

ON THE REAL DAVIES’ CONJECTURE

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We show that every matrix $A \in \mathbb{R}^{n \times n}$ is, at least, $\delta \|A\|$ -close to a *real* matrix $A + E \in \mathbb{R}^{n \times n}$ whose eigenvectors have condition number, at most, $\tilde{O}_n(\delta^{-1})$. In fact, we prove that, with high probability, taking E to be a sufficiently small multiple of an i.i.d. *real* sub-Gaussian matrix of bounded density suffices. This essentially confirms a speculation of Davies and of Banks, Kulkarni, Mukherjee and Srivastava, who recently proved such a result for i.i.d. *complex Gaussian* matrices.

Along the way we also prove nonasymptotic estimates on the minimum possible distance between any two eigenvalues of a random matrix whose entries have arbitrary means; this part of our paper may be of independent interest.

1. Introduction. Recall that a matrix $A \in \mathbb{C}^{n \times n}$ is said to be *diagonalizable* if there exists a diagonal matrix $D \in \mathbb{C}^{n \times n}$ and an invertible matrix $W \in \mathbb{C}^{n \times n}$ such that $A = WDW^{-1}$. Recall also that A is said to be *normal* if $AA^\dagger = A^\dagger A$, where A^\dagger denotes the Hermitian adjoint of A . It is not hard to see that A is normal if and only if it is unitarily diagonalizable, that is, if and only if $A = WDW^{-1}$ for some W satisfying $W^\dagger = W^{-1}$. Since $W^\dagger = W^{-1}$ implies that $\|W\| \|W^{-1}\| = 1$ (here, $\|\cdot\|$ denotes the standard $\ell^2 \rightarrow \ell^2$ operator norm of a matrix), a natural way of quantifying “how far” a matrix is from being normal (equivalently, “how far” it is from being unitarily diagonalizable) is via the *eigenvector condition number* of A ,

$$\kappa_V(A) := \inf_{W: A = WDW^{-1}} \|W\| \|W^{-1}\|.$$

1.1. *Davies’ conjecture.* Any analytic function $f(A)$ of a diagonalizable matrix $A = WDW^{-1}$ may be written as $f(A) = Wf(D)W^{-1}$. Note that, since D is diagonal, $f(D)$ is the diagonal matrix with nontrivial entries given by $f(D)_{ii} = f(D_{ii})$. However, computationally, this may not be an appropriate way of evaluating $f(A)$ when A is “highly nonnormal,” that is, it has a large eigenvector condition number; the reason for this is that, even if all computations are carried to precision ϵ , the overall result may be off by an error of order $\kappa_V(A)\epsilon$. In [4], Davies suggested a way around this obstacle, namely, find a small perturbation $A + E$ of A such that $\kappa_V(A + E)$ is small and compute $f(A + E)$ as a substitute for $f(A)$. In order to quantify the error associated to such a scheme, he introduced a quantity called the *accuracy of approximate diagonalization* defined (for matrices $A \in \mathbb{C}^{n \times n}$ such that $\|A\| \leq 1$) by

$$\underline{\sigma}(A, \epsilon) := \inf_{E \in \mathbb{C}^{n \times n}} (\kappa_V(A + E)\epsilon + \|E\|).$$

Davies conjectured that, for every positive integer n , there exists c_n such that, for all $A \in \mathbb{C}^{n \times n}$ with $\|A\| \leq 1$ and $\epsilon > 0$,

$$(1.1) \quad \underline{\sigma}(A, \epsilon) \leq c_n \sqrt{\epsilon}.$$

Received May 2020; revised March 2021.

MSC2020 subject classifications. Primary 15A12; secondary 60B20.

Key words and phrases. Davies’ conjecture, eigenvector condition number, eigenvalue gaps, pseudospectrum.

While Davies was able to confirm his conjecture for a number of special choices of the matrix A , for general $A \in \mathbb{C}^{n \times n}$, he was only able to obtain the following much weaker bound:

$$(1.2) \quad \underline{\sigma}(A, \varepsilon) \leq (1+n)\varepsilon^{2/(n+1)}.$$

Recently, Davies' conjecture was resolved by Banks, Kulkarni, Mukherjee and Srivastava [1] in a stronger form. They showed that, for every $A \in \mathbb{C}^{n \times n}$ and every $\delta \in (0, 1)$, there exists $E \in \mathbb{C}^{n \times n}$ such that $\|E\| \leq \delta\|A\|$ and

$$(1.3) \quad \kappa_V(A + E) \leq 4n^{3/2}(1 + \delta^{-1}).$$

(1.1) follows easily from this, for instance, by taking $\delta = 2n^{3/4}\sqrt{\varepsilon}$. In fact, [1] showed that choosing E from the distribution $G_n(\mathcal{N}_{\mathbb{C}}(0, 1))$, by which we mean an $n \times n$ random matrix, each of whose entries is an independent copy of a complex Gaussian variable with mean 0 and variance $1/n$, succeeds with high probability.

Banks et al. asked whether similar results continue to hold if one replaces the complex Gaussian perturbations with a different class of random perturbations, noting that their proof relied on special properties of the complex Gaussian distribution. As a warm-up to our main result, we show that one may indeed replace the standard complex distribution with mean 0 and variance $1/n$ by $n^{-1/2}\xi$ for any sub-Gaussian complex random variable ξ with bounded density. This is the content of Proposition 3.1 in Section 3.

1.2. The real Davies' conjecture. Much more challenging is the question of whether one can replace the complex Gaussian distribution by a real distribution, for instance, the real Gaussian distribution with entries of mean 0 and variance $1/n$. This is perhaps most interesting when the matrix A is itself a real matrix. While experimental evidence by Davies suggested that this should be possible, Banks et al. adopted a more cautious position, writing that *a proof (or disproof) remains to be found*.

The key technical challenge in working with real perturbations is that the eigenvalues/eigenvectors of a real matrix are still, in general, complex. A crucial step in [1] uses the fact that the probability of a standard complex Gaussian lying in any ball of radius $\varepsilon > 0$ is $O(\varepsilon^2)$. The exponent 2 in this bound, being the same as the real dimension of the complex plane, allows their argument to go through. However, as noted in [1], Remark 3.4, the analogous argument for real Gaussians completely fails, since the probability of a standard real Gaussian lying in a ball of radius ε centered around the origin in the complex plane is $\Theta(\varepsilon)$. We discuss this in more detail in Section 3.1.

Nevertheless, as our main result we show that real perturbations suffice to regularize the eigenvector condition number (at least when the initial matrix A is real).

THEOREM 1.1. *There is an absolute constant $C > 0$ such that the following holds. Suppose $\delta \in (0, 1/2)$ and $A \in \mathbb{R}^{n \times n}$. Then, there is a matrix $E \in \mathbb{R}^{n \times n}$ with $\|E\| \leq \delta\|A\|$ such that*

$$\kappa_V(A + E) \leq Cn^2\delta^{-1}\sqrt{\log(n\delta^{-1})}.$$

REMARK. As before, this implies Davies' conjecture (1.1), up to an overall factor of $(\log(1/\varepsilon))^{1/4}$.

REMARK. As in [1], our proof actually shows that choosing E to be a suitable rescaling of a real random matrix, each of whose entries is an independent copy of a sub-Gaussian real random variable ξ of bounded density succeeds with high probability. For the precise statement, see Theorem 5.2.

We note that, prior to our work, *no result* on the regularization of the eigenvector condition number of general matrices by a *real* translation seems to be known. Indeed, even the much weaker bound (1.2) relies on a theorem of Friedland [6] which crucially requires working with general complex matrices.

1.3. *Nonasymptotic bounds on eigenvalue gaps.* One of the steps in our proof of Theorem 1.1 utilizes and extends quite recent tools in the nonasymptotic theory of random matrices, most notably those involved in Ge's [7] resolution of the following seemingly innocuous (but long-standing) problem show that, with high probability, all the eigenvalues of a random matrix, each of whose entries is independently ± 1 with equal probability, are distinct. While the (easier) Hermitian analog of this had been known (see [16] and follow-up work), a key challenge in resolving the non-Hermitian case is the same phenomenon we encounter—non-Hermitian matrices, in general, have complex eigenvalues, whereas the probability for a real random variable to lie in a disc of radius ε in the complex plane can very well be $\Omega(\varepsilon)$ (as opposed to the “desired” bound of $O(\varepsilon^2)$).

Building on ideas introduced by Rudelson and Vershynin in [15], Ge found an ingenious way to circumvent this issue, and the ideas in [7] will also play an important role for us (however, note that, by themselves these ideas, unfortunately, still do not suffice to derive the main eigenvector condition number regularization result; see the discussion in Section 3.1). Indeed, a part of our argument involves the derivation of nonasymptotic bounds on the size of the eigenvalue gap (the minimum distance between any two eigenvalues) of a random matrix, much like the main result in [7]. However, unlike in [7], where the entries of the random matrix under consideration are centered, the entries of our random matrices can be *arbitrarily uncentered*. This presents an obstacle in the strategy of [7], which relies on control of the largest singular value (i.e., operator norm) of the matrix (even when the entries are continuous with bounded density, as will be the case here).

However, we show that by exploiting control on the *smallest* singular value of the matrix (which, crucially, is unaffected by the mean profile), we can derive a bound on the eigenvalue gap of arbitrarily uncentered random matrices. Whereas the eigenvalues of uncentered, nonnormal random matrices have attracted much attention (see, for instance, [5, 8] and the references therein), to our knowledge our result is the first to obtain *any* nonasymptotic bound on the eigenvalue gap for unrestricted mean profiles. This result may be of independent interest.

THEOREM 1.2 (Informal, see Proposition 4.3 for a precise version). *Let ξ be a sub-Gaussian real random variable with bounded density, and let G denote an $n \times n$ random matrix, each of whose entries is an independent copy of ξ . Then, for any $A \in \mathbb{R}^{n \times n}$ (such that $\|A\| \geq 1$) and $s \leq 1$,*

$$\mathbb{P}\left[\min_{i \neq j} |\lambda_i(A + G) - \lambda_j(A + G)| \leq s\right] = \tilde{O}(s \cdot \|A\|^4 n^{11/2} + c^n),$$

where $\lambda_1(A + G), \dots, \lambda_n(A + G)$ denote the eigenvalues of $A + G$ and $c \in (0, 1)$ is a constant depending only on ξ .

We remark that, in the case of sub-Gaussian complex random variables with bounded density, we obtain a better bound. In particular, for all mean profiles $A \in \mathbb{C}^{n \times n}$ with norm larger than a small polynomial, our bound improves on a recent result of Banks et al. [2] (which was proved only for complex Gaussians); see Theorem B.2 in Appendix B for details.

1.4. *Organization.* The rest of this paper is organized as follows. In Section 2 we introduce some preliminary notions; in Section 3 we present a generalization of the main result in [1] to general complex random variables; in Section 3.1 we present a more detailed discussion of the difficulties encountered in handling real perturbations as well as a brief overview of our proof. Section 4 presents the formal statements of the three key estimates required for our proof, and Section 5 shows how to deduce Theorem 1.1 from these estimates, using a bootstrapping argument. Finally, Section 6 and Section 7 contain the proofs of these three key estimates. We also include three appendices: Appendix A contains a standard reduction of the problem of bounding the smallest two singular values of a random matrix, Appendix B discusses our improved bounds on the eigenvalue gaps of complex random matrices and Appendix C reproduces some details from [1], needed to prove Theorem 1.1 for the reader's convenience.

1.5. *Concurrent and independent work.* Right before uploading our manuscript to the arXiv, we learned of concurrent and independent work of Banks, Garza-Vargas, Kulkarni and Srivastava [3] with similar main results as ours. Both of our works make use of techniques in [1, 7]. However, beyond this similarity the works are substantially different—in particular, [3] works with the limiting expression in Lemma 2.2 (this requires significant additional ideas, such as developing a version of the restricted invertibility property), whereas our work completely avoids this via a novel bootstrapping scheme. [3] also works with the first moment of the *eigenvalue overlaps* on the real line (i.e., $\sum_{\lambda_i \in \mathbb{R}} \kappa(\lambda_i)$), whereas we work exclusively with the (conditional) second moment of *all* the eigenvalue overlaps (i.e., the conditional expectation of $\sum_{\lambda_i \in \mathbb{C}} \kappa^2(\lambda_i)$) (see [3] for further discussion):

- Compared to [3], our Theorem 1.1 obtains the near optimal dependence of $\delta^{-1} \log(\delta^{-1})$ (indeed, [1] proved a lower bound of $\delta^{-1+1/n}$, even using deterministic complex perturbations, instead of just random real perturbations), whereas [3] obtains a dependence of $\delta^{-3/2}$. For the real Davies' conjecture (1.1), our bound leads to a resolution (up to log-factors), whereas [3] obtain the weaker dependence of $\varepsilon^{2/5}$.
- For quantitative estimates on the eigenvalue gaps, our Theorem 1.2 and Proposition 4.3 have a better rate of $\tilde{O}(s)$ (i.e., near linear in the gap size), whereas [3] achieve a weaker rate of $\tilde{O}(s^{1/3})$.
- Compared to our work, [3] derive logarithmically better quantitative estimates on the first moment of the eigenvalue overlaps on the real line. They also derive nonasymptotic estimates with improved rates for the lower tails of intermediate singular values of random matrices with arbitrary mean profiles; this subject is not considered in our work.
- Finally, we remark that, since we do not work with the limiting expression in Lemma 2.2, our techniques are likely to shed light on [3], Problem 7.1; we intend to return to this in future work.

2. Preliminaries.

2.1. *Random variables.* A (complex) random variable X is said to be *sub-Gaussian* if there exist constants $c, C > 0$ such that, for all $t \geq 0$,

$$\mathbb{P}[|X| > t] \leq C e^{-ct^2}.$$

The collection of sub-Gaussian random variables on an underlying probability space form a normed space under the *sub-Gaussian norm*

$$\|X\|_{\psi_2} := \inf\{t > 0 : \mathbb{E}[e^{(X/t)^2} - 1] \leq 1\}.$$

For more on sub-Gaussian random variables, we refer the reader to [19]. Throughout this paper we will assume that ξ is a sub-Gaussian random variable. In Section 3 it will be complex-valued and admit a density function that is continuous and bounded by some constant K . Otherwise, it will be real-valued, having a density function that is continuous and bounded by some constant K .

Finally, we define $G_n(\xi)$ to be the random matrix such that $n^{1/2}G_n(\xi)$ is distributed as an $n \times n$ matrix, each of whose entries is an independent copy of ξ .

2.2. Hyperplane distances. We will use $\text{dist}(X, H)$ to denote the distance of a vector X to a hyperplane H . Note that we will often be considering the distance between complex vectors X and complex hyperplanes H , so, more formally, the distance is defined as

$$\text{dist}(X, H) := \|\text{proj}_{H^\perp} X\|_2,$$

where H^\perp is the orthogonal complement of H in \mathbb{R}^n or \mathbb{C}^n and proj is the usual Euclidean projection. In particular, if Z has unit length and is normal to H , we have

$$\text{dist}(X, H) \geq |\langle X, Z \rangle|$$

with equality when H has codimension 1 (and, more generally, when $\text{proj}_{H^\perp} X$ and Z are scalar multiples).

2.3. Condition numbers and the pseudospectrum. Given any $n \times n$ invertible matrix M , we define its *condition number* to be $\kappa(M) := \sigma_1(M)/\sigma_n(M) = \|M\| \|M^{-1}\|$, where $\sigma_1(M) \geq \dots \geq \sigma_n(M)$ are the singular values of M , that is, the eigenvalues of $(M^\dagger M)^{1/2}$ (\dagger denotes the conjugate transpose), and $\|M\| = \sigma_1(M)$ is the standard $\ell^2 \rightarrow \ell^2$ operator norm of M .

Given a diagonalizable matrix M , we define its *eigenvector condition number* to be

$$\kappa_V(M) := \inf_{W: M = WDW^{-1}} \|W\| \|W^{-1}\|.$$

Furthermore, suppose M has distinct eigenvalues, and consider its spectral expansion

$$M := \sum_{i=1}^n \lambda_i v_i w_i^\dagger = VDV^{-1},$$

where the right and left eigenvectors v_i and w_i^\dagger are the columns and rows of V and V^{-1} , respectively, normalized so that $w_i^\dagger v_i = 1$. Then, we define the *eigenvalue condition number* of λ_i by

$$\kappa(\lambda_i, M) := \|v_i w_i^\dagger\| = \|v_i\|_2 \|w_i\|_2.$$

We will also make use of the quantity

$$\kappa_2(M) := \sqrt{\sum_{i=1}^n \kappa(\lambda_i, M)^2}.$$

We will often suppress the dependence on M , writing, in particular, κ_2 and $\kappa(\lambda_i)$ when the underlying matrix is clear from context.

Next, for any $\varepsilon \geq 0$, we define the ε -*pseudospectrum* of an $n \times n$ matrix M to be

$$\Lambda_\varepsilon(M) := \{z \in \mathbb{C} : \sigma_n(M - zI) \leq \varepsilon\} = \{z \in \mathbb{C} : \|(M - zI)^{-1}\| \geq \varepsilon^{-1}\}.$$

Note, in particular, that $\Lambda_0(M)$ is the spectrum of M . For a comprehensive treatment of pseudospectra, we refer the reader to the book of Trefethen and Embree [18].

We will also need the following equivalent characterization of the ε -pseudospectrum (see [18] for a proof):

$$\Lambda_\varepsilon(M) = \{z \in \mathbb{C} : M + E \text{ has eigenvalue } z \text{ for some } \|E\| \leq \varepsilon\}.$$

Finally, denote the minimum eigenvalue gap of M by $\eta(M)$, that is,

$$\eta(M) := \min_{i \neq j} |\lambda_i - \lambda_j|,$$

where λ_i are the eigenvalues (counting multiplicity) of M .

We will need some estimates related to the eigenvector condition number and pseudospectral volume from [1].

LEMMA 2.1 ([1], Lemma 3.1). *Let M be an $n \times n$ matrix with distinct eigenvalues, and let V be the matrix whose columns are the eigenvectors of M normalized to have unit norm. Then,*

$$\kappa(V) \leq \sqrt{n \sum_{i=1}^n \kappa(\lambda_i)^2}.$$

REMARK. This implies that $\kappa_V(M) \leq \sqrt{n} \kappa_2(M)$.

LEMMA 2.2 ([1], Lemma 3.2). *Let M be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_n$, and let $B \in \mathbb{C}$ be an open set. Then,*

$$\lim_{\varepsilon \rightarrow 0} \frac{\text{vol}(\Lambda_\varepsilon(M) \cap B)}{\varepsilon^2} = \pi \sum_{\lambda_i \in B} \kappa(\lambda_i)^2.$$

For our treatment of real perturbations, we will also require the following quantitative version of the previous lemma.

LEMMA 2.3. *Let M be an $n \times n$ matrix with distinct eigenvalues λ_i . Furthermore, let*

$$0 < \varepsilon \leq \frac{\eta(M)}{2n\kappa_2(M)}.$$

Then,

$$\frac{\text{vol}(\Lambda_\varepsilon(M) \cap \mathcal{D}(0, 2\|M\|))}{\varepsilon^2} \geq \frac{\pi}{8} \kappa_2(M)^2.$$

REMARK. Recall that $\kappa_2(M)^2 = \sum_{i=1}^n \kappa(\lambda_i, M)^2$.

PROOF. First, note that, for all λ_i , we have $\mathcal{D}(\lambda_i, \kappa(\lambda_i)\varepsilon) \subseteq \mathcal{D}(0, 2\|M\|)$; this follows from the triangle inequality since $|\lambda_i| \leq \|M\|$ and $\varepsilon \leq \eta(M)/\kappa_2 \leq \|M\|/\kappa(\lambda_i)$.

Second, note that $\mathcal{D}(\lambda_i, \kappa(\lambda_i)\varepsilon)$ and $\mathcal{D}(\lambda_j, \kappa(\lambda_j)\varepsilon)$ are disjoint for $i \neq j$; indeed, $|\lambda_i - \lambda_j| \geq \eta(M) \geq 2\varepsilon\kappa_2 \geq (\kappa(\lambda_i) + \kappa(\lambda_j))\varepsilon$.

Third, by [18], equation (52.11), we have that, for $z \in \mathcal{D}(\lambda_j, \kappa(\lambda_j)\varepsilon)$,

$$\left\| (zI - M)^{-1} \right\| - \frac{\kappa(\lambda_j)}{|z - \lambda_j|} \leq \sum_{i \neq j} \frac{\kappa(\lambda_i)}{|z - \lambda_i|}.$$

Let $J := \{j \in [n] : \kappa(\lambda_j)^2 \geq \kappa_2^2/(2n)\}$, and observe that

$$\sum_{j \in J} \kappa(\lambda_j)^2 \geq \frac{1}{2} \kappa_2^2.$$

We claim that, for each $j \in J$,

$$\mathcal{D}(\lambda_j, \kappa(\lambda_j)\varepsilon/2) \subseteq \Lambda_\varepsilon(M) \cap \mathcal{D}(0, 2\|M\|),$$

and, moreover, that these discs are disjoint. Given this claim, we are done, since then,

$$\frac{\text{vol}(\Lambda_\varepsilon(M) \cap \mathcal{D}(0, 2\|M\|))}{\varepsilon^2} \geq \frac{\pi}{4} \sum_{j \in J} \kappa(\lambda_j)^2 \geq \frac{\pi}{8} \kappa_2^2.$$

We now prove the claim. The disjointness of the discs follows from the second point above, and, from the first point above, we know that $\mathcal{D}(\lambda_j, \kappa(\lambda_j)\varepsilon/2) \subseteq \mathcal{D}(0, 2\|M\|)$. Hence, it only remains to prove that $\mathcal{D}(\lambda_j, \kappa(\lambda_j)\varepsilon/2) \subseteq \Lambda_\varepsilon(M)$.

Fix some $j \in J$. If $z \in \mathcal{D}(\lambda_j, \kappa(\lambda_j)\varepsilon/2)$, then, in particular, we have $z \in \mathcal{D}(\lambda_j, |\lambda_i - \lambda_j|/(4n))$ for each $i \neq j$ so that

$$|z - \lambda_j| \leq \frac{|z - \lambda_i|}{4n - 1}.$$

Using this, the assumption that $j \in J$ and Cauchy–Schwarz, we have that

$$\begin{aligned} \sum_{i \neq j} \frac{\kappa(\lambda_i)}{|z - \lambda_i|} &\leq \left(\sum_{i=1}^n \kappa(\lambda_i)^2 \right)^{1/2} \left(\sum_{i \neq j} \frac{1}{|z - \lambda_i|^2} \right)^{1/2} \\ &\leq (2n\kappa(\lambda_j)^2)^{1/2} \left(\frac{(n-1)}{(4n-1)^2 |z - \lambda_j|^2} \right)^{1/2} \\ &\leq \frac{\kappa(\lambda_j)}{2|z - \lambda_j|}. \end{aligned}$$

Thus, by the third point above, we obtain

$$\|(zI - M)^{-1}\| \geq \frac{1}{2} \frac{\kappa(\lambda_j)}{|z - \lambda_j|} \geq \frac{1}{\varepsilon}$$

so that, by definition, $z \in \Lambda_\varepsilon(M)$, as desired. \square

3. Warm-up: General complex perturbations. The proof of Davies' conjecture in [1] shows that, with high probability, the addition of a suitably scaled copy of $G_n(\mathcal{N}_{\mathbb{C}}(0, 1))$, where $\mathcal{N}_{\mathbb{C}}(0, 1)$ is the standard complex Gaussian and regularizes the eigenvector condition number. Here, as a warm-up to the much more involved real case, we provide a brief outline of how to achieve such a regularization using $G_n(\xi)$, where ξ is any sub-Gaussian complex random variable with bounded density. More precisely, we will show the following.

PROPOSITION 3.1. *Let ξ be a sub-Gaussian complex random variable with bounded density. Given $A \in \mathbb{C}^{n \times n}$ with $\|A\| = 1$ and $\delta \in (0, 1)$, there are C_1, C_2 , depending only on ξ , such that we have*

$$\mathbb{P}[\kappa_V(A + \delta G_n) \leq C_1 n^2 \delta^{-1} \text{ and } \|\delta G_n\| \leq C_2 \delta] \geq 1/2$$

for $G_n = G_n(\xi)$.

REMARK. The constant C_2 depends only on the sub-Gaussian moment of ξ , and C_1 depends on the density bound and sub-Gaussian moment of ξ . The sub-Gaussian condition can easily be relaxed to a fourth moment assumption with no cost and to finiteness of the second moment by incurring additional polynomial factors.

Our outline is identical to the one in [1]; the only difference is that [1], Lemma 2.2, is replaced by Lemma 3.2, and [1], Lemma 3.3, is replaced by Lemma 3.4.

For the remainder of this section, we will write $G_n(\xi)$ simply as G_n . We begin with a standard bound on the operator norm of an i.i.d. matrix with sub-Gaussian entries.

LEMMA 3.2 (From [19], Theorem 4.6.1). *For a constant, K depending only on the sub-Gaussian moment of ξ ,*

$$\mathbb{P}[\sigma_1(G_n) \geq K + t] \leq \exp(-nt^2).$$

Next, we need a specialization of a lemma regarding the sum of multidimensional bounded density random variables from [13]. The result we require can also be derived from [10].

LEMMA 3.3 (From [13], Theorem 1.1). *Given real numbers a_1, \dots, a_n with $\sum_{i=1}^n a_i^2 = 1$ and mutually independent \mathbb{R}^2 -valued continuous random variables X_1, \dots, X_n with densities bounded by K , we have that*

$$a_1 X_1 + \dots + a_n X_n$$

is a random variable with density bounded by eK .

As mentioned above, we need an analogue of [1], Lemma 3.3, for general complex distributions with bounded density. This is a standard exercise in using the “invertibility-via-distance” approach (see, e.g., [17]).

LEMMA 3.4 (Small ball estimate for σ_n). *For any fixed $A \in \mathbb{C}^{n \times n}$ and for all $\epsilon \geq 0$,*

$$\mathbb{P}[\sigma_n(A + \delta G_n) \leq \epsilon] \leq \pi e K n^3 \frac{\epsilon^2}{\delta^2}.$$

PROOF. Let $M = A + \delta G_n$ with columns M_i for $1 \leq i \leq n$. Recall that $\sigma_n(M)$ admits the following variational characterization:

$$\sigma_n(M) = \inf_{x \in \mathbb{S}^{n-1}} \|Mx\|_2.$$

Define hyperplanes $H_i = \text{span}\{M_j : j \neq i\}$. The key point is that any vector $x \in \mathbb{S}^{n-1}$ has a coordinate, say x_1 , satisfying $x_1 \geq n^{-1/2}$, in which case

$$\|Mx\|_2 \geq \text{dist}(Mx, H_1) = \text{dist}(x_1 M_1, H_1) \geq n^{-1/2} \text{dist}(M_1, H_1).$$

Therefore, by the union bound,

$$\mathbb{P}[\sigma_n(M) \leq \epsilon] = \mathbb{P}\left[\inf_{x \in \mathbb{S}^{n-1}} \|Mx\|_2 \leq \epsilon\right] \leq \sum_{i=1}^n \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon n^{1/2}].$$

Now, for each $1 \leq i \leq n$,

$$\mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon n^{1/2}] = \mathbb{E}[\mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon n^{1/2} | H_i]] \leq \pi e K n^2 \frac{\epsilon^2}{\delta^2}.$$

We explain the final inequality: $\text{dist}(M_i, H_i) \geq |\langle M_i, N_i \rangle|$, where N_i is any unit vector orthogonal to H_i (in particular, N_i may be chosen measurably with respect to H_i). For such a measurable choice, M_i and N_i are independent after conditioning on H_i , and the random variable $\langle M_i, N_i \rangle$ may be written as $M_{i,1} e^{i\theta_1} |N_{i,1}| + \dots + M_{i,n} e^{i\theta_n} |N_{i,n}|$, where $|N_{i,1}|^2 + \dots + |N_{i,n}|^2 = 1$, and $M_{i,j} e^{i\theta_j}$ are independent random variables with densities bounded

by $Kn\delta^{-2}$. Therefore, by Lemma 3.3 the density of $\langle M_i, N_i \rangle$ is, at most, $eKn\delta^{-2}$ so that $\mathbb{P}[|\langle M_i, N_i \rangle| \leq \epsilon n^{1/2} |H_i|] \leq \pi eKn^2 \epsilon^2 \delta^{-2}$.

Finally, summing over the n events for $1 \leq i \leq n$ gives the desired conclusion. \square

We now deduce the analogue of [1], Theorem 1.5, for general complex random variables ξ ; the proof is identical modulo substituting Lemma 3.4 for [1], Lemma 3.3. We repeat it here in order to highlight several key difficulties in extending the proof to the real case.

LEMMA 3.5. *Suppose $A \in \mathbb{C}^{n \times n}$ with $\|A\| \leq 1$ and $\delta \in (0, 1)$. Let $\lambda_1, \dots, \lambda_n$ be the (random) eigenvalues of $A + \delta G_n$, where $G_n = G_n(\xi)$. Then, for any open set $\mathcal{B} \subseteq \mathbb{C}$,*

$$\mathbb{E} \sum_{\lambda_i \in \mathcal{B}} \kappa(\lambda_i)^2 \leq \frac{eKn^3}{\delta^2} \text{vol}(\mathcal{B}).$$

Note that the λ_i are distinct with probability 1 so that the left-hand side is indeed well defined.

PROOF. For any open set $\mathcal{B} \subseteq \mathbb{C}$, we have

$$\begin{aligned} \mathbb{E} \text{vol}(\Lambda_\epsilon(A + \delta G_n) \cap \mathcal{B}) &= \mathbb{E} \int_{\mathcal{B}} \mathbb{1}_{\{z \in \Lambda_\epsilon(A + \delta G_n)\}} dz \\ &= \int_{\mathcal{B}} \mathbb{P}[z \in \Lambda_\epsilon(A + \delta G_n)] dz \\ (3.1) \qquad &= \int_{\mathcal{B}} \mathbb{P}[\sigma_n(A + \delta G_n - zI) < \epsilon] dz \\ &\leq \pi eKn^3 \frac{\epsilon^2}{\delta^2} \text{vol}(\mathcal{B}) \end{aligned}$$

by Lemma 3.4.

Now, Lemma 2.2 yields

$$\mathbb{E} \sum_{\lambda_i \in \mathcal{B}} \kappa(\lambda_i)^2 \leq \liminf_{\epsilon \rightarrow 0} \frac{\mathbb{E} \text{vol}(\Lambda_\epsilon(A + \delta G_n) \cap \mathcal{B})}{\pi \epsilon^2} \leq \frac{eKn^3}{\delta^2} \text{vol}(\mathcal{B}). \quad \square$$

Given Lemmas 3.2 and 3.5, the deduction of Proposition 3.1 follows exactly as the deduction of [1], Theorem 1.1, given [1], Theorem 1.5, Lemma 2.2; in particular, Lemma 3.5 is only needed when \mathcal{B} is a disk of around constant radius. For the reader's convenience we reproduce the details in Appendix C. The various remarks regarding moment conditions, following Proposition 3.1, follow from standard versions of Lemma 3.2 under weaker moment assumptions

3.1. *Difficulties for real perturbations.* In order to prove that real perturbations suffice, we must overcome significant technical difficulties. As mentioned in the Introduction, the major issue is that the small ball estimate Lemma 3.4 is, in general, weakened from an ϵ^2 rate to an ϵ rate; morally, in the real case we can only study intervals of length ϵ , not disks of radius ϵ . However, since the method of [1] looks at the limiting volume of the pseudospectrum $\Lambda_\epsilon(A + \delta G_n)$, the ϵ^2 rate is required in a fundamental way, as we can see from the proof above.

In order to fix this issue, we generalize results of Ge [7] to random matrices with an arbitrary mean profile—as we will see, this provides a ϵ^2 bound at the additional cost of a factor of $|\text{Im} z|^{-1}$ in (3.1). Therefore, we will split the integral over the disc into two parts: within a strip of width ϵ around the real axis, we use the bound with rate ϵ , and, outside of it, we use our generalization of Ge's [7] results.

However, this still leads to an additional factor of $\log(\varepsilon^{-1})$, which blows up if we take the limit $\varepsilon \rightarrow 0$, as in the proof of Lemma 3.5 (which comes from Lemma 2.2). We may avoid taking the limit by finding an effective value of ε which still makes Lemma 2.2 hold; however, it turns out that such an effective value depends both on a priori estimates on $\eta(A + \delta G_n)$ (which can be obtained using Theorem 1.2) and, more seriously, on $\kappa_V(A + \delta G_n)$ which is precisely the quantity we are trying to control!

As mentioned in the **Introduction**, prior to our work no such estimate seems to be known, not even when G_n is replaced by a deterministic real matrix of bounded operator norm. We overcome this difficulty via a novel bootstrapping scheme which completely eliminates the need for any a priori bound on $\kappa_V(A + \delta G_n)$.

4. Overview of key estimates. For the remainder of the paper, we assume that ξ is a real sub-Gaussian random variable with density bounded by K and $G_n = G_n(\xi)$.

Our proof of Theorem 1.1 relies on three key estimates.

First, we require a bound on the lower tail of the smallest singular value of $G_n(\xi)$ which is invariant with respect to translations by arbitrary complex matrices.

PROPOSITION 4.1. *For any $A \in \mathbb{C}^{n \times n}$, we have*

$$\mathbb{P}[\sigma_n(A + \delta G_n) \leq \epsilon] \leq 2\sqrt{2e}Kn^2\frac{\epsilon}{\delta}.$$

In the case when $A \in \mathbb{R}^{n \times n}$, this essentially appears in [14] and also as [17], Observation 2. We provide the proof for general $A \in \mathbb{C}^{n \times n}$ in Section 6.

Next, we provide a generalization of the main singular value estimate in Ge [7] to the case when the entries of the random matrix have arbitrary means.

PROPOSITION 4.2. *Let $A \in \mathbb{R}^{n \times n}$ with $\|A\| \leq 1$ and $\delta \in (0, 1)$. Let $\mathcal{E}_{K'}$ denote the event that $\|G_n\| \leq K'$, and let $z \in \mathbb{C}$ such that $|z| \leq 3\delta K' + 3$. Finally, let $M = A + \delta G_n - zI$. Then,*

$$\mathbb{P}[\mathcal{E}_{K'} \cap \sigma_n(M) \leq \epsilon] \leq c_{4.2}(1 + \delta K')\frac{\max(K, K^2)n^3\epsilon^2}{\delta^2|\text{Im } z|},$$

where $c_{4.2} > 0$ is an absolute constant.

As in [7], the crucial part of this bound is the rate ϵ^2 (as opposed to just ϵ). The proof of the above proposition requires some essential departures from Ge’s proof which requires strong control over the the ratio $\|M\|/\|G_n\|$ (in particular, the proof in [7] breaks down for $\delta = o(\exp(-O(n)))$). We provide details in Section 6.

From an analogue of the above proposition for the smallest two singular values, we derive the following more precise version of Theorem 1.2; this is similar to the main result in [7] (but again, generalized to an arbitrary mean profile). We prove it in Section 7.

PROPOSITION 4.3. *Let $A \in \mathbb{R}^{n \times n}$ with $\|A\| \leq 1$, let $\delta \in (0, 1)$ and let $M = A + \delta G_n$. Let $\mathcal{E}_{K'}$ denote the event that $\|G_n\| \leq K'$. Then, for $s \leq 1$,*

$$\mathbb{P}[\mathcal{E}_{K'} \cap \eta(M) \leq s] \leq c_{4.3} \log_2(2(1 + \delta K')/s) \cdot (1 + \delta K')^2 \max(K^2, K^3)n^5\frac{s}{\delta^3},$$

where $c_{4.3} > 0$ is an absolute constant.

5. Deduction of Theorem 1.1. In this section, we show how to prove Theorem 1.1, given the estimates in the previous section.

We begin with an immediate corollary of Proposition 4.3.

LEMMA 5.1. *Let $A \in \mathbb{R}^{n \times n}$ with $\|A\| \leq 1$, and let $\delta \in (0, 1)$. Then,*

$$\mathbb{P}[\eta(A + \delta G_n) \geq c_{\xi, 5.1} n^{-6} \delta^4] \geq \frac{3}{4}$$

for some $c_{\xi, 5.1} > 0$, depending only on the density bound K and sub-Gaussian moment of ξ .

REMARK. The $n^{-6} \delta^4$ in the above can be replaced by $n^{-5} \delta^3 / \log(n\delta^{-1})$.

PROOF. Let $\mathcal{E}_{K'} = \{\|G_n\| \leq K'\}$, and choose K' large enough (depending on the sub-Gaussian moment of ξ) so that

$$\mathbb{P}[\mathcal{E}_{K'}] \geq 7/8.$$

This can be done by using Lemma 3.2.

Then, for an appropriate choice of $c_{\xi, 5.1}$, we obtain

$$\mathbb{P}[\mathcal{E}_{K'} \cap \eta(A + \delta G_n) \leq c_{\xi} n^{-6} \delta^4] \leq 1/8$$

by Proposition 4.3. The result follows by the union bound. \square

We are now ready to prove our main technical result which is an analogue of Proposition 3.1 for real perturbations.

THEOREM 5.2. *Let ξ be a sub-Gaussian real random variable with bounded density. Given $A \in \mathbb{R}^{n \times n}$ with $\|A\| \leq 1$ and $\delta \in (0, 1)$, there are C_1, C_2 , depending only on ξ , such that*

$$\mathbb{P}[\{\kappa_V(A + \delta G_n(\xi)) \leq C_1 n^2 \delta^{-1} \sqrt{\log(n\delta^{-1})}\} \cap \{\|\delta G_n\| \leq C_2 \delta\}] \geq 1/2.$$

PROOF. For an appropriate constant K' , depending only on ξ , we have by Lemma 3.2 that

$$\mathbb{P}[\|G_n\| \geq K'] \leq 1/12.$$

Hence, by Lemma 5.1 and the union bound we have that the event

$$\mathcal{E} = \{\eta(A + \delta G_n) \geq c_{\xi, 5.1} n^{-6} \delta^4\} \cap \{\|G_n\| \leq K'\}$$

has probability at least $2/3$.

Define

$$p_t = \mathbb{P}[\kappa_2(A + \delta G_n) \leq t | \mathcal{E}]$$

(recall that $\kappa_2^2 = \sum_{i=1}^n \kappa(\lambda_i)^2$) and

$$\mathcal{E}_t = \mathcal{E} \cap \{\kappa_2 \leq t\}.$$

Fix any real $t \geq 1$ with $p_t \geq 9/10$. We will show that for any such t , conditioned on the event \mathcal{E}_t , the expectation of κ_2^2 is significantly smaller than t (in fact, the dependence on t is logarithmic). By Markov's inequality we can then conclude that $p_{f(t)}$ holds with probability, at least, $p_t(1 - O(f(t)^{-1}))$, where $f(t) = \delta^{-2} \log t$ (up to logarithmic factors and factors of n ; see the precise definition below). In particular, this will allow us to show that, starting

from any (arbitrarily large) t_0 for which $p_{t_0} \geq 99/100$, we can keep iterating this process to find some $t = O(\delta^{-2})$ (suppressing log factors, and factors of n) such that $p_t \geq 9/10$ which suffices for our application.

To accomplish this, we begin by applying Lemma 2.3 to $M = A + \delta G_n$ (conditioned on \mathcal{E}_t) with $\varepsilon = \min(1, c_\xi, 5.1)n^{-6}\delta^4/(2nt)$. Note that, conditioned on \mathcal{E}_t , we indeed have that

$$0 < \varepsilon \leq \frac{\eta(M)}{2n\kappa_2(M)}$$

so that the application of Lemma 2.3 is valid. Then,

$$\begin{aligned} \mathbb{E}[\kappa_2^2|\mathcal{E}_t] &\leq \frac{8}{\pi\varepsilon^2}\mathbb{E}[\text{vol}(\Lambda_\varepsilon(A + \delta G_n) \cap \mathcal{D}(0, 2\|M\|))|\mathcal{E}_t] \\ &= \frac{8}{\pi\varepsilon^2}\mathbb{E}\left[\int_{\mathcal{D}(0, 2\|M\|)} \mathbb{1}_{\{z \in \Lambda_\varepsilon(A + \delta G_n)\}} dz|\mathcal{E}_t\right] \\ &\leq \frac{8}{\pi\varepsilon^2}\mathbb{E}\left[\int_{\mathcal{D}(0, 2+2\delta K')} \mathbb{1}_{\{z \in \Lambda_\varepsilon(A + \delta G_n)\}} dz|\mathcal{E}_t\right] \\ &= \frac{8}{\pi\varepsilon^2}\int_{\mathcal{D}(0, 2+2\delta K')} \mathbb{E}[\mathbb{1}_{\{z \in \Lambda_\varepsilon(A + \delta G_n)\}}|\mathcal{E}_t] dz \\ &= \frac{8}{\pi\varepsilon^2}\int_{\mathcal{D}(0, 2+2\delta K')} \mathbb{P}[\sigma_n(A + \delta G_n - zI) \leq \varepsilon|\mathcal{E}_t] dz \\ &\leq \frac{8}{\pi\varepsilon^2}\int_{\mathcal{D}(0, 2+2\delta K')} \min\left(\frac{10\sqrt{2}eKn^2\varepsilon}{3\delta}, c_{4.2}(1 + \delta K')\frac{5\max(K, K^2)n^3\varepsilon^2}{3\delta^2|\text{Im } z|}\right) dz. \end{aligned}$$

In the last line we used Propositions 4.1 and 4.2, along with $\mathbb{P}[\mathcal{E}_t] = \mathbb{P}[\mathcal{E}_t|\mathcal{E}]\mathbb{P}[\mathcal{E}] \geq (9/10) \cdot (2/3) = 3/5$.

Now, splitting the disk at $|\text{Im } z| = (1 + \delta K') \max(1, K)\varepsilon\delta^{-1}n$ and integrating, we obtain

$$\mathbb{E}[\kappa_2^2|\mathcal{E}_t] \lesssim_\xi (1 + \delta K')^2 \max(K, K^2)n^3\delta^{-2} + (1 + \delta K')^2 \max(K, K^2)n^3\delta^{-2} \log(\delta/(n\varepsilon)).$$

Note that, by our choice of ε , the first term is bounded by the second (up to an absolute constant factor). Thus,

$$\begin{aligned} (5.1) \quad \mathbb{E}[\kappa_2^2|\mathcal{E}_t] &\lesssim_\xi (1 + \delta K')^2 \max(K, K^2)n^3\delta^{-2} \log(\delta/(n\varepsilon)) \\ &\lesssim_\xi (1 + \delta K')^2 \max(K, K^2)n^3\delta^{-2} \log(tn/\delta). \end{aligned}$$

By Markov’s inequality, setting $f(t) = n^3\delta^{-2} \log(tn/\delta)$,

$$\mathbb{P}[\kappa_2 \geq f(t)|\mathcal{E}_t] \lesssim C_\xi(1 + \delta K')^2 \max(K, K^2) f(t)^{-1}.$$

Thus,

$$p_{f(t)} = \mathbb{P}[\kappa_2 \leq f(t)|\mathcal{E}_t]\mathbb{P}[\mathcal{E}_t] \geq p_t(1 - C'_\xi f(t)^{-1}),$$

for $C'_\xi = C_\xi(1 + \delta K')^2 \max(K, K^2)$ (which is of constant order since $\delta \in (0, 1)$).

Choose t_0 such that $p_{t_0} \geq 99/100$, and let $t_{i+1} = f(t_i)$ as long as

$$t_i \geq 8n^3\delta^{-2} \log(1 + n^4\delta^{-3}) \text{ and } f(t_i) \geq 2^{16}C'_\xi;$$

otherwise, terminate. In particular, if

$$(5.2) \quad t_i \geq 8n^3\delta^{-2} \log(1 + n^4\delta^{-3}) + \exp(2^{16}C'_\xi),$$

then the process continues for at least one more step.

We claim that, as long as the process does not terminate,

$$t_i \geq 2n^3\delta^{-2} \log(t_i n/\delta) = 2t_{i+1};$$

in particular, the process terminates in finite time. To see the claimed inequality, note that the function $g(x) = x - 2n^3\delta^{-2} \log(xn/\delta)$ is positive at $x = 8n^3\delta^{-2} \log(1 + n^4\delta^{-3})$, and has derivative $g'(x) = 1 - 2n^2\delta^{-1}x^{-1}$ which is also positive for $x \geq 8n^3\delta^{-2} \log(1 + n^4\delta^{-3})$.

Next, we claim that, at every step before termination, $p_{t_i} \geq 9/10$. Indeed, we know from the above discussion that if $p_{t_i} \geq 9/10$, then

$$p_{t_{i+1}} = p_{f(t_i)} \geq p_{t_i} (1 - C'_\xi t_{i+1}^{-1}).$$

Hence, if the process continues for another step, then

$$p_{t_{i+1}} \geq \frac{99}{100} \prod_{j=1}^{i+1} (1 - C'_\xi t_j^{-1}) \geq \frac{9}{10}$$

so that the lower bound on the probability is maintained throughout the process. The last inequality follows from

$$t_j \geq 2t_{j+1}$$

for $0 \leq j \leq i$ and $t_{i+1} \geq 2^{16}C'_\xi$ which, together, imply

$$\prod_{j=1}^{i+1} (1 - C'_\xi t_j^{-1}) \geq \prod_{j=0}^{\infty} (1 - 2^{-j-16}) \geq \frac{10}{11}.$$

This, together with (5.2), implies that there exists some $t \leq 8n^3\delta^{-2} \log(1 + n^4\delta^{-3}) + \exp(2^{16}C'_\xi)$ such that (5.1) holds for this value t .

Using Lemma 2.1 for such a choice of t , we obtain

$$\mathbb{E}[\kappa_V(A + \delta G_n)^2 | \mathcal{E}_t] \lesssim_\xi (1 + \delta K')^2 \max(K, K^2) n^4 \delta^{-2} \log(tn/\delta).$$

Another application of Markov's inequality gives

$$\mathbb{P}[\kappa_V(A + \delta G_n) \geq C''_\xi (1 + \delta K') \max(K^{1/2}, K) n^2 \delta^{-1} \sqrt{\log(tn/\delta)} | \mathcal{E}_t] \leq 1/6$$

for sufficiently large C''_ξ . Since $\mathbb{P}[\mathcal{E}_t] \geq 3/5$, we conclude that

$$\mathbb{P}[\|G_n\| \leq K' \cap \kappa_V(A + \delta G_n) \leq \sqrt{6}(1 + \delta K') \max(K^{1/2}, K) n^2 \delta^{-1} \sqrt{\log(tn/\delta)}] \geq 1/2.$$

The result now follows from substituting the upper bound on t corresponding to (5.2). \square

PROOF OF THEOREM 1.1. This follows immediately from Theorem 5.2. \square

6. Proof of Propositions 4.1 and 4.2.

6.1. *Invertibility via distance.* We begin with a (by now) standard reduction of the singular value estimates to estimates on the distance of a random vector to an appropriate subspace.

LEMMA 6.1. *Let M be any $n \times n$ random matrix, and fix any $\epsilon \geq 0$. Let M_i be the i th column vector of M and $H_i = \text{span}\{M_j : j \neq i\}$. Then, for any event \mathcal{E} ,*

$$\mathbb{P}[\sigma_n(M) \leq \epsilon \cap \mathcal{E}] \leq \sum_{i=1}^n \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon n^{1/2} \cap \mathcal{E}].$$

PROOF. The argument is identical to the first half of the proof of Lemma 3.4 but intersected with an arbitrary event. \square

LEMMA 6.2. *Let M be any $n \times n$ random matrix, and fix any $\epsilon_2 \geq \epsilon_1 \geq 0$. Let M_i, H_i be as in Lemma 6.1, and let $H_{i,j} = \text{span}\{M_k : k \neq i, j\}$. Then, for any event \mathcal{E} ,*

$$\begin{aligned} &\mathbb{P}[\{\sigma_n(M) \leq \epsilon_1\} \cap \{\sigma_{n-1}(M) \leq \epsilon_2\} \cap \mathcal{E}] \\ &\leq \sum_{i=1}^n \sum_{j \neq i} \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon_1 n^{1/2} \cap \text{dist}(M_j, H_{i,j}) \leq \epsilon_2 n^{1/2} \cap \mathcal{E}]. \end{aligned}$$

The proof of Lemma 6.2 is deferred to Appendix A; this is also completely standard.

6.2. *Proof of Proposition 4.1.* The proof of Proposition 4.1 is quite simple, given the following lemma of Livshyts, Paouris and Pivovarov [11].

LEMMA 6.3 (From [11], Theorem 1.1). *Consider independent \mathbb{R} -valued continuous random variables X_1, \dots, X_n with densities bounded by K , and let $X = (X_1, \dots, X_n)$. Fix an integer $\ell \in \{1, 2\}$. Then, for any $V \in \mathbb{R}^{\ell \times n}$, VX is an \mathbb{R}^ℓ -valued random variable with density bounded by*

$$\frac{e^{\ell/2} K^\ell}{\det(VV^T)^{1/2}}.$$

This has the following immediate corollary.

LEMMA 6.4. *For any $A \in \mathbb{C}^{n \times n}$, let $M = A + \delta G_n$. For all $j \in [n]$, if v_j is a unit normal to $H_j = \text{span}\{M_k : k \neq j\}$, then*

$$\mathbb{P}[|\langle M_j, v_j \rangle| \leq \epsilon |M_{-j}] \leq 2\sqrt{2e} K n^{1/2} \frac{\epsilon}{\delta}.$$

Here, M_{-j} denotes the set of all columns, excluding M_j .

PROOF. After possibly multiplying by an overall phase, we may assume that $v_j = x_j + i y_j$ satisfies $\|x_j\|_2 \geq 1/\sqrt{2}$. Note that multiplication by an overall phase does not affect the random variable $|\langle M_j, v_j \rangle|$. Then,

$$\begin{aligned} \mathbb{P}[\text{dist}(M_j, H_j) \leq \epsilon |M_{-j}] &\leq \mathbb{P}[|\langle M_j, v_j \rangle| \leq \epsilon |M_{-j}] \\ &= \mathbb{P}[|\langle A_j, v_j \rangle + \delta \langle (G_n)_j, v_j \rangle| \leq \epsilon |M_{-j}] \\ &\leq \mathbb{P}[|\text{Re}(\langle A_j, v_j \rangle) + \delta \langle (G_n)_j, x_j \rangle| \leq \epsilon |M_{-j}] \\ &\leq 2\sqrt{2e} K n^{1/2} \frac{\epsilon}{\delta}. \end{aligned}$$

For the last inequality, note that $\delta(G_n)_j$ is a vector of independent real random variables, each with density bounded by $K n^{1/2} \delta^{-1}$, so that applying Lemma 6.3 to this vector with $V = x_j^T$ and using $\|x_j\|_2 \geq 1/\sqrt{2}$ to control the term $\det(VV^T)$ gives the desired conclusion. \square

Now, we are ready to deduce Proposition 4.1.

PROOF OF PROPOSITION 4.1. Let $M = A + \delta G_n$ with columns M_j for $1 \leq j \leq n$. Define hyperplanes $H_j = \text{span}\{M_k : k \neq j\}$. Using Lemma 6.1, we have

$$\mathbb{P}[\sigma_n(M) \leq \epsilon] \leq \sum_{j=1}^n \mathbb{P}[\text{dist}(M_j, H_j) \leq \epsilon n^{1/2}].$$

For each $1 \leq j \leq n$, choose a unit normal vector v_j of H_j (independently of M_j). Then, Lemma 6.4 gives

$$\mathbb{P}[\text{dist}(M_j, H_j) \leq \epsilon n^{1/2} |M_{-j}] \leq 2\sqrt{2e}Kn \frac{\epsilon}{\delta}.$$

Since this probability is uniform over the realization of M_{-j} , we may remove the conditioning using the law of total probability to finish the proof. \square

6.3. *Proof of Proposition 4.2.* The following is the key step in the proof.

LEMMA 6.5. *Let $A \in \mathbb{R}^{n \times n}$ with $\|A\| \leq 1$ and $\delta \in (0, 1)$. Let $z \in \mathbb{C}$ such that $|z| \leq 3\delta K' + 3$ and $M = A + \delta G_n - zI$. Let $\mathcal{E}_{K'}$ denote the event that $\|G_n\| \leq K'$, let $H_j = \text{span}\{M_k : k \neq j\}$ and let v_j be any unit normal vector of H_j . Then,*

$$\mathbb{P}[\mathcal{E}_{K'} \cap |\langle M_j, v_j \rangle| \leq \epsilon |M_{-j}] \leq c_{6.5}(1 + \delta K') \frac{\max(K, K^2)n\epsilon^2}{\delta^2 |\text{Im } z|},$$

where $c_{6.5}$ is an absolute constant.

PROOF. Let $B := A + \delta G_n - \text{Re}(z)I$. For convenience of notation, let $\rho = \text{Im } z$. We split into cases based on $|\rho|$ vs. ϵ .

Case I: $|\rho| \leq 4\epsilon$. From Lemma 6.4, applied to $A - zI$, we find that

$$\mathbb{P}[|\langle M_j, v_j \rangle| \leq \epsilon |M_{-j}] \leq 2\sqrt{2e}Kn^{1/2} \frac{\epsilon}{\delta} \leq 2\sqrt{2e}Kn \frac{4\epsilon^2}{|\rho|\delta^2},$$

since $\delta < 1$. This finishes the proof in this case.

Case II: $|\rho| > 4\epsilon$. We initially show, restricted to $\mathcal{E}_{K'}$, that no $v := x + iy \in \mathbb{S}^{n-1}$ with $\|y\|_2 < |\rho|/(16\delta K' + 16)$ can satisfy

$$\|M^T v\|_2 < \epsilon.$$

Indeed, if this were true, then writing down the equation for the imaginary part of $M^T v$ and using $M^T = B^T - i\rho I$, where B, ρ are real, we get that

$$\|B^T y - \rho x\|_2 < \epsilon.$$

Thus, we would have

$$\frac{|\rho|}{2} < \|\rho x\|_2 < \epsilon + \|B^T y\|_2 < \epsilon + \frac{|\rho|}{4},$$

where the first inequality uses $\|y\|_2 \leq 1/6$ to deduce $\|x\|_2 \geq 1/2$, the second inequality is simply the triangle inequality and the final inequality uses that $\|B^T\| = \|B\| \leq 1 + K'\delta + 3(\delta K' + 1) \leq 4(\delta K' + 1)$. But this is a contradiction to $|\rho| > 4\epsilon$.

Now, note that if $\mathcal{E}_{K'} \cap |\langle M_j, v_j \rangle| \leq \epsilon$ occurs, then

$$\|M^T v_j\|_2 = |\langle M_j, v_j \rangle| \leq \epsilon$$

which implies that $v_j = x_j + iy_j$ satisfies $\|y_j\|_2 \geq |\rho|/(16\delta K' + 16)$. In fact, by rotating v_j by a phase in the above argument, we may conclude that

$$\|\Re(\omega v_j)\|_2 \geq \frac{|\rho|}{16\delta K' + 16}$$

for all $\omega \in \mathbb{C}$ satisfying $|\omega| = 1$, where $\Re(\cdot)$ denotes the real part of a vector $v = x + iy \in \mathbb{R}^n + i\mathbb{R}^n$.

Next, by Lemma 6.3 in the case $\ell = 2$, applied to $V = [x_j, y_j]^T$, we have

$$\mathbb{P}[\mathcal{E}_{K'} \cap |\langle M_j, v_j \rangle| \leq \epsilon |M_{-j}|] \leq \frac{en\delta^{-2}K^2}{\det(VV^T)^{1/2}} \cdot \pi\epsilon^2.$$

This follows since $\langle M_j, v_j \rangle$ (viewed as an \mathbb{R}^2 -valued random variable) is distributed as a deterministic translation of the \mathbb{R}^2 -valued random variable $V(\delta(G_n)_j)$, and $\delta(G_n)_j$ is a random vector, each of whose components is a real random variable with density bounded by $Kn\delta^{-1}$.

It remains to estimate $d := \det(VV^T)^{1/2}$. By [7], Proposition B.0.1, we have

$$\min_{\theta \in \mathbb{R}} \|\Re(e^{i\theta} v_j)\|_2^2 = \frac{1}{2} - \frac{\sqrt{1 - 4d^2}}{2} \leq 2d^2,$$

hence

$$d \geq \frac{|\rho|}{16\sqrt{2}(\delta K' + 1)}$$

which completes the proof. \square

PROOF OF PROPOSITION 4.2. Lemmas 6.1 and 6.5 give

$$\mathbb{P}[\mathcal{E}_{K'} \cap \sigma_n(M) \leq \epsilon] \leq n \left(c_{6.5}(1 + \delta K') \frac{\max(K, K^2)n(\epsilon n^{1/2})^2}{\delta^2 |\text{Im } z|} \right). \quad \square$$

7. Proof of Proposition 4.3. The proof of Proposition 4.3 follows a similar general outline, as in [7].

We first require a statement relating the property that two eigenvalues are contained within a disk to bounds on the two smallest singular values of an appropriately shifted matrix. This follows immediately from the log-majorization theorem which implies that $\sigma_n \sigma_{n-1} \leq |\lambda_n \lambda_{n-1}|$ [9]. This also follows quickly from [12], Lemma A.1, A.2.

LEMMA 7.1 (From [9], Theorem 3.3.2). *Fix $N \in \mathbb{C}^{n \times n}$ and $z \in \mathbb{C}$ with $|z| \leq \|N\|$. Let $M = N - zI$ and suppose there exist $\lambda_i, \lambda_j \in \mathcal{D}(z, s)$ for $i \neq j$. Then, there exist $s \leq t \leq 2\|N\|$ such that*

$$\sigma_n(M) \leq s^2/t \quad \text{and} \quad \sigma_{n-1}(M) \leq 2t.$$

We now prove a variant of Proposition 4.2 for the smallest two singular values.

LEMMA 7.2. *Let $A \in \mathbb{R}^{n \times n}$ with $\|A\| \leq 1$, and let $\delta \in (0, 1)$. Let $z \in \mathbb{C}$ with $|z| \leq 3\delta K' + 3$ and $M = A + \delta G_n - zI$. Let $\mathcal{E}_{K'}$ denote the event that $\|G_n\| \leq K'$. Then,*

$$\mathbb{P}[\sigma_n(M) < \epsilon_1 \cap \sigma_{n-1}(M) \leq \epsilon_2 \cap \mathcal{E}_{K'}] \leq c_{7.2}(1 + \delta K')^2 \max(K^2, K^4)n^6 \frac{\epsilon_1^2 \epsilon_2^2}{\delta^4 (\text{Im } z)^2},$$

where $c_{7.2}$ is an absolute constant.

PROOF. Let G_j the principal minor of G , excluding the j th row and column, and let $\mathcal{E}_{K'}^j$ denote the event that $\|G_j\| \leq K'$. Then, by Lemma 6.2 we have that

$$\begin{aligned} &\mathbb{P}[\sigma_n(M) < \epsilon_1 \cap \sigma_{n-1}(M) \leq \epsilon_2 \cap \mathcal{E}_{K'}] \\ &\leq \sum_{i=1}^n \sum_{j \neq i} \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon_1 n^{1/2} \cap \text{dist}(M_j, H_{i,j}) \leq \epsilon_2 n^{1/2} \cap \mathcal{E}_{K'}^j] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^n \sum_{j \neq i} \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon_1 n^{1/2} \cap \text{dist}(M_j, H_{i,j}) \leq \epsilon_2 n^{1/2} \cap \mathcal{E}_{K'} \cap \mathcal{E}_{K'}^i] \\
 &= \sum_{i=1}^n \sum_{j \neq i} \mathbb{E}[\mathbb{P}[\text{dist}(M_j, H_{i,j}) \leq \epsilon_2 n^{1/2} \cap \mathcal{E}_{K'}^i | M_{-[i,j]}]] \\
 &\quad \cdot \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon_1 n^{1/2} \cap \mathcal{E}_{K'} | M_{-[i,j]} \cap \text{dist}(M_j, H_{i,j}) \leq \epsilon_2 n^{1/2} \cap \mathcal{E}_{K'}^i]] \\
 &\leq \sum_{i=1}^n \sum_{j \neq i} \sup_{M_{-[i,j]}} \mathbb{P}[\text{dist}(M_j, H_{i,j}) \leq \epsilon_2 n^{1/2} \cap \mathcal{E}_{K'}^i | M_{-[i,j]}] \\
 &\quad \cdot \sup_{M_{-i}} \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon_1 n^{1/2} \cap \mathcal{E}_{K'} | M_{-i}].
 \end{aligned}$$

Note that the second term in the product is controlled by Lemma 6.5, using that $\text{dist}(M_i, H_i) \geq |\langle M_i, v_i \rangle|$, where v_i is a unit normal to H_i .

The first term can be controlled by noting that

$$\text{dist}(M_j, H_{i,j}) \geq |\langle M_j, v_{i,j} \rangle|,$$

where $v_{i,j}$ is a unit normal to $H_{i,j}$ with i th coordinate 0 and then applying Lemma 6.5 to the $(n - 1) \times (n - 1)$ minor of M formed by excluding the i th row and column (we must replace δ by $\delta(1 - n^{-1})^{1/2}$ in this application).

To summarize, the last term in the chain of inequalities above can be bounded by

$$\begin{aligned}
 &n(n - 1) \left(c_{6.5}(1 + \delta K') \frac{\max(K, K^2)n(\epsilon_1 n^{1/2})^2}{\delta^2 |\text{Im } z|} \right) \left(c_{6.5}(1 + \delta K') \frac{\max(K, K^2)n(\epsilon_2 n^{1/2})^2}{\delta^2(1 - n^{-1})|\text{Im } z|} \right) \\
 &= c_{6.5}^2(1 + \delta K')^2 \max(K^2, K^4)n^6 \frac{\epsilon_1^2 \epsilon_2^2}{\delta^4 (\text{Im } z)^2}
 \end{aligned}$$

which completes the proof. \square

Finally, we state a variant of Proposition 4.1 for the smallest two singular values, whose proof follows in an identical manner to the proof of Lemma 7.2 with Proposition 4.1 replacing the application of Proposition 4.2. We omit further details.

LEMMA 7.3. *For any $A \in \mathbb{C}^{n \times n}$, we have*

$$\mathbb{P}[\sigma_n(A + \delta G_n) < \epsilon_1 \cap \sigma_{n-1}(A + \delta G_n) \leq \epsilon_2] \leq c_{7.3} K^2 n^4 \frac{\epsilon_1 \epsilon_2}{\delta^2}.$$

Now, we are ready to complete the proof of our eigenvalue spacing result.

PROOF OF PROPOSITION 4.3. Let $M = A + \delta G_n$. We wish to bound the probability that both the events $\mathcal{E}_{K'} = \{\|G_n\| \leq K'\}$ and $\eta(M) \leq s$ occur. Note that, when this happens, the eigenvalues $\lambda_1, \dots, \lambda_n$ of M are contained in $\mathcal{D}(0, 1 + \delta K')$, and there is some pair λ_i, λ_j (with $i \neq j$) within a distance s of each other.

Fix some height $H \geq s$ to be chosen later, and cover the disk $\mathcal{D} = \mathcal{D}(0, 1 + \delta K')$ by regions

$$\mathcal{R}_j = \mathcal{D} \cap \{\text{Im } z \in [(2j - 1)H, (2j + 1)H]\}$$

for all integers j with $|j| \leq (1 + \delta K')/(2H)$. Then, we cover each \mathcal{R}_j by $O(H(1 + \delta K')s^{-2})$ disks of radius s , which we denote by $\mathcal{D}(z_\alpha, s)$ for $\alpha \in A_j$. Thus, $|A_j| \lesssim H(1 + \delta K')s^{-2}$. Since λ_i is in one of these disks, it follows that there exists a pair $i \neq j$ such that $\lambda_i, \lambda_j \in \mathcal{D}(z_\alpha, 2s)$ for some j and some $\alpha \in A_j$.

From this and by Lemma 7.1, we have

$$\begin{aligned} &\mathbb{P}[\mathcal{E}_{K'} \cap \eta(M) \leq s] \\ &\leq \sum_j \sum_{\alpha \in A_j} \mathbb{P}[\exists \lambda_i, \lambda_j \in \mathcal{D}(z_\alpha, 2s)] \\ &\leq \sum_j \sum_{\alpha \in A_j} \mathbb{P}[\exists t \in [s, 2\|M\|] : \sigma_n(M - z_\alpha I) \leq s^2/t \cap \sigma_{n-1}(M - z_\alpha I) \leq 2t] \\ &\leq \sum_j \sum_{\alpha \in A_j} \sum_{\ell=0}^{\lfloor \log_2(2(1+\delta K')/s) \rfloor} \mathbb{P}[\sigma_n(M - z_\alpha I) \leq 2^{-\ell}s \cap \sigma_{n-1}(M - z_\alpha I) \leq 2^{\ell+2}s], \end{aligned}$$

where the last line represents a dyadic chop on the possible values of t , restricting to $t = 2^\ell s$ over the stated range of ℓ . Finally, we apply Lemma 7.2 and Lemma 7.3, with the latter used only in \mathcal{R}_0 (i.e., for $j = 0$).

For $j \neq 0$ and $\alpha \in A_j$, we have $|\text{Im } z_\alpha| \geq |j|H$. Hence, by Lemma 7.2,

$$\begin{aligned} &\mathbb{P}[\sigma_n(M - z_\alpha I) \leq 2^{-\ell}s \cap \sigma_{n-1}(M - z_\alpha I) \leq 2^{\ell+2}s] \\ &\leq c_{7.2}(1 + \delta K')^2 \max(K^2, K^4)n^6 \frac{16s^4}{\delta^4 j^2 H^2}. \end{aligned}$$

For $\alpha \in A_0$, we have by Lemma 7.3 that

$$\mathbb{P}[\sigma_n(M - z_\alpha I) \leq 2^{-\ell}s \cap \sigma_{n-1}(M - z_\alpha I) \leq 2^{\ell+2}s] \leq c_{7.3}K^2n^4 \frac{4s^2}{\delta^2}.$$

Putting this together, we obtain

$$\begin{aligned} &\mathbb{P}[\mathcal{E}_{K'} \cap \eta(M) \leq s] \\ &\leq \sum_j \sum_{\alpha \in A_j} \sum_{\ell=0}^{\lfloor \log_2(2(1+\delta K')/s) \rfloor} \mathbb{P}[\sigma_n(M) \leq 2^{-\ell}s \cap \sigma_{n-1}(M) \leq 2^{\ell+2}s] \\ &\lesssim \log_2(2(1 + \delta K')/s) \left(H(1 + \delta K')s^{-2} \cdot \left(K^2n^4 \frac{s^2}{\delta^2} \right) \right. \\ &\quad \left. + \sum_{j \neq 0} \left[H(1 + \delta K')s^{-2} \cdot \left((1 + \delta K')^2 \max(K^2, K^4)n^6 \frac{s^4}{\delta^4 j^2 H^2} \right) \right] \right) \\ &\lesssim \log_2(2(1 + \delta K')/s) \left((1 + \delta K')^3 \max(K^2, K^4)n^6 \frac{s^2}{\delta^4 H} + (1 + \delta K')K^2n^4 \frac{H}{\delta^2} \right), \end{aligned}$$

where \lesssim indicates a suppressed absolute constant.

Finally, we choose

$$H = (1 + \delta K') \max(1, K) \frac{sn}{\delta} \geq s$$

to find that

$$\mathbb{P}[\mathcal{E}_{K'} \cap \eta(M) \leq s] \lesssim \log_2(2(1 + \delta K')/s) \cdot (1 + \delta K')^2 \max(K^2, K^3)n^5 \frac{s}{\delta^3}. \quad \square$$

APPENDIX A: PROOF OF LEMMA 6.2

We begin with a preliminary lemma.

LEMMA A.1. *Let M be any $n \times n$ random matrix and fix any $\epsilon \geq 0$. Fix an index $\ell \in [n]$. Let M_i, H_i be as in Lemma 6.1, and let $H_{i,j} = \text{span}\{M_k : k \neq i, j\}$. Then, for any event \mathcal{E} ,*

$$\mathbb{P}[\sigma_{n-1}(M) \leq \epsilon \cap \mathcal{E}] \leq \sum_{j \in [n] \setminus \ell} \mathbb{P}[\text{dist}(M_j, H_{j,\ell}) \leq \epsilon n^{1/2} \cap \mathcal{E}].$$

PROOF. If $\sigma_{n-1}(M) \leq \epsilon$, then there is a subspace W of dimension 2 satisfying $\|M|_W\| \leq \epsilon$. Since W has dimension 2, there must exist some vector $w \in W$ with $w_\ell = 0$ and $\|w\|_2 = 1$. In particular, there exists some $j \neq \ell$ such that $|w_j| \geq n^{-1/2}$. Then,

$$\begin{aligned} \epsilon &\geq \|Mw\|_2 \geq \text{dist}(Mw, H_{j,\ell}) = \text{dist}(M_j w_j + M_\ell w_\ell, H_{j,\ell}) \\ &= \text{dist}(M_j w_j, H_{j,\ell}) \geq n^{-1/2} \text{dist}(M_j, H_{j,\ell}). \end{aligned}$$

We have shown that if the event $\sigma_{n-1}(M) \leq \epsilon \cap \mathcal{E}$ occurs, then, at least, one of the events $\text{dist}(M_j, H_{j,\ell}) \leq \epsilon n^{1/2} \cap \mathcal{E}$ occurs for $j \neq \ell$. This clearly implies the desired result. \square

Now, we are ready to deduce Lemma 6.2.

PROOF OF LEMMA 6.2. We have

$$\begin{aligned} &\mathbb{P}[\sigma_n(M) \leq \epsilon_1 \cap \sigma_{n-1}(M) \leq \epsilon_2 \cap \mathcal{E}] \\ &\leq \sum_{i=1}^n \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon_1 n^{1/2} \cap \sigma_{n-1}(M) \leq \epsilon_2 \cap \mathcal{E}] \\ &\leq \sum_{i=1}^n \sum_{j \neq i} \mathbb{P}[\text{dist}(M_i, H_i) \leq \epsilon_1 n^{1/2} \cap \text{dist}(M_j, H_{i,j}) \leq \epsilon_2 n^{1/2} \cap \mathcal{E}]; \end{aligned}$$

the first inequality follows by applying Lemma 6.1 with $\epsilon = \epsilon_1$ and \mathcal{E} replaced by $\sigma_{n-1}(M) \leq \epsilon_2 \cap \mathcal{E}$; the second inequality follows by applying Lemma A.1 with $\epsilon = \epsilon_2$, $\ell = i$, and \mathcal{E} replaced by $\text{dist}(M_i, H_i) \leq \epsilon_1 n^{1/2} \cap \mathcal{E}$. \square

APPENDIX B: EIGENVALUE SPACING FOR COMPLEX RANDOM VARIABLES

In this appendix we sketch a series of estimates regarding the eigenvalue spacing for complex perturbations. We first state the analog of Lemma 7.2 in the complex case. Throughout, $G_n = G_n(\xi)$, where ξ is any sub-Gaussian complex random variable with bounded density, as in Section 3.

LEMMA B.1. *Let $A \in \mathbb{C}^{n \times n}$ and let $M = A + \delta G_n$. Then,*

$$\mathbb{P}[\sigma_n(M) \leq \epsilon_1 \cap \sigma_{n-1}(M) \leq \epsilon_2] \leq c_{B.1} K^2 n^6 \frac{\epsilon_1^2 \epsilon_2^2}{\delta^4},$$

where $c_{B.1}$ is an absolute constant.

SKETCH. This essentially identical to the proof of Lemma 7.2 with Lemma 6.5 replaced by the estimate obtained in Lemma 3.4. \square

THEOREM B.2. *Let $A \in \mathbb{C}^{n \times n}$ with $\|A\| \leq 1$, let $\delta \in (0, 1)$ and let $M = A + \delta G_n$. Let $\mathcal{E}_{K'}$ denote the event that $\|G_n\| \leq K'$. Then, for $s \leq 1$,*

$$\mathbb{P}[\mathcal{E}_{K'} \cap \eta(M) \leq s] \leq c_{B,2} \log_2(2(1 + \delta K')/s) \cdot (1 + \delta K')^2 K^4 n^6 \frac{s^2}{\delta^4},$$

where $c_{B,2} > 0$ is an absolute constant.

SKETCH. This follows by taking an s -net of $\mathcal{D}(0, 1 + \delta K')$ and then, following the proof of Proposition 4.3, using Lemma B.1 instead of Lemma 7.2. \square

Using the above theorem, one can obtain a high-probability spacing between the eigenvalues of $A + \delta G_n(\xi)$ on the order of $\delta^2 n^{-3} / \log(n\delta^{-1})$; for δ sufficiently small, this is sharper than the spacing of order $(\delta/n)^{8/3}$ obtained for $A + \delta G_n(\mathcal{N}_{\mathbb{C}}(0, 1))$ by Banks, Vargas, Kulkarni, and Srivastava [2]. It is quite likely that in the complex Gaussian case, one can obtain a version of Lemma B.1, which has a better dependence on n , and, thereby, obtain an eigenvalue spacing estimate of the order of $(\delta/n)^2 / \log(n\delta^{-1})$ which represents a strict improvement over the estimate in [2].

APPENDIX C: DETAILS FOR PROPOSITION 3.1

Here, we show how to use Lemmas 3.2 and 3.5 to complete the proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. Let $M = A + \delta G_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. By Lemma 3.2

$$\mathbb{P}[\sigma_n(G_n) \leq K'] \geq \frac{3}{4},$$

where K' depends only on the sub-Gaussian moment of ξ . Let $\mathcal{E}_{K'}$ denote the event that $\sigma_n(G_n) \leq K'$. Then, $\mathbb{P}[\mathcal{E}_{K'}] \geq 3/4$.

Furthermore, using Lemma 3.5 for $\mathcal{B} = \mathcal{D}(0, 1 + \delta K')$, we have that

$$\mathbb{E}\left[\sum_{\lambda_i \in \mathcal{B}} \kappa(\lambda_i)^2\right] \leq \frac{\pi e K n^3 (1 + \delta K')^2}{\delta^2}.$$

Recall that K here is an upper bound on the density ξ , which we have assumed.

Under the event $\mathcal{E}_{K'}$ we also have that

$$\sum_{\lambda_i \in \mathcal{B}} \kappa(\lambda_i)^2 = \sum_{i=1}^n \kappa(\lambda_i)^2 = \kappa_2^2,$$

and, therefore, through simple conditioning we obtain that

$$\mathbb{E}[\kappa_2^2 | \mathcal{E}_{K'}] \leq \frac{4\pi e K n^3 (1 + \delta K')^2}{3\delta^2}.$$

Hence, using Markov's inequality, we have that

$$\mathbb{P}\left[\kappa_2^2 \leq \frac{4\pi e K n^3 (1 + \delta K')^2}{\delta^2} | \mathcal{E}_{K'}\right] \geq \frac{2}{3}$$

which implies that

$$\mathbb{P}\left[\kappa_2^2 \leq \frac{4\pi e K n^3 (1 + \delta K')^2}{\delta^2} \cap \mathcal{E}_{K'}\right] \geq \frac{1}{2}.$$

The result now follows immediately from Lemma 2.1. \square

Acknowledgments. V.J. would like to thank Archit Kulkarni and Nikhil Srivastava for introducing him to the problem. The authors would like to thank an anonymous referee for their careful reading of the manuscript and valuable comments.

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