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Poisson approximation. Addendum*

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Abstract: An important feature of a Poisson limit theorem in [4] is the absence of the traditional assumption (A.2). The purpose of this addendum is to explain why assumption (A.2) is not required, and compare the assumptions of the Poisson limit theorem in [4] with traditional ones. We present details of the argument behind Theorem 3 in [4].

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Let $\{X_{n,1},...,X_{n,n}\}_{n\geq 1}$ be a triangle array of dependent 0-1 random variables (r.v.s) such that sequence $X_{n,1},...,X_{n,n}$ is stationary for each $n\in\mathbb{N}$. Set

$$S_0=0,\quad S_n=X_{n,1}+...+X_{n,n}\qquad (n\geq 1).$$

We denote by π_λ a Poisson $\mathbf{\Pi}(\lambda)$ random variable.

A Poisson limit theorem states that as $n\to\infty$,

$$S_n\Rightarrow \pi_\lambda\qquad (\exists\lambda>0). \tag{A.1}$$

Traditionally the formulation of a Poisson limit theorem for a sequence of dependent 0-1 random variables involves the assumption that there exists the limit

$$\lim_{n\to\infty} n\mathbb{P}(X_{n,1}\neq 0)\in(0;\infty), \tag{A.2}$$

cf. the Gnedenko-Marcinkiewicz theorem or Leadbetter et al. [2].

An important feature of Theorem 3 in [4] is that it does not require assumption (A.2).

The proof of Theorem 3 in [4] is brief and does not go into every detail. The purpose of this addendum is to present those details. In particular, we clarify the terms $o(1)$ in [4], formulas (15), (16).

The argument is split into two propositions. First, we recall the definitions of mixing coefficient $\alpha_n(\cdot)$, mixing condition Δ and class $\mathcal{R}$.

Let $\mathcal{F}_{l,m}$ be the σ–field generated by $\{X_{n,i}\}_{l\leq i\leq m}$. Set $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$, and let

$$\alpha_n(l)\ =\ \sup|\,\mathbb{P}(AB)-\mathbb{P}(A)\mathbb{P}(B)|,$$

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where the supremum is taken over $m \geq 1$, $A \in \mathcal{F}_{1,m}$, $B \in \mathcal{F}_{m+l+1,n}$ such that $\mathbb{P}(A) > 0$.

Condition $\Delta \equiv \Delta(\{l_n\})$ is said to hold if $\alpha_n(l_n) \to 0$ as $n \to \infty$ for some sequence $\{l_n\}$ of natural numbers such that $0 \leq l_n \ll n$.

Class $\mathcal{R} \equiv \mathcal{R}(\{l_n\})$. If condition $\Delta(\{l_n\})$ holds, then there exists a sequence $\{r_n\}$ of natural numbers such that

$$n \gg r_n \gg l_n \geq 0, \quad nr_n^{-1}\alpha_n^{2/3}(l_n) \to 0, \quad r_n(l_n+1)^{-1}\alpha_n(l_n) \to 0 \tag{A.3}$$

as $n \to \infty$. For instance, one can take

$$r_n = \left[\max\{\sqrt{n(l_n+1)}\,; n\sqrt{\alpha_n(l_n)}\,; 1\} \wedge (l_n+1)/\sqrt{\alpha_n(l_n)}\,\right].$$

We denote by $\mathcal{R}$ the class of all such sequences $\{r_n\}$.

Proposition 1. *If condition* $\Delta(\{l_n\})$ *holds, then there exists a sequence* $\{\tilde{l}_n\}$ *such that* $\tilde{l}_n \in \mathbb{Z}_+$, $0 \leq \tilde{l}_n \ll n$ *and*

$$\alpha_n(\tilde{l}_n)n/(\tilde{l}_n+1) \to 0 \qquad (n \to \infty). \tag{A.4}$$

In the case of independent r.v.s (A.4) holds with $\tilde{l}_n = 0$. If $\{X_{n,i}\}$ are m-dependent r.v.s, them (A.4) holds with $\tilde{l}_n = m$.

Proof of Proposition 1. If condition $\Delta(\{l_n\})$ holds, then one may choose

$$\tilde{l}_n = [n\sqrt{\alpha_n(l_n)}\,] \vee l_n\,,$$

where $[x]$ denotes the integer part of x. Clearly, $0 \leq \tilde{l}_n \ll n$ as $n \to \infty$, and

$$\alpha_n(\tilde{l}_n)n/(\tilde{l}_n+1) \leq \alpha_n(l_n)n/([n\sqrt{\alpha_n(l_n)}\,]+1) \leq \sqrt{\alpha_n(l_n)} \to 0 \qquad (n \to \infty)$$

as required. $\square$

We denote by $\mathcal{M}$ the class of sequences $\{l_n\}$ such that $l_n \in \mathbb{Z}_+$, $0 \leq l_n \ll n$ and (A.4) holds. Obviously, if $\{l_n\} \in \mathcal{M}$, then condition $\Delta(\{l_n\})$ holds.

If $\{l_n\} \in \mathcal{M}$ and $r_n = [\sqrt{n(l_n+1)}\,]$, then $\{r_n\} \in \mathcal{R}$: (A.4) yields (A.3).

Proposition 2. *If* $\{l_n\} \in \mathcal{M}$ *and*

$$\lim_{n \to \infty} \mathbb{P}(S_n = 0) = e^{-\lambda} \qquad (\exists \lambda \geq 0), \tag{A.5}$$

then

$$\frac{n}{l_n+1}\mathbb{P}(S_{l_n} > 0) \leq \lambda + o(1) \quad (n \to \infty). \tag{A.6}$$

Proof of Proposition 2. Given $l \in \mathbb{N}$, denote

$$S_{<j>} = X_{(j-1)l+1} + \ldots + X_{jl}\,.$$

According to (A.5),

$$e^{-\lambda} + o(1) \;=\; \mathbb{P}(S_n = 0) \le \mathbb{P}(S_{<1>} = S_{<3>} = ... = S_{<k_l>} = 0)$$
$$\le\; \mathbb{P}^{k_l}(S_l = 0) + \alpha_n(l)n/l \qquad (A.7)$$

as $n \to \infty$, where $k_l = 2[n/2l] - 1$.

Let $l = l_n + 1$, where $\{l_n\} \in \mathcal{M}$. Taking into account (A.4) and (A.7),

$$e^{-\lambda} + o(1) \le \exp(-k_l \mathbb{P}(S_l > 0)) \qquad (n \to \infty). \qquad (A.8)$$

If $\lambda = 0$ in (A.5), then (A.8) yields $\mathbb{P}(S_l > 0)n/l \to 0$ as $n \to \infty$, and (A.6) holds. If $\lambda > 0$ in (A.5), then (A.8) entails (A.6). □

The argument of the proof of Proposition 2 shows that the following analogue of (A.6) holds true if $\{r_n\} \in \mathcal{R}$ and (A.5) holds:

$$\frac{n}{r_n + 1} \mathbb{P}(S_{r_n} > 0) \le \lambda + o(1) \qquad (n \to \infty). \qquad (A.6^*)$$

We are now in a position to clarify the terms $o(1)$ in [4], formulas (15), (16).

Let $\{l_n\} \in \mathcal{M}$, $\{r_n\} \in \mathcal{R}$. Splitting $\{X_{n,1}, ..., X_{n,n}\}$ into blocks of length $r := r_n$ separated by sub-blocks of length $l := l_n$ (Bernstein's blocks approach, cf. [2, 3]), we get

$$|\mathbb{P}(S_n = 0) - \mathbb{P}^{[n/r]}(S_r = 0)| \le (\mathbb{P}(S_l > 0) + \alpha_n(l))\, n/r. \qquad (A.9)$$

Here $\mathbb{P}^{[n/r]}(S_r = 0)$ can be replaced with $\exp\big(-[\frac{n}{r}]\mathbb{P}(S_r > 0)\big)$ at a cost of adding extra term $4r/e^2(n-r)$ to the right-hand side of (A.9), cf. (33*) in [4]. Similarly,

$$\left|\mathbb{E}\exp(itS_n) - \left(1 + \mathbb{P}(S_r > 0)\mathbb{E}\left\{e^{itS_r} - 1 | S_r > 0\right\}\right)^{[n/r]}\right|$$
$$\le 2\,(\mathbb{P}(S_l > 0) + 8\alpha_n(l))\, n/r \qquad (A.10)$$

(we have applied the Volkonskiy-Rozanov inequality [5], see also (14.66) in [3]).

Since $l_n \ll r_n \ll n$, (A.4) and (A.6) imply that the right-hand sides of (A.9) and (A.10) are $o(1)$ as $n \to \infty$. Taking into account (A.6*), we conclude that relations (15) and (16) in [4] hold true. □

Theorem 3 in [4] states that (A.5) together with condition (10) in [4] are necessary and sufficient for (A.1), while traditional sufficient for (A.1) conditions are (D') and (A.2), cf. Theorem 5.2.1 in [2]. Recall condition (D'):

$$\lim_{n \to \infty} n \sum_{i=2}^{r_n} \mathbb{P}\left(X_{n,i} \neq 0, X_{n,1} \neq 0\right) = 0 \qquad (D')$$

for any sequence $\{r_n\}$ of natural numbers such that $n \gg r_n \gg 1$, cf. [2, 3]. Condition (D') means that there is no asymptotic clustering of rare events.

Note that (A.5) may hold while condition (D') fails to hold (see, for instance, Example 5.1 in [3]). On the other hand, if (D') and (A.2) hold and $\{r=r_n\}\in\mathcal{R}$, then

$$\mathbb{P}(S_r>0)\sim r\mathbb{P}(X_{n,1}\neq 0) \qquad (n\to\infty) \tag{A.11}$$

in view of inequality (13) in [4]; relations (A.2), (A.9) and (A.11) yield (A.5).

Poisson approximation was apparently introduced by de Moivre [1], problems 5–7.

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