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Distribution dependent SDEs for Navier-Stokes type equations*

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Abstract

To characterize Navier-Stokes type equations where the Laplacian is extended to a singular second order differential operator, we propose a class of SDEs depending on the distribution in future. The well-posedness and regularity estimates are derived for these SDEs.

Keywords: Navier-Stokes type equation; distribution dependent SDE; well-posednes.
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1 Introduction

Let $d \in \mathbb{N}$. Consider the following incompressible Navier-Stokes equation on $E := \mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d$:

$$\partial_t u_t = \kappa \Delta u_t - (u_t \cdot \nabla) u_t - \nabla \wp_t, \quad t \in [0, T] \tag{1.1}$$

with $\nabla \cdot u_t := \sum_{i=1}^d \partial_i u_t^i = 0$, where $T > 0$ is a fixed time,

$$u := (u^1, \cdots, u^d) : [0, T] \times E \to \mathbb{R}^d, \quad \wp : [0, T] \times E \to \mathbb{R},$$

and $u_t \cdot \nabla := \sum_{i=1}^d u_t^i \partial_i$. This equation describes viscous incompressible fluids, where u is the velocity field of a fluid flow, $\wp$ is the pressure, and $\kappa > 0$ is the viscosity constant.

Besides existing probabilistic characterizations on Navier-Stokes equations, see [1] and references therein, in this paper we propose a new type stochastic differential equation (SDE) depending on distributions in the future, such that the solution of (1.1) is explicitly given by the initial datum u_0 and the pressure $\wp$. By proving the well-posedness of the SDE, we derive the well-posedness of (1.1) in $\mathcal{C}_b^n (n \geq 2)$ with given pressure (which is however a part of solution in Navier-Stokes equations), see [3] for an analytic characterization on the pressure to ensure $\nabla \cdot u_t = 0$.

Indeed, we will prove a more general result for the following Navier-Stokes type equation on $E := \mathbb{R}^d$ or $E := \mathbb{R}^d/\mathbb{Z}^d$:

$$\partial_t u_t = L_t u_t - (u_t \cdot \nabla) u_t + V_t, \quad t \in [0, T], \tag{1.2}$$

where

$$L_t := \mathrm{tr}\{a_t \nabla^2\} + b_t \cdot \nabla$$

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and

$$V,\ b:[0,T]\times E\to\mathbb{R}^d,\ \ a:[0,T]\times E\to\mathbb{R}^{d\otimes d}$$

are measurable, and $a_t(x)$ is positive definite for $(t,x)\in[0,T]\times E$.

To characterize (1.2), we consider the following SDE on $\mathbb{R}^d$ where differentials are in $s\in[t,T]$:

$$\begin{aligned}
\mathrm{d}X_{t,s}^x &= \sqrt{2a_{T-s}}(X_{t,s}^x)\mathrm{d}W_s\\
&+\bigg\{b_{T-s}(X_{t,s}^x)-\bigg[\mathbb{E}u_0(X_{s,T}^y)+\mathbb{E}\int_s^T V_{T-r}(X_{s,r}^y)\mathrm{d}r\bigg]_{y=X_{t,s}^x}\bigg\}\mathrm{d}s,\\
&\quad t\in[0,T],s\in[t,T],X_{t,t}^x=x\in\mathbb{R}^d,
\end{aligned}\tag{1.3}$$

where $(W_s)_{s\in[0,T]}$ is a d-dimensional Brownian motion on a complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_s\}_{s\in[0,T]},\mathbb{P})$. When $E=\mathbb{T}^d:=\mathbb{R}^d/\mathbb{Z}^d$, by extending a function f from domain E to domain $\mathbb{R}^d$ as

$$f(x+k)=f(x),\ \ x\in[0,1)^d,k\in\mathbb{Z}^d,\tag{1.4}$$

we also have the SDE (1.3) for the case $E=\mathbb{T}^d$.

Regarding s as the present time, the SDE (1.3) depends on the distribution of $(X_{s,r})_{r\in[s,T]}$ coming from the future. So, this is a future distribution dependent equation, but is essentially different from McKean-Vlasov SDEs which depend on the distribution at present rather than future. We will use $X:=(X_{t,s}^x)_{0\le t\le s\le T,x\in E}$ to formulate the solution to (1.2).

Let $D_T:=\{(t,s):0\le t\le s\le T\}$. We define the solution X of (1.3) as follows.

Definition 1.1. A family $X:=(X_{t,s}^x)_{(t,s,x)\in D_T\times\mathbb{R}^d}$ of random variables on $\mathbb{R}^d$ is called a solution of (1.3), if $X_{t,s}^x$ is $\mathcal{F}_s$-measurable for all $x\in\mathbb{R}^d$ and $0\le t\le s\le T$, $\mathbb{P}$-a.s. continuous in (t,s,x),

$$\mathbb{E}\int_t^T\bigg\{\|a_{T-s}(X_{t,s}^x)\|+\bigg|b_{T-s}(X_{t,s}^x)-\bigg[\mathbb{E}u_0(X_{s,T}^y)+\mathbb{E}\int_s^T V_{T-r}(X_{s,r}^y)\mathrm{d}r\bigg]_{y=X_{t,s}^x}\bigg|\bigg\}\mathrm{d}s<\infty$$

for $(t,x)\in[0,T]\times\mathbb{R}^d$, and $\mathbb{P}$-a.s.

$$\begin{aligned}
X_{t,s}^x&=x+\int_t^s\sqrt{2a_{T-r}}(X_{t,r}^x)\mathrm{d}W_r\\
&+\int_t^s\bigg\{b_{T-r}(X_{t,r}^x)-\bigg[\mathbb{E}u_0(X_{r,T}^y)+\mathbb{E}\int_r^T V_{T-r}(X_{r,\theta}^y)\mathrm{d}\theta\bigg]_{y=X_{t,r}^x}\bigg\}\mathrm{d}r,\ \ (t,s,x)\in D_T\times\mathbb{R}^d.
\end{aligned}$$

We will allow the operator L_t to be singular, where the drift contains a locally integrable term introduced in [4] for singular SDEs. For any $p,q>1$ and $0\le t<s$, we write $f\in\tilde{L}_q^p(t,s)$ if $f=(f_r(x))_{(r,x)\in[t,s]\times\mathbb{R}^d}$ is a measurable function on $[t,s]\times\mathbb{R}^d$ such that

$$\|f\|_{\tilde{L}_q^p(t,s)}:=\sup_{z\in\mathbb{R}^d}\bigg(\int_t^s\|f_r1_{B(z,1)}\|_{L^p}^q\mathrm{d}r\bigg)^{\frac{1}{q}}<\infty,$$

where $B(z,1)$ is the unit ball at z, and $\|\cdot\|_{L^p}$ is the L^p-norm for the Lebesgue measure. We denote $f\in\tilde{H}_q^{2,p}(t,s)$ if $|f|+|\nabla f|+\|\nabla^2 f\|\in\tilde{L}_q^p(t,s)$. When $(t,s)=(0,T)$ we simply denote

$$\tilde{L}_q^p=\tilde{L}_q^p(0,T),\ \ \tilde{H}_q^{2,p}=\tilde{H}_q^{2,p}(0,T).$$

We will take (p,q) from the following class:

$$\mathcal{K}:=\Big\{(p,q):p,q>2,\ \frac{d}{p}+\frac{2}{q}<1\Big\}.$$

We now make the following assumption on the operator L_t.

(H) Let $b_t = b_t^{(0)} + b_t^{(1)}$, and when $E = \mathbb{T}^d$ we extend $a_t, b_t^{(0)}$ and $b_t^{(1)}$ to $\mathbb{R}^d$ as in (1.4).

(1) a is positive definite with

$$\|a\|_\infty + \|a^{-1}\|_\infty := \sup_{(t,x)\in[0,T]\times E} \|a_t(x)\| + \sup_{(t,x)\in[0,T]\times E} \|a_t(x)^{-1}\| < \infty,$$

$$\lim_{\varepsilon\to 0} \sup_{|x-y|\le\varepsilon, t\in[0,T]} \|a_t(x) - a_t(y)\| = 0.$$

(2) There exist $l \in \mathbb{N}$, $\{(p_i, q_i)\}_{0\le i\le l} \subset \mathcal{K}$ and $0 \le f_i \in \tilde{L}_{q_i}^{p_i}, 0 \le i \le l$, such that

$$|b^{(0)}| \le f_0, \quad \|\nabla a\| \le \sum_{i=1}^l f_i.$$

(3) $\|b^{(1)}(0)\|_\infty := \sup_{(t,x)\in[0,T]} |b^{(1)}(0)| < \infty$, and

$$\|\nabla b^{(1)}\|_\infty := \sup_{t\in[0,T]} \sup_{x\ne y} \frac{|b_t^{(1)}(x) - b_t^{(1)}(y)|}{|x-y|} < \infty. \tag{1.5}$$

Under this assumption, we will prove the well-posedness of (1.3) and solve (1.2) in the class

$$\mathcal{U}(p_0, q_0) := \Big\{ u : [0,T] \times E \to \mathbb{R}^d;\ \|u\|_\infty + \|\nabla u\|_\infty + \|\nabla^2 u\|_{\tilde{L}_{q_0}^{p_0}} < \infty \Big\}.$$

Recall that $W^{1,\infty}(E;\mathbb{R}^d)$ is the space of all weakly differentiable functions $f : E \to \mathbb{R}^d$ with $\|f\|_\infty + \|\nabla f\|_\infty < \infty$.

Theorem 1.1. Assume (H). Let $u_0 \in W^{1,\infty}(E;\mathbb{R}^d)$ and $\int_0^T \|V_t\|_\infty^2 \mathrm{d}t < \infty$. Then the following assertions hold.

(1) The SDE (1.3) has a unique solution $X := (X_{t,s}^x)_{(t,s,x)\in D_T\times\mathbb{R}^d}$.

(2) If u solves (1.2) and $u \in \mathcal{U}(p_0, q_0)$, then

$$u_t(x) = \mathbb{E}\bigg[u_0(X_{T-t,T}^x) + \int_{T-t}^T V_{T-s}(X_{T-t,s}^x)\mathrm{d}s \bigg], \ (t,x) \in [0,T] \times E. \tag{1.6}$$

Moreover, there exists a constant $c > 0$ such that for any $i \in \{1,2\}$ and $j, j' \in \{0,1\}$,

$$\|\nabla^i u_t\|_\infty \le c t^{-\frac{i-j}{2}} \|\nabla^j u_0\|_\infty + c \int_{T-t}^T (s+t-T)^{-\frac{i-j'}{2}} \|\nabla^{j'} V_{T-s}\|_\infty \mathrm{d}s, \ t \in (0,T]. \tag{1.7}$$

(3) If $b^{(1)} = 0$ and $u_0, V_t \in \mathcal{C}_b^2$ with $\int_0^T \|V_t\|_{\mathcal{C}_b^2} \mathrm{d}t < \infty$, then u given by (1.6) solves (1.2), and u is in the class $\mathcal{U}(p_0, q_0)$.

In the next two sections, we prove assertions (1) and (2)-(3) of Theorem 1.1 respectively, where in Section 2 the well-posedness is proved for a more general equation than (1.3). Finally, in Section 4 we apply Theorem 1.1 to the equation (1.1).

2 Proof of Theorem 1.1(1)

Let $\mathcal{P}$ be the set of all probability measures on $\mathbb{R}^d$ equipped with the weak topology, let $\mathcal{L}_\xi$ be the distribution of a random variable ξ on $\mathbb{R}^d$. Let

$$\Gamma := C(D_T \times \mathbb{R}^d; \mathcal{P})$$

be the space of continuous maps from $D_T \times \mathbb{R}^d$ to $\mathcal{P}$. For any $\lambda > 0$, Γ is a complete space under the metric

$$\rho_\lambda(\gamma^1,\gamma^2) := \sup_{(t,s,x)\in D_T\times\mathbb{R}^d} \mathrm{e}^{-\lambda(T-t)}\|\gamma^1_{t,s,x}-\gamma^2_{t,s,x}\|_{var}, \ \ \gamma^1,\gamma^2\in\Gamma,$$

where $\|\cdot\|_{var}$ is the total variation norm defined by

$$\|\mu-\nu\|_{var} := \sup_{|f|\le 1}|\mu(f)-\nu(f)|, \ \ \mu,\nu\in\mathcal{P}$$

for $\mu(f) := \int_{\mathbb{R}^d} f\mathrm{d}\mu$. Note that the convergence in $\|\cdot\|_{var}$ is stronger than the weak convergence.

We consider the following more general equation than (1.3):

$$\begin{aligned} \mathrm{d}X^x_{t,s} &= \Big\{b^{(1)}_{T-s}(X^x_{t,s}) + Z_s(X^x_{t,s},\mathcal{L}_X)\Big\}\mathrm{d}s + \sqrt{2a_{T-s}}(X^x_{t,s})\mathrm{d}W_s, \\ &t\in[0,T], s\in[t,T], X^x_{t,t}=x\in\mathbb{R}^d, \end{aligned} \tag{2.1}$$

where $\mathcal{L}_X\in\Gamma$ is defined by $\{\mathcal{L}_X\}_{t,s,x} := \mathcal{L}_{X^x_{t,s}}$, and

$$Z:[0,T]\times\mathbb{R}^d\times\Gamma\to\mathbb{R}^d$$

is measurable.

It is easy to see that (2.1) covers (1.3) for

$$\begin{aligned} Z_t(x,\gamma) &:= b^{(0)}_{T-t}(x) - \int_{\mathbb{R}^d} u_0(y)\gamma_{t,T,x}(\mathrm{d}y) - \int_t^T \mathrm{d}s\int_{\mathbb{R}^d} V_{T-s}(y)\gamma_{t,s,x}(\mathrm{d}y), \\ &(t,x,\gamma)\in[0,T]\times\mathbb{R}^d\times\Gamma. \end{aligned} \tag{2.2}$$

The solution of (2.1) is defined as in Definition 1.1 using $b^{(1)}_{T-s}(X^x_{t,s}) + Z_s(X^x_{t,s},\mathcal{L}_X)$ replacing

$$b_{T-s}(X^x_{t,s}) - \bigg[\mathbb{E}u_0(X^y_{s,T}) + \mathbb{E}\int_s^T V_{T-r}(X^y_{s,r})\mathrm{d}r\bigg]_{y=X^x_{t,s}}.$$

We make the following assumption.

(A) $b^{(1)}$ and a satisfy (H), and there exists $(p_0,q_0)\in\mathcal{K}$ and $f_0\in\tilde{L}^{p_0}_{q_0}$ such that

$$|Z_t(x,\gamma)|\le f_0(t,x), \ \ (t,x,\gamma)\in[0,T]\times\mathbb{R}^d\times\Gamma.$$

Moreover, there exists $0\le g\in L^2([0,T])$ such that

$$\sup_{x\in\mathbb{R}^d}|Z_t(x,\gamma^1)-Z_t(x,\gamma^2)|\le g_t\sup_{(s,x)\in[t,T]\times\mathbb{R}^d}\|\gamma^1_{t,s,x}-\gamma^2_{t,s,x}\|_{var}, \ \ t\in[0,T],\gamma^1,\gamma^2\in\Gamma.$$

When $\|u_0\|_\infty + \int_0^T\|V_t\|^2_\infty\mathrm{d}t < \infty$, (H) implies (A) for Z given by (2.2). So, Theorem 1.1(1) follows from the following result, which also includes regularity estimates on the solution.

Theorem 2.1. Assume (A). Then the following assertions hold.

(1) (2.1) has a unique solution, and the solution has the flow property

$$X^x_{t,r} = X^{X^x_{t,s}}_{s,r}, \ \ 0\le t\le s\le r\le T, \ x\in\mathbb{R}^d. \tag{2.3}$$

(2) For any $j \ge 1$,

$$\nabla_v X_{t,s}^x := \lim_{\varepsilon \downarrow 0} \frac{X_{t,s}^{x+\varepsilon v} - X_{t,s}^x}{\varepsilon}, \quad s \in [t,T]$$

exists in $L^j(\Omega \to C([t,T];\mathbb{R}^d), \mathbb{P})$, and there exists a constant $c(j) > 0$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\bigg[\sup_{s\in[t,T]} |\nabla_v X_{t,s}^x|^j\bigg] \le c(j)|v|^j, \quad v \in \mathbb{R}^d. \tag{2.4}$$

(3) For any $0 \le t < s \le T$, $v \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\nabla_v \big\{\mathbb{E} f(X_{t,s}^\cdot)\big\}(x) = \frac{1}{s-t}\mathbb{E}\bigg[f(X_{t,s}^x)\int_t^s \Big\langle \big(\sqrt{2a_{T-r}}\big)^{-1}(X_{t,r}^x)\nabla_v X_{t,r}^x,\ \mathrm{d}W_r\Big\rangle\bigg]. \tag{2.5}$$

Proof. (a) We first explain the idea of proof using fixed point theorem on Γ. For any $\gamma \in \Gamma$, we consider the following classical SDE

$$\begin{aligned} \mathrm{d}X_{t,s}^{\gamma,x} &= \Big\{b_{T-s}^{(1)}(X_{t,s}^{\gamma,x}) + Z_s(X_{t,s}^{\gamma,x},\gamma)\Big\}\mathrm{d}s + \sqrt{2a_{T-s}}(X_{t,s}^{\gamma,x})\mathrm{d}W_s, \\ & t \in [0,T], s \in [t,T], X_{t,t}^{\gamma,x} = x \in \mathbb{R}^d. \end{aligned} \tag{2.6}$$

By [2, Theorem 2.1] for $[t,T]$ replacing $[0,T]$, see also [4] for $b^{(1)} = 0$, this SDE is well-posed, such that for any $j \ge 1$ and $v \in \mathbb{R}^d$, the directional derivative

$$\nabla_v X_{t,s}^{\gamma,x} := \lim_{\varepsilon\downarrow 0} \frac{X_{t,s}^{\gamma,x+\varepsilon v} - X_{t,s}^{\gamma,x}}{\varepsilon}, \quad s \in [t,T]$$

exists in $L^j(\Omega \to C([t,T];\mathbb{R}^d), \mathbb{P})$, and there exists a constant $c(j) > 0$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\bigg[\sup_{s\in[t,T]} |\nabla_v X_{t,s}^{\gamma,x}|^j\bigg] \le c(j)|v|^j, \quad v \in \mathbb{R}^d, \tag{2.7}$$

and for any $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\nabla_v \big\{\mathbb{E} f(X_{t,s}^{\gamma,\cdot})\big\}(x) = \frac{1}{s-t}\mathbb{E}\bigg[f(X_{t,s}^{\gamma,x})\int_t^s \Big\langle \big(\sqrt{2a_{T-r}}\big)^{-1}(X_{t,r}^{\gamma,x})\nabla_v X_{t,r}^{\gamma,x},\ \mathrm{d}W_r\Big\rangle\bigg]. \tag{2.8}$$

By the pathwise uniqueness of (2.6), the solution satisfies the flow property

$$X_{t,r}^{\gamma,x} = X_{s,r}^{\gamma, X_{t,s}^{\gamma,x}}, \quad 0 \le t \le s \le r \le T,\ x \in \mathbb{R}^d. \tag{2.9}$$

Moreover,

$$\Phi(\gamma)_{t,s,x} := \mathcal{L}_{X_{t,s}^{\gamma,x}}, \quad (t,s,x) \in D_T \times \mathbb{R}^d$$

defines a map $\Phi : \Gamma \to \Gamma$. If Φ has a unique fixed point $\bar\gamma \in \Gamma$, then (2.6) with $\gamma = \bar\gamma$ reduces to (2.1), the well-posedness of (2.6) implies that of (2.1), and the unique solution is given by

$$X_{t,s}^x = X_{t,s}^{\bar\gamma,x}.$$

Then (2.3), (2.4) and (2.5) follow from (2.9), (2.7) and (2.8) for $\gamma = \bar\gamma$ respectively. Therefore, it remains to prove that Φ has a unique fixed point.

(b) By the fixed point theorem, we only need to find constants $\lambda > 0$ and $\delta \in (0,1)$ such that

$$\rho_\lambda(\Phi(\gamma^1), \Phi(\gamma^2)) \le \delta \rho_\lambda(\gamma^1,\gamma^2), \quad \gamma^1,\gamma^2 \in \Gamma. \tag{2.10}$$

Below, we prove this estimate using Girsanov's theorem.

For $i = 1, 2$, consider the SDE

$$\mathrm{d}X^{i,x}_{t,s} = \Big\{ b^{(1)}_{T-s}(X^{i,x}_{t,s}) + Z_s(X^{i,x}_{t,s}, \gamma^i) \Big\} \mathrm{d}s + \sqrt{2a_{T-s}}(X^{i,x}_{t,s}) \mathrm{d}W_s,$$
$$t \in [0,T], s \in [t,T], X^{i,x}_{t,t} = x \in \mathbb{R}^d.$$

By the definition of Φ, we have

$$\Phi(\gamma^i)_{t,s,x} = \mathcal{L}_{X^{i,x}_{t,s}}, \ \ i = 1, 2, \ (t,s,x) \in D_T \times \mathbb{R}^d. \tag{2.11}$$

Let

$$\xi_s := \big(\sqrt{2a_{T-s}}(X^{1,x}_{t,s}) \big)^{-1} \big\{ Z_s(X^{1,x}_{t,s}, \gamma^1) - Z_s(X^{1,x}_{t,s}, \gamma^2) \big\}, \ \ s \in [t,T].$$

By (A), there exists a constant $K > 0$ such that

$$|\xi_s| \le K g_s \sup_{(r,x) \in [s,T] \times \mathbb{R}^d} \|\gamma^1_{s,r,x} - \gamma^2_{s,r,x}\|_{var}. \tag{2.12}$$

By Girsanov theorem,

$$\tilde{W}_s := W_s - \int_t^s \xi_r \mathrm{d}r, \ \ s \in [t,T]$$

is a Brownian motion under the weighted probability $\mathrm{d}\mathbb{Q}_t := R_t \mathrm{d}\mathbb{P}$, where

$$R_t := \mathrm{e}^{\int_t^T \langle \xi_s, \mathrm{d}W_s \rangle - \frac{1}{2} \int_t^T |\xi_s|^2 \mathrm{d}s}.$$

With this new Brownian motion, the SDE for X^1 becomes

$$\mathrm{d}X^{1,x}_{t,s} = \Big\{ b^{(1)}_{T-s}(X^{1,x}_{t,s}) + Z_s(X^{1,x}_{t,s}, \gamma^2) \Big\} \mathrm{d}s + \sqrt{2a_{T-s}}(X^{1,x}_{t,s}) \mathrm{d}\tilde{W}_s, \ \ s \in [t,T].$$

By the (weak) uniqueness for the SDE with $i = 2$, we derive

$$\mathcal{L}_{X^{1,x}_{t,s}|\mathbb{Q}_t} = \mathcal{L}_{X^{2,x}_{t,s}} = \Phi(\gamma^2)_{t,s,x},$$

where $\mathcal{L}_{X^{1,x}_{t,s}|\mathbb{Q}_t}$ is the distribution of $X^{1,x}_{t,s}$ under $\mathbb{Q}_t$. Combining this with (2.11), we get

$$\|\Phi(\gamma^1)_{t,s,x} - \Phi(\gamma^2)_{t,s,x}\|_{var} = \sup_{|f| \le 1} \big| \mathbb{E}[f(X^{1,x}_{t,s}) - f(X^{1,x}_{t,s}) R_t] \big| \le \mathbb{E}|R_t - 1|. \tag{2.13}$$

By Pinsker's inequality and the definition of R_t, we obtain

$$(\mathbb{E}|R_t - 1|)^2 \le 2\mathbb{E}[R_t \log R_t] = 2\mathbb{E}_{\mathbb{Q}_t}[\log R_t] = 2\mathbb{E}_{\mathbb{Q}_t} \int_t^T |\xi_s|^2 \mathrm{d}s, \tag{2.14}$$

where $\mathbb{E}_{\mathbb{Q}_t}$ is the expectation under the probability $\mathbb{Q}_t$. Combining (2.13) and (2.14) with (2.12), and using the definition of ρ_λ, we arrive at

$$\|\Phi(\gamma^1)_{t,s,x} - \Phi(\gamma^2)_{t,s,x}\|_{var} \le \bigg(2K^2 \int_t^T g_s^2 \sup_{(r,y) \in [s,T] \times \mathbb{R}^d} \|\gamma^1_{s,r,y} - \gamma^2_{s,r,y}\|^2_{var} \mathrm{d}s \bigg)^{\frac{1}{2}}$$
$$\le \rho_\lambda(\gamma^1, \gamma^2) \bigg(2K^2 \int_t^T g_s^2 \mathrm{e}^{2\lambda(T-s)} \mathrm{d}s \bigg)^{\frac{1}{2}}, \ \ (t,x) \in [0,T] \times \mathbb{R}^d.$$

Therefore

$$\rho_\lambda(\Phi(\gamma^1), \Phi(\gamma^2)) \le \varepsilon_\lambda \rho_\lambda(\gamma^1, \gamma^2),$$

where

$$\varepsilon_\lambda := \sup_{t \in [0,T]} \bigg(2K^2 \int_t^T g_s^2 \mathrm{e}^{-2\lambda(s-t)} \mathrm{d}s \bigg)^{\frac{1}{2}} \downarrow 0 \text{ as } \lambda \uparrow \infty.$$

By taking large enough $\lambda > 0$, we prove (2.10) for some $\delta < 1$. □

For later use we present the following consequence of Theorem 2.1.

Corollary 2.2. Assume (A) and let

$$P_{t,s}f(x) := \mathbb{E}[f(X_{t,s}^x)],\ \ (t,s,x) \in D_T \times \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d).$$

Then there exists a constant $c > 0$ such that for any function f,

$$\|\nabla P_{t,s}f\|_\infty \le c \min\Big\{(s-t)^{-\frac{1}{2}}\|f\|_\infty,\ \|\nabla f\|_\infty\Big\},$$
$$\|\nabla^2 P_{t,s}f\|_\infty \le c(s-t)^{-\frac{1}{2}}\|\nabla f\|_\infty,\ \ 0 \le t < t \le T.$$

Proof. By (2.5) we have

$$\|\nabla P_{t,s}f\|_\infty \le c(t-s)^{-\frac{1}{2}}\|f\|_\infty$$

for some constant $c > 0$. Next, by chain rule and (2.4),

$$|\nabla P_{t,s}f(x)| = \big|\mathbb{E}[\langle \nabla f(X_{t,s}^x), \nabla X_{t,s}^x\rangle]\big| \le c\|\nabla f\|_\infty,\ \ (t,s,x) \in D_T \times \mathbb{R}^d$$

holds for some constant $c > 0$. Moreover,

$$\nabla P_{t,s}f(x) = \mathbb{E}[\langle \nabla f(X_{t,s}^x), \nabla X_{t,s}^x\rangle] = \mathbb{E}[g(X_{t,s}^x)],$$

where $g(X_{t,s}^x) := \big\langle \nabla f(X_{t,s}^x), \mathbb{E}(\nabla X_{t,s}^x | X_{t,s}^x)\big\rangle$. Combining this with (2.5) and (2.4), we find a constant $c > 0$ such that

$$\begin{aligned}
&\|\nabla^2 P_{t,s}f(x)\| \le \|\nabla \mathbb{E}[g(X_{t,s}^x)]\| \\
&\le \frac{1}{s-t}\mathbb{E}\bigg[\big|g(X_{t,s}^x)\big| \cdot \bigg|\int_s^t \Big\langle \big(\sqrt{2a_{T-r}}\big)^{-1}(X_{t,r}^x)\nabla_v X_{t,r}^x,\ \mathrm{d}W_r\Big\rangle\bigg|\bigg] \\
&\le \frac{1}{t-s}\big(\mathbb{E}|g(X_{t,s}^x)|^2\big)^{\frac{1}{2}}\bigg(\mathbb{E}\int_t^s \|a^{-1}\|_\infty \|\nabla X_{t,r}^x\|^2 \mathrm{d}r\bigg)^{\frac{1}{2}} \le c\|\nabla f\|_\infty.
\end{aligned}$$

Then the proof is finished. □

3 Proofs of Theorem 1.1(2)-(3)

We will need the following lemma implied by [5, Theorem 2.1, Theorem 3.1, Lemma 3.3], see also [4] and references within for the case $b^{(1)} = 0$.

Lemma 3.1. Assume $(A)(1), (A)(3)$ and $\|b^{(0)}\|_{\tilde{L}_{q_0}^{p_0}} < \infty$ for some $(p_0, q_0) \in \mathcal{K}$. Let $\sigma_t = \sqrt{2a_t}$. Then the following assertions hold.

(1) For any $p, q > 1$, $\lambda \ge 0$, $0 \le t_0 < t_1 \le T$ and $f \in \tilde{L}_q^p(t_0, t_1)$, the PDE

$$(\partial_t + L_t)u_t = \lambda u_t + f_t,\ \ t \in [t_0, t_1], u_{t_1} = 0, \tag{3.1}$$

has a unique solution in $\tilde{H}_q^{2,p}(t_0, t_1)$. If $(2p, 2q) \in \mathcal{K}$, then there exist a constant $c > 0$ such that for any $0 \le t_0 < t_1 \le T$ and $f \in \tilde{L}_q^p(t_0, t_1)$, the solution satisfies

$$\|u\|_\infty + \|\nabla u\|_\infty + \|(\partial_t + \nabla_{b^{(1)}})u\|_{\tilde{L}_q^p(t_0,t_1)} + \|\nabla^2 u\|_{\tilde{L}_q^p(t_0,t_1)} \le c\|f\|_{\tilde{L}_q^p(t_0,t_1)}.$$

(2) Let $(X_t)_{t\in[0,T]}$ be a continuous adapted process on $\mathbb{R}^d$ satisfying

$$X_t = X_0 + \int_0^t b_s(X_s)\mathrm{d}s + \int_0^t \sigma_s(X_s)\mathrm{d}W_s,\ \ t \in [0,T]. \tag{3.2}$$

For any $p, q > 1$ with $(2p, 2q) \in \mathcal{K}$, there exists a constant $c > 0$ such that for any X_t satisfying (3.2),

$$\mathbb{E}\bigg(\int_t^s |f_r(X_r)|\mathrm{d}r\bigg|\mathcal{F}_t\bigg) \le c\|f\|_{\tilde{L}_q^p(t,s)},\ \ (t,s) \in D_T, f \in \tilde{L}_q^p(t,s).$$

(3) Let $p,q>1$ with $\frac{d}{p}+\frac{2}{q}<1$. For any $u\in\tilde H^{2,p}_q$ with $\|(\partial_t+b^{(1)})u\|_{\tilde L^p_q}<\infty$, $\{u_t(X_t)\}_{t\in[0,T]}$ is a semimartingale satisfying

$$\mathrm{d}u_t(X_t)=L_tu_t(X_t)\mathrm{d}t+\big\langle\nabla u_t(X_t),\sigma_t(X_t)\mathrm{d}W_t\big\rangle,\ \ t\in[0,T].$$

In the following we consider $E=\mathbb{R}^d$ and $\mathbb{T}^d$ respectively.

3.1 $E=\mathbb{R}^d$

Proof of Theorem 1.1(2). Let $u\in\mathcal{U}(p_0,q_0)$ solve (1.2). Then

$$u\in\tilde H^{2,p_0}_{q_0},\ \ \|(\partial_t+b^{(1)}\cdot\nabla)u\|_{\tilde L^{p_0}_{q_0}}<\infty \tag{3.3}$$

as required by Lemma 3.1(3). It remains to prove (1.6), which together with Corollary 2.2 implies (1.7).

Let

$$\begin{aligned}&\mathcal{L}_t:=\mathrm{tr}\{a_{T-t}\nabla^2\}+\tilde b_t\cdot\nabla,\\&\tilde b_t(x):=b_{T-t}(x)-\mathbb{E}u_0(X^x_{t,T})-\mathbb{E}\int_t^T V_{T-s}(X^x_{t,s})\mathrm{d}s,\ \ (t,x)\in[0,T]\times\mathbb{R}^d.\end{aligned}\tag{3.4}$$

Since $\|u_0\|_\infty+\int_0^T\|V_t\|_\infty\mathrm{d}t<\infty$, $\|b^{(0)}\|_{\tilde L^{p_0}_{q_0}}<\infty$ implies $\tilde b_t(x):=b^{(1)}_{T-t}(x)+\tilde b^{(0)}_t(x)$ with $\|\tilde b^{(0)}\|_{\tilde L^{p_0}_{q_0}}<\infty$. Then (A) holds for $\tilde b$ replacing b, so that by (3.3) and Lemma 3.1(3), the following Itô's formula holds for $X^x_{t,s}$ solving (1.3):

$$\mathrm{d}u_{T-s}(X^x_{t,s})=\big(\partial_s+\mathcal{L}_s\big)u_{T-s}(X^x_{t,s})\mathrm{d}s+\big\{\nabla u_{T-s}(X^x_{t,s})\big\}^*\sqrt{2a_{T-s}(X^x_{t,s})}\mathrm{d}W_s,\ \ s\in[t,T], \tag{3.5}$$

where $(\nabla u)^*_{ij}:=(\partial_ju^i)_{1\le i,j\le d}$. By (1.2) and (3.4), we obtain

$$\begin{aligned}&\big(\partial_s+\mathcal{L}_s\big)u_{T-s}(X^x_{t,s})+V_{T-s}(X^x_{t,s})\\&=\bigg\{\bigg[u_{T-s}(y)-\mathbb{E}u_0(X^y_{s,T})-\mathbb{E}\int_s^T V_{T-r}(X^y_{s,r})\mathrm{d}r\bigg]_{y=X^x_{t,s}}\cdot\nabla\bigg\}u_{T-s}(X^x_{t,s}).\end{aligned}$$

Combining this with the follow property (2.3) and (3.5), we derive

$$\begin{aligned}&\mathbb{E}u_0(X^x_{t,T})-u_{T-t}(x)=\mathbb{E}\big[u_{T-T}(X^x_{t,T})-u_{T-t}(X^x_{t,t})\big]\\&=\mathbb{E}\int_t^T\bigg\{\bigg(u_{T-s}(y)-\mathbb{E}u_0(X^y_{s,T})-\mathbb{E}\int_s^T V_{T-r}(X^y_{s,r})\mathrm{d}r\bigg)_{y=X^x_{t,s}}\cdot\nabla\bigg\}u_{T-s}(X^x_{t,s})\mathrm{d}s\\&\quad-\mathbb{E}\int_t^T V_{T-s}(X^x_{t,s})\mathrm{d}s,\ \ \ (t,x)\in[0,T]\times\mathbb{R}^d.\end{aligned}$$

Letting

$$h_t:=\sup_{x\in\mathbb{R}^d}\bigg|u_{T-t}(x)-\mathbb{E}u_0(X^x_{t,T})-\mathbb{E}\int_t^T V_{T-s}(X^x_{t,s})\mathrm{d}s\bigg|,\ \ t\in[0,T],$$

we arrive at

$$h_t\le\int_t^T h_s\|\nabla u\|_\infty\mathrm{d}s,\ \ t\in[0,T].$$

By Grownwall's inequality we prove $h_t=0$ for $t\in[0,T]$, hence (1.6) holds. □

Proof of Theorem 1.1(3). (a) Let $P_{t,s}f=\mathbb{E}[f(X^x_{t,s})]$ for $f\in\mathcal{B}_b(\mathbb{R}^d)$, where $X^x_{t,s}$ solves (1.3). For u given by (1.6) we have

$$u_t=P_{T-t,T}u_0+\int_{T-t}^T P_{T-t,s}V_{T-s}\mathrm{d}s,\ \ t\in[0,T]. \tag{3.6}$$

By $\|u_0\|_\infty + \int_0^T \|V_t\|_\infty \mathrm{d}t < \infty$ and (1.7), we find a constant $c > 0$ such that

$$\|u\|_\infty + \|\nabla u\|_\infty \le c, \ \ \|\nabla^2 u_t\|_\infty \le ct^{-\frac{1}{2}}, \ \ t \in (0,T]. \tag{3.7}$$

Moreover, the SDE (1.3) becomes

$$\begin{aligned} &\mathrm{d}X_{t,s}^x = \sqrt{2a_{T-s}}(X_{t,s}^x)\mathrm{d}W_s + \big\{b_{T-s} - u_{T-s}\big\}(X_{t,s}^x)\mathrm{d}s, \\ &\quad t \in [0,T], s \in [t,T], X_{t,t}^x = x \in \mathbb{R}^d, \end{aligned} \tag{3.8}$$

and the generator in (3.4) reduces to

$$\mathcal{L}_s := \mathrm{tr}\big\{a_{T-s}\nabla^2\big\} + \big\{b_{T-s} - u_{T-s}\big\} \cdot \nabla, \ \ s \in [0,T].$$

(b) We prove the Kolmogorov backward equation

$$\partial_t P_{t,s} f = -\mathcal{L}_t P_{t,s} f, \ \ f \in \mathcal{C}_b^2, t \in [0,s], s \in (0,T]. \tag{3.9}$$

For any $f \in \mathcal{C}_b^2$, by Itô's formula we have

$$P_{t,s}f(x) = f(x) + \int_t^s P_{t,r}(\mathcal{L}_r f)(x)\mathrm{d}r, \ \ (t,s) \in D_T, \tag{3.10}$$

where $\int_t^s P_{t,r}(\mathcal{L}_r f)(x)\mathrm{d}r = \mathbb{E}\int_t^s \mathcal{L}_r f(X_{t,r}^x)\mathrm{d}r$ exists, since Krylov's estimate in Lemma 3.1(2) holds under (A) and $\|u\|_\infty < \infty$.

By (3.10), we obtain the Kolmogorov forward equation

$$\partial_s P_{t,s} f = P_{t,s}(\mathcal{L}_s f), \quad s \in [t,T]. \tag{3.11}$$

On the other hand, $b^{(1)} = 0$ and (A) imply

$$\|\mathcal{L} f\|_{\tilde{L}_{q_0}^{p_0}} \le c_0 \|f\|_{\mathcal{C}_b^2} \tag{3.12}$$

for some constant $c_0 > 0$. By Lemma 3.1(1), for any $s \in (0,T]$, the PDE

$$(\partial_t + \mathcal{L}_t)\tilde{u}_t = -\mathcal{L}_t f, \ \ t \in [0,s], \tilde{u}_s = 0 \tag{3.13}$$

has a unique solution $\tilde{u} \in \mathcal{U}(p_0,q_0)$, such that

$$\|\nabla^2 \tilde{u}\|_{\tilde{L}_{q_0}^{p_0}(0,s)} \le c_1 \|\mathcal{L} f\|_{\tilde{L}_{q_0}^{p_0}(0,s)} \tag{3.14}$$

holds for some constant $c_1 > 0$ independent of s. By Itô's formula in Lemma 3.1(3),

$$\mathrm{d}\tilde{u}_t(X_{0,t}^x) = -\mathcal{L}_t f(X_{0,t}^x) + \big\langle \nabla f(X_{0,t}^x), \sqrt{2a_{T-t}}(X_{0,t}^x)\mathrm{d}W_t\big\rangle, \ \ t \in [0,s].$$

This and (3.11) imply

$$\begin{aligned} 0 = \tilde{u}_s(x) &= \tilde{u}_t(x) - \int_t^s (P_{t,r}\mathcal{L}_r f)(x)\mathrm{d}r \\ &= \tilde{u}_t(x) - \int_t^s \frac{\mathrm{d}}{\mathrm{d}r}(P_{t,r}f)\mathrm{d}r = \tilde{u}_t(x) - P_{t,s}f(x) + f(x), \ \ t \in [0,s]. \end{aligned}$$

Thus,

$$\tilde{u}_t = P_{t,s} f - f, \ \ t \in [0,s]. \tag{3.15}$$

Combining this with (3.13) we derive (3.9).

(c) By (3.7) and (3.9), we see that u solves (1.6) with $u \in \mathcal{U}(p_0,q_0)$ provided

$$\|\nabla^2 u\|_{\tilde{L}_{q_0}^{p_0}} < \infty. \tag{3.16}$$

By (3.12), (3.14) and (3.15), we find a constant $c_2 > 0$ such that

$$\sup_{t \in [0,s]} \|\nabla^2 P_{\cdot,s} f\|_{\tilde{L}_{q_0}^{p_0}(0,s)} \le c_2 \|f\|_{\mathcal{C}_b^2}, \ \ s \in (0,T], f \in \mathcal{C}_b^2.$$

Combining this with (3.6), $b^{(1)} = 0$ and $\|u_0\|_{\mathcal{C}_b^2} + \int_0^T \|V_t\|_{\mathcal{C}_b^2}\mathrm{d}t < \infty$, we prove (3.16). □

3.2 $E = \mathbb{T}^d$

In this case, all functions on E are extended to $\mathbb{R}^d$ as in (1.4), so that the proof for $E = \mathbb{R}^d$ works also for the present setting if we could verify the following periodic property for the solution of (1.3):

$$X_{t,s}^{x+k} = X_{t,s}^x + k, \ \ (t,s) \in D_T, \ x \in \mathbb{R}^d, \ k \in \mathbb{Z}^d. \tag{3.17}$$

Let $\tilde{X}_{s,t}^x := X_{t,s}^x + k$. Since the coefficients of (1.3) satisfies (1.4), $\tilde{X}_{t,s}^x$ solves (1.3) with $\tilde{X}_{t,t}^x = x + k$. By the uniqueness of (1.3) ensured by Theorem 1.1(1), we derive (3.17).

4 Application to (1.1)

For any $n \in \mathbb{N}$, let $\mathcal{C}_b^n$ be the class of real functions f on E having derivatives up to order n such that

$$\|f\|_{\mathcal{C}_b^n} := \sum_{i=0}^n \|\nabla^i f\|_\infty < \infty,$$

where $\nabla^0 f := f$. Moreover, for $\alpha \in (0,1)$, we denote $f \in \mathcal{C}_b^{n+\alpha}$ if $f \in \mathcal{C}_b^n$ such that

$$\|f\|_{\mathcal{C}_b^{n+\alpha}} := \|f\|_{\mathcal{C}_b^n} + \sup_{x \neq y} \frac{\|\nabla^n f(x) - \nabla^n f(y)\|}{|x-y|^\alpha} < \infty.$$

Consider the following future distribution dependent SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_{t,s}^x = \bigg[\mathbb{E}\int_s^T \nabla \wp_{T-r}(X_{s,r}^y)\mathrm{d}r - \mathbb{E}u_0(X_{s,T}^y)\bigg]_{y = X_{t,s}^x} \mathrm{d}s + \sqrt{2\kappa}\mathrm{d}W_s, \ \ X_{t,t}^x = x, s \in [t,T]. \tag{4.1}$$

See Definition 1.1 below for the definition of solution. When $E = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, we extend u_0 and $\wp_t$ to $\mathbb{R}^d$ periodically, i.e. for a function f on $\mathbb{T}^d$, it is extended to $\mathbb{R}^d$ as in (1.4). With this extension, we also have the SDE (4.1) for the case $E = \mathbb{T}^d$.

Theorem 4.1. If there exists $n \geq 2$ such that $u_0 \in \mathcal{C}_b^n$ and $\wp_t \in \mathcal{C}_b^n$ for a.e. $t \in [0,T]$ with

$$\int_0^T \big(\|\nabla \wp_t\|_\infty^2 + \|\wp_t\|_{\mathcal{C}_b^n}\big)\mathrm{d}t < \infty.$$

Then (4.1) is well-posed and (1.1) has a unique solution satisfying

$$\sup_{t \in [0,T]} \|u_t\|_{\mathcal{C}_b^n} < \infty, \tag{4.2}$$

and the solution is given by

$$u_t(x) = \mathbb{E}u_0(X_{T-t,T}^x) - \mathbb{E}\int_{T-t}^T \nabla \wp_{T-s}(X_{T-t,s}^x)\mathrm{d}s. \tag{4.3}$$

We only prove for $E = \mathbb{R}^d$ as the case for $E = \mathbb{T}^d$ follows by extending functions from $\mathbb{T}^d$ to $\mathbb{R}^d$ as in (1.4).

Let I_d be the $d \times d$ identity matrix. By Theorem 1.1 with $b = 0, a = \kappa I_d$ and $V = -\nabla \wp$, for any $(p_0, q_0) \in \mathcal{K}$, (1.1) has a unique solution in the class $\mathcal{U}(p_0, q_0)$, and by (4.3),

$$\begin{aligned} u_t(x) &:= \mathbb{E}u_0(X_{T-t,T}^x) - \mathbb{E}\int_{T-t}^T \nabla \wp_{T-s}(X_{T-t,s}^x)\mathrm{d}s \\ &= P_{T-t,T}u_0(x) - \int_{T-t}^T P_{T-t,s}\nabla \wp_{T-s}(x)\mathrm{d}s, \ \ (t,x) \in [0,T] \times \mathbb{R}^d. \end{aligned} \tag{4.4}$$

By (3.8) for the present a and b, $X^x_{t,s}$ solves the SDE

$$\mathrm{d}X^x_{t,s} = \sqrt{2\kappa}\mathrm{d}W_s - u_{T-s}(X^x_{t,s})\mathrm{d}s, \quad X^x_{t,t} = x, t \in [0,T], s \in [t,T], \tag{4.5}$$

and the generator is

$$\mathcal{L}_s := \kappa\Delta - u_{T-s}\cdot\nabla, \quad s \in [0,T].$$

It remains to prove (4.2). To this end, we present the following lemma.

Lemma 4.2. Let $P_{t,s}f := \mathbb{E}[f(X^x_{t,s})]$ for the SDE (4.5). Let $m \geq 1$ such that

$$\sup_{t\in[0,T]} \|u_t\|_{\mathcal{C}_b^m} + \|f\|_{\mathcal{C}_b^{m+1}} < \infty, \tag{4.6}$$

then $\sup_{(t,s)\in D_T} \|P_{t,s}f\|_{\mathcal{C}_b^{m+1}} < \infty$.

Proof. By (4.5) and $\sup_{t\in[0,T]} \|u_t\|_{\mathcal{C}_b^m} < \infty$, we have

$$\sup_{(t,s,x)\in D_T\times\mathbb{R}^d} \mathbb{E}\big[\|\nabla^i X^x_{t,s}\|\big] < \infty, \quad 1 \leq i \leq m.$$

By chain rule, this implies that for some constant $c_0 > 0$,

$$\sup_{(t,s)\in D_T} \|P_{t,s}g\|_{\mathcal{C}_b^m} \leq c_0\|g\|_{\mathcal{C}_b^m}, \quad g \in \mathcal{C}_b^m. \tag{4.7}$$

Let $P^0_t = \mathrm{e}^{\kappa\Delta t}$. By $\partial_r P^0_{r-t} = P^0_{r-t}\kappa\Delta$ and (3.9), we have

$$\partial_r P^0_{r-t}P_{r,s}f = P^0_{r-t}\langle\nabla P_{r,s}f, u_{T-r}\rangle, \quad r \in [t,s].$$

So,

$$P_{t,s}f = P^0_{s-t}f - \int_t^s P^0_{r-t}\langle\nabla P_{r,s}f, u_{T-r}\rangle\mathrm{d}r. \tag{4.8}$$

It is well known that for any $\alpha, \beta \geq 0$ there exists a constant $c_{\alpha,\beta} > 0$ such that

$$\|P^0_t g\|_{\mathcal{C}_b^{\alpha+\beta}} \leq c_{\alpha,\beta}t^{-\frac{\alpha}{2}}\|g\|_{\mathcal{C}_b^\beta}, \quad t > 0, g \in \mathcal{C}_b^\beta. \tag{4.9}$$

This together with (4.8) implies that for some constants $c_1, c_2 > 0$,

$$\|P_{t,s}f\|_{\mathcal{C}_b^{m+\frac{1}{2}}} \leq c_1\|f\|_{\mathcal{C}_b^{m+\frac{1}{2}}} + c_1\int_t^s (t+r-s)^{-\frac{3}{4}}\|\langle\nabla P_{r,s}f, u_{T-r}\rangle\|_{\mathcal{C}_b^{m-1}}\mathrm{d}r.$$

Combining this with (4.7) and $\|f\|_{\mathcal{C}_b^m} + \sup_{t\in[0,T]}\|u_t\|_{\mathcal{C}_b^m} < \infty$, we obtain

$$\sup_{(t,s)\in D_T} \|P_{t,s}f\|_{\mathcal{C}_b^{m+\frac{1}{2}}} < \infty.$$

By this together with (4.8) and (4.6), we find a constant $c_2 > 0$ such that

$$\begin{aligned}\sup_{(t,s)\in D_T} &\|P_{t,s}f\|_{\mathcal{C}_b^{m+1}} \leq c_2\|f\|_{\mathcal{C}_b^{m+1}} \\ &+ c_2 \sup_{(t,s)\in D_T}\int_t^s (t+r-s)^{-\frac{3}{4}}\|\langle\nabla P_{r,s}f, u_{T-r}\rangle\|_{\mathcal{C}_b^{m-\frac{1}{2}}}\mathrm{d}r < \infty.\end{aligned}$$

□

We now prove (4.2) as follows. By $u \in \mathcal{U}(p_0, q_0)$, we have

$$\|u\|_\infty + \|\nabla u\|_\infty < \infty.$$

Combining this with (4.4) and Lemma 4.2, we prove (4.2) by inducing in m up to $m = n$.

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