Local and uniform moduli of continuity of chi–square processes

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Abstract

Let \(\{\eta(t), t \in [0,1]\}_{i=1}^{k}\) be independent copies of \(\eta = \{\eta(t), t \in [0,1]\}\), a mean zero continuous Gaussian process. Let

\[Y_k := Y_k(t) = \sum_{i=1}^{k} \eta_i^2(t), \quad t \in [0,1].\]

This paper shows how exact local (at 0) and uniform moduli of continuity (on \([0,1]\)) of \(Y_k\) can be obtained from the exact local and uniform moduli of continuity of \(\eta\).

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1 Introduction

Let \(\eta = \{\eta(t); t \in [0,1]\}\) be a mean zero continuous Gaussian process with covariance \(U = \{U(s,t), s,t \in [0,1]\}\), with \(U(0,0) > 0\). Let \(\{\eta_i; i = 1, \ldots, k\}\) be independent copies of \(\eta\) and set,

\[Y_k(t) = \sum_{i=1}^{k} \eta_i^2(t), \quad t \in [0,1].\]

The stochastic process \(Y_k = \{Y_k(t), t \in [0,1]\}\) is referred to as a chi–square process of order \(k\) with kernel \(U\).

Chi-square processes appear naturally as limiting processes in various statistical models. See e.g. [9, 3, 2, 1, 7] and the references therein. Nevertheless our interest in them is primarily that they are simple examples of permanental processes that are easier to analyze and therefore provide a template for more general results about permanental processes.

Let \(\{K(s,t), s,t \in T\}\) be a kernel, that need not be symmetric, with the property that for all \(t_n = (t_1, \ldots, t_n)\) in \(T^n\), the matrix \(K(t_n) = \{K(t_i, t_j), i,j \in [1,n]\}\) determines an \(n\)-dimensional random variable random variable \((X_\alpha(t_1), \ldots, X_\alpha(t_n))\) with Laplace transform,

\[E\left(e^{-\sum_{i=1}^{n} s_i X_\alpha(t_i)}\right) = \frac{1}{|I + K(t_n)S(s_n)|^\alpha},\]  

(1.2)
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where $S(s_n)$ is a diagonal matrix with positive entries $s_n = (s_1, \ldots, s_n)$, and $\alpha > 0$. It follows from the Kolmogorov Extension Theorem that \( \{K(s, t), s, t \in T\} \) determines a stochastic process which we denote by $X_n = \{X_n(t), t \in T\}$ and refer to as an $\alpha$-permanental process.

Clearly $Y_k$ is a permanental process when $K(t_n) = \{E(Y_i, Y_j), i, j \in [1, n]\}$ and $\alpha = k/2$. The results obtained in this paper are used in our more general study of laws of the iterated logarithm for $k/2$ permanental processes [6].

In this paper we show that when the Gaussian process $\eta$ has a local or uniform modulus of continuity the related $k/2$ chi-square process has a closely related local or uniform modulus of continuity.

**Theorem 1.1.** Let $\phi(t)$ be a non-negative function function on $[0, \delta]$ for some $\delta > 0$. If

$$\limsup_{t \to 0} \frac{\eta(t) - \eta(0)}{\phi(t)} = 1 \quad \text{a.s.,}$$

then for all integers $k \geq 1$,

$$\limsup_{t \to 0} \frac{Y_k(t) - Y_k(0)}{\phi(t)} = 2Y_k^{1/2}(0) \quad \text{a.s.} \quad (1.4)$$

When $k = 1$ this is particularly simple. Since $\eta$ is symmetric it follows from (1.3) that,

$$\liminf_{t \to 0} \frac{\eta(t) - \eta(0)}{\phi(t)} = -1 \quad \text{a.s.} \quad (1.5)$$

Therefore, writing $Y_1(t) - Y_1(0) = (\eta(t) - \eta(0))(\eta(t) + \eta(0))$ and using the continuity of $\eta$, we see that

$$\limsup_{t \to 0} \frac{Y_1(t) - Y_1(0)}{\phi(t)} = 2(\eta(0) \vee -\eta(0)) \quad \text{a.s.,} \quad (1.6)$$

which is (1.4).

A result similar to Theorem 1.1 for the limiting behavior of chi–square sequences at infinity is given in [5, Lemma 6.5].

Set

$$\sigma^2(u, v) = E(\eta(u) - \eta(v))^2 \quad \text{and} \quad \bar{\sigma}^2(x) = \sup_{|u-v| \leq x} \sigma^2(u, v). \quad (1.7)$$

**Theorem 1.2.** Assume that $\inf_{t \in [0, 1]} U(t, t) > 0$ and,

$$\lim_{x \to 0} \bar{\sigma}^2(x) \log 1/x = 0. \quad (1.8)$$

Let $\varphi(t)$ be a non-negative function function on $[0, 1]$. Then if

$$\lim_{h \to 0} \sup_{|u-v| \leq h} \frac{\eta(u) - \eta(v)}{\varphi(|u-v|)} = 1 \quad \text{a.s.,} \quad (1.9)$$

for all intervals $\Delta \subset [0, 1]$, it follows that for all intervals $\Delta \subset [0, 1]$ and all integers $k \geq 1$,

$$\lim_{h \to 0} \sup_{|u-v| \leq h} \frac{Y_k(u) - Y_k(v)}{\varphi(|u-v|)} = 2 \sup_{u \in \Delta} Y_k^{1/2}(u), \quad \text{a.s.} \quad (1.10)$$

Necessary and sufficient conditions for the existence of functions $\phi$ and $\varphi$ that satisfy (1.3) and (1.9), and what they are, are given in [4, Theorem 7.1.4]. However, they are very abstract. When $\sigma^2(u, v)$ satisfies mild regularity conditions, $\phi$ and $\varphi$ are nice functions. An extensive treatment of Gaussian processes satisfying (1.3) and (1.9) is
We show below that
\[ \limsup_{t \to 0} \frac{\eta(t) - \eta(0)}{t} = \eta'(0), \tag{1.11} \]
and
\[ \lim_{h \to 0} \sup_{u,v \in \Delta} \frac{\eta(u) - \eta(v)}{u - v} = \sup_{u \in \Delta} \eta'(u). \tag{1.12} \]

When \( \eta \) is a continuous Gaussian process with stationary increments, \( \sigma^2(u, v) \) in (1.7) can be written as \( \sigma^2(u - v, 0) \). In this case if \( \tilde{\sigma}^2(x) \) is asymptotic to an increasing function at 0, then (1.9) implies (1.8). We discuss this further in Remark 2.3.

It is remarkable that the moduli functions \( \phi \) and \( \varphi \) do not depend on \( k \). This indicates that the extrema of the the increments of \( \eta \) take place on a very sparse set of points.

## 2 Proofs

**Proof of Theorem 1.1.** Let \( \eta_i(t), i = 1, \ldots, k, \) be independent copies of \( \eta(t) \). We write,
\[ \eta^2_i(t) - \eta^2_i(0) = (\eta_i(t) - \eta_i(0))(\eta_i(t) + \eta_i(0)) \]
\[ = (\eta_i(t) - \eta_i(0))(2\eta_i(0) + (\eta_i(t) - \eta_i(0))) \]
\[ = 2(\eta_i(t) - \eta_i(0))\eta_i(0) + (\eta_i(t) - \eta_i(0))^2. \]

By (1.3)
\[ \limsup_{t \to 0} \left( \sum_{i=1}^{k} (\eta_i(t) - \eta_i(0))^2 / \phi(t) \right) \]
\[ \leq \sum_{i=1}^{k} \limsup_{t \to 0} \frac{|\eta_i(t) - \eta_i(0)|}{\phi(t)} \lim_{t \to 0} |\eta_i(t) - \eta_i(0)| = 0. \]

Consequently, using (1.1) we see that,
\[ \limsup_{t \to 0} \frac{Y_k(t) - Y_k(0)}{\phi(t)} = \limsup_{t \to 0} \frac{2 \sum_{i=1}^{k} (\eta_i(t) - \eta_i(0))\eta_i(0)}{\phi(t)}. \tag{2.3} \]

Write,
\[ (\eta_i(t) - \eta_i(0))\eta_i(0) \]
\[ = (\eta_i(t) - U(0,t)\eta_i(0))\eta_i(0) - \left( \frac{U(0,0) - U(0,t)}{U(0,0)} \right) \eta_i^2(0). \tag{2.4} \]

We show below that
\[ \limsup_{t \to 0} \frac{[U(0,0) - U(0,t)]}{U(0,0)\phi(t)} = 0. \tag{2.5} \]

Consequently,
\[ \limsup_{t \to 0} \frac{Y_k(t) - Y_k(0)}{\phi(t)} = \limsup_{t \to 0} \frac{2 \sum_{i=1}^{k} (\eta_i(t) - U(0,t)\eta_i(0))\eta_i(0)}{\phi(t)}. \tag{2.6} \]

The innovation in this proof is to recognize that
\[ \sum_{i=1}^{k} (\eta_i(t) - U(0,t)\eta_i(0))\eta_i(0) \tag{2.7} \]
is actually a one dimensional real valued Gaussian process. Let \( \{ \xi_i(t), t \in [0, 1] \}, i = 1, \ldots, k \), be independent copies of a mean zero Gaussian process \( \{ \xi(t), t \in [0, 1] \} \), and set \( \xi(t) = (\xi_1(t), \ldots, \xi_k(t)) \). Let \( \bar{v} \in \mathbb{R}^k \) with \( \|\bar{v}\|_2 = 1 \). Computing the covariances we see that,

\[
\{ (\bar{v} \cdot \zeta(t)), t \in [0, 1] \} \overset{\text{law}}{=} \{ \xi(t), t \in [0, 1] \}. \tag{2.8}
\]

(This relationship is used by P. Revesz in [8, Theorem 18.1] to obtain LILs for Brownian motion in \( \mathbb{R}^k \).)

Therefore, since \( (\eta_i(t) - (U(0, t)/U(0, 0))\eta_i(0)) \) and \( \eta_i(0) \) are independent for \( i = 1, \ldots, k \), we see that,

\[
\left\{ \left( \eta(t) - \frac{U(0, t)}{U(0, 0)} \eta(0) \right) \right\} \overset{\text{law}}{=} \left\{ \left( \eta(t) - \frac{U(0, t)}{U(0, 0)} \eta(0) \right) \right\}, \tag{2.9}
\]

where \( \eta(t) = (\eta_1(t), \ldots, \eta_k(t)) \) and

\[
\|\eta(t)\|_2 = \left( \sum_{i=1}^{k} \eta_i^2(t) \right)^{1/2} = Y_k^{1/2}(t). \tag{2.10}
\]

Consequently, (2.6) implies that

\[
\limsup_{t \to 0} \frac{Y_k(t) - Y_k(0)}{\phi(t)} \overset{\text{law}}{=} \limsup_{t \to 0} \frac{2 \left( \eta(t) - \frac{U(0, t)}{U(0, 0)} \eta(0) \right)}{\phi(t)}. \tag{2.11}
\]

Using (2.4) again and (2.5) we see that this implies that,

\[
\limsup_{t \to 0} \frac{Y_k(t) - Y_k(0)}{\phi(t)} \overset{\text{law}}{=} \limsup_{t \to 0} \frac{2(\eta(t) - \eta(0))}{\phi(t)} = 2, \tag{2.12}
\]

where the last equality uses (1.3). Using (2.10) we obtain (1.4).

To obtain (2.5) we first note that it follows from (1.3) that,

\[
\phi(t) = (E(\eta(t) - \eta(0))^2)^{1/2} h(t), \tag{2.13}
\]

for some function \( h \) such that \( \lim_{t \to 0} h(t) = \infty \). To see this, suppose that it is false. Then there exists a sequence \( \{ t_k \} \) with \( \lim_{k \to \infty} t_k = 0 \), such that \( \sup_k h(t_k) \leq M \). Therefore, if (1.3) holds, we would have,

\[
\sup_k \frac{\eta(t_k) - \eta(0)}{(E(\eta(t_k) - \eta(0))^2)^{1/2}} \leq M \quad a.s. \tag{2.14}
\]

This is not possible because \( \{ \eta(t_k) - \eta(0)/(E(\eta(t_k) - \eta(0))^2)^{1/2} \} \) is a sequence of standard normal random variables.

Since,

\[
U(0, 0) - U(0, t) = E \left( (\eta(t) - \eta(0)) \eta(0) \right) \leq E \left( (\eta(t) - \eta(0))^2 \right)^{1/2} U^{1/2}(0, 0), \tag{2.15}
\]

we have,

\[
\frac{U(0, 0) - U(0, t)}{\phi(t)} \leq \frac{U^{1/2}(0, 0)}{h(t)}. \tag{2.16}
\]

Using the fact that \( \lim_{t \to 0} h(t) = \infty \) we get (2.5). \hfill \Box
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**Proof of Theorem 1.2.** Note that (1.9) implies that \{\eta(t), t \in [0,1]\} and therefore \{\eta^2(t), t \in [0,1]\} are uniformly continuous on [0,1], which in turn implies that for all \(k \geq 1\), \(\{Y_k(t); t \in [0,1]\}\) is uniformly continuous on [0,1].

To show,

\[
\lim_{h \to 0} \sup_{|u-v| \leq h} \frac{Y_k(u) - Y_k(v)}{\varphi(|u-v|)} \geq 2 \sup_{t \in \Delta} Y_k^{1/2}(t), \quad \text{a.s.}
\]  

(2.17)

it suffices to show that for any \(d \in \Delta\),

\[
\lim_{h \to 0} \sup_{|u-v| \leq h} \frac{Y_k(u) - Y_k(v)}{\varphi(|u-v|)} \geq 2 Y_k^{1/2}(d), \quad \text{a.s.}
\]  

(2.18)

This is because, (2.18) holding almost surely implies that for any countable dense set \(\Delta' \subset \Delta\),

\[
\lim_{h \to 0} \sup_{|u-v| \leq h} \frac{Y_k(u) - Y_k(v)}{\varphi(|u-v|)} \geq 2 \sup_{d \in \Delta'} Y_k^{1/2}(d), \quad \text{a.s.}
\]  

(2.19)

which implies (2.17).

Let \(u, v, d \in \Delta\). We write,

\[
\eta^2(u) - \eta^2(v) = (\eta(u) - \eta(v))(\eta(u) + \eta(v))
\]  

(2.20)

\[
= (\eta(u) - \eta(v)(2\eta(d) + (\eta(u) - \eta(d)) + (\eta(v) - \eta(d))).
\]

Using the line above we see that we can write \(Y_k(u) - Y_k(v)\) as a sum of three terms, two of which contain the product of two differences of \(\eta\). We show below that these two terms, in the limit, are negligible with respect to the uniform modulus of \(Y_k(u) - Y_k(v)\). To this end, using (2.10), we note that

\[
\lim_{h \to 0} \sup_{|u-v| \leq h} \frac{Y_k(u) - Y_k(v)}{\varphi(|u-v|)} Y_k^{1/2}(d)
\]

(2.21)

\[
\geq \lim_{h \to 0} \sup_{|u-v| \leq h} \frac{2 \sum_{i=1}^k (\eta_i(u) - \eta_i(v))\eta_i(d)}{\|\eta(d)\|_2 \varphi(|u-v|)}
\]

\[
- \lim_{h \to 0} \sup_{|u-v| \leq h} \frac{2 \sum_{i=1}^k (\eta_i(u) - \eta_i(v))(\eta_i(u) - \eta_i(d))}{\|\eta(d)\|_2 \varphi(|u-v|)}.
\]

It follows from (1.9) that,

\[
\lim_{h \to 0} \sup_{|u-v| \leq h} \frac{\sum_{i=1}^k (\eta_i(u) - \eta_i(v))(\eta_i(u) - \eta_i(d))}{\varphi(|u-v|)}
\]

(2.22)

\[
\leq \lim_{h \to 0} \sup_{|u-v| \leq h} \sum_{i=1}^k |\eta_i(u) - \eta_i(v)| \sup_{u \in \Delta} |\eta_i(u) - \eta_i(d)| \varphi(|u-v|)
\]

\[
\leq \sum_{i=1}^k |\eta_i(u) - \eta_i(d)| := \Delta^*.
\]

So that,

\[
\lim_{h \to 0} \sup_{|u-v| \leq h} \frac{Y_k(u) - Y_k(v)}{\varphi(|u-v|) Y_k^{1/2}(d)}
\]

(2.23)

\[
\geq \lim_{h \to 0} \sup_{|u-v| \leq h} \frac{2 \sum_{i=1}^k (\eta_i(u) - \eta_i(v))\eta_i(d)}{\|\eta(d)\|_2 \varphi(|u-v|)} - \frac{2\Delta^*}{\|\eta(d)\|_2}.
\]
We now use our critical relationship (2.8) to replace the sum in (2.23) by the difference of a real valued Gaussian process. To do this we write

$$\eta_i(u) - \eta_i(v) = V(v, d)\eta_i(u) - V(u, d)\eta_i(v) + G_i(u, v), \quad (2.24)$$

where, $V(u, v) = U(u, v)/U(d, d)$, and

$$G_i(u, v) = (1 - V(v, d))\eta_i(u) - (1 - V(u, d))\eta_i(v). \quad (2.25)$$

In this notation,

$$\sum_{i=1}^{k}(\eta_i(u) - \eta_i(v))\frac{\eta_i(d)}{||\eta(d)||_2 \varphi(|u - v|)} = \frac{\sum_{i=1}^{k} G_i(u, v)\eta_i(d)}{||\eta(d)||_2 \varphi(|u - v|)}. \quad (2.26)$$

Note that for all $u, v \in [0, 1]$, $E ((V(v, d)\eta_i(u) - V(u, d)\eta_i(v))\eta_i(d)) = 0$.

This shows that $\eta_i(d)$ is independent of $\{V(v, d)\eta_i(u) - V(u, d)\eta_i(v); u, v \in [0, 1]\}$. Therefore by (2.8),

$$\left\{ \sum_{i=1}^{k} (V(v, d)\eta_i(u) - V(u, d)\eta_i(v)) \frac{\eta_i(d)}{||\eta(d)||_2 \varphi(|u - v|)} ; u, v \in [0, 1] \right\} \overset{law}{=} \left\{ V(v, d)\eta(u) - V(u, d)\eta(v); u, v \in [0, 1] \right\}. \quad (2.27)$$

It follows that,

$$\lim_{h \to 0} \sup_{|u - v| < h} \frac{\sum_{i=1}^{k}(\eta_i(u) - \eta_i(v))\eta_i(d)}{||\eta(d)||_2 \varphi(|u - v|)} = \frac{\sum_{i=1}^{k} G_i(u, v)\eta_i(d)}{||\eta(d)||_2 \varphi(|u - v|)} + \lim_{h \to 0} \sup_{|u - v| < h} \frac{\sum_{i=1}^{k} G_i(u, v)\eta_i(d)}{||\eta(d)||_2 \varphi(|u - v|)} \overset{law}{\geq} \lim_{h \to 0} \sup_{|u - v| < h} \frac{\sum_{i=1}^{k} (V(v, d)\eta_i(u) - V(u, d)\eta_i(v))\eta_i(d)}{||\eta(d)||_2 \varphi(|u - v|)} \overset{law}{\geq} \lim_{h \to 0} \sup_{|u - v| < h} \frac{\sum_{i=1}^{k} (V(v, d)\eta_i(u) - V(u, d)\eta_i(v))\eta_i(d)}{||\eta(d)||_2 \varphi(|u - v|)}.$$ 

Using (2.24) we write,

$$\lim_{h \to 0} \sup_{|u - v| < h} \frac{V(v, d)\eta(u) - V(u, d)\eta(v)}{\varphi(|u - v|)} = \frac{G(u, v)}{\varphi(|u - v|)} - \lim_{h \to 0} \sup_{|u - v| < h} \frac{|G(u, v)|}{\varphi(|u - v|)} = 1 - \lim_{h \to 0} \sup_{|u - v| < h} \frac{|G(u, v)|}{\varphi(|u - v|)}, \quad (2.29)$$

where

$$G(u, v) = (1 - V(v, d))\eta(u) - (1 - V(u, d))\eta(v) \quad (2.30)$$

and we use (1.9) for the last line in (2.29).
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It follows from (2.28) and (2.29) that,
\[
\lim_{h \to 0} \sup_{u,v \in \Delta} \frac{\sum_{i=1}^{k} |G_i(u,v)| \eta_i(d)}{||\eta(d)||_2 \varphi(|u-v|)} \overset{law}{=} 1 - \mathcal{H},
\] (2.31)
where,
\[
\mathcal{H} = \lim_{h \to 0} \sup_{u,v \in \Delta} \frac{\sum_{i=1}^{k} |G_i(u,v)||\eta_i(d)|}{||\eta(d)||_2 \varphi(|u-v|)} + \lim_{h \to 0} \sup_{u,v \in \Delta} \frac{|G(u,v)|}{\varphi(|u-v|)}.
\]

We now show that \( \mathcal{H} = o(\Delta) \) almost surely. Using the Schwartz inequality followed by the triangle inequality we note that,
\[
\sum_{i=1}^{k} \frac{|G_i(u,v)||\eta_i(d)|}{\varphi(|u-v|)} \leq \sum_{i=1}^{k} |G_i(u,v)|.
\]
Therefore,
\[
\mathcal{H} \leq \sum_{i=0}^{k} \lim_{h \to 0} \sup_{u,v \in \Delta} \frac{|G_i(u,v)|}{\varphi(|u-v|)}, \quad a.s.,
\] (2.32)
where for notational convenience we have set \( G_0 = G \).

Using the notation in (1.7), it follows from (1.9) that we can write,
\[
\varphi(h) = \tilde{\sigma}(h)g(h), \quad \text{where necessarily, } \lim_{h \to 0} g(h) = \infty.
\] (2.33)

This is shown by a minor modification of the argument used to prove (2.13). Note that for any sequence \( h_k \to 0 \) we can find sequences \( \{u_k\}, \{v_k\} \) in \( \Delta \), with \( |u_k - v_k| \leq h_k \) such that \( \tilde{\sigma}(h_k) \leq 2\sigma(u_k, v_k) \). Now, suppose that \( \limsup_{h \to 0} g(h) = M \). Then by (1.9) we would have,
\[
\sup_{k \to \infty} \frac{\eta(u_k) - \eta(v_k)}{\sigma(u_k, v_k)} \leq 4M \quad a.s.
\]

This is not possible because each term \( (\eta(u_k) - \eta(v_k))/\sigma(u_k, v_k) \) is a standard normal random variable.

We show in Lemma 2.1 below that,
\[
|G(u,v)| \leq \frac{\sigma(u,v)}{U^{1/2}(d,d)} |\eta(u) - \eta(v)| + \frac{\sigma(u,v)}{U^{1/2}(d,d)} |\eta(v)|.
\] (2.34)

Using this and then (1.9) and (2.33) we have,
\[
\lim_{h \to 0} \sup_{u,v \in \Delta} \frac{|G(u,v)|}{\varphi(|u-v|)} \leq \lim_{h \to 0} \sup_{u,v \in \Delta} \frac{|\eta(u) - \eta(v)|\sigma(u,v)}{U^{1/2}(d,d) \varphi(|u-v|)} + \lim_{h \to 0} \sup_{u,v \in \Delta} \frac{\sigma(u,v)}{U^{1/2}(d,d) \varphi(|u-v|)}|\eta(v)|
\]
\[
\leq \sup_{d,v \in \Delta} \frac{\sigma(d,v)}{U^{1/2}(d,d)} + \lim_{h \to 0} \frac{1}{g(h)} \sup_{v \in \Delta} \frac{|\eta(v)|}{U^{1/2}(d,d)} = \frac{\tilde{\sigma}(|\Delta|)}{U^{1/2}(d,d)}.
\]
where \( \tilde{\sigma}(|\Delta|) \) is defined in (1.7). This shows that
\[
\mathcal{H} \leq (k + 1) \frac{\tilde{\sigma}(|\Delta|)}{U^{1/2}(d,d)}.
\] (2.36)

We now use (2.23), (2.31) and (2.35) to see that,
Lemma 2.1. 

\[ |G(u, v)| \leq \frac{\sigma(d, v)}{U^{1/2}(d, d)} |\eta(u) - \eta(v)| + \frac{\sigma(u, v)}{U^{1/2}(d, d)} |\eta(v)|. \]  

(2.42)
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Proof. We write,

\[ G(u, v) = (1 - V(v, d))(\eta(u) - \eta(v)) + (V(u, d) - V(v, d))\eta(v). \]

Note that,

\[ |(V(u, d) - V(v, d))| = \left| \frac{(E(\eta(u) - \eta(v))\eta(d))}{U(d)} \right| \leq \frac{(E(\eta(u) - \eta(v))^2E\eta^2(d))^{1/2}}{U(d)}, \]

and,

\[ (1 - V(v, d)) = \frac{(E(\eta(d) - \eta(v))\eta(d))}{U(d)} \leq \frac{(E(\eta(d) - \eta(v))^2E\eta^2(d))^{1/2}}{U(d)}. \]

\[ \Box \]

Lemma 2.2.

\[ P \left( \Delta^* \geq k((1 + 2C)\tilde{\sigma}^2(\|\Delta\|) \log 1/\|\Delta\|^{1/2}) \right) \leq 2k|\Delta|^C, \quad (2.43) \]

(see (2.22)).

Proof. Let \( a \) be the median of \( \sup_{u \in \Delta} (\eta_1(u) - \eta_1(d)) \). It follows from [4, Lemma 5.4.3] that,

\[ P \left( \sup_{u \in \Delta} |\eta_1(u) - \eta_1(d)| \geq a + \tilde{\sigma}(\|\Delta\|)t \right) \leq 2e^{-t^2/2}. \]

Since by [4, (7.113)],

\[ a = o(\tilde{\sigma}^2(\|\Delta\|) \log 1/\|\Delta\|^{1/2}), \]

we see that

\[ P \left( \sup_{u \in \Delta} |\eta_1(u) - \eta_1(d)| \geq ((1 + 2C)\tilde{\sigma}^2(\|\Delta\|) \log 1/\|\Delta\|^{1/2}) \right) \leq 2|\Delta|^C, \]

and since \( \Delta^* = \sum_{i=1}^k \sup_{u \in \Delta} |\eta_i(u) - \eta_i(d)| \), see (2.22), this gives (2.43). \( \Box \)

Remark 2.3. When \( \eta \) has stationary increments, \( \sigma^2(u, v) \) in (1.7) can be written as \( \sigma^2(u - v) \). For \( x \in [0, \delta] \) define,

\[ \sigma^2(x) = \mu \left( s : \sigma^2(s) \leq x \right) , \]

where \( \mu \) is Lebesgue measure. Clearly, \( \sigma^2(x) \) is an increasing function. It is called the increasing rearrangement of \( \sigma^2(x) \). (See [4, Section 6.4] for more details.) We show in [4, (6.138)] that when \( \eta \) is continuous a.s.,

\[ \lim_{x \to 0} \sigma^2(x) \log(1/x) = 0. \]

This shows that if \( \tilde{\sigma}^2(x) \) is asymptotic to an increasing function at 0, then (1.9), which implies that \( \eta \) is continuous a.s., implies (1.8).
Moduli of continuity of chi–square processes

References