

STEIN'S METHOD, GAUSSIAN PROCESSES AND PALM MEASURES, WITH APPLICATIONS TO QUEUEING

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We develop a general approach to Stein's method for approximating a random process in the path space $\mathbb{D}([0, T] \rightarrow \mathbb{R}^d)$ by a real continuous Gaussian process. We then use the approach in the context of processes that have a representation as integrals with respect to an underlying point process, deriving a general *quantitative* Gaussian approximation. The error bound is expressed in terms of couplings of the original process to processes generated from the reduced Palm measures associated with the point process. As applications, we study certain GI/GI/ ∞ queues in the "heavy traffic" regime.

1. Introduction. Gaussian processes arise as approximations to real processes in a wide variety of applications. Often, the approximation is taken as read, and Gaussian processes become part of the model, as in stochastic integrals in finance. In other circumstances, as in queuing systems, they arise as approximations in the limit; see, for example, Robert (2003) and Pang, Talreja and Whitt (2007). The fundamental example, which forms the basis of many other limiting results, is Donsker's theorem, which states that random walk, after proper normalization, converges weakly in path space to Brownian motion. Then probabilities for systems that converge to a Gaussian process may be approximated by the analogous limiting probabilities, which are typically more tractable, due to the many beautiful and useful properties of Gaussians. A key task in this setting is to estimate the error made in the approximation. For Donsker's theorem, this is well understood, but for more general processes there are few results.

In this paper, we establish a Stein equation, together with properties of its solutions, suitable for use in quantifying the error in approximating a multi-dimensional càdlàg process by a general Gaussian process. The approach generalizes and improves the theory for approximation by Brownian motion presented in Barbour (1990), and dovetails with the companion paper Barbour, Ross and Zheng (2021) to give bounds on the error in terms of the Lévy–Prokhorov distance, which metrizes weak convergence with respect to the Skorokhod topology. As a concrete application of the method, we prove a general result, Theorem 1.4, that gives such bounds when the process being approximated can be expressed as an integral with respect to a point process. Theorem 1.4 is then applied to M/GI/ ∞ and GI/GI/ ∞ queues in the heavy traffic regime, obtaining the first rates of convergence in some settings that are closely related to limiting approximations given in Iglehart (1965), Borovkov (1967), Whitt (1982), Krichagina and Puhalskii (1997) and Puhalskii and Reed (2010), where the limiting processes are typically not Brownian motion.

To give a flavour of the Stein approach, set out in detail in Proposition 2.1 and Theorem 2.2, suppose that Z is a real centered Gaussian process on the interval $[0, T]$, whose covariance function K can be represented in the form

$$K(s, t) := \mathbb{E}\{Z(s)Z(t)\} = \int_0^T \hat{J}_u(s)\hat{J}_u(t)\Lambda(du),$$

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for a measure Λ on $[0, T]$ and a collection of real functions $(\widehat{J}_u, u \in [0, T])$ on $[0, T]$. For instance, if Z is standard Brownian motion, we can take $\widehat{J}_u(t) := \mathbb{I}[u \leq t]$, and if Z is the Brownian bridge on $[0, T]$, we can take $\widehat{J}_u(t) := \mathbb{I}[u \leq t] - t/T$, in either case with Λ being Lebesgue measure. Let W be a random element of $\mathbb{D}[0, T]$. Then, for any g in a particular class of smooth test functionals on $\mathbb{D}[0, T]$, we show that

$$|\mathbb{E}\{g(W)\} - \mathbb{E}\{g(Z)\}| \leq \left| \mathbb{E} \left\{ \int_0^T D^2 f_g(W)[\widehat{J}_u, \widehat{J}_u] \Lambda(du) - Df_g(W)[W] \right\} \right|,$$

where D denotes the Fréchet derivative, and where the functional f_g can be explicitly represented in terms of g . The expression on the right hand side is reminiscent of those that have been exploited in many variants of Stein’s method, and is amenable to a number of the techniques that have previously been developed to bound them. Note that the choices of Λ and \widehat{J}_u that are appropriate in particular applications may only become clear in the course of evaluating the term $\mathbb{E}\{Df_g(W)[W]\}$. For instance, in Theorem 1.2, W is an integral with respect to a point measure, the expectation $\mathbb{E}\{Df_g(W)[W]\}$ is evaluated using Palm theory, and Λ and the functions \widehat{J}_u emerge naturally in the resulting calculations.

1.1. *Setup.* Let N be a simple point process on \mathbb{R}^d with mean measure λ . Let the collection of functions

$$\{J_u : [0, T] \rightarrow \mathbb{R}^p\}_{u \in \mathbb{R}^d},$$

be such that $J_\bullet(s) \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^p, \lambda)$ for all $s \in [0, T]$, and $(u, s) \mapsto J_u(s)$ is jointly measurable on $\mathbb{R}^d \times [0, T]$. In this paper, we focus on Gaussian process approximation for the random process $X : [0, T] \rightarrow \mathbb{R}^p$ of the form

$$X(s) := \int_{\mathbb{R}^d} J_u(s) N(du).$$

As we see shortly in Sections 1.3 and 1.4, many queueing processes can be written in this form. Before going into specific detail, let us establish the general framework.

Define the centered and scaled random measure $\widetilde{N} := \sigma^{-1}(N - \lambda)$, where $\sigma > 0$ is a scaling parameter, and define the process \widetilde{X} by

$$(1.1) \quad \widetilde{X}(s) := \int_{\mathbb{R}^d} J_u(s) \widetilde{N}(du).$$

We are interested in the distribution of \widetilde{X} when λ is at “high intensity” (i.e., the mass of λ is large), and the choice of σ stabilizes \widetilde{N} ; and, in particular, we want to approximate the distribution of \widetilde{X} by that of a Gaussian process $(Z(t), t \in [0, T])$. Informally, we think of \widetilde{N} as close in distribution to a centered Gaussian random measure N with intensity measure $\Lambda \approx \sigma^{-2}\lambda$, and then in turn $Z(s) = \int_{\mathbb{R}^d} \widehat{J}_u(s) N(du)$, for some possibly different family of functions $\{\widehat{J}_u\}_{u \in \mathbb{R}^d}$. Formally, Z is a centered Gaussian process with covariance function

$$\mathbb{E}[Z(s)Z(t)^\top] = \int_{\mathbb{R}^d} \widehat{J}_u(s)\widehat{J}_u(t)^\top \Lambda(du), \quad t, s \in [0, T],$$

where Z and \widehat{J} are column vectors, and \top denotes transpose. If N is a Poisson process, then the natural approximating Gaussian process has $\Lambda = \sigma^{-2}\lambda$ and $\widehat{J}_u = J_u$, but this is not necessarily the case for other point processes.

The corresponding approximation result, Theorem 1.4 below, gives a bound on

$$|\mathbb{E}[g(\widetilde{X})] - \mathbb{E}[g(Z)]|$$

for a certain set \mathcal{M} of “test” functions g . The bound in the approximation result is completely general for X of the form above, but requires the construction of close couplings

$(N, N^{(u)})_{u \in \mathbb{R}^d}$ of N with its “reduced Palm measures” at $u \in \mathbb{R}^d$. In the case where N is a Poisson process, we can set $N^{(u)} = N$, and our bound becomes very simple; see (1.20). The test functions are described in detail in Section 1.2, but are essentially those introduced in Barbour (1990), and include smooth functions of the process at a fixed number of times. Such test functions are now commonly used for Gaussian process approximation in the Stein’s method literature; see, for example Döbler and Kaspzrak (2021) and Kaspzrak (2017, 2020b). For a sequence of processes $(\tilde{X}_n)_{n \geq 1}$, the fact that $|\mathbb{E}[g(\tilde{X}_n)] - \mathbb{E}[g(Z)]| \rightarrow 0$ for all test functions g in \mathcal{M} does not alone imply weak convergence of the processes with respect to either the supremum or the Skorokhod topologies, but with the results of Barbour, Ross and Zheng (2021) and a little extra work, it is not too difficult in our applications to obtain bounds on the Lévy–Prokhorov distance (with respect to the Skorokhod topology) that tend to zero, and hence imply weak convergence. Such bounds can also be used to derive rates of convergence for statistics that are continuous with respect to the Skorokhod topology. We next discuss the test functions in detail.

1.2. *Test functions.* Let $\mathbb{D}^p := \mathbb{D}([0, T] \rightarrow \mathbb{R}^p)$ be the set of functions from $[0, T]$ to \mathbb{R}^p that are right continuous with left limits. Endowed with the sup norm, \mathbb{D}^p is a Banach space (though not separable), and so for a function $g : \mathbb{D}^p \rightarrow \mathbb{R}$, we denote by $D^k g$ its k th Fréchet derivative, $k \in \mathbb{N}$, whenever it exists. Following Barbour (1990) (see also Kaspzrak, Duncan and Vollmer (2017)), for $g : \mathbb{D}^p \rightarrow \mathbb{R}$, we define

$$\|g\|_L := \sup_{w \in \mathbb{D}^p} \frac{|g(w)|}{1 + \|w\|^3},$$

where $\|w\| = \sup_{0 \leq t \leq T} |w(t)|$ denotes the sup-norm, and then define the Banach space

$$L := \{g : \mathbb{D}^p \rightarrow \mathbb{R} : g \text{ is continuous and } \|g\|_L < \infty\}.$$

For g twice Fréchet differentiable, we define

$$\begin{aligned} \|g\|_M &:= \|g\|_L + \sup_{w \in \mathbb{D}^p} \frac{\|Dg(w)\|}{1 + \|w\|^2} + \sup_{w \in \mathbb{D}^p} \frac{\|D^2g(w)\|}{1 + \|w\|} \\ &\quad + \sup_{w, h \in \mathbb{D}^p} \frac{\|D^2g(w+h) - D^2g(w)\|}{\|h\|}, \end{aligned}$$

where $\|A\| := \sup_{w: \|w\|=1} |A[w^{[k]}]|$ for A a k -linear form, and $A[w^{[k]}] := A[w, w, \dots, w]$. This leads to the space

$$M := \{g : \mathbb{D}^p \rightarrow \mathbb{R} : g \text{ is twice Fréchet differentiable and } \|g\|_M < \infty\}.$$

We also work on its subspace

$$M' := \left\{g \in M : \sup_{w \in \mathbb{D}^p} \|D^2g(w)\| < \infty\right\},$$

and for $g \in M'$, we define the norm

$$\begin{aligned} \|g\|_{M'} &:= \sup_{w \in \mathbb{D}^p} \frac{|g(w)|}{1 + \|w\|^2} + \sup_{w \in \mathbb{D}^p} \frac{\|Dg(w)\|}{1 + \|w\|} + \sup_{w \in \mathbb{D}^p} \|D^2g(w)\| \\ &\quad + \sup_{w, h \in \mathbb{D}^p} \frac{\|D^2g(w+h) - D^2g(w)\|}{\|h\|}. \end{aligned}$$

Note that for $g \in M'$, $\|g\|_M \leq \|g\|_{M'}$. Defining $I_t(s) := \mathbf{1}[s \geq t]$, we also typically assume that a test function g satisfies the smoothness condition that, for any r, s, t in $[0, T]$ and

$x_1, x_2 \in \mathbb{R}^p,$

$$(1.2) \quad \sup_{w \in \mathbb{D}^p} |D^2 g(w)[x_1 I_r, x_2(I_s - I_t)]| \leq S_g |x_1| |x_2| |s - t|^{1/2},$$

where S_g is some constant depending on g , and $|\cdot|$ denotes Euclidean norm.

1.3. *M/GI/∞ queue.* Let M_n be a Poisson process on $\mathcal{S} := [0, T] \times \mathbb{R}_+$ with intensity measure

$$\ell_n(dt, dy) := n\alpha(dt)G(dy),$$

where α is a finite measure on $[0, T]$, and G is a distribution supported on a subset of \mathbb{R}_+ . We can view M_n as a measure. Let $(Y_i, i \geq 1)$ be i.i.d. with distribution \tilde{G} supported on a subset of \mathbb{R}_+ . We set

$$N_n := M_n + \sum_{i=1}^{x_n} \delta_{(0, Y_i)},$$

where $x_n \geq 0$ is an integer. Then, N_n is a point process with mean measure

$$\lambda_n := \ell_n + x_n(\delta_0 \times \tilde{G}).$$

For $(t, y) \in \mathcal{S}$, we define

$$J_{t,y}(s) := \mathbf{1}\{t \leq s < t + y\}$$

and the process $X_n : [0, T] \rightarrow \mathbb{R}_+$ by

$$X_n(s) := \int_{\mathcal{S}} J_{t,y}(s) N_n(dt, dy).$$

The process X_n can be regarded as the number of customers in an M/GI/∞ queue: A point $(t, y) \in N_n$ represents a customer arriving at time t with service time y , and such a customer will be in the system at any instant s satisfying $t \leq s < t + y$. We allow the customers initially in the system to have a different service distribution, to model the situation where the process is first observed at a typical time; in such a case, the residual service times would have a distribution derived from G , but not necessarily the same as G . We consider the “heavy traffic” regime, in which n is large, so that the total rate of arrivals $n\alpha(dt)$ is large.

Define the centered and scaled random measure

$$\tilde{N}_n := \sigma_n^{-1}(N_n - \lambda_n) \quad \text{with } \sigma_n^2 = n,$$

and the process \tilde{X}_n by

$$(1.3) \quad \tilde{X}_n(s) := \int_{\mathcal{S}} J_{t,y}(s) \tilde{N}_n(dt, dy).$$

Before we state our main result in this setting, let us fix some notation.

NOTATION.

(i) Given the distribution G and the finite measure α on $[0, T]$ as above, we define the convolution $G * \alpha$ by

$$(1.4) \quad (G * \alpha)(s) := \int_0^s G(s - u)\alpha(du),$$

(where, for brevity, we also use G to denote the cumulative distribution function of the distribution G), and the cumulative intensity A by

$$(1.5) \quad A(s) = \alpha([0, s]).$$

(ii) Define

$$(1.6) \quad \Psi_n(x, x_n, \alpha, T) := 3 \frac{\sqrt{\pi x_n/2} + |x_n - nx|}{2n} + \frac{\alpha([0, T]) + n^{-1}x_n}{2\sqrt{n}}.$$

THEOREM 1.1. *Assume that the convolution $G * \alpha$ defined in (1.4), the cumulative intensity A from (1.5), and the distribution function \tilde{G} are all β -Hölder continuous for some $\beta \in (1/2, 1]$. Define a measure Λ on $\mathcal{S} := [0, T] \times \mathbb{R}_+$ by*

$$\Lambda := (\alpha \times G) + x(\delta_0 \times \tilde{G}),$$

where $x \geq 0$ is fixed. Now set $\hat{J}_{(t,y)} := J_{t,y}$ for $t > 0$ and $\hat{J}_{(0,y)} := J_{0,y} - (1 - \tilde{G})$. Let Z be a real centered Gaussian process with covariance function K given, for $0 \leq s_1 \leq s_2 \leq T$, by

$$(1.7) \quad \begin{aligned} K(s_1, s_2) &= \mathbb{E}[Z(s_1)Z(s_2)] = \int_{\mathcal{S}} \hat{J}_{(t,y)}(s_1)\hat{J}_{(t,y)}(s_2)\Lambda(dt, dy) \\ &= \int_0^{s_1} (1 - G(s_2 - t))\alpha(dt) + x\tilde{G}(s_1)(1 - \tilde{G}(s_2)). \end{aligned}$$

Let \tilde{X}_n be defined as in (1.3). Then, for any $g \in M'$ either satisfying (1.2), or of the form $g(w) = F(w(t_1), \dots, w(t_k))$ for some (twice differentiable) $F: \mathbb{R}^k \rightarrow \mathbb{R}$ and distinct instants $t_1, \dots, t_k \in [0, T]$, we have

$$(1.8) \quad |\mathbb{E}[g(\tilde{X}_n)] - \mathbb{E}[g(Z)]| \leq 2^{3/2} \|g\|_{M'} \Psi_n(x, x_n, \alpha, T),$$

where $\Psi_n(x, x_n, \alpha, T)$ is defined in (1.6).

If $\beta = 1 \leq \alpha([0, T]) \leq \alpha_* T$ for some $\alpha_* < \infty$, and if $T \geq n^{1/2}|x_n n^{-1} - x|$, then, for any $\chi > 0$, there is a constant K_χ such that

$$d_{\text{LP}}(\mathcal{L}(\tilde{X}_n), \mathcal{L}(Z)) \leq K_\chi n^\chi T^{2/5} n^{-1/20},$$

where d_{LP} denotes the Lévy–Prokhorov distance (with respect to Skorokhod topology).

REMARK 1.

(i) The constant S_g from the smoothness condition (1.2) does not appear in the bound (1.8), since the condition (1.2) is only used to apply a technical result; see Lemma 5.3 below.

(ii) As we only consider time intervals $[0, T]$, we only require the Hölder continuity on $[0, T]$.

(iii) Bounds can also be derived under more general assumptions on α and β . These can be deduced from the proof of the theorem.

REMARK 2.

(i) That $\mathcal{L}(\tilde{X}_n) \rightarrow \mathcal{L}(Z)$ with respect to Skorokhod topology for $x_n = 0$ is due to Borovkov (1967)—see the discussion in Whitt (1982)—and, for general x_n , follows from the results of Krichagina and Puhalskii (1997). The only rates of convergence we are aware of are those of Besançon, Decreusefond and Moyal (2020) and Besançon et al. (2021), which give Wasserstein bounds (with respect to the supremum metric) in the special case of the M/M/ ∞ queue. Using Barbour, Ross and Zheng ((2021), Theorem 1.1) with the bounds of this paper would lead to rates in the Wasserstein distance that would be worse than those derived in Besançon et al. (2021) in this case. However, our results apply more generally to M/GI/ ∞ queues, which do not appear to be within the scope of their methods.

(ii) In the best possible case, where $\beta = 1$, our rate of convergence for the Lévy–Prokhorov metric, of $O(T^{2/5}n^{-1/20+\chi})$ for any $\chi > 0$, is, to the best of our knowledge, the first rate of convergence in this metric. An advantage of our bounds is that the dependence on T is explicit, and that T could grow like a small power of n while still yielding a small bound; this would cover transient approximation almost to stationarity.

REMARK 3. We can represent Z as a sum of three independent centered Gaussian processes $Z = Z_1 + Z_2 + Z_3$, where:

(i) Z_1 represents the randomness from the services and has covariance structure

$$\mathbb{E}[Z_1(s_1)Z_1(s_2)] = \int_0^{s_1} G(s_1 - t)(1 - G(s_2 - t))\alpha(dt),$$

for $0 \leq s_1 \leq s_2 \leq T$.

(ii) The second process Z_2 represents the randomness from the arrival process (a kind of weighted renewal functional CLT) and is given by the stochastic integral

$$Z_2(s) = \int_0^s (1 - G(s - t))B(dt),$$

where $B(\cdot)$ is a Gaussian random measure with intensity measure $\alpha(dt)$, that is,

$$\mathbb{E}[Z_2(s_1)Z_2(s_2)] = \int_0^{s_1} (1 - G(s_1 - t))(1 - G(s_2 - t))\alpha(dt);$$

for $0 \leq s_1 \leq s_2 \leq T$.

(iii) The third process Z_3 is a time-changed Brownian bridge: $Z_3(t) = \sqrt{x}B^{br}(\tilde{G}(t))$, where B^{br} is a Brownian bridge with $B^{br}(0) = B^{br}(1) = 0$.

For $x = 0$, this decomposition is identified in Borovkov (1967); see also Whitt ((1982), (2.5) and (2.6)) and the discussion there. The addition of Z_3 is due to the presence of customers initially in the system. The number of those remaining in the system at time t is just the number with service time greater than t , the empirical complementary cumulative distribution function at t . This, after scaling, converges as a function of t to a time-changed Brownian bridge.

REMARK 4. If $g(w) = F(w(t_1), \dots, w(t_k))$, and F is bounded with bounded partial derivatives of order up to three, then $\|g\|_M \leq ck^3$, where c is an upper bound for F and its first three partial derivatives. Assuming that $|x_n - nx| = O(n^{1/2})$, the bound (1.8) is of order $O(k^3n^{-1/2})$, and standard smoothing arguments, as in Götze et al. ((2019), Section 1.1.4), imply that for any convex $\mathcal{K} \subseteq \mathbb{R}^k$ and $\varepsilon > 0$,

$$|\mathbb{P}((\tilde{X}_n(t_i))_{i=1}^k \in \mathcal{K}) - \mathbb{P}((Z(t_i))_{i=1}^k \in \mathcal{K})| \leq C\{\varepsilon^{-3}k^3n^{-1/2} + k^{1/4}\varepsilon\}.$$

Choosing $\varepsilon = k^{11/16}n^{-1/8}$, leads to a uniform upper bound of order $k^{15/16}n^{-1/8}$ on the difference of convex set probabilities for any k -dimensional distributions of \tilde{X}_n and Z . The power of k in the bound is likely not optimal, but still leads to meaningful results for k growing like a small power of n .

1.4. GI/GI/ ∞ queue. Consider a stationary renewal process V_n on $[0, T]$, whose renewal distribution v_n is that of R/n , where R is a positive integer valued random variable with aperiodic support having mean m , variance v^2 , and $\mathbb{E}[R^r] < \infty$ for some fixed $r \geq 5$. Let G be a distribution function supported on a subset of \mathbb{R}_+ , and let $(Y_i, i \geq 1)$ be an i.i.d.

sequence with distribution G that is also independent of the renewal process V_n . Now define the random measure N_n on $\mathcal{S} := [0, T] \times \mathbb{R}_+$ by

$$N_n := \sum_{i=1}^{\lfloor nT \rfloor} M_n(i/n) \delta_{(i/n, Y_i)},$$

where now

$$M_n(s) := V_n(s) - V_n(s-) := \mathbf{1}[\text{a renewal occurs at time } s], \quad s \in [0, T],$$

and Y_i represents the service time of a customer arriving at time i/n . Due to the stationarity, the mean measure of N_n is

$$\lambda_n := \frac{1}{m} \sum_{i=1}^{\lfloor nT \rfloor} (\delta_{i/n} \times G).$$

For $(t, y) \in \mathcal{S}$, we define $J_{t,y}(s) := \mathbf{1}[t \leq s < t + y]$ and

$$X_n(s) := \int_{\mathcal{S}} J_{t,y}(s) N_n(dt, dy).$$

As for the $M/GI/\infty$ queue, the process X_n can be regarded as the number of customers in a $GI/GI/\infty$ queue with arrival times given by a stationary renewal process driven by v_n and service times distributed according to G . Note that because v_n (the law of R/n) is discrete, the results of the previous section are not a special case of those derived here.

Define the centered and scaled random measure

$$\tilde{N}_n := \sigma_n^{-1} (N_n - \lambda_n),$$

where

$$(1.9) \quad \sigma_n^2 := nv^2/m^3,$$

and define the process \tilde{X}_n by

$$(1.10) \quad \tilde{X}_n(s) := \int_{\mathcal{S}} J_{t,y}(s) \tilde{N}_n(dt, dy).$$

Writing $I_t(\cdot) := \mathbf{1}\{\cdot \geq t\}$ and $\bar{G} := 1 - G$, our main result of the section is the following.

THEOREM 1.2. *Recall the notation just above. Assume that $\mathbb{E}R^5 < \infty$ and that, for some $\beta \in (0, 1]$ and $0 < \eta < 1$, the distribution function G satisfies*

$$(1.11) \quad G(t) - G(s) \leq g_G(s)(t - s)^\beta, \quad 0 < s < t \leq T,$$

where $g_G: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is bounded and nonincreasing and such that $\int_0^\infty g_G^\eta(s) ds < \infty$. Now define a measure Λ on $\mathcal{S} := [0, T] \times \mathbb{R}_+$ by

$$\Lambda(dt, dy) := dtG(dy),$$

and set

$$\hat{J}_{t,y}(s) := \frac{m}{v} J_{t,y}(s) - \frac{m + v}{v} \bar{G}(s - t) I_t(s).$$

Let Z be a Gaussian process with covariance function K given, for $0 \leq s_1 \leq s_2 \leq T$, by

$$(1.12) \quad \begin{aligned} K(s_1, s_2) &= \mathbb{E}[Z(s_1)Z(s_2)] = \int_{\mathcal{S}} \hat{J}_{t,y}(s_1) \hat{J}_{t,y}(s_2) \Lambda(dt, dy) \\ &= \frac{m^2}{v^2} \int_0^{s_1} \bar{G}(s_2 - t) G(s_1 - t) dt + \int_0^{s_1} \bar{G}(s_1 - t) \bar{G}(s_2 - t) dt. \end{aligned}$$

Let \tilde{X}_n be as defined in (1.10). Then, for any $g \in M'$ satisfying the smoothness condition (1.2) for some S_g , there is a constant C depending on $\mathcal{L}(R)$ and $g_G(0)$ such that, with $\bar{\beta} := \min\{\beta, 1/2\}$, we have

$$(1.13) \quad |\mathbb{E}[g(\tilde{X}_n)] - \mathbb{E}[g(Z)]| \leq CT(S_g n^{-1/2} + \|g\|_{M'} n^{-\bar{\beta}}).$$

If $g \in M'$ is of the form $g(w) = F(w(t_1), \dots, w(t_k))$ for some (twice differentiable) $F: \mathbb{R}^k \rightarrow \mathbb{R}$ and distinct instants $t_1, \dots, t_k \in [0, 1]$, then the same bound holds, but with $S_g n^{-1/2}$ replaced by $\|g\|_{M'} k^3 n^{-1}$.

Moreover, assuming that $T \geq 1$, taking

$$(1.14) \quad l_r := \lceil (r - \eta(r - 1))/2 \rceil - 1, \quad \beta_r := \beta(r - 2)/(r - 1),$$

we have

$$(1.15) \quad d_{LP}(\mathcal{L}(\tilde{X}_n), \mathcal{L}(Z)) \leq C \sqrt{\log n} \{ (T^4 n^{-\bar{\beta}})^{(l_r \beta_r - 1)} T^3 \}^{1/(6l_r + 4l_r \beta_r - 1)},$$

where d_{LP} denotes the Lévy–Prokhorov distance (with respect to the Skorokhod topology). For instance, if the distribution of R has all positive moments, and if G has a finite moment and a bounded and ultimately monotone density, and if $T \leq n^\psi$ for some $\psi < 1/8$, then, for any $\chi > 0$, there is a constant K_χ such that

$$d_{LP}(\mathcal{L}(\tilde{X}_n), \mathcal{L}(Z)) \leq K_\chi n^\chi T^{2/5} n^{-1/20}.$$

REMARK 5. The expression (1.12) for the covariance agrees (up to scaling) with Whitt ((1982), (2.5) and (2.7)). Note also that choosing G supported on (T, ∞) corresponds to the renewal CLT, where the limiting Gaussian process is the standard Brownian motion. The decomposition in Remark 3 still applies here with $Z = (m^2/v^2)Z_1 + Z_2$ and $\alpha(dt) = dt$. The addition of initial customers would contribute to the limit in the same way as in Theorem 1.1; that is, it would add a Brownian bridge component as described in Remark 3, but we omit this for the sake of clarity. Weak convergence with respect to the Skorokhod topology was (essentially) shown in Borovkov (1967), and here we can view (1.15) as a rate of convergence. We are not aware of previous results with rates of convergence. The same considerations as in Remark 4 lead to a uniform upper bound of order $k^{15/16} n^{-\bar{\beta}/4}$ on the difference of convex set probabilities for any k -dimensional distributions of \tilde{X}_n and Z . Again, we remark that an appealing aspect of our bound is the explicit incorporation of T .

REMARK 6. For a process X'_n defined analogously to X_n , but driven by V'_n defined to be a delayed (rather than stationary) renewal process with inter-renewal distribution $\mathcal{L}(R/n)$, it is easy to see that X_n and X'_n can be constructed on the same space so that $\|X_n - X'_n\|$ is stochastically dominated by the coupling time

$$T_c := \inf\{i \geq 1: M_n(i/n) = M'_n(i/n) = 1\},$$

where $M'_n(s) = V'_n(s) - V'_n(s-)$ is defined in analogy with M_n (here, we view V_n and V'_n as renewal processes on $[0, \infty)$, and so T_c does not depend on n). Defining the scaled process

$$\tilde{X}'_n := \frac{X'_n - \int_S J_{t,y} \lambda_n(dt, dy)}{\sigma_n},$$

we thus easily find that $\|\tilde{X}_n - \tilde{X}'_n\|$ is stochastically dominated by $(m^{3/2} T_c/v) n^{-1/2}$. Under the hypotheses of the theorem, Pitman ((1974), Proposition (6.10)) implies that T_c is finite with probability one, and if the delay distribution has finite $(r + 1)$ -moment, that $\mathbb{E}[T'_c] < \infty$.

Under this moment condition, for any function $g \in M'$ with $\|Dg(w)\| < \infty$, we then easily have that

$$|\mathbb{E}[g(\tilde{X}'_n)] - \mathbb{E}[g(\tilde{X}_n)]| \leq \|Dg(w)\| \frac{m^{3/2}\mathbb{E}[T_c]}{v\sqrt{n}}.$$

Combining this with the bounds of the theorem and the triangle inequality gives bounds on $|\mathbb{E}[g(\tilde{X}'_n)] - \mathbb{E}[g(Z)]|$ for $g \in M'$ with $\|Dg(w)\| < \infty$, and subsequent bounds of the same order on d_{LP} (the restriction that $\|Dg(w)\| < \infty$ is no problem, since the key result Theorem 5.1 only requires bounds on a smaller class of test functions with bounded derivatives).

1.5. *General approximation theorem.* Here we state the general approximation theorem used to prove Theorems 1.1 and 1.2. We first need a definition.

DEFINITION 1.3. For a point process $\Xi \subseteq \mathbb{R}^d$ with mean measure κ , we say that $\Xi^{(u)}$ is distributed as the reduced Palm measure of Ξ , if

$$\mathbb{E}\left[\int g(\Xi, u)\Xi(du)\right] = \mathbb{E}\left[\int g(\Xi^{(u)} + \delta_u, u)\kappa(du)\right]$$

for all functions g such that the integral on the left hand side exists.

For simple point processes (meaning that there is a.s. at most one point at any location), we think of $\Xi^{(u)} + \delta_u$ as having the distribution of Ξ conditional on there being a point at u , which explains why we can take $\Xi^{(u)} = \Xi$ if Ξ is a Poisson process. For rigorous background on reduced Palm measures, see Daley and Vere-Jones ((2008), Chapter 13).

THEOREM 1.4. Recall the notation and definitions of Section 1.1 leading up to (1.1). Let $(N, N^{(u)})_{u \in \mathbb{R}^d}$ be a collection of couplings of N with its reduced Palm measures and define

$$X^{(u)}(s) := \int_{\mathbb{R}^d} J_v(s)N^{(u)}(dv), \quad s \in [0, T].$$

Let Z be a centered Gaussian process with almost surely continuous sample paths having covariance function

$$(1.16) \quad K(s, t) = \mathbb{E}[Z(s)Z(t)^\top] = \int_{u \in \mathbb{R}^d} \hat{J}_u(s)\hat{J}_u(t)^\top \Lambda(du),$$

where the function $(u, s) \mapsto \hat{J}_u(s) \in \mathbb{R}^p$ is measurable such that

$$\hat{J}_u(\cdot) \in L^2([0, T] \rightarrow \mathbb{R}^p) \quad \text{and} \quad \hat{J}(\cdot) \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^p, \Lambda) \quad \text{for all } s \in [0, T] \text{ and all } u \in \mathbb{R}^d.$$

Now suppose that $g \in M$, and define $f := f_g$ to be the Stein solution given in Theorem 2.2. Then, for \tilde{X} as defined in (1.1), we have

$$\begin{aligned} & |\mathbb{E}[g(\tilde{X})] - \mathbb{E}[g(Z)]| \\ & \leq \left| \mathbb{E}\left[\int_{\mathbb{R}^d} D^2 f(\tilde{X})[\hat{J}_u^{[2]}]\Lambda(du) \right. \right. \\ (1.17) & \quad \left. \left. - \int_{\mathbb{R}^d} D^2 f(\tilde{X})[J_u, \mathbb{E}[X^{(u)} - X + J_u]](\sigma^{-2}\lambda(du))\right] \right| \\ (1.18) & \quad + \left| \mathbb{E}\left[\int_{\mathbb{R}^d} D^2 f(\tilde{X})[J_u, (X^{(u)} - X) - \mathbb{E}[X^{(u)} - X]](\sigma^{-2}\lambda(du))\right] \right| \\ (1.19) & \quad + \frac{\|g\|_M}{2\sigma} \mathbb{E}\left[\int_{\mathbb{R}^d} \|J_u\| \|X^{(u)} - X + J_u\|^2 (\sigma^{-2}\lambda(du))\right]. \end{aligned}$$

If N is a Poisson point process, $\Lambda = \sigma^{-2}\lambda$ and $\widehat{J}_u = J_u$, then for any $g \in M$,

$$(1.20) \quad |\mathbb{E}[g(\widetilde{X})] - \mathbb{E}[g(Z)]| \leq \frac{\|g\|_M}{2\sigma} \left[\int_{\mathbb{R}^d} \|J_u\|^3 (\sigma^{-2}\lambda(du)) \right].$$

REMARK 7.

(i) Although our focus in this work is about real Gaussian process approximation, there is no essential extra difficulty in establishing the above multivariate Gaussian approximation result. See Section 3 for an illustration in the multivariate setting.

(ii) To check the hypothesis that Z has continuous sample paths, it is enough to establish that, for some positive constants C, b , we have

$$(1.21) \quad \mathbb{E}[|Z(s) - Z(t)|^2] \leq C|s - t|^b,$$

which, using Gaussianity, implies the Kolmogorov continuity criterion.

(iii) In general, the terms in (1.19) and (1.20) are not automatically finite. In our applications, the integrand J_u is uniformly bounded, $\mathbb{E}[\|X\|^2]$ is finite, and N is a simple point process over a subset \mathcal{S} of \mathbb{R}^d with finite intensity measure such that $N(\mathcal{S})$ has finite second moment. Under these extra assumptions, the aforementioned terms are finite.

REMARK 8. In practice, the choice of Λ and $\{\widehat{J}_u\}_{u \in \mathbb{R}^d}$ arises from computing the quantity $\mathbb{E}[X^{(u)} - X + J_u]$, plugging the resulting expression into the second term of the difference of (1.17), and then discarding the asymptotically negligible terms. This is an appealing aspect of the theorem, as it suggests a candidate limit, while also providing an intuitive expression for the covariance of the limit in the form (1.16).

REMARK 9. Bounding (1.17) and (1.18) in applications requires using the structure of $(X^{(u)} - X)$ and its mean, along with the bounds and “smoothness” properties of f given in Theorem 2.2 below. Bounding (1.18) is the main difficulty in applying the theorem, and typically requires constructing intermediate couplings that exploit local or weak global dependence.

To apply the theorem for the M/GI/ ∞ queue, we need to define the reduced Palm couplings of the arrival/service point process. Away from time zero, the arrivals/services are a Poisson point process, and so we can take the reduced Palm measure to be the original process. For the customers in the system at time zero, the Palm measure corresponds to removing a point at random, which is only a small perturbation of the original process. For the GI/GI/ ∞ queue, the arrival/service point process is no longer Poisson. However, because the service times are i.i.d., constructing the reduced Palm measure coupling at a point (s, y) comes down to constructing a close coupling of a stationary renewal process to one conditioned to have a renewal at s , which in turn is similar to coupling a stationary renewal sequence to a zero-delayed renewal process, and this is well understood. The details are in Sections 3 and 4.

Theorem 1.4 follows from a new development of Stein’s method (Stein (1972, 1986)), formulated as Theorem 2.2. Stein’s method provides a general framework for bounding the error when approximating a complicated distribution of interest by a well-understood target distribution; see Ross (2011) for a basic introduction. By now, Stein’s method has been developed for a large number of univariate distributions, as in the monographs Chen, Goldstein and Shao (2011) for the normal and Barbour, Holst and Janson (1992) for the Poisson. Stein’s method for multivariate distributions other than the normal is not so well developed, and even less is known for random processes. Poisson process approximation is a notable exception,

with a succession of papers going back to [Barbour \(1988\)](#) and [Arratia, Goldstein and Gordon \(1989\)](#). There is also work on approximation by Brownian motion, which began with [Barbour \(1990\)](#), and on some closely related Gaussian processes, such as time changes of Brownian motion ([Kasprzak \(2017, 2020b\)](#)) and multivariate correlated Brownian motions ([Kasprzak \(2020a\)](#), [Döbler and Kasprzak \(2021\)](#)); there has also been recent work for Dirichlet process approximation ([Gan and Ross \(2021\)](#)).

All of the Gaussian process approximation results just cited are derived using Barbour's generator approach ([Barbour \(1990\)](#)), which identifies a "characterizing" operator of a Gaussian process as the generator of the Ornstein–Uhlenbeck semigroup. To avoid working with the generator in the continuum, these papers first approximate the process of interest by a discretized version of the Gaussian process, which has an Ornstein–Uhlenbeck generator of a simple form. After this is achieved, the problem is reduced to showing that the discretized Gaussian process is close to the true Gaussian process. Here we avoid this two step procedure, by developing the relevant properties of the "Stein solution" for any Gaussian process with continuous sample paths; see [Theorem 2.2](#) below. Thus, the development of Stein's method here is technically different from that in the finite-dimensional setting; in particular, our results rely on the Karhunen–Loève expansion of the Gaussian process. [Theorem 2.2](#) can be used to prove approximation bounds in quite general settings, in which the dependency structures are amenable to Stein's method, such as those exhibiting an exchangeable pair. The formulation is particularly useful in our applications, where the jumps of the processes that we study occur at random times. Note that, in the context of [Theorem 1.4](#), any discretization error between the process and the target Gaussian process is captured by [\(1.17\)](#), and bounding this term typically relies on the smoothness property [\(1.2\)](#).

To compare our approach to others developing Stein's method for Gaussian processes, first note that the smooth function metric used here is not simply related to weak convergence with respect to Skorokhod topology, and that the test functions that we use do not yield natural statistics of the process. However, the companion paper [Barbour, Ross and Zheng \(2021\)](#) develops infinite-dimensional Gaussian smoothing inequalities that can be used to convert bounds on the smooth function metric to those on the Lévy–Prokhorov metric (with respect to Skorokhod topology), as is done here. The paradigm of using a smooth function metric that is natural for Stein's method, and then applying a smoothing inequality to obtain bounds in a more useful metric (here, the Lévy–Prokhorov metric), is frequently useful.

The recent papers [Coutin and Decreusefond \(2020\)](#) and [Besançon et al. \(2021\)](#) use Stein's method to obtain bounds in the bounded Wasserstein distance for Donsker's theorem, and for Lipschitz functionals of Poisson measures. They obtain rates of convergence in this restricted setting of better order than those that our method typically yields, though less good than those obtainable using strong approximation. However, in their approach, they make use of the independence structure within the process being approximated, and of the fact that the limiting process is Brownian motion; we need neither of these simplifications.

There is also an approach to Stein's method on Hilbert and abstract Wiener spaces, initiated in [Shih \(2011\)](#) and developed further in [Coutin and Decreusefond \(2013\)](#), [Besançon, Decreusefond and Moyal \(2020\)](#), [Bourguin and Campese \(2020\)](#), and [Bourguin, Campese and Dang \(2021\)](#). These papers view the Gaussian process as an element of a functional space equipped with an integral metric. Thus the probability metrics that they work with are not strong enough to imply weak convergence with respect to the Skorokhod topology, and hence do not imply rates for such convergence, either. In particular, convergence in the metrics used in these papers does not imply convergence of finite-dimensional distributions. See also the discussions of these different approaches in [Döbler and Kasprzak \(\(2021\), Section 1.5\)](#) and [Barbour, Ross and Zheng \(\(2021\), Section 1.1\)](#).

The remainder of the paper is organized as follows. In [Section 2](#), we develop Stein's method in the general context of Gaussian process approximation, establishing [Theorem 2.2](#),

together with some ancillary results. We then prove Theorem 1.4. In Sections 3 and 4, we apply Theorem 1.4 to the M/GI/∞ and GI/GI/∞ queue examples given above, proving Theorems 1.1 and 1.2.

2. Stein’s method for Gaussian processes. Our first step in developing Stein’s method for Gaussian processes is to establish a useful form for the characterizing operator.

PROPOSITION 2.1. *Let Z be a centered continuous \mathbb{R}^p -valued Gaussian process on $[0, T]$ with*

$$(2.1) \quad K(s_1, s_2) := \mathbb{E}[Z(s_1)Z(s_2)^\top] = \int_{\mathbb{R}^d} J_u(s_1)J_u(s_2)^\top \Lambda(du),$$

where Λ is a measure on \mathbb{R}^d and the function $(u, s) \mapsto J_u(s) \in \mathbb{R}^p$ is measurable such that

$$J_u(\bullet) \in L^2([0, T] \rightarrow \mathbb{R}^p) \quad \text{and} \quad J_\bullet(s) \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^p, \Lambda) \quad \text{for all } (s, u) \in [0, T] \times \mathbb{R}^d.$$

Then for any function $f \in M$,

$$(2.2) \quad \mathbb{E}(D^2 f(w)[Z^{[2]}]) = \int_{\mathbb{R}^d} D^2 f(w)[J_u^{[2]}] \Lambda(du),$$

and

$$(2.3) \quad \mathbb{E}\left(\int_{\mathbb{R}^d} D^2 f(Z)[J_u^{[2]}] \Lambda(du) - Df(Z)[Z]\right) = 0.$$

PROOF. First we show that all the expectations exist. Since $f \in M$, we have

$$|D^2 f(w)[Z^{[2]}]| \leq \|D^2 f\|(1 + \|w\|)\|Z\|^2 \quad \text{and} \quad |Df(Z)[Z]| \leq \|Df\|(1 + \|Z\|^2)\|Z\|,$$

so that we need to show that $\|Z\|$ has finite third moment, which is guaranteed by Fernique’s theorem. To establish the expressions for the moments, we use the multivariate Karhunen–Loève expansion of Z ; see [Happ and Greven \(\(2018\), Section 2.2\)](#). Define the linear operator T on the Hilbert space $L^2([0, T] \rightarrow \mathbb{R}^p)$ by setting

$$(Tf)(s) := \int_0^T K(s, t) f(t) dt.$$

It is easy to see that T is a positive and compact self-adjoint operator on $L^2([0, T] \rightarrow \mathbb{R}^p)$, so that by the spectral theorem, we can find $\{h_k, k \in \mathbb{N}\}$ that is, an orthonormal basis of $L^2([0, T] \rightarrow \mathbb{R}^p)$ formed by the eigenvectors of T with respective eigenvalues $\{\ell_k, k \in \mathbb{N}\} \subset \mathbb{R}_+$. As a result, $Z(t)$ admits the following representation:

$$(2.4) \quad Z(t) = \sum_{k \in \mathbb{N}} X_k h_k(t),$$

where

$$X_k := \int_0^T Z(t)^\top h_k(t) dt = \int_0^T h_k(t)^\top Z(t) dt,$$

and the convergence in (2.4) can be taken in $L^2(\Omega)$, uniformly in $t \in [0, T]$, and, because of the assumption of continuity of sample paths, can also be taken with respect to sup norm; see [Adler and Taylor \(\(2007\), Theorem 3.1.2\)](#).

For $k, j \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{E}[X_k X_j] &= \mathbb{E}\left[\int_0^T \int_0^T h_j(s)^\top Z(s) Z(t)^\top h_k(t) dt ds\right] \\ &= \int_0^T \int_0^T h_j(s)^\top K(s, t) h_k(t) dt ds \\ &= \int_0^T h_j(s)^\top (Th_k)(s) ds = \ell_k \mathbf{1}[k = j], \end{aligned}$$

and so the variables $\{X_k\}_{k \in \mathbb{N}}$ are independent centered Gaussian random variables.

Using the representation (2.4), we have, for the symmetric bilinear form $A = D^2 f(w)$,

$$A[Z, Z] = A\left[\sum_{k \in \mathbb{N}} X_k h_k, \sum_{j \in \mathbb{N}} X_j h_j\right] = \sum_{k, j \in \mathbb{N}} X_k X_j A[h_k, h_j],$$

so that

$$(2.5) \quad \mathbb{E}[A[Z, Z]] = \mathbb{E}\left[\sum_{k, j \in \mathbb{N}} X_k X_j A[h_k, h_j]\right] = \sum_{k \in \mathbb{N}} \ell_k A[h_k, h_k].$$

On the other hand, expanding J_u in the orthonormal basis $\{h_k\}$ implies

$$J_u = \sum_{k \in \mathbb{N}} h_k J_u^{(k)},$$

where $J_u^{(k)} := \int_0^T J_u(s)^\top h_k(s) ds \in \mathbb{R}$ for each $k \in \mathbb{N}$. It follows that

$$(2.6) \quad \begin{aligned} \int_{\mathbb{R}^d} A[J_u, J_u] \Lambda(du) &= \int_{\mathbb{R}^d} A\left[\sum_{k \in \mathbb{N}} h_k J_u^{(k)}, \sum_{j \in \mathbb{N}} h_j J_u^{(j)}\right] \Lambda(du) \\ &= \sum_{k, j \in \mathbb{N}} A[h_k, h_j] \int_{\mathbb{R}^d} J_u^{(k)} J_u^{(j)} \Lambda(du). \end{aligned}$$

Continuing with straightforward calculations, we have

$$\begin{aligned} \int_{\mathbb{R}^d} J_u^{(k)} J_u^{(j)} \Lambda(du) &= \int_{\mathbb{R}^d} \left(\int_0^T h_k(s)^\top J_u(s) ds\right) \left(\int_0^T J_u(t)^\top h_j(t) dt\right) \Lambda(du) \\ &= \int_0^T \int_0^T h_k(s)^\top \left\{ \int_{\mathbb{R}^d} J_u(s) J_u(t)^\top \Lambda(du) \right\} h_j(t) ds dt \\ &= \int_0^T \int_0^T h_k(s)^\top K(s, t) h_j(t) ds dt = \int_0^T h_k(s)^\top (Th_j)(s) ds \\ &= \ell_j \int_0^T h_k(s)^\top h_j(s) ds = \ell_j \mathbf{1}[k = j]. \end{aligned}$$

Plugging this into (2.6), and noting (2.5) gives (2.2):

$$(2.7) \quad \int_{\mathbb{R}^d} A[J_u, J_u] \Lambda(du) = \sum_{k \in \mathbb{N}} \ell_k A[h_k, h_k] = \mathbb{E}[A[Z, Z]].$$

For (2.3), the first equality in (2.7) implies that it is enough to establish that

$$(2.8) \quad \mathbb{E}[Df(Z)[Z]] = \sum_{k \in \mathbb{N}} \ell_k \mathbb{E}[D^2 f(Z)[h_k, h_k]].$$

Writing $Z_k := Z - X_k h_k$, we have

$$\begin{aligned} \mathbb{E}[Df(Z)[Z]] &= \sum_{k \in \mathbb{N}} \mathbb{E}[X_k Df(Z_k + X_k h_k)[h_k]] \\ &= \sum_{k \in \mathbb{N}} \mathbb{E}[\mathbb{E}(X_k Df(Z_k + X_k h_k)[h_k] | Z_k)]. \end{aligned}$$

Since Z_k is independent of X_k , we can apply the one-dimensional Stein identity

$$\mathbb{E}[X_k g(X_k) | Z_k] = \ell_k \mathbb{E}[g'(X_k) | Z_k]$$

with $g(x) = Df(Z_k + x h_k)[h_k]$ to each term in this sum. Then (2.8) easily follows by noting that $g'(x) = D^2 f(Z_k + x h_k)[h_k, h_k]$ and thus $g'(X_k) = D^2 f(Z)[h_k, h_k]$. \square

The next result represents the foundation of Stein’s method for continuous Gaussian processes.

THEOREM 2.2. *Let Z be a centered continuous Gaussian process with covariance function given by (2.1). Given $g \in M$, we define $f_g : \mathbb{D}^p \rightarrow \mathbb{R}$ by*

$$(2.9) \quad f_g(w) := - \int_0^\infty (\mathbb{E}[g(w e^{-s} + \sqrt{1 - e^{-2s}} Z)] - \mathbb{E}[g(Z)]) ds.$$

Then $f_g \in M$ and for $k \in \{1, 2\}$,

$$(2.10) \quad D^k f_g(w) = - \mathbb{E} \int_0^\infty e^{-ks} D^k g(w e^{-s} + \sqrt{1 - e^{-2s}} Z) ds.$$

Furthermore, for $w, w', w_1, w_2 \in \mathbb{D}^p$, we have

$$(2.11) \quad |D^2 f_g(w + w')[w_1, w_2] - D^2 f_g(w)[w_1, w_2]| \leq \|g\|_M \|w_1\| \|w_2\| \|w'\|,$$

and if $g \in M'$, then

$$(2.12) \quad |D^2 f_g(w)[w_1, w_2]| \leq (3/2) \|g\|_{M'} \|w_1\| \|w_2\|.$$

Finally, f_g satisfies the Stein’s equation

$$(2.13) \quad Af_g(w) := \int_{\mathbb{R}^d} D^2 f_g(w)[J_u^{[2]}] \Lambda(du) - Df_g(w)[w] = g(w) - \mathbb{E}[g(Z)].$$

PROOF. That $f_g \in M$ and that (2.10) holds follow from the arguments of [Kasprzak, Duncan and Vollmer \(\(2017\), Lemma 4.1\)](#) (see also [Barbour \(1990\)](#)) for the special case of Brownian motion. Their argument only relies on the supremum of the Gaussian process having finite third moment, which is also valid in our setting.

The bounds on the derivatives also more or less follow along the same lines as existing work, see [Kasprzak \(\(2020b\), Proposition 3.2\)](#) or [Kasprzak \(\(2020a\), Proposition 5.5\)](#), but our setting is different enough that we include a proof. To show (2.12), we use equation (2.10) and Lemma 2.3 below, which relates the absolute value of a bilinear form at a given argument to its norm, to find that

$$\begin{aligned} |D^2 f_g(w)[w_1, w_2]| &\leq \int_0^\infty e^{-2s} \mathbb{E}[|D^2 g(w e^{-s} + \sqrt{1 - e^{-2s}} Z)[w_1, w_2]|] ds \\ &\leq 3 \|w_1\| \|w_2\| \int_0^\infty e^{-2s} \mathbb{E}[\|D^2 g(w e^{-s} + \sqrt{1 - e^{-2s}} Z)\|] ds \\ &\leq 3 \|g\|_{M'} \|w_1\| \|w_2\| \int_0^\infty e^{-2s} ds = (3/2) \|g\|_{M'} \|w_1\| \|w_2\|, \end{aligned}$$

where the third inequality uses that $g \in M'$. The proof of (2.11) follows from similar arguments using equation (2.10), Lemma 2.3, and the Lipschitz continuity of D^2g .

The usual path to show (2.13) is to view the family of operators $P_s: g \mapsto g(we^{-s} + \sqrt{1 - e^{-2s}}Z)$ as an Ornstein–Uhlenbeck semigroup with generator equal to the characterizing operator \mathcal{A} , and then the result follows essentially from strong continuity of the semigroup. However, the semigroup is not strongly continuous, even for Z a Brownian motion, and so an alternative approach is to follow the proof of the relevant result for strongly continuous semigroups; see [Kasprzak, Duncan and Vollmer \(2017\)](#). While such a strategy could work in our setting, we provide a direct proof that is simpler than existing approaches, using Gaussian calculations and (2.10).

Putting $\mathcal{W}_s = we^{-s} + \sqrt{1 - e^{-2s}}Z$ and using (2.2), (2.10), we can rewrite the left hand side of (2.13) as

$$(2.14) \quad \mathbb{E} \int_0^\infty e^{-s} Dg(\mathcal{W}_s)[w] ds - \mathbb{E} \int_0^\infty e^{-2s} D^2g(\mathcal{W}_s)[Z', Z'] ds,$$

where Z' is an independent copy of Z . The right hand side of (2.13) can be written as

$$\mathbb{E}[g(w) - g(Z)] = -\mathbb{E} \int_0^\infty \frac{d}{ds} g(\mathcal{W}_s) ds,$$

and, since $g \in M$,

$$\frac{d}{ds} g(\mathcal{W}_s) = Dg(\mathcal{W}_s) \left[-e^{-s}w + \frac{e^{-2s}}{\sqrt{1 - e^{-2s}}}Z \right].$$

Thus, using linearity of the derivative,

$$\begin{aligned} g(w) - \mathbb{E}[g(Z)] &= -\mathbb{E} \int_0^\infty Dg(\mathcal{W}_s) \left[-e^{-s}w + \frac{e^{-2s}}{\sqrt{1 - e^{-2s}}}Z \right] ds \\ &= \mathbb{E} \int_0^\infty e^{-s} Dg(\mathcal{W}_s)[w] ds - \mathbb{E} \int_0^\infty \frac{e^{-2s}}{\sqrt{1 - e^{-2s}}} Dg(\mathcal{W}_s)[Z] ds. \end{aligned}$$

Comparing with (2.14), it only remains to show

$$(2.15) \quad \mathbb{E} \int_0^\infty \frac{e^{-2s}}{\sqrt{1 - e^{-2s}}} Dg(\mathcal{W}_s)[Z] ds = \mathbb{E} \int_0^\infty e^{-2s} D^2g(\mathcal{W}_s)[Z', Z'] ds.$$

We claim that, after swapping the order of integration, the integrands are equal. To see this, for fixed $w \in \mathbb{D}^p$, and $s \in [0, \infty)$, we write

$$h(\hat{w}) = g(e^{-s}w + \sqrt{1 - e^{-2s}}\hat{w}).$$

Then for any $x, y \in \mathbb{D}^p$,

$$\begin{aligned} Dh(\hat{w})[x] &= \sqrt{1 - e^{-2s}} Dg(e^{-s}w + \sqrt{1 - e^{-2s}}\hat{w})[x], \\ D^2h(\hat{w})[x, y] &= (1 - e^{-2s}) D^2g(e^{-s}w + \sqrt{1 - e^{-2s}}\hat{w})[x, y]. \end{aligned}$$

Now, the Stein equation (2.3) with (2.2) implies

$$\mathbb{E}[Dh(Z)[Z]] = \mathbb{E}[D^2h(Z)[Z', Z']],$$

which, using the definition of h , is the same as

$$\mathbb{E}[Dg(\mathcal{W}_s)[Z]] = \sqrt{1 - e^{-2s}} \mathbb{E}[D^2g(\mathcal{W}_s)[Z', Z']].$$

This implies (2.15), and thus (2.13). \square

REMARK 10. Theorem 2.2 can be used to establish quantitative approximation of a process $W \in \mathbb{D}^p$ by a continuous Gaussian process Z in ways typical of Stein’s method. Taking any test function $g \in M$, the difference $\mathbb{E}[g(W)] - \mathbb{E}[g(Z)]$ can be bounded by

$$\left| \mathbb{E} \left\{ \int_{\mathbb{R}^d} D^2 f_g(W) [J_u^{[2]}] \Lambda(du) - Df_g(W)[W] \right\} \right|,$$

where the functions J_u and the measure Λ are as in the representation (2.1) of the covariance function of Z . The quantity $\mathbb{E}\{Df_g(W)[W]\}$ can then be treated in one of a number of standard ways, depending on the context. In applying Theorem 1.4, in which W is a centered and normalized version of an integral $X(\cdot) := \int_{\mathbb{R}^d} J_u^{(X)}(\cdot) N(du)$ with respect to a point process N , the expectation $\mathbb{E}\{Df_g(W)[W]\}$ is evaluated using Palm theory, and Λ and the functions J_u emerge from the resulting calculations. In particular, the functions J_u are not in general the same as the functions $J_u^{(X)}$.

For making estimates when exploiting the above approach, the following two lemmas are often useful. They are needed, for example, in proving Theorem 1.4.

LEMMA 2.3. *If $f \in M$ and $w, w', w_1, w_2 \in \mathbb{D}^p$, then*

$$|D^2 f(w)[w_1, w_2]| \leq 3 \|w_1\| \|w_2\| \|D^2 f(w)\|,$$

and

$$|D^2 f(w + w')[w_1, w_2] - D^2 f(w)[w_1, w_2]| \leq 3 \|w_1\| \|w_2\| \|D^2 f(w + w') - D^2 f(w)\|$$

PROOF. Using bilinearity, we have

$$\begin{aligned} D^2 f(w)[w_1, w_2] &= \frac{1}{2} (D^2 f(w)[w_1 + w_2, w_1 + w_2] \\ &\quad - D^2 f(w)[w_1, w_1] - D^2 f(w)[w_2, w_2]). \end{aligned}$$

Taking the absolute value and using the triangle inequality implies

$$\begin{aligned} |D^2 f(w)[w_1, w_2]| &\leq \frac{1}{2} \|D^2 f(w)\| (\|w_1\|^2 + \|w_2\|^2 + \|w_1 + w_2\|^2) \\ &\leq \frac{3}{2} \|D^2 f(w)\| (\|w_1\|^2 + \|w_2\|^2), \end{aligned}$$

and we deduce from the bilinearity that for any $t > 0$

$$|D^2 f(w)[w_1, w_2]| = |D^2 f(w)[tw_1, t^{-1}w_2]| \leq \frac{3}{2} \|D^2 f(w)\| (t^2 \|w_1\|^2 + t^{-2} \|w_2\|^2).$$

Taking $t^2 = \|w_2\|/\|w_1\|$ yields the first inequality. The second inequality follows from the same arguments, with $D^2 f(w)$ replaced by $D^2 f(w + w') - D^2 f(w)$. \square

LEMMA 2.4. *If $f \in M$ and $w_1, w_2, J \in \mathbb{D}^p$, then*

$$\begin{aligned} &Df(w_2)[J] - Df(w_1)[J] \\ &= D^2 f(w_1)[J, w_2 - w_1] \\ &\quad + \int_0^1 (D^2 f(w_1 + t(w_2 - w_1))[J, w_2 - w_1] - D^2 f(w_1)[J, w_2 - w_1]) dt. \end{aligned}$$

PROOF. Set $h(t) = Df(w_1 + t(w_2 - w_1))[J]$, and note that

$$h'(t) = D^2 f(w_1 + t(w_2 - w_1))[J, w_2 - w_1]$$

is continuous on $[0, 1]$. Therefore,

$$h(1) - h(0) = h'(0) + \int_0^1 (h'(t) - h'(0)) dt,$$

which is the lemma. \square

We now turn to proving our second main result.

PROOF OF THEOREM 1.4. Recall that

$$\tilde{X}(s) = \int_{\mathbb{R}^d} J_v(s) \frac{(N - \lambda)(dv)}{\sigma},$$

and let $f = f_g$ be the Stein solution in Theorem 2.2. In view of (2.13), it suffices to show the bound for

$$(2.16) \quad |\mathbb{E} \mathcal{A}f(\tilde{X})| = \left| \mathbb{E} \int_{\mathbb{R}^d} D^2 f(\tilde{X})[\hat{J}_u^{[2]}] \Lambda(du) - \mathbb{E}[Df(\tilde{X})[\tilde{X}]] \right|.$$

Using the definition of \tilde{X} , we first write

$$(2.17) \quad \begin{aligned} \mathbb{E} Df(\tilde{X})[\tilde{X}] &= \mathbb{E} \int_{\mathbb{R}^d} Df(\tilde{X})[J_u] \tilde{N}(du) \\ &= \sigma^{-1} \mathbb{E} \left[\int_{\mathbb{R}^d} Df(\tilde{X})[J_u] N(du) - \int_{\mathbb{R}^d} Df(\tilde{X})[J_u] \lambda(du) \right]. \end{aligned}$$

Now, with $g(N, u) := Df(\tilde{X})[J_u]$, we can write

$$\begin{aligned} \mathbb{E} \int_{\mathbb{R}^d} g(N, u) N(du) &= \mathbb{E} \int_{\mathbb{R}^d} g(N^{(u)} + \delta_u, u) \lambda(du) \\ &= \mathbb{E} \int_{\mathbb{R}^d} Df(\sigma^{-1}(X^{(u)} + J_u - \mathbb{E}[X]))[J_u] \lambda(du). \end{aligned}$$

Combining this with (2.17), we find that

$$\mathbb{E}[Df(\tilde{X})[\tilde{X}]] = \sigma^{-1} \mathbb{E} \left[\int_{\mathbb{R}^d} (Df(\sigma^{-1}(X^{(u)} + J_u - \mathbb{E}[X]))[J_u] - Df(\tilde{X})[J_u]) \lambda(du) \right].$$

Applying Lemma 2.4 with $w_1 = \tilde{X}$, $w_2 = \sigma^{-1}(X^{(u)} + J_u - \mathbb{E}[X])$ and $J = J_u$ yields

$$(2.18) \quad \begin{aligned} &\mathbb{E} Df(\tilde{X})[\tilde{X}] \\ &= \frac{1}{\sigma^2} \mathbb{E} \left[\int_{\mathbb{R}^d} D^2 f(\tilde{X})[J_u, X^{(u)} - X + J_u] \lambda(du) \right] \\ &\quad + \frac{1}{\sigma^2} \mathbb{E} \left[\int_{\mathbb{R}^d} \int_0^1 \left(D^2 f(\tilde{X} + t\sigma^{-1}(X^{(u)} + J_u - X)) [J_u, X^{(u)} + J_u - X] \right. \right. \\ &\quad \left. \left. - D^2 f(\tilde{X}) [J_u, X^{(u)} + J_u - X] \right) dt \lambda(du) \right]. \end{aligned}$$

Now the first bound of the theorem easily follows by adding and subtracting

$$\frac{1}{\sigma^2} \mathbb{E} \left[\int_{\mathbb{R}^d} D^2 f(\tilde{X})(J_u, \mathbb{E}[X^{(u)} - X + J_u]) \lambda(du) \right]$$

to the right hand side of (2.18), plugging the resulting expression for $\mathbb{E} Df(\tilde{X})[\tilde{X}]$ into (2.16), and then applying (2.11) to arrive at (1.19).

The second assertion (1.20) follows from the first, after observing that we can set $N^{(u)} = N$, and hence $X^{(u)} = X$. \square

3. M/GI/∞ queue: Proof of Theorem 1.1. Let us first recall some notation: $\mathcal{S} := [0, T] \times \mathbb{R}_+$, α is a finite measure on $[0, T]$ with $\alpha([0, T]) \geq 1$, and G and \tilde{G} are distribution functions on \mathbb{R}_+ . The point process N_n that we consider has the following form:

$$N_n = M_n^{(1)} + M_n^{(2)} \quad \text{where } M_n^{(2)} := \sum_{k=1}^{x_n} \delta_{(0, Y_i)},$$

where $x_n \geq 1$ is an integer, $(Y_i, i \geq 1)$ is a sequence of *i.i.d.* random variables with distribution \tilde{G} , and $M_n^{(1)}$ is a Poisson point process on \mathcal{S} with intensity measure $\ell_n(dt, dy) := n\alpha(dt)G(dy)$ that is independent of $(Y_i, i \geq 1)$. Λ denotes the measure $(\alpha \times G) + x(\delta_0 \times \tilde{G})$.

In order to illustrate the use of Theorem 1.4 in a multivariate context, we define

$$J_{t,y}(s) := \begin{cases} \mathbf{1}[t \leq s < t + y]\mathbf{e}^{(1)} & \text{if } t > 0; \\ \mathbf{1}[0 \leq s < y]\mathbf{e}^{(2)} & \text{if } t = 0, \end{cases}$$

where $\mathbf{e}^{(i)}, i = 1, 2$, denotes the coordinate vectors in \mathbb{R}^2 , and then define

$$U_n^{(i)}(s) := \int_{\mathcal{S}} J_{t,y}(s) M_n^{(i)}(dt, dy), \quad i = 1, 2; \quad U_n := U_n^{(1)} + U_n^{(2)}.$$

Then $X_n := (1, 1)^\top U_n$ models the number of customers in an M/G/∞ queue, and U_n distinguishes those who were in the queue at time 0 and those who arrived afterwards. We quantify the convergence of $\tilde{U}_n := n^{-1/2}(U_n - \mathbb{E}U_n)$ to the bivariate centered Gaussian process \tilde{Z} with covariance matrix

$$\begin{aligned} & \mathbb{E}\{\widehat{Z}(s_1)\widehat{Z}(s_2)^\top\} \\ & := \int_0^{s_1} (1 - G(s_2 - t))\alpha(dt)\mathbf{e}^{(1)}(\mathbf{e}^{(1)})^\top + x\tilde{G}(s_1)(1 - \tilde{G}(s_2))\mathbf{e}^{(2)}(\mathbf{e}^{(2)})^\top, \end{aligned}$$

and use this to deduce Theorem 1.1; in particular, see (3.4).

We start with the following proposition, which states the well-known families of reduced Palm couplings $(M_n^{(1,t,y)}, M_n^{(1)}_{(t,y) \in \mathcal{S}})$ and $(M_n^{(2,y)}, M_n^{(2)}_{y \in \mathbb{R}_+})$.

PROPOSITION 3.1. *Let $M_n^{(i)}, i = 1, 2$, be defined as above. For $(t, y) \in \mathcal{S}$, and given $M_n^{(1)}$, let $M_n^{(1,t,y)} := M_n^{(1)}, t > 0$. Then $M_n^{(1,t,y)}$ has the reduced Palm distribution of $M_n^{(1)}$ at (t, y) . Similarly, given $M_n^{(2)}$, let Y be a point uniformly and independently chosen from $\{Y_1, \dots, Y_{x_n}\}$. Then $M_n^{(2,y)} := M_n^{(2)} - \delta_{(0,Y)}$ has the reduced Palm distribution of $M_n^{(2)}$ at $(0, y)$.*

Because $M_n^{(1)}$ and $M_n^{(2)}$ are independent, it follows that the reduced Palm distributions of N_n are given by $N_n^{(t,y)} = N_n - \delta_{(0,Y)}\mathbf{1}\{t = 0\}$, and hence that $U_n^{(t,y)} := \int_{\mathcal{S}} J_{t,y}N_n^{(t,y)}(dt, dy)$ satisfies

$$(3.1) \quad U_n^{(t,y)} - U_n = -J_{0,Y}\mathbf{1}\{t = 0\}.$$

PROOF OF THEOREM 1.1. Suppose that $G * \alpha, A$, and \tilde{G} are β -Hölder continuous with constants $c_{G,\alpha}, c_\alpha$, and $c_{\tilde{G}}$, respectively. We start by computing the bounds in Theorem 1.4 on the difference $|\mathbb{E}g(\tilde{U}_n) - \mathbb{E}g(\tilde{Z})|$. We first show that the process \tilde{Z} has a continuous modification. Since, by assumption, neither $G * \alpha$ nor \tilde{G} have atoms in $[0, T]$, it is clear from (1.7) that the covariance function is continuous. Moreover, for any $s \geq 0$ and $0 \leq h \leq (T - s) \wedge 1$,

(1.7) and the Hölder continuity of $G * \alpha$ easily imply that

$$\begin{aligned} & |\mathbb{E}[\widehat{Z}(s+h)^\top \widehat{Z}(s+h) - \widehat{Z}(s)^\top \widehat{Z}(s+h)]| \\ &= \int_s^{s+h} (1 - G(s+h-t))\alpha(dt) + x(1 - \widetilde{G}(s+h))[\widetilde{G}(s+h) - \widetilde{G}(s)] \\ &\leq (c_{G,\alpha} + xc_{\widetilde{G}})h^\beta, \end{aligned}$$

and, similarly, that

$$|\mathbb{E}[\widehat{Z}(s)^\top \widehat{Z}(s) - \widehat{Z}(s)^\top \widehat{Z}(s+h)]| \leq (c_{G,\alpha} + xc_{\widetilde{G}})h^\beta,$$

so that (1.21) is satisfied. Therefore, in view of Remark 7, the Gaussian process \widehat{Z} has a continuous modification. In what follows, we will work with this continuous Gaussian process, that we still denote by \widehat{Z} .

The contributions to the integrals in (1.17)–(1.19) from $\{\{0\} \times \mathbb{R}_+\}$ and $\{(0, T] \times \mathbb{R}_+\}$ can be separately bounded, and the results added for the overall bounds. First, on $\{(0, T] \times \mathbb{R}_+\}$, $U_n^{((t,y))} - U_n = 0$, in view of (3.1), so that there is no contribution from (1.18), or from (1.17) either, since $\Lambda = \alpha \times G = n^{-1} \ell_n$ on $\{(0, T] \times \mathbb{R}_+\}$; and (1.19) contributes at most

$$(3.2) \quad \frac{\|g\|_M}{2\sqrt{n}} \int_S \|J_{t,y}\|^3 \alpha(dt) G(dy) \leq \frac{\|g\|_M \alpha([0, T])}{2\sqrt{n}}.$$

Next, on $\{\{0\} \times \mathbb{R}_+\}$, $U_n^{((0,y))} - U_n = -J_{0,y}$, and so

$$(3.3) \quad \mathbb{E}[U_n^{(0,y)} - U_n] = - \int J_{0,y} \widetilde{G}(dy) = -(1 - \widetilde{G})\mathbf{e}^{(2)}.$$

Now consider the contribution to (1.17), with $\Lambda = x(\delta_0 \times \widetilde{G})$ and $\lambda = x_n(\delta_0 \times \widetilde{G})$ on $\{\{0\} \times \mathbb{R}_+\}$, taking $\widehat{J}_{(0,y)} := J_{(0,y)} - (1 - \widetilde{G})\mathbf{e}^{(2)}$. The contribution can be written as

$$\begin{aligned} & x \int_{\mathbb{R}_+} D^2 f(\widetilde{U}_n)[(J_{0,y} - (1 - \widetilde{G})\mathbf{e}^{(2)})^{[2]}] \widetilde{G}(dy) \\ & \quad - \frac{x_n}{n} \int_{\mathbb{R}_+} D^2 f(\widetilde{U}_n)[J_{0,y}, J_{0,y} - (1 - \widetilde{G})\mathbf{e}^{(2)}] \widetilde{G}(dy) \\ &= \frac{nx - x_n}{n} \int_{\mathbb{R}_+} D^2 f(\widetilde{U}_n)[(J_{0,y} - (1 - \widetilde{G})\mathbf{e}^{(2)})^{[2]}] \widetilde{G}(dy) \\ & \quad - \frac{x_n}{n} \int_{\mathbb{R}_+} D^2 f(\widetilde{U}_n)[(1 - \widetilde{G})\mathbf{e}^{(2)}, J_{0,y} - (1 - \widetilde{G})\mathbf{e}^{(2)}] \widetilde{G}(dy) \\ &= \frac{nx - x_n}{n} \int_{\mathbb{R}_+} D^2 f(\widetilde{U}_n)[(J_{0,y} - (1 - \widetilde{G})\mathbf{e}^{(2)})^{[2]}] \widetilde{G}(dy), \end{aligned}$$

where the last line uses (3.3), as well as Lemma 5.3 below (noting in particular that $J_{0,y}(s) = (\mathbf{1}\{s \geq 0\} - \mathbf{1}\{s \geq y\})\mathbf{e}^{(2)}$ and that f inherits from g either its smoothness property (1.2) or its being a function of a finite number of values of its argument, using (2.10) of Theorem 2.2). Therefore, using (2.12) of Theorem 2.2, the contribution to (1.17) on $\{\{0\} \times \mathbb{R}_+\}$ is bounded by

$$(3/2)\|g\|_{M'} |n^{-1}x_n - x|.$$

For the contribution to (1.18) on $\{\{0\} \times \mathbb{R}_+\}$, we use (2.12) and (3.1), giving

$$\begin{aligned} & \mathbb{E}\left[\int_{\mathbb{R}_+} |D^2 f(\widetilde{U}_n)[J_{0,y}, (U_n^{(0,y)} - U_n) - \mathbb{E}[U_n^{(0,y)} - U_n]]| n^{-1}x_n \widetilde{G}(dy)\right] \\ & \leq (3/2)\|g\|_{M'} \int_{\mathbb{R}_+} \|J_{0,y}\| \mathbb{E}\left[\left\|\frac{1}{x_n} \sum_{i=1}^{x_n} (J_{0,Y_i} - \mathbb{E}[J_{0,Y}])\right\|\right] n^{-1}x_n \widetilde{G}(dy). \end{aligned}$$

Then (3.3) implies that

$$\begin{aligned} & \int_{\mathbb{R}_+} \|J_{0,y}\| \mathbb{E} \left[\left\| \frac{1}{x_n} \sum_{i=1}^{x_n} (J_{0,Y_i} - \mathbb{E}[J_{0,Y}]) \right\| \right] n^{-1} x_n \tilde{G}(dy) \\ &= \frac{x_n}{n} \int_{\mathbb{R}_+} \mathbb{E} \left[\left\| \frac{1}{x_n} \sum_{i=1}^{x_n} J_{0,Y_i} - (1 - \tilde{G}) \mathbf{e}^{(2)} \right\| \right] \tilde{G}(dy) \\ &= \frac{x_n}{n} \mathbb{E} \left[\left\| \frac{1}{x_n} \sum_{i=1}^{x_n} \mathbf{1}[Y_i > \cdot] - (1 - \tilde{G}) \right\| \right] \\ &= \frac{x_n}{n} \mathbb{E} \left[\left\| \frac{1}{x_n} \sum_{i=1}^{x_n} \mathbf{1}[Y_i \leq \cdot] - \tilde{G} \right\| \right]. \end{aligned}$$

Thus we must bound the mean of the sup-norm of the difference between an empirical CDF and its limit. According to Massart ((1990), Corollary 1) (improving on Dvoretzky, Kiefer and Wolfowitz (1956)), for any $y > 0$,

$$\mathbb{P} \left(\left\| \frac{1}{x_n} \sum_{i=1}^{x_n} \mathbf{1}[Y_i \leq \cdot] - \tilde{G} \right\| > y \right) \leq 2e^{-2x_n y^2},$$

so that

$$\mathbb{E} \left[\left\| \frac{1}{x_n} \sum_{i=1}^{x_n} \mathbf{1}[Y_i \leq \cdot] - \tilde{G} \right\| \right] \leq 2 \int_0^\infty e^{-2x_n y^2} dy = \sqrt{\frac{\pi}{2x_n}},$$

giving a contribution to (1.18) on $\{0\} \times \mathbb{R}_+$ of at most $(3/2)\|g\|_{M'} \sqrt{\pi/(2x_n)}$.

For the contribution to (1.19) on $\{0\} \times \mathbb{R}_+$, we use (3.3) and (3.1) to easily find that

$$\begin{aligned} \int_{\mathbb{R}_+} \|J_{0,y}\| \|U_n^{(0,y)} - U_n + J_{0,y}\|^2 n^{-1} x_n \tilde{G}(dy) &= \int_{\mathbb{R}_+} \|J_{0,y}\| \|J_{0,y} - J_{0,Y}\|^2 n^{-1} x_n \tilde{G}(dy) \\ &\leq n^{-1} x_n, \end{aligned}$$

giving a contribution to (1.19) on $\{0\} \times \mathbb{R}_+$ of at most $(1/2)\|g\|_{M'} n^{-3/2} x_n$. Collecting these bounds, we deduce that, for any $g \in M'$,

$$(3.4) \quad |\mathbb{E}[g(\tilde{U})] - \mathbb{E}[g(\hat{Z})]| \leq \|g\|_{M'} \Psi(x, x_n, \alpha, T),$$

where $\Psi_n(x, x_n, \alpha, T)$ is as given in (1.6). Noting that, for $\tilde{g}: \mathbb{D}^1 \rightarrow \mathbb{R}$ and $g: \mathbb{D}^2 \rightarrow \mathbb{R}$ defined by $g(w_1, w_2) := \tilde{g}(w_1 + w_2)$, we have $\|\tilde{g}\|_{M'} \leq 2^{3/2}\|g\|_{M'}$, the bound (1.8) in Theorem 1.1 follows.

To prove the bound on the Lévy–Prokhorov distance, we use the main results of Barbour, Ross and Zheng (2021), as stated in Theorem 5.1 below. The first hypothesis of the theorem is satisfied with $\kappa_2 = 0$, and with κ_1 upper bounded by a quantity of order $O(|n^{-1}x_n - x| + \alpha([0, T])n^{-1/2})$, read from the bound (1.8) just established (noting that $\|g\|_{M'} \leq \|g\|_{M^0}$).

To bound the modulus of continuity terms, we use Lemma 5.2, treating the components $\tilde{X}_n^{(1)}$ and $\tilde{X}_n^{(2)}$ of \tilde{U}_n separately (so that $\tilde{U}_n^{(i)} = \tilde{X}_n^{(i)} \mathbf{e}^{(i)}$). To verify (5.1) for $\tilde{X}_n^{(1)}$, let

$$(3.5) \quad \begin{aligned} \mathcal{R}_s &:= \{(u, y) : 0 \leq u \leq s, 0 < y < s - u\}, \\ \mathcal{R}_1(s_1, s_2) &:= \mathcal{R}_{s_2} \setminus \mathcal{R}_{s_1} \quad \text{and} \\ \mathcal{R}_2(s_1, s_2) &:= (s_1, s_2] \times \mathbb{R}_+, \quad s_1 < s_2. \end{aligned}$$

Fix $0 \leq s < t \leq T$ with $1/(2n) \leq (t - s) \leq 1/2$. Recalling the definition of the random measure $M_n^{(1)}$, we have

$$\tilde{X}_n^{(1)}(t) - \tilde{X}_n^{(1)}(s) = \tilde{Y}_n(2; s, t) - \tilde{Y}_n(1; s, t),$$

where

$$(3.6) \quad \tilde{Y}_n(i; s, t) := n^{-1/2} \{M_n^{(1)}(\mathcal{R}_i(s, t)) - \ell_n(\mathcal{R}_i(s, t))\}, \quad i = 1, 2,$$

are (dependent) centered and normalized Poisson random variables with means

$$(3.7) \quad \ell_n(\mathcal{R}_1(s, t)) \leq nc_{G,\alpha}(t - s)^\beta \quad \text{and} \quad \ell_n(\mathcal{R}_2(s, t)) \leq nc_\alpha(t - s)^\beta,$$

by the Hölder continuity of $G * \alpha$ and A . Now, for W_n a sum of n independent Bernoulli random variables with success probability $\tilde{p} \leq \mu$, it follows from Rosenthal's inequality that, for any $l \geq 1$,

$$(3.8) \quad n^{-l} \mathbb{E}|W_n - \mathbb{E}[W_n]|^{2l} \leq C_{2l} n^{-l} \max\{(n\mu)^l, n\mu\} \leq C_{2l} \max\{\mu^l, n^{-l+1}\mu\},$$

where C_r is the Rosenthal constant for exponent r . A limiting argument shows that the inequality (3.8) holds also for $W_n \sim \text{Po}(n\mu)$. Thus it follows that, for any $l \geq 1$ and $|t - s| \geq (1/2)n^{-1/\beta}$,

$$(3.9) \quad \begin{aligned} & \mathbb{P}[|\tilde{Y}_n(i; s, t)| \geq \theta/2] \\ & \leq C_{2l} 2^{2l} \theta^{-2l} \max\{(c_{G,\alpha} \vee c_\alpha)^l |t - s|^{l\beta}, (c_{G,\alpha} \vee c_\alpha) n^{-l+1} |t - s|^\beta\} \\ & \leq K_l^{(1)} \theta^{-2l} |t - s|^{l\beta}, \quad i = 1, 2, \end{aligned}$$

which implies (5.1) for $\tilde{X}_n^{(1)}$ with $M = n^{1/\beta}$, $K = 2K_l^{(1)}$, $a = l\beta - 1$, and $b = 2l$, for any $l \geq 1$.

To establish (5.2) for $\tilde{X}_n^{(1)}$, with $M = n^{1/\beta}$, observe that, for all s such that $(k - 1)/M \leq s \leq k/M$,

$$\begin{aligned} & |\tilde{X}_n^{(1)}(s) - \tilde{X}_n^{(1)}((k - 1)/M)| \\ & \leq n^{-1/2} \sum_{i=1}^2 \left\{ M_n^{(1)}\left(\mathcal{R}_i\left(\frac{k-1}{M}, \frac{k}{M}\right)\right) + \ell_n\left(\mathcal{R}_i\left(\frac{k-1}{M}, \frac{k}{M}\right)\right) \right\} \\ & = \sum_{i=1}^2 \tilde{Y}_n\left(i; \frac{k-1}{M}, \frac{k}{M}\right) + 2n^{-1/2} \sum_{i=1}^2 \ell_n\left(\mathcal{R}_i\left(\frac{k-1}{M}, \frac{k}{M}\right)\right), \end{aligned}$$

and that $\ell_n(\mathcal{R}_i(\frac{k-1}{M}, \frac{k}{M})) \leq c_{G,\alpha} \vee c_\alpha$, see (3.7) and (3.6). Hence, if $\theta \geq 4(c_{G,\alpha} \vee c_\alpha)n^{-1/2}$, then

$$\begin{aligned} \mathbb{P}\left[\sup_{\frac{k-1}{M} \leq s \leq \frac{k}{M}} |\tilde{X}_n^{(1)}(s) - \tilde{X}_n^{(1)}((k - 1)/M)| \geq \theta\right] & \leq \mathbb{P}\left(\sum_{i=1}^2 \left|\tilde{Y}_n\left(i; \frac{k-1}{M}, \frac{k}{M}\right)\right| \geq \theta/2\right) \\ & \leq \sum_{i=1}^2 \mathbb{P}\left[\left|\tilde{Y}_n\left(i; \frac{k-1}{M}, \frac{k}{M}\right)\right| \geq \theta/4\right], \end{aligned}$$

and this probability is bounded by $2^{2l'+1} K_{l'}^{(1)} \theta^{-2l'} n^{-l'}$, for any $l' \geq 1$, in view of (3.9) as established above. Hence, for $\theta \geq 4(c_{G,\alpha} \vee c_\alpha)n^{-1/2}$, we can take

$$(3.10) \quad \varphi_M^{(1)}(\theta) := n^{\frac{1}{\beta}-l'} 2^{2l'+1} K_{l'}^{(1)} \theta^{-2l'} \leq 2^{2l'+1} K_{l'}^{(1)} \theta^{-2l'} \epsilon^{l'\beta-1}$$

in (5.2), for any $l' \geq 1/\beta$ and $\epsilon \geq n^{-1/\beta}$, to be compared with the bound in (5.1). In particular, taking $\theta \geq 4(c_{G,\alpha} \vee c_\alpha)n^{-1/2}$ and $l' = l$, for any $l > 1/\beta$, and applying Lemma 5.2 with $M = n^{1/\beta}$, it follows that, for any $\epsilon \in (n^{-1/\beta}, 1]$ and for any $\theta > 4(c_{G,\alpha} \vee c_\alpha)n^{-1/2}$, we have

$$(3.11) \quad \mathbb{P}[\omega_{\tilde{X}_n^{(1)}}(\epsilon) \geq \theta/2] \leq TC\theta^{-2l}\epsilon^{l\beta-1},$$

for a suitable constant C that does not depend on (ϵ, n, θ) . By observing that, for $\epsilon \geq n^{-1/\beta}$ and $\theta \leq 4(c_{G,\alpha} \vee c_\alpha)n^{-1/2}$, the bound (3.11) is comparable to or larger than 1, the constant C can be chosen in such a way that the bound is valid for all $\theta > 0$.

Turning to $\tilde{X}_n^{(0)}$, we can first assume that $x_n/n \leq 2x$ without loss of generality, because $T \geq n^{1/2}|x_n n^{-1} - x|$ and the final bound is only meaningful for $T \ll n^{1/8}$. Note that, for $s < t$,

$$\tilde{X}_n^{(0)}(t) - \tilde{X}_n^{(0)}(s) = n^{-1/2} \sum_{i=1}^{x_n} \{\mathbf{1}[s < Y_i \leq t] - (\tilde{G}(t) - \tilde{G}(s))\}$$

is a normalized sum of independent centered Bernoulli random variables, and that, by assumption, $x_n(\tilde{G}(t) - \tilde{G}(s)) \leq 2nxc_{\tilde{G}}(t - s)^\beta$. Arguing exactly as for $\tilde{X}^{(1)}$ now yields (5.1) for $\tilde{X}_n^{(0)}$, for any $l \geq 1$, with $1 + a = l\beta$ and $b = 2l$, and with

$$K = K_l^{(0)} = 2^{l+1}x c_{2l} c_{\tilde{G}} \max\{x c_{\tilde{G}}, 1\}^{l-1}.$$

For (5.2), the argument is again as for $\tilde{X}_n^{(1)}$. We first write

$$\begin{aligned} & \sup \left\{ \left| \tilde{X}_n^{(0)}(s) - \tilde{X}_n^{(0)}((k-1)/M) \right| : \frac{k-1}{M} \leq s \leq \frac{k}{M} \right\} \\ & \leq n^{-1/2} \sum_{i=1}^{x_n} \left\{ \mathbf{1} \left[\frac{k-1}{M} < Y_i \leq \frac{k}{M} \right] + \left[\tilde{G} \left(\frac{k}{M} \right) - \tilde{G} \left(\frac{k-1}{M} \right) \right] \right\} \\ & \leq n^{-1/2} |W_n - \mathbb{E}[W_n]| + 4xn^{1/2} \{ \tilde{G}(k/M) - \tilde{G}((k-1)/M) \}, \end{aligned}$$

where W_n is a sum of x_n i.i.d. Bernoulli random variables with success probability $\tilde{p} \leq c_{\tilde{G}}M^{-\beta}$. Therefore, with $M = n^{1/\beta}$, it follows from (3.8) and by first considering $\theta > 8xc_{\tilde{G}}n^{-1/2}$ that we can take

$$(3.12) \quad \varphi_M^{(0)}(\theta) := n^{-l'+\frac{1}{\beta}} 2^{2l'} K_l^{(0)} \theta^{-2l'}$$

for $\varphi_M(\theta)$ in (5.2) for any $l' \geq 1$. Hence, from Lemma 5.2, for any $\epsilon \in (n^{-1/\beta}, 1]$ and any $\theta > 0$, we have

$$(3.13) \quad \mathbb{P}[\omega_{\tilde{X}_n^{(0)}}(\epsilon) \geq \theta/2] \leq TC\theta^{-2l}\epsilon^{l\beta-1},$$

for any $l \geq 1/\beta$, for a suitable constant C .

For the analogous inequality for \hat{Z} , an easy calculation shows that, for any $0 \leq u < s \leq T$,

$$\mathbb{E}[|\hat{Z}(s) - \hat{Z}(u)|^2] \leq 2(1+x)c(s-u)^\beta,$$

and so Barbour, Ross and Zheng ((2021), Remark 1.6) implies there is a constant C depending on x, c, β such that, for each component $\hat{Z}^{(i)}$ of \hat{Z} ,

$$(3.14) \quad \mathbb{P}[\omega_{\hat{Z}^{(i)}}(\epsilon) \geq \theta] \leq CT\theta^{-2l}\epsilon^{l\beta-1}, \quad i = 1, 2,$$

for any $l \geq 1$.

A bound on the Lévy–Prokhorov distance between $\mathcal{L}(\tilde{U}_n)$ and $\mathcal{L}(\hat{Z})$ now follows by using (3.11), (3.13), and (3.14) in Theorem 5.1 below, with $\kappa_1 = O(|x_n n^{-1} - x| + \alpha([0, T])n^{-1/2})$ (from (3.4)) and $\kappa_2 = 0$, giving a bound of order

$$(3.15) \quad O(\theta + \delta\sqrt{T \log n} + (\epsilon\delta)^{-3}\kappa_1 T^{3/2} + T\theta^{-2l}\epsilon^{l\beta-1})$$

for the Lévy–Prokhorov distance, for any $\delta, \theta > 0$ and $\epsilon \in (n^{-1/\beta}, 1)$. Taking $\delta\sqrt{T} = \theta$ and matching $(\epsilon\delta)^{-3}\kappa_1 T^{3/2} = T\theta^{-2l}\epsilon^{l\beta-1}$ reduces the bound (3.15) to

$$O(\theta\sqrt{\log n} + \kappa_1 T^3(\epsilon\theta)^{-3}) \quad \text{with} \quad \epsilon = (\kappa_1 T^2\theta^{2l-3})^{\frac{1}{2+l\beta}} > n^{-1/\beta}$$

and then balancing θ with $\kappa_1 T^3(\epsilon\theta)^{-3}$ yields the bound $O(\theta\sqrt{\log n})$ with

$$\theta = (\kappa_1 T^3)^{(l\beta-1)/(6l+4l\beta-1)} T^{3/(6l+4l\beta-1)}.$$

That is, we have a bound of order

$$O(\sqrt{\log n}(T^4 n^{-\varphi})^{(l\beta-1)/(6l+4l\beta-1)} T^{3/(6l+4l\beta-1)}),$$

for any $l \geq 1/\beta$, where

$$\varphi := \min \left\{ \frac{1}{2} + \log_n \left(\frac{T}{\alpha([0, T])} \right), \log_n \left(\frac{T}{|x_n n^{-1} - x|} \right) \right\}.$$

The simplified bound given in the statement of Theorem 1.1, for $\beta = 1$ and $n^{1/2}|x_n/n - x| \leq \alpha([0, T]) = O(T)$, follows, for any $\chi > 0$, by taking l large enough. \square

4. GI/GI/ ∞ queue: Proof of Theorem 1.2. Let us first recall some notation from Section 1.4. The stationary renewal process V_n is driven by R/n , and the point process N_n that we consider has the form

$$N_n := \sum_{i=1}^{\lfloor nT \rfloor} M_n(i/n)\delta_{(i/n, Y_i)},$$

where:

- (i) $M_n(i/n)$ marks the arrival of a customer at time i/n , and can be represented as

$$M_n(i/n) := \sum_{j=1}^{\infty} \mathbf{1}[R_0 + R_1 + \dots + R_j = i],$$

where $(R_j, j \geq 1)$ are independent copies of R , and R_0 has the delay distribution

$$\mathbb{P}(R_0 = k) = m^{-1}\mathbb{P}(R \geq k), \quad k \in \mathbb{N};$$

- (ii) the service times $Y := (Y_i, i \geq 1)$ are i.i.d. with distribution G ,
- (iii) M_n and Y are independent.

As in the previous section, we begin with a coupling lemma.

LEMMA 4.1. *With the above notation, let $(R_{ik}, 1 \leq i, k < \infty)$ and $(R'_{ik}, 1 \leq i, k < \infty)$ be independent i.i.d. sequences with the same distribution as R that are also independent of V_n and Y . Define*

$$S_{ij} := \sum_{k=1}^j R_{ik} \quad \text{and} \quad S'_{ij} := \sum_{k=1}^j R'_{ik}, \quad i, j \in \{1, 2, \dots\};$$

$$\widehat{M}_n^{(i/n)} := \sum_{j=1}^{\infty} (\delta_{(i+S_{ij})/n} + \delta_{(i-S'_{ij})/n}), \quad i \in \{0, 1, \dots, n\};$$

then set

$$T_i := \min\{\lfloor nT \rfloor - i + 1, \inf\{j \geq 1: M_n((i + j)/n) = \widehat{M}_n^{(i/n)}((i + j)/n) = 1\}\},$$

$$T'_i := \min\{i, \inf\{j \geq 1: M_n((i - j)/n) = \widehat{M}_n^{(i/n)}((i - j)/n) = 1\}\};$$

and finally define

$$M_n^{(i/n)}(j/n) := \begin{cases} \widehat{M}_n^{(i/n)}(j/n) & i - T'_i < j < i + T_i, \\ M_n(j/n) & \text{otherwise.} \end{cases}$$

Then

$$N_n^{(i/n)} := \sum_{j=1}^{\lfloor nT \rfloor} M_n^{(i/n)}(j/n) \delta_{(j/n, Y_j)}$$

has the reduced Palm distribution of N_n at $(i/n, y) \in \mathcal{S}$.

PROOF. It is well known that $\widehat{M}_n^{(i/n)}$ has the reduced Palm measure of M_n at (i/n) ; see, for example, Daley and Vere-Jones ((2008), Chapter 13). Then note that $M_n^{(i/n)} \stackrel{d}{=} \widehat{M}_n^{(i/n)}$, since T_i is the first time there is a renewal in both the zero delayed renewal process started from $i: (i + S_{ij})_{j \geq 1}$ appearing in the definition of $\widehat{M}_n^{(i/n)}$, and the analogous stationary process induced by M_n , at which point we can continue using either process without changing the distribution. A similar statement holds for T'_i but now moving backwards in time. From this observation, it is clear that $N_n^{(i/n)}$ is distributed as claimed. \square

REMARK 11. We write $N_n^{(i/n)}$ for the reduced Palm distribution at $(i/n, y)$, for all y .

The next result gives an expression for $\mathbb{E}[X_n^{(i/n)} - X_n]$, to be used in (1.17) of Theorem 1.4.

LEMMA 4.2. With the notation above, let

$$X_n^{(i/n)} := \int_{\mathcal{S}} J_{t,y}(s) N_n^{(i/n)}(dt, dy).$$

If $\mathbb{E}[R^r] < \infty$ for some $r > 3$, and if G satisfies the assumptions (1.11), then there is a constant C , depending only on $\mathcal{L}(R)$, β and $g_G(0)$, such that

$$A_{n,i}(s) := \mathbb{E}[X_n^{(i/n)}(s)] - \mathbb{E}[X_n(s)] - \mathbf{1}\{s \geq in^{-1}\} \overline{G}(s - in^{-1}) \left(\frac{v^2 - m^2}{m^2} \right)$$

satisfies

$$|A_{n,i}(s)| \leq C \{n^{-\beta} + (|i - \lfloor ns \rfloor| + 1)^{-(r-2)} + i^{-(r-2)}\}.$$

PROOF. First note that

$$X_n^{(i/n)}(s) - X_n(s) = \sum_{j=1}^{\lfloor nT \rfloor} (M_n^{(i/n)}(j/n) - M_n(j/n)) \mathbf{1}\{j/n \leq s < Y_j + j/n\},$$

so that, using the independence of Y and $(M_n, M_n^{(i/n)})$, we have

$$(4.1) \quad \mathbb{E}[X_n^{(i/n)}(s)] - \mathbb{E}[X_n(s)] = \sum_{j=1}^{\lfloor ns \rfloor} (\mathbb{E}[M_n^{(i/n)}(j/n)] - \mathbb{E}[M_n(j/n)]) \overline{G}(s - jn^{-1}).$$

By stationarity, $\mathbb{E}[M_n(j/n)] = 1/m$; and by Lemma 4.1,

$$\mathbb{E}[M_n^{(i/n)}(j/n)] = u_{|j-i|}^0 := u_{|j-i|} \mathbf{1}\{j \neq i\},$$

where u_l is the probability that there is a renewal at time l in a renewal process with inter-arrival distribution $\mathcal{L}(R)$, started from zero. Hence we can write (4.1) in the form

$$\begin{aligned} & \mathbb{E}[X_n^{(i/n)}(s)] - \mathbb{E}[X_n(s)] \\ &= \sum_{j=1}^{\lfloor ns \rfloor} (u_{|j-i|}^0 - m^{-1}) \overline{G}(s - jn^{-1}) \\ (4.2) \quad &= \sum_{j=1}^{\lfloor ns \rfloor} (u_{|j-i|}^0 - m^{-1}) \overline{G}(s - in^{-1}) \mathbf{1}\{s \geq in^{-1}\} \\ &+ \sum_{j=1}^{\lfloor ns \rfloor} (u_{|j-i|}^0 - m^{-1}) (\overline{G}(s - jn^{-1}) - \overline{G}(s - in^{-1}) \mathbf{1}\{s \geq in^{-1}\}). \end{aligned}$$

The remainder of the proof consists of showing that the first term in (4.2) is close to $\mathbf{1}\{s \geq in^{-1}\} \overline{G}(s - in^{-1})(v^2 - m^2)/m^2$, and that the second term is small. The main tool is the inequality

$$(4.3) \quad |\mathbb{E}[M_n^{(i/n)}(j/n)] - 1/m| \leq C_R (|j - i| + 1)^{-(r-1)},$$

for a suitable constant C_R , which follows from Pitman ((1974), Corollary (6.21)), together with the observation that

$$(4.4) \quad \sum_{j=-\infty}^{\infty} (u_{|j-i|}^0 - m^{-1}) = \frac{v^2 - m^2}{m^2}.$$

For the first term in (4.2), using (4.3) and (4.4), for $s \geq i/n$, we have

$$\begin{aligned} & \left| \sum_{j=1}^{\lfloor ns \rfloor} (u_{|j-i|}^0 - m^{-1}) - \frac{v^2 - m^2}{m^2} \right| \leq C_R \left\{ \sum_{j=-\infty}^0 (|j - i| + 1)^{-(r-1)} \right. \\ (4.5) \quad & \left. + \sum_{j=\lfloor ns \rfloor + 1}^{\infty} (|j - i| + 1)^{-(r-1)} \right\} \\ & \leq \frac{C_R}{r - 2} \{i^{-(r-2)} + |\lfloor ns \rfloor - i + 1|^{-(r-2)}\}; \end{aligned}$$

for $s < i/n$, the term is zero because of the factor $\mathbf{1}\{s \geq in^{-1}\}$. For the second term in (4.2), for $s \geq i/n$, using (1.11),

$$|\overline{G}(s - i/n) - \overline{G}(s - j/n)| \leq g_G(0) |n^{-1}(i - j)|^\beta, \quad 1 \leq j \leq \lfloor ns \rfloor,$$

so that

$$\begin{aligned} & \left| \sum_{j=1}^{\lfloor ns \rfloor} (u_{|j-i|}^0 - m^{-1}) (\overline{G}(s - jn^{-1}) - \overline{G}(s - in^{-1}) \mathbf{1}\{s \geq in^{-1}\}) \right| \\ (4.6) \quad & \leq \sum_{j=1}^{\lfloor ns \rfloor} C_R (|j - i| + 1)^{-(r-1)} g_G(0) n^{-\beta} |i - j|^\beta \leq n^{-\beta} C_R g_G(0) \left\{ \frac{2}{r - 2 - \beta} \right\}. \end{aligned}$$

Finally, for $s < i/n$, we have

$$\begin{aligned}
 (4.7) \quad \sum_{j=1}^{\lfloor ns \rfloor} |u_{|j-i|}^0 - m^{-1} \overline{G}(s - jn^{-1})| &\leq C_R \sum_{j=1}^{\lfloor ns \rfloor} (|j-i| + 1)^{-(r-1)} \\
 &\leq C_R \left\{ \frac{r-1}{r-2} \right\} |\lfloor ns \rfloor - i + 1|^{-(r-2)}.
 \end{aligned}$$

Combining (4.5)–(4.7) with (4.2) proves the lemma. \square

We now use the coupling of Lemma 4.1 and Theorem 1.4 to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Following the notation in Theorem 1.4, we set

$$\Lambda(dt, dy) := dtG(dy) \quad \text{and} \quad \widehat{J}_{t,y} := \frac{m}{v} J_{t,y} - \frac{m+v}{v} \overline{G}_t I_t,$$

where $\overline{G}_t(s) := \overline{G}(s - t)$ and $I_t(s) := \mathbf{1}\{s \geq t\}$.

From the assumptions (1.11) on G , it is clear from (1.12) that the covariance function is continuous. Moreover, for any $s \geq 0$ and $0 \leq h \leq (T - s) \wedge 1$, they easily imply, with (1.12), that there is a constant c' such that

$$\mathbb{E}[(Z(s+h) - Z(s))^2] \leq c'h^\beta,$$

so that (1.21) is satisfied. Therefore, in view of Remark 7, we can assume that Z has continuous sample paths.

To bound the term corresponding to (1.17), there are two issues. The first is that we need to compare integration against the atom-less Λ to the atoms of λ_n/σ_n^2 corresponding to the renewals of X_n occurring on a discrete lattice. The second is that there are error terms in the differences of the means of $X_n^{(i/n)}$ and X_n , as given in Lemma 4.2. To handle the first issue, we introduce $\Lambda_n := \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} (\delta_{i/n} \times G)$ as a discretized version of Λ . Then we can compute

$$\begin{aligned}
 (4.8) \quad &\mathbb{E} \left[\int D^2 f(\tilde{X}_n) [\widehat{J}_{t,y}^{[2]}] \Lambda(dt, dy) - \int D^2 f(\tilde{X}_n) [\widehat{J}_{t,y}^{[2]}] \Lambda_n(dt, dy) \right] \\
 &= \sum_{i=1}^{\lfloor nT \rfloor} \int_{\mathbb{R}_+} \mathbb{E} \left[\int_{(i-1)/n}^{i/n} (D^2 f(\tilde{X}_n) [\widehat{J}_{t,y}^{[2]}] - D^2 f(\tilde{X}_n) [\widehat{J}_{i/n,y}^{[2]}]) dt \right] G(dy) \\
 &= \sum_{i=1}^{\lfloor nT \rfloor} \int_{\mathbb{R}_+} \mathbb{E} \left[\int_{(i-1)/n}^{i/n} (D^2 f(\tilde{X}_n) [\widehat{J}_{t,y} - \widehat{J}_{i/n,y}, \widehat{J}_{t,y}] \right. \\
 &\quad \left. + D^2 f(\tilde{X}_n) [\widehat{J}_{i/n,y}, \widehat{J}_{t,y} - \widehat{J}_{i/n,y}]) dt \right] G(dy).
 \end{aligned}$$

We work on

$$(4.9) \quad |D^2 f(\tilde{X}_n) [\widehat{J}_{t,y} - \widehat{J}_{i/n,y}, \widehat{J}_{s,y}]|,$$

where $t \in ((i-1)/n, i/n]$, and $s \geq 0$. First, note that we can write

$$(4.10) \quad \widehat{J}_{r,y} = \frac{m}{v} (I_r - I_{r+y}) - \frac{m+v}{v} (I_r - G_r),$$

so that

$$\widehat{J}_{t,y} - \widehat{J}_{i/n,y} = \frac{m}{v} ((I_t - I_{i/n}) - (I_{t+y} - I_{i/n+y})) - \frac{m+v}{v} ((I_t - I_{i/n}) - (G_t - G_{i/n})),$$

and thus bilinearity implies that (4.9) is bounded by

$$(4.11) \quad C\{|D^2 f(\tilde{X}_n)[I_t - I_{i/n}, \hat{J}_{s,y}]| + |D^2 f(\tilde{X}_n)[I_{t+y} - I_{i/n+y}, \hat{J}_{s,y}]| + |D^2 f(\tilde{X}_n)[G_t - G_{i/n}, \hat{J}_{s,y}]\};$$

here and below we allow C to change from line to line, but only depending on $\mathcal{L}(R)$ and $g_G(0)$.

To bound the last term of (4.11), equation (2.12) of Theorem 2.2 and the assumption (1.11) on G (noting that $|t - i/n| \leq 1/n$) imply that

$$(4.12) \quad |D^2 f(\tilde{X}_n)[G_t - G_{i/n}, \hat{J}_{s,y}]| \leq C \|g\|_{M'} n^{-\beta}.$$

To bound the first two terms, we apply (2.10) and, noting that $J_{s,y} = I_s - I_{s+y}$, we have

$$\begin{aligned} & |D^2 f_g(w)[I_t - I_{i/n}, J_{s,y}]| \\ & \leq \int_0^\infty e^{-2z} \mathbb{E}[|D^2 g(we^{-z} + \sqrt{1 - e^{-2z}}Z)[I_t - I_{i/n}, J_{s,y}]|] dz \\ & \leq \begin{cases} S_g n^{-1/2} & \text{if } g \text{ satisfies (1.2),} \\ k^2 \|g\|_{M'} \sum_{j=1}^k \mathbf{1}\{t_j \in ((i-1)/n, i/n)\} & \text{if } g(w) = F(w(t_1), \dots, w(t_k)), \end{cases} \end{aligned}$$

where, in the first case, we use the smoothness condition (1.2), and in the second the explicit expression

$$D^2 g(w)[w_1, w_2] = \sum_{j,\ell=1}^k F_{j\ell}(w(t_1), \dots, w(t_k)) w_1(t_j) w_2(t_\ell),$$

where we write $F_{j\ell}$ for the mixed partial derivative of F in the coordinates j and ℓ . Similarly, using Lemma 5.3, we can write

$$\begin{aligned} & |D^2 f(\tilde{X}_n)[I_t - I_{i/n}, G_s]| \\ & = \left| \int D^2 f(\tilde{X}_n)[I_t - I_{i/n}, I_{s+y'}] G(dy') \right| \\ & \leq \begin{cases} S_g n^{-1/2} & \text{if } g \text{ satisfies (1.2),} \\ k^2 \|g\|_{M'} \sum_{j=1}^k \mathbf{1}\{t_j \in ((i-1)/n, i/n)\} & \text{if } g(w) = F(w(t_1), \dots, w(t_k)), \end{cases} \end{aligned}$$

and there are analogous bounds for the last two displays, when replacing t by $t + y$ and i/n by $i/n + y$. Noting that $\sum_{i=1}^{\lfloor nT \rfloor} \sum_{j=1}^k \mathbf{1}\{t_j \in ((i-1)/n, i/n)\} = k$, we can apply these last inequalities with the representation (4.10) for $\hat{J}_{s,y}$ to see that the absolute value of (4.8) is bounded by

$$(4.13) \quad \begin{aligned} & \left| \mathbb{E} \left[\int D^2 f(\tilde{X}_n)[\hat{J}_{t,y}^{[2]}] \Lambda(dt, dy) - \int D^2 f(\tilde{X}_n)[\hat{J}_{t,y}^{[2]}] \Lambda_n(dt, dy) \right] \right| \\ & \leq CT \begin{cases} \|g\|_{M'} n^{-\beta} + S_g n^{-1/2} & \text{if } g \text{ satisfies (1.2),} \\ \|g\|_{M'} (n^{-\beta} + k^3 n^{-1}) & \text{if } g(w) = F(w(t_1), \dots, w(t_k)). \end{cases} \end{aligned}$$

To finish bounding (1.17), we can apply Lemma 5.3 (after rewriting \widehat{J} as per (4.10)) to find

$$\begin{aligned}
 \mathbb{E}\left[\int D^2 f(\tilde{X}_n)[\widehat{J}_{t,y}^{[2]}]\Lambda_n(dt, dy)\right] &= \frac{m^2}{v^2n} \sum_{i=1}^{\lfloor nT \rfloor} \int_{\mathbb{R}_+} \mathbb{E}[D^2 f(\tilde{X}_n)[J_{i/n,y}^{[2]}]]G(dy) \\
 (4.14) \qquad \qquad \qquad &+ \frac{v^2 - m^2}{v^2n} \sum_{i=1}^{\lfloor nT \rfloor} \mathbb{E}[D^2 f(\tilde{X}_n)[(I_{i/n}\overline{G}_{i/n})^{[2]}]].
 \end{aligned}$$

Since $\lambda_n/\sigma_n^2 = (m^2/v^2)\frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} (\delta_{i/n} \times G)$, we have

$$\begin{aligned}
 \mathbb{E}\left[\int D^2 f(\tilde{X}_n)[J_{t,y}, \mathbb{E}[X_n^{(t,y)} - X_n + J_{t,y}]](\sigma_n^{-2}\lambda_n(dt, dy))\right] \\
 (4.15) \qquad \qquad \qquad &= \frac{m^2}{v^2n} \sum_{i=1}^{\lfloor nT \rfloor} \int_{\mathbb{R}_+} \mathbb{E}[D^2 f(\tilde{X}_n)[J_{i/n,y}, \mathbb{E}[X_n^{(i/n)} - X_n + J_{i/n,y}]]]G(dy).
 \end{aligned}$$

From Lemma 4.2 (and using the notation there), we have

$$(4.16) \qquad \mathbb{E}[X_n^{(i/n)} - X_n + J_{i/n,y}] = J_{i/n,y} + \frac{v^2 - m^2}{m^2} I_{i/n}\overline{G}_{i/n} + A_{n,i}.$$

Hence (4.14)–(4.16) imply that the contribution from equation (1.17) is bounded by

$$\begin{aligned}
 &\left| \mathbb{E}\left[\int D^2 f_g(\tilde{X}_n)[\widehat{J}_{t,y}^{[2]}]\Lambda_n(dt, dy) \right. \right. \\
 &\qquad \qquad \qquad \left. \left. - \int D^2 f_g(\tilde{X}_n)[J_{t,y}, \mathbb{E}[X_n^{(t,y)} - X_n + J_{t,y}]]\frac{\lambda_n(dt, dy)}{\sigma_n^2} \right] \right| \\
 (4.17) \qquad &\leq \frac{C}{n} \sum_{i=1}^{\lfloor nT \rfloor} \int_{\mathbb{R}_+} |\mathbb{E}[D^2 f_g(\tilde{X}_n)[J_{i/n,y}, A_{n,i}]]|G(dy)
 \end{aligned}$$

$$(4.18) \qquad \leq CT \begin{cases} \|g\|_{M'n^{-\beta}} + S_g n^{-1/2} & \text{if } g \text{ satisfies (1.2),} \\ \|g\|_{M'}(n^{-\beta} + k^3 n^{-1}) & \text{if } g(w) = F(w(t_1), \dots, w(t_k)), \end{cases}$$

where we have used the bounds of Lemma 4.2, writing

$$(|i - \lfloor ns \rfloor| + 1)^{-(r-2)} = \sum_{j=0}^{\lfloor nT \rfloor - 1} \mathbf{1}[s \in [j/n, (j+1)/n)](|i - j| + 1)^{-(r-2)},$$

and using the same smoothness/finite number of instants arguments leading to (4.13).

To bound (1.18), we first construct processes $\widehat{X}_n^{[i/n]}$ and $X_n^{[i/n]}$ such that $\widehat{X}_n^{[i/n]}$ is independent of $X_n^{(i/n)} - X_n$, the pair of processes $X_n^{[i/n]}$ and $\widehat{X}_n^{[i/n]}$ are close to one another, and $\mathcal{L}(X_n^{(i/n)} - X_n, X_n^{[i/n]}) = \mathcal{L}(X_n^{(i/n)} - X_n, X_n)$. To do so, recalling the notation of Lemma 4.1, let M_n^* be a copy of M_n that is independent of both M_n and $M_n^{(i/n)}$. Define

$$\begin{aligned}
 \widehat{T}_i &:= \min\{n - i + 1, \inf\{j > T_i : M_n((i+j)/n) = M_n^*((i+j)/n) = 1\}\}, \\
 \widehat{T}'_i &:= \min\{i, \inf\{j > T'_i : M_n((i-j)/n) = M_n^*((i-j)/n) = 1\}\}.
 \end{aligned}$$

Now set

$$M_n^{[i/n]}(j/n) := \begin{cases} M_n(j/n) & i - \widehat{T}'_i < j < i + \widehat{T}_i, \\ M_n^*(j/n) & \text{otherwise} \end{cases}$$

and, for $(Y'_j, j \geq 1)$ i.i.d. with distribution function G and independent of the previous variables, set

$$N_n^{[i/n]} := \sum_{j=i-\widehat{T}'_i}^{i+\widehat{T}_i} M_n^{[i/n]}(j/n)\delta_{(j/n, Y_j)} + \sum_{j \notin [i-\widehat{T}'_i, i+\widehat{T}_i]} M_n^{[i/n]}(j/n)\delta_{(j/n, Y'_j)};$$

$$N_n^* := \sum_{j=1}^{\lfloor nT \rfloor} M_n^*(j/n)\delta_{(j/n, Y'_j)},$$

and then set

$$X_n^{[i/n]} := \int_S J_{t,y} N_n^{[i/n]}(dt, dy); \quad X_n^* := \int_S J_{t,y} N_n^*(dt, dy).$$

It is clear that $\mathcal{L}(N_n^{[i/n]}, N_n^{(i/n)} - N_n) = \mathcal{L}(N_n, N_n^{(i/n)} - N_n)$, because N_n and $N_n^{[i/n]}$ differ only by having different choices of independent and identically distributed zero-delayed renewal processes defining their continuations outside the interval $[i - \widehat{T}'_i, i + \widehat{T}_i]$, and these are independent of $N_n^{(i/n)} - N_n$, which is determined by events defined only on $[i - \widehat{T}'_i, i + \widehat{T}_i]$.

Now, defining

$$\widetilde{X}_n^{[i/n]} := \sigma_n^{-1}(X_n^{[i/n]} - \lambda_n); \quad \widetilde{X}_n^* := \sigma_n^{-1}(X_n^* - \lambda_n),$$

we observe that

$$(4.19) \quad \mathbb{E} \left[\int D^2 f(\widetilde{X}_n^*) [J_{i/n,y}, (X_n^{(i/n)} - X_n) - \mathbb{E}[X_n^{(i/n)} - X_n]] G(dy) \right] = 0.$$

This follows primarily because M_n^* is independent of both M_n and $M_n^{(i/n)}$, and hence \widetilde{X}_n^* and $X_n^{(i/n)} - X_n$ are independent. In more detail, because $M_n^{(i/n)}(j/n) = M_n(j/n)$ for $j \notin (i - T'_i, i + T_i)$, and because of the independence of $(M_n^{(i/n)} - M_n)$, $(Y_j, j \geq 1)$ and \widetilde{X}_n^* , we can use Lemma 5.3 to show that

$$\begin{aligned} & \mathbb{E}[D^2 f(\widetilde{X}_n^*) [J_{i/n,y}, (X_n^{(i/n)} - X_n)]] \\ &= \mathbb{E} \left[\sum_{j=i-T'_i+1}^{i+T_i-1} (M_n^{(i/n)}(j/n) - M_n(j/n)) \mathbb{E}\{D^2 f(\widetilde{X}_n^*) [J_{i/n,y}, I_{j/n} - I_{j_n^{-1}+Y_j}]\} \right] \\ &= \mathbb{E} \left[\sum_{j=i-T'_i+1}^{i+T_i-1} (M_n^{(i/n)}(j/n) - M_n(j/n)) \int_{\mathbb{R}_+} D^2 f(\widetilde{X}_n^*) [J_{i/n,y}, I_{j/n} - I_{j_n^{-1}+y'}] G(dy') \right] \\ &= \mathbb{E} \left[\sum_{j=i-T'_i+1}^{i+T_i-1} (M_n^{(i/n)}(j/n) - M_n(j/n)) D^2 f(\widetilde{X}_n^*) [J_{i/n,y}, I_{j/n} \overline{G}_{j/n}] \right] \\ &= \sum_{j=1}^{\lfloor nT \rfloor} \mathbb{E}[M_n^{(i/n)}(j/n) - M_n(j/n)] \mathbb{E}(D^2 f(\widetilde{X}_n^*) [J_{i/n,y}, I_{j/n} \overline{G}_{j/n}]) \\ &= \mathbb{E} \left(D^2 f(\widetilde{X}_n^*) \left[J_{i/n,y}, \sum_{j=1}^{\lfloor nT \rfloor} \mathbb{E}[M_n^{(i/n)}(j/n) - M_n(j/n)] I_{j/n} \overline{G}_{j/n} \right] \right) \\ &= \mathbb{E}(D^2 f(\widetilde{X}_n^*) [J_{i/n,y}, \mathbb{E}[X_n^{(i/n)} - X_n]]), \end{aligned}$$

and integrating with respect to y gives (4.19). Thus, we can bound the term corresponding to (1.18) as follows. First observe that, because $\mathcal{L}(N_n^{[i/n]}, N_n^{(i/n)} - N_n) = \mathcal{L}(N_n, N_n^{(i/n)} - N_n)$,

$$(4.20) \quad \left| \frac{m^2}{v^2 n} \sum_{i=1}^{\lfloor nT \rfloor} \int_{\mathbb{R}_+} \mathbb{E}[D^2 f(\tilde{X}_n)[J_{i/n,y}, (X_n^{(i/n)} - X_n) - \mathbb{E}[X_n^{(i/n)} - X_n]]]G(dy) \right| \\ = \left| \frac{m^2}{v^2 n} \sum_{i=1}^{\lfloor nT \rfloor} \int_{\mathbb{R}_+} \mathbb{E}[D^2 f(\tilde{X}_n^{[i/n]})[J_{i/n,y}, (X_n^{(i/n)} - X_n) - \mathbb{E}(X_n^{(i/n)} - X_n)]]G(dy) \right|.$$

Now we can use (4.19) to give

$$(4.20) \leq \frac{m^2}{v^2 n} \sum_{i=1}^{\lfloor nT \rfloor} \int_{\mathbb{R}_+} |\mathbb{E}[D^2 f(\tilde{X}_n^{[i/n]})[J_{i/n,y}, (X_n^{(i/n)} - X_n) - \mathbb{E}[X_n^{(i/n)} - X_n]] \\ - D^2 f(\tilde{X}_n^*)[J_{i/n,y}, (X_n^{(i/n)} - X_n) - \mathbb{E}(X_n^{(i/n)} - X_n)]]|G(dy) \\ (4.21) \leq \frac{C \|g\|_M}{n} \sum_{i=1}^{\lfloor nT \rfloor} \mathbb{E}[\|\tilde{X}_n^{[i/n]} - \tilde{X}_n^*\| \|(X_n^{(i/n)} - X_n) - \mathbb{E}(X_n^{(i/n)} - X_n)\|] \\ \leq \frac{C \|g\|_M}{n^{3/2}} \sum_{i=1}^{\lfloor nT \rfloor} \mathbb{E}[(1 + \hat{T}_i + \hat{T}'_i)(1 + T_i + T'_i)] \leq \frac{CT \|g\|_M}{\sqrt{n}},$$

where the second inequality follows from (2.11), the third is because $N_n^{[i/n]}(j/n) = N_n^*(j/n)$ for $j \notin [i - \hat{T}'_i, i + \hat{T}_i]$ and $N_n^{(i/n)}(j/n) = N_n(j/n)$ for $j \notin [i - T'_i, i + T_i]$, and the final inequality is obtained by using Cauchy–Schwarz and then noting that, by Pitman ((1974), Proposition (6.10)), under the assumption $\mathbb{E}[R^3] < \infty$, T_i, T'_i, \hat{T}_i and \hat{T}'_i all have finite second moments, whose values depend only on $\mathcal{L}(R)$.

Similarly, to bound (1.19), note that

$$\mathbb{E}[\|X_n^{(i/n)} - X_n + J_{i/n,y}\|^2] \leq \mathbb{E}[(1 + T_i + T'_i)^2] \leq C,$$

again if $\mathbb{E}[R^3] < \infty$. Thus

$$(4.22) \quad \frac{\|g\|_M}{2\sigma_n} \mathbb{E}\left[\int_S \|J_{t,y}\| \|X_n^{(t,y)} - X_n + J_{t,y}\|^2 (\sigma_n^{-2} \lambda_n(dt, dy)) \right] \leq \frac{CT \|g\|_M}{\sqrt{n}}.$$

Combining (4.13), (4.18), (4.21), and (4.22) yields the bound given in (1.13).

To prove the bound on the Lévy–Prokhorov distance, we follow the template for the M/GI/∞ queue, and use the main results of Barbour, Ross and Zheng (2021), as stated in Theorem 5.1 below. The first hypothesis of Theorem 5.1 is satisfied, with κ_1 and κ_2 read from the bound (1.13) (noting that $\|g\|_M \leq \|g\|_{M'} \leq \|g\|_{M^0}$).

To bound the modulus of continuity terms, we again use Lemma 5.2. To verify (5.1), for any $0 \leq s_1 < s_2 \leq T$, define the regions

$$\mathcal{R}_1(s_1, s_2) := \mathcal{R}_{s_2} \setminus \mathcal{R}_{s_1} \quad \text{and} \quad \mathcal{R}_2(s_1, s_2) := (s_1, s_2] \times \mathbb{R}_+,$$

as before at (3.5), so that

$$\tilde{X}_n(t) - \tilde{X}_n(s) = \tilde{Y}_n(2; s, t) - \tilde{Y}_n(1; s, t),$$

where $\tilde{Y}_n(i; s_1, s_2) := \sigma_n^{-1/2} \{N_n(\mathcal{R}_i(s_1, s_2)) - \lambda_n(\mathcal{R}_i(s_1, s_2))\}$, $i = 1, 2$. We now use Markov’s inequality to bound each term in

$$\mathbb{P}(|\tilde{X}_n(t) - \tilde{X}_n(s)| \geq \theta) \leq \mathbb{P}(|\tilde{Y}_n(1; s, t)| \geq \theta/2) + \mathbb{P}(|\tilde{Y}_n(2; s, t)| \geq \theta/2).$$

First, $\sigma_n \tilde{Y}_n(2; s, t) = \sum_{ns < i \leq nt} [M_n(i/n) - m^{-1}]$ is the centered number of renewals in the interval $(ns, nt]$. By the usual renewal theory coupling arguments, as in Glynn ((1982), Proposition 6.10), writing $\widehat{M}_i := M_n(i/n)$, the sequence $(\widehat{M}_i, 1 \leq i \leq nT)$ is strong mixing as introduced in Rosenblatt (1956) with coefficients $\alpha_j \leq k_R j^{-(r-1)}$, $j = 1, 2, \dots$, for a constant $k_R < \infty$, depending only on $\mathcal{L}(R)$, that we can choose to be at least 1, and $\alpha_0 := 1/2$. Thus, for $0 < u \leq 1$, as in Rio ((2013), (1.21)),

$$\alpha^{-1}(u) := \sum_{j \geq 0} \mathbf{1}\{u < \alpha_j\} \leq (k_R/u)^{1/(r-1)} + \mathbf{1}\{u < 1/2\} \leq 2(k_R/u)^{1/(r-1)}.$$

Applying Rio ((2013), Theorem 2.2), it follows that, for any $l \in \mathbb{N}$, there is a constant $C_{l,R}$ depending only on $\mathcal{L}(R)$ and l such that

$$\begin{aligned} & \mathbb{E}[(\tilde{Y}_n(2; s, t))^{2l}] \\ & \leq C_{l,R} \sigma_n^{-2l} \left\{ \left(\sum_{ns < i \leq nt} \int_0^1 \alpha^{-1}(u) Q_i^2(u) du \right)^l + \sum_{ns < i \leq nt} \int_0^1 [\alpha^{-1}(u)]^{2l-1} Q_i^{2l}(u) du \right\}, \end{aligned}$$

where $Q_i := q_{1/m}$ and, for $w \in [0, 1]$ and $u \in (0, 1]$,

$$(4.23) \quad q_w(u) = \begin{cases} w \mathbf{1}_{(0, 1-w]}(u) + (1-w) \mathbf{1}_{(1-w, 1]}(u) & w \geq 1/2, \\ (1-w) \mathbf{1}_{(0, w]}(u) + w \mathbf{1}_{(w, 1]}(u) & w < 1/2. \end{cases}$$

Straightforward computing now shows that, for $l < r/2$,

$$\mathbb{E}[(\tilde{Y}_n(2; s, t))^{2l}] \leq K_{l,R} \sigma_n^{-2l} [(\lfloor nt \rfloor - \lfloor ns \rfloor)^l + (\lfloor nt \rfloor - \lfloor ns \rfloor)],$$

for a constant $K_{l,R} < \infty$. Markov's inequality and (1.9) thus imply that, if $n(t - s) \geq 1/2$ and for $l < r/2$,

$$(4.24) \quad \mathbb{P}(|\tilde{Y}_n(2; s, t)| \geq \sigma_n \theta / 2) \leq K_{l,R} \left(\frac{m^3}{v^2} \right)^l 2^{2l+1} \theta^{-2l} 3^l (t - s)^l =: C_l^{(1)} \theta^{-2l} (t - s)^l.$$

For $\tilde{Y}_n(1; s, t)$, we observe that

$$N_n(\mathcal{R}_1(s, t)) - \lambda_n(\mathcal{R}_1(s, t)) = \sum_{1 \leq i \leq nt} [M_n(i/n) B_{i,n} - m^{-1} p_{i,n}],$$

where

$$\begin{aligned} B_{i,n} &:= \mathbf{1}[(s - in^{-1})_+ < Y_i \leq t - in^{-1}] \sim \text{Be}(p_{i,n}); \\ p_{i,n} &:= G(t - i/n) - G((s - i/n)_+). \end{aligned}$$

The fact that the random variables $(B_{i,n}, 1 \leq i \leq nT)$ are independent of N_n implies that the mixing properties of the sequence $(\widehat{M}_i, 1 \leq i \leq nT)$ are inherited by the sequence $(\widehat{M}_i B_{i,n}, 1 \leq i \leq nT)$, so that Rio ((2013), Theorem 2.2) can be applied with the same function α^{-1} , giving

$$(4.25) \quad \begin{aligned} \mathbb{E}[(\tilde{Y}_n(1; s, t))^{2l}] & \leq C_{l,R} \sigma_n^{-2l} \left\{ \left(\sum_{1 \leq i \leq nt} \int_0^1 \alpha^{-1}(u) Q_i^2(u) du \right)^l \right. \\ & \quad \left. + \sum_{1 \leq i \leq nt} \int_0^1 [\alpha^{-1}(u)]^{2l-1} Q_i^{2l}(u) du \right\}, \end{aligned}$$

where now $Q_i := q_{p_{i,n}/m}$. Using (4.23), we have that for $1 \leq l < r/2$,

$$(4.26) \quad \int_0^1 (\alpha^{-1}(u))^{2l-1} q_w^{2l}(u) du \leq c_1(r, R, l) w^{(r-2l)/(r-1)}.$$

From the assumptions on G , by comparing sums and integrals, it follows that, for $1/(2n) \leq (t - s) \leq 1$,

$$(4.27) \quad n^{-1} \sum_{1 \leq i \leq nt} p_{i,n} \leq \int_s^t G(v) dv + 2n^{-1} \leq 5(t - s),$$

and, using (4.26), for $1 \leq l < r/2$ such that $(r - 2l)/(r - 1) \geq \eta$, we have

$$(4.28) \quad \begin{aligned} & n^{-1} \sum_{1 \leq i \leq nt} p_{i,n}^{(r-2l)/(r-1)} \\ & \leq (t - s)^{\beta(r-2l)/(r-1)} \left\{ \int_0^\infty (g_G(v))^{(r-2l)/(r-1)} dv + 2(g_G(0))^{(r-2l)/(r-1)} \right\} \\ & \leq c_3(r, R, l)(t - s)^{\beta(r-2l)/(r-1)}. \end{aligned}$$

Hence, from (4.25), if $r(1 - \beta) \geq 1$, writing $\beta_r := \beta(r - 2)/(r - 1)$, we find that, for $0 \leq s < t \leq T$ and for $n(t - s) \geq 1/2$,

$$\mathbb{E}[(\tilde{Y}_n(1; s, t))^{2l}] \leq C_{l,R}(n/\sigma_n^2)^l (t - s)^{l\beta_r},$$

for any $l < (r - \eta(r - 1))/2$. The assumption $r(1 - \beta) \geq 1$ ensures the exponent $l\beta_r$ is no larger than those appearing when applying (4.25), (4.26), and (4.28). It now follows, by Markov’s inequality and (1.9), that, for such l , and if $r(1 - \beta) \geq 1$,

$$(4.29) \quad \mathbb{P}(|\tilde{Y}_n(1, s, t)| \geq \theta/2) \leq C_l^{(1)} \theta^{-2l} (t - s)^{l\beta_r}, \quad n(t - s) \geq 1/2.$$

If $r(1 - \beta) < 1$, the inequality (4.29) holds only for $0 \leq s < t \leq T$ such that $(t - s)n^{(r-1)/r\beta} \geq 1/2$.

To verify (5.2) for \tilde{X}_n , we note that, for $(k - 1)/M \leq u \leq k/M$,

$$|\tilde{Y}_n(i; (k - 1)/M, u)| \leq |\tilde{Y}_n(i, (k - 1)/M, k/M)| + 2\sigma_n^{-1} \lambda_n(\mathcal{R}_i((k - 1)/M, k/M)),$$

$i = 1, 2$, where $\lambda_n(\mathcal{R}_2((k - 1)/M, k/M)) = m^{-1}n/M$ and, because of (4.27), $\lambda_n(\mathcal{R}_1((k - 1)/M, k/M)) \leq 5m^{-1}n/M$. Writing $\rho := \rho(r, \beta) := \min\{1, (r - 1)/r\beta\}$, and taking $M := n^\rho$, it follows from (4.24) and (4.29) that we can take

$$(4.30) \quad \varphi_M(\theta) := M^{1-l\beta_r} C_l^{(3)} \theta^{-2l}$$

in (5.2), for a suitable constant $C_l^{(3)}$, if $\theta > 48m^{-1}n^{1-\rho}/\sigma_n$. We can now apply Lemma 5.2 with $b = 2l$ and $a = l\beta_r - 1$, for $l = l_r := \lceil (r - \eta(r - 1))/2 \rceil - 1$, and with $M := n^\rho$, to find that, for $\theta > 48(1 - 2^{-(l_r\beta_r-1)/4l_r})^{-1}m^{-1}n^{1-\rho}/\sigma_n$ and $\epsilon \in (n^{-\rho}, 1)$,

$$(4.31) \quad \mathbb{P}(\omega_{\tilde{X}_n}(\epsilon) \geq \theta/2) \leq CT\theta^{-2l_r} \epsilon^{l\beta_r-1},$$

for some suitable constant C .

For the modulus of continuity of Z , an easy calculation shows that for any $0 \leq r < s \leq T$,

$$\mathbb{E}[(Z(s) - Z(r))^2] \leq C(s - r)^\beta,$$

and so Barbour, Ross and Zheng ((2021), Remark 1.6) implies that for any $l \geq 1$,

$$(4.32) \quad \mathbb{P}[\omega_Z(\epsilon) \geq \theta] \leq CT\theta^{-2l} \epsilon^{l\beta-1}.$$

From (4.13), (4.18), (4.21), and (4.22), and for $T \geq 1$, we can now apply Theorem 5.1 with

$$\kappa_1 = O(Tn^{-\bar{\beta}}) \quad \text{where } \bar{\beta} := \min\{\beta, 1/2\}; \quad \kappa_2 = O(Tn^{-1/2}).$$

This, using (4.31) and (4.32), for any choice of $\epsilon, \delta > 0$ and $\theta > cn^{1/2-\rho(r,\beta)}$, and for any $l \geq 1$, implies a bound of

$$C(\delta\sqrt{T \log n} + \theta + T^{5/2}n^{-\bar{\beta}}(\epsilon\delta)^{-3} + T^{3/2}n^{-1/2}(\epsilon\delta)^{-2} + T\epsilon^{l_r\beta_r-1}\theta^{-2l_r} + T\epsilon^{l\beta-1}\theta^{-2l}),$$

for a suitable constant C , where we recall that $\beta_r := \beta(r - 2)/(r - 1)$ and that $\rho(r, \beta) := \min\{1, (r - 1)/r\beta\}$. Taking

$$\begin{aligned} \theta &= \sqrt{T}\delta = \{(T^4n^{-\bar{\beta}})^{l_r\beta_r-1}T^3\}^{1/(6l_r+4l_r\beta_r-1)}, \\ \epsilon &= \{(T^4n^{-\bar{\beta}})^{2l_r+1}T^{-4}\}^{1/(6l_r+4l_r\beta_r-1)}, \end{aligned}$$

gives a bound of order

$$O(\sqrt{\log n}\{(T^{5/2}n^{-\bar{\beta}})^{l_r\beta_r-1}T^3\}^{1/(6l_r+4l_r\beta_r-1)}).$$

Here, we note that if $\epsilon \leq 1$, for example, if $T \leq n^{\psi'}$ for $\psi' < \bar{\beta}(2l_r + 1)/(8l_r)$, then the term $T\epsilon^{l\beta-1}\theta^{-2l}$ can be made smaller order than the others by choosing $l = l_r$, noting $\beta_r < \beta$. A calculation also shows that this choice of θ indeed satisfies $\theta > 48(1 - 2^{-(l_r\beta_r-1)/4l_r})^{-1}m^{-1}n^{1-\rho}/\sigma_n$ for all n sufficiently large. \square

5. Smoothing result. We state a specific consequence of the main results of Barbour, Ross and Zheng (2021), which we use above to prove weak convergence. Let $M^0 \subset M'$ be the set of functions $h: \mathbb{D}^p \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \|h\|_{M^0} &:= \sup_{w \in \mathbb{D}^p} |h(w)| + \sup_{w \in \mathbb{D}^p} \|Dh(w)\| + \sup_{w \in \mathbb{D}^p} \|D^2h(w)\| \\ &+ \sup_{w, v \in \mathbb{D}^p} \frac{\|D^2h(w+v) - D^2h(w)\|}{\|v\|} \end{aligned}$$

is finite. Note that $\|h\|_{M'} \leq \|h\|_{M^0}$. For $x \in \mathbb{D}^p$, let $\omega_x(\epsilon) := \sup_{0 \leq s, t \leq T: |s-t| < \epsilon} |x(t) - x(s)|$ denote the modulus of continuity of x .

THEOREM 5.1 (Corollary 1.3 of Barbour, Ross and Zheng (2021)). *Let Y, Z be random elements of $\mathbb{D}^p := \mathbb{D}([0, T], \mathbb{R}^p)$, with $T \geq 1$, such that Z has almost surely continuous sample paths. Suppose that there are $\kappa_1, \kappa_2 \geq 0$ such that for any $g \in M^0$ satisfying the smoothness condition (1.2), we have*

$$|\mathbb{E}g(Y) - \mathbb{E}g(Z)| \leq \kappa_1 \|h\|_{M^0} + S_g \kappa_2.$$

Letting d_{LP} denote the Lévy–Prokhorov metric, we have that for any positive $\delta, \epsilon, \theta, \gamma$ with $\epsilon, \delta \in (0, 1)$,

$$\begin{aligned} d_{LP}(\mathcal{L}(X), \mathcal{L}(Z)) &\leq \widehat{C} \max \left\{ \theta + \gamma, \frac{\kappa_1 T^{3/2}}{(\epsilon\delta)^3} + \frac{\kappa_2 T^{1/2}}{(\epsilon\delta)^2} + \mathbb{P}(\omega_Y(\epsilon) \geq \theta) + \mathbb{P}(\omega_Z(\epsilon) \geq \theta) + pe^{-\frac{\gamma^2}{8\rho T\delta^2}} \right\}, \end{aligned}$$

where \widehat{C} is a universal constant.

To bound the modulus of continuity terms appearing in the previous theorem, we use the following lemma, also noted in Barbour, Ross and Zheng ((2021), Lemma 1.4 and Remark 1.5(1)), applied to each component.

LEMMA 5.2. *Let $X \in \mathbb{D}$ be such that there are positive constants a, b , and K such that*

$$(5.1) \quad \begin{aligned} \mathbb{P}(|X(s) - X(t)| \geq \theta) &\leq K\theta^{-b}|s - t|^{1+a} \\ \text{for } \frac{1}{2}M^{-1} \leq |s - t| &\leq 1/2, \end{aligned}$$

and that

$$(5.2) \quad \begin{aligned} M\mathbb{P}\left(\sup_{(k-1)/M \leq s \leq k/M} |X(s) - X((k-1)/M)| \geq \theta\right) &\leq \varphi_M(\theta) \\ \text{for } 1 \leq k \leq \lceil MT \rceil. \end{aligned}$$

Then, for any $\epsilon \in (M^{-1}, 1)$,

$$\mathbb{P}(\omega_X(\epsilon) \geq \theta) \leq 2T\{\varphi_M(\theta(1 - 2^{-a/(2b)}))/18\} + C'(K, a, b)\theta^{-b}\epsilon^a,$$

for a constant $C'(K, a, b) < \infty$.

We have also used the following, technical lemma.

LEMMA 5.3. *Assume $f \in M'$ either satisfies the smoothness condition (1.2), or is a function of the form $f(w) = F(w(t_1), \dots, w(t_k))$ for some $F: (\mathbb{R}^p)^k \rightarrow \mathbb{R}$ and $\{t_1, \dots, t_k\} \subseteq [0, T]$. Letting $I_r(s) := \mathbf{1}\{s \geq r\}$ and $G_t(s) := G(s - t)$ for a distribution function G , then for any $w, x_1, x_2 \in \mathbb{D}^p$, and $r, t \geq 0$, we have*

$$\begin{aligned} \int_{\mathbb{R}_+} D^2 f(w)[x_1 I_r, x_2 I_{t+y}]G(dy) &= D^2 f(w)\left[x_1 I_r, \int_{\mathbb{R}_+} x_2 I_{t+y}G(dy)\right] \\ &= D^2 f(w)[x_1 I_r, x_2 G_t] \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}_+} D^2 f(w)[x_1 G_r, x_2 I_{t+y}]G(dy) &= D^2 f(w)\left[x_1 G_r, \int_{\mathbb{R}_+} x_2 I_{t+y}G(dy)\right] \\ &= D^2 f(w)[x_1 G_r, x_2 G_t]. \end{aligned}$$

PROOF. If $f(w) = F(w(t_1), \dots, w(t_k))$, then a simple calculation shows that

$$D^2 f(w)[w_1, w_2] = \sum_{i,j=1}^k w_1(t_i)^\top F_{ij}(w(t_1), \dots, w(t_k))w_2(t_j),$$

where we write F_{ij} for the $p \times p$ matrix corresponding to the mixed partial of F in coordinates i and j . The result now follows directly after noting that $\int I_{t+y}G(dy) = G_t$.

Now assume that f satisfies the smoothness condition (1.2). Both results are obviously true from bilinearity if G is a discrete distribution function. By considering the atoms separately, we can without loss of generality assume G is continuous. We show the result by approximating G by a discretised version. Let $Y \sim G$ and define $Y_m := \lfloor mY \rfloor / m$, noting that Y_m converges almost surely (so in distribution) to Y , as $m \rightarrow \infty$. For the first assertion, the function

$$y \mapsto D^2 f(w)[x_1 I_r, x_2 I_{t+y}]$$

is continuous for $y \geq 0$, since

$$\begin{aligned} D^2 f(w)[x_1 I_r, x_2 I_{t+y+\epsilon}] - D^2 f(w)[x_1 I_r, x_2 I_{t+y}] \\ = D^2 f(w)[x_1 I_r, x_2(I_{t+y+\epsilon} - I_{t+y})] \rightarrow 0, \end{aligned}$$

by (1.2). By applying Lemma 2.3 and noting that $f \in M'$, it is also easy to see the function is bounded. Thus, because of weak convergence, as $m \rightarrow \infty$,

$$\begin{aligned} \mathbb{E}[D^2 f(w)[x_1 I_r, x_2 I_{t+Y_m}]] &\rightarrow \mathbb{E}[D^2 f(w)[x_1 I_r, x_2 I_{t+Y}]] \\ &= \int_{\mathbb{R}_+} D^2 f(w)[x_1 I_r, x_2 I_{t+y}] G(dy). \end{aligned}$$

On the other hand, because Y_m is discrete, we have

$$\mathbb{E}[D^2 f(w)[x_1 I_r, x_2 I_{t+Y_m}]] = D^2 f(w)[x_1 I_r, x_2 \mathbb{E}[I_{t+Y_m}]],$$

and, again because of Lemma 2.3 and because $f \in M'$, we have

$$\begin{aligned} &|D^2 f(w)[x_1 I_r, x_2 \mathbb{E}[I_{t+Y_m}]] - D^2 f(w)[x_1 I_r, x_2 \mathbb{E}[I_{t+Y}]]| \\ &\leq 3 \|f\|_{M'} \|x_1\| \|x_2\| \|\mathbb{E}[I_{t+Y_m}] - \mathbb{E}[I_{t+Y}]\| \end{aligned}$$

which converges to zero, since

$$\begin{aligned} \|\mathbb{E}[I_{t+Y_m}] - \mathbb{E}[I_{t+Y}]\| &= \sup_{y \in [0, T]} |\mathbb{P}(Y \leq y - t) - \mathbb{P}(Y_m \leq y - t)| \\ &\leq \sup_{y \in [0, T]} \left| \mathbb{P}\left(Y \in \left(y, \frac{\lfloor my \rfloor + 1}{m}\right)\right) \right| \rightarrow 0, \end{aligned}$$

by the continuity of G . For the second assertion, the function $y \mapsto D^2 f(w)[x_1 G_r, x_2 I_{t+y}]$ is bounded continuous on \mathbb{R}_+ , since, using the first assertion as well as the condition (1.2),

$$\begin{aligned} &D^2 f(w)[x_1 G_r, x_2 I_{t+y+\varepsilon}] - D^2 f(w)[x_1 G_r, x_2 I_{t+y}] \\ &= \int_{\mathbb{R}_+} D^2 f(w)[x_1 I_{r+y'}, x_2 (I_{t+y+\varepsilon} - I_{t+y})] G(dy') \xrightarrow{\varepsilon \downarrow 0} 0. \end{aligned}$$

The rest of the proof follows in exactly the same way as for the first assertion, replacing I_r by G_r . \square

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