{(11)} & -\hat{G}_{1, \xi}^{(12)}
\hat{G}_{2, \xi}^{(21)} & \hat{G}_{2, \xi}
\end{pmatrix}
\]

where

\[
\hat{G}_{2, \xi}(z) = (\hat{K}^{(22)}_{\xi} - z)^{-1},
\]

\[
\hat{G}_{1, \xi}(z) = (\hat{K}^{(11)}_{\xi} - z) - \hat{K}^{(12)}_{\xi} \hat{K}^{(21)}_{\xi} (\hat{K}^{(11)}_{\xi} - z)^{-1} = (1 - \hat{G}_{1, \xi}^{(12)} \hat{G}_{2, \xi} \hat{K}_{\xi}^{(21)})^{-1} \hat{G}_{1, \xi}.
\]

Thus

\[
\int_{\omega_A} z^{-n-1}((\hat{K}_{\xi} - z)^{-1}f, \tilde{f})dz = \int_{\omega_A} z^{-n-1}((\hat{G}_{1, \xi}^{(11)}(z)^{-1}f_{11}, \tilde{f}_{11}) - (\hat{G}_{1, \xi}^{(11)}(z)^{-1} \hat{G}_{2, \xi} f_{21}, \tilde{f}_{21})) dz + O\left( l^2 n \right) \quad (6.22)
\]

\[
- \int_{\omega_A} z^{-n-1}(\hat{G}_{2, \xi} \hat{K}_{\xi}^{(21)}(z)^{-1}f_{12}, \tilde{f}_{12})dz
\]

\[
+ \int_{\omega_A} z^{-n-1}((\hat{G}_{2, \xi} + \hat{G}_{2, \xi} \hat{K}_{\xi}^{(21)}(z)^{-1}f_{21}, \tilde{f}_{21})) dz.
\]
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Denoting
\[ R = (1 - \hat{G}_{1,\xi} \hat{\mathcal{K}}^{(12)}_{\xi} \hat{G}_{2,\xi} \hat{\mathcal{K}}^{(21)}_{\xi})^{-1}, \]
we get
\[ \hat{G}^{(11)}_{\xi} = R \hat{G}_{1,\xi}. \]
According to (6.19)–(6.21), we obtain
\[ \| \hat{G}_{1,\xi} \hat{\mathcal{K}}^{(12)}_{\xi} \hat{G}_{2,\xi} \hat{\mathcal{K}}^{(21)}_{\xi} \| \leq C n \frac{W}{t^2} = \frac{CW}{t^2 n}. \]
Therefore,
\[ \| 1 - R \| \leq \frac{CW}{t^2 n}, \]
which together with (6.20) implies
\[ \left| \int_{\omega_A} z^{n-1} \left( (\hat{G}^{(11)}_{\xi} - \hat{G}_{1,\xi}) f_1, f_1 \right) \right| = \left| \int_{\omega_A} z^{n-1} \left( (1 - R)G_{1,\xi} f_1, f_1 \right) \right| \leq C \| 1 - R \| \cdot \| f_1 \|^2 \cdot \int_{\omega_A} \frac{|dz|}{|z - 1|} \leq \frac{CW \log n}{t^2 n}. \]
It is easy to see also that
\[ \| f_2 \| \leq C/n, \]
and because of the consideration above
\[ \| \hat{G}_{2,\xi} + \hat{G}_{2,\xi} \hat{\mathcal{K}}^{(21)}_{\xi} \hat{G}^{(11)}_{\xi} \hat{G}^{(12)}_{\xi} \hat{G}_{2,\xi} \| \leq CW/t^2, \]
\[ \| \hat{G}_{2,\xi} \hat{\mathcal{K}}^{(21)}_{\xi} \hat{G}^{(11)}_{\xi} \| \leq CW/t^2, \quad \| \hat{G}^{(11)}_{\xi} \hat{\mathcal{K}}^{(12)}_{\xi} \hat{G}_{2,\xi} \| \leq CW/t^2, \]
so other terms in (6.22) are also small. □

The next step is to show that we can consider only the projection of \( \hat{K}^{(11)}_{\xi}, \hat{\mathcal{K}}^{(11)}_{\xi} \) on the linear span of \( \{ \Psi_{0,j} \}_{j \leq l} \) (see (6.16)). We prove

**Lemma 6.4.** Let \( P_l \) be the projection on \( \{ \phi_j \}_{j=0}^{l-1} \) of (4.11), \( \Delta_l = P_l \Delta P_l \), and \( \hat{\nu}_l = P_l \hat{\nu} P_l \) with \( \hat{\nu} \) defined in (1.9). Then
\[ \int_{\omega_A} z^{n-1} (\hat{G}_{1,\xi}(z) f_1, f_1) \, dz = O \left( \frac{(l - 1)^2 n}{W^{3/2}} \right) + O \left( \frac{1}{W^{1/2}} \right) \]
\[ + \int_{\omega_A} \zeta^{n-1} \left( \left( P_l - \frac{1}{t_s W} \Delta_l - \frac{i \pi \xi}{n} \hat{\nu}_l - \zeta + O \left( \frac{(l - 1)^4}{W^2} \right) \right)^{-1} f_0, f_0 \right) d\zeta, \]
where \( O(x) \) is an operator whose norm is bounded by \( Cx \) which does not depend on \( \zeta \), and
\[ f_0 = (f, \Psi_0). \]
Recall \( l = 1 \) for \( W \ll n \) and \( (l - 1)^2 n/W^{3/2} \sim \log^2 n/W^{1/2} \) for \( W \geq Cn \), and \( t_s = (2\pi \rho(E))^2 \).
A similar formula is true for \( \hat{G}_1 \) (i.e. for \( \hat{K}^{(11)}_{\xi} \) instead of \( \hat{\mathcal{K}}^{(11)}_{\xi} \)).

**Proof.** Write \( \hat{K}^{(11)}_{\xi} - z, \hat{\mathcal{K}}^{(11)}_{\xi} - z \) in the block form
\[ \hat{K}^{(11)}_{\xi} - z = \begin{pmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{pmatrix}, \quad \hat{\mathcal{K}}^{(11)}_{\xi} - z = \begin{pmatrix} M_{1,\xi} & M_{1,2,\xi} \\ M_{2,1,\xi} & M_{2,\xi} \end{pmatrix} \]
according to decomposition
\[ H_1 = M_1 \oplus M_2, \]
where $\mathcal{M}_1$ is a linear span of $\Psi_{0}^{(l)} = \{\Psi_{0,j}\}_{j \leq l}$ (see (6.16)).

Set
\[ G_1^0(z) = (M_{11}z - M_{12}z M_{22}z M_{21}z)^{-1}, \quad \tilde{G}_1^0(z) = (M_{11} - M_{12} M_{21}z)^{-1}. \]

Then, using Proposition 5.2, we get
\[ \int \omega A z^{n-1}(G(z)f_1, \tilde{f}_1)dz = \int \omega A z^{n-1}(G_1^0(z), g^{(1)}(z))dz + O(e^{-nc}), \]
\[ \int \omega A z^{n-1}(G(z)f_1, \tilde{f}_1)dz = \int \omega A z^{n-1}(\tilde{G}_1^0(z), g^{(1)}(z))dz + O(e^{-nc}), \]

where $f^{(1)}$, $g^{(1)}$ are defined as in (5.5).

Recall that
\[ \tilde{K}^{(1)} - z = (\hat{A}P_l \hat{K}_* P_l \Psi_{0}^{(l)}, \Psi_{0}^{(l)}) - z. \]

Set
\[ P_l \hat{K}_* P_l = P_l - \tilde{K}_l. \]

Then $\tilde{K}_l$, according to (6.15), is a diagonal matrix with eigenvalues
\[ \tilde{\lambda}_j(t) = j(j + 3)/tW + O((j^2/tW)^2), \quad j = 0, \ldots, l - 1. \]

Since (4.3) and (6.1) imply
\[ t = \left( a_+ + \frac{\theta + a_+ \frac{\pi}{2}}{\sqrt{W}} \right) \left( a_+ - \frac{\theta - a_+ \frac{\pi}{2}}{\sqrt{W}} \right), \]

$\tilde{K}_l$ can be rewritten as
\[ \tilde{K}_l = \Delta_l/t_* W + O_{a} \left( \frac{(l - 1)^2}{W^{3/2}} \right) + O \left( \frac{(l - 1)^4}{W} \right) \]

with $t_* = (a_+ - a_-)^2 = (2\pi \rho(E))^2$. Here $O_{a}(X)$ is a diagonal in $\{\phi_j\}_{j=0}^{l-1}$ operator of the type $O(X)$ whose eigenvalues are linear in $a, a'$.

Now, since, according to Lemma 6.1, $\hat{A}_{00} = 1 + O(1/W)$, substituting (6.23), we get
\[ (\hat{A}\tilde{K}_l \Psi_{0}^{(l)}, \Psi_{0}^{(l)})_{jj} = (\tilde{\lambda}_j(t)\hat{A}\psi_j^+ \psi_j^-, \psi_j^+ \psi_j^-) = \frac{j(j + 3)}{t_* W} \cdot (1 + O \left( \frac{j^2}{W} \right)) \]

Therefore,
\[ \tilde{K}^{(1)} - z = \hat{A}_{00} P_l - z - \Delta_l/t_* W + O((l - 1)^4/W^2). \]

Similarly
\[ M_{12} = \hat{A}_{12} \otimes P_l + O \left( \frac{(l - 1)^2}{W^{3/2}} \right), \]
\[ M_{21} = \hat{A}_{21} \otimes P_l + O \left( \frac{(l - 1)^2}{W^{3/2}} \right), \]
\[ M_{22} = \hat{A}_{22} \otimes P_l - z + O \left( \frac{(l - 1)^2}{W} \right). \]

Notice also that because of Lemma 6.1
\[ \|\hat{A}_{12}\| \leq W^{-1/2}, \quad \|\hat{A}_{21}\| \leq W^{-1/2}, \quad \|\hat{A}_{22} - z\|^{-1} \leq C. \]
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Hence

\[ M_{11} - M_{12}M_{22}^{-1}M_{21} = A_{00}P_1 - \Delta_l/t_sW - z - \hat{A}_{12}(\hat{A}_{22} - z)^{-1}\hat{A}_{21}P_1 + O_z\left(\frac{(l-1)^2}{W^2}\right) + O\left(\frac{(l-1)^4}{W^2}\right) \]

where

\[ g_1(z) = \hat{A}_{00} - 1 - \hat{A}_{12}(\hat{A}_{22} - z)^{-1}\hat{A}_{21} + O_z((l-1)^2/W) - O_z((l-1)^2/W) \]

is analytic and bounded in \( \{ z : |z - 1| < \delta \} \) for small enough \( \delta \) (recall \( l^2/W \leq \log^2 n/n \) and \( \|\hat{A}_{22}\| \leq |q_{\pm}| < 1 \) with \( q_{\pm} \) of (6.8) according to Lemma 6.1). Here \( O_z(\cdot) \) is an operator of type \( O(\cdot) \) which may depend on \( z \), and \( O_z(\cdot) \) is an operator \( O_z(\cdot) \) with substitution \( z = 1 \). Lemma 6.1 implies also

\[ g_1(1) = 0. \tag{6.24} \]

Now set

\[ \zeta(z) = z + W^{-1}g_1(z). \tag{6.25} \]

Since \( g_1(z) \) has a bounded derivative in \( \{ z : |z - 1| < \delta \} \), we get

\[ \zeta'(z) = 1 + O(1/W), \]

and the Implicit Function Theorem implies that there exists the inverse function \( z(\zeta) \) with a derivative of order \( 1 + O(1/W) \). In addition, by (6.24), \( \zeta(1) = 1 \), so the image of \( \{ z : |z - 1| < \delta \} \) lies in \( \{ \zeta : \delta/2 < |\zeta - 1| < 2\delta \} \), and it is easy to show that

\[ z(\zeta) = \zeta + W^{-1}\tilde{g}_1(\zeta), \]

where \( \tilde{g}_1 \) is a bounded analytic in \( \{ \zeta : |\zeta - 1| < 2\delta \} \), and \( \tilde{g}_1(1) = 0 \).

Now we consider the contour \( \tilde{\omega}_A = \{ z : \text{dist}\{ z : (1 - C(l-1)(l+2)/W; 1) \} \leq A/n \} \) and the contour \( \mathcal{L}_2 = \{ |z| \leq \frac{|q_{\pm}| + 1}{2} < 1 \} \) with \( q_{\pm} \) of (6.8). It is easy to see that \( \tilde{\omega}_A \cup \mathcal{L}_2 \) encircles all the eigenvalues of \( \hat{K}^{(11)}, \hat{K}_z^{(11)} \) (see (4.14) – (4.15) and Lemma 6.1). But for \( z \in \mathcal{L}_2 \)

\[ |z|^{n-1} \leq e^{-cn}, \]

so the contribution of the integral over \( \mathcal{L}_2 \) is small, and we need to consider integral over \( \tilde{\omega}_A \) only. It follows from (6.24)–(6.25) and the consideration above that \( \zeta(z), z \in \tilde{\omega}_A \) will be inside \( \tilde{\omega}_2A \), and so, since \( l = 1 \) for \( W \ll n \) and \( l^2/W \sim \log^2 n/n \) for \( W \geq Cn \), we get

\[ z(\zeta) = \zeta + O((l-1)^2/W^2) + O(1/nW), \]

hence

\[ z^{n-1} = \begin{cases} \zeta^{n-1} + O(\log^2 n/W), & W \geq Cn, \\ \zeta^{n-1} + O(1/W), & W \ll n. \end{cases} \tag{6.26} \]

Notice also that for \( z \in \tilde{\omega}_A \)

\[ \| (P_l - \Delta_l/t_sW \zeta(z))^{-1} \| \leq Cn, \]

thus

\[ \| G_l^2 \| = \| (P_l - \Delta_l/t_sW \zeta(z) + O((l-1)^4/W^2))^{-1} \| \leq \| (P_l - \Delta_l/t_sW \zeta(z))^{-1} \| \]

\[ \times \| (1 + O((l-1)^4/W^2) \cdot (P_l - \Delta_l/t_sW \zeta(z))^{-1}) \| \leq Cn. \]
Hence, recalling \( l = 1 \) and \( \|\tilde{\omega}_A\| = C/n \) for \( W \ll n \) and \( |\tilde{\omega}_A| \leq C(l - 1)^2/W \) for \( W \geq Cn \), and \( \|f^{(1)}(z) - f_0\| \leq C/\sqrt{W} \), we obtain

\[
\int_{\omega_A} z^{-1} (G_1^0(z) f^{(1)}(z), g^{(1)}(z)) \, dz = \int_{\omega_A} z^{-1} (G_1^0(z) f_0, f_0) \, dz + O((l - 1)^2 n/W^{3/2}) + O(1/W^{1/2}).
\]

According to (6.26), this can be further transformed as

\[
\int_{\omega_A} z^{-1} (G_1^0(z) f_0, f_0) \, dz = O \left( \frac{(l - 1)^4 n^2}{W^3} \right) + O \left( \frac{1}{W} \right) + \int_{\zeta(\omega_A)} \zeta^{-1} ((P_1 - \Delta_k \bar{t}, W - \zeta + O((l - 1)^4/W^2))^{-1} f_0, f_0) \, d\zeta,
\]

and the contour now can be changed back to \( \omega_A \) (notice \( l^4 n^2/W^3 = \log^4 n/W, l^4/W^2 = \log^2 n/W \) for \( W \geq Cn \).

In order to perform the same analysis for \( \tilde{K}_k^{(1)} \) notice that

\[
\|M_{12}\| \leq C/\sqrt{W}, \quad M_{12,k} = M_{12} + O \left( \frac{1}{n\sqrt{W}} \right);
\]

\[
\|M_{21}\| \leq C/\sqrt{W}, \quad M_{21,k} = M_{21} + O \left( \frac{1}{n\sqrt{W}} \right);
\]

\[
\|M_{22}^{-1}\| \leq C, \quad M_{22,k} = M_{22} + O \left( \frac{1}{n} \right),
\]

and

\[
M_{11,k} = (AP_k F_{k,k} K_k F_{k-k} P_l \Psi_0, \Psi_0) - z = M_{11} - A_{00} \cdot \frac{i \pi \xi}{n} P_l \bar{\nu}_l + O \left( \frac{1}{n\sqrt{W}} \right).
\]

Thus, since \( A_{00} = 1 + O(1/W) \), we have

\[
M_{1,k} - M_{12,k} M_{22,k}^{-1} M_{21,k} = M_{11} - M_{12} M_{22}^{-1} M_{21} - \frac{i \pi \xi}{n} P_l \bar{\nu}_l + O \left( \frac{1}{n\sqrt{W}} \right),
\]

and hence we can apply same consideration as above.

\[\Box\]

Now let us analyze

\[
\int_{\omega_A} \zeta^{-1} ( (P_1 - 1/\bar{t}_j \bar{\nu}_1 - \zeta + O((l - 1)^4/W^2))^{-1} f_0, f_0) \, d\zeta.
\]

- **Localized regime:** \( W \ll n \). In this regime \( l = 1 \), so we need to study

\[
\int_{\omega_A} \zeta^{-1} ( (P_1 - \frac{i \pi \xi}{n} \bar{\nu}_1 - \zeta)^{-1} f_0, f_0) \, d\zeta = -2\pi i \cdot \|f_0\|^2 ( (P_1 - \frac{i \pi \xi}{n} \bar{\nu}_1)^{-1} 1_{1,1}).
\]

But since \( \phi_0 = 1 \) and \( \bar{\nu}_1 = P_l \bar{\nu}_1, \bar{\nu} \cdot 1 = \phi_1 \) (see (1.9) and Proposition 4.2), we obtain

\[
(P_1 - \frac{i \pi \xi}{n} \bar{\nu}_1) 1 = 1 - \frac{i \pi \xi}{n} P_l \phi_1 = 1,
\]

which implies

\[
((P_1 - \frac{i \pi \xi}{n} \bar{\nu}_1)^{-1} 1) = 1,
\]

thus Theorem 1.1 in the regime \( W \ll n \).
• **Critical regime:** \( n = C_* W \).

Again we need to study \( (K_{01}^{n-1}, 1) \) with

\[
K_0 = P_l - \frac{1}{t_* W} \Delta_l - \frac{i \pi \xi}{n} \hat{\nu}_l + O(l^4/W^2) = P_l - \frac{C^*}{n} \Delta_l - \frac{i \pi \xi}{n} \hat{\nu}_l + O(\log^4 n/n^2),
\]

where \( C^* = C_*/t_* \). It is enough to prove

**Lemma 6.5.** Given (4.18), if \( n = C_* W \), \( l = \lceil \log W \rceil \) we have

\[
(K_{01}^{n-1}, 1) \to (e^{-C^* \Delta - i \xi \hat{\nu}_1}, 1), \quad n, W \to \infty,
\]

with \( \hat{\nu}, \Delta \) as in Theorem 1.1.

Similar Lemma is proved in [32], but for the sake of completeness we repeat the proof here.

**Proof of Lemma 6.5.** Notice that

\[
K_0 = P_l - n^{-1} C^* \Delta_l - \frac{i \pi \xi}{n} \hat{\nu}_l + O(\log^4 n/n^2) = P_l e^{-n^{-1}(C^* \Delta_l + i \xi \hat{\nu}_l)} + O(\log^4 n/n^2) P_l.
\]

Thus

\[
K_0^{n-1} = P_l e^{-C^* \Delta_l - i \xi \hat{\nu}_l} P_l + O(\log^4 n/n),
\]

so

\[
(K_0^{n-1}, 1) = (e^{-C^* \Delta_l - i \xi \hat{\nu}_l}, 1) + O(\log^4 n/n).
\]

Consider the basis \( \{ \phi_j \} \) of (4.11). In this basis the Laplace-Bertamini operator \( \Delta \) is diagonal, and the operator \( \hat{\nu} \) is three diagonal (since it corresponds to the multiplication by \( 2x^2 - 1 \), see (1.9) and (4.12)). To simplify notations, let \( F \) be an operator of multiplication by \( (i \pi \xi \nu) \) and \( \tilde{\Delta} = C^* \Delta \). Set

\[
D = \tilde{\Delta} + F, \\
D^{(l)} = \Delta + F^{(l)},
\]

where \( F^{(l)} \) be the matrix \( F \) where we put \( F_{l-1,l} = F_{l,l-1} = 0 \). It is evident that (recall \( \phi_0 = 1 \))

\[
(e^{-D^{(l)}} \phi_0, \phi_0) = (e^{-P_l D P_l} \phi_0, \phi_0) = (e^{-P_l (C^* \Delta_l + i \xi \hat{\nu}_l)} P_l, 1).
\]

Thus, we are left to prove that

\[
\left( (e^{-D} - e^{-D^{(l)}}) \phi_0, \phi_0 \right) \to 0.
\]

(6.28)

Notice that both \( e^{-D} \) and \( e^{-D^{(l)}} \) are bounded operators, and \( |F| \leq C, |F^{(l)}| \leq C \). We will use the well-known Duhamel formula

\[
e^{-tA_1} - e^{-tA_2} = \int_0^t e^{-(t-s)A_2} (A_1 - A_2) e^{-sA_1} ds.
\]

(6.29)
For $A_1 = D$, $A_2 = D^{(l)}$ and $t = 1$ it gives
\[
\left| (e^{-D} - e^{-D^{(l)}}) \phi_0 \right| = \left| \int_0^1 e^{-(1-s)D^{(l)}} (F - F^{(l)}) e^{-sD} \phi_0 \, ds \right|
\]
\[
= \left| \int_0^1 e^{-(1-s)D^{(l)}} (F_{l-1} \cdot E_{l-1} + F_l \cdot E_{l,l-1}) e^{-sD} \phi_0 \, ds \right|
\]
\[
= \left| \int_0^1 e^{-(1-s)D^{(l)}} \left( F_{l-1} \phi_0 (e^{-sD} \phi_0, \phi_{l-1}) + F_{l-1} \phi_{l-1} (e^{-sD} \phi_0, \phi_l) \right) \, ds \right|
\]
\[
\leq C \left( \left| (e^{-sD} \phi_0, \phi_{l-1}) \right| + \left| (e^{-sD} \phi_0, \phi_l) \right| \right).
\]

Here $E_{l-1,l}$ is an operator whose matrix in the basis $\{ \phi_j \}$ has 1 at $(l-1, l)$ place and zeros everywhere else, and $E_{l,l-1}$ is defined in a similar way. $F_{l-1}, F_l$ are $(l-1, l)$ and $(l, l-1)$ elements of the matrix $F$ in the same basis.

Now let us bound $\left| (e^{-sD} \phi_0, \phi_l) \right|$. To this end, apply Duhamel’s formula (6.29) $p = [l/2]$ times with $A_1 = D$ and $A_2 = \hat{A}$. We obtain
\[
(e^{-sD} \phi_0, \phi_l) = \sum_{j=1}^p \int_{s_1 + \ldots + s_j \leq s} (e^{-s_1 \Delta_l} F e^{-s_2 \Delta_l} F \ldots e^{-s_j \Delta_l} \phi_0, \phi_l) \, ds_1 \ldots ds_j
\]
\[+ \int_{s_1 + \ldots + s_p \leq s} (e^{-s_1 \Delta_l} F e^{-s_2 \Delta_l} F \ldots e^{-s_p \Delta_l} \phi_0, \phi_l) \, ds_1 \ldots ds_p.
\]

Since $e^{-s \Delta_l}$ is diagonal in the basis $\{ \phi_j \}$, and $F$ is only three diagonal, the expression $e^{-s_1 \Delta_l} F e^{-s_2 \Delta_l} F \ldots e^{-s_j \Delta_l} \phi_0$ is in the linear span of $\{ \phi_k \}_{k=0}^j$, and thus the sum above is 0. Hence
\[
\left| (e^{-sD} \phi_0, \phi_l) \right| \leq \left| \int_{s_1 + \ldots + s_p \leq s} (e^{-s_1 \Delta_l} F e^{-s_2 \Delta_l} F \ldots e^{-s_p \Delta_l} \phi_0, \phi_l) \, ds_1 \ldots ds_p \right|
\]
\[
\leq C \left| \int_{s_1 + \ldots + s_p \leq s} \, ds_1 \ldots ds_p \right| = C \left[ \frac{s^l}{l!} \right] \leq C e^{-l \log l} \to 0,
\]

which finishes the proof of (6.28).

**• delocalized regime:** $W \gg n$. Since in this regime $l^4/W^2 = C \log^4 n/n^2$, we get
\[
\int_A \zeta^{n-1} \left( \left( P_l - \frac{C}{W} \Delta_l - \frac{i \pi \xi}{n} \hat{v}_l - \zeta + O(l^4/W^2) \right)^{-1} f_0, \tilde{f}_0 \right) d\zeta
\]
\[
= \int_A \zeta^{n-1} \left( \left( P_l - \frac{C}{W} \Delta_l - \frac{i \pi \xi}{n} \hat{v}_l - \zeta \right)^{-1} f_0, \tilde{f}_0 \right) d\zeta + O(l^4/n),
\]

Hence we need to study
\[
|f_0|^2 \int_A \zeta^{n-1} \left( \left( P_l - \frac{C}{W} \Delta_l - \frac{i \pi \xi}{n} \hat{v}_l - \zeta \right)^{-1} 1, 1 \right) d\zeta
\]

Now let us define
\[
m = \sqrt{\frac{W}{n}}, \quad m \to \infty,
\]
in order to get
\[
m^2 n / W = \frac{1}{m} \to 0.
\]
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Set
\[ G(z) = \left( P_l - \frac{C_s}{W} \Delta_l - \frac{i\pi \xi}{n} \nu_l - z \right)^{-1}, \quad G^{(m)}(z) = \left( P_m - \frac{C_s}{W} \Delta_m - \frac{i\pi \xi}{n} \nu_m - z \right)^{-1}, \]
\[ G^{(m,l)}(z) = \left( P_l - \frac{C_s}{W} \Delta_l - \frac{i\pi \xi}{n} \nu_{l,m} - z \right)^{-1}, \]
where \( \nu_{l,m} \) has the same matrix as a tridiagonal operator \( \nu_l \) but with \( (m, m+1) \) and \( (m+1, m) \) elements equal to 0, and \( P_m \) is a projection on \( \{ \phi_j \}_{j \leq m}, \Delta_m = P_m \Delta P_m, \nu_m = P_m \nu P_m. \) Notice that \( \nu_{l,m} \) has a block diagonal structure with blocks \( m \times m \) and \( (l-m) \times (l-m) \), thus
\[ (G^{(m,l)}(z)1,1) = (G^{(m)}(z)1,1). \] (6.30)

We are going to prove
\[ \int_{\omega_A} z^{n-1} (G(z) \cdot 1,1) \, dz = \int_{\omega_A} z^{n-1} (G^{(m,l)}(z) \cdot 1,1) \, dz + O(1/m). \] (6.31)
Then, if we define
\[ C_0^{(m)}(z) = \left( P_m - \frac{i\pi \xi}{n} \nu_m - z \right)^{-1}, \] (6.32)
we can write using (6.30) and the standard resolvent identity
\[ (G^{(m,l)}(z) \cdot 1,1) = (G^{(m)}(z)1,1) = (G^{(m)}(z)1,1) + (G^{(m)}(z)\left( \frac{C_s}{W} \Delta_m \right) G_0^{(m)}(z)1,1). \]
But
\[ \| \frac{C_s}{W} \Delta_m \| \leq \frac{C m^2}{W} \leq \frac{1}{mn}, \]
hence, since both resolvent can be bounded by \( |z-1|^{-1} \), we get
\[ \left| \int_{\omega_A} z^{n-1} (G^{(m)}(z)\left( \frac{C_s}{W} \Delta_m \right) G_0^{(m)}(z)1,1) \, dz \right| \leq \frac{C}{mn} \int_{\omega_A} \frac{|dz|}{|z-1|^2} \leq \frac{C}{m}, \] (6.33)
where we have used
\[ \int_{\omega_A} \frac{|dz|}{|z-1|^2} \leq Cn. \] (6.34)
Now (6.30)–(6.33) imply
\[ |f_0|^2 \int_{\omega_A} z^{n-1} (G(z)1,1) \, dz = |f_0|^2 \int_{\omega_A} z^{n-1} (G_0^{(m)}(z)1,1) \, dz + O(1/m) \]
\[ = -2\pi i \cdot |f_0|^2 \cdot \left( P_m - \frac{i\pi \xi}{n} \nu_m \right)^{n-1}(1,1) + O(1/m). \]
Since \( \nu \) is bounded, we can easily change \( \nu_m \) to \( \hat{\nu} \) and use
\[ \left( 1 - \frac{i\pi \xi}{n} \hat{\nu} \right)^{n-1} = e^{-i\pi \xi \hat{\nu}} + O(1/n), \]
which implies Theorem 1.1.

Therefore we are left to prove (6.31). First we will need a bound

**Lemma 6.6.** For \( |z| \geq 1 + A/n \) we have
\[ |G_{ij}(z)| \leq \frac{C}{|z-1|} e^{-\delta|i-j|}, \]
where \( C \) and \( \delta \) depends only on \( A \). Same is true for \( G^{(m,l)}(z) \).
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Notice that since \( \hat{\nu} \) is bounded 3-diagonal matrix (see (4.12)), and \( P_l - C_+ \Delta_t \) is diagonal, Lemma 6.6 follows from the standard Combes-Thomas arguments (see, e.g., [25], Ch 13, Proposition 13.13.1).

Using the resolvent identity we can write

\[
\int_{\varpi_A} z^{n-1} (G(z) \cdot 1, 1) dz = \int_{\varpi_A} z^{n-1} (G^{(m,l)}(z) \cdot 1, 1) dz
\]

\[
+ \int_{\varpi_A} z^{n-1} (G(z) (\frac{\delta \hat{v}}{n}) G^{(m,l)}(z) \cdot 1, 1) dz,
\]

where \( \delta \hat{v} = i \pi (\hat{v} - \hat{v}_{m,m}) \), i.e. the matrix with only two non-zero elements \((m, m+1)\) and \((m+1, m)\). Rewrite

\[
\left( G(z) (\frac{\delta \hat{v}}{n}) G^{(m,l)}(z), 1, 1 \right)
\]

\[
= \frac{(\delta \hat{v})_{m,m+1}}{n} G^{(m,l)}_{m+1,0}(z) G_{0m}(z) + \frac{(\delta \hat{v})_{m+1,m}}{n} G^{(m,l)}_{m,0}(z) G_{0,m+1}(z).
\]

But according to Lemma 6.6

\[
|G^{(m,l)}_{m+1,0}(z)| \leq \frac{C}{|z - 1| e^{-\delta_m}},
\]

and similar bounds hold for other resolvent elements in (6.36). Thus

\[
\int_{\varpi_A} |z| z^{n-1} (G(z) (\frac{\delta \hat{v}}{n}) G^{(m,l)}(z), 1, 1) dz \leq \frac{Ce^{-2\delta_m}}{n} \int_{\varpi_A} \frac{|dz|}{|z - 1|^2} \leq Ce^{-2\delta_m},
\]

where we have used (6.34). This and (6.35) yield (6.31).

7 Analysis of \( I_+ \) and \( I_- \)

Since the integrals \( I_+ \) and \( I_- \) are similar, we can consider \( I_+ \) only. In this case we will consider \( \{ F_i \} \) of Proposition 2.1 as a \( \hat{S}_p(2) \) matrix which is in \( W^{-1/2} \)-neighbourhood of \( a_+ I_4 \). Then \( F_i \) can be parametrized as \( F_i = a_+ (I + i \theta_i X_i) / \sqrt{W} \), where \( X_i \) is a quaternion Hermitian matrix

\[
X_j = \begin{pmatrix}
\hat{a}_{i,j} & \hat{w}_{j,1} & 0 & \hat{w}_{j,2} \\
\hat{w}_{j,1} & \hat{a}_{j,1} & -\hat{w}_{j,2} & 0 \\
0 & -\hat{w}_{j,2} & \hat{a}_{j,1} & \hat{w}_{j,1} \\
\hat{w}_{j,2} & 0 & \hat{w}_{j,1} & \hat{a}_{j,2} \\
\end{pmatrix}
\]

where \( \hat{w}_{j,1} = (x_j + i y_j) / \sqrt{2}, \hat{w}_{j,2} = (p_j + i q_j) / \sqrt{2} \). This change transforms the measure \( dF_i \) to

\[
\left( \frac{ia_+ \theta_i}{4} \right)^6 W^{-3} d\hat{a}_{i,j} d\hat{w}_{i,j} dx_j dy_j dp_j dq_j.
\]

We need to keep the same \( \tilde{C}_n \) as in (4.4), so in the parametrization above the operator \( K^+_n \) has the form

\[
K^+_n (X, X') = \beta^2 A^+_n (\tilde{a}, x, y, p, q; \tilde{a}', x', y', p', q')(1 + o(1)),
\]

where

\[
A^+_n (\tilde{a}, x, y, p, q; \tilde{a}', x', y', p', q') = A^+ (\tilde{a}_1, \tilde{a}'_1) A^+ (\tilde{a}_2, \tilde{a}'_2) A^+ (x, x') A^+ (y, y') A^+ (p, p') A^+ (q, q').
\]
with $A^+$ of (6.4).

Similarly to Lemma 6.1 one can get that the largest eigenvalue of $A^+_0$ is $\beta^2 (\lambda_0^+)^6 + O(W^{-1})$ (see (6.8)), and the next eigenvalue is smaller then $\beta^2 (\lambda_0^+)^6 (1 - \delta)$. Remember that we have normalization $\lambda_0(K_\pm)^{-1}$, and $\lambda_0(K_\pm) = \lambda_0^+ \lambda_0^- + O(1/W)$ (see Lemma 6.1). But, according to (6.9), $\beta |\lambda_0^+|^2 < 1$, thus

$$
\|\lambda_0(K_\pm)^{-1} K_+\| < 1 - \delta,
$$

and so

$$
I_+ = O(e^{-cn}).
$$

\section{SUSY techniques}

Here we provide the basic formulas and definitions of SUSY approach used in Section 2.

Let us consider two sets of formal variables $\{\psi_j\}_{j=1}^n, \{\overline{\psi}_j\}_{j=1}^n$, which satisfy the anticommutation conditions

$$
\psi_j \psi_k + \psi_k \psi_j = \psi_j \overline{\psi}_k + \psi_k \overline{\psi}_j = 0, \quad j, k = 1, \ldots, n. \quad (A.1)
$$

Note that this definition implies $\psi_j^2 = \overline{\psi}_j^2 = 0$. These two sets of variables $\{\psi_j\}_{j=1}^n$ and $\{\overline{\psi}_j\}_{j=1}^n$ generate the Grassmann algebra $\mathfrak{A}$. Taking into account that $\psi_j^2 = 0$, we have that all elements of $\mathfrak{A}$ are polynomials of $\{\psi_j\}_{j=1}^n$ and $\{\overline{\psi}_j\}_{j=1}^n$ of degree at most one in each variable. We can also define functions of the Grassmann variables. Let $\chi$ be an element of $\mathfrak{A}$, i.e.

$$
\chi = a + \sum_{j=1}^n (a_j \psi_j + b_j \overline{\psi}_j) + \sum_{j \neq k} (a_{j,k} \psi_j \psi_k + b_{j,k} \psi_j \overline{\psi}_k + c_{j,k} \overline{\psi}_j \psi_k) + \ldots. \quad (A.2)
$$

For any sufficiently smooth function $f$ we define by $f(\chi)$ the element of $\mathfrak{A}$ obtained by substituting $\chi - a$ in the Taylor series of $f$ at the point $a$. Since $\chi$ is a polynomial of $\{\psi_j\}_{j=1}^n, \{\overline{\psi}_j\}_{j=1}^n$ of the form (A.2), according to (A.1) there exists such $l$ that $(\chi - a)^l = 0$, and hence the series terminates after a finite number of terms and so $f(\chi) \in \mathfrak{A}$.

For example, we have

$$
\exp\{a \overline{\psi}_1 \psi_1\} = 1 + a \overline{\psi}_1 \psi_1 + (a \overline{\psi}_1 \psi_1)^2/2 + \ldots = 1 + a \overline{\psi}_1 \psi_1,
$$

$$
\exp\{a_{11} \overline{\psi}_1 \psi_1 + a_{12} \overline{\psi}_1 \psi_2 + a_{21} \overline{\psi}_2 \psi_1 + a_{22} \overline{\psi}_2 \psi_2\} = 1 + a_{11} \overline{\psi}_1 \psi_1 + a_{12} \overline{\psi}_1 \psi_2 + a_{21} \overline{\psi}_2 \psi_1 + a_{22} \overline{\psi}_2 \psi_2 + a_{11} a_{12} \overline{\psi}_1 \psi_1 \overline{\psi}_2 \psi_2 \quad (A.3)
$$

Following Berezin [2], we define the operation of integration with respect to the anticommuting variables in a formal way:

$$
\int d \psi_j = \int d \overline{\psi}_j = 0, \quad \int \psi_j d \psi_j = \int \overline{\psi}_j d \overline{\psi}_j = 1, \quad (A.4)
$$

and then extend the definition to the general element of $\mathfrak{A}$ by linearity. A multiple integral is defined to be a repeated integral. Assume also that the “differentials” $d \psi_j$ and $d \overline{\psi}_k$ anticommute with each other and with the variables $\psi_j$ and $\overline{\psi}_k$. Thus, according to the definition, if

$$
f(\psi_1, \ldots, \psi_k) = p_0 + \sum_{j_1=1}^k p_{j_1} \psi_{j_1} + \sum_{j_1 < j_2} p_{j_1,j_2} \psi_{j_1} \psi_{j_2} + \ldots + p_{1,2,\ldots,k} \psi_1 \ldots \psi_k,
$$

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then

\[ \int f(\psi_1, \ldots, \psi_k) d\psi_k \ldots d\psi_1 = p_{1,2,\ldots,k}. \]  
\[ \text{(A.5)} \]

Let \( A \) be an ordinary Hermitian matrix with positive real part. The following Gaussian integral is well-known

\[ \int \exp \left\{ -\sum_{j,k=1}^n A_{jk} z_j z_k \right\} n \prod_{j=1}^n d \Re z_j d \Im z_j = \frac{1}{\det A}. \]
\[ \text{(A.6)} \]

One of the important formulas of the Grassmann variables theory is the analog of this formula for the Grassmann algebra (see [2]):

\[ \int \exp \left\{ -\sum_{j,k=1}^n A_{jk} \psi_j \psi_k \right\} n \prod_{j=1}^n d \psi_j d \bar{\psi}_j = \det A, \]
\[ \text{(A.7)} \]

where \( A \) now is any \( n \times n \) matrix.

For \( n = 1 \) and 2 this formula follows immediately from (A.3) and (A.5).

We will also need the following proposition

**Proposition A.1** (see [21] and references therein). Let \( \psi_j = (\psi_j^1, \ldots, \psi_j^m)^t, \ j = 1, \ldots, p \) be the Grassman vectors, and let \( F \) be some function that depends only on combinations

\[ \psi^+ \psi := \left\{ \sum_{\alpha=1}^m \bar{\psi}_{j\alpha} \psi_{k\alpha} \right\}^p_{j,k=1}, \quad \psi^t \psi := \left\{ \sum_{\alpha=1}^m \psi_{j\alpha} \bar{\psi}_{k\alpha} \right\}^p_{j,k=1}, \]

and set

\[ d\Psi = \prod_{j=1}^p \prod_{\alpha=1}^m d\bar{\psi}_{j\alpha} d\psi_{j\alpha}. \]

Assume also that \( m \geq p \). Then

\[ \int F \left( \psi^+ \psi, \psi^t \psi, \psi^+ \bar{\psi}, \psi^t \bar{\psi} \right) d\Psi d\bar{\Psi} = C_{p,m} \int F(Q) \cdot \det^{-m/2} Q d\mu(Q), \]

where \( C_{p,m} \) is some constant depending on \( p \) and \( m \), \( Q \in Sp(p) \), and \( d\mu(Q) \) is a Haar measure over \( Sp(p) \).

**B Proof of Proposition 4.2**

The first part of Proposition 4.2 follows from the standard representation theory arguments and can be found e.g. in [19], Ch.5. The recurrence relation (4.12) follows from the recurrence relation for hypergeometric functions, see e.g. [1].

Notice also that operator \( \hat{\nu} \) correspond to the multiplication on \( c(2x^2 - 1) \) with \( x = \sqrt{S(Q)} \) (see (4.8)). Thus, (4.12) gives that \( \hat{\nu} \phi_0 \) is proportional to \( \phi_1 \), which implies (4.13).

To get the asymptotic expression (4.14) for the eigenvalues of \( K^* \) we need the Itzykson-Zuber formula of the integration over \( \hat{Sp}(2) \) (for the proof see, e.g., [33])

**Proposition B.1.** If \( p \neq 0 \), then

\[ \int_{\hat{Sp}(2)} \exp \{-p S(Q(Q^*)^*)\} d\mu(Q) = \frac{6}{p^3} \left( 1 - 2/p + e^{-p}(1 + 2/p) \right). \]
\[ \text{(B.1)} \]

Moreover,

\[ \int_{\hat{Sp}(2)} \exp \{i\pi \xi - 2i\pi \xi S(Q)\} d\mu(Q) = DS(\pi \xi). \]
\[ \text{(B.2)} \]
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Given (4.12), it is easy to check that $P_{2n}(0) = (-1)^n$, and the coefficient at $x^2$ of $P_{2n}$ is $(-1)^{n-1}n(n+3)/2$. Therefore

$$
\lambda_j(t) = \frac{p^2}{6} \int_{Sp(2)} \exp\{-p S(Q)\} \left(-1\right)^j P_j(Q) d\mu(Q)
$$

$$
= \frac{p^2}{6} \int_{Sp(2)} \exp\{-p S(Q)\} \left(1 - \frac{j(j+3)}{2} S(Q) + \ldots\right) d\mu(Q)
$$

$$
= 1 - \frac{2}{p} + \frac{p^2}{6} \left(-\frac{12}{p^3} \left(1 - \frac{2}{p}\right) + \frac{6}{p^2} \cdot \frac{2}{p^2}\right) \frac{j(j+3)}{2} + O((j^2/Wt)^2)
$$

$$
= 1 - \frac{(j+1)(j+2)}{Wt} + O(j^2/Wt^2)
$$

with $p = Wt$. Here we used $j(j+3) + 2 = (j+1)(j+2)$, (B.1), and

$$
\int_{Sp(2)} \exp\{-p S(Q)\} S(Q)^k d\mu(Q) = (-1)^k \left(\frac{d}{dp}\right)^k \int_{Sp(2)} \exp\{-p S(Q)\} d\mu(Q).
$$

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