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Hydrodynamic limits of interacting particle systems on crystal lattices in periodic realizations

Zehao Guan*

Abstract

We study the hydrodynamic limits of the simple exclusion processes and the zero range processes on crystal lattices. For a periodic realization of crystal lattice, we derive the hydrodynamic limit for the exclusion processes and the zero range processes, which depends on both the structure of crystal lattice and its embedding into $\mathbb{R}^d$. Even through the crystal lattices have inhomogeneous local structure, for all periodic realizations, we apply the entropy method to derive the hydrodynamic limits. Also, we discuss how the limit equation depends on the choices of the realizations.

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1 Introduction

The purpose of this paper is to discuss the hydrodynamic limits of the interacting particle systems on the crystal lattices. We can regard the interacting particle systems as the interacting random walks of lots of particles. Roughly speaking, the hydrodynamic limit, that is to deduce the macroscopic behavior of the system from the microscopic interacting particles, can be regarded as the law of large numbers for these stochastic processes through a proper space-time scaling limit. The limiting macroscopic behavior is described by a deterministic evolution equation, which is called the hydrodynamic equation. The hydrodynamic limits of interacting particle systems have been investigated intensively in the square lattice $\mathbb{Z}^d$, which have their origins in mathematics and physics (see [5]). It is interesting to study the scaling limits of interacting particle systems in much more general spaces. In this direction, Jara [4] shows the hydrodynamic limit for the zero range process in the Sierpinski gasket. In [1], Faggionato studies the exclusion process on the percolation clusters.

In this paper, we focus on the crystal lattice, such as the triangular lattice and the hexagonal lattice, which is the simplest extension of the square lattice. The crystal lattice

*Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1, Meguro-Ku, Tokyo, 153-8914, Japan.

has been studied from the view of discrete geometric analysis by Kotani and Sunada (see [6, 7]). They study the random walks on crystal lattices and discuss the relationship between the asymptotic behaviors of the random walks and the geometric structures of the crystal lattices.We mention a recent work by Ishiwata, Kawabi and Kotani [3], in which they study the asymptotic behaviors of the non-symmetric random walks on the crystal lattices. They establish two kinds of functional central limit theorems for the random walks.

A crystal lattice is defined as an infinite graph $X = (V, E)$ which admits a free action of a free abelian group Γ with a finite quotient graph X_0. Here V is the set of vertices and E is the set of all oriented edges. For an oriented edge $e \in E$, we denote by oe the origin of e, by te the end point of e. For each positive integer N, the subgroup $N\Gamma$ also acts freely on X and we call the finite graph $X_N := X/N\Gamma$ the N-scaling finite graph. Let $p(\cdot)$ be a symmetric Γ-periodic weight function on E, that is, $p(e) = p(\bar{e}) > 0$ and $p(\sigma e) = p(e)$ for all $e \in E$, $\sigma \in \Gamma$.

Here we consider the exclusion processes and the zero range processes on X_N associated to a symmetric weight function. To observe these processes in the continuous space, we embed X into Euclidean space $\mathbb{R}^d$ through an embedding map Φ which respects the group action. We call such an embedding map Φ a periodic realization of the crystal lattice X. For each periodic realization Φ, we construct an embedding map Φ_N from X_N into a torus such that the image $\Phi_N(X_N)$ converges to a torus as N tends to infinity. We introduce two specific classes of the periodic realizations used in our paper: the harmonic realizations and the standard realizations. Intuitively, the periodic realizations only have the periodic structure and the harmonic realizations minimize the energy among the periodic realizations with the fixed lattice group and the standard realization minimize the energy among all the periodic realizations. Roughly speaking one has the implications: ‘standard $\Rightarrow$ harmonic $\Rightarrow$ periodic’, (for more details, see Section 2.2).

In this paper, we deal with the exclusion processes and the zero range processes on crystal lattices and investigate the influence of the geometric structure to the hydrodynamic equation. More precisely, consider the zero range process $\eta(t)$ on $X_N := (V_N, E_N)$ with generator

$$L_N f(\eta) := \sum_{e \in E_N} p(e) g(\eta_{oe})[f(\eta^e) - f(\eta)], \tag{1.1}$$

where $\eta := (\eta_x)_{x \in V_N} \in \mathbb{N}^{V_N}$ is the configuration, $g : \mathbb{N} \to \mathbb{R}_+$ with $g(0) = 0$ is the jump rate, $\mathbb{N} := \{0, 1, 2, \dots, \}$ and η^e is given by

$$\eta^e_x = \begin{cases} \eta_{oe} - 1 & x = oe \\ \eta_{te} + 1 & x = te \\ \eta_x & otherwise. \end{cases}$$

This means, if there are η_{oe} particles at site oe, independently with the number of particles on other sites, at rate $p(e)g(\eta_{oe})$ one of the particles at oe jumps to te. For each periodic realization Φ, define the empirical density by

$$\pi^{\Phi,N}_t(du) := \frac{1}{|V_N|} \sum_{x \in V_N} \eta_x(t) \delta_{\Phi_N(x)}(du). \tag{1.2}$$

In Theorem 3.4, we prove that $\pi^{\Phi,N}_t(du)$ converges in a weak sense to the solution of the equation (3.19) as $N \to \infty$. This equation is called the hydrodynamic equation and we also discuss the influence of the realization Φ on the diffusion coefficients of the hydrodynamic equation. We observe that the diffusion coefficient matrix can be computed by the finite quotient graph X_0 and the harmonic realization Φ_h associated to Φ (see Theorem 3.4). The exclusion process is defined in a similar way (see Section 3.1).

According to the types of interactions, interacting particle systems on the square lattice are categorized into *gradient systems* and *non-gradient systems*. We call the system a gradient system when the current of particles through each bond can be represented by the difference of a local function and its shift. Otherwise, we call the system a non-gradient system. The gradient condition allows us to do the integration by parts twice to get the hydrodynamic limit equation, which is not possible in the case of non-gradient system (see [5]). For non-gradient systems, Varadhan [9] proposed an approach to show the hydrodynamic limits and it has been applied to various non-gradient models. However, the explicit calculations of the diffusion coefficients for non-gradient models is not clear yet.

In contrast to the square lattice, for the systems on the crystal lattices, generally we can not perform the integration by parts even once because of the inhomogeneous local structure. The first integration by parts is allowed only when the quotient graph has just one vertex, which is the case for the square lattice. In [8], to overcome this difficulty, Tanaka transfers the Laplacian to the test function through the harmonic realizations. Tanaka considers the weakly asymmetric simple exclusion process where the weight function is identical to 1 and a harmonic realization is fixed, then he discusses the influence of the weakly asymmetric part and the macroscopic geometric structures to the macroscopic behaviors of particles.

In this paper, as in [8], the entropy method, harmonic realization and the concept of local function bundles developed in [8] are used. However, there are clear contrasts between our results and [8]. First, we deal with the case with general symmetric periodic weight functions through periodic realizations (including harmonic realizations). Second, we investigate the influence of the geometric structure to the hydrodynamic equation. We obtain an explicit formula between the diffusion matrix of the hydrodynamic limits under different periodic realizations (see Proposition 3.2). With Proposition 3.2, we reduce the general periodic realizations to the harmonic realizations, obtain the standard realizations (see Proposition 5.1) and give an explicit calculations of the diffusion coefficients for the non-gradient models with periodic weight functions on $\mathbb{Z}^d$.

Next we give an example how to apply our results to the non-gradient models on $\mathbb{Z}^d$. Consider the exclusion process on the discrete torus $\mathbb{T}_{2N} := \mathbb{Z}/2N\mathbb{Z}$, represented by $\{0, 1, 2, \ldots, 2N-1\}$, with generator

$$L_N f(\eta) := \sum_{x \in \mathbb{T}_{2N}} p(x, x+1) \left\{ f(\eta^{x,x+1}) - f(\eta) \right\},$$

where

$$p(x, x+1) = \begin{cases} \alpha & x \ \ even \\ \\ \beta & x \ \ odd. \end{cases}$$

We have that

$$L_N \eta_x = \begin{cases} \beta(\eta_{x-1} - \eta_x) - \alpha(\eta_x - \eta_{x+1}) & x \ \ even \\ \\ \alpha(\eta_{x-1} - \eta_x) - \beta(\eta_x - \eta_{x+1}) & x \ \ odd \end{cases}$$

Even though $p(\cdot)$ does not rely on the configuration η, it turns out to be inhomogeneous and non-gradient. If we consider a new process $\{\xi_x := (\eta_{2x}, \eta_{2x+1})\}_{x \in \mathbb{Z}}$, then ξ is a homogeneous process with more complex state space $\{(0,0), (0,1), (1,0), (1,1)\}$. It also might be possible to show the hydrodynamic limit for this model with non-gradient method. However, regarding this model as a simple exclusion process on crystal lattice, we can show the hydrodynamic limit directly without using the non-gradient method and give the explicit diffusion coefficients (see Section 6 Example 1).

We now give an outline of the proof for the zero range processes on crystal lattices. Following the entropy method developed in [2], we establish the local ergodic theorem, called the replacement lemma. The local ergodic theorem is the key step of the proof since it enables us to replace the local averages with the global averages. The proof of the replacement lemma is based on the one block estimate and the two blocks estimate. We first give the proof of Theorem 3.4 for the harmonic realization Φ. Then the proof of Theorem 3.4 is completed by applying Proposition 3.2.

The rest of the paper is organized as follows: In Section 2, we introduce the crystal lattice and construct the N-scaling finite graph. In Section 3, we introduce the exclusion process and the zero range process and state our main results. In Section 4, we prove the replacement lemma through the one-block estimate and two-blocks estimate. In Section 5, we discuss how to get the standard realization via the diffusion matrix. In Section 6, we give two examples. In Appendix A, we prove some lemmas. Throughout this paper, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $f(N) = o_N$ means that $f(N) \to 0$ as $N \to \infty$.

2 Crystal Lattice

2.1 Crystal lattice

In this section, we introduce the crystal lattice and fix some notations.

Let $X = (V, E)$ be a locally finite connected graph, where V is the set of vertices and E is the set of all oriented edges. For an oriented edge $e \in E$, we denote by oe the origin of e, by te the terminus of e and by $\bar{e}$ the inverse edge of e. We call $X = (V, E)$ a Γ-crystal lattice if a group Γ, which is isomorphic to $\mathbb{Z}^d$, acts on X freely and the quotient graph X/Γ is a finite graph, denoted by $X_0 = (V_0, E_0)$. More precisely, each $\sigma \in \Gamma$ defines a graph isomorphism $\sigma : X \to X$ and the graph isomorphism is fixed point-free except for $\sigma = id$. Let $p(\cdot)$ be a symmetric Γ-periodic weight function on E, that is, $p(e) = p(\bar{e}) > 0$ and $p(\sigma e) = p(e)$ for all $e \in E$, $\sigma \in \Gamma$. The dimension of X, symbolically dim$X$, is defined to be the rank of Γ. We will embed X into the Euclidean space $\mathbb{R}^d$ of dimension $d = \mathrm{rank}\Gamma$.

Since the crystal lattice is an abstract graph, to observe the processes in the continuous space, we embed X into Euclidean space $\mathbb{R}^D$ through an embedding map which respects the group action.

Let ϕ be an injective homomorphism: $\Gamma \to \mathbb{R}^d$ such that there exists a basis $u_1, \cdots, u_d \in \mathbb{R}^d$,

$$\phi(\Gamma) = \left\{ \sum_{i=1}^{d} k_i u_i \mid k_i \ integers \right\}.$$

Together with the vector translation as the group action, we call the image $\phi(\Gamma)$ a lattice group.

Definition 2.1. *We call an embedding $\Phi : V \to \mathbb{R}^d$ is a periodic realization if there exists some homomorphism: $\phi : \Gamma \to \mathbb{R}^d$, with the form stated above, such that Φ is ϕ-periodic, means, $\Phi(\sigma x) = \Phi(x) + \phi(\sigma)$, for every $x \in V$ and every $\sigma \in \Gamma$. Furthermore, we call Φ a harmonic realization if it is periodic and harmonic in the sense $\sum_{e \in E_x} p(e)[\Phi(te) - \Phi(oe)] = 0$, for every $x \in V$, where $E_x := \{e \in E | oe = x\}$.*

In this paper, realizations are always assumed to be periodic. Note that Φ depends on ϕ, so we call $\phi(\Gamma)$ the lattice group of Φ. Given a periodic realization Φ, define $v(e) := \Phi(te) - \Phi(oe)$ for $e \in E$. By the periodicity, v induces a map on E_0, also denoted by v. Define a $d \times d$ symmetric and positive definite matrix by

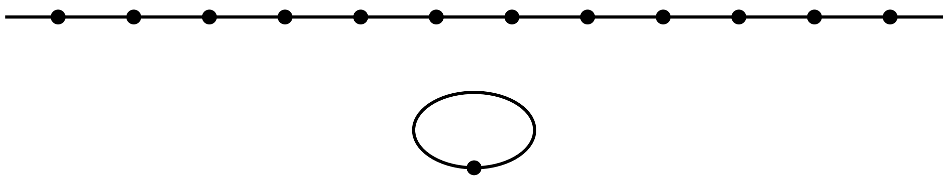

Figure 1: The image and the quotient graph of Φ in Example 1a.

$$\mathbb{D}_\Phi := \frac{1}{|V_0|}\left(\sum_{e\in E_0} p(e)v_i(e)v_j(e)\right)_{i,j=1,\ldots,d}, \tag{2.1}$$

which is called the diffusion coefficient matrix of Φ.

Let $\phi(\Gamma) = \left\{\sum_{i=1}^d k_i u_i \mid k_i \ integers\right\}$ be a lattice group, define the fundamental parallelotope for $\phi(\Gamma)$, by setting

$$D_\phi := \left\{\sum_{i=1}^d t_i u_i \mid 0 \le t_i < 1, i = 1,\ldots,d\right\}. \tag{2.2}$$

We also define the fundamental parallelotope D_Φ for realization Φ by setting $D_\Phi := D_\phi$, where $\phi(\Gamma)$ is the lattice group of Φ.

For each periodic realization Φ, define the energy of Φ by setting

$$E(\Phi) := \frac{1}{2}\sum_{e\in E_0} p(e)||v(e)||^2. \tag{2.3}$$

Here $||\cdot||$ represents the length of the vector in $\mathbb{R}^d$.

Definition 2.2. *We call Φ a standard realization if Φ minimizes the energy among the periodic realizations with fixed volume of fundamental parallelotope, i.e., for all Φ' with $\mathrm{vol}(D_{\Phi'}) = \mathrm{vol}(D_\Phi)$, it holds that*

$$E(\Phi) \le E(\Phi').$$

Remark 2.3. It has been shown that the harmonic realization minimizes the energy in the family of periodic realizations with the same lattice group and it is unique up to a translation. Furthermore, for a fixed lattice group $\phi(\Gamma)$, the harmonic realization can be obtained by solving the equations

$$\sum_{e\in E_x} p(e)[\Phi(te) - \Phi(oe)] = 0, \quad \Phi(\sigma x) = \Phi(x) + \phi(\sigma), \quad x \in V_0, \ \ \sigma \in \Gamma. \tag{2.4}$$

Let Φ_ϕ^h be the unique harmonic realization (up to a translation) associated to the lattice group $\phi(\Gamma)$. Thus, to find the standard realization, it suffices to find the lattice group $\phi(\Gamma)$ such that Φ_ϕ^h minimizes the energy with fixed volume. As an application of our results, we obtain the standard realizations (see Section 5). For much more details about standard realizations, see [6, 7].

Let us see some examples of crystal lattices.

1. One dimensional lattice

1a. The one dimensional standard lattice $X = (V, E)$, where the set of vertices $V = \mathbb{Z}$, the set of edges $E = \{(x, x+1), (x+1, x)|\ x \in \mathbb{Z}\}$. The group $\mathbb{Z}$ acts freely on X by the additive operation in $\mathbb{Z}$ and the quotient graph consists of one vertex and one loop. We define $\phi : \mathbb{Z} \to \mathbb{R}$ by $\phi(\sigma) := \sigma$ for all $\sigma \in \mathbb{Z}$ and define the embedding map $\Phi(x) := x$ for all $x \in \mathbb{Z}$. Then Φ is a $\mathbb{Z}$-periodic realization. Furthermore, for the weight function $p(\cdot)$ identically equals to 1, Φ is harmonic(and also standard) and $\mathbb{D}_\Phi = 2$ (see Figure 1).

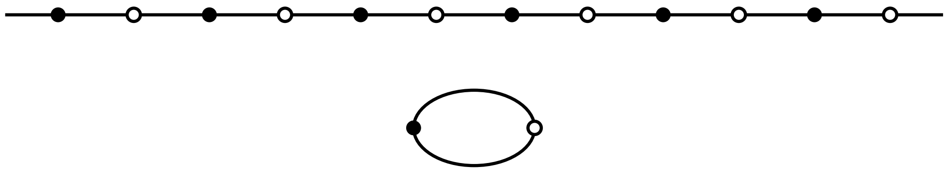

Figure 2: The image and the quotient graph of Φ in Example 1b.

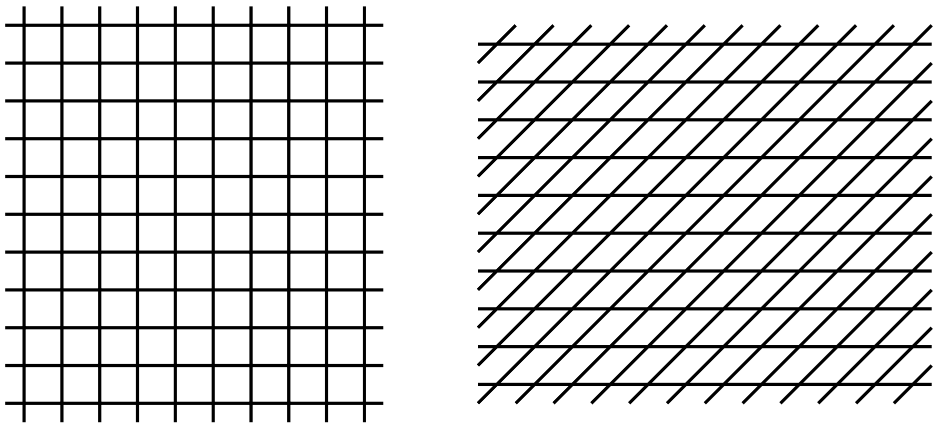

Figure 3: The images of Φ in Example 2a and 2b.

1b. We give another group action on the above X. The group $\mathbb{Z}$ acts freely on X by defining $\sigma x := x + 2\sigma$ for $\sigma \in \mathbb{Z}, x \in V$, then the quotient graph consists of two vertices and two unoriented edges between them. We define $\phi : \mathbb{Z} \to \mathbb{R}$ by $\phi(\sigma) := 2\sigma$ and the embedding $\Phi : X \to \mathbb{R}$ by $\Phi(\sigma 0) := 0 + \phi(\sigma)$, $\Phi(\sigma 1) := 1 + \phi(\sigma)$. Then Φ is a periodic realization. Furthermore, for the weight function $p(\cdot)$ identically equals to 1, Φ is harmonic(and also standard) and $\mathbb{D}_\Phi = 2$ (see Figure 2).

2. The square lattice.

2a. The standard square lattice $X = (V, E)$, where the set of vertices $V = \mathbb{Z}^2$, the set of unoriented edges $E = \{(x, x + (0,1)),\ (x, x + (1,0)) |\ x \in \mathbb{Z}^2\}$. Group $\mathbb{Z}^2$ acts freely on X by the additive operation in $\mathbb{Z}^2$ and the quotient graph consists of one vertex and two unoriented loops. We define $\phi : \mathbb{Z}^2 \to \mathbb{R}^2$ by setting $\phi(\sigma) := \sigma$ for $\sigma \in \mathbb{Z}^2$ and define the embedding $\Phi : X \to \mathbb{R}^2$ by $\Phi(\sigma(0,0)) := (0,0) + \phi(\sigma)$ for $\sigma \in \mathbb{Z}^2$. Then Φ is a periodic realization. Furthermore, for the weight function $p(\cdot)$ identically equals to 1, Φ is harmonic (and also standard) and $\mathbb{D}_\Phi = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (see Figure 3).

2b. We will give another realization for the square lattice X. Take a basis $u_1 = (1,0), u_2 = (1,1)$ in $\mathbb{R}^2$ and define $\phi : \mathbb{Z}^2 \to \mathbb{R}^2$ by $\phi(\sigma) := x u_1 + y u_2$ for $\sigma = (x,y) \in \mathbb{Z}^2$. We define the embedding map $\Phi(\sigma(0,0)) := (0,0) + \phi(\sigma)$ for $\sigma \in \mathbb{Z}^2$. Then, for the weight function $p(\cdot)$ identically equals to 1, Φ is a harmonic realization (not standard) and $\mathbb{D}_\Phi = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$ (see Figure 3).

3. The hexagonal lattice.

3a. The hexagonal lattice is a $\mathbb{Z}^2$-crystal lattice. V_0 consists of two vertices $\{x_0, x_1\}$ and the corresponding E_0 consists of six edges $\{(x_0, x_1), (x_0, x_2), (x_0, x_3), (x_1, x_0), (x_2, x_0), (x_3, x_0)\}$, where $x_2 = [(0,-1)]x_1, x_3 = [(1,-1)]x_1$. Take a basis $\{u_1 = (\sqrt{3}, 0), u_2 = (\sqrt{3}/2, 3/2)\}$ of $\mathbb{R}^2$ and define $\phi : \mathbb{Z}^2 \to \mathbb{R}^2$ by $\phi(\sigma) := x u_1 + y u_2$ for $\sigma = (x,y) \in \mathbb{Z}^2$. We define the embedding $\Phi : X \to \mathbb{R}^2$ by setting $\Phi(\sigma x_0) := (0,0) + \phi(\sigma)$, $\Phi(\sigma x_1) :=$

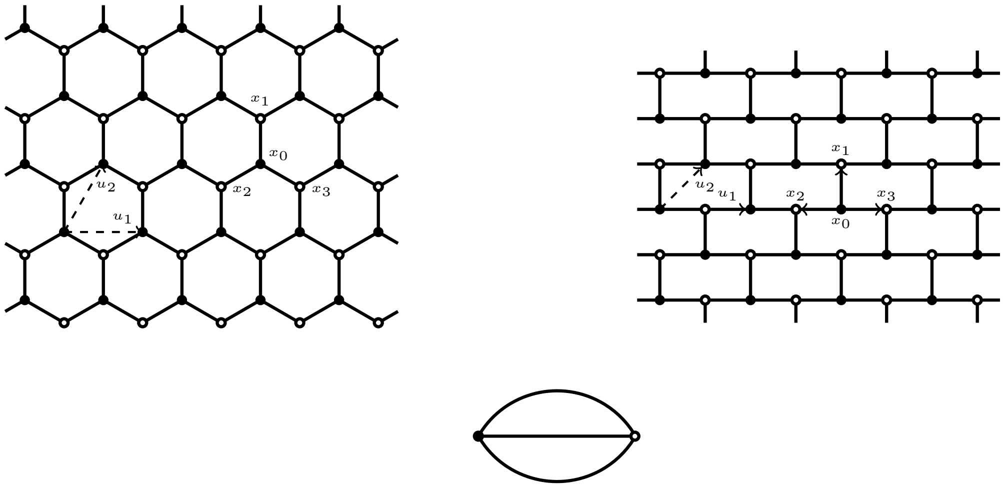

Figure 4: The images and quotient graph of Φ in Example 3a an 3b.

$(0,1)+\phi(\sigma)$, for $\sigma \in \mathbb{Z}^2$. Then, for the weight function $p(\cdot)$ identically equals to 1, Φ is a harmonic realization(and also standard) and $\mathbb{D}_\Phi = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$ (see Figure 4).

3b. We consider another realization of the hexagonal lattice. We choose the basis $\{u_1 = (2,0), u_2 = (1,1)\}$ of $\mathbb{R}^2$ and define $\phi : \mathbb{Z}^2 \to \mathbb{R}^2$ by $\phi((x,y)) := xu_1 + yu_2$ for $(x,y) \in \mathbb{Z}^2$. We define the embedding $\Phi : X \to \mathbb{R}^2$ by setting $\Phi(\sigma x_0) := (0,0)+\phi(\sigma)$, $\Phi(\sigma x_1) := (0,1)+\phi(\sigma)$, for $\sigma \in \mathbb{Z}^2$. Then, for the weight function $p(\cdot)$ identically equals to 1, Φ is not a harmonic realization. Indeed, for $x = (0,0) \in \mathbb{Z}^2$, $\sum_{e \in E_x} p(e)[\Phi(te) - \Phi(oe)] = (0,1)+(-1,0)+(1,0) = (0,1) \neq (0,0)$ (see Figure 4).

For much more examples, also see [8].

2.2 N-scaling finite graph

Recall that Γ is isomorphic to $\mathbb{Z}^d$. For every positive integer $N \geq 1$, $N\Gamma$ is isomorphic to $N\mathbb{Z}^d$. The subgroup $N\Gamma$ acts also freely on X and its quotient graph $X/N\Gamma$ is also a finite graph, denoted by $X_N = (V_N, E_N)$. Then $\Gamma_N := \Gamma/N\Gamma \cong \mathbb{Z}^d/N\mathbb{Z}^d$ acts freely on X_N. We call X_N the N-scaling finite graph. Since Φ is periodic, the map

$$\frac{1}{N}\Phi : X \to \mathbb{R}^d,$$

satisties $\frac{1}{N}\Phi(\sigma^N x) = \frac{1}{N}\Phi(x) + \phi(\sigma)$, where $\phi(\Gamma)$ is the lattice group of Φ. Let $\mathbb{T}^d_\phi := \mathbb{R}^d/\phi(\Gamma)$, equipped with the flat metric induced from the Euclidean metric. Then the map $\frac{1}{N}\Phi$ induces the map

$$\Phi_N : X_N \to \mathbb{T}^d_\phi.$$

We call Φ_N the N-scaling map. We can think about Φ_N as a discrete approximation of the continuous torus $\mathbb{T}^d_\phi$.

Next we introduce the fundamental domain of V, which is used throughout the proof. The fundamental domain of V is a connected lift of the quotient graph V_0. More precisely, for a fixed $x_0 \in V$, we can take a subset D_{x_0} of V, such that D_{x_0} is a lift of V_0, $x_0 \in D_{x_0} \subset V$ and D_{x_0} is connected in the following sense: For any $x, y \in D_{x_0}$ there exist a path $e_1, ..., e_l$ in E such that $oe_1 = x, te_1 = oe_2, \ldots, te_{l-1} = oe_l, te_l = y$ and $oe_1, te_1, ..., oe_l, te_l$ are all in D_{x_0}. This kind of set D_{x_0} always exists if we take a spanning tree in X_0 and its lift in X. We call D_{x_0} a fundamental domain of V. To abuse the notation, we denote by $x_0, D_{x_0} \subset V_N$ the image of x_0, D_{x_0} by the covering map respectively.

We will explain the fundamental domain D_{x_0} through a simple example. In the hexagonal lattice (see Figure 4), V_0 consists of two points, then the fundamental domain D_{x_0} can be equal to $\{x_0, x_1\}, \{x_0, x_2\}$ or $\{x_0, x_3\}$. However, as a lift of V_0, $\{x_0, x_3 + u_2\}$ is not connected in the sense of the fundamental domain, thus $\{x_0, x_3 + u_2\}$ is not a fundamental domain.

3 Interacting particle systems on crystal lattices

3.1 Simple exclusion process on crystal lattices

Let $X_N := (V_N, E_N)$ be the N-scaling finite graph. Let $Z_N = \{0,1\}^{V_N}$ be the configuration space and $\eta = \{\eta_x\}_{x\in V_N} \in Z_N$ be the configuration. Let $\nu_\rho^N (0 \le \rho \le 1)$ be the product Bernoulli measure with density ρ, i.e., $\nu_\rho^N(\eta_x = 1) = \rho$. The generator acting on $L^\infty(Z_N, \nu_\rho^N)$ as

$$L_N f(\eta) := \sum_{e\in E_N} p(e)[f(\eta^e) - f(\eta)], \qquad f \in L^\infty(Z_N, \nu_\rho^N), \tag{3.1}$$

where $p(\cdot)$ is a symmetric Γ-periodic weight function on E and

$$\eta_x^e = \begin{cases} \eta_{te} & x = oe \\ \eta_{oe} & x = te \\ \eta_x & otherwise, \end{cases}$$

defines a Markov process $\eta(t)$ on Z_N called the simple exclusion process.

For an arbitrary fixed time $T > 0$, let $D([0,T], Z_N)$ be the path space and for a probability measure μ^N on Z_N, let P_{μ^N} be the distribution on $D([0,T], Z_N)$ of the continuous time Markov process with generator $N^2 L_N$ and initial measure μ^N and E_{μ^N} be the expectation with respect to P_{μ^N}.

For the exclusion processes, we have the following result:

Theorem 3.1. *Let Φ be a periodic realization with lattice group $\phi(\Gamma)$ and Φ_ϕ^h be the harmonic realization associated to the lattice group $\phi(\Gamma)$. Let $\rho_0 : \mathbb{T}_\phi^d \to [0,1]$ be a measurable function. Assume that the initial measures $\{\mu^N\}$ satisfy that*

$$\lim_{N\to\infty} \mu^N \left[\left| \frac{1}{|V_N|} \sum_{x\in V_N} G(\Phi_N(x))\eta_x - \int_{\mathbb{T}_\phi^d} G(u)\rho_0(u) \frac{du}{\mathrm{vol}(\mathbb{T}_\phi^d)} \right| > \delta \right] = 0 \tag{3.2}$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T}_\phi^d \to \mathbb{R}$, then for every $t > 0$,

$$\lim_{N\to\infty} P_{\mu^N} \left[\left| \frac{1}{|V_N|} \sum_{x\in V_N} G(\Phi_N(x))\eta_x(t) - \int_{\mathbb{T}_\phi^d} G(u)\rho(t,u) \frac{du}{\mathrm{vol}(\mathbb{T}_\phi^d)} \right| > \delta \right] = 0 \tag{3.3}$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T}_\phi^d \to \mathbb{R}$, where $\rho(t,u)$ is the unique weak solution of the following linear heat equation

$$\frac{\partial}{\partial t}\rho = \nabla \mathbb{D}_{\Phi_\phi^h} \nabla \rho, \quad \rho(0,\cdot) = \rho_0(\cdot). \tag{3.4}$$

In [8], Tanaka obtains the hydrodynamic limit for the simple exclusion process in the case where the weight function $p(\cdot)$ is identical to 1 and the realization Φ is harmonic associated with $p(\cdot) \equiv 1$. In Theorem 3.1, we obtain the hydrodynamic limit for the exclusion process in the case of general periodic realizations and general symmetric periodic weight functions $p(\cdot)$.

The proof has two parts. Firstly, we obtain the hydrodynamic limit when Φ is harmonic associated with a given symmetric periodic weight function $p(\cdot)$. The proof of this part is similar to [8], and also similar to the zero range case we discuss later in the paper, so we omit it here. Secondly, as mentioned in Remark 2.3, for a fixed lattice group, the harmonic realization always exists by solving the equations (2.4). For a general symmetric periodic weight function $p(\cdot)$, we extend the result to the case of general periodic realization from harmonic realization. This part is proved by the following proposition, that shows that the hydrodynamic limit can be transferred through different realizations.

Proposition 3.2. *Let Φ be a periodic realization with lattice group $\phi(\Gamma) = \{\sum_{i=1}^d k_i u_i\}$. Assume for every $t > 0$, it holds that,*

$$\lim_{N\to\infty} P_{\mu^N}\left[\left|\frac{1}{|V_N|}\sum_{x\in V_N} G(\Phi_N(x))\eta_x(t) - \int_{\mathbb{T}^d_\phi} G(u)\rho(t,u)\frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)}\right| > \delta\right] = 0, \tag{3.5}$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T}^d_\phi \to \mathbb{R}$, where $\rho(t,u)$ is the unique weak solution of the following linear heat equation

$$\frac{\partial}{\partial t}\rho = \nabla \mathbb{D}\nabla\rho, \quad \rho(0,\cdot) = \rho_0(\cdot), \tag{3.6}$$

where $\mathbb{D}$ is the diffusion matrix of Φ. Then, for any periodic realization $\widetilde{\Phi}$ with lattice group $\widetilde{\phi}(\Gamma) = \{\sum_{i=1}^d k_i \widetilde{u}_i\}$ and every $t > 0$, we have

$$\lim_{N\to\infty} P_{\mu^N}\left[\left|\frac{1}{|V_N|}\sum_{x\in V_N} G(\widetilde{\Phi}_N(x))\eta_x(t) - \int_{\mathbb{T}^d_{\widetilde{\phi}}} G(u)\widetilde{\rho}(t,u)\frac{du}{\mathrm{vol}(\mathbb{T}^d_{\widetilde{\phi}})}\right| > \delta\right] = 0, \tag{3.7}$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T}^d_{\widetilde{\phi}} \to \mathbb{R}$, where $\widetilde{\rho}(t,u)$ is the unique weak solution of the following linear heat equation

$$\frac{\partial}{\partial t}\widetilde{\rho} = \nabla A\mathbb{D}A^T\nabla\widetilde{\rho}, \quad \widetilde{\rho}(0,\cdot) = \rho_0(A^{-1}\cdot). \tag{3.8}$$

Here A is the basis transformation matrix from $\{u_1,\dots,u_d\}$ to $\{\widetilde{u}_1,\dots,\widetilde{u}_d\}$.

Proof. First, we note that a realization Φ is uniquely determined by its values on D_{x_0} and its lattice group, i.e., $\{\Phi(x), x \in D_{x_0}\}$ and $\phi(\Gamma) = \{\sum_{i=1}^d k_i u_i\}$. For every $G \in C^2(\mathbb{T}^d_{\widetilde{\phi}})$,

$$\begin{aligned}\frac{1}{|V_N|}\sum_{x\in V_N} G(\widetilde{\Phi}_N(x))\eta_x &= \frac{1}{|V_N|}\sum_{x\in D_{x_0}}\sum_{\sigma\in\Gamma_N} G\left(\frac{A\phi(\sigma)+\widetilde{\Phi}(x)}{N}\right)\eta_{\sigma x}\\ &= \frac{1}{|V_N|}\sum_{x\in D_{x_0}}\sum_{\sigma\in\Gamma_N} \widetilde{G}\left(\frac{\phi(\sigma)+A^{-1}\widetilde{\Phi}(x)}{N}\right)\eta_{\sigma x},\end{aligned}$$

where $\widetilde{G}(x) := G(Ax)$ is a function on $\mathbb{T}^d_\phi$. Notice that

$$\begin{aligned}&\left|\frac{1}{|V_N|}\sum_{x\in D_{x_0}}\sum_{\sigma\in\Gamma_N}\widetilde{G}\left(\frac{\phi(\sigma)+A^{-1}\widetilde{\Phi}(x)}{N}\right)\eta_{\sigma x} - \frac{1}{|V_N|}\sum_{x\in D_{x_0}}\sum_{\sigma\in\Gamma_N}\widetilde{G}\left(\frac{\phi(\sigma)+\Phi(x)}{N}\right)\eta_{\sigma x}\right|\\ &\quad\le \sup_{\substack{x\in D_{x_0}\\ \sigma\in\Gamma_N}}\left|\widetilde{G}\left(\frac{\phi(\sigma)+A^{-1}\widetilde{\Phi}(x)}{N}\right) - \widetilde{G}\left(\frac{\phi(\sigma)+\Phi(x)}{N}\right)\right|\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x,\end{aligned}$$

which converges to 0 in probability as $N \to \infty$. By assumption, we have that

$$\begin{aligned}\frac{1}{|V_N|}\sum_{x\in V_N}\widetilde{G}(\Phi_N(x))\eta_x(t) \to &\int_{\mathbb{T}^d_\phi}\widetilde{G}(u)\rho_0(u)\frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)}\\ &+\int_0^t ds\int_{\mathbb{T}^d_\phi}\nabla\mathbb{D}\nabla\widetilde{G}(u)\rho(s,u)\frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)}\\ =&\int_{\mathbb{T}^d_{\tilde{\phi}}}G(u)\rho_0(A^{-1}u)\frac{1}{|A|}\frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)}\\ &+\int_0^t\int_{\mathbb{T}^d_{\tilde{\phi}}}\nabla A\mathbb{D}A^T\nabla G(u)\rho(s,A^{-1}u)\frac{1}{|A|}\frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)}\\ =&\int_{\mathbb{T}^d_{\tilde{\phi}}}G(u)\widetilde{\rho}_0(u)\frac{du}{\mathrm{vol}(\mathbb{T}^d_{\tilde{\phi}})}\\ &+\int_0^t ds\int_{\mathbb{T}^d_{\tilde{\phi}}}\nabla A\mathbb{D}A^T\nabla G(u)\widetilde{\rho}(s,u)\frac{du}{\mathrm{vol}(\mathbb{T}^d_{\tilde{\phi}})}\end{aligned}$$

in probability as $N \to \infty$. By the triangle inequality, the proof is completed. □

Proof of Theorem 3.1. For any given periodic realization Φ, we take the harmonic realization Φ^h whose lattice group coincides with Φ. Firstly, we obtain the hydrodynamic limit of the exclusion process for Φ^h through the strategy as that for the zero range process given in Section 3.4. below. Note that for the exclusion process, we do not need the replacement lemma, nor the entropy estimate. Then, applying Proposition 3.2 with $A = I_d$, we obtain the hydrodynamic limit for Φ and Theorem 3.1 is proved. □

3.2 Zero range process on crystal lattices

Let $X_N := (V_N, E_N)$ be the N-scaling finite graph. Let $Z_N = \mathbb{N}^{V_N}$ be the configuration space and $\eta = \{\eta_x\}_{x\in V_N} \in Z_N$ be the configuration. For each $\eta \in Z_N$, η^e is obtained from η where one particle jumped from oe to te, i.e.

$$\eta^e_x = \begin{cases}\eta_{oe}-1 & x = oe\\ \eta_{te}+1 & x = te\\ \eta_x & otherwise.\end{cases}$$

Let $g : \mathbb{N} \to \mathbb{R}_+$ be a function with $g(0) = 0$. We assume that $g(k) > 0$ for $k > 0$ and satisfies the condition

$$g^* := \sup_{k\geq 0}|g(k+1)-g(k)| < \infty.$$

Define

$$Z(\varphi) := \sum_{k\geq 0}\frac{\varphi^k}{g(k)!},$$

where $g(k)! := g(1)\cdots g(k)$ and $g(0)! := 1$ and let $\varphi^* := \{\limsup_{k\to\infty}\sqrt[k]{g(k)!}\}^{-1}$ be the radius of convergence. Furthermore, we assume that

$$\lim_{\varphi\uparrow\varphi^*}Z(\varphi) = \infty.$$

For each $\varphi \geq 0$, let $\bar{\nu}_\varphi$ be the product measure on Z_N with the marginals given by

$$\bar{\nu}_\varphi\{\eta|\eta_x = k\} = \frac{1}{Z(\varphi)}\frac{\varphi^k}{g(k)!}. \tag{3.9}$$

The generator on $L^2(Z_N, \bar{\nu}_\varphi)$ with action

$$L_N f(\eta) := \sum_{e \in E_N} p(e) g(\eta_{oe})[f(\eta^e) - f(\eta)], \tag{3.10}$$

defines a Markov process $\eta(t)$ on Z_N called zero range process with parameters (p, g). Here $p(\cdot)$ is a symmetric Γ-periodic weight function on E.

To apply the entropy method to the zero range processes on crystal lattices, we assume that the zero range processes satisfying one of the following assumptions:

$(\mathbf{FEM})$(finite exponential moments) $Z(\cdot)$ is finite on $\mathbb{R}_+$, equivalently, $\varphi^* = \infty$.

$(\mathbf{SLG})$(sub-linear growth) $g(\cdot)$ has sub-linear growth, i.e.,

$$\overline{\lim_{k \to \infty}} \frac{g(k)}{k} = 0.$$

Next we give a family of invariant measures parametrized by density for the zero range processes. The proof of the next Proposition 3.3 are close to [5], Section 2, Page 29, so we omit its proof here.

Proposition 3.3. *For each $\varphi \geq 0$, the product measure $\bar{\nu}_\varphi$ is invariant for the zero range process with parameter (p, g).*

Let $R(\varphi) = E_{\bar{\nu}_\varphi}[\eta_x]$ denote the expection of particles per site. Notice that $R(\varphi)$ is strictly increasing with $R(0) = 0$. Denote by Ψ the inverse mapping of R. For each $\alpha \geq 0$, define the product measure ν_α by

$$\nu_\alpha(\cdot) = \bar{\nu}_{\Psi(\alpha)}(\cdot). \tag{3.11}$$

Then we obtain a family of invariant measures $\{\nu_\alpha, \alpha \geq 0\}$ parametrized by density, i.e., for every $\alpha \geq 0$,

$$E_{\nu_\alpha}[\eta_x] = \alpha. \tag{3.12}$$

Moreover, it is not hard to see that

$$\Psi(\alpha) = E_{\nu_\alpha}[g(\eta_x)]. \tag{3.13}$$

Furthermore, the function Ψ is uniformly Lipschitz on $\mathbb{R}_+$ with Lipschitz constant g^*. (See [5])

For arbitrary fixed time $T > 0$, let $D([0,T], Z_N)$ be the path space and for a probability measure μ^N on Z_N, let P_{μ^N} be the distribution on $D([0,T], Z_N)$ of the continuous time Markov process with generator $N^2 L_N$ and initial measure μ^N and E_{μ^N} be the expectation with respect to P_{μ^N}.

3.3 Relative entropy and Dirichlet form

Let $\mathcal{P}(Z_N)$ be the space of probability measures on Z_N. For an invariant measure ν_{α^*} with $\alpha^* > 0$, the relative entropy and Dirichlet form of $\mu \in \mathcal{P}(Z_N)$, which is absolutely continuous with respect to ν_{α^*} is defined as the following:

$$H_N(\mu | \nu_{\alpha^*}) := \int f \log f d\nu_{\alpha^*} \tag{3.14}$$

$$D_N(\mu) := -\int \sqrt{f} L_N \sqrt{f} d\nu_{\alpha^*} \tag{3.15}$$

where $f := \frac{d\mu}{d\nu_{\alpha^*}}$, i.e. $f(\eta) = \frac{\mu(\eta)}{\nu_{\alpha^*}(\eta)}$. To keep notation simple, we shall denote the entropy by $H_N(f)$ and the Dirichlet form by $D_N(f)$.
For the zero range process, we have the following result:

Theorem 3.4. *Let Φ be a periodic realization with lattice group $\phi(\Gamma)$ and Φ^h_ϕ be the harmonic realization associated to the lattice group $\phi(\Gamma)$. Let $\rho_0 : \mathbb{T}^d_\phi \to \mathbb{R}_+$ be an integrable function. Assume that there exist some positive constants K_0, α^* such that*

$$H_N(\mu^N|\nu_{\alpha^*}) \leq K_0|V_N|, \tag{3.16}$$

$$\lim_{N\to\infty} \mu^N \left[\left| \frac{1}{|V_N|} \sum_{x\in V_N} G(\Phi_N(x))\eta_x - \int_{\mathbb{T}^d_\phi} G(u)\rho_0(u) \frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)} \right| > \delta \right] = 0, \tag{3.17}$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T}^d_\phi \to \mathbb{R}$. Then, for every $t > 0$,

$$\lim_{N\to\infty} P_{\mu^N} \left[\left| \frac{1}{|V_N|} \sum_{x\in V_N} G(\Phi_N(x))\eta_x(t) - \int_{\mathbb{T}^d_\phi} G(u)\rho(t,u) \frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)} \right| > \delta \right] = 0, \tag{3.18}$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T}^d_\phi \to \mathbb{R}$, where $\rho(t,u)$ is assumed to be the unique weak solution of the following equation

$$\frac{\partial}{\partial t}\rho = \nabla \mathbb{D}_{\Phi^h_\phi} \nabla \Psi(\rho), \quad \rho(0,\cdot) = \rho_0(\cdot). \tag{3.19}$$

Remark 3.5. Though the processes on the square lattice are categorized into nearest neighbor interaction models and more general finite range interaction models, the processes on the crystal lattice are assumed to have nearest neighbor interaction without loss of generality. In fact, for a given set of vertices V and symmetric jump rate $\{p(x,y)\}_{x,y\in V}$, we define $E := \{(x,y); p(x,y) = p(y,x) > 0\}$, then the jumps occur only between nearest neighbor sites by definition and moreover, $p(e) > 0$ for all $e \in E$. In particular, any finite range simple exclusion process (zero range process) on the square lattice can be regarded as a nearest neighbor simple exclusion process (zero range process) on a crystal lattice.

Remark 3.6. We have assumed the uniqueness of the weak solution of equation (3.19) with initial condition ρ_0. In the case of square lattice $\mathbb{Z}^d$, results on uniqueness can be found in Chapter 5 of [5]. Proceeding as in [5], we can prove that all limit points of the sequence $\{\pi^{\Phi,N}_\cdot\}$ are concentrated on paths satisfying an energy estimate. If $|V_0| = 1$, we can obtain the uniqueness through the L^2 estimate as in [5]. But more generally($|V_0| \geq 2$) it is open.

3.4 Proof of Theorem 3.4

First, we give the proof of Theorem 3.4 for the harmonic realization Φ. Then the proof of Theorem 3.4 is completed by applying Proposition 3.2, following the same argument as in the proof of Theorem 3.1.

Fix a harmonic realization Φ and recall that the empirical density is given by

$$\pi^N_t(du) = \pi^{\Phi,N}_t(du) := \frac{1}{|V_N|} \sum_{x\in V_N} \eta_x(t)\delta_{\Phi_N(x)}(du), \tag{3.20}$$

where δ_x is the Dirac measure at $x \in \mathbb{T}^d_\phi$, then π^N_t is a process taking value in $\mathcal{M}_+$, the space of all finite positive measures on $\mathbb{T}^d_\phi$. Let $D([0,T],\mathcal{M}_+)$ be the path space equipped with the Skorohod topology and Q^N be the measure on $D([0,T],\mathcal{M}_+)$ associated to π^N_t starting from μ^N. For every $G \in C^2(\mathbb{T}^d_\phi)$, we define

$$\langle \pi^N_t, G\rangle := \frac{1}{|V_N|} \sum_{x\in V_N} \eta_x(t) G(\Phi_N(x)). \tag{3.21}$$

Referring to [5], we will consider the martingale M_t and its quadratic process N_t defined by

$$M_t := \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^t N^2 L_N \langle \pi_s^N, G\rangle ds, \tag{3.22}$$

$$N_t := M_t^2 - \int_0^t N^2 \left\{ L_N \langle \pi_s^N, G\rangle^2 - 2\langle \pi_s^N, G\rangle L_N \langle \pi_s^N, G\rangle \right\} ds. \tag{3.23}$$

Note that M_t and N_t can be rewritten as

$$M_t = \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^t \frac{1}{|V_N|} \sum_{x\in V_N} \Delta_N G(\Phi_N(x)) g(\eta_x) ds, \tag{3.24}$$

$$N_t = M_t^2 - \int_0^t \frac{N^2}{|V_N|^2} \sum_{e\in E_N} p(e) g(\eta_{oe})(G(\Phi_N(te)) - G(\Phi_N(oe)))^2 ds, \tag{3.25}$$

where the discrete Laplacian is defined by

$$\Delta_N G(\Phi_N(x)) := 2N^2 \sum_{e\in E_x} p(e)[G(\Phi_N(te)) - G(\Phi_N(oe))], \tag{3.26}$$

for $G \in C^2(\mathbb{T}^d_\phi)$ and $x \in V_N$. Since Φ is harmonic and by Taylor's formula, we have that

$$\begin{aligned}
\Delta_N G(\Phi_N(x)) &= \sum_{e\in E_x} p(e) \left\{ 2N \sum_{i=1}^d \frac{\partial G}{\partial x_i}(\Phi_N(x)) v_i(e) + \sum_{i,j=1}^d \frac{\partial^2 G}{\partial x_i \partial x_j}(\Phi_N(x)) v_i(e) v_j(e) \right\} + o_N \\
&= \sum_{e\in E_x} \sum_{i,j=1}^d \frac{\partial^2 G}{\partial x_i \partial x_j}(\Phi_N(x)) p(e) v_i(e) v_j(e) + o_N \\
&= F_N(x) + o_N,
\end{aligned} \tag{3.27}$$

where

$$F_N(x) := \sum_{e\in E_x} \sum_{i,j=1}^d \frac{\partial^2 G}{\partial x_i \partial x_j}(\Phi_N(x)) p(e) v_i(e) v_j(e). \tag{3.28}$$

The following Lemma 3.7 and Lemma 3.8 claims that he sequence of $\{Q^N\}$ is relatively compact and all limit points Q^* are concentrated on trajectories of absolutely continuous measures with respect to the Lebesgue measure on the torus. The proofs of both lemmas appear very closely in [5], Section 4, page 55. Thus, we just give a brief proof here.

Lemma 3.7.

$$\lim_{N\to\infty} E_{\mu^N} \left[\sup_{0\le t\le T} |M_t|^2 \right] = 0 \tag{3.29}$$

Proof. First, by entropy inequality, we have that

$$\begin{aligned}
E_{\mu^N} \left[\frac{1}{|V_N|} \sum_{x\in V_N} \eta_x \right] &\le \frac{1}{\gamma |V_N|} \log E_{\nu_{\alpha^*}} \left[e^{\gamma \sum_{x\in V_N} \eta_x} \right] + \frac{H_N(\mu^N|\nu_{\alpha^*})}{\gamma |V_N|} \\
&\le \frac{1}{\gamma} \log E_{\nu_{\alpha^*}} \left[e^{\gamma \eta_x} \right] + \frac{K_0}{\gamma}
\end{aligned}$$

for all $\gamma > 0$. Furthermore, by Doob's inequality,

$$\begin{aligned} E_{\mu^N}\left[\sup_{0\le t\le T}|M_t|^2\right] &\le 4E_{\mu^N}\left[|M_T|^2\right] \\ &= E_{\mu^N}\left[\int_0^T \frac{N^2}{|V_N|^2}\sum_{e\in E_N} p(e)g(\eta_{oe})(G(\Phi_N(te))-G(\Phi_N(oe)))^2 ds\right] \\ &\le \frac{4C(p,G)g^*T}{|V_N|}E_{\mu^N}\left[\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x\right]\to 0 \end{aligned}$$

as $N\to\infty$, where $C(p,G)$ is a constant depending only on $p(\cdot)$ and $G(\cdot)$. □

Lemma 3.8. *The sequence of $\{Q^N\}$ is relatively compact and all limit points Q^* are concentrated on trajectories of absolutely continuous measures with respect to the Lebesgue measure:*

$$Q^*\left[\pi|\pi_t(du)=\pi(t,u)du\right]=1 \tag{3.30}$$

Proof. To show $\{Q^N\}$ is relatively compact, it suffices to show that, for each $G\in C^2(\mathbb{T}^d_\phi)$, the following (i) and (ii) holds.(See [5])

(i) For any $t>0$ and any $\epsilon>0$, there exists a constant $A>0$ such that

$$\sup_N Q^N\left[\pi \mid |\langle\pi_t,G\rangle|>A\right].$$

(ii) For any $\epsilon>0$, it holds that

$$\lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,Q^N\left[\pi \mid \sup_{|s-t|<\gamma}|\langle\pi_t,G\rangle-\langle\pi_s,G\rangle|>\epsilon\right]=0.$$

To show (i), note that

$$\begin{aligned} \sup_N Q^N\left[\pi \mid |\langle\pi_t,G\rangle|>A\right] &= \sup_N P_{\mu^N}\left[|\langle\pi^N_t,G\rangle|>A\right] \\ &\le \sup_N P_{\mu^N}\left[\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x(t)>\frac{A}{||G||_\infty}\right] \\ &= \sup_N \mu^N\left[\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x>\frac{A}{||G||_\infty}\right]. \end{aligned}$$

In the last equation, we used the conservation of the number of total particles. Then it is not hard to show (i) through the assumption (3.17). For the proof of (ii), note that

$$|\langle\pi^N_t,G\rangle-\langle\pi^N_s,G\rangle|\le|M_t-M_s|+\int_s^t\left|\frac{1}{|V_N|}\sum_{x\in V_N}\Delta_N G(\Phi_N(x))g(\eta_x)\right|dr.$$

Furthermore, we have

$$\begin{aligned} &\lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,P_{\mu^N}\left[\sup_{|s-t|<\gamma}\int_s^t\left|\frac{1}{|V_N|}\sum_{x\in V_N}\Delta_N G(\Phi_N(x))g(\eta_x)\right|dr>\epsilon\right] \\ &\qquad\le \lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,\mu^N\left[C(p,G)g^*\gamma\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x>\epsilon\right]=0. \end{aligned}$$

By Chebyshev's inequality, Doob's inequality and Lemma 3.7, we have

$$\lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,P_{\mu^N}\left[\sup_{|s-t|<\gamma}|M_t-M_s|>\epsilon\right]\le\lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,\frac{4}{\epsilon^2}E_{\mu^N}[M_T^2]=0.$$

Then, by triangle inequality, (ii) holds. Thus, $\{Q^N\}$ is relatively compact. For the left part and more details of the proof, see Section 5.1 of [5]. □

We now prove Theorem 3.4 using the following replacement lemma, whose proof is postponed to Section 4. Before giving a statement of this lemma, we introduce two local averages, which are crucial to our proof and introduced in [8] first.

Take a basis $\{\sigma_1, \dots, \sigma_d\}$ of Γ and identify Γ with $\mathbb{Z}^d$. We define the standard generator system of Γ by setting

$$S = \{\sigma_1, \dots, \sigma_d, -\sigma_1, \dots, -\sigma_d\}.$$

We introduce the length function associated to S, $|\cdot| : \Gamma \to \mathbb{N}$ by

$$|\sigma| := \min\left\{ l \mid \sigma = \sum_{k=1}^{l} \epsilon_{i_k}\sigma_{i_k}, \epsilon_{i_k} \in \{-1, 1\}, i_k \in \{1, \cdots, d\} \right\},$$

for $\sigma \in \Gamma$. Then the map $(\sigma, \sigma') \in \Gamma \times \Gamma \to |\sigma - \sigma'| \in \mathbb{N}$ induces a metric in Γ, which is called the word metric associated to S. By the natural homomorphism from Γ to Γ_N, the length function and metric are defined in the same way. To abuse the notation, we use the same symbol. Since Γ acts on X freely, for each $x \in V$, there exists a unique $\sigma_x \in \Gamma$ such that $x \in \sigma_x D_{x_0}$. Define the map $[\cdot] : V \to \Gamma$ by setting $[x] = \sigma_x$ for $x \in V$.

Next we introduce the local function bundles developed in [8]. For $R > 0$, define the R-ball by setting

$$B(D_{x_0}, R) := \bigcup_{\sigma \in \Gamma, |\sigma| \le R} \sigma D_{x_0}. \tag{3.31}$$

Define a local average of η for $x \in V$ by

$$\bar{\eta}_{x,R} := \frac{1}{|[x]B(D_{x_0}, R)|} \sum_{z \in [x]B(D_{x_0}, R)} \eta_z \tag{3.32}$$

and a local average of g by

$$\widetilde{g}_{x,R}(\eta) := \frac{1}{|\{\underline{\sigma} \mid |\underline{\sigma}| \le R\}|} \sum_{|\underline{\sigma}| \le R} g_{\underline{\sigma}x}(\eta), \tag{3.33}$$

where $g_x(\eta) := g(\eta_x)$ for $x \in V$ and $|U|$ stands for the number of the elements in U. Note that $\bar{\eta}_{x,R} = \bar{\eta}_{x_0,R}$ for every $x \in D_{x_0}$, which is used technically in our proofs.

Lemma 3.9 (Replacement lemma)**.** *For every $x \in D_{x_0}$ and every $\delta > 0$,*

$$\varlimsup_{\epsilon \to 0} \varlimsup_{N \to \infty} P_{\mu^N}\left[\int_0^T \frac{1}{|\Gamma_N|} \sum_{\sigma \in \Gamma_N} V_{\sigma x, \epsilon N}(\eta(t)) dt > \delta \right] = 0. \tag{3.34}$$

where

$$V_{\sigma x, \epsilon N}(\eta) := |\widetilde{g}_{\sigma x, \epsilon N}(\eta) - \Psi(\bar{\eta}_{\sigma x_0, \epsilon N})|. \tag{3.35}$$

Now completing the proof of Theorem 3.4. For $G : [0, T] \times \mathbb{T}^d_\phi \to \mathbb{R}$ of class $C^{1,2}$, consider the martingale $M_t = M_t^{G,N}$ given by

$$M_t := \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t (\partial_s + N^2 L_N) \langle \pi_s^N, G_s \rangle ds.$$

As the above Lemma 3.7, we have that

$$\lim_{N \to \infty} P_{\mu^N}\left[\sup_{0 \le t \le T} \left| \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, \partial_s G_s \rangle ds \right.\right.$$
$$\left.\left. - \int_0^t \frac{1}{|V_N|} \sum_{x \in V_N} \Delta_N G(\Phi_N(x)) g(\eta_x) ds \right| > \delta \right] = 0$$

Note that $g(\eta_x)$ is not a function of the empirical process $\pi_\cdot^N$, to close the equation, we replace $g(\eta_x)$ in the following steps.

(i) Firstly, we can replace $g_x(\eta)$ by $\widetilde{g}_{x,\epsilon N}(\eta)$.
Since

$$\begin{aligned}
&P_{\mu^N}\left[\left|\int_0^T \frac{1}{|V_N|}\sum_{x\in V_N}\Delta_N G(\Phi_N(x))\left\{g_x(\eta(s))-\widetilde{g}_{x,\epsilon N}(\eta(s))\right\}ds\right|>\delta\right]\\
&\leq \frac{1}{\delta}E_{\mu^N}\left[\left|\int_0^T \frac{1}{|V_N|}\sum_{x\in V_N} g_x(\eta(s))\left\{\Delta_N G(\Phi_N(x))-\frac{1}{|\{\underline{\sigma}||\underline{\sigma}|\leq\epsilon N\}|}\sum_{|\underline{\sigma}|\leq\epsilon N}\Delta_N G(\Phi_N(\underline{\sigma}x))\right\}\right|\right]\\
&\leq \frac{1}{\delta}E_{\mu_N}\left[\int_0^T g^*\sup_{\substack{x\in D_{x_0}\\ \sigma\in\Gamma_N,|\underline{\sigma}|\leq\epsilon N}}|\Delta_N G(\Phi_N(\sigma x))-\Delta_N G(\Phi_N(\underline{\sigma}\sigma x))|\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x(s)ds\right]\\
&=\frac{g^*T}{\delta}E_{\mu_N}\left[\sup_{\substack{x\in D_{x_0}\\ \sigma\in\Gamma_N,|\underline{\sigma}|\leq\epsilon N}}|\Delta_N G(\Phi_N(\sigma x))-\Delta_N G(\Phi_N(\underline{\sigma}\sigma x))|\,\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x(0)\right],
\end{aligned}$$

which tends to 0 as $N\to\infty,\epsilon\to 0$ by Lebesgue dominated convergence theorem. Note that, we use the conservation of the total number of particles in last equation.

(ii) By the replacement lemma, we replace $\widetilde{g}_{x,\epsilon N}(\eta)$ by $\Psi(\bar{\eta}_{[x]x_0,\epsilon N})$,

$$\varlimsup_{\epsilon\to 0}\varlimsup_{N\to\infty}P_{\mu^N}\left[\left|\int_0^T \frac{1}{|V_N|}\sum_{x\in V_N}\Delta_N G(\Phi_N(x))\{\widetilde{g}_{x,\epsilon N}(\eta(s))-\Psi(\bar{\eta}_{[x]x_0,\epsilon N})\}ds\right|>\delta\right]=0.$$

(iii) Using (3.27) and (3.28), we replace the discrete Laplacian $\Delta_N G(\Phi_N(x))$ by $F_N(x)$, and obtain that

$$\lim_{\epsilon\to 0}\varlimsup_{N\to\infty}P_{\mu^N}\left[\int_0^T\left|\frac{1}{|V_N|}\sum_{x\in V_N}\left\{\Delta_N G(\Phi_N(x))-F_N(x)\right\}\Psi(\bar{\eta}_{[x]x_0,\epsilon N})\right|ds>\delta\right]=0.$$

Note that

$$F_N(\sigma x)=\sum_{e\in E_x}\sum_{i,j=1}^d\frac{\partial^2 G}{\partial x_i\partial x_j}(\Phi_N(\sigma x_0))p(e)v_i(e)v_j(e)+o_N,\quad \forall x\in D_{x_0},\ \forall\sigma\in\Gamma_N.$$

Together with (i), (ii) and (iii), we have

$$\begin{aligned}
\varlimsup_{\epsilon\to 0}\varlimsup_{N\to\infty}Q^N\Bigg[\Bigg|&\langle\pi_T^N,G\rangle-\langle\pi_0^N,G\rangle-\int_0^T\langle\pi_s^N,\partial_s G_s\rangle ds\\
&-\int_0^T\frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N}\nabla\mathbb{D}_\Phi\nabla G(\Phi_N(\sigma x_0))\Psi(\bar{\eta}_{\sigma x_0,\epsilon N}))ds\Bigg|>\delta\Bigg]=0.
\end{aligned}$$

By Lemma A.2, we can replace $\Psi(\bar{\eta}_{\sigma x_0,\epsilon N}))$ by $\Psi(\langle\pi_t^N,\chi_{\Phi_N(\sigma x_0),\epsilon}\rangle)$, where $\chi_{\mathbf{z},\epsilon}:\mathbb{T}_\phi^d\to\mathbb{R}$ is the characteristic function of the ϵ-ball in $\mathbb{T}_\phi^d$ centered on $\mathbf{z}\in\mathbb{T}_\phi^d$, see Appendix A. We have that,

$$\begin{aligned}
\varlimsup_{\epsilon\to 0}\varlimsup_{N\to\infty}Q^N\Bigg[\Bigg|&\langle\pi_T^N,G\rangle-\langle\pi_0^N,G\rangle-\int_0^T\langle\pi_s^N,\partial_s G_s\rangle ds\\
&-\int_0^T\frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N}\nabla\mathbb{D}_\Phi\nabla G(\Phi_N(\sigma x_0))\Psi(\langle\pi_s^N,\chi_{\Phi_N(\sigma x_0),\epsilon}\rangle)ds\Bigg|>\delta\Bigg]=0.
\end{aligned}$$

By Lemma 3.8, for any limit point Q^* of $\{Q^N\}$, we have that

$$\overline{\lim_{\epsilon\to 0}}\, Q^* \Bigg[\Bigg|\langle \rho_T, G\rangle - \langle \rho_0, G\rangle - \int_0^T \int_{\mathbb{T}^d_\phi} \{\rho(s,u)\partial_s G(u) + \nabla \mathbb{D}_\Phi \nabla G(u)\cdot \Psi(\langle \rho_s, \chi_{\cdot,\epsilon}\rangle)\frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)} ds\}\Bigg| > \delta\Bigg] = 0.$$

By the dominated convergence theorem, as $\epsilon \to 0$, we obtain that,

$$Q^* \Bigg[\Bigg|\langle \rho_T, G\rangle - \langle \rho_0, G\rangle - \int_0^T \{\rho(s,u)\partial_s G(u) + \langle \Psi(\rho_t), \nabla \mathbb{D}_\Phi \nabla G\rangle dt\}\Bigg| > \delta\Bigg] = 0.$$

Up to now, we have proved that Q^* concentrates on path $\{\rho_t, 0 \le t \le T\}$, which is the weak solution of (3.4). In particular, we assume such a weak solution is unique, then we have that, for every $\delta > 0$,

$$\lim_{N\to\infty} Q^N \left[d_{S_k}\left(\pi^N, \frac{\rho(\cdot,u)du}{\mathrm{vol}(\mathbb{T}^d_\phi)}\right) > \delta\right] = 0,$$

where d_{S_k} is the Skorohod distance on $D([0,T],\mathcal{M}_+)$. Since the limit measure is concentrated on weakly continuous trajectories, for fixed time $t \in (0,T)$, π^N_t converges in distribution to the deterministic measure $\rho(t,u)du/vol(\mathbb{T}^d_\phi)$. The convergence in distribution to a deterministic variable implies the convergence in probability. In particular, since $T > 0$ is arbitrary, we obtain that for every $\delta > 0$, every $t > 0$ and every continuous function $G \in C(\mathbb{T}^d_\phi)$,

$$\lim_{N\to\infty} P_{\mu^N}\left[\left|\frac{1}{|V_N|}\sum_{x\in V_N} G(\Phi_N(x))\eta_x(t) - \int_{\mathbb{T}^d_\phi} G(u)\Psi(\rho(t,u))\frac{du}{\mathrm{vol}(\mathbb{T}^d_\phi)}\right| > \delta\right] = 0,$$

which concludes the Theorem 3.2. □

4 The replacement lemma

In this section, we prove the replacement lemma through one block estimate and two blocks estimate. To prove these theorems and lemmas, we apply the classical entropy method (see [5], Section 5). To apply the entropy method smoothly, some adaptations are needed, like, the discrete Laplacian on crystal lattices, two different local averages in the replacement lemma, the graph distance and some estimates on the crystal lattices.

4.1 Proof of replacement lemma

Let μ^N_t be the distribution of $\eta(t)$ on Z_N and set

$$f^N_t := \frac{d\mu^N_t}{d\nu_{\alpha^*}}, \qquad \bar{f}^N_T := \frac{1}{T}\int_0^T f^N_t dt. \tag{4.1}$$

Then f^N_t satisfies the Kolmogorov equation

$$\begin{cases} \partial_t f^N_t &= N^2 L_N f^N_t, \\ f^N_0 &= \frac{d\mu^N}{d\nu_{\alpha^*}}. \end{cases}$$

As in Section 5.2 of [5], we have that

$$H_N(\bar{f}^N_T) \le H_N(f^N_0), \qquad D_N(\bar{f}^N_T) \le \frac{H_N(f^N_0)}{2TN^2}. \tag{4.2}$$

Note that

$$\begin{aligned} & P_{\mu^N}\left[\int_0^T \frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} V_{\sigma x,\epsilon N}(\eta(t))dt > \delta\right] \\ & \le \frac{1}{\delta}E_{\mu^N}\left[\int_0^T \frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} V_{\sigma x,\epsilon N}(\eta(t))dt\right] \\ & = \frac{T}{\delta}\int \frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} V_{\sigma x,\epsilon N}(\eta)\bar{f}_T^N(\eta)\nu_{\alpha^*}(d\eta). \end{aligned}$$

Thus, to show the replacement lemma, it remains to show that, for every $C > 0$ and every $x \in D_{x_0}$,

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{\substack{H_N(f)\le C|V_N|,\\ D_N(f)\le C\frac{|V_N|}{N^2}}} \int \frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} V_{\sigma x,\epsilon N}(\eta)f(\eta)\nu_{\alpha^*}(d\eta) = 0. \tag{4.3}$$

The equation (4.3) will be shown by the following one block estimate and two blocks estimate.

Lemma 4.1 (The one block estimate)**.** *For every $C > 0$ and every $x \in D_{x_0}$,*

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{\substack{H_N(f)\le C|V_N|\\ D_N(f)\le C\frac{|V_N|}{N^2}}} \int \frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} V_{\sigma x,l}(\eta)f(\eta)\nu_{\alpha^*}(d\eta) = 0. \tag{4.4}$$

Lemma 4.2 (The two blocks estimate)**.** *For every $C > 0$ and every $x \in D_{x_0}$,*

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{\substack{H_N(f)\le C|V_N|\\ D_N(f)\le C\frac{|V_N|}{N^2}}} \sup_{|\underline{\sigma}|\le\epsilon N} \int \frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} |\bar{\eta}_{\underline{\sigma}\sigma x,l} - \bar{\eta}_{\sigma x,\epsilon N}|f(\eta)\nu_{\alpha^*}(d\eta). \tag{4.5}$$

We first prove (4.3) by using the one block estimate and two blocks estimate. Recall that

$$V_{\sigma x,\epsilon N}(\eta) = \left|\frac{1}{|\{\underline{\sigma} \mid |\underline{\sigma}| \le \epsilon N\}|}\sum_{|\underline{\sigma}|\le\epsilon N} g_{\underline{\sigma}\sigma x} - \Psi(\bar{\eta}_{\sigma x_0,\epsilon N})\right|.$$

Add and subtract the following expression

$$\frac{1}{|\{\underline{\sigma} \mid |\underline{\sigma}| \le \epsilon N\}|}\sum_{|\underline{\sigma}|\le\epsilon N} \frac{1}{|\{\sigma' \mid |\sigma'| \le l\}|}\sum_{|\sigma'|\le l} \{g_{\sigma'\underline{\sigma}\sigma x}(\eta) - \Psi(\bar{\eta}_{\underline{\sigma}\sigma x_0,l})\}.$$

By the triangle inequality, we can estimate the integral term appeared in (3.34) by three parts separately.

$$\int \frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} V_{\sigma x,\epsilon N}(\eta)f(\eta)\nu_{\alpha^*}(d\eta) \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 := \int \frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} \Bigg| & \widetilde{g}_{\sigma x,\epsilon N}(\eta) \\ & - \frac{1}{|\{\underline{\sigma} \mid |\underline{\sigma}| \le \epsilon N\}|}\sum_{|\underline{\sigma}|\le\epsilon N} \frac{1}{|\{\sigma' \mid |\sigma'| \le l\}|}\sum_{|\sigma'|\le l} g_{\sigma'\underline{\sigma}\sigma x}(\eta)\Bigg| f(\eta)\nu_{\alpha^*}(d\eta) \end{aligned}$$

$$
\begin{aligned}
&= \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} \left| \frac{1}{|\{\underline{\sigma} \mid |\underline{\sigma}| \le \epsilon N\}|} \sum_{|\underline{\sigma}|\le \epsilon N} \left[g_{\underline{\sigma}\sigma x} - \frac{1}{|\{\sigma' \mid |\sigma'| \le l\}|} \sum_{|\sigma'|\le l} g_{\sigma'\underline{\sigma}\sigma x}(\eta) \right] \right| f(\eta)\nu_{\alpha^*}(d\eta) \\
&\le \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} \frac{1}{(2\epsilon N+1)^d} \sum_{\epsilon N - l \le \underline{\sigma} \le \epsilon N + l} (2l+1)^d g_{\underline{\sigma}\sigma x}(\eta) f(\eta)\nu_{\alpha^*}(d\eta) \\
&\le \frac{C_1(l, g^*, d)}{\epsilon N} \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} \eta_{\sigma x} f(\eta)\nu_{\alpha^*}(d\eta),
\end{aligned}
$$

and C_1 is a constant depending on l, d and g^*. By the entropy inequality, I_1 is bounded above by

$$
\begin{aligned}
I_1 &\le \frac{C_1(l, g^*, d)}{\epsilon N |\Gamma_N| \gamma} \left\{ H_N(f) + \log E_{\nu_{\alpha^*}} \left[e^{\sum_{\sigma\in\Gamma_N} \gamma \eta_{\sigma x}} \right] \right\} \\
&\le \frac{C_1(l, g^*, d)}{\epsilon N \gamma} \left\{ C|D_{x_0}| + \log E_{\nu_{\alpha^*}} \left[e^{\gamma \eta_{\sigma x}} \right] \right\},
\end{aligned}
$$

which tends to 0 as $N \to \infty$.

$$
\begin{aligned}
I_2 &:= \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} \left| \Psi(\bar{\eta}_{\sigma x_0, \epsilon N}) - \frac{1}{|\{\underline{\sigma} \mid |\underline{\sigma}| \le \epsilon N\}|} \sum_{|\underline{\sigma}|\le \epsilon N} \Psi(\bar{\eta}_{\underline{\sigma}\sigma x_0, l}) \right| f(\eta)\nu_{\alpha^*}(d\eta) \\
&\le \sup_{|\underline{\sigma}|\le \epsilon N} \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} \left| \Psi(\bar{\eta}_{\sigma x_0, \epsilon N}) - \Psi(\bar{\eta}_{\underline{\sigma}\sigma x_0, l}) \right| f(\eta)\nu_{\alpha^*}(d\eta) \\
&\le g^* \sup_{|\underline{\sigma}|\le \epsilon N} \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} |\bar{\eta}_{\sigma x_0, \epsilon N} - \bar{\eta}_{\underline{\sigma}\sigma x_0, l}| f(\eta)\nu_{\alpha^*}(d\eta),
\end{aligned}
$$

which tends to 0 as $N \to \infty, \epsilon \to 0, l \to \infty$ by the two blocks estimate.

$$
\begin{aligned}
I_3 &:= \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} \left| \frac{1}{|\{\underline{\sigma} \mid |\underline{\sigma}| \le \epsilon N\}|} \sum_{|\underline{\sigma}|\le \epsilon N} \frac{1}{|\{\sigma' \mid |\sigma'| \le l\}|} \sum_{|\sigma'|\le l} \Big\{ g_{\sigma'\underline{\sigma}\sigma x}(\eta) \right. \\
&\qquad\qquad \left. - \Psi(\bar{\eta}_{\underline{\sigma}\sigma x_0, l}) \Big\} \right| f(\eta)\nu_{\alpha^*}(d\eta) \\
&\le \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} |\widetilde{g}_{\sigma x, l}(\eta) - \Psi(\bar{\eta}_{\sigma x_0, l})| f(\eta)\nu_{\alpha^*}(d\eta),
\end{aligned}
$$

which tends to 0 as $N \to \infty, l \to \infty$ by one block estimate. □

4.2 Proof of the one block estimate

Before the proof of the one block estimate, we prove the following lemma first, which allows us to cut off the large density.

Lemma 4.3. *For every $C > 0$,*

$$
\overline{\lim_{A\to\infty}}\ \overline{\lim_{l\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{H_N(f)\le C|V_N|} \int \frac{1}{|\Gamma_N|} \sum_{\sigma\in\Gamma_N} \bar{\eta}_{\sigma x_0, l} 1_{\{\bar{\eta}_{\sigma x_0, l} \ge A\}} f(\eta)\nu_{\alpha^*}(d\eta) = 0. \tag{4.6}
$$

Proof. By the entropy inequality, the integral part is bounded above by

$$
\frac{1}{\gamma|\Gamma_N|} \{ H_N(f) + \log E_{\nu_{\alpha^*}} [e^{\gamma \sum_{\sigma\in\Gamma_N} \bar{\eta}_{\sigma x_0, l} 1_{\{\bar{\eta}_{\sigma x_0, l} \ge A\}}}] \}. \tag{4.7}
$$

Recall that

$$
\bar{\eta}_{\sigma x_0, l} = \frac{1}{|D_{x_0}||\{\underline{\sigma} \mid |\underline{\sigma}| \le l\}|} \sum_{x\in D_{x_0}} \sum_{|\underline{\sigma}|\le l} \eta_{\underline{\sigma}\sigma x},
$$

then we have $\bar{\eta}_{\sigma x_0,l}$ and $\bar{\eta}_{\sigma' x_0,l}$ are independent for $|\sigma-\sigma'|>2l$ under the product measure ν_{α^*}. For each $\sigma\in\{\sigma||\sigma|\le l\}$, denote $\Omega_\sigma:=\{\underline{\sigma}|\underline{\sigma}-\sigma\in(2l+1)\Gamma\}$. Notice that $\{\bar{\eta}_{\underline{\sigma}x_0,l}1_{\{\bar{\eta}_{\underline{\sigma}x_0,l\}}\ge A}\}_{\underline{\sigma}\in\Omega_\sigma}$ are independent. By Hölder's inequality, Chebyshev exponential inequality and $\log(1+x)\le x$ for all $x\ge 0$, we have that

$$\begin{aligned}
&\frac{1}{\gamma|\Gamma_N|}\log E_{\nu_{\alpha^*}}\left[e^{\gamma\sum_{\sigma\in\Gamma_N}\bar{\eta}_{\sigma x_0,l}1_{\{\bar{\eta}_{\sigma x_0,l}\ge A\}}}\right]\\
&\quad\le\frac{1}{\gamma|\Gamma_N|}\log\prod_{|\sigma|\le l}\left\{E_{\nu_{\alpha^*}}\left[e^{\gamma(2l+1)^d\sum_{\underline{\sigma}\in\Omega_\sigma}\bar{\eta}_{\underline{\sigma}x_0,l}1_{\{\bar{\eta}_{\underline{\sigma}x_0,l}\ge A\}}}\right]\right\}^{\frac{1}{(2l+1)^d}}\\
&\quad\le\frac{1}{\gamma(2l+1)^d}\log\left(1+E_{\nu_{\alpha^*}}\left[1_{\{\bar{\eta}_{x_0,l}\ge A\}}e^{\gamma(2l+1)^d\bar{\eta}_{x_0,l}}\right]\right)\\
&\quad\le\frac{1}{\gamma(2l+1)^d}\left\{\nu_{\alpha^*}\left[\eta|\bar{\eta}_{x_0,l}\ge A\right]^{\frac{1}{2}}\left(E_{\nu_{\alpha^*}}\left[e^{\frac{2\gamma}{|D_{x_0}|}\sum_{x\in D_{x_0}}\sum_{|\sigma|\le l}\eta_{\sigma x}}\right]\right)^{\frac{1}{2}}\right\}\\
&\quad\le\frac{1}{\gamma(2l+1)^d}\left\{e^{-A(2l+1)^d}E_{\nu_{\alpha^*}}\left[e^{(2l+1)^d\bar{\eta}_{x_0,l}}\right]E_{\nu_{\alpha^*}}\left[e^{\frac{2\gamma}{|D_{x_0}|}\sum_{x\in D_{x_0}}\sum_{|\sigma|\le l}\eta_{\sigma x}}\right]\right\}^{\frac{1}{2}}\\
&\quad=\frac{1}{\gamma(2l+1)^d}\left\{e^{-\frac{1}{2}(2l+1)^d\left[A-|D_{x_0}|\log M_{\alpha^*}\left(\frac{1}{|D_{x_0}|}\right)-|D_{x_0}|\log M_{\alpha^*}\left(\frac{2\gamma}{|D_{x_0}|}\right)\right]}\right\}
\end{aligned}$$

where $M_{\alpha^*}(\theta):=E_{\alpha^*}[e^{\theta\eta_{x_0}}]$. Then (4.7) is bounded by

$$\frac{C|V_N|}{\gamma|\Gamma_N|}+\frac{1}{\gamma(2l+1)^d}\left\{e^{-\frac{1}{2}(2l+1)^d\left[A-|D_{x_0}|\log M_{\alpha^*}\left(\frac{1}{|D_{x_0}|}\right)-|D_{x_0}|\log M_{\alpha^*}\left(\frac{2\gamma}{|D_{x_0}|}\right)\right]}\right\}. \tag{4.8}$$

We first fix γ large enough so that $\frac{C|V_N|}{\gamma|\Gamma_N|}=\frac{C}{\gamma|D_{x_0}|}$ is small, then pick A large enough such that $A-|D_{x_0}|\log M_{\alpha^*}(\frac{1}{|D_{x_0}|})-|D_{x_0}|\log M_{\alpha^*}(\frac{2\gamma}{|D_{x_0}|})>0$. For such γ and A, (4.8) tends to 0 as $l\to\infty$. The proof is completed. □

Now we are ready for the proof of one block estimate.

Proof of Lemma 4.1. Notice that

$$\begin{aligned}
V_{\sigma x,l}&=\left|\frac{1}{|\{\underline{\sigma}\mid|\underline{\sigma}|\le l\}|}\sum_{|\underline{\sigma}|\le l}g_{\underline{\sigma}\sigma x}(\eta)-\Psi(\bar{\eta}_{\sigma x_0,l})\right|\\
&\le(g^*|D_{x_0}|+g^*)\bar{\eta}_{\sigma x_0,l}.
\end{aligned}$$

Together with Lemma 4.3, to prove the one block estimate, it suffices to show that, for every $C_1>0$,

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{N\to\infty}}\sup_{D_N(f)\le\frac{C|V_N|}{N^2}}\int\frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N}V_{\sigma x,l}(\eta)1_{\{\bar{\eta}_{\sigma x_0,l}\le C_1\}}f(\eta)\nu_{\alpha^*}(d\eta)=0. \tag{4.9}$$

Since ν_{α^*} is Γ_N invariant, we have the integral part above is equal to

$$\int V_{x,l}(\eta)1_{\{\bar{\eta}_{x_0,l}\le C_1\}}\bar{f}(\eta)\nu_{\alpha^*}(d\eta),$$

where $\bar{f}(\eta):=\frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N}\sigma f(\eta)$, where we set $\sigma f(\eta):=f(\sigma^{-1}\eta)$ and $(\sigma\eta)_x:=\eta_{\sigma^{-1}x}$ for $\sigma\in\Gamma_N$ and $x\in V_N$. Notice that $V_{x,l}(\eta)1_{\{\bar{\eta}_{x_0,l}\le C_1\}}$ only depends on the coordinates $\{\eta_x,\ x\in B(D_{x_0},l)\}$. For fixed l, we will consider a subgraph $\Lambda_l:=(V_l,E_l)$, where $V_l:=B(D_{x_0},l)$ and $E_l:=\{e\in E|oe,te\in V_l\}$. Let $X^l:=\mathbb{N}^{V_l}$ be the configuration space and $\nu^l_{\alpha^*}$ be the restriction of $\nu_{\alpha*}$ to X^l, i.e., $\nu^l_{\alpha^*}(\xi):=\nu_{\alpha^*}\{\eta|\eta_x=\xi_x,x\in V_l\}$. For a density function f, we represent by f_l the conditional expectation of f to the σ-algebra generated by $\{\eta_x, x \in V_l\}$, i.e.,

$$f_l(\xi) := \frac{1}{\nu^l_{\alpha^*}(\xi)} E_{\nu_{\alpha^*}}[f(\eta) 1_{\{\eta | \eta_x = \xi_x, x \in V_l\}}], \qquad \xi \in X^l.$$

With these notations, we can rewrite (4.9) as

$$\overline{\lim_{l\to\infty}} \, \overline{\lim_{N\to\infty}} \sup_{D_N(f) \le \frac{C|V_N|}{N^2}} \int V_{x,l}(\xi) 1_{\{\bar{\xi}_{x_0,l} \le C_1\}} \bar{f}_l(\xi) \nu^l_{\alpha^*}(d\xi). \tag{4.10}$$

Next we estimate the Dirichlet form of $\bar{f}_l$. For each $e \in E_N$, define

$$L_{oe,te} f(\eta) := \frac{1}{2}\{g(\eta_{oe})[f(\eta^e) - f(\eta)] + g(\eta_{te})[f(\eta^{\bar{e}}) - f(\eta)]\}, \tag{4.11}$$

$$I_{oe,te}(f) := \frac{1}{2} \int g(\eta_{oe}) \{\sqrt{f(\eta^e)} - \sqrt{f(\eta)}\}^2 \nu_{\alpha^*}(d\eta). \tag{4.12}$$

Then we have

$$L_N f(\eta) = \sum_{e \in E_N} p(e) L_{oe,te} f(\eta),$$
$$D_N(f) = \sum_{e \in E_N} p(e) I_{oe,te}(f).$$

Let us restrict the above definition to X^l: for every density function $h : X^l \to \mathbb{R}$,

$$I^l_{oe,te}(h) := \frac{1}{2} \int g(\xi_{oe}) \{\sqrt{h(\xi^e)} - \sqrt{h(\xi)}\}^2 \nu^l_{\alpha^*}(d\xi),$$
$$D^l(h) := \sum_{e \in E_l} I^l_{oe,te}(h).$$

By Cauchy-Schwarz inequality,

$$I^l_{oe,te}(\bar{f}_l) \le I_{oe,te}(\bar{f}).$$

Note that $I_{oe,te}(\sigma f) = I_{\sigma oe, \sigma te}(f)$, which implies $D_N(\sigma f) = D_N(f)$ and combine with the convexity of the Dirichlet form, it holds that,

$$D^l(\bar{f}_l) \le \sum_{e \in E_l} I_{oe,te}(\bar{f}) \le |\{\sigma | |\sigma| \le l\}| \frac{1}{p^* |\Gamma_N|} D_N(\bar{f}) \le \frac{C|D_{x_0}|(2l+1)^d}{p^* N^2},$$

where $p^* := \min_{e \in E} p(e) > 0$ is a constant. Since we cut off the density by the indicator function, we can restrict the supremum to the densities concentrated on the set $\{\xi | \bar{\xi}_{x_0,l} \le C_1\}$. This subset of $\mathcal{P}(X^l)$ is compact for the weak topology. Furthermore, by the continuity of the Dirichlet form, as $N \to \infty$, (4.10) is bounded above by

$$\overline{\lim_{l\to\infty}} \sup_{D^l(f) = 0} \int V_{x,l}(\xi) 1_{\{\bar{\xi}_{x_0,l} \le C_1\}} f(\xi) \nu^l_{\alpha^*}(\xi) = 0. \tag{4.13}$$

For each $j \ge 0$, let $\nu^{l,j}$ be the measure $\nu^l_{\alpha^*}$ conditioned to the hyperplane $\{\xi | \sum_{x \in V_l} \xi_x = j\}$, i.e.,

$$\nu^{l,j}(\cdot) := \frac{\nu^l_{\alpha^*}(\cdot)}{\nu^l_{\alpha^*}\{\xi | \sum_{x \in V_l} \xi_x = j\}}.$$

Note that $D^l(f) = 0$ implies that f is constant on each hyperplane $\{\xi | \sum_{x \in V_l} \xi_x = j\}$ for each $j \le 0$.

We have that

$$\begin{aligned}\int V_{x,l}(\xi)1_{\{\bar{\xi}_{x_0,l}\le C_1\}}f(\xi)\nu^l_{\alpha^*}(\xi) &= \sum_{j=0}^{C_1|V_l|}\int_{\{\sum_{x\in V_l}\xi_x=j\}} V_{x,l}(\xi)f(\xi)\nu^l_{\alpha^*}(d\xi)\\ &= \sum_{j=0}^{C_1|V_l|}\int_{\{\sum_{x\in V_l}\xi_x=j\}} f(\xi)\nu^l_{\alpha^*}(\xi)\int V_{x,l}(\xi)\nu^{l,j}(d\xi).\end{aligned}$$

Since $\sum_{j=0}^{C_1|V_l|}\int_{\{\sum_{x\in V_l}\xi_x=j\}} f(\xi)\nu^l_{\alpha^*}(\xi)=1$, it is enough to show that

$$\overline{\lim_{l\to\infty}}\sup_{j\le C_1|V_l|}\int V_{x,l}(\xi)\nu^{l,j}(d\xi)=0. \tag{4.14}$$

For a fixed positive integer k, define $A:=A_{l-k}\cap\{(2k+1)\sigma,\sigma\in\Gamma\}:=\{\sigma_i\}_{i=1}^n$, where $A_m:=\{|\sigma|\le m\}$ for each positive integer m. For each $1\le i\le n$, let $B_i=\sigma_i+A_k$. Then $\{B_i\}_{i=1}^n$ are pairwise disjoint and belong to A_l. Let $B_0:=A_l\setminus\cup_{i=1}^n B_i$, then $|B_0|\le|A_l|-|A_{l-k}|\le c(k)l^{d-1}$. Notice that $\nu^{l,j}$ is concentrated on the configurations with j particles, then

$$\begin{aligned}\int V_{x,l}(\xi)\nu^{l,j}(d\xi) &= \int\left|\frac{1}{|\{\sigma||\sigma|\le l\}|}\sum_{|\sigma|\le l}g(\eta_{\sigma x})-\Psi(\bar{\xi}_{x_0,l})\right|\nu^{l,j}(d\xi)\\ &\le \sum_{i=1}^n\frac{|B_i|}{|A_l|}\int\left|\frac{1}{|B_i|}\sum_{\sigma\in B_i}g(\xi_{\sigma x})-E_{\nu_{\frac{j}{|V_l|}}}[g(\xi_{x_0})]\right|\nu^{l,j}(d\xi)\\ &\quad+\frac{|B_0|}{|A_l|}\int\left|\frac{1}{|B_0|}\sum_{\sigma\in B_0}g(\xi_{\sigma x})-E_{\nu_{\frac{j}{|V_l|}}}[g(\xi_{x_0})]\right|\nu^{l,j}(d\xi).\end{aligned}$$

The last term tends to 0 as $l\to\infty$ since $|B_0|\le c(k)l^{d-1}$. Since $\{\xi_x,x\in B_i\}_{i=1}^n$ have the same distribution, the previous summation is bounded by

$$\int\left|\frac{1}{|\{\sigma||\sigma|\le k\}|}\sum_{|\sigma|\le k}g(\eta_{\sigma x})-E_{\nu_{\frac{j}{|V_l|}}}[g(\xi_{x_0})]\right|\nu^{l,j}(d\xi).$$

By the equivalence of ensembles(see Appendix 2 of [5]), as $l\to\infty$ and $\frac{j}{|V_l|}\to\alpha$, the above integral converges (uniformly in α on each interval of $\mathbb{R}_+$) to

$$\int\left|\frac{1}{|\{\sigma||\sigma|\le k\}|}\sum_{|\sigma|\le k}g(\eta_{\sigma x})-E_{\nu_\alpha}[g(\xi_{x_0})]\right|\nu_\alpha(d\xi).$$

By the law of large numbers, the last integral converges (uniformly in α on each interval of $\mathbb{R}_+$) to 0 as $k\to\infty$. This completes the proof of the one block estimate. □

4.3 Proof of the two blocks estimate

As in Section 4.3, for fixed l, we consider the subgraph $\Lambda_l:=(V_l,E_l)$, where $V_l:=B(D_{x_0},l)$ and $E_l:=\{e\in E|oe,te\in V_l\}$. Let $X^l\times X^l:=\mathbb{N}^{V_l}\times\mathbb{N}^{V_l}$ be the configuration space and $\nu^l_{\alpha^*}\otimes\nu^l_{\alpha^*}$ be the product measure on $X^l\times X^l$. For a density function $f:Z_N\to\mathbb{R}_+$, $f_{\sigma,l}$ is the conditional expectation of f with respect to the σ-algebra generated by $\{\eta_x,x\in V_l\cup\sigma V_l\}$, i.e., for every $(\xi,\xi')\in X^l\times X^l$,

$$f_{\sigma,l}(\xi,\xi')=\frac{1}{\nu^l_{\alpha^*}\otimes\nu^l_{\alpha^*}(\xi,\xi')}E_{\nu_{\alpha^*}}[f(\eta)1_{\{\eta\mid\eta_x=\xi_x,\eta_{\sigma x}=\xi'_x,\ x\in V_l\}}].$$

For a density $h : X^l \times X^l \to \mathbb{R}_+$, define the Dirichlet form of h by

$$D^{l,l}(h) := \sum_{e \in E_l} (I^{1,l}_{oe,te} + I^{2,l}_{oe,te}) + I^l_{x_0,x'_0}(h), \tag{4.15}$$

where

$$\begin{aligned} I^{1,l}_{oe,te} &:= \frac{1}{2}\int g(\xi_{oe}) \left[\sqrt{h(\xi^e,\xi')} - \sqrt{h(\xi,\xi')}\right]^2 \nu^l_{\alpha^*} \otimes \nu^l_{\alpha^*}(d\xi, d\xi'), \\ I^{2,l}_{oe,te} &:= \frac{1}{2}\int g(\xi_{oe}) \left[\sqrt{h(\xi,\xi'^e)} - \sqrt{h(\xi,\xi')}\right]^2 \nu^l_{\alpha^*} \otimes \nu^l_{\alpha^*}(d\xi, d\xi'), \\ I^l_{x_0,x'_0}(h) &:= \frac{1}{2}\int g(\xi_{x_0}) \left[\sqrt{h(\xi^{x_0,-},\xi'^{x_0,+})} - \sqrt{h(\xi,\xi')}\right]^2 \nu^l_{\alpha^*} \otimes \nu^l_{\alpha^*}(d\xi, d\xi'), \end{aligned} \tag{4.16}$$

where

$$\xi^{x,\pm}_z = \begin{cases} \xi_x \pm 1 & z = x \\ \xi_z & otherwise. \end{cases}$$

For any $x, y \in V_N$, define

$$I_{x,y} := \frac{1}{2}\int g(\eta_x) \left\{\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)}\right\}^2 \nu_{\alpha^*}(d\eta), \tag{4.17}$$

where $\eta^{x,y}$ stands for the configuration of η where a particle jumped from x to y, i.e.,

$$\eta^{x,y}_z = \begin{cases} \eta_x - 1 & z = x \\ \eta_y + 1 & z = y \\ \eta_z & otherwise. \end{cases}$$

We first give two lemmas needed later.

Lemma 4.4. *There exists a constant $c_1 > 0$ such that for all $\sigma \in \Gamma$, it holds that*

$$d(x_0, \sigma x_0) \le c_1 |\sigma|, \tag{4.18}$$

where d is the graph distance of X.

This lemma is easy to show by taking $c_1 = \sup_{|\sigma|=1} d(x_0, \sigma x_0)$ and induction, so we omit the proof.

Lemma 4.5. *For every $\sigma \in \Gamma_N$,*

$$I_{x_0,\sigma x_0}(\bar{f}) \le d(x_0, \sigma x_0)^2 \frac{D_N(\bar{f})}{p^* |\Gamma_N|}. \tag{4.19}$$

where $p^* := \min_{e \in E} p(e) > 0$ is a constant.

Proof. For $x_0, \sigma x_0$, there exists a path $c = (e_1, \dots, e_n)$ such that $x_0 = oe_1, te_1 = oe_2, \dots,$ $te_{n-1} = oe_n, te_n = \sigma x_0$ and $n = d(x_0, \sigma x_0)$. After a change of variable $\eta = \xi^{x_0,+}$, we have

$$\begin{aligned} I_{x_0,\sigma x_0}(\bar{f}) &= \frac{1}{2}\Psi(\alpha^*) \int \left(\sqrt{\bar{f}(\eta^{\sigma x_0,+})} - \sqrt{\bar{f}(\eta^{x_0,+})}\right)^2 \nu_{\alpha^*}(d\eta) \\ &= \frac{1}{2}\Psi(\alpha^*) \int \left(\sum_{k=1}^n (\sqrt{\bar{f}(\eta^{te_k,+})} - \sqrt{\bar{f}(\eta^{oe_k,+})})\right)^2 \nu_{\alpha^*}(d\eta) \\ &\le \frac{1}{2}\Psi(\alpha^*) n \sum_{k=1}^n \int \left(\sqrt{\bar{f}(\eta^{te_k,+})} - \sqrt{\bar{f}(\eta^{oe_k,+})}\right)^2 \nu_{\alpha^*}(d\eta) \\ &= n \sum_{k=1}^n I_{oe_k,te_k}(\bar{f}) \le d(x_0, \sigma x_0)^2 \frac{D_N(\bar{f})}{p^* |\Gamma_N|}. \end{aligned}$$

□

Proof of the two blocks estimate. As in the proof of the one block estimate, we can cut off of large density first. It is enough to show that

$$\begin{aligned}\overline{\lim_{l\to\infty}}\ \overline{\lim_{\epsilon\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{D_N(f)\le\frac{C|V_N|}{N^2}} \sup_{2l\le|\sigma|\le\epsilon N}&\\ \int |\bar{\eta}_{x_0,l}-\bar{\eta}_{\sigma x_0,l}|1_{\{\bar{\eta}_{x_0,l}+\bar{\eta}_{\sigma x_0,l}\le A\}}\bar{f}(\eta)\nu_{\alpha^*}(d\eta)=0.&\end{aligned} \tag{4.20}$$

(4.20) can be written as

$$\begin{aligned}\overline{\lim_{l\to\infty}}\ \overline{\lim_{\epsilon\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{D_N(f)\le\frac{C|V_N|}{N^2}} \sup_{2l\le|\sigma|\le\epsilon N}&\\ \int |\bar{\xi}_{x_0,l}-\bar{\xi'}_{x_0,l}|1_{\{\bar{\xi}_{x_0,l}+\bar{\xi'}_{x_0,l}\le A\}}\bar{f}_{\sigma,l}(\xi,\xi')\nu^l_{\alpha^*}\otimes\nu^l_{\alpha^*}(d\xi,d\xi')=0.&\end{aligned} \tag{4.21}$$

As in the proof of Lemma 4.1, the next step consists in the estimation of the Dirichlet form $D^{l,l}(\bar{f}_{\sigma,l})$. Note that

$$\sum_{e\in E_l}(I^{1,l}_{oe,te}(\bar{f}_{\sigma,l})+I^{2,l}_{oe,te}(\bar{f}_{\sigma,l}))\le\sum_{e\in E_l\cup\sigma E_l}I_{oe,te}(\bar{f})\le 2|E_l|\frac{D_N(\bar{f})}{p^*|\Gamma_N|}\le\frac{2C|E_l||V_0|}{p^*N^2} \tag{4.22}$$

and by Lemma 4.5

$$I^l_{x_0,x_0'}(\bar{f}_{\sigma,l})\le I_{x_0,\sigma x_0}(\bar{f})\le d(x_0,\sigma x_0)^2\frac{D_N(\bar{f})}{p^*|\Gamma_N|}\le p^{*-1}c_1^2\epsilon^2C|V_0|. \tag{4.23}$$

For the same reason of Lemma 4.1, it is enough to prove that

$$\overline{\lim_{l\to\infty}}\sup_{D^{l,l}(f)=0}\int|\bar{\xi}_{x_0,l}-\bar{\xi'}_{x_0,l}|1_{\{\bar{\xi}_{x_0,l}+\bar{\xi'}_{x_0,l}\le A\}}f(\xi,\xi')\nu^l_{\alpha^*}\otimes\nu^l_{\alpha^*}(d\xi,d\xi')=0. \tag{4.24}$$

The proof will be completed in the way as mentioned in the proof of the one block estimate. □

5 Standard realization

In this section, we apply our results to find the standard realization of the crystal lattices. For a crystal lattice $X=(V,E)$, fix a harmonic realization Φ_0 with lattice group $\phi_0(\Gamma)=\{\sum_{i=1}^d k_iu_i\ |k_i\in\mathbb{Z}\ \}$. Since the diffusion matrix $\mathbb{D}_{\Phi_0}$ is strictly positive definite, all eigenvalues $\{\lambda_1,\cdots,\lambda_d\}$ of $\mathbb{D}_{\Phi_0}$ are strictly positive. We can write $\mathbb{D}_{\Phi_0}$ as

$$\mathbb{D}_{\Phi_0}=P^T diag(\lambda_1,\dots,\lambda_d)P. \tag{5.1}$$

Here P is an orthogonal matrix and $diag(\lambda_1,\dots,\lambda_d)$ is a diagonal matrix.

With the statement above we have the following proposition, which allows us to get the standard realization from a fixed harmonic realization.

Proposition 5.1. *For all Φ' with $vol(D_{\Phi'})=vol(D_\Phi)$, it holds that*

$$E(\Phi)\le E(\Phi'),$$

where Φ is the harmonic realization with the lattice group $\{\sum_{i=1}^d k_iAu_i\ |k_i\in\mathbb{Z}\ \}$ and A is given by

$$A=diag\left(\left(\frac{\lambda_1\cdots\lambda_d}{\lambda_1^d}\right)^{\frac{1}{2d}},\dots,\left(\frac{\lambda_1\cdots\lambda_d}{\lambda_d^d}\right)^{\frac{1}{2d}}\right)P. \tag{5.2}$$

Proof. For any harmonic realization Φ with lattice group $\phi(\Gamma) = \{\sum_{i=1}^{d} k_i \widetilde{u}_i \mid k_i \in \mathbb{Z}\}$, let A be the basis transformation from $\{u_1, \ldots, u_d\}$ to $\{\widetilde{u}_1, \ldots, \widetilde{u}_d\}$. By Theorem 3.1 and Proposition 3.2, we have that

$$\mathbb{D}_{\Phi} = A\mathbb{D}_{\Phi_0}A^T.$$

Note that the energy is nothing but the trace of diffusion matrix, i.e.,

$$E(\Phi) = tr(\mathbb{D}_{\Phi}) = tr(A\mathbb{D}_{\Phi_0}A^T).$$

Thus, for a fixed volume of fundamental parallelotope, to find the standard realization, it suffices to find A with $|A| = 1$ such that

$$tr(A\mathbb{D}_{\Phi_0}A^T) = \min_{|A|=1} tr(A\mathbb{D}_{\Phi_0}A^T).$$

By Schwarz inequality, we have that

$$tr(A\mathbb{D}_{\Phi_0}A^T) \geq d\sqrt[d]{|A\mathbb{D}_{\Phi_0}A^T|} = d\sqrt[d]{|\mathbb{D}_{\Phi_0}|}.$$

Then the above inequality takes the equal when A is equal to

$$A = diag\left(\left(\frac{\lambda_1 \cdots \lambda_d}{\lambda_1^d}\right)^{\frac{1}{2d}}, \ldots, \left(\frac{\lambda_1 \cdots \lambda_d}{\lambda_d^d}\right)^{\frac{1}{2d}}\right) P.$$

As mentioned in Remark 2.3, the harmonic realization with lattice group $\{\sum_{i=1}^{d} k_i A u_i \mid k_i \in \mathbb{Z}\ \}$ is also the standard realization. The proof is completed. ☐

6 Examples

In this section, we give two concrete examples where we can apply our main results. In particular, both models are given as a process on the square lattice with inhomogeneous jump rates. We study theses models as a homogeneous model on a crystal lattice and obtain explicit hydrodynamic equations.

Example 1. Consider the exclusion process on discrete torus $\mathbb{T}_{2N} := \{0, 1, \ldots, 2N-1\}$ with generator:

$$L_N f(\eta) := \sum_{x \in \mathbb{T}_{2N}} p(x, x+1) \left\{ f(\eta^{x,x+1}) - f(\eta) \right\}, \tag{6.1}$$

where

$$p(x, x+1) = \begin{cases} \alpha & x \ \ even \\ \\ \beta & x \ \ odd. \end{cases} \tag{6.2}$$

We can regard it as a process on the crystal $X = (V, E)$, where $V = \mathbb{Z}, E = \{(x, x+1), (x+1, x) \mid z \in \mathbb{Z}\}$ and the group $\Gamma = \mathbb{Z}$ with the group action: $\sigma x := 2\sigma + x$ and for $\sigma \in \Gamma$. Then $V_0 = \{0, 1\}$, $\mathbb{T}_{\phi} = [0, 2)$, $\Gamma_N = \{0, 1, \cdots, N-1\}$, $V_N = \{0, 1, \ldots, 2N-1\}$ with $2N = 0$ and $E_N = \{(x, x+1), (x+1, x) \mid x \in V_N\}$. The generator is given by

$$L_N f(\eta) := \sum_{e \in E_N} p(e)\{f(\eta^e) - f(\eta)\}. \tag{6.3}$$

Take the realization Φ as in Example 1b, the associated empirical process is

$$\pi_t^{\Phi,N}(du) = \frac{1}{2N} \sum_{x \in \{0,1\}} \sum_{y=1}^{N-1} \eta_{2y+x}(t) \delta_{(2y+x)/N}(du).$$

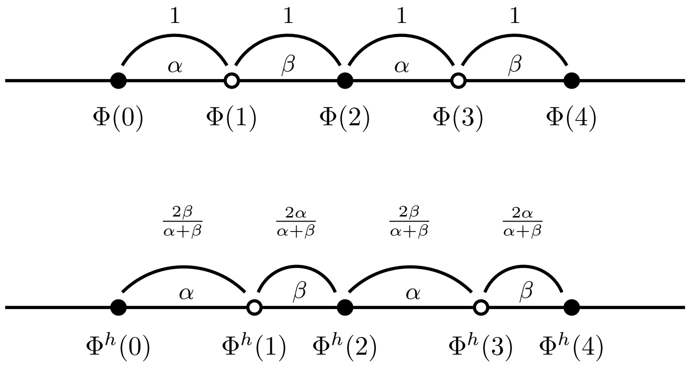

Figure 5: The images of Φ (above) and Φ^h (below). The numbers in the first line show the length between vertices in each realization, while those in the second line show the jump rates between vertices.

Note that this realization is not harmonic for the lattice group $\{\phi(\sigma) = 2\sigma,\ \sigma \in \Gamma\}$ and the weight function $p(\cdot)$. We obtain the harmonic realization Φ^h (See Figure 5) associated to the lattice group $\phi(\Gamma)$ by (2.4). Precisely,

$$\Phi^h(\sigma 0) := 0 + \phi(\sigma),\ \Phi^h(\sigma 1) := \frac{2\beta}{\alpha+\beta} + \phi(\sigma). \tag{6.4}$$

By Theorem 3.1, we have that

$$\varlimsup_{N\to\infty} P_{\mu^N}\left[\left|\frac{1}{2N}\sum_{x\in\mathbb{T}_{2N}} G\left(\frac{x}{N}\right)\eta_x(t) - \frac{1}{2}\int_0^2 G(u)\rho(t,u)du\right| > \delta\right] = 0, \tag{6.5}$$

for every $\delta > 0$, every continuous $G : [0,2) \to \mathbb{R}$ and $\rho(t,u)$ is the unique weak solution of

$$\partial_t \rho = \frac{4\alpha\beta}{\alpha+\beta}\partial_x^2 \rho, \quad \rho(0,\cdot) = \rho_0(\cdot), \tag{6.6}$$

where $\frac{4\alpha\beta}{\alpha+\beta}$ is just the diffusion coefficient (matrix) of Φ^h.

Remark 6.1. Note that if β is much bigger than α, the diffusion coefficient is close to 4α. This means: from microscopic view, particle jumps very fast between site 1 and site 2; in macroscopic view, the diffusion speed is close to 4α, where α is the jump rate between site 0 and site 1.

Example 2. Let $\Gamma := \{k_1(2,0) + k_2(1,1) | k_1, k_2 \in \mathbb{Z}\}$ be a lattice group on $\mathbb{Z}^2$. Let $p(\cdot,\cdot) : \mathbb{Z}^2 \times \mathbb{Z}^2 \to [0,1]$ be a weight function satisfying:
(i) $p(x,y) = p(y,x)$ for all $x, y \in \mathbb{Z}^2$.
(ii) $p(\sigma x, \sigma y) = p(x,y)$ for all $x, y \in \mathbb{Z}^2$ and all $\sigma \in \Gamma$, where $\sigma x := x + \sigma$.
(iii) $p(x,y) = 0$ if $|x-y| := |x_1 - y_1| + |x_2 - y_2| \neq 1$.
(iv) $p((0,0),(1,0)) = \frac{1}{6}, p((0,0),(0,1)) = \frac{1}{3}, p((0,0),(-1,0)) = \frac{1}{2}, p((0,0),(0,-1)) = 0$.
Consider the zero range process on the discrete torus $\mathbb{T}_N^2 := \mathbb{Z}^2/N\Gamma$ with generator

$$L_N f(\eta) := \sum_{x,y\in\mathbb{T}_N^2} p(x,y) g(\eta_x)[f(\eta^{x,y}) - f(\eta)]. \tag{6.7}$$

Note that weight function $p(\cdot,\cdot)$ is inhomogeneous, namely not invariant under the group action of $\mathbb{Z}^2$. As mentioned in Remark 3.5, to investigate the hydrodynamic limit on $\mathbb{T}_N^2$,

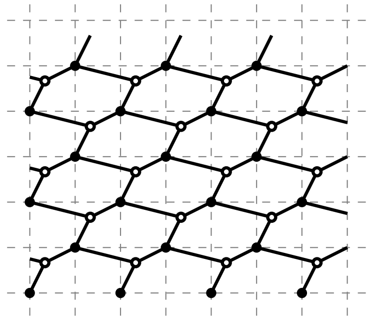

Figure 6: The image of the harmonic realization Φ^h in Example 2.

we can regard this process as a homogeneous process on the hexagonal lattice with the realization Φ in Example 3b. More precisely, consider the zero range process on the hexagonal lattice, which is introduced in Section 2 Example 3a, with generator

$$L_N f(\eta) := \sum_{e \in E_N} p(e) g(\eta_{oe})[f(\eta^e) - f(\eta)], \tag{6.8}$$

where the symmetric periodic weight function $p(\cdot)$ is given by

$$p(e_1) = \frac{1}{3}, \;\; p(e_2) = \frac{1}{2}, \;\; p(e_3) = \frac{1}{6}. \tag{6.9}$$

Here e_1, e_2, e_3 are the edges of the fundamental graph defined in Section 2 Example 3a. Choosing the realization Φ in Example 3b, the empirical process associated with Φ is

$$\pi_t^{\Phi,N}(du) = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_x(t) \delta_{\Phi_N(x)}(du).$$

Note that Φ is not harmonic for $p(\cdot)$ and the associated harmonic realization Φ^h is obtained by shifting every white vertex along (1/3,-1/3)(See Figure 6). By Theorem 3.4, we obtain that $\{\pi_t^{\Phi,N}\}_N$ converges to $\rho(t,u)du/\mathrm{vol}(\mathbb{T}^d_\phi)$ in probability. Here $\mathrm{vol}(\mathbb{T}^d_\phi) = 2$ and ρ is the unique weak solution of $\partial_t \rho = \nabla \mathbb{D}_{\Phi^h} \nabla \Psi(\rho)$, where $\mathbb{D}_{\Phi^h}$ is the diffusion matrix of Φ^h,

$$\mathbb{D}_{\Phi^h} = \begin{pmatrix} 5/9 & 1/9 \\ 1/9 & 2/9 \end{pmatrix}.$$

Remark 6.2. Both the above two examples are non-gradient systems, and if we apply the non-gradient method, the diffusion coefficients are not so clear since they are given by a variational formula. However, we can compute the diffusion coefficients explicitly by applying our main theorems.

A Two estimates on crystal lattices

In Appendix A, we proved that $\bar{\eta}_{\sigma x_0, \epsilon N}$ can be replaced by $\langle \pi^N, \chi_{\Phi_N(\sigma x_0),\epsilon} \rangle$, as mentioned in the proof of Theorem 3.4, which allows us to close the equation in the empirical process $\pi^N_\cdot$.

For a given lattice group

$$\phi(\Gamma) = \left\{ \sum_{i=1}^d k_i u_i \mid k_i \; integers \right\},$$

where $\{u_1, \cdots, u_d\}$ is a basis in $\mathbb{R}^d$. For $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{x}$ can be written uniquely as $\mathbf{x} = \sum_{i=1}^d x_i u_i$. Define a norm on $\mathbb{R}^d$ by

$$||\mathbf{x}||_1 := \sum_{i=1}^d |x_i|,$$

for $\mathbf{x} = \sum_{i=1}^d x_i u_i \in \mathbb{R}^d$. Define the distance d_1 by setting $d_1(\mathbf{x}, \mathbf{x}') = ||\mathbf{x} - \mathbf{x}'||_1$ for $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$ and the induced metric in $\mathbb{T}^d_\phi$ from d_1 is also denoted by d_1.

Since Φ is periodic, there exists a constant $C_0 := \max_{x \in V} ||\Phi([x]x_0) - \Phi(x)||_1 < \infty$ such that

$$||\Phi([x]x_0) - \Phi([z]x_0)||_1 - 2C_0 \le ||\Phi(x) - \Phi(z)||_1 \le ||\Phi([x]x_0) - \Phi([z]x_0)||_1 + 2C_0, \quad \text{(A.1)}$$

for $x, z \in V$. Furthermore, since $||\Phi([x]x_0) - \Phi([z]x_0)||_1 = |[x] - [z]|$, (A.1) can be written as

$$|[x] - [z]| - 2C_0 \le ||\Phi(x) - \Phi(z)||_1 \le |[x] - [z]| + 2C_0. \quad \text{(A.2)}$$

Let $D_\phi := \{\sum_{i=1}^d t_i u_i \mid 0 \le t_i < 1, i = 1, \ldots, d\}$ be the fundamental parallelotope. For $\mathbf{x} \in \mathbb{R}^d$, there exists a unique $\sigma_{\mathbf{x}} \in \Gamma$ such that $\mathbf{x} \in \phi(\sigma_{\mathbf{x}}) + D_\phi$. Define the map $[\cdot] : \mathbb{R}^d \to \Gamma$ by setting $[\mathbf{x}] = \sigma_{\mathbf{x}}$. Since Φ is periodic, there exists a constant $C_1 := \max_{\mathbf{x} \in \mathbb{R}^d} ||\Phi([\mathbf{x}]x_0) - \mathbf{x}||_1 < \infty$ such that

$$||\Phi([\mathbf{x}]x_0) - \Phi([\mathbf{z}]x_0)||_1 - 2C_1 \le ||\Phi(\mathbf{x}) - \Phi(\mathbf{z})||_1 \le ||\Phi([\mathbf{x}]x_0) - \Phi([\mathbf{z}]x_0)||_1 + 2C_1, \quad \text{(A.3)}$$

for $\mathbf{x}, \mathbf{z} \in \mathbb{R}^d$. Furthermore, we have that

$$|[\mathbf{x}] - [\mathbf{z}]| - 2C_1 \le ||\Phi(\mathbf{x}) - \Phi(\mathbf{z})||_1 \le |[\mathbf{x}] - [\mathbf{z}]| + 2C_1. \quad \text{(A.4)}$$

For $\epsilon > 0$, define the ϵ-ball in $\mathbb{T}^d_\phi$ centered on $\mathbf{z} \in \mathbb{T}^d_\phi$ by setting

$$B_{\mathbf{z}}(\epsilon) := \{\mathbf{x} \in \mathbb{T}^d_\phi \mid d_1(\mathbf{x}, \mathbf{z}) \le \epsilon\}.$$

and let $\chi_{\mathbf{z},\epsilon} : \mathbb{T}^d_\phi \to \mathbb{R}$ be a characteristic function defined by

$$\chi_{\mathbf{z},\epsilon} := \frac{\mathrm{vol}(\mathbb{T}^d_\phi)}{\mathrm{vol}(B_{\mathbf{z}}(\epsilon))} 1_{B_{\mathbf{z}}(\epsilon)}.$$

Lemma A.1. *There exists a constant $C_3(\epsilon) > 0$ depending only on ϵ such that for any $\mathbf{z} \in \mathbb{T}^d_\phi$ and any $N \ge 1$,*

$$\left| \frac{\mathrm{vol}(B_{\mathbf{z}}(\epsilon))}{\mathrm{vol}(\mathbb{T}^d_\phi)} - \frac{|\cup_{|\sigma| \le \epsilon N} \sigma D_{x_0}|}{|V_N|} \right| \le \frac{C_2(\epsilon)}{N}, \quad \text{(A.5)}$$

where $vol(A)$ stands for the volume of a Borel set A and $|B|$ the cardinality of a set B.

Proof. For any $\mathbf{z} \in \mathbb{T}^d_\phi$, take a lift $\widetilde{\mathbf{z}} \in \mathbb{R}^d$. For sufficiently small $\epsilon > 0$, take a lift $B_{\widetilde{\mathbf{z}}}(\epsilon) \subset \mathbb{R}^d$ of $B_{\mathbf{z}}(\epsilon) \subset \mathbb{T}^d_\phi$. By (6.3) and (6.4), we have that

$$\bigcup_{|\sigma| \le \epsilon N - 2C_1} \sigma[\widetilde{\mathbf{z}}] D_\phi \subset B_{\widetilde{\mathbf{z}}}(\epsilon N) \subset \bigcup_{|\sigma| \le \epsilon N + 2C_1} \sigma[\widetilde{\mathbf{z}}] D_\phi,$$

which implies that

$$\left| N^d \mathrm{vol}(B_{\widetilde{\mathbf{z}}}(\epsilon)) - \mathrm{vol}(\cup_{|\sigma| \le \epsilon N} \sigma[\widetilde{\mathbf{z}}] D_\phi) \right| \le \mathrm{vol}(D_\phi) 2^d [(\epsilon N + 2C_1)^d - (\epsilon N - 2C_1)^d].$$

Note that $\mathrm{vol}(B_{\widetilde{\mathbf{z}}}(\epsilon)) = \mathrm{vol}(B_{\mathbf{z}}(\epsilon))$, $\mathrm{vol}(D_\phi) = \mathrm{vol}(\mathbb{T}^d_\phi)$ and $|V_N| = N^d |D_{x_0}|$, we obtain that there exists a constant $C_2(\epsilon)$ depending only on ϵ such that

$$\left| \frac{\mathrm{vol}(B_{\mathbf{z}}(\epsilon))}{\mathrm{vol}(\mathbb{T}^d_\phi)} - \frac{|\cup_{|\sigma|\le \epsilon N}\, \sigma D_{x_0}|}{|V_N|} \right| \le \frac{C_2(\epsilon)}{N}.$$

For the empirical density $\pi^N := \frac{1}{|V_N|}\sum_{x\in V_N} \eta_x \delta_{\Phi_N(x)}$ on $\mathbb{T}^d_\phi$, $\eta \in Z_N$, we have the following lemma. ☐

Lemma A.2. *There exists a constant $C_4(\epsilon) > 0$ depending only on ϵ such that*

$$\frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} \left| \langle \pi^N, \chi_{\Phi_N(\sigma x_0),\epsilon}\rangle - \bar{\eta}_{\sigma x_0,\epsilon N}\right| \le \frac{C_3(\epsilon)}{N}\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x. \tag{A.6}$$

Proof. For $\sigma \in \Gamma_N$, take a lift $\widetilde{\sigma x_0} \in V$ of σx_0 and a lift $B_{(1/N)\Phi(\widetilde{\sigma x_0})}(\epsilon) \subset \mathbb{R}^d$ of $B_{\Phi_N(\sigma x_0)}(\epsilon) \subset \mathbb{T}^d_\phi$. Similar to Lemma A.1, it holds that

$$\bigcup_{|\sigma'|\le \epsilon N - 2C_0} \sigma'\sigma D_{x_0} \subset \left\{ x \in V \;\middle|\; \left\| \frac{1}{N}\Phi(x) - \frac{1}{N}\Phi(\widetilde{\sigma x_0})\right\|_1 \le \epsilon \right\} \subset \bigcup_{|\sigma'|\le \epsilon N + 2C_0} \sigma'\sigma D_{x_0}.$$

Furthermore, take a lift $\widetilde{\eta} \in Z$ of $\eta \in Z_N$, then it holds that

$$\sum_{\substack{x\in V_N \\ \Phi_N(x)\in B_{\Phi_N(\sigma x_0)}(\epsilon)}} \eta_x = \sum_{\substack{x\in V \\ \frac{1}{N}\Phi(x)\in B_{(1/N)\Phi(\widetilde{\sigma x_0})}(\epsilon)}} \widetilde{\eta}_x.$$

We have that,

$$\left| \frac{1}{|V_N|} \sum_{\substack{x\in V \\ \|\frac{1}{N}\Phi(x) - \frac{1}{N}\Phi(\widetilde{\sigma x_0})\|_1 \le \epsilon}} \widetilde{\eta}_x - \frac{1}{|V_N|}\sum_{\substack{x\in\sigma'\sigma D_{x_0} \\ |\sigma'|\le \epsilon N}} \widetilde{\eta}_x \right| \le \frac{1}{|V_N|} \sum_{\substack{x\in\sigma'\sigma D_{x_0} \\ \epsilon N - 2C_0 \le |\sigma'| \le \epsilon N + 2C_0}} \widetilde{\eta}_x.$$

By Lemma A.1, it holds that

$$\begin{aligned}\left| \frac{\mathrm{vol}(\mathbb{T}^d_\phi)}{|V_N|\mathrm{vol}(B_{\Phi_N(\sigma x_0)}(\epsilon))} - \frac{1}{|\cup_{|\sigma'|\le\epsilon N}\, \sigma'\sigma D_{x_0}|}\right| &\le \frac{\mathrm{vol}(\mathbb{T}^d_\phi)}{\mathrm{vol}(B_{\Phi_N(\sigma x_0)}(\epsilon))|\cup_{|\sigma'|\le\epsilon N}\, \sigma'\sigma D_{x_0}|}\frac{C_3(\epsilon)}{N} \\ &\le \frac{C_5(\epsilon)}{N^{d+1}},\end{aligned}$$

where $C_5(\epsilon)$ is a constant depending only on ϵ. By triangle inequaltiy, we have that

$$\begin{aligned}\left| \langle \pi^N, \chi_{\Phi(\sigma x_0),\epsilon}\rangle - \bar{\eta}_{\sigma x_0,\epsilon N}\right| &\le \frac{\mathrm{vol}(\mathbb{T}^d_\phi)}{|V_N|\mathrm{vol}(B_{\Phi_N(\sigma x_0)}(\epsilon))} \sum_{\substack{x\in\sigma'\sigma D_{x_0} \\ \epsilon N - 2C_0 \le |\sigma'| \le \epsilon N + 2C_0}} \eta_x \\ &+ \left| \frac{\mathrm{vol}(\mathbb{T}^d_\phi)}{|V_N|\mathrm{vol}(B_{\Phi_N(\sigma x_0)}(\epsilon))} \sum_{\substack{x\in\sigma'\sigma D_{x_0} \\ |\sigma'|\le\epsilon N}} \eta_x - \frac{1}{|\cup_{|\sigma'|\le\epsilon N}\, \sigma'\sigma D_{x_0}|} \sum_{\substack{x\in\sigma'\sigma D_{x_0} \\ |\sigma'|\le\epsilon N}} \eta_x \right| \\ &\le \frac{\mathrm{vol}(\mathbb{T}^d_\phi)}{|V_N|\mathrm{vol}(B_{\Phi_N(\sigma x_0)}(\epsilon))} \sum_{\substack{x\in\sigma'\sigma D_{x_0} \\ \epsilon N - 2C_0 \le |\sigma'| \le \epsilon N + 2C_0}} \eta_x + \frac{C_5(\epsilon)}{N^{d+1}} \sum_{\substack{x\in\sigma'\sigma D_{x_0} \\ |\sigma'|\le\epsilon N}} \eta_x.\end{aligned}$$

Sum on $\sigma \in \Gamma_N$, it concludes that there exists a constant $C_4(\epsilon)$ depending only on ϵ such that

$$\frac{1}{|\Gamma_N|}\sum_{\sigma\in\Gamma_N} \left| \langle \pi^N, \chi_{\Phi(\sigma x_0),\epsilon}\rangle - \bar{\eta}_{\sigma x_0,\epsilon N}\right| \le \frac{C_3(\epsilon)}{N}\frac{1}{|V_N|}\sum_{x\in V_N}\eta_x.$$ ☐

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