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Moderate deviations for the self-normalized random walk in random scenery*

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Abstract

Let G be an infinite connected graph with vertex set V. Let $\{S_n : n \in \mathbb{N}_0\}$ be the simple random walk on G and let $\{\xi(v) : v \in V\}$ be a collection of i.i.d. random variables which are independent of the random walk. Define the random walk in random scenery as $T_n = \sum_{k=0}^{n} \xi(S_k)$, and the normalization variables $V_n = (\sum_{k=0}^{n} \xi^2(S_k))^{1/2}$ and $L_{n,2} = (\sum_{v\in V} \ell_n^2(v))^{1/2}$. For $G = \mathbb{Z}^d$ and $G = \mathbb{T}_d$, the d-ary tree, we provide large deviations results for the self-normalized process $T_n\sqrt{n}/(L_{n,2}V_n)$ under only finite moment assumptions on the scenery.

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1 Introduction

Let G be an infinite connected graph with vertex set V and let $\{S_n : n \in \mathbb{N}_0\}$ be the simple random walk on G started at a distinguished vertex $o \in V$. Denote the law and expectation of this walk by P and E. Let $\{\xi(v) : v \in V\}$ be independent copies of a random variable ξ, which we denote as the scenery. Denote the law of the scenery by $\mathbb{P}$ and the expectation with respect to this law by $\mathbb{E}$. We will always assume $\mathbb{E}\xi = 0$ and $\sigma^2 = \mathbb{E}\xi^2 > 0$. The random walk in random scenery (RWRS) is the process $\{T_n : n \in \mathbb{N}_0\}$ defined by

$$T_n = \sum_{k=0}^{n} \xi(S_k) = \sum_{v\in V} \ell_n(v)\xi(v),$$

where $\ell_n(v) = \sum_{k=0}^{n} \mathbb{1}\{S_k = v\}$ is the local time of v at time n. This process was introduced for the case $G = \mathbb{Z}^d$ by Kesten and Spitzer [11], and by Borodin [5, 6]

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independently and at the same time in order to introduce new scaling and self-similar laws. However for $d \geq 3$, when the random walk by time n visits most points a constant amount of times, [11] showed under appropriate assumptions on the distribution of ξ that $T_n/\sqrt{n}$ converges in distribution to a Gaussian random variable. More recently, large and moderate deviations of T_n have been studied for $G = \mathbb{Z}^d$ in [2, 3, 8, 9, 10]. Fleischmann, Mortërs and Wachtel [9] proved a moderate deviations principle (MDP) for $d \geq 3$. Assuming Cramér's condition, i.e. that there exists $\theta > 0$ such that $\mathbb{E}e^{\theta|\xi|} < \infty$, they showed that

$$\lim_{n\to\infty} y_n^{-2} \log \mathbb{P} \otimes P(T_n/\sqrt{n} \geq y_n) = -\frac{1}{2\sigma^2(2G(0)-1)} \tag{1.1}$$

for $y_n = o(n^{1/6})$, where $G(\cdot)$ is the Green's function of the random walk. We write $a_n = o(b_n)$ or $a_n \ll b_n$ if $\lim_{n\to\infty} a_n/b_n = 0$ for positive sequences a_n and b_n. In contrast with moderate deviations of sums of i.i.d. random variables, in [3] it was shown that this regime is maximal under Cramér's condition. That is, more assumptions need to be made on the scenery in order to get moderate deviations when y_n grows faster than $n^{1/6}$.

There has been a recent interest in proving large deviations for sums of i.i.d. random variables under minimal moment assumptions. It is well understood that if one replaces the normalization constant by self-normalization, this is possible. In [14], Shao provided self-normalized large and moderate deviations for the partial sum of i.i.d. random variables, while only making assumptions on the second moment. In [8], Feng, Shao and Zeitouni extended this framework to RWRS by proving a Crámer type moderate deviations. Define

$$V_n^2 = \sum_{k=0}^{n} \xi^2(S_k) = \sum_{v\in V} \ell_n(v)\xi^2(v)$$

and

$$L_{n,2}^2 = \sum_{v\in V} \ell_n^2(v).$$

For the simple random walk on $\mathbb{Z}^d$ for $d \geq 3$, it is known (see [9, 11]) that

$$T_n/\sqrt{n} \xrightarrow{d} N(0, \sigma^2(2G(0)-1)),\ \ L_{n,2}^2/n \xrightarrow{p} 2G(0)-1 \text{ and } V_n^2/n \xrightarrow{p} \sigma^2,$$

so that we have the self-normalized central limit theorem

$$\frac{T_n\sqrt{n}}{V_n L_{n,2}} \xrightarrow{d} N(0,1).$$

In [8], they proved that if $d \geq 4$ and there exists $\alpha > 0$ and $c_\alpha > 0$ such that $\mathbb{P}(\xi \geq t) \leq 2e^{-c_\alpha t^\alpha}$ for $t > 0$, then

$$\mathbb{P} \otimes P\left(\frac{T_n\sqrt{n}}{V_n L_{n,2}} \geq x\right) \sim 1 - \Phi(x) \tag{1.2}$$

uniformly for $x \in [0, O(n^\tau)]$ and any $0 < \tau < \alpha/(6\alpha+4)$. Here $\Phi(\cdot)$ is the standard normal distribution function, and we write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$ for positive sequences a_n and b_n. By self-normalizing, [8] was able to achieve a MDP while only assuming sub-exponential tails on the scenery, which contrasts (1.1). However, for a suitable range of y_n, one would expect

$$-\log \mathbb{P} \otimes P\left(\frac{T_n\sqrt{n}}{V_n L_{n,2}} \geq y_n\right) \asymp y_n^2 \tag{1.3}$$

under only finite moment assumptions on the scenery, i.e. $\mathbb{E}|\xi|^\kappa < \infty$ for some fixed $\kappa > 0$. Here we write $a_n \asymp b_n$ if $0 < \liminf_{n\to\infty} a_n/b_n \le \limsup_{n\to\infty} a_n/b_n < \infty$ for positive sequences a_n and b_n. In the same paper, the authors showed that if ξ has the probability density function $\frac{1}{2}\alpha(1+|t|)^{-1-\alpha}$ for some $\alpha > 0$ and $(\log n)^{1/2} \ll y_n \ll n^{1/2}$, then

$$\liminf_{n\to\infty} y_n^{-2d/(d+2)} (\log n)^{-2/(d+2)} \log \mathbb{P} \otimes P\left(\frac{T_n\sqrt{n}}{V_n L_{n,2}} \ge y_n\right) > -\infty. \tag{1.4}$$

Since $y_n^2 \gg y_n^{2d/(d+2)} (\log n)^{2/(d+2)}$, this lower bound shows that moderate deviations of the form (1.3) does not hold when $y_n \gg (\log n)^{1/2}$ if one is only to assume finite moment conditions on ξ.

The lower bound is achieved by considering the event in which the random walk occupies the ball $\{z \in \mathbb{Z}^d : |z| \le R\}$ for time y_n^2, and taking each of the scenery values inside the ball to be of size $O(n)$. The probability of the former event is roughly $\exp(-y_n^2/R^2)$, and for the latter event $n^{-R^d} = \exp(-R^d \log n)$. Since the random walk and the scenery are independent of one other, we get the lower bound $\exp(-y_n^2/R^2 - R^d \log n)$ which, optimized over R, yields (1.4). From this example, we observe that deviations of the self-normalized RWRS depend on the interplay between the scenery and the random walk. Furthermore, we see that this process is sensitive to the correlation of the local times due to the heavy tails of the scenery.

Motivated by this phenomenon, in this paper we study self-normalized moderate deviations for graphs other than the lattice. We expect that for graphs where the simple random walk has weakly-dependent local times, moderate deviations of the form (1.3) are attainable. A natural candidate for such graphs are trees, since regeneration epochs of the random walk on the tree have exponential tails (see Section 3 for definitions). Lastly, we also provide an upper bound that complements (1.4).

2 Main results

Let $\mathbb{T}_d$ be the d-ary tree rooted at o, i.e. $\deg(v) = d+1$ for $v \ne o$ and $\deg(o) = d$. In this paper we will always assume $d \ge 2$. Since the simple random walk on $\mathbb{T}_d$ is transient, it follows that for $G = \mathbb{T}_d$ we have $T_n/\sqrt{n}$ converges weakly to a normal random variable. The following theorem provides precise asymptotics for the self-normalized RWRS on $\mathbb{T}_d$ while only making finite moment assumptions on the scenery.

Theorem 2.1. *Let $G = \mathbb{T}_d$ and let y_n be a positive sequence such that $y_n \to \infty$.*

1. *Suppose that $\mathbb{E}\xi^4 < \infty$ and $y_n = o(n^{1/6})$. We then have*

$$\limsup_{n\to\infty} y_n^{-2} \log \mathbb{P} \otimes P\left(\frac{T_n\sqrt{n}}{V_n L_{n,2}} \ge y_n\right) \le -\frac{c_d}{2}, \tag{2.1}$$

   where c_d is a positive constant independent of n, and $c_d \uparrow 1$ as $d \to \infty$.
2. *Suppose $\mathbb{E}\xi^6 < \infty$ and $y_n = o(n^{1/6})$. We then have*

$$\liminf_{n\to\infty} y_n^{-2} \log \mathbb{P} \otimes P\left(\frac{T_n\sqrt{n}}{V_n L_{n,2}} \ge y_n\right) \ge -\frac{1}{2}. \tag{2.2}$$

In light of (1.4), the following theorem provides a sharp upper bound for the self-normalized RWRS on the lattice when only assuming finite moment conditions on the scenery.

Theorem 2.2. *Let $G = \mathbb{Z}^d$ for $d \ge 3$ and let y_n be a positive sequence such that $(\log n)^{1/2} \ll y_n \ll n^{1/6}$. If $\mathbb{E}\xi^4 < \infty$, then*

$$\limsup_{n\to\infty} y_n^{-2d/(d+2)} (\log n)^{-2/(d+2)} \log \mathbb{P} \otimes P\left(\frac{T_n\sqrt{n}}{V_n L_{n,2}} \ge y_n\right) < 0. \tag{2.3}$$

Remark 2.3. For Theorems 2.1 and 2.2 we assume that $y_n \ll n^{1/6}$. In this regime, the deviation comes from the moderate deviation of the scenery. When $y_n \gg n^{1/6}$, the deviation comes from large deviations for the local time statistics $L_{n,2}$ and $L_{n,3}$, see Section 3 for definitions.

The rest of the paper is organized as follows. In Section 3, we study concentration inequalities for different local time statistics that will aid us in the proof of Theorem 2.1. Our main tool will be the regeneration structure of the random walk on the tree. In Sections 4 and 5, we prove the upper and lower bound of Theorem 2.1. In Section 6 we review the necessary concentration inequalities for local time in the lattice, and in Section 7 we prove Theorem 2.2.

3 Local time for $\mathbb{T}_d$

For the rest of the paper will write P_v and E_v when the random walk is conditioned on starting at $v \in V$.

3.1 Regeneration times

Our proofs for the concentration of local times will utilize the regenerative structure of the random walk on the tree. For $v \in V$, denote the level of v by $|v|$, which is the length of the unique geodesic between v and o. We define regeneration times as

$$\tau_1 = \inf\{n \in \mathbb{N} : |S_n| \neq |S_k| \text{ for all } k < n \text{ and } |S_k| \neq |S_{n-1}| \text{ for all } k > n\}$$

and for $j \in \mathbb{N}$

$$\tau_{j+1} = \inf\{n > \tau_j : |S_n| \neq |S_k| \text{ for all } k < n \text{ and } |S_k| \neq |S_{n-1}| \text{ for all } k > n\}.$$

Regeneration times were studied in [7, 12] for biased random walks on Galton-Watson trees. Rerunning the same proofs, we can conclude that the simple random walk on $\mathbb{T}_d$ has infinitely many regeneration times such that $\{\tau_{j+1} - \tau_j\}_{j \geq 1}$ is an i.i.d. sequence which is independent of τ_1, and that τ_1 and $\tau_2 - \tau_1$ have exponential moments. The following lemma shows that as d tends to infinity, the tails of the regeneration times become lighter.

Lemma 3.1. *Define*

$$s_d = \sup\{\lambda \geq 0 : Ee^{\lambda \tau_1} < \infty \text{ and } Ee^{\lambda(\tau_2 - \tau_1)} < \infty\}.$$

We then have $s_d \geq \frac{1}{3}\log((d+1)/3) + \frac{1}{3} - \frac{1}{d+1}$.

Proof. Since we are only concerned with the levels of $\mathbb{T}_d$, we can consider our random walk as a Markov chain on $\mathbb{N}_0$ starting at 0, with transition probabilities $p_{0,1} = 1$, $p_{j,j+1} = d/(d+1)$ and $p_{j,j-1} = 1/(d+1)$ for $j \in \mathbb{N}$. We claim that

$$\{\tau_1 = k\} \subset \{\text{RW took at least } \lfloor k/3 \rfloor \text{ steps backwards by time } k\}.$$

Observe that $|S_{\tau_1}| \leq \lfloor \tau_1/3 \rfloor$. This is because for every $m \in \mathbb{N}_0$ such that $m < |S_{\tau_1}|$, the random walk must visit m at least twice. Now suppose by contradiction the random walk took less than $\lfloor k/3 \rfloor$ steps backwards, which means the random walk took more than $k - \lfloor k/3 \rfloor$ steps forward. This implies $|S_{\tau_1}| > \lfloor k/3 \rfloor$, which is a contradiction. Applying the Chernoff inequality to the binomial random variable Z with parameters $k \in \mathbb{N}_0$ and $1/(d+1)$, it follows that for all $x \geq k/(d+1)$

$$P(Z > x) \leq \exp\left(-x \log\left(\frac{x}{k/(d+1)}\right) + x - \frac{k}{d+1}\right).$$

By our inclusion and the above inequality, we have

$$P(\tau_1 = k) \le P(Z \ge k/3) \le \exp\left(-\frac{k}{3}\log((d+1)/3) + \frac{k}{3} - \frac{k}{d+1}\right).$$

Hence when $\lambda < \frac{1}{3}\log((d+1)/3) + \frac{1}{3} - \frac{1}{d+1}$, which is strictly positive when $d > 2$, we have $Ee^{\lambda\tau_1} < \infty$. We are left to bound $P(\tau_2 - \tau_1 = k)$. Rerunning the proof of Lemma 4.3 in [7], we conclude that there exists a constant $C > 0$ such that for every k we have $P(\tau_2 - \tau_1 = k) \le C \cdot P(\tau_1 = k)$. This finishes the proof. □

3.2 Concentration inequalities

We introduce the various local time statistics of the simple random walk on $\mathbb{T}_d$ used throughout the remainder of this paper. The one we are most interested in is the size of the level sets of the local time

$$L_n(t) = |\{v \in V : \ell_n(v) > t\}|.$$

Let $\mathcal{R}_n = \{S_0, \dots, S_{n-1}\}$ be the range of the random walk at time n. Setting $\lambda_d = s_d/2$, we define

$$\mathcal{L}^{(n)} = \mathcal{L} = \left\{v \in \mathcal{R}_n : \ell_n(v) < \frac{4}{\lambda_d}\log n\right\}$$

to be the set of vertices with small local time. Another statistic that appears throughout the proof is

$$\mathscr{L}_n = \sum_{v \in \mathcal{L}^c} \ell_n(v).$$

For $q \in \mathbb{N}$, denote the q-fold self-intersection local time by

$$L_{n,q} = \left(\sum_{v \in V} \ell_n^q(v)\right)^{1/q}.$$

The q-fold self-intersection local times often appear in the study of RWRS because they quantify the number of times the random walk visits the same sites, see [3] for a discussion for the case $\mathbb{Z}^d$. Throughout the paper, we will frequently use the fact that

$$L_{n,q}^q \ge n. \tag{3.1}$$

Lastly, denote the maximum of the local times by

$$L_{n,\infty} = \max_{v \in \mathcal{R}_n} \ell_n(v).$$

We will denote $\theta_k = \tau_k - \tau_{k-1}$ for $k \ge 1$ and $\theta_1 = \tau_1$ to be the regeneration epochs. Our main ingredient for deriving concentration inequalities for the local time will be the existence of regeneration epochs, and that they have exponential moments.

Lemma 3.2. *Suppose $u \ge 1$ and $t \ge 1$. There exists a constant $M > 0$ independent of u and t such that for $\beta \in (0, \lambda_d/2]$ we have*

$$E\exp\left(\beta\sum_{k=1}^{n}\theta_k \cdot \mathbb{1}\{\theta_k > t\}\right) \le \exp(Mn\exp(-\beta t/2)), \tag{3.2}$$

which implies

$$P(L_n(t) \ge u) \le \exp(Mn\exp(-\beta t/2) - \beta t u). \tag{3.3}$$

Proof. We begin by making a few observations. Firstly, at the kth regeneration epoch θ_k, there are at most $\lfloor \theta_k/t \rfloor$ vertices v satisfying $\ell_n(v) > t$. Secondly, any visited site is visited during a single regeneration epoch. Since by time n there are at most n regeneration times, we get

$$L_n(t) \le \sum_{k=1}^{n} \lfloor \theta_k/t \rfloor \cdot \mathbb{1}\{\theta_k > t\}.$$

Combining this inequality with the Chebyshev inequality, we have for any $\beta \in (0, \lambda_d]$

$$\begin{aligned} P(L_n(t) \ge u) &\le P\left(\sum_{k=1}^{n} \theta_k \cdot \mathbb{1}\{\theta_k > t\} \ge ut \right) \\ &\le \exp(-\beta u t) \prod_{k=1}^{n} E \exp\left(\beta \theta_k \cdot \mathbb{1}\{\theta_k > t\}\right), \end{aligned}$$

where the last inequality uses the fact that the regeneration epochs are independent. We are left to bound the exponential moment. Again by the Chebyshev inequality, we have $P(\theta_k > t) \le M \exp(-\beta t)$ for some positive constant M. Assuming $\beta \in (0, \lambda_d/2]$, we have

$$\begin{aligned} \prod_{k=1}^{n} E \exp\left(\beta \theta_k \cdot \mathbb{1}\{\theta_k > t\}\right) &\le \prod_{k=1}^{n} \left(E[\exp(\beta\theta_k)\, \mathbb{1}\{\theta_k > t\}] + 1 \right) \\ &\le \prod_{k=1}^{n} \left((E \exp(2\beta\theta_k))^{1/2} P(\theta_k > t)^{1/2} + 1 \right) \\ &\le \exp(Mn \exp(-\beta t/2)). \end{aligned}$$

□

Lemma 3.3. *There exists an $M > 0$ such that for any $u \ge 1$*

$$P\left(\mathscr{L}_n \ge u\right) \le M \exp(-\lambda_d u/2).$$

Proof. We have the inequality $\mathscr{L}_n \le \sum_{k=1}^{n} \theta_k \cdot \mathbb{1}\{\theta_k > \frac{4}{\lambda_d} \log n\}$. The Chebyshev inequality combined with an application of (3.2) finishes the proof. □

Lemma 3.4. *Suppose $x > 1$. Then there exists a constant $c_1 > 0$ independent of n and x such that*

$$P(L_{n,\infty} \ge x) \le n \exp(-c_1(x-1)).$$

Proof. For $v \in V$, define $T_v^+ = \inf\{n \ge 1 : S_n = v\}$ to be the return time of v and let $p_v = P_v(T_v^+ = +\infty)$ be the escape probability starting at v. Observe that the probability of escaping from o is the same as the probability of escaping from v conditioned on the event of $\{|S_1| > |v|\}$. Therefore,

$$p_o = P_v(T_v^+ = +\infty || S_1| > |v|) = \frac{P_v(T_v^+ = +\infty, |S_1| > |v|)}{P_v(|S_1| > |v|)} \le \frac{P_v(T_v^+ = +\infty)}{P_v(|S_1| > |v|)} = \frac{d+1}{d} p_v.$$

We now have the uniform lower bound $p_v \ge p_o d/(d+1)$. By the strong Markov property, we have

$$P_v(\ell_n(v) \ge x) \le (1 - p_v)^{x-1} \le \left(1 - \frac{d}{d+1} p_o\right)^{x-1}.$$

Using the same proof as Lemma 18 in [9], we have

$$\begin{aligned} P(L_{n,\infty} \geq x) \leq \sum_{v \in V} P(\ell_n(v) \geq x) &= \sum_{v \in V} \sum_{k=1}^{n} P(T_v^+ = k) P_v(\ell_{n-k}(v) \geq x) \\ &\leq \sum_{v \in V} P_v(\ell_n(v) \geq x) \sum_{k=1}^{n} P(T_v^+ = k) \\ &\leq \left(1 - \frac{d}{d+1} p_o\right)^{x-1} \sum_{v \in V} P(T_v^+ \leq n). \end{aligned}$$

We finish by observing $\sum_{v \in V} P(T_v^+ \leq n) \leq \sum_{v \in V} \sum_{k=1}^{n} P(S_k = v) = n$. □

Lemma 3.5. *Fix an integer $q \geq 2$. There exist positive constants B_q and c_q such that*

$$\limsup_{n \to \infty} n^{-1/q} \log P\left(L_{n,q}^q \geq B_q n\right) \leq -c_q.$$

Proof. Fix $B_q > 0$ and define the events $E = \{L_{n,q}^q \geq B_q n\}$ and $F = \{L_{n,\infty} > n^{1/q}\}$. By Lemma 3.4, we are left to bound $P(E \cap F^c)$, which we will do by bounding the probability that the level sets of the local time are large. Define the sets

$$\mathcal{D}_k = \left\{v \in V : 2^{k-1} < \ell_n(v) \leq 2^k\right\}$$

for $k = 0, \ldots, \lceil \log_2(n^{1/q}) \rceil = K_2$, and define the events

$$\begin{aligned} D_k &= \left\{|\mathcal{D}_k| > \frac{8Mn}{\lambda_d} \frac{e^{-\lambda_d 2^{k-1}/8}}{2^{k-1}}\right\} && \text{for } k = 0, \ldots, K_1 = \lfloor \log_2(\frac{8}{\lambda_d} \log n) \rfloor \\ D_k &= \left\{|\mathcal{D}_k| > \frac{n^{1/q}}{2^{k-1}}\right\} && \text{for } k = K_1 + 1, \ldots, K_2. \end{aligned}$$

For $k = 1, \ldots, K_1$, we apply (3.3) for $\beta = \lambda_d/4$ and get

$$P(D_k) \leq P\left(L_n(2^{k-1}) \geq \frac{8Mn}{\lambda_d} \frac{e^{-\lambda_d 2^{k-1}/8}}{2^{k-1}}\right) \leq \exp\left(-Mne^{-\lambda_d 2^{k-1}/8}\right) \leq \exp(-Mn^{1/2}).$$

For $v \in \mathcal{D}_k$ for $k = K_1 + 1, \ldots, K_2$ we have $\ell_n(v) \geq 4 \log n/\lambda_d$, so we can apply (3.3) with parameter $\beta = \lambda_d/2$ to get

$$P(D_k) \leq P\left(L_n(2^{k-1}) \geq \frac{n^{1/q}}{2^{k-1}}\right) \leq \exp\left(M - \frac{\lambda_d}{2} n^{1/q}\right).$$

By the union bound it follows that there exists $c_q > 0$ independent of n such that

$$\limsup_{n \to \infty} n^{-1/q} \log P\left(\cup_{k=0}^{K_2} D_k\right) \leq -c_q.$$

It is left to show $E \cap F^c \subset \left(\cup_{k=0}^{K_2} D_k\right) \cap F^c$. On the event F^c, we have $V = \cup_{k=0}^{K_2} \mathcal{D}_k$. Therefore, on the event $\left(\cup_{k=0}^{K_2} D_k\right)^c \cap F^c$, we get

$$L_{n,q}^q \leq \sum_{k=0}^{K_2} 2^{qk} |\mathcal{D}_k| \leq \frac{8Mn}{\lambda_d} \sum_{k=0}^{K_1} 2^{(q-1)k+1} e^{-\lambda_d 2^{k-1}/8} + n^{1/q} \sum_{k=K_1+1}^{K_2} 2^{(q-1)k+1} \leq B_q n$$

for some constant B_q. □

4 Proof of the upper bound (2.1)

Our strategy will be to decompose T_n/V_n into summands according to the size of the local times and scenery values. The probability that T_n/V_n is large will be rewritten as the probability that each summand is large. The probability that the summand over small local time and scenery value is large essentially reduces to the regime of bounded i.i.d. random variables, see Lemma 4.1. The probability that the summands over large local time and large scenery value is large will be bounded by the events $\mathscr{L}_n$ and $|\mathcal{E}^c|$ are large, see definitions below.

Without loss of generality, assume $\mathbb{E}\xi^2 = 1$. Before we continue, we introduce notation for the rest of the proof. Define the sets

$$\mathcal{L}^{(n)} = \mathcal{L} = \left\{ v \in \mathcal{R}_n : \ell_n(v) < \frac{4}{\lambda_d} \log n \right\}$$

and

$$\mathcal{E}^{(n)} = \mathcal{E} = \left\{ v \in \mathcal{R}_n : \xi(v) < \frac{\sqrt{n}}{y_n \log^2 n} \right\},$$

as well as the partial sums

$$\begin{aligned} T_{n,1} &= \sum_{v \in \mathcal{L} \cap \mathcal{E}} \ell_n(v)\xi(v), & V_{n,1}^2 &= \sum_{v \in \mathcal{L} \cap \mathcal{E}} \ell_n(v)\xi^2(v), \\ T_{n,2} &= \sum_{v \in \mathcal{L} \cap \mathcal{E}^c} \ell_n(v)\xi(v), & V_{n,2}^2 &= \sum_{v \in \mathcal{L} \cap \mathcal{E}^c} \ell_n(v)\xi^2(v) \\ T_{n,3} &= \sum_{v \in \mathcal{L}^c} \ell_n(v)\xi(v), & V_{n,3}^2 &= \sum_{v \in \mathcal{L}^c} \ell_n(v)\xi^2(v), \end{aligned}$$

so that $T_n = T_{n,1} + T_{n,2} + T_{n,3}$ and $V_n^2 = V_{n,1}^2 + V_{n,2}^2 + V_{n,3}^2$. Before we prove the upper bound, we need a few auxiliary results.

Lemma 4.1. *Suppose $1 \ll y_n \ll n^{1/6}$ and $\mathbb{E}\xi^4 < \infty$. We then have*

$$\limsup_{n \to \infty} y_n^{-2} \log \mathbb{P} \otimes P \left(\frac{T_{n,1}\sqrt{n}}{V_{n,1} L_{n,2}} \geq y_n \right) \leq -\frac{1}{2}.$$

Lemma 4.2. *Suppose that $\mathbb{E}|\xi^m| < \infty$ for some fixed $m \in \mathbb{N}$. For $x > 0$, we have*

$$\mathbb{P} \otimes P(|\mathcal{E}^c| \geq x) \leq \left(e\mathbb{E}|\xi^m| \frac{y_n^m \log^{2m}(n)}{x n^{m/2-1}} \right)^x.$$

Proof of (2.1). Fix $\epsilon \in (0,1)$. We have

$$\begin{aligned} \mathbb{P} \otimes P \left(\frac{T_n\sqrt{n}}{V_n L_{n,2}} \geq y_n \right) &\leq \mathbb{P} \otimes P \left(\frac{T_{n,1}\sqrt{n}}{V_{n,1} L_{n,2}} \geq \epsilon y_n \right) \\ &\quad + \mathbb{P} \otimes P \left(\frac{T_{n,2}\sqrt{n}}{V_{n,2} L_{n,2}} \geq (1-\epsilon) y_n/2 \right) \\ &\quad + \mathbb{P} \otimes P \left(\frac{T_{n,3}\sqrt{n}}{V_{n,3} L_{n,2}} \geq (1-\epsilon) y_n/2 \right) \\ &=: P_1 + P_2 + P_3. \end{aligned}$$

To bound P_1, we simply apply Lemma 4.1

$$P_1 = \mathbb{P} \otimes P \left(\frac{T_{n,1}\sqrt{n}}{V_{n,1} L_{n,2}} \geq \epsilon y_n \right) \leq \exp\left(-\frac{\epsilon^2}{2} y_n^2 (1 - o(1)) \right).$$

To bound P_2, we observe that by (3.1) and the Cauchy-Schwartz inequality,

$$\frac{T_{n,2}\sqrt{n}}{V_{n,2}L_{n,2}} \le \frac{T_{n,2}}{V_{n,2}} = \frac{\sum_{v\in\mathcal{L}\cap\mathcal{E}^c}\ell_n(v)\xi(v)}{\sqrt{\sum_{v\in\mathcal{L}\cap\mathcal{E}^c}\ell_n(v)\xi^2(v)}} \le \sqrt{\sum_{v\in\mathcal{L}\cap\mathcal{E}^c}\ell_n(v)} \le \sqrt{\frac{4}{\lambda_d}\log n|\mathcal{E}^c|}.$$

Combining the above inequality and Lemma 4.2 yields

$$\begin{aligned}
P_2 = \mathbb{P}\left(\frac{T_{n,2}\sqrt{n}}{V_{n,2}L_{n,2}} \ge (1-\epsilon)y_n/2\right) &\le \mathbb{P}\left(|\mathcal{E}^c| \ge (1-\epsilon)^2\frac{\lambda_d}{16}\frac{y_n^2}{\log n}\right) \\
&\le \left(\frac{e\mathbb{E}\xi^4}{(1-\epsilon)^2\lambda_d/16}\frac{y_n^2\log^9 n}{n}\right)^{(1-\epsilon)^2\frac{\lambda_d}{16}\frac{y_n^2}{\log n}} \\
&= \exp\left(-(1-\epsilon)^2\frac{\lambda_d}{16}y_n^2\left(\frac{\log(n/y_n^2)}{\log n} - o(1)\right)\right) \\
&\le \exp\left(-(1-\epsilon)^2\frac{\lambda_d}{24}y_n^2(1-o(1))\right).
\end{aligned}$$

To bound P_3, we again use (3.1) and the Cauchy-Schwartz inequality

$$\frac{T_{n,3}\sqrt{n}}{V_{n,3}L_{n,2}} \le \frac{T_{n,3}}{V_{n,3}} = \frac{\sum_{v\in\mathcal{L}^c}\ell_n(v)\xi(v)}{\sqrt{\sum_{v\in\mathcal{L}^c}\ell_n(v)\xi^2(v)}} \le \sqrt{\sum_{v\in\mathcal{L}^c}\ell_n(v)} = \sqrt{\mathscr{L}_n}.$$

Applying this as well as Lemma 3.3 gives us

$$P_3 = P\left(\frac{T_{n,3}\sqrt{n}}{V_{n,3}L_{n,2}} \ge (1-\epsilon)y_n/2\right) \le P\left(\mathscr{L}_n \ge (1-\epsilon)^2y_n^2/4\right) \le M\exp\left(-(1-\epsilon)^2\frac{\lambda_d}{8}y_n^2\right).$$

We conclude that for $d \ge 2$

$$\limsup_{n\to\infty} y_n^{-2}\log\mathbb{P}\otimes P\left(\frac{T_n\sqrt{n}}{V_nL_{n,2}} \ge y_n\right) < -\min\left\{\frac{\epsilon^2}{2}, (1-\epsilon)^2\frac{\lambda_d}{24}, (1-\epsilon)^2\frac{\lambda_d}{8}\right\}.$$

Recall from Lemma 3.1 that $\lambda_d \uparrow \infty$ as $d \to \infty$. Hence for d large enough we can let $\epsilon = 1 - (24/\lambda_d)^{1/2}$, and get

$$\limsup_{n\to\infty} y_n^{-2}\log\mathbb{P}\otimes P\left(\frac{T_n\sqrt{n}}{V_nL_{n,2}} \ge y_n\right) \le -\frac{1}{2}(1-(24/\lambda_d)^{1/2})^2.$$

Letting $c_d = (1-(24/\lambda_d)^{1/2})^2$ finishes the proof. □

We are left to prove Lemmas 4.1 and 4.2. The first lemma is a self-normalized moderate deviation result; since our sum is over vertices with small local time, the proof is very similar to the i.i.d. regime. The proof of the second lemma is a straightforward Chernoff inequality.

Proof of Lemma 4.1. Recall the local time statistics $\mathscr{L}_n = \sum_{v\in\mathcal{L}^c}\ell_n(v)$, $L_{n,2}^2 = \sum_{v\in V}\ell_n^2(v)$ and $L_{n,3}^3 = \sum_{v\in V}\ell_n^3(v)$. We begin by defining the atypical event

$$A = \left\{L_{n,2}^2 > B_2n\right\} \cup \left\{L_{n,3}^3 > B_3n\right\} \cup \{\mathscr{L}_n \ge y_n^2\log n\}.$$

Since $y_n = o(n^{1/6})$, by Lemmas 3.5 and 3.3 we have

$$\limsup_{n\to\infty} y_n^{-2}\log P(A) = -\infty.$$

Hence we are left to bound

$$\mathbb{P}\otimes P\left(\left\{\frac{T_{n,1}\sqrt{n}}{V_{n,1}L_{n,2}}\geq y_n\right\}\cap A^c\right).$$

Fix $\delta\in(0,1)$. We decompose our probability with respect to the size of $V_{n,1}$:

$$\begin{aligned}&\mathbb{P}\otimes P\left(\left\{\frac{T_{n,1}\sqrt{n}}{V_{n,1}L_{n,2}}\geq y_n\right\}\cap A^c\right)\\&\quad\leq \mathbb{P}\otimes P\left(\{V_{n,1}^2\leq \delta n\}\cap A^c\right)+\mathbb{P}\otimes P\left(\left\{\frac{T_{n,1}}{L_{n,2}}\geq \delta^{1/2}y_n\right\}\cap A^c\right)\\&\quad=: I_1+I_2.\end{aligned}$$

We first show that I_1 is negligible. By the Chebyshev inequality, for any $\kappa>0$ we have

$$\begin{aligned}\mathbb{P}(V_{n,1}^2<\delta n)&\leq \exp(\kappa\delta n)\mathbb{E}[\exp(-\kappa V_{n,1}^2)]\\&=\exp(\kappa\delta n)\prod_{v\in\mathcal{L}}\mathbb{E}[\exp(-\kappa\ell_n(v)\xi^2(v)\mathbb{1}\{v\in\mathcal{E}\})].\end{aligned}$$

By monotone convergence, we have $\mathbb{E}[\xi^2(v)\mathbb{1}\{v\in\mathcal{E}\}]=1-o(1)$. Using this, and that $e^{-x}\leq 1-x+x^2/2$ for $x\geq 0$, we have

$$\begin{aligned}\mathbb{E}[\exp(-\kappa\ell_n(v)\xi^2(v)\mathbb{1}\{v\in\mathcal{E}\})]&\leq \mathbb{E}[1-\kappa\ell_n(v)\xi^2(v)\mathbb{1}_{v\in\mathcal{E}}+\kappa^2\ell_n^2(v)\xi^4(v)\mathbb{1}\{v\in\mathcal{E}\}/2]\\&\leq 1-\kappa\ell_n(v)(1-o(1))+\mathbb{E}[\xi^4]\kappa^2\ell_n^2(v)/2\\&\leq \exp(-\kappa\ell_n(v)(1-o(1))+\mathbb{E}[\xi^4]\kappa^2\ell_n^2(v)/2).\end{aligned}$$

On the event A^c, we have $\sum_{v\in\mathcal{L}}\ell_n(v)\geq n-y_n^2\log n$. We then get

$$\begin{aligned}I_1&\leq \exp(\kappa\delta n)E\left[\exp\left(-\kappa\sum_{v\in\mathcal{L}}\ell_n(v)(1-o(1))+\mathbb{E}[\xi^4]\kappa^2\sum_{v\in\mathcal{L}}\ell_n^2(v)/2\right)\mathbb{1}_{A^c}\right]\\&\leq \exp(-\kappa(1-\delta)n(1-o(1))+B_2\mathbb{E}[\xi^4]\kappa^2 n/2).\end{aligned}$$

Optimizing over κ gives the bound

$$I_1\leq\exp\left(-\frac{(1-\delta)^2}{2B_2\mathbb{E}[\xi^4]}n(1-o(1))\right),$$

and since $y_n=o(n^{1/6})$, we have for ever $\delta\in(0,1)$

$$\limsup_{n\to\infty}y_n^{-2}\log I_1=-\infty.$$

The rest of the proof is left to bound I_2. Applying Chebyshev's inequality with $\delta^{1/2}y_n$, we have

$$\begin{aligned}I_2&\leq\exp(-\delta y_n^2)\cdot\mathbb{E}\otimes E\left[\exp(\delta^{1/2}y_nT_{n,1}/L_{n,2})\mathbb{1}_{A^c}\right]\\&=\exp(-\delta y_n^2)\cdot E\left[\prod_{v\in\mathcal{L}}\mathbb{E}\left[\exp\left(\delta^{1/2}y_n\frac{\ell_n(v)}{L_{n,2}}\xi(v)\mathbb{1}\{v\in\mathcal{E}\}\right)\right]\mathbb{1}_{A^c}\right].\end{aligned}$$

Since $L_{n,2}\geq n^{1/2}$, for $v\in\mathcal{L}$ we have

$$\delta^{1/2}y_n\frac{\ell_n(v)}{L_{n,2}}\xi(v)\mathbb{1}\{v\in\mathcal{E}\}\leq\frac{4\delta^{1/2}}{\lambda_d\log n}.$$

There exists a constant $C > 0$ such that for any $x \leq 1$,

$$e^x \leq 1 + x + x^2/2 + C|x|^3.$$

By this, and that $\mathbb{E}[\xi(v)\mathbb{1}\{v \in \mathcal{E}\}] \leq 0$, we have for large enough n

$$\begin{aligned} I_2 &\leq \exp(-\delta y_n^2) E\left[\prod_{v\in\mathcal{L}} \left(1 + \frac{\delta}{2}\frac{\ell_n^2(v)}{L_{n,2}^2} y_n^2 + O(1)\frac{\ell_n^3(v)}{L_{n,2}^3} y_n^3\right) \mathbb{1}_{A^c}\right] \\ &\leq \exp(-\delta y_n^2) E\left[\prod_{v\in\mathcal{L}} \exp\left(\frac{\delta}{2}\frac{\ell_n^2(v)}{L_{n,2}^2} y_n^2 + O(1)\frac{\ell_n^3(v)}{L_{n,2}^3} y_n^3\right) \mathbb{1}_{A^c}\right] \\ &\leq \exp(-\delta y_n^2/2) E\left[\exp\left(O(1)\frac{L_{n,3}^3}{L_{n,2}^3} y_n^3\right) \mathbb{1}_{A^c}\right]. \end{aligned}$$

Since $L_{n,3}^3 \asymp n$ and $L_{n,2}^2 \asymp n$ on the event A^c, as well as that $y_n = o(n^{1/6})$, we have the bound

$$\limsup_{n\to\infty} y_n^{-2} \log I_2 \leq -\frac{\delta}{2}.$$

Taking $\delta \uparrow 1$ finishes the proof. □

Proof of Lemma 4.2. Observe that

$$\left\{\mathbb{1}\left\{|\xi(v)| > \frac{\sqrt{n}}{y_n \log^2 n}\right\} : v \in \mathcal{R}_n\right\}$$

are i.i.d. random variables with respect to $\mathbb{P}$, and that by Markov's inequality we have

$$\mathbb{P}\left(|\xi(v)| \geq \frac{\sqrt{n}}{y_n \log^2 n}\right) \leq \mathbb{E}|\xi^m| \frac{y_n^m \log^{2m}(n)}{n^{m/2}}.$$

Applying the Chernoff inequality to the binomial random variable Z with parameters $n \in \mathbb{N}_0$ and $p \in [0,1]$, it follows that for all $x > 0$

$$P(Z > x) \leq \left(\frac{enp}{x}\right)^x.$$

Using this, and that $|\mathcal{R}_n| \leq n$ P-a.s, we have

$$\begin{aligned} \mathbb{P}\otimes P\left(|\mathcal{E}^c| \geq x\right) &= E\left[\mathbb{P}\left(\sum_{v\in\mathcal{R}_n} \mathbb{1}\left\{|\xi(v)| > \frac{\sqrt{n}}{y_n \log^2 n}\right\} > x\right)\right] \\ &\leq E\left[\left(\frac{e|\mathcal{R}_n|\mathbb{P}\left(|\xi(v)| \geq \frac{\sqrt{n}}{y_n \log^2 n}\right)}{x}\right)^x\right] \\ &\leq \left(\frac{en\mathbb{P}\left(|\xi(v)| \geq \frac{\sqrt{n}}{y_n \log^2 n}\right)}{x}\right)^x \\ &\leq \left(e\mathbb{E}|\xi^m| \frac{y_n^m \log^{2m}(n)}{x n^{m/2-1}}\right)^x. \end{aligned}$$

□

5 Proof of the lower bound (2.2)

The following proof is a straightforward application of the techniques used in the proof of Theorem 2.2 in [8]. Without loss of generality, assume $\mathbb{E}\xi^2 = 1$. Let x, y and b be positive numbers. By the the Cauchy-Schwartz inequality we have

$$xy \le \frac{1}{2}\left(\frac{x^2}{b} + y^2 b\right).$$

Letting $b = y_n L_{n,2}/n$, $x = y_n L_{n,2}/\sqrt{n}$ and $y = V_n$, we get

$$\begin{aligned}
&\mathbb{P}\otimes P\left(T_n \ge V_n \frac{y_n L_{n,2}}{\sqrt{n}}\right)\\
&\quad\ge \mathbb{P}\otimes P\left(T_n \ge \frac{1}{2b}\left(b^2 V_n^2 + y_n^2 \frac{L_{n,2}^2}{n}\right)\right)\\
&\quad= \mathbb{P}\otimes P\left(\sum_{v\in V}\ell_n(v)(2b\xi(v) - b^2\xi^2(v)) \ge y_n^2\frac{L_{n,2}^2}{n}\right)\\
&\quad\ge E\left[\mathbb{P}\left(\sum_{v\in V}\ell_n(v)(2b\xi(v) - b^2\xi^2(v)) \ge y_n^2\frac{L_{n,2}^2}{n}\right)\mathbb{1}\{L_{n,2}^2 \le B_2 n, L_{n,3}^3 \le B_3 n\}\right].
\end{aligned}$$

We are left to bound the inner probability, for which we will use Theorem 2 from [13]. Assume the random walk is fixed such that

$$L_{n,2}^2 \le B_2 n \qquad \text{and} \qquad L_{n,3}^3 \le B_3 n. \tag{5.1}$$

Defining $\eta(v) = 2b\xi(v) - b^2\xi^2(v)$, we have

$$\mathbb{P}\left(\sum_{v\in V}\ell_n(v)(2b\xi(v) - b^2\xi^2(v)) \ge y_n^2\frac{L_{n,2}^2}{n}\right) = \mathbb{P}\left(\sum_{v\in V}\ell_n(v)(\eta(v) - \mathbb{E}\eta(v)) \ge 2y_n^2\frac{L_{n,2}^2}{n}\right).$$

Define

$$\begin{aligned}
M_n^2 &= \sum_{v\in V}\ell_n^2(v)\mathbb{E}(\eta(v) - \mathbb{E}\eta(v))^2, & \Gamma_n &= \sum_{v\in V}\ell_n^3(v)\mathbb{E}|\eta(v) - \mathbb{E}\eta(v)|^3,\\
Q_n &= \frac{\Gamma_n}{M_n^3}, & x &= \frac{2y_n^2 L_{n,2}^2}{nM_n}.
\end{aligned}$$

Since $L_{n,q}^q \ge n$ for $q = 2$ and $q = 3$, by (5.1) we have $L_{n,2}^2 \asymp n$ and $L_{n,3}^3 \asymp n$. We thus get

$$M_n^2 = L_{n,2}^2(4b^2 - 4b^3\mathbb{E}\xi^3 + b^4(\mathbb{E}\xi^4 - 1)) \asymp y_n^2 \qquad \text{and} \qquad \Gamma_n \asymp L_{n,3}^3 b^3 \asymp \frac{y_n^3}{n^{1/2}},$$

which in turn implies

$$Q_n \asymp \frac{1}{n^{1/2}} \qquad \text{and} \qquad x \asymp y_n \ll n^{1/2} \asymp Q_n^{-1}.$$

It now follows from Theorem 2 in [13] that there exists positive constants c_1 and c_2 independent of n such that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{v\in V}\ell_n(v)(\eta(v) - \mathbb{E}\eta(v)) \ge \frac{2y_n^2 L_{n,2}^2}{n}\right)\\
&\quad\ge \left(1 - \Phi\left(\frac{2y_n^2 L_{n,2}^2}{nM_n}\right)\right)(1 - c_1 Q_n x)\exp\left(-c_2 Q_n x^3\right)\\
&\quad= \left(1 - \Phi\left(\frac{2y_n^2 L_{n,2}^2}{nM_n}\right)\right)(1 - o(1))\exp\left(-o(1)\right),
\end{aligned}$$

since $Q_n x^3 \asymp y_n^3/n^{1/2}$ and $y_n = o(n^{1/6})$. Since $L_{n,2} \asymp n^{1/2}$ and $y_n = o(n^{1/4})$, we have

$$M_n = 2\frac{L_{n,2}^2 y_n}{n}(1 - o(1)).$$

By Lemma 3.5 and that $y_n = o(n^{1/4})$, we get $P\left(\{L_{n,2}^2 \le B_2 n\} \cap \{L_{n,3}^3 \le B_3 n\}\right) \sim 1$. Putting everything together, we have

$$\begin{aligned}\mathbb{P} \otimes P\left(\frac{T_n\sqrt{n}}{V_n L_{n,2}} \ge y_n\right) &\ge (1 - o(1))(1 - \Phi(y_n))P\left(\{L_{n,2}^2 \le B_2 n\} \cap \{L_{n,3}^3 \le B_3 n\}\right) \\ &\sim 1 - \Phi(y_n).\end{aligned}$$

This finishes the proof.

6 Local time for $\mathbb{Z}^d$

We review the necessary concentration inequalities required for the proof of Theorem 2.2. The following inequality was provided in Proposition 3.3 in [4].

Lemma 6.1. *Define $L_n(t) = |\{z \in \mathbb{Z}^d : \ell_n(z) > t\}|$. There exists positive constants c_1, c_2, c_3 such that for $t > c_1 \log n$ and $u \ge 1$, we have*

$$P(L_n(t) > u) \le c_3 \exp(-c_2 \cdot t u^{1-2/d}).$$

Based on this last lemma, we easily get the following estimate which will be needed in the proof of Theorem 2.2.

Lemma 6.2. *Suppose $y_n^2 \gg \log n$ and let $t_* = y_n^{4/(d+2)}(\log n)^{d/(d+2)}$. Then there exists a positive constant C_1 such that*

$$\limsup_{n\to\infty} y_n^{-2d/(d+2)}(\log n)^{-2/(d+2)} P\left(\sum_{z:\ell_n(z)>t_*} \ell_n(z) \ge y_n^2\right) \le -C_1.$$

Proof. Define the sets

$$\mathcal{D}_k = \{z \in \mathbb{Z}^d : 2^k t_* < \ell_n(z) \le 2^{k+1} t_*\}$$

for $k = 0, \dots, K$, where K satisfies $2^{K+1} t_* = y_n^{2d/(d+2)}(\log n)^{2/(d+2)}$. Let $a_k = \epsilon \cdot y_n^2 2^{-2k/(d-2)}$, where $\epsilon > 0$ is chosen such that $\sum_{k=0}^{\infty} a_k \le y_n^2$. We have

$$P\left(\sum_{z:\ell_n(z)>t_*} \ell_n(z) \ge y_n^2\right) \le \sum_{k=0}^{K} P\left(\sum_{z\in\mathcal{D}_k} \ell_n(z) \ge a_k\right) + P\left(L_{n,\infty} > 2^{K+1} t_*\right).$$

Reviewing the proof of Lemma 3.4, we see that the conclusion holds for any graph G that satisfies $\inf_{v\in V} P_v(T_v^+ = \infty) > 0$. Since this applies to the graph $\mathbb{Z}^d$ for $d \ge 3$, we can bound the second term on the right-hand side by Lemma 3.4. The first term on the right-hand side is bounded by

$$\begin{aligned}\sum_{k=0}^{K} P\left(|\mathcal{D}_k| \ge \frac{a_k}{2^{k+1} t_*}\right) &\le \sum_{k=0}^{K} \exp\left(-\left(\frac{a_k}{2}\right)^{1-2/d} (2^k t_*)^{2/d}\right) \\ &\le (K+1)\exp(-C_2 y_n^{2d/(d+2)}(\log n)^{2/(d+2)}),\end{aligned}$$

where the first inequality follows from Lemma 6.1 and the fact that $t_* \gg \log n$ since we assume $y_n^2 \gg \log n$. We finish by observing that combinatorial factor is negligible since $K = O(\log y_n)$. □

As in the proof of (2.1), we will need large deviations for the self-intersection local time $L_{n,2}^2 = \sum_{z\in\mathbb{Z}^d} \ell_n^2(z)$. The following result follows from Proposition 1.1 and Remark 1.3 in [1], as well as Proposition 1.5 in [3].

Lemma 6.3. *Let $\{S_n : n \in \mathbb{N}_0\}$ be the simple random walk on $\mathbb{Z}^d$ and suppose $d \geq 3$. For $y > 2G(0) - 1$, we have*

$$\limsup_{n\to\infty} n^{-1/3} \log P(L_{n,2}^2 \geq y \cdot n) < 0.$$

7 Proof of Theorem 2.2

We begin by defining the sets

$$\mathcal{E}^{(n)} = \mathcal{E} = \left\{ z \in \mathcal{R}_n : \xi(z) \leq \frac{\sqrt{n}}{y_n \log^2 n} \right\}$$

and

$$\mathcal{L}^{(n)} = \mathcal{L} = \left\{ z \in \mathbb{Z}^d : \ell_n(z) \leq y_n^{4/(d+2)} (\log n)^{d/(d+2)} \right\}.$$

We define the partial sums

$$\begin{aligned}
T_{n,1} &= \sum_{z\in\mathcal{L}\cap\mathcal{E}} \ell_n(z)\xi(z), & V_{n,1}^2 &= \sum_{z\in\mathcal{L}\cap\mathcal{E}} \ell_n(z)\xi^2(z),\\
T_{n,2} &= \sum_{z\in\mathcal{L}\cap\mathcal{E}^c} \ell_n(z)\xi(z), & V_{n,2}^2 &= \sum_{z\in\mathcal{L}\cap\mathcal{E}^c} \ell_n(z)\xi^2(z)\\
T_{n,3} &= \sum_{z\in\mathcal{L}^c} \ell_n(z)\xi(z), & V_{n,3}^2 &= \sum_{z\in\mathcal{L}^c} \ell_n(z)\xi^2(z).
\end{aligned}$$

Applying (3.1), we have

$$\begin{aligned}
&\mathbb{P}\otimes P\left(T_n\sqrt{n}/(V_n L_{n,2}) \geq y_n\right) \leq \mathbb{P}\otimes P\left(T_n/V_n \geq y_n\right)\\
&\leq \mathbb{P}\otimes P(T_{n,1}/V_n \geq y_n/3) + \mathbb{P}\otimes P(T_{n,2}/V_n \geq y_n/3) + \mathbb{P}\otimes P(T_{n,3}/V_n \geq y_n/3).
\end{aligned}$$

As in the proof of (2.1), we apply the Cauchy-Schwartz inequality and get that this bounded by

$$\begin{aligned}
&\mathbb{P}\otimes P(T_{n,1}/V_n \geq y_n/3) + \mathbb{P}\otimes P\left(|\mathcal{E}^c| \geq y_n^{2d/(d+2)} \log n^{-d/(d+2)}/9\right)\\
&\qquad + P\left(\sum_{z\in\mathcal{L}^c} \ell_n(z) \geq y_n^2/9\right).
\end{aligned}$$

Reviewing the proof of Lemma 4.2, we see that the conclusion holds for any random walk that satisfies $|\mathcal{R}_n| \leq n$. Since this holds for the simple random walk on $\mathbb{Z}^d$, we can apply Lemma 4.2 with the assumption $\mathbb{E}\xi^4 < \infty$, to get

$$\limsup_{n\to\infty} y_n^{-2d/(d+2)} (\log n)^{-2/(d+2)} \log \mathbb{P}\otimes P(|\mathcal{E}^c| \geq y_n^{2d/(d+2)} \log n^{-d/(d+2)}/9) < 0.$$

By Lemma 6.2, we have

$$\limsup_{n\to\infty} y_n^{-2d/(d+2)} (\log n)^{-2/(d+2)} \log P\left(\sum_{z\in\mathcal{L}^c} \ell_n(z) \geq y_n^2/9\right) < 0.$$

The rest of the proof is nearly identical to the proof of Lemma 4.1. For $B > 2G(0) - 1$, we have

$$\begin{aligned}\mathbb{P} \otimes P\left(T_{n,1}/V_n \geq y_n\right) &\leq \mathbb{P} \otimes P\left(\{V_n^2 \leq n/4\} \cap \{L_{n,2}^2 \leq Bn\}\right) \\ &\quad + \mathbb{P} \otimes P\left(\{T_{n,1}/n^{1/2} \geq y_n/2\} \cap \{L_{n,2}^2 \leq Bn\}\right) + P(L_{n,2}^2 > Bn) \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

By Lemma 6.3 and that $y_n = o(n^{1/6})$, we have

$$\limsup_{n\to\infty} y_n^{-2d/(d+2)} (\log n)^{-2/(d+2)} \log I_3 = -\infty,$$

and by the same proof as in Lemma 4.1, we have

$$\limsup_{n\to\infty} y_n^{-2d/(d+2)} (\log n)^{-2/(d+2)} \log I_1 = -\infty,$$

and so we are left to bound I_2. By Chebyshev's inequality, for any $\alpha_n > 0$ we have

$$I_2 \leq \exp(-y_n\alpha_n/2) \cdot E\left[\prod_{z\in\mathcal{L}} \mathbb{E}\left[\exp\left(\ell_n(z)\xi(z)\frac{\alpha_n}{n^{1/2}}\mathbb{1}\{z \in \mathcal{E}\}\right)\right] \mathbb{1}\{L_{n,2}^2 \leq B_2 n\}\right].$$

We set $\alpha_n = y_n^{(d-2)/(d+2)} (\log n)^{2/(d+2)}$, and observe that there exists $C > 0$ such that for $z \in \mathcal{L}$, we have

$$\ell_n(z)\xi(z)\frac{\alpha_n}{n^{1/2}}\mathbb{1}\{z \in \mathcal{E}\} \leq C/\log n.$$

There exists a constant $C > 0$ such that for $x \leq 1$ we have $e^x \leq 1 + x + Cx^2$. Combining both facts yields the inequality

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\ell_n(z)\xi(z)\frac{\alpha_n}{n^{1/2}}\mathbb{1}\{z \in \mathcal{E}\}\right)\right] \\ &\quad \leq \mathbb{E}\left[1 + \ell_n(z)\xi(z)\frac{\alpha_n}{n^{1/2}}\mathbb{1}\{z \in \mathcal{E}\} + C\frac{\ell_n^2(z)}{n}\xi^2(z)\alpha_n^2\mathbb{1}\{z \in \mathcal{E}\}\right] \\ &\quad \leq \exp\left(C\frac{\ell_n^2(z)}{n}\alpha_n^2\right).\end{aligned}$$

We thus have

$$I_2 \leq \exp(-y_n^{2d/(d+2)} (\log n)^{2/(d+2)}/2 + CB_2 y_n^{(2d-4)/(d+2)} (\log n)^{4/(d+2)}).$$

Observe that $y_n^{2d/(d+2)} (\log n)^{2/(d+2)} \gg y_n^{(2d-4)/(d+2)} (\log n)^{4/(d+2)}$ precisely when $y_n^2 \gg \log n$, which completes the proof.

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